DIRECT TENSOR EXPRESSION BY EULERIAN APPROACH FOR CONSTITUTIVE RELATIONS BASED ON STRAIN INVARIANTS IN TRANSVERSELY ISOTROPIC GREEN ELASTICITY: FINITE EXTENSION AND TORSION

A Thesis

by

MIN JAE SONG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

December 2006

Major Subject: Mechanical Engineering

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James E.Moore Jr.

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ABSTRACT

Direct Tensor Expression by Eulerian Approach for Constitutive Relations based on Strain Invariants in Transversely Isotropic Green Elasticity: Finite Extension and Torsion. (December 2006)

Min Jae Song, B.S. Hanyang University

Chair of Advisory Committee: Dr. K.R.Rajagopal

It has been proven by J.C.Criscione that constitutive relations(mixed approach) based on a set of five strain invariants (β_1, β_2, β_3,β_4, β_5) are useful and stable for experimentally determining response terms for transversely isotropic material. On the other hand, Rivlin's classical model is an unsuitable choice for determining response terms due to the co-alignment of the five invariants (I_1, I_2, I_3, I_4, I_5). Despite this, however, a mixed (Lagrangian and Eulerian) approach causes unnecessary computational time and requires intricate calculation in the constitutive relation. Through changing the way to approach the derivation of a constitutive relation, we have verified that using an Eulerian approach causes shorter computational time and simpler calculation than using a mixed approach does. We applied this approach to a boundary value problem under specific deformation, i.e. finite extension and torsion to a fiber reinforced circular cylinder. The results under this deformation show that the computational time by Eulerian is less than half of the time by mixed. The main reason for the difference is that we have to determine two unit vectors on the cross fiber direction from the right Cauchy Green deformation tensor at every radius of the cylinder when we use a mixed approach. On the contrary, we directly use the left Cauchy Green deformation tensor in the constitutive relation by the Eulerian approach without defining the two cross fiber vectors. Moreover, the computational time by the Eulerian approach is

not influenced by the degree of deformation even in the case of computational time by the Eulerian approach, possibly becoming the same as the computational time by the mixed approach. This is from the theoretical thought that the mixed approach is almost the same as the Eulerian approach under small deformation. This new constitutive relation by Eulerian approach will have more advantages with regard to saving computational time as the deformation gets more complicated. Therefore, since the Eulerain approach effectively shortens computational time, this may enhance the computational tools required to approach the problems with greater degrees of anisotropy and viscoelasticity.

To My parents, J.G.SONG and S.A.JUNG, and my wife, JIYEON YANG

ACKNOWLEDGMENTS

I am deeply grateful to my advisor, Professor K.R.Rajagopal, for the guidance and invariable help I received during my master's thesis research. I wish to express my warm and sincere thanks to Professor J.C.Criscione for his detailed and constructive comments, and for his support throughout this work. Also I would like to thank Professor James E. Moore for agreeing to serve on my committee. Finally, I thank all my colleagues who helped and encouraged me to research this work.

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CHAPTER I

INTRODUCTION

Many materials which exhibit isotropic or anisotropic behavior have been studied with strain energy function based on invariants, which are kinematic scalars that are invariant under permissible symmetry transformation, in Green(hyper) Elasticity. Through many years, I_1, $I_2 \ldots$ of invariants have been chosen, but it was shown by J.C.Criscione that these invariants were unsuitable choices for experimental analysis due to co-alignment. Though these do not seem to be theoretically inaccurate, a small experimental error causes significant error in response functions, and hence we can say the invariants were poorly chosen for experimental determination of constitutive relation.Before considering anisotropy, it is helpful to consider isotropy because it is a basic way to understand material symmetry.

A. Isotropy

Isotropic material has the same responses for all direction at a point belonging to the body. In isotropic material, the functional form of strain energy is based on 3 invariants, which is calculated from characteristic equation for Right Cauchy Green deformation tensor($\mathbf{C}$). Corresponding to these three invariants ($I_1 \equiv tr(\mathbf{C})$, $I_2 \equiv \frac{1}{2}((tr(\mathbf{C}))^2 - tr(\mathbf{C}^2))$, $I_3 \equiv det(\mathbf{C})$), constitutive relation was modeled as[11]

$$\mathbf{T} = \frac{2}{I_3^{\frac{1}{2}}}\Big((\frac{\partial \mathbf{W}}{\partial I_1} + I_1 \frac{\partial \mathbf{W}}{\partial I_2})\mathbf{B} - \frac{\partial \mathbf{W}}{\partial I_2}\mathbf{B}^2 + I_3 \frac{\partial \mathbf{W}}{\partial I_3}\mathbf{I}\Big) \tag{1.1}$$

The journal model is *IEEE Transactions on Automatic Control.*

For incompressible material,

$$\mathbf{T} = \frac{2}{I_3^{\frac{1}{2}}}\Big((\frac{\partial \mathbf{W}}{\partial I_1} + I_1\frac{\partial \mathbf{W}}{\partial I_2})\mathbf{B} - \frac{\partial \mathbf{W}}{\partial I_2}\mathbf{B}^2\Big) - p\mathbf{I} \tag{1.2}$$

Finite deformation of isotropic materials was studied by R.S.Rivlin and D.W.Saunders[3]. They showed a way to find response functions($\frac{\partial W}{\partial I_1}$,$\frac{\partial W}{\partial I_2}$)[1] by biaxial testing of highly incompressible elastic materials. The choice of invariants, however, is suitable for only rubber like materials where I_1, I_2> 5 and then only for shearing type deformations which are not axially symmetric.[5] Because of co-alignment of invariants ,the response terms are largely perturbed by a small experimental measurement error. In [5], in order to show co-alignment of invariants, "covariance" was defined between two tensors such that,

$$R_C(\mathbf{A}_1, \mathbf{A}_2) = \frac{abs(tr(\mathbf{A}_1^T\mathbf{A}_2))}{|\mathbf{A}_1||\mathbf{A}_2|} \tag{1.3}$$

The range of this covariance is from zero to one, and two tensors are said to be co-linear or orthogonal, if covariance is one or zero. By this covariance, it has been shown that error is magnified by the meaningful factor$(1 - R_C(dev(\mathbf{B}), dev(\mathbf{B}^{-1}))^2)^{-1/2}$[2]. Thus, if covariance is one, error will go to infinity. In fact, from[5], $R_C(dev(\mathbf{B}), dev(\mathbf{B}^{-1}))$ for biaxial stretch is very high $(0.6 \sim 1.0)$, so small experimental error causes significant error for response terms.

To solve the problem in co-alignment, the idea of Hencky(Natural) Strain(ln($\mathbf{V}$)) was used and choice of three invariants($K_1 \equiv tr(\eta)$,$K_2 \equiv |dev(\eta)|$,and $K_3 \equiv 3\sqrt{6}det(\Phi)$)[3]

[1]W is strain energy function in terms of I_1 and I_2.

[2]$dev(\mathbf{B}) = \mathbf{B} - \frac{1}{3}tr(\mathbf{B})\mathbf{I}$

[3]$\eta = ln(\mathbf{V})$ and $\Phi = \frac{dev(\eta)}{K_2}$

proved mutual orthogonality.[1] And constitutive relation was modeled as [1],[6]

$$\mathbf{T} = \frac{1}{J}\frac{\partial \mathbf{W}}{\partial K_1}\mathbf{I} + \frac{1}{J}\frac{\partial \mathbf{W}}{\partial K_2}\mathbf{udev}(\ln \mathbf{V}) - \frac{3}{J}\frac{\partial \mathbf{W}}{\partial K_3}\frac{\sqrt{1-K_3^2}}{K_2}\mathbf{cudev}(\ln \mathbf{V})^4 \tag{1.4}$$

Moreover, direct tensor expression for natural strain showed fast and accurate approach to exact natural strain for many tests.[6]

B. Transverse Isotropy

In a study for transversely isotropic behavior which is the simplest anisotropic behavior, and for transversely isotropic material, which has preferred direction and responses that are independent on rotation by preferred direction[5], Spencer showed that strain energy depends on preferred fiber direction in reference configuration and right Cauchy Green deformation tensor($\mathbf{C}$).[2] Thus, strain energy function was defined as W(I_1,I_2,...,I_5) corresponding to five invariants [6]. Moreover, constitutive relation was determined as

$$\mathbf{T} = \frac{1}{J}\sum_{i=1}^{5}\frac{\partial \mathbf{W}}{\partial I_i}\tilde{\mathbf{A}}_i, \quad \mathbf{T} = -p\mathbf{I} + \sum_{i=2}^{5}\frac{\partial \mathbf{W}}{\partial I_i}\tilde{\mathbf{A}}_i \tag{1.5}$$

4

$$\mathbf{udev}(\mathbf{A}) = (\sqrt{\mathbf{dev}(A) : \mathbf{dev}(A)})^{-1}\mathbf{dev}(A)$$
$$\mathbf{cudev}(A) = \frac{\sqrt{6}\mathbf{I} + 9\sqrt{6}det(\mathbf{udev}(\mathbf{A}))\mathbf{udev}(\mathbf{A}) - 3\sqrt{6}\mathbf{udev}^2(\mathbf{A})}{3\sqrt{1-54(det(\mathbf{udev}(\mathbf{A})))^2}}$$

5 This explanation, in fact, is for rotationally transversely isotropic material. The meaning of general transversely isotropic material is that, in addition to rotational transverse isotropy, if body is changed as the front parts to the rear parts along the preferred direction, the body still has same responses which are independent on rotation by the preferred axis.(i.e., reflection with mirror plane normal to axis.)

6 $I_1 \equiv tr(\mathbf{C})$, $I_2 \equiv \frac{1}{2}((tr(\mathbf{C}))^2 - tr(\mathbf{C}^2))$, $I_3 \equiv det(\mathbf{C})$, $I_4 \equiv M \cdot \mathbf{C}M$, and $I_5 \equiv M \cdot \mathbf{C}^2 M$

$$
\begin{aligned}
\tilde{\mathbf{A}}_1 &= 2\mathbf{B} \\
\tilde{\mathbf{A}}_2 &= 2(I_1\mathbf{B} - \mathbf{B}^2) \\
\tilde{\mathbf{A}}_3 &= 2I_3\mathbf{I} \\
\tilde{\mathbf{A}}_4 &= 2I_4\mathbf{m} \otimes \mathbf{m} \\
\tilde{\mathbf{A}}_5 &= 2I_4(\mathbf{Bm} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{Bm})
\end{aligned} \tag{1.6}
$$

If we put $\tilde{\mathbf{A}}_i$ to both right and left sides in this constitutive relation, and then we obtain matrix system such as

$$[tr(\mathbf{T}^T\tilde{\mathbf{A}}_i)]_{5\times1} = [tr(\tilde{\mathbf{A}}_j^T\tilde{\mathbf{A}}_i)]_{5\times5}[\frac{\partial\mathbf{W}}{\partial I_j}]_{5\times1} \tag{1.7}$$

Because this system has high condition number[7], small measurement error in stress will make huge error of response terms. Furthermore, in order to determine functional form of strain energy by test, we fix four of five invariants, but even simple uniaxial test perturbs four of five invariants. In order to solve the problem in co-alignment of invariants[5], the strain energy function was redetermined in chapter 3.C based on five strain invariants in chapter 3.B, which are first one is related to incompressibility, the second one is fiber stretch, the third one is stretch on cross fiber directions, the fourth one is shear along fiber direction, and the fifth one is relation between cross fiber stretch and along fiber shear. And based on these invariants, functional form of strain energy was determined as polynomial form (3.21). From constitutive relation(3.31), $\mathbf{A}_i$ have been verified to be orthogonal to each other, and using these five invariants have been shown to have an advantage of experimental relevance whereas W, which is function of I_1,I_2,...,I_5, doesn't.[4]

[7]Condition number is to provide sensitivity of perturbation of a matrix system, which is expressed as $k = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$

C. Orthotropy

Orthotropic material is macroscopically considered as having a large number of laminae which are stacked with the fiber aligned in two different directions, that is, a laminate has two preferred directions.($\mathbf{M}_1$,$\mathbf{M}_2$)[2] Strain invariants are based on seven invariants such as $\mathbf{W} = \mathbf{W}(I_1, I_2, I_3, I_4, I_5, I_6, I_7)$[8]. Based on these, constitutive relation was determined as[8]

$$\mathbf{T} = \frac{1}{J}\sum_{i=1}^{7}\frac{\partial \mathbf{W}}{\partial I_i}\hat{\mathbf{A}}_i, \quad \mathbf{T} = -p\mathbf{I} + \sum_{i=2}^{7}\frac{\partial \mathbf{W}}{\partial I_i}\hat{\mathbf{A}}_i \tag{1.8}$$

$$\begin{aligned} &\hat{\mathbf{A}}_1 = 2\mathbf{B}, \quad \hat{\mathbf{A}}_2 = 2(I_1\mathbf{B} - \mathbf{B}^2), \quad \hat{\mathbf{A}}_3 = 2I_3\mathbf{I} \\ &\hat{\mathbf{A}}_4 = 2I_4\mathbf{m}_1 \otimes \mathbf{m}_1, \quad \hat{\mathbf{A}}_5 = 2I_4(\mathbf{B}\mathbf{m}_1 \otimes \mathbf{m}_1 + \mathbf{m}_1 \otimes \mathbf{B}\mathbf{m}_1) \\ &\hat{\mathbf{A}}_6 = 2I_6\mathbf{m}_2 \otimes \mathbf{m}_2, \quad \hat{\mathbf{A}}_7 = 2I_6(\mathbf{B}\mathbf{m}_2 \otimes \mathbf{m}_2 + \mathbf{m}_2 \otimes \mathbf{B}\mathbf{m}_2) \end{aligned} \tag{1.9}$$

In this case, similarly because of co-alignments of invariants, significant error of response terms occurs for experimental analysis. In order to minimize "covariance"(co-alignment), six strain invariants were introduced as

$$\begin{aligned} &\alpha_1 = \ln J, \quad \alpha_2 = \ln(\lambda_M^{\frac{3}{2}}), \quad \alpha_3 = \ln(\zeta^2) \\ &\alpha_4 = \phi_{MS}, \quad \alpha_5 = \phi_{MN}, \quad \alpha_6 = \phi_{SN} \\ &\mathbf{W} = \mathbf{W}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \end{aligned} \tag{1.10}$$

$\alpha_{1,2,3}$ are related to normal strain and $\alpha_{4,5,6}$ are related to shear strain. Also constitutive relation is

$$\mathbf{T} = \frac{1}{J}\sum_{i=1}^{6}\frac{\partial \mathbf{W}}{\partial \alpha_i}\check{\mathbf{A}}_i, \quad \mathbf{T} = -p\mathbf{I} + \sum_{i=2}^{6}\frac{\partial \mathbf{W}}{\partial \alpha_i}\check{\mathbf{A}}_i \tag{1.11}$$

$$\check{\mathbf{A}}_1 = \mathbf{I}, \quad \check{\mathbf{A}}_2 = \mathbf{m} \otimes \mathbf{m} - \frac{1}{2}(\mathbf{n} \otimes \mathbf{n} + \mathbf{s} \otimes \mathbf{s}) \tag{1.12}$$

---

[8] $I_6 = \mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2$, $I_7 = \mathbf{M}_2 \cdot \mathbf{C}^2\mathbf{M}_2$

$$\check{\mathbf{A}}_3 = \mathbf{n} \otimes \mathbf{n} + \mathbf{s} \otimes \mathbf{s}, \quad \check{\mathbf{A}}_4 = \lambda_M^{-\frac{3}{2}} \zeta^{-1} (\mathbf{m} \otimes \mathbf{s} + \mathbf{s} \otimes \mathbf{m}) \tag{1.13}$$

$$\check{\mathbf{A}}_5 = \lambda_M^{-\frac{3}{2}} (\alpha_6 \zeta^{-1} (\mathbf{m} \otimes \mathbf{s} + \mathbf{s} \otimes \mathbf{m}) + \zeta (\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m})) \tag{1.14}$$

$$\check{\mathbf{A}}_6 = \zeta^2 (\mathbf{s} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{s}) \tag{1.15}$$

For orthonormal vectors in reference configuration, $\mathbf{M}$ is on fiber direction, $\mathbf{N}$ is normal to the laminate, and $\mathbf{S}$ is orthogonal to $\mathbf{M}$ in the laminate plane. Corresponding to these three vectors, $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{s}$ are orthonormal vectors in current configuration. This have been shown that $\check{\mathbf{A}}_i$ are mutually orthogonal, so it is better choice than (1.9) for experimentally determining response terms.

D. Purpose of this study

In these types of material symmetry, we focus on transverse isotropy because, as the case of isotropy, for transversely isotropic behavior, constitutive relation need to be changed to constitutive relation by Eulerian approach. Of course, the reformulated constitutive relation(3.31) of x-iso behavior with strain invariants was well posed , but, due to this becoming complicated for inhomogeneous deformation such as torsion, we expect changing constitutive relation by Eulerian approach will be much better than using(3.31). Thus, first purpose of this study is to find constitutive relation by Eulerian approach, which is calculated from the constitutive relation by mixed(Lagrangian-Eulerian)(3.31) approach, for transversely isotropic behavior based on five strain invariants set. The second purpose is to compare computational time(Matlab) for using both constitutive relations under finite extension and torsion to fiber reinforced circular cylinder. This new method of approach will possibly cause shortening in computational time.

CHAPTER II

CONTINUUM MECHANICS

A. Basic Kinematics

A body is said to be a continuum if every particle belonging to the body(the set of particles of interest) has a neighborhood such that all the particles in the neighborhood belonging to the body. By an abstract body($\mathbf{B}$), placer κ maps to three dimensional Euclidean space as one to one. The $\kappa_R(\mathbf{B})$ indicates referential configuration, which is usually at time($t = 0$), and the $\kappa_t(\mathbf{B})$ indicates current configuration, which is usually at time($t = t_1$).

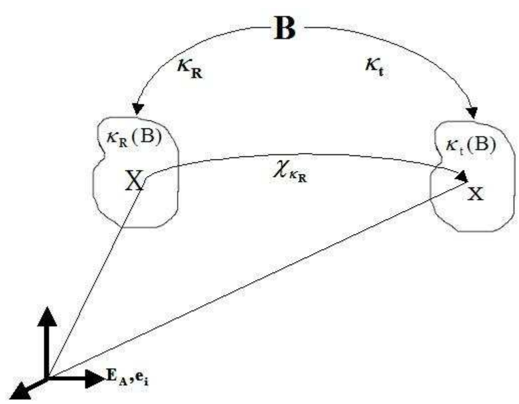

Fig. 1. Schematic illustration of reference configuration and current configuration

Since the placers are one to one, this mapping can be defined such that a position X in the reference configuration($\kappa_R(\mathbf{B})$) is mapped to position x in current configuration($\kappa_t(\mathbf{B})$), and this mapping also can be called motion of the body.
In the Fig.1, X serves as indicating which particle of $\mathbf{B}$, which means attention is concentrated on what is happening at a particle and in a neighborhood, and x serves as identifying the location which is occupied by different particle in time, which means we are interested in what is happening in location. Thus, in the former case, we call it

"Lagrangian or referential approach" and the latter case is called "Eulerian or spatial approach". In addition, if the way to the solution has both approaches, we call it "Mixed approach".

In order to understand how these two configuration are related to each other, we need to see below.

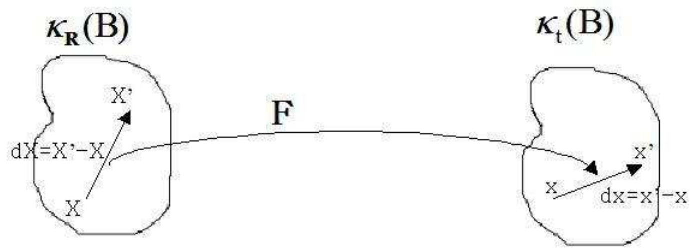

Fig. 2. Local deformation gradient tensor

Since $x = \chi(X,t)$, we can find relation, by Taylor Expansion, such as $dx = \frac{\partial \chi}{\partial X} dX$. From this relation, we define $F = \frac{\partial \chi}{\partial X}$ called "Local Deformation Gradient Tensor". The reason to call "local" is this relation is only for very small range of one particle such as neighborhood, but usually it is called "Deformation Gradient". This deformation gradient is the second order tensor to transform a infinitesimal vector in $\kappa_R(\mathbf{B})$ to a infinitesimal vector in $\kappa_t(\mathbf{B})$, and two point tensor because bases of $\mathbf{F}$ are in both $\kappa_R(\mathbf{B})$ and $\kappa_t(\mathbf{B})$.

Let us assume there are three different infinitesimal vectors in $\kappa_R(\mathbf{B})$, and, corresponding to those three vectors, there are three infinitesimal vectors in $\kappa_t(\mathbf{B})$.

From $dv = dx^1 \cdot dx^2 \times dx^3 = det(\mathbf{F})dV$, $det(\mathbf{F})$ is known as volume change. Thus, $det(\mathbf{F})$ becomes 1 for incompressible material or under an isochoric motion. If we let dA an infinitesimal surface area with normal unit vector $\mathbf{N}$ in $\kappa_R(\mathbf{B})$ and da an infinitesimal surface area with normal unit vector $\mathbf{n}$ in $\kappa_t(\mathbf{t})$, we have a relation between

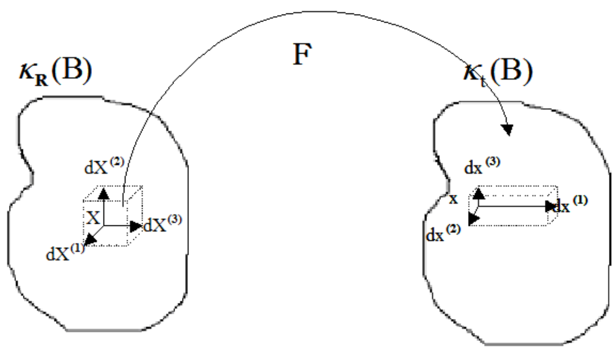

Fig. 3. Volume change in $\kappa_R(\mathbf{B})$ and $\kappa_t(\mathbf{B})$

surface areas such that,

$$\mathbf{n}da = det(\mathbf{F})\mathbf{F}^{-T}\mathbf{N}dA \tag{2.1}$$

which is called Nanson's Realtion[7]. From Fig. 3, Nanson's relation maps an area and normal vector from $\kappa_R(\mathbf{B})$ to $\kappa_t(\mathbf{B})$.

Now, from Fig.2, it can be shown that there is a rigid body motion [1] between dx and dX, and, if $\mathbf{F}$ is non singular, then this deformation gradient tensor can be expressed as,

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \quad ^{2} \tag{2.2}$$

which is called "Polar Decomposition". This equation expresses that $\mathbf{RU}$ is described as stretch first and then rotation whereas $\mathbf{VR}$ indicates rotation first and stretch. From (2.2),

$$\mathbf{C} := \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 \tag{2.3}$$

$$\mathbf{B} := \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 \tag{2.4}$$

[1] Rigid body motion means distance between two points belonging to a body is always same through deformation.

[2] $\mathbf{R}$ is Orthogonal transformation and rotation tensor. $\mathbf{U}$ and $\mathbf{V}$ are stretch tensors which are positive definite.

wherein, $\mathbf{C}$ is called "The Right Cauchy-Green deformation tensor", and $\mathbf{B}$ is called "The Right Cauchy-Green deformation tensor". From this equations, Green-St.Venant strain tensor $\mathbf{E}$ and Almansi-Hamel strain tensor $\mathbf{e}$ have been defined as

$$\mathbf{E} := \frac{1}{2}(\mathbf{C} - \mathbf{I}) \tag{2.5}$$

$$\mathbf{e} := \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) \tag{2.6}$$

In these relations, we know $\mathbf{U}$ is rotation invariant and $\mathbf{V}$ depends on rotation, and $\mathbf{B}$ and $\mathbf{C}$ can be shown to be symmetric tensors. It follows $\mathbf{U}$ and$\mathbf{V}$ are also symmetric. Because a symmetric tensor in $3-D$ has 3 real principal values(eigenvalues) and orthogonal principal directions(eigenvectors), from definition of eigenvalues(λ) and eigenvectors($\mathbf{v}$) such as

$$(\mathbf{U} - \lambda\mathbf{I})\mathbf{v} = 0 \tag{2.7}$$

we get characteristic equation for $\mathbf{U}$ by $det((\mathbf{U} - \lambda\mathbf{I})) = 0$ as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{2.8}$$

$$I_1 = tr(\mathbf{U}), I_2 = \frac{1}{2}[(tr(\mathbf{U}))^2 - tr(\mathbf{U}^2)], I_3 = det(\mathbf{U}) \tag{2.9}$$

I_1, I_2, and I_3 are called "Principal invariants of $\mathbf{U}$", which are independent on change of coordinate system. Let us consider three principal values as λ_e(major axis such as the most stretch direction), λ_q, λ_c(minor axis such as the least stretch direction))[3]. Also, corresponding to the three principal values, we denote three principal directions as $\mathbf{A}_{e,q,c}$ in $\kappa_R(\mathbf{B})$ and $\mathbf{a}_{e,q,c}$ in $\kappa_t(\mathbf{B})$. Thus, $\mathbf{U}$ and $\mathbf{V}$ have the same principal values, but different principal directions. In addition, $\mathbf{U}$ and $\mathbf{C}$ have the same principal directions but different principal values such as square root of it. The below figure is easy to understand the different between $\mathbf{U}$, $\mathbf{V}$, $\mathbf{C}$, and $\mathbf{B}$. One can say $\mathbf{U}$ and $\mathbf{C}$ are

[3]Sometimes λ_e=λ_q=λ_c happens, so we consider $\lambda_e \geq \lambda_q \geq \lambda_c$

living in reference configuration whereas **V** and **B** are living in current configuration because of principal directions.

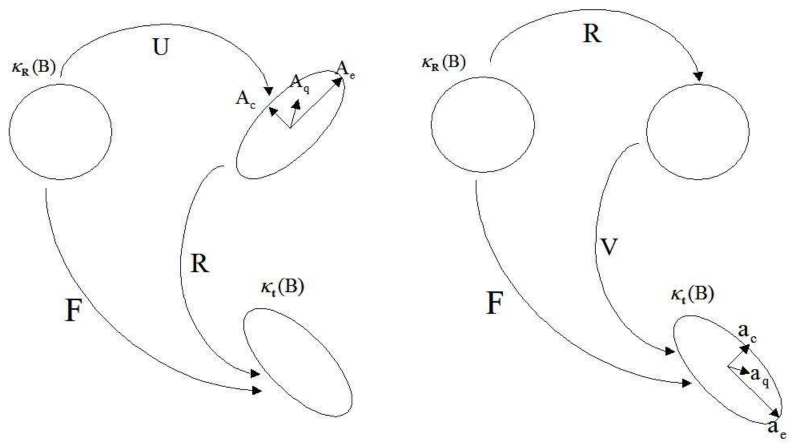

Fig. 4. Motions by polar decomposition (Every configuration represents infinitely small part in the body.)

We can easily express with Tensor expression from Fig 4., in a summary as

$$\mathbf{U} = \lambda_e \mathbf{A}_e \otimes \mathbf{A}_e + \lambda_q \mathbf{A}_q \otimes \mathbf{A}_q + \lambda_c \mathbf{A}_c \otimes \mathbf{A}_c \tag{2.10}$$

$$\mathbf{C} = \lambda_e^2 \mathbf{A}_e \otimes \mathbf{A}_e + \lambda_q^2 \mathbf{A}_q \otimes \mathbf{A}_q + \lambda_c^2 \mathbf{A}_c \otimes \mathbf{A}_c \tag{2.11}$$

$$\mathbf{V} = \lambda_e \mathbf{a}_e \otimes \mathbf{a}_e + \lambda_q \mathbf{a}_q \otimes \mathbf{a}_q + \lambda_c \mathbf{a}_c \otimes \mathbf{a}_c \tag{2.12}$$

$$\mathbf{B} = \lambda_e^2 \mathbf{a}_e \otimes \mathbf{a}_e + \lambda_q^2 \mathbf{a}_q \otimes \mathbf{a}_q + \lambda_c^2 \mathbf{a}_c \otimes \mathbf{a}_c \tag{2.13}$$

$$\mathbf{F} = \lambda_e \mathbf{a}_e \otimes \mathbf{A}_e + \lambda_q \mathbf{a}_q \otimes \mathbf{A}_q + \lambda_c \mathbf{a}_c \otimes \mathbf{A}_c \tag{2.14}$$

$$\mathbf{E} = \frac{1}{2}[(\lambda_e^2 - 1)\mathbf{A}_e \otimes \mathbf{A}_e + (\lambda_q^2 - 1)\mathbf{A}_q \otimes \mathbf{A}_q + (\lambda_c^2 - 1)\mathbf{A}_c \otimes \mathbf{A}_c] \tag{2.15}$$

$$\mathbf{e} = \frac{1}{2}[(1 - \lambda_e^{-2})\mathbf{a}_e \otimes \mathbf{a}_e + (1 - \lambda_q^{-2})\mathbf{a}_q \otimes \mathbf{a}_q + (1 - \lambda_c^{-2})\mathbf{a}_c \otimes \mathbf{a}_c] \tag{2.16}$$

Thus, we call **B**, **V**, and $\mathbf{a}_{e,q,c}$ “Eulerian Description”, and **C**, **U**, and $\mathbf{A}_{e,q,c}$ “Lagrangian Description”.

B. Balance Laws

1. Balance of Mass

Let P_t a part of current configuration corresponding to P_R in $\kappa_R(\mathbf{B})$, and balance of mass tells,

$$\int_{P_R} \rho_R dV = \int_{P_t} \rho dv \quad ^4 \tag{2.17}$$

From $dv = det(\mathbf{F})dV$, we can rewrite this as,

$$\int_{P_t} \rho dv = \int_{P_R} \rho det(\mathbf{F}) dV \tag{2.18}$$

Since P_R is arbitrary and integrand is continuous, it follows that,

$$\int_{P_R} (\rho_R - \rho det(\mathbf{F})) dV = 0 \tag{2.19}$$

$$\Rightarrow \rho_R = \rho det(\mathbf{F}) \tag{2.20}$$

2. Balance of Linear Momentum

On the boundary of P_t, we have tractions($\mathbf{t}$) and body force($\mathbf{b}$). Balance of linear momentum states

$$\int_{\partial P_t} \mathbf{t} da + \int_{P_t} \rho \mathbf{b} dv = \frac{d}{dt} \int_{P_t} \rho \boldsymbol{v} dv \tag{2.21}$$

This shows that time rate of change of the linear momentum of P_t equals to summation of forces on the boundary and forces due to specific body force. By Gauss-Green Divergence theorem[5] and relation between stress tensor($\mathbf{T}$) and traction vector($\mathbf{t}$)[6],

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[4] ρ_R and ρ denote density in $\kappa_R(\mathbf{B})$ and $\kappa_t(\mathbf{B})$

[5] If F is a continuously differentiable vector field defined on a neighborhood of V, then we have $\int_v div(F) dv = \int_{\partial v} F \cdot n da$

[6] $\mathbf{t} = \mathbf{T}^T \mathbf{n}$

it follows

$$\int_{P_t} div(\mathbf{T}^T)dv + \int_{P_t} \rho\mathbf{b}dv = \frac{d}{dt}\int_{P_t} \rho\boldsymbol{v}dv \tag{2.22}$$

$$\int_{P_t} [div(\mathbf{T}^T) + \rho\mathbf{b}]dv = \int_{P_t} \rho\frac{d\boldsymbol{v}}{dt}dv \tag{2.23}$$

$$\Rightarrow div(\mathbf{T}^T) + \rho\mathbf{b} = \rho\frac{d\boldsymbol{v}}{dt} \tag{2.24}$$

3. Balance of Angular Momentum

The density of Angular momentum is defined as $x \times \rho v$ in unit volume, so total angular momentum is

$$\int_{\partial P_t} x \times \mathbf{t}da + \int_{P_t} x \times \rho\mathbf{b}dv = \frac{d}{dt}\int_{P_t} x \times \rho\boldsymbol{v}dv \tag{2.25}$$

By balance of linear momentum, this shows that stress tensor is symmetric as $\mathbf{T} = \mathbf{T}^T$.(To see in detail, see [10].)

4. Equilibrium Equation

Result equations from (2.17) through (2.25) are called "Equilibrium Equations" as

$$div(\mathbf{T}) + \rho\mathbf{b} = \rho\frac{d\boldsymbol{v}}{dt} \tag{2.26}$$

In cartesian coordinate system and cylindrical coordinate system, it can be shown in elastostatic state as

$$\begin{aligned} \frac{\partial\mathbf{T}_{xx}}{\partial x} + \frac{\partial\mathbf{T}_{xy}}{\partial y} + \frac{\partial\mathbf{T}_{xz}}{\partial z} + \rho\mathbf{b}_x = 0 \\ \frac{\partial\mathbf{T}_{yx}}{\partial x} + \frac{\partial\mathbf{T}_{yy}}{\partial y} + \frac{\partial\mathbf{T}_{yz}}{\partial z} + \rho\mathbf{b}_y = 0 \\ \frac{\partial\mathbf{T}_{zx}}{\partial x} + \frac{\partial\mathbf{T}_{zy}}{\partial y} + \frac{\partial\mathbf{T}_{zz}}{\partial z} + \rho\mathbf{b}_z = 0 \end{aligned} \tag{2.27}$$

$$\begin{aligned} \frac{\partial\mathbf{T}_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\mathbf{T}_{\theta r}}{\partial\theta} + \frac{\partial\mathbf{T}_{zr}}{\partial z} + \frac{1}{r}(\mathbf{T}_{rr} - \mathbf{T}_{\theta\theta}) + \rho\mathbf{b}_r = 0 \\ \frac{\partial\mathbf{T}_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\mathbf{T}_{\theta\theta}}{\partial\theta} + \frac{\partial\mathbf{T}_{z\theta}}{\partial z} + \frac{2\mathbf{T}_{r\theta}}{r} + \rho\mathbf{b}_\theta = 0 \\ \frac{\partial\mathbf{T}_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\mathbf{T}_{\theta z}}{\partial\theta} + \frac{\partial\mathbf{T}_{zz}}{\partial z} + \frac{\mathbf{T}_{rz}}{r} + \rho\mathbf{b}_z = 0 \end{aligned} \tag{2.28}$$

C. Constitutive Relations and Strain Energy Functions in Green Elasticity

We associate the notion of stored energy($\mathbf{W}$) with the solid and assume that $\mathbf{W}$ depends on only deformation gradient tensor $\mathbf{F}$.Let us consider spherical neighborhood with radius(r) in a body as Fig. 5.

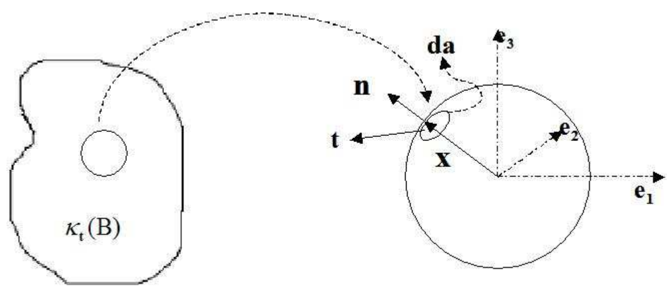

Fig. 5. Spherical neighborhood. da is infinitesimal surface area on boundary of sphere with unit normal vector($\mathbf{n}$). e_i are bases, and $\mathbf{t}$ denotes applied surface traction ($\mathbf{T}^T\mathbf{n}$=$\mathbf{Tn}$).$\mathbf{x}$ is position vector.

Let us consider a velocity vector as $\dot{\mathbf{x}}$, and, without body force, power($\dot{\Omega}$), which is derivative work with respect to time can be expressed as rate of which work is done the boundary of sphere due to $\mathbf{t}$.

$$\dot{\Omega} = \int_{\partial P_t} \mathbf{Tn} \cdot \dot{\mathbf{x}} da \tag{2.29}$$

By Taylor expansion,

$$\mathbf{T}_{ij} = \mathbf{T}_{ij}|_{x=0} + \frac{\partial \mathbf{T}_{ij}}{\partial \mathbf{x}_1}|_{x=0}\mathbf{x}_1 + \frac{\partial \mathbf{T}_{ij}}{\partial \mathbf{x}_2}|_{x=0}\mathbf{x}_2 + \frac{\partial \mathbf{T}_{ij}}{\partial \mathbf{x}_3}|_{x=0}\mathbf{x}_3 + h.o.t \tag{2.30}$$

$$\dot{\mathbf{x}}_i = \dot{\mathbf{x}}_i|_{x=0} + \frac{\partial \dot{\mathbf{x}}_i}{\partial \mathbf{x}_1}|_{x=0}\mathbf{x}_1 + \frac{\dot{\mathbf{x}}_i}{\partial \mathbf{x}_2}|_{x=0}\mathbf{x}_2 + \frac{\partial \dot{\mathbf{x}}_i}{\partial \mathbf{x}_3}|_{x=0}\mathbf{x}_3 + h.o.t \tag{2.31}$$

By (2.30) and (2.31), (2.29) is found with spherical integration.

$$\dot{\Omega} = \int_0^{2\pi}\int_0^{\pi} [(\mathbf{T}_{11}\mathbf{n}_1 + \mathbf{T}_{12}\mathbf{n}_2 + \mathbf{T}_{13}\mathbf{n}_3)\dot{\mathbf{x}}_1 + +(\mathbf{T}_{21}\mathbf{n}_1 + \mathbf{T}_{22}\mathbf{n}_2 + $$

$$+ \mathbf{T}_{23}\mathbf{n}_3)\dot{\mathbf{x}}_2 + (\mathbf{T}_{31}\mathbf{n}_1 + \mathbf{T}_{32}\mathbf{n}_2 + \mathbf{T}_{33}\mathbf{n}_3)\dot{\mathbf{x}}_3] R^2 \sin\phi d\phi d\theta \tag{2.32}$$

It follows that[7]

$$\dot{\Omega} = \frac{4}{3}\pi R^3[\mathbf{T} : \dot{\mathbf{F}}\mathbf{F}^{-1}] \tag{2.33}$$

$$\Rightarrow \dot{\mathbf{W}} = \frac{\dot{\Omega}}{volume(\kappa_R(\mathbf{B}))} = \mathbf{T} : \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{2.34}$$

By (2.2), (2.3), and separation of $d\mathbf{F}\mathbf{F}^T$ with symmetric and Skew-symmetric part, it follows,

$$d\mathbf{W} = (det\mathbf{F})\mathbf{F}^{-1}\mathbf{T}\mathbf{F}\mathbf{F}^{-T} : d\mathbf{E} \Rightarrow d\mathbf{W} = \mathbf{S} : d\mathbf{E} \quad ^8 \tag{2.35}$$

Now, we can see strain energy is composed of components of $\mathbf{E}$, and, finally, we get

$$\mathbf{T} = \frac{1}{det\mathbf{F}}\mathbf{F}\frac{\partial \mathbf{W}}{\partial \mathbf{E}}\mathbf{F}^T \tag{2.36}$$

$$\mathbf{T} = -p\mathbf{I} + \frac{1}{det\mathbf{F}}\mathbf{F}\frac{\partial \mathbf{W}}{\partial \mathbf{E}}\mathbf{F}^T \tag{2.37}$$

It essentially tells us $\mathbf{T}$ and $\mathbf{W}$ are functions of $\mathbf{F}$. p is Lagrangian multiplier, so (2.37) is used under isochoric motion. For isotropic material, Spencer showed strain energy depends on $\mathbf{C}$, which is right-Cauchy Green Deformation Tensor. And strain energy is composed of three invariants which is found from characteristic equation for $\mathbf{C}$.[2]

$$\mathbf{W} = \mathbf{W}(\boldsymbol{I}_1, \boldsymbol{I}_2, \boldsymbol{I}_3) \tag{2.38}$$

$$\boldsymbol{I}_1 = tr(\mathbf{C}), \boldsymbol{I}_2 = \frac{1}{2}[(tr\mathbf{C})^2 - tr(\mathbf{C}^2)], \boldsymbol{I}_3 = det(\mathbf{C}) \tag{2.39}$$

For transversely isotropic material, strain energy function can be expressed as .[2]

$$\mathbf{W} = \mathbf{W}(\boldsymbol{I}_1, \boldsymbol{I}_2, \boldsymbol{I}_3, \boldsymbol{I}_4, \boldsymbol{I}_5) \tag{2.40}$$

$$\boldsymbol{I}_4 = \mathbf{M} \cdot \mathbf{C}\mathbf{M}, \boldsymbol{I}_5 = \mathbf{M} \cdot \mathbf{C}^2\mathbf{M} \tag{2.41}$$

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[7] ":" denotes $\mathbf{A} : \mathbf{B} = \mathbf{A}_{ij}\mathbf{B}_{ij} \quad (i, j = 1, 2, 3)$, and R is radius in $\kappa_R(\mathbf{B})$

[8] $\mathbf{S}$ is called "Second Piola-Kirchhoff stress tensor", which is existing in $\kappa_R(\mathbf{B})$

However, as we see in Chapter I, even if theses invariants are well-posed in theoretical approach, they are not smart choices of invariants for experimentally determining response terms because of co-alignment of invariants. Thus, we use these basic conceptions, and ,in this study which is considered as a transversely isotropic material, we will use five strain invariants such as $\mathbf{W} = \mathbf{W}(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ under finite deformation.[4]

D. Frame Indifference

We usually see events in a body from different coordinate systems. Thus, it is possible what happens under deformation depends on each frame. There is a translation($c(t)$) and rotation($Q(t)$) between two frame. Let two positions for each frames denoted by $\mathbf{x}^*$ and $\mathbf{x}$.

$$\mathbf{x}^* = c(t) + Q(t)(\mathbf{x} - 0) \tag{2.42}$$

Using (2.42), Scalar(a), vector($\mathbf{a}$), and tenors($\mathbf{A}$) are said to be frame indifferent if

$$a^* = a, \mathbf{a}^* = Q\mathbf{a}, \mathbf{A}^* = Q\mathbf{A}Q^T \tag{2.43}$$

Also, if Q doesn't depend on time(t), these are said to be Galilean Invariants. Frame indifference is more restrictive than Galilean invariant. And, as a matter of fact, Galilean invariance is enough to explain the relation between frames. For displacement($\mathbf{u}$) and $\mathbf{F}$, we can easily find $\mathbf{u}$ is Galilean invariant, but $\mathbf{F}$ is not such as,

$$\mathbf{u}^* = Q\mathbf{u}, \quad \mathbf{F}^* = Q\mathbf{F} \tag{2.44}$$

It follows that

$$\mathbf{C}^* = \mathbf{C}, \quad \mathbf{B}^* = Q\mathbf{B}Q^T \tag{2.45}$$

From (2.44), Cauchy stress tensor($\mathbf{T}$) is Galilean invariant such as $\mathbf{T}^* = Q\mathbf{T}Q^T$ We have already known $\mathbf{T} = f(\mathbf{F})$, and, from this relation and (2.2),[9]

$$\mathbf{T}^* = f(Q\mathbf{F}) = f(\mathbf{U}) = \mathbf{R}^T f(\mathbf{R}\mathbf{U})\mathbf{R} \tag{2.46}$$

$$\Rightarrow \mathbf{T} = f(\mathbf{R}\mathbf{U}) = \mathbf{R} f(\mathbf{U})\mathbf{R}^T \tag{2.47}$$

This is the most general restriction for Cauchy stress, and, in fact, we really didn't use frame indifference in generality. All we used is restriction for Galilean Invariance. In this case, even if Q depends on time(t), Cauchy Stress($\mathbf{t}$) is frame indifferent as well as Galilean invariant.

[9] we pick $Q = \mathbf{R}^T$ because $\mathbf{R} \in Ortho^+$

CHAPTER III

FINITE ELASTICITY

In this chapter, we see constitutive by Mixed and Eulerian approach for transversely isotropic material under finite extension and torsion to fiber reinforced circular cylinder. This theoretical approach is based on assumptions that, first, fibers are axial symmetrically arranged and continuously distributed in the body, and, second, this material is macroscopically considered as a transversely isotropic material and an incompressible material.

A. Kinematics

Under finite extension and torsion, let original domain defined by $R \in [0, R_{out}]$, $\Theta \in [0, 2\pi]$, and $Z \in [0, H]$. In finite extension, (R, Θ, Z) denotes a position in reference configuration, and, corresponding to this position, $[r', \theta', z']$ denotes the position after the deformation. The mapping in cylindrical coordinate system is,

$$r' = r'(R), \quad \theta' = \Theta, \quad z' = \Lambda Z \tag{3.1}$$

In cylindrical coordinate system, deformation gradient tensor can be expressed as

$$\mathbf{F}_{extension} = \begin{pmatrix} \frac{\partial r'}{\partial R} & \frac{1}{R}\frac{\partial r'}{\partial \Theta} & \frac{\partial r'}{\partial Z} \\ r'\frac{\partial \theta'}{\partial R} & \frac{r'}{R}\frac{\partial \theta'}{\partial \Theta} & r'\frac{\partial \theta'}{\partial Z} \\ \frac{\partial z'}{\partial R} & \frac{1}{R}\frac{\partial z'}{\partial \Theta} & \frac{\partial z'}{\partial Z} \end{pmatrix} = \begin{pmatrix} \frac{\partial r'}{\partial R} & 0 & 0 \\ 0 & \frac{r'}{R} & 0 \\ 0 & 0 & \Lambda \end{pmatrix} \tag{3.2}$$

For incompressible material, $det(\mathbf{F}) = 1$, which follows,

$$\Lambda \frac{r'}{R}\frac{\partial r'}{\partial R} = 1 \Rightarrow r' = R\frac{1}{\sqrt{\Lambda}} \tag{3.3}$$

$$\mathbf{F}_{extension} = \begin{pmatrix} \frac{1}{\sqrt{\Lambda}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\Lambda}} & 0 \\ 0 & 0 & \Lambda \end{pmatrix} \tag{3.4}$$

For the second deformation, which is torsion(twist per unit lengthγ'), mapping is

$$r = r(r'), \quad \theta = \theta' + \gamma' z', \quad z = z' \tag{3.5}$$

$$\mathbf{F}_{torsion} = \begin{pmatrix} \frac{\partial r}{\partial r'} & \frac{1}{r'}\frac{\partial r}{\partial \theta'} & \frac{\partial r}{\partial z'} \\ r\frac{\partial \theta}{\partial r'} & \frac{r}{r'}\frac{\partial \theta}{\partial \theta'} & r\frac{\partial \theta}{\partial z'} \\ \frac{\partial z}{\partial r'} & \frac{1}{r'}\frac{\partial z}{\partial \theta'} & \frac{\partial z}{\partial z'} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial r'} & 0 & 0 \\ 0 & \frac{r}{r'} & r\gamma' \\ 0 & 0 & 1 \end{pmatrix} \tag{3.6}$$

Similarly, by incompressibility and (3.3),

$$\frac{r}{r'}\frac{\partial r}{\partial r'} = 1 \Rightarrow \frac{\partial r}{\partial r'} = 1 (r > 0, r' > 0) \tag{3.7}$$

$$\mathbf{F}_{torsion} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r\gamma' \\ 0 & 0 & 1 \end{pmatrix} \tag{3.8}$$

Finally, total deformation gradient for first extension and second torsion,

$$\mathbf{F} = \begin{pmatrix} \frac{1}{\sqrt{\Lambda}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\Lambda}} & \Lambda\gamma' r \\ 0 & 0 & \Lambda \end{pmatrix} \tag{3.9}$$

From (2.3) and (2.4),

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\Lambda} & 0 & 0 \\ 0 & \frac{1}{\Lambda} & \sqrt{\Lambda}\gamma' r \\ 0 & \sqrt{\Lambda}\gamma' r & \Lambda^2(\gamma'^2 r^2 + 1) \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \frac{1}{\Lambda} & 0 & 0 \\ 0 & \frac{1}{\Lambda} + \Lambda^2\gamma'^2 r^2 & \Lambda^2\gamma' r \\ 0 & \Lambda^2\gamma' r & \Lambda^2 \end{pmatrix} \tag{3.10}$$

Let $\mathbf{M}$ and $\mathbf{m}$ denotes unit vector on preferred direction(fiber direction) in $\kappa_R(\mathbf{B})$ and $\kappa_t(\mathbf{B})$, and $\mathbf{N}_1$ and $\mathbf{N}_2$ are unit vectors on cross fiber directions in $\kappa_R(\mathbf{B})$.(Similarly, $\mathbf{n}_1$, $\mathbf{n}_2 \in \kappa_t(\mathbf{B})$) Relationship between $\kappa_R(\mathbf{B})$ and $\kappa_t(\mathbf{B})$ can be expressed by $\mathbf{F}$ as

$$\mathbf{m} = (\mathbf{M} \cdot \mathbf{C}\mathbf{M})^{-\frac{1}{2}}\mathbf{F}\mathbf{M} \tag{3.11}$$

$$\mathbf{n}_1 = \frac{\mathbf{F}^{-T}\mathbf{N}_1}{\left\|\mathbf{F}^{-T}\mathbf{N}_1\right\|}, \quad \mathbf{n}_2 = \frac{\mathbf{F}^{-T}\mathbf{N}_2}{\left\|\mathbf{F}^{-T}\mathbf{N}_2\right\|} \tag{3.12}$$

Because there must be at least two perpendicular planes including $\mathbf{M}$, we can use Nanson's relation(2.1), (3.12) is relation between $\mathbf{N}$ and $\mathbf{n}$ by it, and there are some conditions for this relation.

$$\mathbf{M} \cdot \mathbf{N}_1 = \mathbf{M} \cdot \mathbf{N}_2 = \mathbf{N}_1 \cdot \mathbf{N}_2 = 0 \tag{3.13}$$

$$\mathbf{N}_1 \cdot \mathbf{C}^{-1}\mathbf{N}_2 = 0, \quad \mathbf{N}_1 \cdot \mathbf{C}^{-1}\mathbf{N}_1 \leq \mathbf{N}_2 \cdot \mathbf{C}^{-1}\mathbf{N}_2 \tag{3.14}$$

From (3.14), we can see the surface area with normal vector($\mathbf{N}_1$) is smaller than one with normal vector($\mathbf{N}_2$), which means $\mathbf{n}_1$ has maximum spaces in $\kappa_t(\mathbf{B})$ whereas $\mathbf{n}_2$ has minimum spaces. Thus we can say $\mathbf{N}_1$ is unit vector on the most stretch direction and $\mathbf{N}_2$ is unit vector on the least stretch direction in $\kappa_R(\mathbf{B})$. To find these two vectors, as (2.11), we use $\mathbf{C}_{2D}$ on $\mathbf{N}_1\mathbf{N}_2$ plane, and find eigenvectors of $\mathbf{C}_{2D}$ as $\mathbf{N}_1$ and $\mathbf{N}_2$.

Using these relation, we find each components of stress tensor from constitutive relation, which will be showed on Section C and D, and the equilibrium equations in cylindrical coordinate system(2.28)[7] such that

$$\int \frac{d\mathbf{T}_{rr}}{dr}dr = \int (\mathbf{T}_{rr} - \mathbf{T}_{\theta\theta})\frac{1}{r}dr \tag{3.15}$$

For axial load(F_z) and moment(M_z),

$$F_z = 2\pi \int_0^{R_{out}} \mathbf{T}_{zz} r dr, \quad M_z = 2\pi \int_0^{R_{out}} \mathbf{T}_{z\theta} r^2 dr \tag{3.16}$$

In axial load, we can express this as

$$F_z = 2\pi \int_0^{R_{out}} \mathbf{T}_{zz} r dr + 2\pi \int_0^{R_{out}} \mathbf{T}_{rr} r dr - 2\pi \int_0^{R_{out}} \mathbf{T}_{rr} r dr \tag{3.17}$$

By (2.28),

$$2\pi \int_0^{R_{out}} \mathbf{T}_{rr} r dr = \pi R_{out}^2 \mathbf{T}_{rr}|_{R_{out}} - \pi \int_0^{R_{out}} \mathbf{T}_{\theta\theta} r dr + \pi \int_0^{R_{out}} \mathbf{T}_{rr} r dr \tag{3.18}$$

It immediately follows,

$$F_z = \pi \int_0^{R_{out}} (2\mathbf{T}_{zz} - \mathbf{T}_{\theta\theta} - \mathbf{T}_{rr}) r dr \quad {}^{1} \tag{3.19}$$

This equation will be used in computation because we don't have to consider Lagrangian Multiplier(p).[2]

B. Strain Invariants

Five strain invariants were defined, each with physical meaning [4],

$$\begin{aligned} \beta_1 &= ln(J) \\ \beta_2 &= ln(\lambda_M^{\frac{3}{2}}) \\ \beta_3 &= ln(\xi^2) \\ \beta_4 &= \psi \\ \beta_5 &= \cos^2\gamma - \sin^2\gamma \end{aligned} \tag{3.20}$$

[1]Boundary condition tells us stress free on lateral surface, so $\mathbf{T}_{rr}|_{R_{out}} = 0$

[2]For incompressibility, constitutive relation is expressed as $\mathbf{T} = -p\mathbf{I} + \Sigma_{i=2}^{5} \frac{\partial \mathbf{W}}{\partial \beta_i} \mathbf{A}_i$

The first strain invariant β_1 is dilatation part which causes volume change and others are distortional parts without volume changes. In detail, the second strain invariant β_2 is a stretch ratio (λ_M) on the fiber direction, which means distortional fiber strestch, and the third β_3 is a pure shear (ξ) on the cross fiber directions. The fourth β_4 is a (ψ)shear on along-fiber direction, and finally β_5 indicates direction of simple shear related to cross fiber shear axis. (γ is an angle between the line where the plane of simple shear along fiber direction intersects cross fiber section and maximum stretch direction on the cross fiber section.)

C. Strain Energy Function

Functional form of strain energy was determined as polynomial form after considering cases of vanishing pure shear on cross fiber directions and simple shear along fiber direction as [4]

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}_0 - q_1\beta_1 + q_2\beta_2 + \frac{1}{2}g_1\beta_1^2 + \frac{1}{2}g_2\beta_2^2 + g_{12}\beta_1\beta_2 + \mathbf{H}_0(\beta_1, \beta_2) \\ +\beta_3^2(\frac{1}{2}g_3 + \mathbf{H}_3(\beta_1, \beta_2, \beta_3)) \\ +\beta_4^2[\frac{1}{2}g_4 + \mathbf{H}_4(\beta_1, \beta_2, \beta_4) + \beta_3\beta_5\mathbf{H}_5(\beta_1, \beta_2, \beta_4) \\ +\beta_3^2\mathbf{H}_6(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)] \end{pmatrix} \tag{3.21}$$

The functions $\mathbf{H}_i$ are high order functions based on strain invariants, and $\mathbf{W}_0$ is strain energy in reference configuration, which is caused by prestresses. When deformation gradient equals to one, only the first derivative of β_1 and β_2 are non-vanishing. Thus, q_1 indicates the pressure in reference configuration, and q_2 indicates distortional part without volume change in reference configuration. These are also made by prestresses(residual stress).

In this study, we consider stress free in reference configuration and incompressible material, which means $\mathbf{W}_0$, q_1, q_2, and β_1 are removed. Moreover, in order to pick the simplest strain energy function to determine response terms, we ignore all higher order terms $\mathbf{H}_i$. Finally, we obtain

$$\mathbf{W} = \frac{1}{2}g_2\beta_2^2 + \frac{1}{2}g_3\beta_3^2 + \frac{1}{2}g_4\beta_4^2 \tag{3.22}$$

$$g_2 = 4g_3 = 4g_4 \tag{3.23}$$

This strain energy function can be applied to muscle tissue, which means fiber direction is four times stiffer than other directions such as cross fiber directions.

D. Mixed Approach

The reason to be called “Mixed” is that, even if this expression of constitutive relation is Eulerian, the way to approach is first finding $\mathbf{N}_1$ and $\mathbf{N}_2$, which means Lagrangian approach, and the way to apply these to constitutive relation based on $\mathbf{m}$ and $\mathbf{n}_{1,2}$ is Eulerian approach. Thus we call it “Mixed”.

1. Strain Invariants

Once we find $\mathbf{N}_1$ and $\mathbf{N}_2$, strain invariants, which were introduced in Chapter III, section B, can be expressed as,[4]

$$\beta_1 = \ln J = \ln(det(\mathbf{C}^{\frac{1}{2}})) \tag{3.24}$$

$$\beta_2 = \ln(\lambda_M) = \ln(J^{-1/3}(\mathbf{M}\cdot\mathbf{CM})^(1/2)) \tag{3.25}$$

$$\beta_3 = \ln\xi = \ln\left(\left(\frac{\mathbf{N}_2\cdot\mathbf{C}^{-1}\mathbf{N}_2}{\mathbf{N}_1\cdot\mathbf{C}^{-1}\mathbf{N}_1}\right)^{\frac{1}{4}}\right) \tag{3.26}$$

$$\beta_4 = \psi = \left(\frac{\mathbf{M}\cdot\mathbf{C}^2\mathbf{M}}{(\mathbf{M}\cdot\mathbf{CM})^2} - 1\right)^{\frac{1}{2}} \tag{3.27}$$

$$\beta_5 = \cos^2\gamma - \sin^2\gamma = \frac{(\mathbf{N}_1\cdot\mathbf{CM})^2 - (\mathbf{N}_2\cdot\mathbf{CM})^2}{(\mathbf{N}_1\cdot\mathbf{CM})^2 + (\mathbf{N}_2\cdot\mathbf{CM})^2} \tag{3.28}$$

2. Constitutive Relation

$$\mathbf{T} = \frac{1}{J}\sum_{i=1}^{5}\frac{\partial W}{\partial \beta_i}\mathbf{A}_i \tag{3.29}$$

For incompressible material,

$$\mathbf{T} = -p\mathbf{I} + \sum_{i=2}^{5}\frac{\partial W}{\partial \beta_i}\mathbf{A}_i \tag{3.30}$$

$$\begin{aligned}
\mathbf{A}_1 &= \mathbf{I} \\
\mathbf{A}_2 &= m \otimes m - \tfrac{1}{2}(\mathbf{I} - m \otimes m) \\
\mathbf{A}_3 &= n_1 \otimes n_1 - n_2 \otimes n_2 \\
\mathbf{A}_4 &= \lambda_M^{-\frac{3}{2}}[\xi \cos\gamma(m \otimes n_1 + n_1 \otimes m) + \xi^{-1}\sin\gamma(m \otimes n_2 + n_2 \otimes m)] \\
\mathbf{A}_5 &= \tfrac{4\cos\gamma\sin\gamma}{\lambda_M^{\frac{3}{2}}\beta_4}\begin{pmatrix} \xi\sin\gamma(m \otimes n_1 + n_1 \otimes m) - \xi^{-1}\cos\gamma(m \otimes n_2 + n_2 \otimes m) \\ +\lambda^{\frac{3}{2}}(\xi^2 - \xi^{-2})^{-1}(n_2 \otimes n_1 + n_1 \otimes n_2) \end{pmatrix}
\end{aligned} \tag{3.31}$$

To prove the stress from this constitutive relation as Galilean Invariant or frame indifferent, as Chapter II, let us denote Q is rotation tensor between frames. Using (3.11) by $\mathbf{F}^* = Q\mathbf{F}$,

$$\mathbf{m}^* = (\mathbf{M}\cdot\mathbf{CM})^{-\frac{1}{2}}\mathbf{F}^*\mathbf{M} = (\mathbf{M}\cdot\mathbf{CM})^{-\frac{1}{2}}Q^T\mathbf{FM} \tag{3.32}$$

It follows $\mathbf{m} = Q\mathbf{m}^*$, and from (3.12),

$$\mathbf{n}^* = \frac{(Q^T\mathbf{F})^{-T}\mathbf{N}}{\left\|(Q^T\mathbf{F})^{-T}\mathbf{N}\right\|} = \frac{Q^T\mathbf{F})^{-T}\mathbf{N}}{\left\|Q^T\mathbf{F}^{-T}\mathbf{N}\right\|} \tag{3.33}$$

It shows $\mathbf{n}^*_{1,2} = Q\mathbf{n}_{1,2}$[3]. Thus all these vectors are Galilean Invariants as well as frame indifference. Because this constitutive relation is composed of $\mathbf{m}$ and $\mathbf{n}_{1,2}$, it

[3]Denominator is magnitude of vector, so magnitude of vector doesn't depend on rigid body motion

is obvious that stress($\mathbf{T}^*$) should be $Q\mathbf{T}Q^T$. Moreover, $\mathbf{A}_i$ have been verified to be mutually orthogonal.[4]

3. Computational Method - Matlab

1. From the motion, find deformation gradient and left and right Cauchy Green deformation tensors in terms of radius.

2. Use the loop for radius with step size.

(1) Recalculate $\mathbf{F}$, $\mathbf{C}$, and $\mathbf{B}$ at every step size of radius, which must have components as numbers because, in order to find $\mathbf{N}_1$ and $\mathbf{N}_2$, we have to compare the magnitude of two eigenvalues of $\mathbf{C}_{2D}$.(See 3.A)

(2) After finding $\mathbf{C}_{2D}$ on $R\Theta$-plane[4], compare magnitude of two eigenvalues. Corresponding to the larger eigenvalue, the eigenvector is considered as $\mathbf{N}_1$, and another one is $\mathbf{N}_2$.

(3) Once we find $\mathbf{N}_1$ and $\mathbf{N}_2$[5], now use originally found $\mathbf{F}$, $\mathbf{C}$, and $\mathbf{B}$ because , later, in applying to constitutive relation, we should prevent from error "divided by zero". And using (3.23) through (3.27), find five strain invariants.

(4) Find components of stress($\mathbf{T}$) from constitutive relation(3.29) and by equilibrium equation in terms of radius(r). At this point we need to store all results from each radius step size which have to be called at every step size of another loop inside of main loop for trapezoidal rule. And then plot $\mathbf{T}_{zz}$ and $\mathbf{T}_{z\theta}$ to confirm the results.

3. Find axial load and moment by trapezoidal rule.(3.16,19) To compare the computational time, ways to find results have to be same, so Matlab code is needed to

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[4] Under finite extension and torsion to circular cylinder, $\mathbf{N}_1\mathbf{N}_2$ plane is on $R\Theta$-plane

[5] In fact, $\mathbf{N}_1$ and $\mathbf{N}_2$ don't depend on radius under this specific deformation, but we need to make this for general cases.

be showed. And for better way to under standing, the source code is provided as Appendix.A.

E. Eulerian Approach

1. Strain Invariants

β_1 through β_5 have been expressed in terms of $I_{1,2,3,4,5}$[4] because it is easier way to find strain invariants by $\mathbf{m}$ and $\mathbf{B}$.

$$\beta_1 = \frac{(\ln I_3)}{2} \tag{3.34}$$

$$\beta_2 = (3\ln I_4 - \ln I_3) \tag{3.35}$$

$$\beta_3 = \ln\left(\left(\frac{I_4 I_1 - I_5}{2\sqrt{I_3 I_4}}\right) + \sqrt{\left(\frac{I_4 I_1 - I_5}{2\sqrt{I_3 I_4}}\right)^2 - 1}\right) \tag{3.36}$$

$$\beta_4 = \sqrt{\frac{I_5}{I_4^2} - 1} \tag{3.37}$$

$$\beta_5 = \frac{I_1 I_4 I_5 + I_1 I_4^3 + 2I_3 I_4 - I_5^2 - 2I_2 I_4^2 - I_5 I_4^2}{(I_5 - I_4^2)\sqrt{I_1^2 I_4^2 + I_5^2 - 2I_1 I_4 I_5 - 4I_3 I_4}} \tag{3.38}$$

From deformation gradient tensor and right-left Cauchy Green deformation tensors based on $\mathbf{M}$, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{m}$, $\mathbf{n}_1$, and $\mathbf{n}_2$ such as

$$\mathbf{F} = J^{\frac{1}{3}} \begin{pmatrix} \xi\lambda_M^{-\frac{1}{2}} & 0 & 0 \\ 0 & \xi^{-1}\lambda_M^{-\frac{1}{2}} & 0 \\ \lambda_M\psi\cos\gamma & \lambda_M\psi\sin\gamma & \lambda_M \end{pmatrix} \tag{3.39}$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = J^{\frac{2}{3}} \begin{pmatrix} \xi^2\lambda_M^{-1} + \lambda_M^2\psi^2\cos^2\gamma & \lambda_M^2\psi^2\cos\gamma\sin\gamma & \lambda_M^2\psi\cos\gamma \\ \lambda_M^2\psi^2\cos\gamma\sin\gamma & \xi^{-2}\lambda_M^{-1} + \lambda_M^2\psi^2\sin^2\gamma & \lambda_M^2\psi\sin\gamma \\ \lambda_M^2\psi\cos\gamma & \lambda_M^2\psi\sin\gamma & \lambda_M^2 \end{pmatrix} \tag{3.40}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = J^{\frac{2}{3}} \begin{pmatrix} \xi^2\lambda_M^{-1} & 0 & \lambda_M^{\frac{1}{2}}\xi\psi\cos\gamma \\ 0 & \xi^{-2}\lambda_M^{-1} & \lambda_M^{\frac{1}{2}}\xi^{-1}\psi\sin\gamma \\ \lambda_M^{\frac{1}{2}}\xi\psi\cos\gamma & \lambda_M^{\frac{1}{2}}\xi^{-1}\psi\sin\gamma & \lambda_M^2(\psi^2+1) \end{pmatrix} \tag{3.41}$$

$$\begin{aligned} \mathbf{B}^{-1} &= (\mathbf{F}^{-1})^T\mathbf{F}^{-1} \\ &= J^{-\frac{2}{3}} \begin{pmatrix} \xi^{-2}\lambda_M + \lambda_M\xi^{-2}\psi^2\cos^2\gamma & \lambda_M\psi^2\cos\gamma\sin\gamma & -\lambda_M^{-\frac{1}{2}}\xi^{-1}\psi\cos\gamma \\ \lambda_M\psi^2\cos\gamma\sin\gamma & \xi^2\lambda_M + \lambda_M\xi^2\psi^2\sin^2\gamma & -\lambda_M^{-\frac{1}{2}}\xi\psi\sin\gamma \\ -\lambda_M^{-\frac{1}{2}}\xi^{-1}\psi\cos\gamma & -\lambda_M^{-\frac{1}{2}}\xi\psi\sin\gamma & \lambda_M^{-2} \end{pmatrix} \end{aligned} \tag{3.42}$$

$I_{1,2,3,4,5}$ can be expressed as

$$I_1 = tr(\mathbf{B}) \tag{3.43}$$

$$I_2 = (det\mathbf{B})(tr\mathbf{B}^{-1}) \tag{3.44}$$

$$I_3 = det\mathbf{B} \tag{3.45}$$

$$I_4 = (\mathbf{m}\cdot\mathbf{B}^{-1}\mathbf{m})^{-1} \tag{3.46}$$

$$I_5 = \frac{\mathbf{m}\cdot\mathbf{B}\mathbf{m}}{\mathbf{m}\cdot\mathbf{B}^{-1}\mathbf{m}} \tag{3.47}$$

Finally, it follows

$$\begin{aligned} \beta_1 &= \frac{1}{2}\ln(det\mathbf{B}) \\ \beta_2 &= -\frac{1}{2}\beta_1 - \frac{3}{4}\ln(\mathbf{m}\cdot\mathbf{B}^{-1}\mathbf{m}) \\ \beta_3 &= \ln\left(\frac{tr\mathbf{B} - \mathbf{m}\cdot\mathbf{B}\mathbf{m}}{2(det\mathbf{B})^{\frac{1}{2}}(\mathbf{m}\cdot\mathbf{B}^{-1}\mathbf{m})^{\frac{1}{2}}} + \sqrt{\frac{(tr\mathbf{B} - \mathbf{m}\cdot\mathbf{B}\mathbf{m})^2}{4(det\mathbf{B})(\mathbf{m}\cdot\mathbf{B}^{-1}\mathbf{m})} - 1}\right) \\ \beta_4 &= \sqrt{(\mathbf{m}\cdot\mathbf{B}\mathbf{m})(\mathbf{m}\cdot\mathbf{B}^{-1}\mathbf{m}) - 1} \\ \beta_5 &= \frac{(tr\mathbf{B} - \mathbf{m}\cdot\mathbf{B}\mathbf{m})(\mathbf{m}\cdot\mathbf{B}\mathbf{m} + (\mathbf{m}\cdot\mathbf{B}^{-1}\mathbf{m})^{-1}) - 2det\mathbf{B}(tr\mathbf{B}^{-1} - \mathbf{m}\cdot\mathbf{B}^{-1}\mathbf{m})}{(\mathbf{m}\cdot\mathbf{B}\mathbf{m} - (\mathbf{m}\cdot\mathbf{B}^{-1}\mathbf{m})^{-1})\sqrt{(tr\mathbf{B} - \mathbf{m}\cdot\mathbf{B}\mathbf{m})^2 - 4det\mathbf{B}(\mathbf{m}\cdot\mathbf{B}^{-1}\mathbf{m})}} \end{aligned}$$

2. Constitutive Relation

$$\mathbf{T} = \frac{1}{J}\sum_{i=1}^{5}\frac{\partial W}{\partial \beta_i}\mathbf{A}_i \tag{3.48}$$

$$\mathbf{T} = -p\mathbf{I} + \sum_{i=2}^{5}\frac{\partial W}{\partial \beta_i}\mathbf{A}_i \tag{3.49}$$

$$
\begin{aligned}
\mathbf{A}_1 &= \mathbf{I}\\
\mathbf{A}_2 &= m\otimes m - \tfrac{1}{2}(\mathbf{I} - m\otimes m)\\
\mathbf{A}_3 &= \tfrac{\lambda_M}{\xi^2-\xi-2}\begin{pmatrix} 2J^{-\frac{2}{3}}\mathbf{B} - 2J^{-\frac{2}{3}}(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}) + 2\lambda_M^2(\beta_4^2+1)\mathbf{m}\otimes\mathbf{m}\\ -\lambda_M^{-1}(\xi^2+\xi^{-2})(\mathbf{I}-\mathbf{m}\otimes\mathbf{m}) \end{pmatrix}\\
\mathbf{A}_4 &= J^{-\frac{2}{3}}\lambda_M^{-2}\beta_4^{-1}(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}) - 2(\beta_4 + \beta_4^{-1})\mathbf{m}\otimes\mathbf{m}\\
\mathbf{A}_5 &= \begin{matrix} \tfrac{4}{\lambda_M\beta_4^2(\xi^2-\xi^{-2})}\begin{pmatrix} J^{-\frac{2}{3}}\mathbf{B}^{-1} + \lambda_M^2 J^{-\frac{2}{3}}\mathbf{B} - J^{-\frac{2}{3}}(\lambda_M^2 - \lambda_M^{-1}(\xi^2\sin^2\gamma + \xi^{-2}\cos^2\gamma))\\ (\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}) - \lambda_M(\xi^2+\xi^{-2})(\mathbf{I}-\mathbf{m}\otimes\mathbf{m})\\ +[\lambda_M^4(\beta_4^2+1) - 2\lambda_M(\beta_4^2+1)(\xi^2\sin^2\gamma + \xi^{-2}\cos^2\gamma) - \lambda_M^{-2}]\mathbf{m}\otimes\mathbf{m} \end{pmatrix}\\ + \tfrac{4}{\lambda_M(\xi^2-\xi^{-2})^2}\begin{pmatrix} (\xi^2\sin^2\gamma - \xi^{-2}\cos^2\gamma)\begin{pmatrix} \lambda_M^2 J^{-\frac{2}{3}}(\mathbf{B} - \mathbf{m}\otimes\mathbf{Bm} - \mathbf{Bm}\otimes\mathbf{m})\\ +\lambda_M^4(\beta_4^2+1)\mathbf{m}\otimes\mathbf{m} \end{pmatrix}\\ -\lambda_M(\xi^4\sin^2\gamma - \xi^{-4}\cos^2\gamma)(\mathbf{I}-\mathbf{m}\otimes\mathbf{m}) \end{pmatrix} \end{matrix}
\end{aligned} \tag{3.50}
$$

For Galilean Invariance or frame indifference, as we showed before, $\mathbf{m}^* = Q\mathbf{m}$, and left Cauchy Green deformation tensor($\mathbf{B}$) is also Galilean invariant as well as frame indifferent.

$$\mathbf{B}^* = \mathbf{F}^*\mathbf{F}^{*T} = Q\mathbf{F}(Q\mathbf{F})^T = Q\mathbf{B}\mathbf{Q}^T \tag{3.51}$$

Thus it should be true that stress tensor from these relations is Galilean invariant as well as frame indifference because these equations are based on $\mathbf{m}$ and $\mathbf{B}$.

3. Computational Method - Matlab

1. From the motion, find deformation gradient and left and right Cauchy Green deformation tensors in terms of radius.

2. Once we find **B** and **m**, find five strain invariants.

3. Find components of stress(**T**) from constitutive relation(3.29) and by equilibrium equation in terms of radius(r) by trapezoidal rule with same step size as mixed.

4. Find axial load and moment by direct integration.[6]

[6] To see in detail, see Appendix.B

CHAPTER IV

RESULTS

A. Computational Time

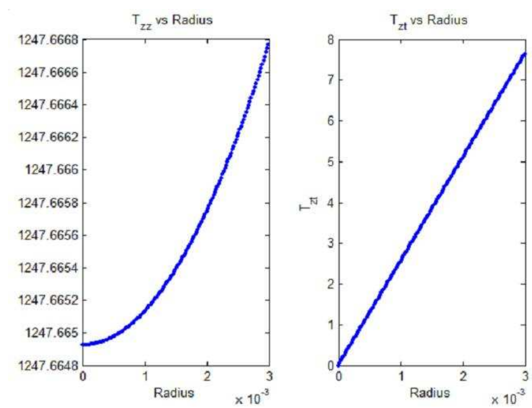

Fig. 6. Stresses. Units of the component of stress and radius are $[N/m^2]$ and $[m]$

Table I. MATLAB profile report: Summary

	Mixed	Eulerian
Total time	1083 sec	511 sec
Clock Speed	2600 MHz	2600 MHz
Number of M−function	57	59
Number of M−subfunctions	17	16
Number of MEX functions	1	1
Step size of radius	$5*10^{-6}$m	$5*10^{-6}$m

In Fig.6, as a matter of course, results from both computations must be same under stretch ratio(2.00) and twist per unit length(2.00), but computational time is different in Table I. Computational time for Eulerian approach is less than half of the time for Mixed approach. The reason that, as you read before, computation for mixed has main loop with step size of radius from first to the end. Moreover, inside of loop, we have to use "if" statement to distinct the larger eigenvalue of $\mathbf{C}_{2D}$. On the other hands, Eulerian approach doesn't need any "if" statement and we don't have to find N1 and N2. All equations are in terms of radius and, using these functions, we can directly find components of stress as functions of radius. If we use smaller step size than this for accuracy, based on this matlab code, computation for mixed faces to limitation of personal computer as "too large integer" where as computation for Eulerian does not. Of course, it depends on how making code efficiently, but algorithms for these computations doesn't change, so it is obvious that Eulerian approach is much better than Mixed approach. Furthermore, because deformation in reality is much more complicated than simple extension and torsion, time difference will be much larger.

B. Small Deformation and Large Deformation

One might think that, on small deformation, computational time for mixed will be getting close to computational time for Eulerian. Theoretically, it is correct because Lagrangian approach is almost same as Eulerain approach, which means there is almost no distinction between Mixed and Eulerain description. So, in Table II, we

Table II. Computational time for small and large deformation

	Stretch: 2.00 Twist: 2.00	Stretch: 1.02 Twist: 0.02
Time for Mixed Approach	1083 s	1170 s
Time for Eulerian Approach	511 s	549 s

used two computations under small deformation ($stretch = 1.02$, $twist = 0.02$) and large deformation ($stretch = 2.00$, $twist = 2.00$). As you see, small deformation doesn't influence computational time because computational time depends on the number of functions, length of equations, and step size of radius. The reason that there are small time differences such as $90s$ for mixed and $40s$ for Eulerain is because of length difference between the results from constitutive relations. In other words, because the results have to be used at every step size of radius inside of trapezoidal rule, a little different length, which is caused by change of parameters, of the results makes that time different.

CHAPTER V

CONCLUSION AND FUTURE WORK

Constitutive relations on five strain invariants by mixed approach has been already shown to have experimental advantages.[4] The constitutive relation by Eulerain approach can be applied to experimental analysis. Because five strain invariants($\beta_{1,2,3,4,5}$) are good for experimentally determining response terms, it is quite necessary to compare constitutive relation by Eulerian approach with Rivlin's model for data reduction. For the future work, this constitutive relations by the Eulerian approach is only useful for determining response terms by homogeneous deformations, so it will be necessary to develop this model in order to determine the response terms by inhomogeneous deformations.

The goal of this research is to extend this concept to biological tissues such as myocardium and vascular tissues. In order to do this, we need to apply this to stronger anisotropy such as orthotropy. In orthotropic material, six strain invariants have been already introduced by J.C.Criscione for cardiac tissue.[8] Through changing this elasticity to viscoelasticity, we can get more accurate solution for the analysis of heart. Therefore, it is necessary to find new constitutive models in general anisotropic and inhomogeneous material for modeling all kinds of biological tissues. This research will cause significant contribution to modeling in biomechanics.

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APPENDIX A

SOURCE CODE FOR MIXED APPROACH IN MATLAB

```
% Extension and Torsion to fiber reinforced cylinder
% Cylindrical Coordinate System
clear all
profile on
ro = 3*10^(-3); %outer radius[m]
r = sym('r','positive');
%r=radius,

format short e
gam = 0.02;          %0.02 twist per unit length
stretch = 1.02;      %1.02 stretch of extension
M = [0; 0; 1];  %fiber direction in reference configuration
i_1 = 5*10^(-6);
i_2 = 2*10^(-5);
k=1;
Fd = [1/sqrt(stretch) 0 0;0 1/sqrt(stretch) gam*r*stretch;0 0
      stretch]; %deformation gradient tensor
C = Fd' * Fd;
B = Fd * Fd';
C_i =  inv(C);
for re = 0:i_1:ro
    re
    Fd_v = [1/sqrt(stretch) 0 0;0 1/sqrt(stretch) gam*re*stretch
           ;0 0 stretch]; %deformation gradient tensor
    C_v = Fd_v' * Fd_v;
    B_v = Fd_v * Fd_v';
    C_v_2D = [C_v(1,1) C_v(1,2); C_v(2,1) C_v(2,2)];
    [V,E] = eig(C_v_2D);
    if E(1,1) >= E(2,2)
        N_1 = [V(1,1);V(2,1);0];
        % Largest stretch on the cross fiber direction in ref.
        N_2 = [V(1,2);V(2,2);0];
        % Smallest stretch on the cross fiber direction in ref.
    elseif E(1,1) < E(2,2)
        N_2 = [V(1,1);V(2,1);0];
        N_1 = [V(1,2);V(2,2);0];
    end
    m = (dot(M, C*M))^(-0.5)*(Fd*M); % fiber Direction in cur.
```

```
n_1_1 = (inv(Fd')*N_1);
n_1 = n_1_1/sqrt(n_1_1(1,1)^2 + n_1_1(2,1)^2 + n_1_1(3,1)^2);
n_2_1 = (inv(Fd')*N_2);
n_2 = n_2_1/sqrt(n_2_1(1,1)^2 + n_2_1(2,1)^2 + n_2_1(3,1)^2);

lamda =(dot(M,C*M))^0.5;% stretch in torsion
zeta = (dot(N_2,C_i*N_2)/dot(N_1,C_i*N_1))^(1/4);
%pure shear on the cross fiber direction
sai = sqrt((dot(M,C^2*M)/((dot(M,C*M))^2))-1);
%simple shear along fiber direction
gem = 0.5*acos(((dot(N_1,C*M))^2-(dot(N_2,C*M))^2)
                /((dot(N_1,C*M))^2+(dot(N_2,C*M))^2));
%relation between simple shear and ross fiber shear
%strain Invariants
beta_2 = log(lamda^(3/2));
beta_3 = log(zeta^2);
beta_4 = sai;
beta_5 = cos(2*gem);

for i = 1:3
    for j = 1:3
        m_m(i,j) = m(i,1)*m(j,1);% m tensor m
        n_11(i,j) = n_1(i,1)*n_1(j,1);% n_1 tensor n_1
        n_22(i,j) = n_2(i,1)*n_2(j,1);% n_2 tensor n_2
        mn_1(i,j) = m(i,1)*n_1(j,1);% m tensor n_1
        mn_2(i,j) = m(i,1)*n_2(j,1);% m tensor n_2
        n_12(i,j) = n_1(i,1)*n_2(j,1);% n_1 tensor n_2
    end
end
n_1m = mn_1';
n_2m = mn_2';
n_21 = n_12';

co = cos(gem);
si = sin(gem);
%Constitutive Relation
A_2 = m_m - 0.5*(eye(3)-m_m);
A_3 = n_11-n_22;
A_4 = lamda^(-3/2)*((zeta*co*(mn_1+n_1m)) +
                   (zeta^(-1)*si*(mn_2+n_2m)));
A_5 = (4*co*si)/(lamda^(3/2)*sai)*(((zeta*si*(mn_1+n_1m))
      - (zeta^(-1)*co*(mn_2+n_2m))) + ((lamda^(3/2)*sai)
      *((zeta^2-zeta^(-2))^(-1))*(n_21+n_12)));
%Linearized elasticity without prestress in reference conf.
%W=400beta_2^2+100beta_3^2+100beta_4^2
```

```
W_2 = 800*beta_2;
W_3 = 200*beta_3;
W_4 = 200*beta_4;
W_5 = 0;
Tp{k} = W_2*A_2 + W_3*A_3 + W_4*A_4 + W_5*A_5; %T+pI
s_1(k) =(Tp{k}(1,1) - Tp{k}(2,2))/r;



if rem(re,i_2) == 0
    if re ==0
        Tp_1 = simplify(Tp{k});
        P = subs(Tp_1(1,1),r,re);
        Trr = subs(Tp_1(1,1),r,re)-P;
        Tzz = subs(Tp_1(3,3),r,re)-P;
        subplot(1,2,1)
        plot(re,Tzz,'.'), title('T_z_z vs Radius ')
        ,xlabel('Radius'),ylabel('T_z_z')
        hold on
        Tzt = subs(Tp_1(3,2),r,0);
        subplot(1,2,2)
        plot(re,Tzt,'.'), title('T_z_t vs Radius ')
        ,xlabel('Radius'),ylabel('T_z_t')
        hold on
    else
        i_4 = 1;
        s_3 = 0;
        for rs = 0:i_1:re-i_1
            if rs == 0
                s_2 = (0 + subs(s_1(i_4+1),r,rs+i_1))*i_1/2;
                s_3 = s_3 + s_2;
            else
                s_2 = (subs(s_1(i_4),r,rs)
                      + subs(s_1(i_4+1),r,rs+i_1))*i_1/2;
                s_3 = s_3 + s_2;
            end
            i_4 = i_4+1;
        end
        P = subs(Tp{k}(1,1),r,re) + s_3;
        Trr = subs(Tp{k}(1,1),r,re) - P;
        Tzz = subs(Tp{k}(3,3),r,re) - P;
        subplot(1,2,1)
        plot(re,Tzz,'.')
        hold on
        Tzt = subs(Tp{k}(3,2),r,re);
```

```
                subplot(1,2,2)
                plot(re,Tzt,'.')
                hold on
            end
        end
        k= k+1;
end
hold off
% To find axial load and Moment
k_1 =1;
F_z = 0;
M_z = 0;
for rp =0:i_1:ro-i_1
        rp
        f_1_1=simplify(r*(2*Tp{k_1}(3,3)-Tp{k_1}(1,1)-Tp{k_1}(2,2)));
        g_1_1=simplify(Tp{k_1}(2,3)*r^2);
        f_1_2=simplify(r*(2*Tp{k_1+1}(3,3)-Tp{k_1+1}(1,1)-Tp{k_1+1}(2,2)));
        g_1_2=simplify(Tp{k_1+1}(2,3)*r^2);
        f_1=(subs(f_1_1,r,rp)+subs(f_1_2,r,rp+i_1))*i_1/2;
        F_z=F_z + f_1;
        g_1=(subs(g_1_1,r,rp)+subs(g_1_2,r,rp+i_1))*i_1/2;
        M_z=M_z + g_1;
        k_1=k_1+1;
end
F_z_f = pi*F_z;
M_z_f = 2*pi*M_z;
F_z_f
M_z_f
profile report
```

APPENDIX B

SOURCE CODE FOR EULERIAN APPROACH IN MATLAB

```
% By using strain invariant set with B,m
% Cylindrical Coordinate System
clear all
profile on
ro = 3*10^(-3); %outer radius[cm]
r = sym('r','positive');
%r=radius, p=Lagrangean multiplier
format short e
gam = 2.00;          %0.02 twist per unit length
stretch = 2.00;      %1.02 stretch of extension
M = [0; 0; 1];  %fiber direction in reference configuration
Fd = [1/sqrt(stretch) 0 0;0 1/sqrt(stretch) gam*r*stretch
      ;0 0 stretch]; %deformation gradient tensor
B = Fd * Fd';
C = Fd'* Fd;
B_i = inv(B);
m = (dot(M, C*M))^(-0.5)*(Fd*M); % fiber Direction in cur.
lamda =(dot(M,C*M))^0.5;%  stretch=1 in torsion
%strain Invariants
beta_2 = log ((lamda)^(3/2));
beta_3_1 = ((trace(B)-dot(m,B*m))/(2*(dot(m,B_i*m))^(1/2)))
        +sqrt((((trace(B)-dot(m,B*m))^2)/(4*(dot(m,B_i*m))))-1);
beta_3 = simplify(log(beta_3_1));
beta_4 = simplify(sqrt((dot(m,B*m)*dot(m,B_i*m))-1));
beta_5 = simplify(((trace(B)-dot(m,B*m))*(dot(m,B*m)
          +(dot(m,B_i*m))^(-1))-2*(trace(B_i)-dot(m,B_i*m)))
          /((dot(m,B*m)-(dot(m,B_i*m))^(-1))*sqrt((trace(B)
          -dot(m,B*m))^2 - 4*(dot(m,B_i*m)))));
zeta = sqrt(beta_3_1);%pure shear on the cross fiber direction
sai = beta_4;%simple shear along fiber direction
gem = 0.5*acos(beta_5);
%relation between simple shear and cross fiber shear
B_m = B*m;
B_im = B_i*m;
for i=1:3
     for j=1:3
          m_m(i,j) = m(i,1)*m(j,1);
          m_Bm(i,j) = m(i,1)*B_m(j,1);
```

```
            Bm_m(i,j) = B_m(i,1)*m(j,1);
        end
    end

            co = cos(gem);
            si = sin(gem);
%Constitutive Relation

A_2=simplify(m_m - 0.5*(eye(3)-m_m));
A_3=simplify(lamda/(zeta^2-zeta^(-2))*(2*B-2*(m_Bm+Bm_m)
    +2*lamda^2*(sai^2+1)*m_m-(1/lamda)*(zeta^2+zeta^(-2))
    *(eye(3)-m_m)));
A_4=simplify(lamda^(-2)*sai^(-1)*(m_Bm+Bm_m)
             - 2*(sai+sai^(-1))*m_m);
A_5=simplify((4/(lamda*sai^2*(zeta^2-zeta^(-2)))) * (B_i
              + lamda^2*B - (lamda^2-lamda^(-1)*(zeta^2*(si)^2
              +zeta^(-2)*(co)^2))*(m_Bm+Bm_m) - lamda*(zeta^2
              +zeta^(-2))*(eye(3)-m_m) + (lamda^4*(sai^2+1)-2*lamda
              *(sai^2+1)*(zeta^2*(si)^2+zeta^(-2)*(co)^2)-lamda^(-2))
              *m_m)+(4/(lamda*(zeta^2 - zeta^(-2))^2)) * (((zeta^2
              *(si)^2 -zeta^(-2)*(co)^2)*(lamda^2*(B-m_Bm-Bm_m)
              + lamda^4*(sai^2+1)*m_m)) - lamda*(zeta^4*(si)^2
              -zeta^(-4)*(co)^2)*(eye(3)-m_m)));

%Linearized elasticity without prestress in reference conf.
%W=400beta_2^2+100beta_3^2+100beta_4^2
W_2 = 800*beta_2;
W_3 = 200*beta_3;
W_4 = 200*beta_4;
W_5 = 0;
Tp = W_2*A_2 + W_3*A_3 + W_4*A_4 + W_5*A_5; %T+pI
s_1 =(Tp(1,1) - Tp(2,2))/r;

i = 5*10^(-6);
    for re = 0:2*10^(-5):ro
        if re == 0
            Tp_1 = simplify(Tp);
            P = subs(Tp_1(1,1),r,0);
            Trr = subs(Tp_1(1,1),r,0)-P;
            Tzz = subs(Tp_1(3,3),r,0)-P;
            subplot(1,2,1)
            plot(re,Tzz,'.'), title('T_z_z vs Radius ')
            ,xlabel('Radius'),ylabel('T_z_z')
            hold on
            Tzt = subs(Tp_1(3,2),r,0);
```

```
            subplot(1,2,2)
            plot(re,Tzt,'.'), title('T_z_t vs Radius ')
            ,xlabel('Radius'),ylabel('T_z_t')
            hold on
        else
            s_3 = 0;
            for rs = 0:i:(re-i)
                if rs ==0
                    s_2 = (0 + subs(s_1,r,rs+i))*i/2;
                    s_3 = s_3 + s_2;
                else
                    s_2 = (subs(s_1,r,rs) + subs(s_1,r,rs+i))*i/2;
                    s_3 = s_3 + s_2;
                end
            end
            P = subs(Tp(1,1),r,re) + s_3;
            Trr = subs(Tp(1,1),r,re) - P;
            Tzz = subs(Tp(3,3),r,re) - P;
            subplot(1,2,1)
            plot(re,Tzz,'.')
            hold on
            Tzt = subs(Tp(3,2),r,re);
            subplot(1,2,2)
            plot(re,Tzt,'.')
            hold on
        end
        re
    end
    hold off

% To find axial load and Moment
f_1 = simplify(r*(2*Tp(3,3) - Tp(1,1) - Tp(2,2)));
g_1 = simplify(Tp(2,3)*r^2);
f = simplify(int(f_1,r));
g = simplify(int(g_1,r));
F_z = pi*(subs(f,r,ro)-subs(f,r,0))
M_z = 2*pi*(subs(g,r,ro)-subs(g,r,0))
profile report
```

VITA

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The typist for this thesis was Min Jae Song.