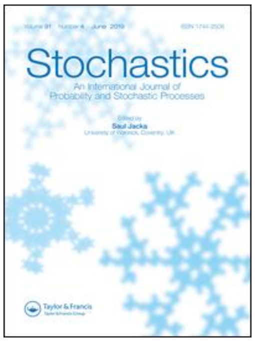



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# Discounted optimal stopping problems in continuous hidden Markov models


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# Discounted optimal stopping problems in continuous hidden Markov models

Pavel V. Gapeev

Department of Mathematics, London School of Economics, London, UK

**ABSTRACT**

We study a two-dimensional discounted optimal stopping problem related to the pricing of perpetual commodity equities in a model of financial markets in which the behaviour of the underlying asset price follows a generalized geometric Brownian motion and the dynamics of the convenience yield are described by an unobservable continuous-time Markov chain with two states. It is shown that the optimal time of exercise is the first time at which the commodity spot price paid in return to the fixed coupon rate hits a lower stochastic boundary being a monotone function of the running value of the filtering estimate of the state of the chain. We rigorously prove that the optimal stopping boundary is regular for the stopping region relative to the resulting two-dimensional diffusion process and the value function is continuously differentiable with respect to the both variables. It is verified by means of a change-of-variable formula with local time on surfaces that the value function and the boundary are determined as a unique solution of the associated parabolic-type free-boundary problem. We also give a closed-form solution to the optimal stopping problem for the case of an observable Markov chain.

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## 1. Introduction

The main aim of this paper is to study the analytic properties of the value function of the discounted optimal stopping problem:

$$V_* = \sup_{\tau} E\left[\int_0^{\tau} \mathrm{e}^{-rs}(X_s - L)\,\mathrm{d}s\right] \tag{1}$$

for a given constant $L > 0$. Here, for a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{G}, P)$ with a standard Brownian motion $B = (B_t)_{t\geq 0}$ and a continuous-time Markov chain $\Theta = (\Theta_t)_{t\geq 0}$ with two states, 0 and 1 (the processes $B$ and $\Theta$ are supposed to be independent under the probability measure $P$). We define the process $X = (X_t)_{t\geq 0}$ by

$$X_t = x \exp\left(\int_0^t \left(r - \frac{\sigma^2}{2} - \delta_0 - (\delta_1 - \delta_0)\Theta_s\right) \mathrm{d}s + \sigma\, B_t\right) \tag{2}$$

**CONTACT** Pavel V. Gapeev p.v.gapeev@lse.ac.uk

which solves the stochastic differential equation

$$dX_t = \big(r - \delta_0 - (\delta_1 - \delta_0)\Theta_t\big)X_t\,dt + \sigma\,X_t\,dB_t \quad (X_0 = x) \tag{3}$$

where $x > 0$ is fixed, and $r > 0$, $\delta_i > 0$, $i = 0, 1$, and $\sigma > 0$ are some given constants. In our application, the process $X$ describes the risk-neutral dynamics of the market spot price of a commodity under a risk-neutral (or martingale) probability measure $P$ in a model of financial markets, in which $r$ is the interest rate of a riskless bank account and $\sigma$ is the volatility coefficient. Suppose that $\Theta$ reflects the state of the economy, that is, the economy is either in the so-called *good* state when $\Theta = 0$, or in the so-called *bad* state when $\Theta = 1$, that also has an influence on the convenience yield of the commodity being equal to $\delta_0(1 - \Theta) + \delta_1\Theta$, which is usually unobservable to small investors trading in the market. We further assume that the process $\Theta$ has the same distribution with respect to the martingale measure $P$ as well as with respect to the initial or physical probability measure. This property enables us to specify the pricing measure $P$ from the set of all martingale measures in the incomplete market model defined in (2)–(3).

Suppose that the supremum in (1) is taken over all stopping times $\tau$ with respect to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the commodity spot price process $X$, and the expectation there is taken with respect to the risk-neutral probability measure $P$. Then, the value of (1) can be interpreted as the rational (or no-arbitrage) net present value of a perpetual commodity equity given that the equity holder issued a perpetual defaultable bond (debt) in an extension of the Black-Merton-Scholes model with an unobservable convenience yield. According to this contract, the equity holder receives the cash flow (generated by assets of the commodity producing firm) at the rate $X$ and pays the fixed coupon rate $L$ to the holder of the bond up to the time of default $\tau$ chosen by the former. The rational valuation of contracts of such type on the infinite time horizon and with the default opportunities in the classical model based on a one-dimensional geometric Brownian motion with constant coefficients was studied in Leland [34]. Other related problems include the rational valuation of perpetual American warrants in the classical Black-Merton-Scholes model of financial markets were formulated and solved by Samuelson [48] and McKean [37] (see, e.g. Shiryaev [49, Chapter VIII; Section 2a], Peskir and Shiryaev [45, Chapter VII; Section 25], or Detemple [11], for an extensive overview of other related results in the area).

Optimal stopping problems for two-dimensional diffusion processes have attracted a considerable attention in the literature on optimal stochastic control theory. One class of such problems is formed by the optimal stopping problems for one-dimensional continuous (time-homogeneous strong) Markov processes on finite time intervals initiated and studied by van Moerbeke [50], Jacka [28], Broadie and Detemple [4], and Carr et al. [6] among others (see also Myneni [38] for the review of contemporary results in the area). It turned out that such problems for one-dimensional diffusions with finite-time horizon or time-dependent rewards are inherently two-dimensional, and thus, they are analytically more difficult than those for time-independent ones on infinite time intervals. A standard approach for handling such a problem is to analyze the equivalent free-boundary problem for the infinitesimal (parabolic) operator of the underlying diffusion process. It was shown by Peskir [41,42] (see also Gapeev and Peskir [20,21] and other following related articles on optimal stopping problems for one-dimensional diffusion processes on finite time intervals), by using the change-of-variable formula with local time on curves derived by Peskir [40], that the value functions and optimal stopping boundaries are uniquely characterized by the parabolic free-boundary problems which are equivalent to the (systems of) nonlinear integral equations for the boundaries.

Another class of such problems is formed by the optimal stopping problems for two-dimensional diffusion processes with time-independent rewards on infinite-time intervals. Such problems particularly appear in relation to the Bayesian sequential hypothesis testing and quickest change-point (disorder) detection problems for observable diffusion processes. It was shown in Gapeev and Shiryaev [23,24] that the sufficient statistics processes containing the observable processes and the appropriated posterior probabilities as their state space components are driven by the same (one-dimensional innovation) standard Brownian motion, so that the original problems are equivalent to free-boundary problems for partial differential operators of parabolic type. These problems of statistical sequential analysis were taken further and solved by Johnson and Peskir [31,32] for the cases of models with observable Bessel processes. It was shown, by using the change-of-variable formula with local time on surfaces derived by Peskir [43], that the value functions and optimal stopping boundaries are uniquely characterized by the parabolic-type free-boundary problems which are equivalent to the (systems of) nonlinear Fredholm integral equations for the boundaries. Other optimal stopping problems for two-dimensional diffusion processes were studied by Assing et al. [2], where the monotonicity and continuity of the value functions were proved by using time-change and coupling techniques. More recently, Ernst et al. [16] solved the optimal stopping problem for a two-dimensional diffusion process related to the optimal real-time detection of a drifting Brownian coordinate which is is equivalent to a free-boundary problem the associated partial differential operator of elliptic type. The important recent results in the area comprise the continuity of the optimal stopping boundaries in optimal stopping problems for two-dimensional diffusions proved by Peskir [44] and the global $C^1$-regularity of the value function in two-dimensional optimal stopping problems studied by De Angelis and Peskir [9].

In the present paper, we study the necessarily two-dimensional optimal stopping problem of (7) which is associated with the one of (1) for the commodity spot price expressed by a generalized geometric Brownian motion having the drift rate described by the filtering estimate of an unobservable continuous-time Markov chain with two states (Wonham filter). Such a hidden Markov model was proposed by Shiryaev [49, Chapter III, Section 4a] for the description of interest rate dynamics, and then applied by Elliott and Wilson [14] for the computation of zero-coupon bond prices and other quantities in the interest rate framework. First one-dimensional optimal stopping problems for the filtering estimate processes were studied by Beibel and Lerche [18]. Some preliminary results including the monotonicity of the optimal stopping boundary and the verification assertion related to a different optimal stopping problem in the same model with the resulting two-dimensional continuous (strong Markov) diffusion process were presented in Gapeev [19] (see also Gapeev and Rodosthenous [22] for a study of another optimal stopping problem in the same model). In contrast to the previous results, we give a rigorous proof of regularity of the optimal stopping boundary as well as establish the continuous differentiability of the value function for the optimal stopping problem of (7) that is crucial for the proof of the appropriate verification assertion. Note that such analytic properties of the candidate value function and the optimal stopping boundary were not proved but just assumed in [19]. Observe that closed-form solutions of other optimal stopping problems in the corresponding model with an observable two-state Markov chain were obtained by Guo [26], Guo and Zhang [27], and Gapeev et al. [25], for the perpetual American standard and lookback (Russian) option problems, respectively. Such problems in models with observable regime-switching parameters were studied by Jobert and Rogers [30] for an exponential diffusion-type model with several states for the Markov chain, by Dalang and Hongler [8] for a model with a two-state continuous-time Markov chain and no diffusion part, and by Jiang and Pistorius [29] for an exponential jump-diffusion model, among others. Various analytic properties of value functions of optimal stopping problems in models with geometric Brownian motions having unobservable regime-switching parameters were studied by Buffington and Elliott [5] for the numerical approximation of the value function of an American option in the corresponding extension of the Black-Merton-Scholes model, by Décamps et al. [10] for the investment timing problem in a model with a geometric Brownian motion having a random (Bernoulli) drift rate. Further properties of the value functions (including numerical approximations for the optimal stopping boundaries) of the problems of optimal liquidation of risky assets described by generalized geometric Brownian motions with unobservable random drift rates were studied by Ekström and Lu [12] for the case of a finite time horizon, and by Ekström and Vaicenavicius [13] for the case of a general random drift rate.

The rest of the paper is organized as follows. In Section 2, we embed the original problem of (1) into the optimal stopping problem of (7) for the two-dimensional continuous Markov diffusion process $(X, \Pi)$ defined in (2)–(3) and (4)–(6). It is shown that the optimal stopping time $\tau_*$ is expressed as the first time at which the commodity spot price process $X$ hits a lower stochastic boundary $a_*(\Pi)$ which represents a decreasing function of the running value of the filtering economic state estimate process $\Pi$ (Lemma 2.1). In Section 3, we formulate the equivalent free-boundary problem for a partial differential equation of parabolic type and present rigorous proofs of the facts that the optimal stopping boundary is regular for the stopping region relative to the process $(X, \Pi)$ and the value function of the optimal stopping problem is continuously differentiable in both variables (Lemmata 3.1–3.4). In Section 4, in order to be able to apply the change-of-variable formula derived in [43], we introduce an appropriate change of variables that allows to reduce the resulting parabolic-type partial differential equation to the normal (or canonic) form. We verify that the solution of the associated free-boundary problem provides the solution of the initial optimal stopping problem (Lemma 4.1). We state the main result concerning the rational valuation of the perpetual commodity equities in the considered hidden Markov model (Theorem 4.2). We also give a closed-form solution to the optimal stopping problem of (7) in terms of Gauss' hypergeometric functions under certain relations on the parameters of the model, for the case of unobservable random (Bernoulli) drift rates (Corollary 4.3). Finally, in Appendix, we derive a closed-form solution to the optimal stopping problem of (A1) in the model with an observable Markov chain $\Theta$ (Corollary A.1), which gives bounds for the value function and the optimal stopping boundary for the original problem in the hidden Markov model.

## 2. Formulation of the problem

In this section, we introduce the setting and notation of the optimal stopping problem which is related to the pricing of perpetual commodity equities in the underlying diffusion-type model with *unobservable* convenience yield dynamics described by a continuous-time Markov chain with two states.

### 2.1. The model with partial information

Let us assume that the process $\Theta$ has the initial distribution $\{1-\pi, \pi\}$, for $\pi \in [0,1]$, the transition probability matrix $\{(\lambda_0\, \mathrm{e}^{-(\lambda_0+\lambda_1)t} + \lambda_1)/(\lambda_0+\lambda_1), \lambda_0(1-\mathrm{e}^{-(\lambda_0+\lambda_1)t})/(\lambda_0+\lambda_1); \lambda_1(1-\mathrm{e}^{-(\lambda_0+\lambda_1)t})/(\lambda_0+\lambda_1), (\lambda_1\, \mathrm{e}^{-(\lambda_0+\lambda_1)t} + \lambda_0)/(\lambda_0+\lambda_1)\}$, and thus, the intensity-matrix $\{-\lambda_0, \lambda_0; \lambda_1, -\lambda_1\}$, for all $t \geq 0$ and some $\lambda_i > 0$, $i = 0, 1$, fixed. In other words, the Markov chain $\Theta$ changes its state from $i$ to $1-i$ at exponentially distributed times of intensity $\lambda_i$, for every $i = 0, 1$, which are independent of the dynamics of the standard Brownian motion $B$. Such a process $\Theta$ is called a *telegraphic signal* in the literature (see, e.g. [36, Chapter IX, Section 4] or [15, Chapter VIII]). It is shown by means of standard arguments (see, e.g. [36, Chapter IX] or [15, Chapter VIII]) that the commodity spot price process $X$ from (2)–(3) admits the representation

$$\mathrm{d}X_t = \big(r - \delta_0 - (\delta_1 - \delta_0)\Pi_t\big)X_t\, \mathrm{d}t + \sigma\, X_t\, \mathrm{d}\overline{B}_t \quad (X_0 = x) \tag{4}$$

on its natural filtration $(\mathcal{F}_t)_{t\geq 0}$, while the filtering economic state estimate $\Pi = (\Pi_t)_{t\geq 0}$ defined by $\Pi_t = E[\Theta_t \,|\, \mathcal{F}_t] \equiv P(\Theta_t = 1 \,|\, \mathcal{F}_t)$ solves the stochastic differential equation

$$\mathrm{d}\Pi_t = \big(\lambda_0(1-\Pi_t) - \lambda_1\, \Pi_t\big)\, \mathrm{d}t - \frac{\delta_1 - \delta_0}{\sigma}\Pi_t(1-\Pi_t)\, \mathrm{d}\overline{B}_t \quad (\Pi_0 = \pi) \tag{5}$$

for some $(x, \pi) \in (0, \infty) \times [0, 1]$ fixed. Here, the innovation process $\overline{B} = (\overline{B}_t)_{t\geq 0}$ defined by

$$\overline{B}_t = \int_0^t \frac{\mathrm{d}X_s}{\sigma X_s} - \frac{1}{\sigma}\int_0^t \big(r - \delta_0 - (\delta_1 - \delta_0)\Pi_s\big)\, \mathrm{d}s \tag{6}$$

is a standard Brownian motion, according to P. Lévy's characterization theorem (see, e.g. [36, Theorem 4.1]). It can be verified that $(X, \Pi)$ is a (time-homogeneous strong) Markov process, under $P$ with respect to its natural filtration $(\mathcal{F}_t)_{t\geq 0}$, as a unique strong solution of the system of stochastic differential equations in (4)–(5) (see, e.g. [39, Theorem 7.2.4]). Note that the model presented above which contains the observable process $X$ and the estimate process $\Pi$ of the continuous-time Markov chain $\Theta$ given running observations is known as the Wonham filter in the literature (see [36, Chapter IX] for the derivation of stochastic differential equations for the filtering estimates of continuous-time Markov chains).

### 2.2. The optimal stopping problem

Suppose that an equity holder issues a perpetual defaultable bond (debt) at time 0. Assume that the equity holder receives the cash flow (generated by assets of the commodity producing firm) being equal to the commodity spot price $X$ and pays in return the fixed coupon rate $L$ to the holder of the bond for the opportunity to declare a default to the latter at some random time $\tau$ which the former can choose. In this respect, the equity holder looks for the exercise time $\tau_*$ maximizing the expected cumulative discounted net payout related to the contract, so that the rational (or no-arbitrage) net present value of such a contingent claim is given by the value $V_*(x,\pi)$ of the optimal stopping problem

$$V_*(x,\pi) = \sup_{\tau} E_{x,\pi}\left[\int_0^{\tau} e^{-rs}(X_s - L)\,ds\right] \tag{7}$$

for some $L>0$ fixed, where $E_{x,\pi}$ denotes the expectation with respect to the probability measure $P_{x,\pi}$ under which the two-dimensional (time-homogeneous strong) Markov process $(X,\Pi)$ starts at some $(x,\pi)\in(0,\infty)\times[0,1]$. We assume that the supremum in (7) is taken over all stopping times $\tau$ with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$. It follows from the results of [7, Theorem 4.1] based on the solutions of the associated (doubly) reflected backward stochastic differential equations that the optimal stopping problem of (7) has a value, for each $(x,\pi)\in(0,\infty)\times[0,1]$ fixed. In this paper, we study analytic properties of the value function of (7) in the two-dimensional diffusion model of $(X,\Pi)$ defined in (2)–(3) and (4)–(6). It follows from the explicit form of the process $X$ in (2)–(3) and the representations for the processes $X$ and $\Pi$ in (4)–(5) that the discounted process $(e^{-rt}X_t)_{t\ge 0}$ is a strict supermartingale closed at zero, under the assumption that $\delta_i>0$, for $i=0,1$, and thus, the value function in (7) is finite, for each $(x,\pi)\in(0,\infty)\times[0,1]$ fixed. By virtue of the results of general theory of optimal stopping problems for Markov processes, the optimal stopping time $\tau_*$ in the problem of (7) should have the form

$$\tau_* = \inf\left\{t\ge 0 \,\middle|\, V_*(X_t,\Pi_t)=0\right\} \tag{8}$$

so that the corresponding continuation and stopping regions $C_*$ and $D_*$ are given by

$$C_* = \left\{(x,\pi)\in(0,\infty)\times[0,1] \,\middle|\, V_*(x,\pi)>0\right\} \tag{9}$$

and

$$D_* = \left\{(x,\pi)\in(0,\infty)\times[0,1] \,\middle|\, V_*(x,\pi)=0\right\} \tag{10}$$

respectively (see, e.g. [45, Chapter I, Subsection 2.2]). We prove in Subsection 3.1 below that $V_*(x,\pi)$ is a continuous function, so that the set $C_*$ is open and the set $D_*$ is closed.

### 2.3. The structure of optimal stopping times

Let us now show the form of the optimal stopping time $\tau_*$ in (8) and clarify the structure of the associated continuation and stopping regions $C_*$ and $D_*$ in (9) and (10), respectively. From now on, we assume without loss of generality that the inequality $\delta_0>\delta_1$ holds. In this case, it follows from the explicit form of the process $X$ in (2)–(3) and the representations for the processes $X$ and $\Pi$ in (4)–(5) as well as from the comparison results for solutions of (one-dimensional time-homogeneous) stochastic differential equations (see, e.g. [17, Theorem 1]) that the function $V_*(x,\pi)$ in (7) is increasing in the both variables $x$ and $\pi$ on $(0,\infty)$ and $[0,1]$, respectively. In the rest of this section, we indicate by $(X^{(x,\pi)},\Pi^{(\pi)})$ the dependence of the process $(X,\Pi)$ defined in (4) and (5) on the starting point $(x,\pi)\in(0,\infty)\times[0,1]$.

**Lemma 2.1:** *Suppose that the processes $X$ and $\Pi$ are defined by* (2)–(3) *and* (4)–(6), *with $r>0$, $\delta_0>\delta_1>0$, $\sigma>0$, and $\lambda_i\ge 0$, for $i=0,1$. Then, the optimal stopping time $\tau_*$*

*from* (8) *in the problem of* (7)*, for some $L > 0$ fixed, admits the representation*

$$\tau_* = \inf \left\{ t \geq 0 \,\middle|\, X_t \leq a_*(\Pi_t) \right\} \tag{11}$$

*so that the continuation and stopping regions $C_*$ and $D_*$ in* (9) *and* (10) *take the forms*

$$C_* = \left\{ (x, \pi) \in (0, \infty) \times [0, 1] \,\middle|\, x > a_*(\pi) \right\} \tag{12}$$

*and*

$$D_* = \left\{ (x, \pi) \in (0, \infty) \times [0, 1] \,\middle|\, x \leq a_*(\pi) \right\} \tag{13}$$

*respectively. Here, $a_*(\pi)$ is a function satisfying the properties*

$$a_*(\pi) : [0, 1] \to (0, L] \text{ is decreasing and } f_*(i) \leq a_*(i) \leq L \text{ holds, for } i \in \{0, 1\}, \tag{14}$$

*where $f_*(i)$, for $i = 0$, 1, are determined from the expressions in* (A25) *and* (A26) *with* (A12) *below.*

***Proof:*** We first observe from the form of the integrand in the expression of (7) that it is not optimal for the equity holder to exercise the contract when $X_t \geq L$, for $t \geq 0$. This fact means that the points $(x, \pi) \in [L, \infty) \times [0, 1]$ belong to the continuation region $C_*$ in (9). On the other hand, the structure of the reward in the expression of (7) also implies that the equity holder should exercise the contract at some time when $X_t \leq L$, for $t \geq 0$. This fact means that the points $(x, \pi) \in (0, L] \times [0, 1]$ cover the stopping region $D_*$ in (10).

Let us now fix some $(x, \pi) \in C_*$ such that $x < L$ holds, and consider the optimal stopping time $\tau_* = \tau_*(x, \pi)$ for the equity holder. Then, by means of the results of general optimal stopping theory for Markov processes (see, e.g. [45, Chapter I, Section 2.2]), we conclude from the structure of the continuation region $C_*$ in (9) and the form of the stopping time in (8) as well as from the equality in (7) that the expression

$$V_*(x, \pi) = E_{x,\pi} \left[ \int_0^{\tau_*} \mathrm{e}^{-rs} \big( X_s^{(x,\pi)} - L \big) \, \mathrm{d}s \right] > 0 \tag{15}$$

holds. Hence, taking any $x'$ such that $x < x' \leq L$ and $\pi < \pi'$ and using the property that the function $V_*(x, \pi)$ is increasing in $x$ and $\pi$ on $(0, \infty)$ and $[0, 1]$, we obtain from the expression in (15) that the inequalities $V_*(x', \pi') \geq V_*(x, \pi) > 0$ are satisfied, so that $(x', \pi') \in C_*$ too. On the other hand, if we assume that $(x, \pi) \in D_*$ such that $x < L$, using arguments similar to the ones above, we obtain that $V_*(x'', \pi'') \leq V_*(x, \pi) = 0$ holds, for all $x'' \leq x < L$ and $\pi'' \leq \pi$, so that $(x'', \pi'') \in D_*$. Therefore, we may conclude that the optimal stopping time $\tau_*$ has the form of (11), where the left-hand boundary function $a_*(\pi)$ is decreasing on $[0, 1]$, thus proving the claim. Note that the existence of such a boundary $a_*(\pi)$ can also be deduced from the convexity of the function $x \mapsto V_*(x, \pi)$ on $(0, \infty)$, for each $\pi \in [0, 1]$ fixed.

It is shown in Appendix below that the function $W_*(x, i)$ from (A1) admits the explicit expressions in (A27) and (A28) with (A22)–(A24) and the associated optimal stopping time $\zeta_*$ is given by (A2), where the numbers $f_*(i)$, for $i = 0$, 1, are uniquely determined from the expressions in (A25) and (A26) with (A12). If we suppose that $a_*(i) < f_*(i)$ holds, for some $i = 0$, 1, then, for each $x \in (a_*(i), f_*(i))$ given and fixed, we would have

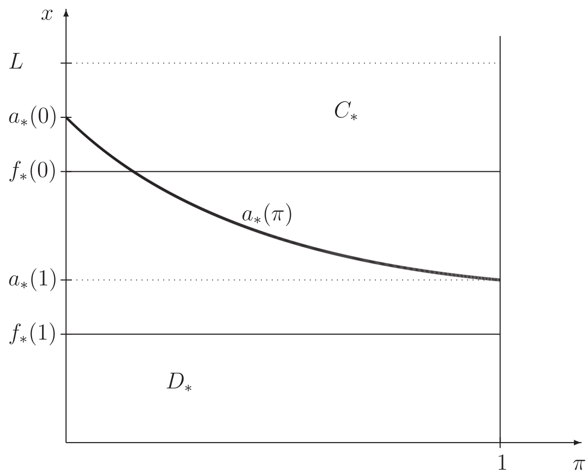


**Figure 1.** A computer drawing of the optimal exercise boundary $a_*(\pi)$.

$V_*(x,i) > 0 = W_*(x,i)$, contradicting the obvious fact that $V_*(x,i) \le W_*(x,i)$, for all $x > 0$ and every $i = 0, 1$. Note that the latter inequality holds, since the supremum in (7) is taken over all stopping times $\tau$ with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$ which is smaller than the corresponding filtration $(\mathcal{G}_t)_{t\ge 0}$ for the supremum over all stopping times $\zeta$ in (A1). Thus, we may conclude that the inequality $f_*(i) \le a_*(i)$ should hold, for every $i = 0, 1$ (see Figure 1 below for a computer drawing of the optimal stopping boundaries $a_*(\pi)$, for $\pi \in [0,1]$, and $f_*(i)$, for $i = 0, 1$).

■

## 3. Preliminaries

In this section, we derive analytic properties for the value function of the optimal stopping problem which are necessary for the proof of the main results of the paper stated below. We start with the formulation of a free-boundary problem which is equivalent to the original optimal stopping problem.

### 3.1. The free-boundary problem

By means of standard arguments based on an application of Itô's formula (see, e.g. [Chapter V, Section 5.1, 33] or [Theorem 7.5.4, 39]), it is shown that the infinitesimal operator $\mathbb{L}_{(X,\Pi)}$ of the process $(X, \Pi)$ solving the stochastic differential equations in (4)–(5) has the structure

$$\mathbb{L}_{(X,\Pi)} = \big(r - \delta_0 - (\delta_1 - \delta_0)\pi\big)x\,\partial_x + \frac{\sigma^2 x^2}{2}\partial_{xx} - (\delta_1 - \delta_0)x\pi(1-\pi)\partial_{x\pi} + \big(\lambda_0(1-\pi) - \lambda_1\pi\big)\partial_\pi + \frac{1}{2}\left(\frac{\delta_1 - \delta_0}{\sigma}\right)^2 \pi^2(1-\pi)^2\partial_{\pi\pi} \tag{16}$$

for all $(x,\pi) \in (0,\infty) \times (0,1)$. In order to characterize the unknown value function $V_*(x,\pi)$ from (7) and the unknown boundary $a_*(\pi)$ from (11), we may use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [45, Chapter IV, Section 8]) and formulate the following associated free-boundary problem

$$(\mathbb{L}_{(X,\Pi)} V - rV)(x,\pi) = -(x-L) \quad \text{for } x > a(\pi) \tag{17}$$
$$V(x,\pi)\big|_{x=a(\pi)} = 0 \quad \text{(instantaneous stopping)} \tag{18}$$
$$V_x(x,\pi)\big|_{x=a(\pi)} = 0 \quad \text{and} \quad V_\pi(x,\pi)\big|_{x=a(\pi)} = 0 \quad \text{(smooth fit)} \tag{19}$$
$$V(x,\pi) = 0 \quad \text{for } x < a(\pi) \tag{20}$$
$$V(x,\pi) > 0 \quad \text{for } x > a(\pi) \tag{21}$$
$$(\mathbb{L}_{(X,\Pi)} V - rV)(x,\pi) < -(x-L) \quad \text{for } x < a(\pi) \tag{22}$$

for $\pi \in (0,1)$. Observe that the superharmonic characterization of the value function (see, e.g. [45, Chapter IV, Section 9]) implies that $V_*(x,\pi)$ is the smallest function satisfying (17)–(18) and (20)–(21) with the boundary $a_*(\pi)$. Note that the inequality in (22) follows directly from the assertion of Lemma 2.1 which was proved in Subsection 2.3 above.

### 3.2. Continuity of the value function

Let us now show that the value function $V_*(x,\pi)$ in (7) is continuous at any point $(x,\pi) \in (0,\infty) \times [0,1]$. In order to deduce this property, it is enough to prove the following assertion.

**Lemma 3.1:** *The value function $V_*(x,\pi)$ of the optimal stopping problem in (7) has the properties*

$$x \mapsto V_*(x,\pi) \text{ is continuous at } x' \text{ uniformly over } \pi \in [\pi' - \varepsilon, \pi' + \varepsilon] \tag{23}$$
$$\pi \mapsto V_*(x',\pi) \text{ is continuous at } \pi' \tag{24}$$

*for every $(x',\pi') \in (0,\infty) \times (0,1)$ given and fixed, with some $\varepsilon > 0$ small enough.*

***Proof:*** In order to deduce the property of (23), let us fix some $x_1 \le x_2$ in $[x' - \varepsilon, x' + \varepsilon]$ and $\pi \in [\pi' - \varepsilon, \pi' + \varepsilon]$ such that the associated square belongs to $(0,\infty) \times [0,1]$. We consider $\tau_* = \tau_*(x_1,\pi)$ the optimal stopping time in (7) for the starting point $(x_1,\pi)$ of the process $(X,\Pi)$. Then, taking into account the explicit form of the process $X$ in (2)–(3) and the representations for the processes $X$ and $\Pi$ in (4)–(5) as well as the fact that the left-hand boundary $a_*(\pi)$ from (11) is a decreasing function, under the assumption that $\delta_0 > \delta_1$, we get

$$\begin{aligned} 0 &\le V_*(x_2,\pi) - V_*(x_1,\pi) \\ &\le E\left[\int_0^{\tau_*} \mathrm{e}^{-rs}\big(X_s^{(x_2,\pi)} - L\big)\,\mathrm{d}s\right] - E\left[\int_0^{\tau_*} \mathrm{e}^{-rs}\big(X_s^{(x_1,\pi)} - L\big)\,\mathrm{d}s\right] \\ &= E\left[\int_0^{\tau_*} \mathrm{e}^{-rs}\big(X_s^{(x_2,\pi)} - X_s^{(x_1,\pi)}\big)\,\mathrm{d}s\right] = (x_2 - x_1) E\left[\int_0^{\tau_*} \mathrm{e}^{-rs} X_s^{(1,\pi)}\,\mathrm{d}s\right] \end{aligned} \tag{25}$$

where the last expectation is finite, for all $x_1$ and $x_2$ from $[x' - \varepsilon, x' + \varepsilon]$. Here, we recall that $(X^{(x,\pi)}, \Pi^{(\pi)})$ indicates the dependence of the process $(X,\Pi)$ on the starting point

$(x, \pi) \in (0, \infty) \times [0, 1]$. Observe that the right-hand side in (25) converges monotonically to zero as $x_1$ approaches $x_2$, independently of $\pi \in [\pi' - \varepsilon, \pi' + \varepsilon]$ for any $\varepsilon > 0$ fixed, so that the property in (23) holds.

In order to deduce the property of (24), let us fix $\pi_1 \le \pi_2$ in $[\pi' - \varepsilon, \pi' + \varepsilon]$ such that the associated interval belongs to $[0, 1]$. We now denote by $\tau_* = \tau_*(x', \pi_1)$ the optimal stopping time in (7) for the starting point $(x', \pi_1)$ of the process $(X, \Pi)$. Then, taking into account the explicit form of the process $X$ in (2)–(3) and the representations for the processes $X$ and $\Pi$ in (4)–(5), we get

$$\begin{aligned} 0 &\le V_*(x', \pi_2) - V_*(x', \pi_1) \\ &\le E\left[\int_0^{\tau_*} \mathrm{e}^{-rs}\big(X_s^{(x',\pi_2)} - L\big)\, \mathrm{d}s\right] - E\left[\int_0^{\tau_*} \mathrm{e}^{-rs}\big(X_s^{(x',\pi_1)} - L\big)\, \mathrm{d}s\right] \\ &= E\left[\int_0^{\tau_*} \mathrm{e}^{-rs}\big(X_s^{(x',\pi_2)} - X_s^{(x',\pi_1)}\big)\, \mathrm{d}s\right] = x'\, E\left[\int_0^{\tau_*} \mathrm{e}^{-rs}\big(X_s^{(1,\pi_2)} - X_s^{(1,\pi_1)}\big)\, \mathrm{d}s\right] \end{aligned} \tag{26}$$

where the last expectation is finite as the one in (25), for all $\pi_1$ and $\pi_2$ from $[\pi' - \varepsilon, \pi' + \varepsilon]$, and we have, under the assumption that $\delta_0 > \delta_1$, that

$$\begin{aligned} X_t^{(1,\pi_2)} - X_t^{(1,\pi_1)} &= X_t^{(1,\pi_1)} \left(\frac{X_t^{(1,\pi_2)}}{X_t^{(1,\pi_1)}} - 1\right) \\ &= X_t^{(1,\pi_1)} \left(\exp\left(\int_0^t (\delta_0 - \delta_1)\big(\Pi_s^{(\pi_2)} - \Pi_s^{(\pi_1)}\big)\, \mathrm{d}s\right) - 1\right) \end{aligned} \tag{27}$$

for all $t \ge 0$. Here, $\Pi^{(\pi)}$ indicates the dependence of the process $\Pi$ on the starting point $\pi \in [0, 1]$. Hence, by using the comparison results for strong solutions of stochastic differential equations from [17, Theorem 1] and applying the Lebesgue dominated convergence theorem, we may conclude that the right-hand side in (26) converges to zero as $\pi_1$ approaches $\pi_2$, for any $\varepsilon > 0$ fixed, so that the property in (24) holds. This fact particularly means that the instantaneous-stopping condition of (18) above is satisfied. ■

### 3.3. Regularity of the optimal stopping boundary

Let us now prove the regularity of the boundary $\partial C_*$ of the continuation region in (12).

**Lemma 3.2:** *The boundary $\partial C_*$ of the continuation region in* (12) *is regular for the stopping region $D_*$ in* (13) *relative to the process $(X, \Pi)$ defined in* (2)–(3) *and* (4)–(6)*, under $\delta_0 > \delta_1$.*

***Proof:*** By virtue of the sample path structure and distributional properties of the two-dimensional diffusion processes $(X, \Pi)$ from (4)–(5) started at a point $(x', \pi')$, for some $(x', \pi') \in (0, \infty) \times (0, 1)$ fixed, and an infinitesimally small deterministic time interval $\Delta$, we observe that the representations

$$X_\Delta = x' + \big(r - \delta_0 - (\delta_1 - \delta_0)\pi'\big)x'\, \Delta + \sigma\, x'\, \Delta\overline{B} + o(\Delta) \quad \text{as } \Delta \downarrow 0 \; (P\text{—a.s.}) \tag{28}$$

and

$$\Pi_\Delta = \pi' + \big(\lambda_0(1 - \pi') - \lambda_1\, \pi'\big)\Delta - \frac{\delta_1 - \delta_0}{\sigma}\pi'(1 - \pi')\Delta\overline{B} + o(\Delta) \quad \text{as } \Delta \downarrow 0 \; (P\text{—a.s.}) \tag{29}$$

hold, where we set $\Delta\overline{B} = \overline{B}_\Delta$, and $o(\Delta)$ denotes a random function satisfying $o(\Delta)/\Delta \to 0$ as $\Delta \downarrow 0$ ($P$-a.s.). Recall that $\Delta\overline{B} \equiv \overline{B}_\Delta \sim Z\sqrt{\Delta}$ as $\Delta \downarrow 0$ ($P$-a.s.), where $Z$ is a standard normal random variable. Then, starting from the point $(x', \pi')$ and letting the process $(X, \Pi)$ evolve for an infinitesimal amount of time $\Delta$, we see that the process moves infinitesimally along the direction

$$\big(X_\Delta - x', \Pi_\Delta - \pi'\big) \sim \left(\sigma\, x'\, Z\sqrt{\Delta}, -\frac{\delta_1 - \delta_0}{\sigma}\pi'(1-\pi')Z\sqrt{\Delta}\right) \quad \text{as } \Delta \downarrow 0 \ (P\text{—a.s.}) \tag{30}$$

for any point $(x', \pi') \in (0, \infty) \times (0, 1)$ fixed. In other words, the two-dimensional process $(X, \Pi)$ moves along either the South-West or the North-East direction in the plane, under the assumption that $\delta_0 > \delta_1$. Hence, combining this fact with the fact that the boundary $a_*(\pi)$ is decreasing and taking into account the fact that the local drift of the process $(X, \Pi)$ has the order of $\Delta$, we may conclude that the first hitting time $\tau_*(x', \pi')$ to the stopping set $D_*$ converges to zero ($P$-a.s.) as the point $(x', \pi')$ approaches the point $(x, \pi)$ such that $x = a_*(\pi)$. This fact exactly means that all the points of the boundary $\partial C_*$ are regular for the stopping region $D_*$ relative to the process $(X, \Pi)$ (see, e.g. [39, Subsection 9.2] for an extensive discussion on this point and other references to the related literature). ■

### 3.4. Smooth-fit conditions for the value function

Let us now show that the value function $V_*(x, \pi)$ in (7) satisfies the smooth-fit conditions of (19) above.

**Lemma 3.3:** *The value function $V_*(x, \pi)$ of the optimal stopping problem in* (7) *satisfies the smooth-fit conditions of* (19).

***Proof:*** Let us further consider a point $(x, \pi) \in (0, L] \times (0, 1)$ at the boundary $\partial C_*$, so that $x = a_*(\pi)$ and $V_*(x, \pi) = 0$ holds. In order to derive the property in the left-hand part of (19), we first observe directly from the structure of the continuation region in (9) and (12) that the inequality

$$\liminf_{\varepsilon \downarrow 0} \frac{V_*(x+\varepsilon, \pi) - V_*(x, \pi)}{\varepsilon} \ge 0 \tag{31}$$

is satisfied, due to the fact that $V_*(x+\varepsilon, \pi) \ge 0$ holds. Let us now denote by $\tau^1_\varepsilon = \tau_*(x + \varepsilon, \pi)$ the optimal stopping time in (7) for the starting point $(x+\varepsilon, \pi)$ of the process $(X, \Pi)$, with some $\varepsilon > 0$ small enough. Then, taking into account the explicit form of the process $X$ in (2)–(3) and the representations for the processes $X$ and $\Pi$ in (4)–(5) as well as the fact that the left-hand boundary $a_*(\pi)$ from (11) is a decreasing function, we get

$$\begin{aligned} &V_*(x+\varepsilon, \pi) - V_*(x, \pi) \\ &\le E\left[\int_0^{\tau^1_\varepsilon} \mathrm{e}^{-rs}\big(X_s^{(x+\varepsilon,\pi)} - L\big)\,\mathrm{d}s\right] - E\left[\int_0^{\tau^1_\varepsilon} \mathrm{e}^{-rs}\big(X_s^{(x,\pi)} - L\big)\,\mathrm{d}s\right] \\ &= E\left[\int_0^{\tau^1_\varepsilon} \mathrm{e}^{-rs}\big(X_s^{(x+\varepsilon,\pi)} - X_s^{(x,\pi)}\big)\,\mathrm{d}s\right] = \varepsilon\, E\left[\int_0^{\tau^1_\varepsilon} \mathrm{e}^{-rs}\, X_s^{(1,\pi)}\,\mathrm{d}s\right] \end{aligned} \tag{32}$$

where the last expectation is positive and finite, for any $\varepsilon > 0$ small enough. Hence, by using the fact that $\tau^1_\varepsilon \to 0$ ($P$-a.s.) as $\varepsilon \downarrow 0$ due to the regularity of the boundary $\partial C_*$ for the region $D_*$ relative to $(X, \Pi)$, and applying the Lebesgue dominated convergence theorem, we obtain that the inequality

$$\limsup_{\varepsilon \downarrow 0} \frac{V_*(x+\varepsilon, \pi) - V_*(x, \pi)}{\varepsilon} \leq 0 \tag{33}$$

holds. Thus, getting the inequalities in (31) and (33) together, we conclude that the smooth-fit condition in the left-hand part of (19) above is satisfied.

In order to derive the property in the right-hand part of (19), we first observe directly from the structure of the continuation region in (9) and (12) that the inequality

$$\liminf_{\varepsilon \downarrow 0} \frac{V_*(x, \pi + \varepsilon) - V_*(x, \pi)}{\varepsilon} \geq 0 \tag{34}$$

is satisfied, due to the fact that $V_*(x, \pi + \varepsilon) \geq 0$ holds. Let us finally denote by $\tau^2_\varepsilon = \tau_*(x, \pi + \varepsilon)$ the optimal stopping time in (7) for the starting point $(x, \pi + \varepsilon)$ of the process $(X, \Pi)$, with some $\varepsilon > 0$ small enough. Then, taking into account the explicit form of the process $X$ in (2)–(3) and the representations for the processes $X$ and $\Pi$ in (4)–(5), by applying Itô's formula to the process $\Pi/(1 - \Pi)$, we get

$$\begin{aligned}
&V_*(x, \pi + \varepsilon) - V_*(x, \pi) \\
&\quad \leq E\left[\int_0^{\tau^2_\varepsilon} \mathrm{e}^{-rs}\big(X_s^{(x,\pi+\varepsilon)} - L\big)\, \mathrm{d}s\right] - E\left[\int_0^{\tau^2_\varepsilon} \mathrm{e}^{-rs}\big(X_s^{(x,\pi)} - L\big)\, \mathrm{d}s\right] \\
&\quad = E\left[\int_0^{\tau^2_\varepsilon} \mathrm{e}^{-rs}\big(X_s^{(x,\pi+\varepsilon)} - X_s^{(x,\pi)}\big)\, \mathrm{d}s\right] \\
&\quad = x\, E\left[\int_0^{\tau^2_\varepsilon} \mathrm{e}^{-rs}\big(X_s^{(1,\pi+\varepsilon)} - X_s^{(1,\pi)}\big)\, \mathrm{d}s\right]
\end{aligned} \tag{35}$$

where the last expectation is positive and finite, for any $\pi \in (0, 1)$ and $\varepsilon > 0$ small enough, under the assumption that $\delta_0 > \delta_1$, and we have

$$\begin{aligned}
&X_t^{(1,\pi+\varepsilon)} - X_t^{(1,\pi)} \\
&\quad = X_t^{(1,\pi)} \left(\frac{X_t^{(1,\pi+\varepsilon)}}{X_t^{(1,\pi)}} - 1\right) \\
&\quad = X_t^{(1,\pi)} \left(\exp\left(\int_0^t (\delta_0 - \delta_1)\big(\Pi_s^{(\pi+\varepsilon)} - \Pi_s^{(\pi)}\big)\, \mathrm{d}s\right) - 1\right) \\
&\quad = X_t^{(1,\pi)} \left(\exp\left(\int_0^t (\delta_0 - \delta_1)\Pi_s^{(\pi)}(1 - \Pi_s^{(\pi+\varepsilon)}) \left(\frac{\Pi_s^{(\pi+\varepsilon)}(1 - \Pi_s^{(\pi)})}{(1 - \Pi_s^{(\pi+\varepsilon)})\Pi_s^{(\pi)}} - 1\right) \mathrm{d}s\right) - 1\right)
\end{aligned} \tag{36}$$

and

$$\frac{\Pi_t^{(\pi-\varepsilon)}(1-\Pi_t^{(\pi)})}{(1-\Pi_t^{(\pi-\varepsilon)})\Pi_t^{(\pi)}} = \frac{(\pi-\varepsilon)(1-\pi)}{(1-\pi-\varepsilon)\pi} \times \exp\left(\int_0^t \left(\frac{\lambda_0(1-\Pi_s^{(\pi-\varepsilon)})-\lambda_1\Pi_s^{(\pi-\varepsilon)}}{\Pi_s^{(\pi-\varepsilon)}(1-\Pi_s^{(\pi-\varepsilon)})} - \frac{\lambda_0(1-\Pi_s^{(\pi)})-\lambda_1\Pi_s^{(\pi)}}{\Pi_s^{(\pi)}(1-\Pi_s^{(\pi)})} + \frac{(\delta_0-\delta_1)^2}{\sigma^2}(\Pi_s^{(\pi-\varepsilon)}-\Pi_s^{(\pi)})\right) ds\right) \tag{37}$$

for all $t \geq 0$. Note that we can take into account the dependence structure of the solution $\Pi$ of the stochastic differential equation in (5) on its starting point and provide the appropriate Taylor's expansions for the exponential functions in (36) and (37). Hence, by using the fact that $\tau_\varepsilon^2 \to 0$ ($P$-a.s.) as $\varepsilon \downarrow 0$ due to the regularity of the boundary $\partial C_*$ for the region $D_*$ relative to $(X, \Pi)$ and applying the Lebesgue dominated convergence theorem, we obtain that the inequality

$$\limsup_{\varepsilon\downarrow 0} \frac{V_*(x,\pi+\varepsilon)-V_*(x,\pi)}{\varepsilon} \leq 0 \tag{38}$$

holds. Thus, getting the inequalities in (34) and (38) together, we conclude that the smooth-fit condition in the right-hand part of (19) above is satisfied. ■

### 3.5. Continuous differentiability of the value function

Let us now show that the value function $V_*(x,\pi)$ in (7) is continuously differentiable on the whole state space of the process $(X, \Pi)$.

**Lemma 3.4:** *The value function $V_*(x,\pi)$ of the optimal stopping problem in (7) belongs to the class $C^{1,1}$ on $(0,\infty)\times(0,1)$.*

***Proof:*** We first recall that, by virtue of the strong Markov property of the process $(X, \Pi)$, it is shown that the value function $V_*(x,\pi)$ solves the parabolic-type partial differential equation of (16)+(17) (with degenerate coefficients), so that $V_*(x,\pi)$ surely belongs at least to the class $C^{1,1}$ on the continuation region $C_*$ in (12). In this respect, it remains for us to prove that the partial derivatives $(V_*)_x(x,\pi)$ and $(V_*)_\pi(x,\pi)$ are continuous functions at the boundary $\partial C_*$. For this purpose, we will show the existence of other directional derivatives of $V_*(x,\pi)$ along the boundary $\partial C_*$ following the schema of arguments used in [31, Section 11] combined with the ones of Lemma 3.3 above. Note that the method applied below allows to prove the assertion for any point $(x,\pi)$ of the state space $(0,\infty)\times(0,1)$ of the process $(X, \Pi)$.

On one hand, we need to show that the property

$$\lim_{n\to\infty} (V_*)_x(x_n,\pi_n) = 0 \tag{39}$$

holds, for any sequence $(x_n,\pi_n)_{n\in\mathbb{N}}$ tending to $(x,\pi)$ as $n\to\infty$ such that $x = a_*(\pi)$. Since we have $V_*(x_n,\pi_n) = 0$, for $(x_n,\pi_n)\in D_*$, and the conditions of (19) hold at $x = a_*(\pi)$, there is no restriction to assume that $(x_n, \pi_n) \in C_*$, for every $n \in \mathbb{N}$. Let us first show that the inequality

$$\liminf_{n\to\infty} (V_*)_x(x_n, \pi_n) = \liminf_{n\to\infty} \lim_{\varepsilon\downarrow 0} \frac{V_*(x_n+\varepsilon, \pi_n) - V_*(x_n, \pi_n)}{\varepsilon} \geq 0 \tag{40}$$

holds. In this case, we observe from the first identity in (40) that one can choose subsequences $(x_{n_k}, \pi_{n_k})_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ such that

$$\liminf_{n\to\infty} (V_*)_x(x_n, \pi_n) = \lim_{k\to\infty} \frac{V_*(x_{n_k}+\varepsilon_k, \pi_{n_k}) - V_*(x_{n_k}, \pi_{n_k})}{\varepsilon_k} \tag{41}$$

with $(x_{n_k}+\varepsilon_k, \pi_{n_k})_{k\in\mathbb{N}}$ tending to $(x, \pi)$ as $k\to\infty$. Let us consider $\tau_k^1 = \tau_*(x_{n_k}, \pi_{n_k})$ the optimal stopping time for the value function $V_*(x_{n_k}, \pi_{n_k})$, for every $k \in \mathbb{N}$. Then, taking into account the structure of the continuation region in (9) and (12) as well as the explicit form of the process $X$ in (2)–(3) and (4), we find that

$$\begin{aligned}
&V_*(x_{n_k}+\varepsilon_k, \pi_{n_k}) - V_*(x_{n_k}, \pi_{n_k}) \\
&\leq E\left[\int_0^{\tau_k^1} e^{-rs}\big(X_s^{(x_{n_k}+\varepsilon_k,\pi_{n_k})} - L\big)\, ds\right] - E\left[\int_0^{\tau_k^1} e^{-rs}\big(X_s^{(x_{n_k},\pi_{n_k})} - L\big)\, ds\right] \\
&= E\left[\int_0^{\tau_k^1} e^{-rs}\big(X_s^{(x_{n_k}+\varepsilon_k,\pi_{n_k})} - X_s^{(x_{n_k},\pi_{n_k})}\big)\, ds\right] = \varepsilon_k\, E\left[\int_0^{\tau_k^1} e^{-rs}\, X_s^{(1,\pi_{n_k})}\, ds\right]
\end{aligned} \tag{42}$$

where the last expectation is positive and finite, for every $k \in \mathbb{N}$. Hence, letting $k\to\infty$ and recalling the fact that $\tau_k^1 \to 0$ ($P$-a.s.) as $k\to\infty$ due to the regularity of the boundary $\partial C_*$ for the region $D_*$ in (13) relative to $(X, \Pi)$, we see by the Lebesgue dominated convergence theorem that the expression in (42) combined with the one in (41) implies the desired one in (40). Thus, it remains for us to show that the inequality

$$\limsup_{n\to\infty} (V_*)_x(x_n, \pi_n) = \limsup_{n\to\infty} \lim_{\varepsilon\downarrow 0} \frac{V_*(x_n+\varepsilon, \pi_n) - V_*(x_n, \pi_n)}{\varepsilon} \leq 0 \tag{43}$$

holds too. For this purpose, we observe from the first identity in (43) that one can choose subsequences $(x_{n_k}, \pi_{n_k})_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} (V_*)_x(x_n, \pi_n) = \lim_{k\to\infty} \frac{V_*(x_{n_k}+\varepsilon_k, \pi_{n_k}) - V_*(x_{n_k}, \pi_{n_k})}{\varepsilon_k} \tag{44}$$

with $(x_{n_k}+\varepsilon_k, \pi_{n_k})_{k\in\mathbb{N}}$ tending to $(x, \pi)$ as $k\to\infty$. Let us consider $\tau_k^2 = \tau_*(x_{n_k} + \varepsilon_k, \pi_{n_k})$ the optimal stopping time for the value function $V_*(x_{n_k}+\varepsilon_k, \pi_{n_k})$, for every $k \in \mathbb{N}$. Then, by virtue of the explicit form of the process $X$ in (2)–(3) and the representations for the processes $X$ and $\Pi$ in (4)–(5), we find in the same way as in (32) above that

$$V_*(x_{n_k}+\varepsilon_k, \pi_{n_k}) - V_*(x_{n_k}, \pi_{n_k}) \leq \varepsilon_k\, E\left[\int_0^{\tau_k^2} e^{-ru}\, X_u^{(1,\pi_{n_k})}\, du\right] \tag{45}$$

where the last expectation is positive and finite, for every $k \in \mathbb{N}$. Hence, letting $k\to\infty$ and recalling the fact that $\tau_k^2 \to 0$ ($P$-a.s.) as $k\to\infty$ due to the regularity of the boundary

$\partial C_*$ for the region $D_*$ relative to $(X, \Pi)$, we see by the Lebesgue dominated convergence theorem that the expression in (45) combined with the one in (44) implies the desired one in (43). Therefore, getting the inequalities in (40) and (43) together, we obtain the property of (39).

On the other hand, we need to show that the property

$$\lim_{n\to\infty} (V_*)_\pi(x_n, \pi_n) = 0 \tag{46}$$

holds, for any sequence $(x_n, \pi_n)_{n\in\mathbb{N}}$ tending to $(x, \pi)$ as $n \to \infty$. Since we have $V_*(x_n, \pi_n) = 0$, for $(x_n, \pi_n) \in D_*$ and the conditions of (19) hold at $x = a_*(\pi)$, we assume again that $(x_n, \pi_n) \in C_*$, for every $n \in \mathbb{N}$. Then, we conclude from the structure of the continuation region in (9) and (12) that the inequality

$$\liminf_{n\to\infty} (V_*)_\pi(x_n, \pi_n) = \liminf_{n\to\infty} \lim_{\varepsilon\downarrow 0} \frac{V_*(x_n, \pi_n + \varepsilon) - V_*(x_n, \pi_n)}{\varepsilon} \geq 0 \tag{47}$$

holds. Thus, it remains for us to show that the inequality

$$\limsup_{n\to\infty} (V_*)_\pi(x_n, \pi_n) = \limsup_{n\to\infty} \lim_{\varepsilon\downarrow 0} \frac{V_*(x_n, \pi_n + \varepsilon) - V_*(x_n, \pi_n)}{\varepsilon} \leq 0 \tag{48}$$

holds too. In this case, we observe from the first identity in (48) that one can choose subsequences $(x_{n_k}, \pi_{n_k})_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} (V_*)_\pi(x_n, \pi_n) = \lim_{k\to\infty} \frac{V_*(x_{n_k}, \pi_{n_k} + \varepsilon_k) - V_*(x_{n_k}, \pi_{n_k})}{\varepsilon_k} \tag{49}$$

with $(x_{n_k}, \pi_{n_k} + \varepsilon_k)_{k\in\mathbb{N}}$ tending to $(x, \pi)$ as $k \to \infty$. Let us consider $\tau_k^3 = \tau_*(x_{n_k}, \pi_{n_k} + \varepsilon_k)$ the optimal stopping time for the value function $V_*(x_{n_k}, \pi_{n_k} + \varepsilon_k)$, for every $k \in \mathbb{N}$. Then, taking into account the explicit form of the process $X$ in (2)–(3) and the representations for the processes $X$ and $\Pi$ in (4)–(5), by applying Itô's formula to the process $\Pi/(1 - \Pi)$, we find in the same way as in (35) with (36)–(37) above that

$$V_*(x_{n_k}, \pi_{n_k} + \varepsilon_k) - V_*(x_{n_k}, \pi_{n_k}) \leq x_{n_k} E\left[\int_0^{\tau_k^3} e^{-rs}\big(X_s^{(1,\pi_{n_k}+\varepsilon_k)} - X_s^{(1,\pi_k)}\big)\, du\right] \tag{50}$$

where the last expectation is positive and finite, under $\delta_i > 0$, for $i = 0, 1$, for every $k \in \mathbb{N}$. Hence, letting $k \to \infty$ and recalling the fact that $\tau_k^3 \to 0$ ($P$-a.s.) as $k \to \infty$ due to the regularity of the boundary $\partial C_*$ for the region $D_*$ relative to $(X, \Pi)$, by means of arguments similar to the ones used in the end of the previous subsection, we see by the Lebesgue dominated convergence theorem that the expression in (50) combined with the one in (49) implies the desired one in (48). Therefore, getting the inequalities in (47) and (48) together, we obtain the property of (46). ■

## 4. Main results and proofs

In this section, we make the appropriate change of variables and provide the appropriate verification assertion which constitute the proof of the main result of the paper stated below.

### 4.1. The change of variables

In order to provide the analysis of the free-boundary problem in (17)–(22) and be able to apply the change-of-variable formula from [43] in the verification assertion below, we introduce an appropriate change of variables to reduce the infinitesimal operator of the process $(X, \Pi)$ from (16) to the normal (or canonic) form and reformulate the initial optimal stopping problem within the new coordinates. For this purpose, let us define the process $Y = (Y_t)_{t \geq 0}$ by

$$Y_t = \frac{X_t^{-\eta} \Pi_t}{1 - \Pi_t} \quad \text{with} \quad \eta = \frac{\delta_0 - \delta_1}{\sigma^2} \quad \text{so that} \quad \Pi_t = \frac{X_t^{\eta} Y_t}{1 + X_t^{\eta} Y_t} \tag{51}$$

for all $t \geq 0$. Then, by applying Itô's formula to the expressions in (51), we get from the representations in (4)–(6) that the process $(X, Y)$ solves the system of stochastic differential equations

$$dX_t = \left( r - \delta_0 - (\delta_1 - \delta_0) \frac{X_t^{\eta} Y_t}{1 + X_t^{\eta} Y_t} \right) X_t \, dt + \sigma \, X_t \, d\overline{B}_t \quad (X_0 = x) \tag{52}$$

and

$$dY_t = \left( \frac{(\lambda_0 - \lambda_1 X_t^{\eta} Y_t)(1 + X_t^{\eta} Y_t)}{X_t^{\eta} Y_t} - \xi \right) Y_t \, dt \quad \left( Y_0 = y \equiv \frac{x^{-\eta} \pi}{1 - \pi} \right) \tag{53}$$

with

$$\xi = \frac{\eta}{2} (2r - \delta_0 - \delta_1 - \sigma^2) \tag{54}$$

for any $(x, y) \in (0, \infty)^2$ (see, e.g. [19,24], or [31] among others for similar transformations of variables). It is seen from the form of the stochastic differential equation in (53) that the process $Y$ started at $y > 0$ is of bounded variation. More precisely, if the inequality $X_t^{\eta} Y_t < \nu$ holds with

$$\nu = \frac{1}{2\lambda_1} \Big( \sqrt{(\lambda_1 - \lambda_0 + \xi)^2 + 4\lambda_0 \lambda_1} - (\lambda_1 - \lambda_0 + \xi) \Big) > 0 \tag{55}$$

for $t \geq 0$, then the process $Y$ is increasing, while if the inequality $X_t^{\eta} Y_t > \nu$ holds, for $t \geq 0$, then the process $Y$ is decreasing.

Observe that, for any $(x, \pi) \in (0, \infty) \times (0, 1)$ fixed, the value function of the optimal stopping problem in (7) takes the form $V_*(x, \pi) = U_*(x, x^{-\eta} \pi / (1 - \pi))$ with

$$U_*(x, y) = \sup_{\tau} E_{x,y} \left[ \int_0^{\tau} e^{-rs} (X_s - L) \, ds \right] \tag{56}$$

where $E_{x,y}$ denotes the expectation taken under the assumption that the two-dimensional Markov process $(X, Y)$ satisfying the stochastic differential equations in (52)–(53) starts at some $(x, y) \in (0, \infty)^2$. We note from the relations in (51) that there exists a one-to-one correspondence between the processes $(X, \Pi)$ and $(X, Y)$, so that the supremum in (56) is equivalently taken over all stopping times $\tau$ with respect to the natural filtration of $(X, Y)$

which coincides with $(\mathcal{F}_t)_{t\geq 0}$. It thus follows from the expression in (11) that the stopping time $\tau_*$ in (56) can be expressed as

$$\tau_* = \inf\left\{t \geq 0 \,\middle|\, X_t \leq g_*(Y_t)\right\} \tag{57}$$

so that the continuation $C_*$ in (9) corresponds to

$$F_* = \left\{(x, y) \in (0, \infty)^2 \,\middle|\, x > g_*(y)\right\} \tag{58}$$

with some function $0 < g_*(y) \leq L$, for $y > 0$. In order to provide relations between the functions $a_*(\pi)$ and $g_*(y)$, we follow the line of arguments around [31, Formula (11.20)] to see that, for each $y > 0$ fixed, there exists a unique $\pi \in (0, 1)$ such that the equality

$$x = a_*(\pi) = \left(\frac{\pi}{y(1-\pi)}\right)^{1/\eta} = g_*(y) \tag{59}$$

holds. In this case, we observe that the first equality in (59) takes the form $\pi = a_*^{-1}(x)$, where $a_*^{-1}(x)$ is understood as a generalized inverse of the function $a_*(\pi)$. Then, it follows from the change of variables introduced in (51) that the latter equality is equivalent to

$$\frac{x^\eta y}{1 + x^\eta y} = a_*^{-1}(x) \quad \text{so that} \quad y = \frac{a_*^{-1}(x)}{x^\eta(1 - a_*^{-1}(x))} \equiv h_*(x) \tag{60}$$

for each $y > 0$ fixed. Hence, we may conclude that the function $h_*(x)$ from (60) is positive and decreasing, and thus, we can define $g_*(y) = h_*^{-1}(y)$, for each $y > 0$ fixed, where $h_*^{-1}(y)$ is a generalized inverse of the function $h_*(x)$. In this view, by virtue of the fact proved in Lemma 2.1 that the boundary $a_*(\pi)$ in the expressions of (11) and (12)–(13) is monotone, we may conclude that the boundary $g_*(y)$ in the expressions of (57) and (58) is of bounded variation. On the other hand, we obtain from the change of variables in (51) that the last equality in (59) takes the form

$$\frac{x^{-\eta}\pi}{1-\pi} = g_*^{-1}(x) \quad \text{so that} \quad \pi = \frac{x^\eta g_*^{-1}(x)}{1 + x^\eta g_*^{-1}(x)} \equiv b_*(x) \tag{61}$$

for each $\pi \in (0, 1)$ fixed. Therefore, we may conclude that the function $b_*(x)$ from (61) is positive and decreasing, and thus, we can define $a_*(\pi) = b_*^{-1}(\pi)$, for each $\pi \in (0, 1)$ fixed, where $b_*^{-1}(\pi)$ is a generalized inverse of the function $b_*(x)$.

By means of standard arguments based on an application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}_{(X,Y)}$ of the process $(X, Y)$ solving the stochastic differential equations in (52)–(53) has the structure

$$\begin{aligned} \mathbb{L}_{(X,Y)} = &\left(r - \delta_0 - (\delta_1 - \delta_0)\frac{x^\eta y}{1 + x^\eta y}\right) x\,\partial_x + \frac{\sigma^2 x^2}{2}\,\partial_{xx} \\ &+ \left(\frac{(\lambda_0 - \lambda_1 x^\eta y)(1 + x^\eta y)}{x^\eta y} - \xi\right) y\,\partial_y \end{aligned} \tag{62}$$

with $\xi$ given by (54), for all $(x, y) \in (0, \infty)^2$. In this case, according to the system in (17)–(22) above, the unknown value function $U_*(x, y)$ from (56) and the unknown boundary $g_*(y)$ from (57) can be characterized by the following associated free-boundary problem

$$(\mathbb{L}_{(X,Y)}U - rU)(x,y) = -(x-L) \quad \text{for } x > g(y) \tag{63}$$

$$U(x,y)\big|_{x=g(y)} = 0 \quad \text{(instantaneous stopping)} \tag{64}$$

$$U_x(x,y)\big|_{x=g(y)} = 0 \quad \text{and} \quad U_y(x,y)\big|_{x=g(y)} = 0 \quad \text{(smooth fit)} \tag{65}$$

$$U(x,y) = 0 \quad \text{for } x < g(y) \tag{66}$$

$$U(x,y) > 0 \quad \text{for } x > g(y) \tag{67}$$

$$(\mathbb{L}_{(X,Y)}U - rU)(x,y) < -(x-L) \quad \text{for } x < g(y) \tag{68}$$

for $y > 0$. We recall that the existence of the value of the optimal stopping problem in (56) follows from the results of [7, Theorem 4.1] and the one-to-one correspondence between the processes $(X, \Pi)$ and $(X, Y)$. Then, by virtue of the strong Markov property of the process $(X, Y)$, it is shown that the value function $U_*(x, y)$ from (56) solves the parabolic-type partial differential equation in (62)+(63). Hence, taking into account the regularity of the points of the optimal exercise boundary $\partial C_*$ for the stopping region $D_*$ relative to the process $(X, \Pi)$, we may conclude from the results on parabolic-type partial differential equations (see, e.g. [35, Chapter V]) combined with standard applications of Itô's formula and Doob's optional sampling theorem (see, e.g. [45, Chapter III, Subsection 7.3]) that the value function $U_*(x, y)$ belongs to the class $C^{2,1}$ in the region $F_* \cap ((0,\infty)^2 \setminus E)$ which naturally coincides with the closure of $F_*$, because the set $E = \{(x,y) \in (0,\infty)^2 \,|\, x^\eta y = \nu\}$ is thin, where the region $F_*$ has the form of (58) and $\nu$ is given by (55). Moreover, by virtue of the regularity of the value function proved in Lemmata 3.1–3.4 and the bijective and smooth change of variables introduced in (51), it follows that the instantaneous-stopping and smooth-fit conditions of (64) and (65) hold for the value function $U_*(x, y)$ too.

### 4.2. The verification lemma

In order to formulate and prove the main results of the paper, taking into account the structure of the partial differential equation in (62)+(63) as well as the instantaneous-stopping and smooth-fit conditions in (64) and (65), we observe that the second-order derivative $(U_*)_{xx}(x, y)$ admits a continuous extension to the closure of the appropriate continuation region $F_*$ from (58). This fact means that, by virtue of the regularity of the boundary $\partial C_*$ for $D_*$ relative to $(X, \Pi)$ proved in Lemma 3.2 above, the function $U_*(x, y)$ admits a natural extension in the class $C^{2,1}$ in the closure of the region $F_* \cap ((0,\infty)^2 \setminus E)$ which naturally coincides with the closure of $F_*$ with $E = \{(x,y) \in (0,\infty)^2 \,|\, x^\eta y = \nu\}$ and $\nu$ given by (55). Moreover, by virtue of the results of [44, Theorem 3], it follows from the regularity of the value function $U_*(x, y)$ and the expressions in (60) and (61) that the boundary $g_*(y)$ in (58) is a continuous function (of bounded variation). Recall the property that the process $Y$ is monotone outside the curve $E = \{(x,y) \in (0,\infty)^2 \,|\, x^\eta y = \nu\}$ with $\nu$ given by (55). In this case, it can be shown by means of arguments similar to the ones applied in part 1 of the proof of [31, Theorem 19] that there exists a sequence of piecewise-monotone processes $Y^n = (Y^n_t)_{t \geq 0}$, $n \in \mathbb{N}$, which converges to $Y$ in $P$-probability on compact time intervals from $[0,\infty)$, and the sequence of total variations of $Y^n$, $n \in \mathbb{N}$, also converges in $P$-probability to the one of $Y$, on each interval $[0, T]$, for any $T > 0$ fixed, as $n \to \infty$. Note that, without loss of generality, the processes $Y^n$, $n \in \mathbb{N}$, can be assumed to be continuous, by virtue of possible applications of standard straight-line approximations. Since each of the resulting continuous processes $g_*(Y^n)$ is of bounded variation, so that it is a continuous semimartingale, the change-of-variable formula from [43, Theorem 3.1] can be applied to the process $\mathrm{e}^{-rt}U(X, Y^n)$, for every $n \in \mathbb{N}$, and thus, to the process $\mathrm{e}^{-rt}U(X, Y)$, by virtue of the appropriate convergence relations mentioned above and the assumed regularity of the candidate value function $U(x, y)$ which is defined by the right-hand side of the expression in (69) below (see part 1 of the proof of [31, Theorem 19] for further arguments).

We continue with the following verification assertion related to the free-boundary problem in (62)+(63)–(68).

**Lemma 4.1:** *Suppose that the processes $X$ and $Y$ solve the stochastic differential equations in* (52)–(53) *with $r > 0$, $\delta_0 > \delta_1 > 0$, $\sigma > 0$, and $\lambda_i \geq 0$, for $i = 0, 1$. Then, the value function $U_*(x, y)$ of the optimal stopping problem in* (56) *admits the representation*

$$U_*(x, y) = \begin{cases} U(x, y; g_*(y)), & \text{if } x > g_*(y) \\ 0, & \text{if } 0 < x \leq g_*(y) \end{cases} \tag{69}$$

*and the stopping time $\tau_*$ from* (57) *is optimal, where the continuous function $U(x, y; g_*(y))$ and the continuous boundary $0 < g_*(y) \leq L$, for $y > 0$, of bounded variation are determined as the unique solution to the system in* (62)+(63)–(65) *and* (67).

***Proof:*** **(i)** In order to verify the assertions stated above, let us denote by $U(x, y)$ the right-hand side of the expression in (69). There, the candidate value function $U(x, y; g_*(y))$ belongs to the class $C^{1,1}$ at the appropriate boundary $\partial F_*$, and thus, to the class $C^{2,1}$ in the closure of the appropriate region $F_*$, where the region $F_*$ is given by (58). Hence, we can apply the change-of-variable formula with local time on surfaces from [43] (see also [45, Chapter II, Section 3.5] for a summary of the related results and further references) to the process $\mathrm{e}^{-rt}U(X_t, Y_t)$ to obtain

$$\mathrm{e}^{-rt}\, U(X_t, Y_t) = U(x, y) + \int_0^t \mathrm{e}^{-rs}(\mathbb{L}_{(X,Y)}U - rU)(X_s, Y_s)I\big(X_s < g_*(Y_s)\big)\, \mathrm{d}s + M_t \tag{70}$$

where the process $M = (M_t)_{t \geq 0}$ defined by

$$M_t = \int_0^t \mathrm{e}^{-rs}\, U_x(X_s, Y_s)I\big(X_s \neq g_*(Y_s)\big)\sigma X_s\, \mathrm{d}\overline{B}_s \tag{71}$$

is a continuous local martingale with respect to the probability measure $P_{x,y}$. Note that, since the time spent by the process $X$ at the surface $g_*(Y)$ of bounded variation is of Lebesgue measure zero, the indicator which appears in the integral in the expression of (71) can be ignored.

It follows from the strong Markov property of the process $(X, Y)$ that the function $U(x, y)$ satisfies the parabolic-type partial differential equation in (62)+(63), which together with the conditions of (64)–(65) and the equality in (66) meaning directly that the inequality in (68) holds, imply that the inequality $(\mathbb{L}_{(X,Y)}U - rU)(x, y) \leq -(x - L)$ is satisfied, for any $(x, y) \in (0, \infty)^2$ such that $x \neq g_*(y)$, as well. Moreover, by virtue of the inequality in (67), which follows from the superharmonic characterization of the value function (see, e.g. [45, Chapter IV, Section 9]), we see from the equality in (66) that the inequality $U(x, y) \geq 0$ holds, for all $(x, y) \in (0, \infty)^2$, too. Let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence for the process $M$ such that $\tau_n = \inf\{t \geq 0 \,|\, |M_t| \geq n\}$, for each $n \in \mathbb{N}$. Then, the expression in (70) yields that the inequalities

$$\int_0^{\tau \wedge \tau_n} \mathrm{e}^{-rs} (X_s - L)\, \mathrm{d}s \leq \mathrm{e}^{-r(\tau \wedge \tau_n)}\, U(X_{\tau \wedge \tau_n}, Y_{\tau \wedge \tau_n}) \leq U(x, y) + M_{\tau \wedge \tau_n} \tag{72}$$

hold, for any stopping times $\tau$ of the process $(X, Y)$ started at $(x, y) \in (0, \infty)^2$. In this case, taking the expectations with respect to the probability measure $P_{x,y}$ in (72), by means of Doob's optional sampling theorem (see, e.g. [Chapter III, Theorem 3.6, 36] or [Chapter II, Theorem 3.2, 46]), we get that the inequalities

$$\begin{aligned} E_{x,y}\left[\int_0^{\tau \wedge \tau_n} \mathrm{e}^{-rs} (X_s - L)\, \mathrm{d}s\right] &\leq E_{x,y}\big[\mathrm{e}^{-r(\tau \wedge \tau_n)}\, U(X_{\tau \wedge \tau_n}, Y_{\tau \wedge \tau_n})\big] \\ &\leq U(x, y) + E_{x,y}\big[M_{\tau \wedge \tau_n}\big] = U(x, y) \end{aligned} \tag{73}$$

hold, for any stopping time $\tau$ of the process $(X, Y)$ and each $n \in \mathbb{N}$. Hence, taking into account the obvious fact that the expectations in (73) are bounded below, letting $n$ go to infinity and using Fatou's lemma, we obtain that the inequalities

$$E_{x,y}\left[\int_0^{\tau} \mathrm{e}^{-rs} (X_s - L)\, \mathrm{d}s\right] \leq E_{x,y}\big[\mathrm{e}^{-r\tau}\, U(X_\tau, Y_\tau)\big] \leq U(x, y) \tag{74}$$

are satisfied, for any stopping time $\tau$ and all $(x, y) \in (0, \infty)^2$.

In order to prove the fact that the boundary $g_*(y)$ is optimal, taking into account the fact that the function $U(x, y)$ and the continuous boundary $g_*(y)$ solve the parabolic-type partial differential equation in (63) and satisfy the conditions of (64), inserting $\tau_*$ in place of $\tau$ into the expression of (72), we obtain that the equalities

$$\int_0^{\tau_* \wedge \tau_n} \mathrm{e}^{-rs} (X_s - L)\, \mathrm{d}s = \mathrm{e}^{-r(\tau_* \wedge \tau_n)}\, U(X_{\tau_* \wedge \tau_n}, Y_{\tau_* \wedge \tau_n}) = U(x, y) + M_{\tau_* \wedge \tau_n} \tag{75}$$

are satisfied, for all $(x, y) \in (0, \infty)^2$ and each $n \in \mathbb{N}$. Therefore, taking into account the fact that the variable $\int_0^{\tau_*} \mathrm{e}^{-rs} (X_s - L)\, \mathrm{d}s$ is finite on the event $\{\tau_* = \infty\}$ ($P_{x,y}$-a.s.), because the process $(\mathrm{e}^{-rt} X_t)_{t \geq 0}$ is a strict supermartingale closed at zero, under the assumption $\delta_i > 0$, for $i = 0, 1$, we can apply the Lebesgue dominated convergence theorem to the expression of (75) to obtain the equalities

$$E_{x,y}\left[\int_0^{\tau_*} \mathrm{e}^{-rs} (X_s - L)\, \mathrm{d}s\right] = E_{x,y}\big[\mathrm{e}^{-r\tau_*}\, U(X_{\tau_*}, Y_{\tau_*})\big] = U(x, y) \tag{76}$$

for all $(x, y) \in (0, \infty)^2$. The latter fact means that the candidate function $U(x, y)$ coincides with the value function $U_*(x, y)$ of the optimal stopping problem in (56).

**(ii)** In order to prove the uniqueness of the candidate value function $U(x, y)$ and the boundary $g_*(y)$ as solutions to the free-boundary problem in (63)–(68), let us assume that there exist another positive continuous boundary of bounded variation $\widetilde{g}(y)$ such that the inequality in (68) is satisfied, which means that $0 < \widetilde{g}(y) \le L$ holds, for all $y > 0$. Then, introduce the function

$$\widetilde{U}(x,y) = \begin{cases} \widetilde{U}(x,y;\widetilde{g}(y)), & \text{if } x > \widetilde{g}(y) \\ 0, & \text{if } 0 < x \le \widetilde{g}(y) \end{cases} \tag{77}$$

with $\widetilde{U}(x,y;\widetilde{g}(y))$ being another solution to the system in (62)+(63)–(65) and (67) and the stopping time

$$\widetilde{\tau} = \inf\left\{t \ge 0 \,\middle|\, X_t \le \widetilde{g}(Y_t)\right\} \tag{78}$$

respectively. We can now follow the arguments from the previous part of the proof and use the facts that the function $\widetilde{U}(x,y)$ belongs to the class $C^{2,1}$ in the closure of the appropriate region $\widetilde{F}$ given by

$$\widetilde{F} = \{(x,y) \in (0,\infty)^2 \,|\, x > \widetilde{g}(y)\}. \tag{79}$$

Moreover, we observe that the function $\widetilde{U}(x,y)$ solves the partial differential equation in (62)+(63) as well as satisfies the conditions of (64)–(67) at $\widetilde{g}(y)$ instead of $g_*(y)$, by construction. Hence, we can apply the change-of-variable formula from [43] to get

$$e^{-rt}\,\widetilde{U}(X_t,Y_t) = \widetilde{U}(x,y) + \int_0^t e^{-rs}(\mathbb{L}_{(X,Y)}\widetilde{U} - r\widetilde{U})(X_s,Y_s)I\big(X_s < \widetilde{g}(Y_s)\big)\,ds + N_t \tag{80}$$

where the process $N = (N_t)_{t\ge 0}$ defined by

$$N_t = \int_0^t e^{-rs}\,\widetilde{U}_x(X_s,Y_s)I\big(X_s \ne \widetilde{g}(Y_s)\big)\sigma X_s\,d\overline{B}_s \tag{81}$$

is a continuous local martingale with respect to the probability measure $P_{x,y}$. Let $(\varkappa_n)_{n\in\mathbb{N}}$ be a localizing sequence for the process $N$ such that $\varkappa_n = \inf\{t \ge 0 \,|\, |N_t| \ge n\}$, for each $n \in \mathbb{N}$. Thus, inserting $\widetilde{\tau} \wedge \varkappa_n$ and instead of $t$ into (80) and applying arguments similar to the ones used for the derivations of the formulas in (72)–(76) above, we obtain the equalities

$$E_{x,y}\left[\int_0^{\widetilde{\tau}} e^{-rs}(X_s - L)\,ds\right] = E_{x,y}\big[e^{-r\widetilde{\tau}}\,\widetilde{U}(X_{\widetilde{\tau}},Y_{\widetilde{\tau}})\big] = \widetilde{U}(x,y) \tag{82}$$

for all $(x,y) \in (0,\infty)^2$. Therefore, recalling the fact that $\tau_*$ is the optimal stopping time in (7), and comparing the expressions in (76) and (82), we see that the inequality $\widetilde{U}(x,y) \le U(x,y)$ should hold, for all $(x,y) \in (0,\infty)^2$.

In order to prove the fact that the inequality $g_*(y) \le \widetilde{g}(y)$ holds, for all $y > 0$, let us take a point $(x,y)$ such that $0 < x < g_*(y)$ and $y > 0$, for which we have $\widetilde{U}(x,y) = U(x,y) = 0$. For this purpose, we consider the stopping time

$$\varkappa_* = \inf\left\{t \ge 0 \,\middle|\, X_t \ge g_*(Y_t)\right\}. \tag{83}$$

Then, inserting $\varkappa_* \wedge \varkappa_n$ into (70) and (80) in place of $t$, respectively, and using the fact that the variable $\int_0^{\varkappa_*} e^{-rs}(X_s - L)\,ds$ is finite on the event $\{\varkappa_* = \infty\}$ ($P_{x,y}$-a.s.), by means of arguments similar to the ones applied above, we obtain

$$E_{x,y}\big[\mathrm{e}^{-r\varkappa_*}\, U(X_{\varkappa_*}, Y_{\varkappa_*})\big] = E_{x,y}\left[\int_0^{\varkappa_*} \mathrm{e}^{-rs}(\mathbb{L}_{(X,Y)}U - rU)(X_s, Y_s)\, \mathrm{d}s\right] \tag{84}$$

and

$$E_{x,y}\big[\mathrm{e}^{-r\varkappa_*}\, \widetilde{U}(X_{\varkappa_*}, Y_{\varkappa_*})\big] = E_{x,y}\left[\int_0^{\varkappa_*} \mathrm{e}^{-rs}(\mathbb{L}_{(X,Y)}\widetilde{U} - r\widetilde{U})(X_s, Y_s) I\big(X_s < \widetilde{g}(Y_s)\big)\, \mathrm{d}s\right] \tag{85}$$

for all $(x, y) \in (0, \infty)^2$. Hence, taking into account the fact that the inequality $\widetilde{U}(x, y) \leq U(x, y)$ holds, for all $(x, y) \in (0, \infty)^2$, we get from the expressions in (84) and (85) that the inequality

$$\begin{aligned} E_{x,y}&\left[\int_0^{\varkappa_*} \mathrm{e}^{-rs}(\mathbb{L}_{(X,Y)}U - rU)(X_s, Y_s)\, \mathrm{d}s\right] \\ &\geq E_{x,y}\left[\int_0^{\varkappa_*} \mathrm{e}^{-rs}(\mathbb{L}_{(X,Y)}\widetilde{U} - r\widetilde{U})(X_s, Y_s)\, I\big(X_s < \widetilde{g}(Y_s)\big)\, \mathrm{d}s\right] \end{aligned} \tag{86}$$

is satisfied. Thus, by virtue of the continuity of $g_*(y)$ as well as the property that $\widetilde{U}(x, y) = U(x, y) = 0$ holds, for $(x, y)$ such that $0 < x < g_*(y)$ and $y > 0$, we see from the expressions in (86) and (68) that the inequality $g_*(y) \leq \widetilde{g}(y)$ is satisfied, for all $y > 0$.

We finally show that $\widetilde{g}(y)$ should coincide with $g_*(y)$. For this purpose, we take a point $(x, y)$ such that $x \in (g_*(y), \widetilde{g}(y))$, for some $y > 0$. Hence, inserting $\tau_* \wedge \varkappa_n$ into (80) in place of $t$ and using the fact that the variable $\int_0^{\tau_*} \mathrm{e}^{-rs}(X_s - L)\, \mathrm{d}s$ is finite on the event $\{\tau_* = \infty\}$ ($P_{x,y}$-a.s.), by means of arguments similar to the ones applied above, we obtain

$$\begin{aligned} &E_{x,y}\big[\mathrm{e}^{-r\tau_*}\, \widetilde{U}(X_{\tau_*}, Y_{\tau_*})\big] \\ &\quad = \widetilde{U}(x, y) + E_{x,y}\left[\int_0^{\tau_*} \mathrm{e}^{-rs}(\mathbb{L}_{(X,Y)}\widetilde{U} - r\widetilde{U})(X_s, Y_s) I\big(X_s < \widetilde{g}(Y_s)\big)\, \mathrm{d}s\right] \end{aligned} \tag{87}$$

for all $(x, y) \in (0, \infty)^2$. Thus, since we have $\widetilde{U}(x, y) = U(x, y) = 0$ for $x = g_*(y)$, and the inequality $\widetilde{U}(x, y) \leq U(x, y)$ holds, we see from the expressions in (76) and (87) that the inequality

$$E_{x,y}\left[\int_0^{\tau_*} \mathrm{e}^{-rs}(\mathbb{L}_{(X,Y)}\widetilde{U} - r\widetilde{U})(X_s, Y_s) I\big(X_s < \widetilde{g}(Y_s)\big)\, \mathrm{d}s\right] \geq 0 \tag{88}$$

should be satisfied. However, the strict inequality in (88) cannot be satisfied due to the continuity of $g_*(y)$ and the expression in (68). We may therefore conclude that $g_*(y) = \widetilde{g}(y)$ holds, so that $\widetilde{U}(x, y)$ coincides with $U(x, y)$, for all $(x, y) \in (0, \infty)^2$. ∎

### 4.3. The main result

Getting the assertions of Lemmata 2.1, 3.1–3.4, and 4.1 together, we conclude the paper by formulating its main result concerning the optimal stopping problem related to the pricing of perpetual commodity equities in the considered hidden Markov model.

**Theorem 4.2:** *Let the processes $X$ and $\Pi$ be defined by* (2)–(3) *and* (4)–(6)*, with $r > 0$, $\delta_0 > \delta_1 > 0$, $\sigma > 0$, and $\lambda_i \geq 0$, for $i = 0, 1$. Then, the value function of the optimal stopping problem in* (7) *takes the form $V_*(x, \pi) = U_*(x, x^{-\eta}\pi/(1-\pi))$ and the stopping time $\tau_*$ from* (11) *is optimal with the decreasing boundary $a_*(\pi) = b_*^{-1}(\pi)$, for $(x, \pi) \in (0, \infty) \times [0, 1]$, where $b_*^{-1}(\pi)$ is a generalized inverse of the function $b_*(x)$ from* (61)*. Here, the function $U_*(x, y)$ admits the representation in* (69)*, where the continuous function $U(x, y; g_*(y))$ and the continuous boundary $0 < g_*(y) \leq L$, for $y > 0$, of bounded variation are determined as the unique solution to the system in* (62)+(63)–(65) *and* (67)*.*

## 4.4. Solution in a particular case

In order to underline the complexity in the structure of solutions to the optimal stopping problem of (7) in the two-dimensional diffusion model defined by (2)–(3) and (4)–(6), we present a specific choice of parameters under which the original problem admits a closed-form solution. More precisely, let us assume for this subsection that $\lambda_0 = \lambda_1 = 0$ and $\delta_0 + \delta_1 = 2r - \sigma^2$ holds in (53). The first equality means that $\Theta_t = \Theta_0$, for all $t \geq 0$, where $P(\Theta_0 = 1) = \pi$ and $P(\Theta_0 = 0) = 1 - \pi$, for $\pi \in [0, 1]$ (see, e.g. [45, Chapter VI, Section 21] and [19, Section 4]). Such a situation occurs when the unknown convenience yield of the commodity does not change its value during the whole infinite time interval (see also [10,12,13] for solutions of other optimal stopping problems in the models with random drift rates). In this case, the parabolic-type partial differential equation in (62)+(63) is degenerated into an ordinary one, and the general solution to the resulting equation takes the form

$$U(x, y) = G_1(y)H_1(x, y) + G_2(y)H_2(x, y) + H_0(x, y) \tag{89}$$

where $G_i(y)$, $i = 1, 2$, are some arbitrary continuous functions. Here, we assume that the functions $H_j(x, y)$, $j = 1, 2$, represent fundamental solutions to the homogeneous second-order ordinary differential equation corresponding to the one of (62)+(63) under $\lambda_0 = \lambda_1 = 0$ and $\delta_0 + \delta_1 = 2r - \sigma^2$ given by

$$H_j(x, y) = x^{\gamma_{0,j}} F(\psi_{j,1}, \psi_{j,2}; \varphi_j; -x^\eta y) \tag{90}$$

for every $j = 1, 2$ and each $y > 0$ fixed. We denote by $F(\alpha, \beta; \gamma; x)$ Gauss' hypergeometric function, which is defined by means of the expansion

$$F(\alpha, \beta; \gamma; x) = 1 + \sum_{m=1}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \frac{x^m}{m!} \tag{91}$$

for $\gamma \neq 0, -1, -2, \ldots$, and $(\gamma)_m = \gamma(\gamma+1)\cdots(\gamma+m-1)$, $m \in \mathbb{N}$ (see, e.g. [1, Chapter XV] and [3, Chapter II]), and additionally set

$$\psi_{k,l} = \frac{\chi_{0,k} - \chi_{1,l}}{\eta} \quad \text{and} \quad \varphi_k = 1 + \frac{2}{\eta}\left(\chi_{0,k} - \frac{1}{2} + \frac{r - \delta_0}{\sigma^2}\right) \tag{92}$$

and

$$\chi_{i,k} = \frac{1}{2} - \frac{r - \delta_i}{\sigma^2} - (-1)^k \sqrt{\left(\frac{1}{2} - \frac{r - \delta_i}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{93}$$

so that $\chi_{i,2} < 0 < 1 < \chi_{i,1}$ holds, for every $k, l = 1, 2$ and $i = 0, 1$. Thus, $H_j(x, y), j = 1, 2$, in the expression of (90) are (strictly) increasing and decreasing (convex) functions satisfying the properties $H_1(0+, y) = +0, H_1(\infty, y) = \infty$ and $H_2(0+, y) = \infty, H_2(\infty, y) = +0$, for each $y > 0$ fixed (see, e.g. [47, Chapter V, Section 50] for further details). Observe that $G_1(y) = 0$ should hold in (89), for each $y > 0$, since otherwise, we would have $U(x, y) \to \pm\infty$ with more than a linear growth in $x$ as $x \uparrow \infty$, that must be excluded by virtue of the obvious fact that the value function in (56) is of a linear growth in $x$ under $x \uparrow \infty$. Moreover, the function $H_0(x, y)$ in the expression of (89) represents a particular solution to the equation in (62)+(63) given by

$$H_0(x, y) = H_2(x, y) \int_x \frac{D_0(z, y)}{H_2^2(z, y)} \int_z \frac{2(L - w)}{\sigma^2 w^2} \frac{H_2(w, y)}{D_0(w, y)} \, \mathrm{d}w \, \mathrm{d}z \tag{94}$$

where $D_0(x, y)$ is the appropriate Wronskian determinant having the form

$$D_0(x, y) = H_1(x, y)\partial_x H_2(x, y) - \partial_x H_1(x, y) H_2(x, y) \tag{95}$$

for $y > 0$.

Finally, by applying the condition of (64) and the left-hand condition of (65) to the function in (89), we obtain that the equalities

$$G_2(y) H_2(g(y), y) + H_0(g(y), y) = 0 \tag{96}$$

and

$$G_2(y)\partial_x H_2(g(y), y) + \partial_x H_0(g(y), y) = 0 \tag{97}$$

should hold, for each $y > 0$ fixed. Hence, solving the system of transcendental equations in (96)–(97), we obtain that the function

$$U(x, y; g(y)) = H_0(x, y) - \frac{H_0(g(y), y)}{H_2(g(y), y)} H_2(x, y) \tag{98}$$

for $x > g(y)$, satisfies the system in (62)+(63)–(64), while the left-hand condition of (65) is also satisfied when the equality

$$\frac{\partial_x H_2(g(y), y)}{H_2(g(y), y)} = \frac{\partial_x H_0(g(y), y)}{H_0(g(y), y)} \tag{99}$$

holds, for $y > 0$. Observe that it can be shown by means of straightforward computations based on an application of the implicit function theorem that the candidate value function in (98) such that the equality in (99) holds inherently satisfies the right-hand condition of (65) (see also [19, Remark 4.1] for similar arguments applied for another optimal stopping and free-boundary problem in this model). Note that the uniqueness of the solution $g_*(y)$ to the transcendental equation of (99) follows from the uniqueness of the solution to the system in (62)+(63)–(65) and (67) which is proved in Lemma 4.1 above. Recall that the existence of the solution to the system in (62)+(63)–(65) and (67) follows from the results of [7, Theorem 4.1].

We now summarize the above results in the following assertion.

**Corollary 4.3:** *Suppose that the assumptions of Lemma* 4.1 *are satisfied with* $\lambda_0 = \lambda_1 = 0$ *and* $\delta_0 + \delta_1 = 2r - \sigma^2$. *Then, the value function* $U_*(x, y)$ *of the optimal stopping problem in* (56) *admits the representation of* (69), *where the function* $U(x, y; g_*(y))$ *is given by the expression in* (98), *and the equation in* (99) *admits a unique solution* $0 < g_*(y) \leq L$, *for each* $y > 0$ *fixed.*

## 5. Concluding remarks

It follows from the structure of the reward in (7) and the results on the dependence of strong solutions of (one-dimensional time-homogeneous) stochastic differential equations on the initial points (see, e.g. [17]) that the corresponding value function $V_*(x, \pi)$ is increasing and convex in $\pi$ on the interval $[0, 1]$, for each $x > 0$ fixed. Moreover, we recall that the supremum in (7) is taken over all stopping times with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ which is smaller than the corresponding filtration $(\mathcal{G}_t)_{t \geq 0}$ for the supremum in (A1).

Hence, we may conclude that the function $\overline{W}(x, \pi)$ defined by

$$\overline{W}(x, \pi) = W_*(x, 0)(1 - \pi) + W_*(x, 1)\pi \tag{100}$$

satisfies the inequalities

$$\overline{W}(x, \pi) \geq V_*(x, 0)(1 - \pi) + V_*(x, 1)\pi \geq V_*(x, \pi) \tag{101}$$

for all $(x, \pi) \in (0, \infty) \times [0, 1]$. We finally note that it can be verified by means of straightforward calculations that the function $\overline{W}(x, \pi)$ from (100) satisfies the partial differential equation in (16)–(17) for $(x, \pi) \in (f_*(0), \infty) \times (0, 1)$.

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## ORCID

*Pavel V. Gapeev* http://orcid.org/0000-0002-1346-2074

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## Appendix

In this section, we derive a closed-form solution of the optimal stopping problem which is related to the pricing of perpetual commodity equities in the underlying diffusion-type model with *observable* convenience yield dynamics described by a continuous-time Markov chain with two states.

### A.1 The optimal stopping and free-boundary problem

Let us finally introduce the value function $W_*(x,i)$ of the optimal stopping problem

$$W_*(x,i) = \sup_{\zeta} E_{x,i}\left[\int_0^{\zeta} \mathrm{e}^{-rs}(X_s - L)\,\mathrm{d}s\right] \tag{A1}$$

for some $L > 0$ fixed, where $E_{x,i}$ denotes the expectation taken under the probability measure $P_{x,i}$ under which the two-dimensional Markov process $(X, \Theta)$ starts at some $(x,i) \in (0,\infty) \times \{0,1\}$. Suppose that the supremum in (A1) is taken over all stopping times $\zeta$ with respect to the natural filtration $(\mathcal{G}_t)_{t\geq 0}$ of the process $(X,\Theta)$. Since the continuous-time Markov chain $\Theta$ is observable in this formulation, the optimal stopping time for the problem of (A1) should be of the form

$$\zeta_* = \inf\{t \geq 0 \mid X_t \leq f_*(\Theta_t)\} \tag{A2}$$

for some numbers $0 < f_*(i) \leq L$, for $i = 0, 1$, to be determined.

## A.2 Solution to the free-boundary problem

By means of standard arguments based on an application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}_{(X,\Theta)}$ of the process $(X,\Theta)$ from (2)–(3) acts on an arbitrary locally bounded function $F(x,i)$, which is of the class $C^2$ in $x$ on $(0,\infty)$ under $\Theta = i$, for $i = 0, 1$ fixed, according to the rule

$$(\mathbb{L}_{(X,\Theta)}F)(x,i) = \big(r - \delta_0 - (\delta_1 - \delta_0)i\big)x\,F_x(x,i) + \frac{\sigma^2 x^2}{2}F_{xx}(x,i) + \lambda_i\big(F(x,1-i) - F(x,i)\big) \quad \text{(A3)}$$

for all $(x,i) \in (0,\infty) \times \{0,1\}$. Following the way of arguments from [27] (see also [29] for a more general model), we conclude that the unknown value function $W_*(x,i)$ from (A1) and the unknown numbers $f_*(i)$, for $i = 0, 1$, from (A2) solve the following free-boundary problem for a coupled system of second-order ordinary differential equations

$$(\mathbb{L}_{(X,\Theta)} - rW)(x,i) = -(x - L) \quad \text{for } x > f(i) \quad \text{(A4)}$$

$$W(x,i)\big|_{x=f(i)} = 0 \quad \text{(instantaneous stopping)} \quad \text{(A5)}$$

$$W_x(x,i)\big|_{x=f(i)} = 0 \quad \text{(smooth fit)} \quad \text{(A6)}$$

$$W(x,i) = 0 \quad \text{for } x < f(i) \quad \text{(A7)}$$

$$W(x,i) > 0 \quad \text{for } x > f(i) \quad \text{(A8)}$$

$$(\mathbb{L}_{(X,\Theta)} - rW)(x,i) < -(x - L) \quad \text{for } x < f(i) \quad \text{(A9)}$$

for every $i = 0, 1$.

By means of straightforward computations, we obtain that the general solutions of the two-dimensional system of second order ordinary differential equations in (A3)+(A4) are given by

$$W(x,1) = D_1(1)x^{\gamma_1} + D_2(1)x^{\gamma_2} + G(x,1) \quad \text{(A10)}$$

for $f(1) < x \le f(0)$, and

$$W(x,i) = C_1(i)x^{\beta_1} + C_2(i)x^{\beta_2} + C_3(i)x^{\beta_3} + C_4(i)x^{\beta_4} + A(x,i) \quad \text{(A11)}$$

for $x > f(0)$ and $i = 0, 1$, with

$$G(x,1) = \frac{x}{\delta_1 + \lambda_1} - \frac{L}{r + \lambda_1} \quad \text{and} \quad A(x,i) = \frac{(\delta_{1-i} + \lambda_0 + \lambda_1)x}{\delta_0\delta_1 + \delta_0\lambda_1 + \delta_1\lambda_0} - \frac{L}{r} \quad \text{(A12)}$$

for $x > 0$. Here, $\gamma_j$, for $j = 1, 2$, are explicitly given by

$$\gamma_j = \frac{1}{2} - \frac{r - \delta_1}{\sigma^2} - (-1)^j\sqrt{\left(\frac{1}{2} - \frac{r - \delta_1}{\sigma^2}\right)^2 + \frac{2(r + \lambda_1)}{\sigma^2}} \quad \text{(A13)}$$

so that $\gamma_2 < 0 < 1 < \gamma_1$ holds, and $\beta_k$, for $k = 1, 2, 3, 4$, are the roots of the corresponding characteristic equation

$$Q_0(\beta)Q_1(\beta) = \lambda_0\lambda_1 \quad \text{with} \quad Q_i(\beta) = r + \lambda_i - (r - \delta_i)\beta - \frac{\sigma^2}{2}\beta(\beta - 1) \quad \text{(A14)}$$

for $i = 0, 1$, so that $\beta_4 < \beta_3 < 0 < 1 < \beta_2 < \beta_1$ holds (see, e.g. [26, Section 2] and [27, Section 2] for similar arguments related to other optimal stopping problems in models with observable continuous-time Markov chains). Observe that $C_1(i) = C_2(i) = 0$ should hold in (A11), since otherwise, we would have $W(x,i) \to \pm\infty$ of more than a linear growth as $x \uparrow \infty$, that must be excluded by virtue of the obvious fact that the value function in (A1) is of at most linear growth under $x \uparrow \infty$, for any $i = 0, 1$ fixed.

Then, using the structure of the coupled system of ordinary differential equations in (A3)+(A4) and applying the conditions of (A5) and (A6) to the functions in (A10) and (A11) with (A12) and

$C_1(i) = C_2(i) = 0$, for $i = 0, 1$, we obtain that the following equalities

$$D_1(1)f^{\gamma_1}(1) + D_2(1)f^{\gamma_2}(1) + G(f(1), 1) = 0 \tag{A15}$$

$$D_1(1)\gamma_1 f^{\gamma_1}(1) + D_2(1)\gamma_2 f^{\gamma_2}(1) + G_x(f(1), 1)f(1) = 0 \tag{A16}$$

$$C_3(0)f^{\beta_3}(0) + C_4(0)f^{\beta_4}(0) + A(f(0), 0) = 0 \tag{A17}$$

$$C_3(0)\beta_3 f^{\beta_3}(0) + C_4(0)\beta_4 f^{\beta_4}(0) + A_x(f(0), 0)f(0) = 0 \tag{A18}$$

$$C_3(0)Q_0(\beta_3) = C_3(1)\lambda_1 \quad \text{and} \quad C_4(0)Q_0(\beta_4) = C_4(1)\lambda_1 \tag{A19}$$

should hold, where we have $Q_0(\beta_k)/\lambda_1 = \lambda_0/Q_1(\beta_k)$, for $k = 3, 4$, according to the expressions in (A14). Observe that, since the point $f_*(0)$ belongs to the corresponding continuation region when $\Theta = 1$, the function in (A11) should be (at least) continuously differentiable, and thus, the equalities

$$C_3(1)f^{\beta_3}(0) + C_4(1)f^{\beta_4}(0) + A(f(0), 1) = D_1(1)f^{\gamma_1}(0) + D_2(1)f^{\gamma_2}(0) + G(f(0), 1) \tag{A20}$$

$$\begin{aligned} &C_3(1)\beta_3 f^{\beta_3}(0) + C_4(1)\beta_4 f^{\beta_4}(0) + A_x(f(0), 1)f(0) \\ &\quad = D_1(1)\gamma_1 f^{\gamma_1}(0) + D_2(1)\gamma_2 f^{\gamma_2}(0) + G_x(f(0), 1)f(0) \end{aligned} \tag{A21}$$

should be satisfied. Hence, solving the systems of arithmetic equations in (A15)–(A19) and (A20)–(A21), by means of straightforward computations, we obtain that the functions

$$W(x, 1; f(1)) = \sum_{j=1}^{2} \frac{G_x(f(1), 1)f(1) - \gamma_{3-j}G(f(1), 1)}{\gamma_{3-j} - \gamma_j} \left(\frac{x}{f(1)}\right)^{\gamma_j} + G(x, 1) \tag{A22}$$

for $f(1) < x \le f(0)$, as well as

$$W(x, 0; f(0)) = \sum_{k=3}^{4} \frac{A_x(f(0), 0)f(0) - \beta_{7-k}A(f(0), 0)}{\beta_{7-k} - \beta_k} \left(\frac{x}{f(0)}\right)^{\beta_k} + A(x, 0) \tag{A23}$$

and

$$W(x, 1; f(0)) = \sum_{k=3}^{4} \frac{A_x(f(0), 0)f(0) - \beta_{7-k}A(f(0), 0)}{\beta_{7-k} - \beta_k} \frac{Q_0(\beta_k)}{\lambda_1} \left(\frac{x}{f(0)}\right)^{\beta_k} + A(x, 1) \tag{A24}$$

for $x > f(0)$, satisfy the system in (A3)+(A4) and (A5), while the condition of (A6) is also satisfied when the equalities

$$\begin{aligned} &\sum_{k=3}^{4} \frac{A_x(f(0), 0)f(0) - \beta_{7-k}A(f(0), 0)}{\beta_{7-k} - \beta_k} \frac{Q_0(\beta_k)}{\lambda_1} + A(f(0), 1) \\ &\quad = \sum_{j=1}^{2} \frac{G_x(f(1), 1)f(1) - \gamma_{3-j}G(f(1), 1)}{\gamma_{3-j} - \gamma_j} \left(\frac{f(0)}{f(1)}\right)^{\gamma_j} + G(f(0), 1) \end{aligned} \tag{A25}$$

and

$$\begin{aligned} &\sum_{k=3}^{4} \frac{A_x(f(0), 0)f(0) - \beta_{7-k}A(f(0), 0)}{\beta_{7-k} - \beta_k} \frac{Q_0(\beta_k)}{\lambda_1} \beta_k + A_x(f(0), 1)f(0) \\ &\quad = \sum_{j=1}^{2} \frac{G_x(f(1), 1)f(1) - \gamma_{3-j}G(f(1), 1)}{\gamma_{3-j} - \gamma_j} \gamma_j \left(\frac{f(0)}{f(1)}\right)^{\gamma_j} + G_x(f(0), 1)f(0) \end{aligned} \tag{A26}$$

hold (see also [26, Section 2] and [27, Section 2] for similar calculations related to other optimal stopping problems in models with observable continuous-time Markov chains).

Summarizing the facts proved above, we formulate the following result which can be proved by means of the same arguments as Lemma 4.1 and Theorem 4.2 above (see also the corresponding verification assertions from [26, Theorem 2.1] and [27, Theorem 2]).

**Corollary A.1:** *Suppose that the process X is defined by* (2)–(3), *with* $r > 0$, $\delta_0 > \delta_1 > 0$, *and* $\sigma > 0$, *where* $\Theta$ *be a two-state continuous-time Markov chain with the state space* $\{0, 1\}$ *and transition intensities* $\lambda_i > 0$, *for* $i = 0, 1$. *Then, the value function* $W_*(x, i)$ *of the optimal stopping problem in* (A1) *admits the representations*

$$W_*(x,0) = \begin{cases} W(x,0;f_*(0)), & \text{if } x > f_*(0) \\ 0, & \text{if } 0 < x \le f_*(0) \end{cases} \tag{A27}$$

*and*

$$W_*(x,1) = \begin{cases} W(x,1;f_*(0)), & \text{if } x > f_*(0) \\ W(x,1;f_*(1)), & \text{if } f_*(1) < x \le f_*(0) \\ 0, & \text{if } 0 < x \le f_*(1) \end{cases} \tag{A28}$$

*and the optimal stopping time* $\zeta_*$ *has the form of* (A2), *where the functions* $W(x, i; f(0))$, *for* $i = 0$, *1, are given by* (A23) *and* (A24), *the function* $W(x, 1; f(1))$ *is given by* (A22), *and the numbers* $0 < f_*(1) \le f_*(0) \le L$ *are uniquely determined by the system of arithmetic equations in* (A25)–(A26).

## *A.3 The solution in a particular case*

Let us finally present explicit solutions of the optimal stopping problem in (A1) under the assumption $\lambda_0 = \lambda_1 = 0$. In this case, the general solutions of the second-order ordinary differential equations in (A3)+(A4) are given by

$$W(x,i) = C_1(i)\, x^{\chi_{i,1}} + C_2(i)\, x^{\chi_{i,2}} + \frac{x}{\delta_i} - \frac{L}{r} \tag{A29}$$

where $C_j(i)$, for $i = 0, 1$ and $j = 1, 2$, are some arbitrary constants, while $\chi_{i,j}$, for $i = 0, 1$ and $j = 1, 2$, are given by (93) above. Observe that $C_1(i) = 0$ should hold in (A29), since otherwise, we would have $W(x, i) \to \pm\infty$ of more than a linear growth as $x \uparrow \infty$, that must be excluded by virtue of the obvious fact that the value function in (A1) is of at most linear growth under $x \uparrow \infty$, for any $i = 0, 1$ fixed. Then, by applying the conditions of (A5) and (A6) to the function in (A29), we obtain that the equalities

$$C_2(i) f^{\chi_{i,2}}(i) + \frac{f(i)}{\delta_i} - \frac{L}{r} = 0 \quad \text{and} \quad C_2(i)\, \chi_{i,2} f^{\chi_{i,2}}(i) + \frac{f(i)}{\delta_i} = 0 \tag{A30}$$

should hold, for $i = 0, 1$. Hence, solving the system of equations in (A30), we obtain that the function

$$W(x,i;f_*(i)) = \left(\frac{L}{r} - \frac{f_*(i)}{\delta_i}\right)\left(\frac{x}{f_*(i)}\right)^{\chi_{i,2}} + \frac{x}{\delta_i} - \frac{L}{r} \quad \text{with} \quad f_*(i) = \frac{\chi_{i,2}}{\chi_{i,2}-1}\,\frac{\delta_i L}{r} \tag{A31}$$

for $x > f_*(i)$, satisfies the system of (A3)+(A4) with (A5)–(A6), for every $i = 0, 1$.

We summarize the facts shown above in the following assertion.

**Corollary A.2:** *Suppose that the process X is defined by (2)–(3), with* $r > 0$, $\delta_0 > \delta_1 > 0$, *and* $\sigma > 0$, *and* $\Theta \equiv \Theta_0$ *is a Bernoulli random variable with the values in* $\{0, 1\}$, *so that* $\lambda_0 = \lambda_1 = 0$ *holds. Then, the value function* $W_*(x, i)$ *of the optimal stopping problem in (A1) admits the representation*

$$W_*(x,i) = \begin{cases} W(x,i;f_*(i)), & \text{if } x > f_*(i) \\ 0, & \text{if } x \le f_*(i) \end{cases} \tag{A32}$$

*and the optimal stopping time* $\zeta_*$ *has the form of (A2), where the function* $W(x, i; f_*(i))$ *as well as the number* $f_*(i)$ *are given by (A31), for* $i = 0, 1$.