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Any set of irregular points has full Hausdorff dimension and full topological entropy

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ABSTRACT. We prove, for subshifts of finite type, conformal repellers, and two-dimensional horseshoes, that the set of points where both the pointwise dimension, local entropy, Lyapunov exponents, and Birkhoff averages do not exist carries *full* topological entropy and *full* Hausdorff dimension. This follows from a much stronger statement formulated for a class of symbolic dynamical systems which includes subshifts with the specification property. Our proofs strongly rely on the multifractal analysis of dynamical systems and constitute the first mathematical application of this theory.

1. Introduction

1.1. Typical points and non-typical points. In the numerical study of dynamical systems we are naturally interested in the asymptotic behavior of *typical points*, with respect to some invariant measure. This study gives important information about the observable properties of a dynamical system, from the point of view of that measure. Moreover, typical points with respect to different measures (for example, measure of maximal entropy and measure of maximal dimension) give complementary information. We believe that this information can be put together in order to reconstruct the dynamical system (see [4, 5]). However, we may need a huge number of typical points, each corresponding to a different measure, to effect this reconstruction.

In this paper, we show that surprisingly *all* the information about the dynamical system is hidden in the set of *non-typical points*. In particular, this set carries *full* topological entropy and *full* Hausdorff dimension. Since the non-typical points belong to zero measure sets with respect to every invariant measure, the random choice of points privileges those which are typical. Thus, our result would be of little interest in applications without an algorithm to find non-typical points. We provide such an algorithm; namely, choosing two typical points with respect to two *different* ergodic invariant measures, we combine their symbolic representations to produce a non-typical point.

1.2. Multifractal analysis of dynamical systems. The existence of different ergodic invariant measures strongly relies on the multifractal analysis of dynamical systems. Its main constituent component — dimension spectra — capture information about various dimensions associated with the dynamics. Among them are Hausdorff dimension, correlation dimension, and information dimension of invariant measures. The typical orbit distribution observed by a computer is non-uniform, and clearly depicts hots and cold spots where the density of points of the orbit is higher or lower, respectively, than the average. Those spectra are new powerful tools which give a mathematical description of this phenomenon.

Dimension spectra are examples of more general multifractal spectra introduced by Pesin and the authors in [4] (see also [5]). They provide information on the distribution of pointwise dimensions, local entropies, Lyapunov exponents, etc. Namely, we consider the level sets of functions other then the pointwise dimension (for example, the local entropy or the Lyapunov exponents) — called *multifractal decompositions*, and then compute the Hausdorff dimension, topological entropy, etc, of each level sets

— obtained the so-called *multifractal spectra*. This gives rise to dimension spectra and entropy spectra, respectively, of pointwise dimensions, local entropies, Lyapunov exponents, etc.

Furthermore, we can construct families of ergodic invariant measures, one for each level set, of full Hausdorff dimension, full topological entropy, etc. These *full* measures contain all the information about the multifractal spectra and their use seems to be the most effective way of studying multifractal decompositions. Our proofs strongly rely on the existence of full measures and to the best of our knowledge constitute the first mathematical application of the multifractal analysis of dynamical systems.

1.3. **Sets of irregular points.** We now illustrate the type of results obtained in this paper (see Sections 4, 5, and 6 for definitions and more details). We say that a set $Z \subset X$ has *full* Hausdorff dimension if $\dim_H Z = \dim_H X$.

Theorem 1.1. *For a subshift of finite type σ, the set of points x for which the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(\sigma^k x) \tag{1}$$

does not exist for some continuous function g, has full Hausdorff dimension.

Theorem 1.2. *For a repeller of a conformal $C^{1+\varepsilon}$ expanding map f, if the measure of maximal entropy and the measure of maximal dimension are distinct, then the set of points x for which the limit*

$$\lim_{n\to\infty} \frac{1}{n} \log\|d_x f^n\| \tag{2}$$

does not exist, has full Hausdorff dimension.

Theorem 1.3. *For a locally maximal hyperbolic set Λ of a $C^{1+\varepsilon}$ surface diffeomorphism, if μ is an equilibrium measure other than the measure of maximal dimension, then the set of points x for which the limit*

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{3}$$

does not exist, has full Hausdorff dimension.

These are special cases of our results. Notice that the limits in (1), (2), and (3) (when they exist), define the Birkhoff average of g, the (upper) Lyapunov exponent of f, and the pointwise dimension of μ, respectively. The three theorems above indicate that even though the "negligible" sets of zero measure composed of the points where the Birkhoff Ergodic Theorem, Oseledets Multiplicative Ergodic Theorem (or simply Kingman Sub-Additive Ergodic Theorem in this case), and the affirmative solution of the Eckmann–Ruelle conjecture (see [3]), respectively, does not hold, in fact carries *full* Hausdorff dimension. Similar statements can be proved for the Shannon–McMillan–Breiman Theorem.

We emphasize that the above results are only special cases of much stronger results. In particular, these results extend (with natural non-avoidable exceptions) to intersection of sets where *both* Birkhoff averages, Lyapunov exponents, local entropies, pointwise dimension, etc, do not exist. Moreover, besides the Hausdorff dimension we consider the topological entropy and prove that all those sets have also full topological entropy. We refer to the sections below for a precise formulation of the results.

An important element of unification in our approach is the use of Carathéodory dimension characteristics. These were introduced by Pesin (see [11] for a comprehensive description). We introduce a new Carathéodory dimension characteristic, the so-called *u-dimension* for each positive Hölder continuous function u (see Section 2.2 below). This allows us to treat simultaneously all the results mentioned above for each class of maps. Moreover, we provide a new description of Bowen's pressure formula based on the notion of u-dimension. We believe that our results and techniques are well adapted to obtain obtain proof that sets of irregular points are full with respect to other natural quantities such as the topological pressure of a fixed potential.

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2. Sets of Irregular Points

2.1. **Definitions.** Let S be a finite set, and $\sigma\colon S^{\mathbb{N}} \to S^{\mathbb{N}}$ the shift map given by $\sigma(i_1 i_2 \cdots) = (i_2 i_3 \cdots)$. We fix a number $\beta > 1$ and define a metric on $S^{\mathbb{N}}$ by

$$d(i_1 i_2 \cdots, j_1 j_2 \cdots) = \sum_{k=1}^{\infty} \beta^{-k} |i_k - j_k|. \tag{4}$$

We consider a *subshift* $\Sigma \subset S^{\mathbb{N}}$, i.e., a closed σ-invariant subset of $S^{\mathbb{N}}$ (i.e., $\sigma\Sigma = \Sigma$), and assume that $\sigma|\Sigma$ is topologically transitive. Let Z_Σ be the family of *cylinder sets*

$$C = \{(j_1 j_2 \cdots) \in \Sigma : (j_1 \cdots j_k) = (i_1 \cdots i_k)\}$$

for $i_1, \ldots, i_k \in S$. We call the integer k the *length* of C and denote it by $|C|$. We denote by CC' the cylinder set corresponding to the juxtaposition of the tuples specifying the cylinder sets C and C', in this order. Let $C_n(x) \in Z_\Sigma$ denote the cylinder set of length n which contains the point $x \in \Sigma$.

Given sequences of functions $F^i = \{f_n^i\colon \Sigma \to \mathbb{R}^+\}_{n\in\mathbb{N}}$ for $i = 1, \ldots, m$, we define the set

$$\widehat{\mathfrak{F}}(F^1, \ldots, F^m) = \left\{ x \in \Sigma : \varliminf_{n\to\infty} f_n^k(x) < \varlimsup_{n\to\infty} f_n^k(x) \text{ for } k = 1, \ldots, m \right\},$$

and call it a *set of irregular points*. Clearly, for every $1 \le k < m$,

$$\widehat{\mathfrak{F}}(F^1, \ldots, F^m) = \widehat{\mathfrak{F}}(F^1, \ldots, F^k) \cap \widehat{\mathfrak{F}}(F^{k+1}, \ldots, F^m).$$

We use the hat in $\widehat{\mathcal{F}}$ and in any other set to indicate that they are subsets of Σ.

We will show that, under general natural assumptions, that any set of irregular points carries full topological entropy.

We now describe several natural examples of such sets. Let $C(\Sigma)$ be the space of continuous functions on Σ. For each function $g \in C(\Sigma)$, set

$$\widehat{\mathcal{B}}(g) = \widehat{\mathcal{F}}\left(\left\{\frac{1}{n}S_n g\right\}_{n\in\mathbb{N}}\right) = \left\{x \in \Sigma : \lim_{n\to\infty} \frac{1}{n} S_n g(x) \text{ does not exist}\right\}.$$

where $S_n g(x) = \sum_{k=0}^{n-1} g(\sigma^k x)$ and, for each probability measure μ on Σ, and each continuous function $u \colon \Sigma \to \mathbb{R}^+$,

$$\widehat{\mathcal{H}}(\mu; u) = \widehat{\mathcal{F}}\left(\left\{\frac{f_n}{S_n u}\right\}_{n\in\mathbb{N}}\right) = \left\{x \in \Sigma : \lim_{n\to\infty} -\frac{\log \mu(C_n(x))}{S_n u(x)} \text{ does not exist}\right\}, \quad (5)$$

where $f_n(x) = -\log \mu(C_n(x))$ for each n. We write

$$\widehat{\mathcal{H}}(\mu) = \widehat{\mathcal{H}}(\mu; 1) \quad \text{and} \quad \widehat{\mathcal{H}}(\mu_1, \dots, \mu_m; u) = \bigcap_{i=1}^{m} \widehat{\mathcal{H}}(\mu_i; u).$$

The sets $\widehat{\mathcal{B}}(g)$ and $\widehat{\mathcal{H}}(\mu; u)$ are σ-invariant but may not be compact.

We also define the set

$$\widehat{\mathcal{B}} = \left\{x \in \Sigma : \lim_{n\to\infty} \frac{1}{n} S_n g(x) \text{ does not exist for some } g \in C(\Sigma)\right\}. \quad (6)$$

Note that

$$\widehat{\mathcal{B}} = \bigcup_{g\in C(\Sigma)} \widehat{\mathcal{B}}(g).$$

For each measure μ on Σ, we define the set of *typical points for* μ by

$$\widehat{\mathcal{G}}(\mu) = \left\{x \in \Sigma : \lim_{n\to\infty} \frac{1}{n} S_n g(x) = \int_\Sigma g \, d\mu \text{ for every } g \in C(\Sigma)\right\}.$$

Clearly, $\widehat{\mathcal{B}} \subset \Sigma \setminus \bigcup_\mu \widehat{\mathcal{G}}(\mu)$. If $x \in \Sigma \setminus \widehat{\mathcal{B}}$, then the map

$$g \mapsto \lim_{n\to\infty} \frac{1}{n} S_n g(x)$$

defines a σ-invariant bounded linear functional on $C(\Sigma)$, and, by the Riesz Representation Theorem, $x \in \widehat{\mathcal{G}}(\mu)$ for some invariant measure μ. Hence, $\widehat{\mathcal{B}} = \Sigma \setminus \bigcup_{\mu\in\mathcal{M}} \widehat{\mathcal{G}}(\mu)$, where $\mathcal{M}$ is the set of σ-invariant Borel probability measures on Σ. It is not hard to see that if $\mathcal{M}_{\text{ergodic}} \subset \mathcal{M}$ is the subset of ergodic measures, then

$$\widehat{\mathcal{B}} = \Sigma \setminus \bigcup_{\mu\in\mathcal{M}_{\text{ergodic}}} \widehat{\mathcal{G}}(\mu).$$

Remarks.

1. If μ is σ-invariant, then $\mu(\widehat{\mathcal{B}}) = 0$ (by the Birkhoff Ergodic Theorem and the separability of $C(\Sigma)$), and $\mu(\widehat{\mathcal{H}}(\mu)) = 0$ (by the Shannon–McMillan–Breiman Theorem).
2. If g_1 and g_2 are cohomologous, i.e., $g_1 - g_2 = \psi - \psi \circ \sigma + c$, where ψ is some continuous function and c some constant, then $\widehat{\mathcal{B}}(g_1) = \widehat{\mathcal{B}}(g_2)$.
3. If μ is a Gibbs measure, then there is a cohomology class of functions in $C(\Sigma)$ such that $\widehat{\mathcal{H}}(\mu) = \widehat{\mathcal{B}}(g)$ if and only if g belongs to this cohomology class.
4. If $\sigma|\Sigma$ is uniquely ergodic, then the set $\widehat{\mathcal{B}}$ is empty.
5. Let μ be a σ-invariant measure of maximal entropy; if μ is a Gibbs measure (in particular, when $\sigma|\Sigma$ is a topologically mixing subshift, this holds for subshifts of finite type, sofic subshifts, i.e., factors of subshifts of finite type, and, more generally, subshifts with the specification property), then $\widehat{\mathcal{H}}(\mu)$ is empty.

2.2. **The notion of u-dimension.** Let $u\colon \Sigma \to \mathbb{R}^+$ be a Hölder continuous function. For every set $Z \subset \Sigma$, we define

$$m_u(Z, \alpha) = \lim_{n\to\infty} \inf_{\mathcal{U}} \sum_i \sup_{x \in C_{m_i}} \exp(-S_{m_i} u(x)\alpha),$$

where the infimum is taken over all finite or countable covers $\mathcal{U}$ of Z by cylinder sets C_{m_i} such that $|C_{m_i}| = m_i \geq n$ for each i. Since u is Hölder continuous, there is a constant $L > 0$ such that $|S_m u(x) - S_m u(y)| \leq L$ whenever $y \in C_m(x)$.

By the general theory of Carathéodory dimension characteristics (see [11]), there is a unique critical value $\alpha = \dim_u Z$ at which $m_u(Z, \cdot)$ jumps from $+\infty$ to 0. We call $\dim_u Z$ the *u-dimension* of Z. For every probability measure μ on Z, we define

$$\dim_u \mu = \inf\{\dim_u Z : \mu(Z) = 1\}$$

and call $\dim_u \mu$ the *u-dimension* of μ.

We formulate here some preliminary results.

Proposition 2.1. *If μ_1 and μ_2 are probability measures on Σ, and $u\colon \Sigma \to \mathbb{R}^+$ is Hölder continuous, then, for every $\delta > 0$,*

$$\mu_1\left(\left\{x : \varliminf_{n\to\infty} -\frac{\log \mu_2(C_n(x))}{S_n u(x)} > \dim_u \mu_1 - \delta\right\}\right) > 0. \tag{7}$$

Proof. If (7) does not hold, then the set

$$\Gamma_\delta = \left\{x \in \Sigma : \varliminf_{n\to\infty} -\frac{\log \mu_2(C_n(x))}{S_n u(x)} \leq \dim_u \mu_1 - \delta\right\} \tag{8}$$

has full μ_1-measure. For each $x \in \Gamma_\delta$, let $\{n_k(x)\}_{k\in\mathbb{N}}$ be an increasing sequence of positive integers such that

$$-\frac{\log \mu_2(C_{n_k(x)}(x))}{S_{n_k(x)} u(x)} \leq \dim_u \mu_1 - \delta$$

for each k. Observe that two cylinder sets are either disjoint, or one is contained in the other. Hence, for each $L > 0$ there is a finite or countable cover $\{C_{m_i}(x_i) : i \in \mathbb{N}\}$

of Γ_δ formed by disjoint cylinders sets, for some points $x_i \in \Gamma_\delta$ and integers $m_i \in \{n_k(x_i) : k \in \mathbb{N}\}$ such that $m_i > L$ for each $i \in \mathbb{N}$. We obtain

$$\begin{aligned}\mu_2(\Gamma_\delta) &= \sum_{i=1}^{\infty} \mu_2(C_{m_i}(x_i)) \\ &\geq \sum_{i=1}^{\infty} \exp[-(\dim_u \mu_1 - \delta) S_{m_i} u(x_i)] \\ &\geq c(\theta) \sum_{i=1}^{\infty} \sup_{x \in C_{m_i}(x_i)} \exp[-(\dim_u \mu_1 - \delta) S_{m_i} u(x)],\end{aligned}$$

where $c(\theta)$ is a constant depending only on the Hölder exponent of u. Hence, $\dim_u \mu_1 - \delta \geq \dim_u \Gamma_\delta \geq \dim_u \mu_1$, because $\mu_1(\Gamma_\delta) = 1$. This contradiction implies the desired result. □

Corollary 2.2. *Let μ_1 and μ_2 be two probability measures on Σ, and $u\colon \Sigma \to \mathbb{R}^+$ is Hölder continuous. If μ_1 is an ergodic σ-invariant measure, then*

$$\mu_1 \left(\left\{ x \in \Sigma : \varliminf_{n\to\infty} -\frac{\log \mu_2(C_n(x))}{S_n u(x)} \geq \dim_u \mu_1 \right\} \right) = 1.$$

Proof. For each $\delta > 0$, the set Γ_δ defined by (8) is σ-invariant. By Proposition 2.1, $\mu_1(\Sigma \setminus \Gamma_\delta) = 1$ for every $\varepsilon > 0$, and hence, the set

$$\bigcap_{\delta>0} (\Sigma \setminus \Gamma_\delta) = \left\{ x \in \Sigma : \varliminf_{n\to\infty} -\frac{\log \mu_2(C_n(x))}{S_n u(x)} \geq \dim_u \mu_1 \right\}$$

has also full μ_1-measure. □

The following is an immediate consequence of Birkhoff Ergodic Theorem, Shannon–McMillan–Breiman Theorem, and Theorem 4.1 in [11].

Proposition 2.3. *If μ is an ergodic σ-invariant probability measure on Σ, and the function $u\colon \Sigma \to \mathbb{R}^+$ is Hölder continuous, then, for μ-almost every $x \in \Sigma$,*

$$\lim_{n\to\infty} -\frac{\log \mu(C_n(x))}{S_n u(x)} = \frac{h_\mu(\sigma)}{\int_\Sigma u \, d\mu} = \dim_u \mu. \tag{9}$$

We define the *lower* and *upper u-pointwise dimensions* of μ at the point x by

$$\underline{d}_{\mu,u}(x) = \varliminf_{n\to\infty} -\frac{\log \mu(C_n(x))}{S_n u(x)} \quad \text{and} \quad \overline{d}_{\mu,u}(x) = \varlimsup_{n\to\infty} -\frac{\log \mu(C_n(x))}{S_n u(x)}.$$

If these two numbers coincide, i.e., if the limit in (9) exists, we call the common value the *u-pointwise dimension* of μ at the point x, and denote it by $d_{\mu,u}(x)$.

Let g be a continuous function, and $Z \in \Sigma$ a not necessarily compact or invariant set. For each real number β, we set

$$p_g(Z, \beta) = \lim_{n\to\infty} \inf_{\mathcal{U}} \sum_i \sup_{x \in C_{m_i}} \exp(-m_i \beta + S_{m_i} g(x)),$$

where the infimum is taken over all finite or countable covers $\mathcal{U}$ of Z by cylinder sets C_{m_i} such that $m_i \geq n$ for each i. The pressure of g on the set Z (see [11]) is the unique critical value $\beta = P_Z(g)$ at which $p_g(Z, \beta)$ jumps from $+\infty$ to 0.

Proposition 2.4 (Bowen's pressure formula). *We have* $\dim_u Z = \alpha$, *where* α *is the unique root of the equation* $P_Z(-\alpha u) = 0$.

Proof. Set $g = -\alpha u$. Then $p_g(Z, 0) = m_u(Z, \alpha)$ and we obtain the desired result. □

Bowen's pressure formula was introduced by Bowen in [8] in the context of quasi-circles. In [15], Ruelle considered the same equation in the context of real analytic maps.

We are now able to prove the next proposition.

Proposition 2.5. *The following property holds:*

$$\dim_u \bigcup_\mu \widehat{\mathcal{G}}(\mu) = \sup\{\dim_u \mu : \mu \in \mathcal{M}_{\mathrm{ergodic}}\}. \tag{10}$$

We note that the union in (10) is in general not countable; otherwise, Proposition 2.5 would follow immediately from the general theory of Charathéodory dimension characteristics (see [11]).

Proof. By the variational principle for the topological pressure [12],

$$P_{\bigcup_\mu \widehat{\mathcal{G}}(\mu)}(-\alpha u) = \sup_{\mu \in \mathcal{M}_{\mathrm{ergodic}}} \left(h_\mu(\sigma) - \alpha \int_\Sigma u \, d\mu \right).$$

By Proposition 2.4, if $\alpha = \dim_u \bigcup_\mu \widehat{\mathcal{G}}(\mu)$, then $\sup_{\mu \in \mathcal{M}_{\mathrm{ergodic}}} \left(h_\mu(\sigma) - \alpha \int_\Sigma u \, d\mu \right) = 0$. Moreover, by Proposition 2.3, for any ergodic measure μ, $h_\mu(\sigma) - \gamma \int_\Sigma u \, d\mu < 0$ for all $\gamma > \dim_u \mu$. This shows that $\alpha \leq \sup_{\mu \in \mathcal{M}_{\mathrm{ergodic}}} \dim_u \mu$.

On the other hand, we can prove that $\dim_u \widehat{\mathcal{G}}(\mu) \geq \dim_u \mu$ for every ergodic measure μ. This completes the proof. □

Examples. We will consider mainly two expressions for the function u.

1. $u \equiv 1$ then the u-dimension of a set $Z \in \Sigma$ is simply the topological entropy of this set.
2. $u = \log a$ where a is defined as the lift to Σ of the norm of the derivative of an expanding conformal map. Then the u-dimension of a lifted set equals the Hausdorff dimension of this set (see Section 5).

3. Main Result

Consider a non-decreasing sequence $\Psi = \{\psi_n\}_{n \in \mathbb{N}}$ of positive numbers such that $\psi_n/n \to 0$ as $n \to \infty$. Define the subset $\Sigma_\Psi \subset \Sigma$ of points $x \in \Sigma$ such that for each $n \in \mathbb{N}$ and $C \subset Z_\Sigma$ with $|C| < \psi_n$ satisfying $CC_n(x) \in Z_\Sigma$, given $\overline{C} \in Z_\Sigma$ there exists $\underline{C} \in Z_\Sigma$ such that

$$CC_n(x)\underline{C}\overline{C} \in Z_\Sigma \quad \text{and} \quad |\underline{C}| \leq |CC_n(x)| + \psi_{|CC_n(x)|}.$$

We note that $\Sigma_\Psi \subset \sigma\Sigma_\Psi$, but presumably Σ_Ψ need not be σ-invariant in general.

For each measure μ on the subshift Σ we consider the following property:

$$\text{There exists a sequence } \Psi \text{ such that } \mu(\Sigma_\Psi) > 0. \tag{11}$$

This holds, for example, for σ-invariant measures on subshifts of finite type, sofic subshifts, and, more generally, subshifts with the specification property; in each of these cases $\Sigma_\Psi = \Sigma$ for some constant sequence Ψ.

Definition 3.1. A system of measures $\mu_1, \ldots, \mu_k$ is called *distinguishing* for $F^1, \ldots, F^m$ if for every $1 \le i \le m$, there exist distinct integers $j_1 = j_1(i)$, $j_2 = j_2(i) \in [1, k]$ and numbers $a^i_{j_1} \ne a^i_{j_2}$ such that

$$\lim_{n\to\infty} f^i_n(x) = a^i_{j_1} \text{ for } \mu_{j_1}\text{-almost all } x,$$

$$\lim_{n\to\infty} f^i_n(x) = a^i_{j_2} \text{ for } \mu_{j_2}\text{-almost all } x.$$

We can always assume that $k \le 2m$ in the definition. For example, let μ_1 and μ_2 be two distinct ergodic σ-invariant probability measures on Σ. Then, there is a function $g \in C(\Sigma)$ such that $\int_\Sigma g\, d\mu_1 \ne \int_\Sigma g\, d\mu_2$, and, by the Birkhoff Ergodic Theorem, the measures μ_1, μ_2 are distinguishing for $\{\frac{1}{n}S_n g\}_{n\in\mathbb{N}}$.

The following is our main result. It gives lower bounds for the u-dimension of sets of irregular points.

Theorem 3.2. *Let $\mu_1, \ldots, \mu_k$ be a distinguishing system of ergodic measures for $F^1, \ldots, F^m$ such that the condition* (11) *holds for each measure μ_i with respect to some sequence Ψ^i. Then, for any Hölder continuous function $u\colon \Sigma \to \mathbb{R}^+$ we have*

$$\dim_u \widehat{\mathfrak{F}}(F^1, \ldots, F^m) \ge \min\{\dim_u \mu_1, \ldots, \dim_u \mu_k\}.$$

Proof. For the sake of clarity we first present the proof in the case $m = 1$. The general case is discussed at the end.

When $m = 1$, we write $f_n = f^1_n$ for each $n \in \mathbb{N}$, and, without loss of generality, we may assume that μ_1, μ_2 is a distinguishing system of measures for $F = \{f_n\}_{n\in\mathbb{N}}$ (see Definition 3.1); we write $a^1_j = a_j$ for $j = 1, 2$. We may also assume that $a_j \ne 0$ for $j = 1, 2$. Otherwise we can consider the sequence of functions $F + a = \{f_n + a\}_{n\in\mathbb{N}}$, where a is a non-zero constant, since $\widehat{\mathfrak{F}}(F + a) = \widehat{\mathfrak{F}}(F)$. Without loss of generality we assume that $\dim_u \mu_1 \ge \dim_u \mu_2$. Choose a positive number δ such that

$$|a_1 - a_2| > 4\delta. \tag{12}$$

We consider the sequence $\Psi = \{\max\{\psi^1_n, \psi^2_n\}\}_{n\in\mathbb{N}}$, where $\Psi^i = \{\psi^i_n\}_{n\in\mathbb{N}}$ for $i = 1, 2$. For each integer $k \ge 1$, we set

$$p_k = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 2 & \text{if } k \text{ is even} \end{cases}.$$

For each integer $\ell \geq 1$, let $\widehat{\Gamma}_1^\ell \subset \Sigma_\Psi$ be the set of points $x \in \Sigma_\Psi$ such that for all $m \geq \ell$ and $i = 1, 2$, we have

$$|f_m(x) - a_1| < \delta \quad \text{and} \quad -\frac{\log \mu_i(C_m(x))}{S_m u(x)} > \dim_u \mu_1 - \delta. \tag{13}$$

For each $\ell \geq 1$, let $\widehat{\Gamma}_2^\ell \subset \Sigma_\Psi$ be the set of points $x \in \Sigma_\Psi$ such that for all $m \geq \ell$,

$$|f_m(x) - a_2| < \delta \quad \text{and} \quad -\frac{\log \mu_2(C_m(x))}{S_m u(x)} > \dim_u \mu_2 - \delta. \tag{14}$$

Clearly $\widehat{\Gamma}_i^{\ell+1} \supset \widehat{\Gamma}_i^\ell$ for each $\ell \geq 1$, and $i = 1, 2$.

Let ν_1 and ν_2 be the normalized measures obtained from the restrictions of μ_1 and μ_2 to the set Σ_Ψ. Fix $\varepsilon \in (0,1)$, and for each integer $k \geq 1$ set

$$\ell_k = \min\left(\left\{\ell \in \mathbb{N} : \nu_{p_k}(\widehat{\Gamma}_{p_k}^\ell) > 1 - \varepsilon/2^{k+1}\right\} \cup \{\ell_{k-1}\}\right), \tag{15}$$

where $\ell_0 = \infty$. We note that $\ell_k \geq \ell_{k-1}$. By Corollary 2.2 and Proposition 2.3, we have $\ell_k < \infty$ for every $k \geq 1$.

For $j = 1, 2$, since μ_j is invariant, the set of points $x \in \Sigma$ such that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(\sigma^m x)$$

for every $m \in \mathbb{N}$ has full μ_j-measure. For these points we can define the number

$$D_{n,m}(x) = \max_{y,z \in \sigma^{-n}x} \left\{\left|\frac{f_{n+m}(y)}{f_m(x)}\right|, \left|\frac{f_m(x)}{f_{n+m}(z)}\right|\right\},$$

and by Lusin's Theorem, for each $j = 1, 2$, and $\delta > 0$ there is an integer $r_j(n, \delta) \geq n$ such that $D_{n,m}(x) < 1 + \delta$ for all $m > r_j(n, \delta)$ and all x outside a set $Y_j^n(\varepsilon)$ of μ_j-measure at least $1 - \delta$.

For each $k \geq 1$, we define inductively the increasing sequences of positive integers $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ by $m_1 = n_1 = \ell_1$, and, for every $k \geq 2$, by

$$m_k = r_{p_k}(n_{k-1} + \lfloor\psi_{n_{k-1}}\rfloor, \varepsilon/2^{k+1}) + \ell_{k+1}! \quad \text{and} \quad n_k = n_{k-1} + \lfloor\psi_{n_{k-1}}\rfloor + m_k + 1. \tag{16}$$

We set

$$\Gamma_{p_k}^{\ell_k} = \widehat{\Gamma}_{p_k}^{\ell_k} \cap Y_{p_k}^{n_{k-1}}(\varepsilon/2^{k+1}).$$

Then $\nu_{p_k}(\Gamma_{p_k}^{\ell_k}) > 1 - \varepsilon/2^k$. For each $k \geq 1$, we define a family of cylinder sets by

$$\mathfrak{C}_k = \{C_{m_k}(x) : x \in \Gamma_{p_k}^{\ell_k}\};$$

moreover, we set $\mathfrak{D}_1 = \mathfrak{C}_1$, and

$$\mathfrak{D}_k = \{\underline{C}C\overline{C} \in Z_\Sigma : \underline{C} \in \mathfrak{D}_{k-1}, \overline{C} \in \mathfrak{C}_k, \text{ and } C \in Z_\Sigma \text{ is minimal}\}.$$

Here, minimality refers to the order $<$ in Z_Σ defined by: if C, $C' \in Z_\Sigma$ are distinct, we write $C < C'$ if $|C| < |C'|$, or if $|C| = |C'|$ but C is smaller than C' in the lexicographical order.

We now prove that for each $\underline{C}C\overline{C} \in \mathfrak{D}_k$ with $\underline{C} \in \mathfrak{D}_{k-1}$ and $\overline{C} \in \mathfrak{C}_k$, we have $|\underline{C}| \le n_{k-1}$ and $|C| < \psi_{n_{k-1}}$ for each $k \ge 2$. For $k = 2$ this is clear because $n_1 = m_1$. Using (16) and induction on $k > 2$, we obtain

$$|\underline{C}C\overline{C}| \le n_{k-1} + \psi_{n_{k-1}} + m_k < n_k$$

and hence, $|C'| \le \psi_{n_k}$ for each $\underline{C}'C'\overline{C}' \in \mathfrak{D}_{k+1}$ with $\underline{C}' \in \mathfrak{D}_k$ and $\overline{C}' \in \mathfrak{C}_{k+1}$, because Ψ is non-decreasing.

Set

$$\Lambda = \bigcap_{k\ge 1} \bigcup_{C\in\mathfrak{D}_k} C.$$

We define a measure μ on Λ by $\mu(C) = \nu_1(C)$ if $C \in \mathfrak{D}_1$, and by

$$\mu(\underline{C}C\overline{C}) = \mu(\underline{C})\nu_{p_k}(\overline{C})$$

if $\underline{C}C\overline{C} \in \mathfrak{D}_k$ for some $k > 1$. We extend μ to Σ by $\mu(A) = \mu(A \cap \Lambda)$ for each measurable set $A \subset \Sigma$. For each $k \ge 1$ and every $\underline{C} \in \mathfrak{D}_{k-1}$, it follows from (15) that

$$\mu\left(\bigcup_{\overline{C}\in\mathfrak{D}_k} \underline{C} \cap \overline{C}\right) \ge \mu(\underline{C})\left(1 - \frac{\varepsilon}{2^k}\right),$$

and hence,

$$\mu(\Lambda) \ge \prod_{k=1}^{\infty}\left(1 - \frac{\varepsilon}{2^k}\right) > 0$$

for all sufficiently small ε.

Observe that $m_k \le |C| \le n_k$ for each $k \ge 1$. Let now $x \in C \in \mathfrak{D}_k$. Then $\sigma^{|C|-m_k}x \in \Gamma_{p_k}^{\ell_k}$. By (13) and (14), we obtain

$$\begin{aligned} |f_{|C|}(x) - a_{p_k}| &\le \frac{f_{|C|}(x)}{f_{m_k}(\sigma^{|C|-m_k}x)} \times |f_{m_k}(\sigma^{|C|-m_k}x) - a_{p_k}| \\ &\quad + \left|1 - \frac{f_{|C|}(x)}{f_{m_k}(\sigma^{|C|-m_k}x)}\right| \times |a_{p_k}| \\ &\le D_{|C|-m_k,m_k}(x) \times |f_{m_k}(\sigma^{|C|-m_k}x) - a_{p_k}| \\ &\quad + (D_{|C|-m_k,m_k}(x) - 1)\times|a_{p_k}|. \end{aligned}$$

Hence, for all sufficiently large k and every $x \in C \in \mathfrak{D}_k$, we have

$$|f_{|C|}(x) - a_{p_k}| < 2\delta. \tag{17}$$

It follows from (12) and (17) that

$$\widehat{\mathfrak{F}}(F) \supset \Lambda. \tag{18}$$

Lemma 3.3. *If $x \in \Lambda$, then*

$$\varliminf_{n\to\infty} -\frac{\log \mu(C_n(x))}{S_n u(x)} \ge \dim_u \mu_2 - 2\delta.$$

Proof of the lemma. Let $x \in \Lambda$. We prove that for all sufficiently large integer q, there exists $\delta' = \delta'(q) \in (\delta, 2\delta)$ such that

$$-\frac{\log \mu(C_q(x))}{S_q u(x)} \geq \dim_u \mu_2 - \delta'.$$

First, observe that by (13) there exists an integer $q_0 = q_0(x) \leq m_1$ such that for each integer $q \in [q_0, m_1]$, we have $\mu(C_q(x)) = \nu_1(C_q(x)) \leq \mu_1(C_q(x))$, and

$$-\frac{\log \mu(C_q(x))}{S_q u(x)} \geq -\frac{\log \mu_1(C_q(x))}{S_q u(x)} \geq \dim_u \mu_2 - \delta.$$

We now proceed by induction on $q \geq q_0$. For each $q \in \mathbb{N}$, choose an integer k_q such that $|C^{k_q}| \leq q < |C^{k_q+1}|$, where

$$\mathfrak{D}_{k_q+1} \ni C^{k_q+1} \subset C_q(x) \subset C^{k_q} \in \mathfrak{D}_{k_q}.$$

Assume that

$$|C^{k_q}| \leq q \leq |C^{k_q}| + \psi_{|C^{k_q}|} + \ell_{k_q+1}. \tag{19}$$

We have

$$\frac{S_q u(x)}{S_{|C^{k_q}|} u(x)} \leq \frac{S_{|C^{k_q}|+\psi_{|C^{k_q}|}+\ell_{k_q+1}} u(x)}{S_{|C^{k_q}|} u(x)} = 1 + \frac{\psi_{|C^{k_q}|} + \ell_{k_q+1}}{|C^{k_q}|} \times \frac{\max_{x\in\Sigma} u(x)}{\min_{x\in\Sigma} u(x)},$$

and the last fraction approaches zero as $q \to \infty$.

Using induction and the positivity of u, we obtain

$$\begin{aligned} -\frac{\log \mu(C_q(x))}{S_q u(x)} &\geq -\frac{\log \mu(C^{k_q})}{S_q u(x)} \\ &\geq -\frac{\log \mu(C^{k_q})}{S_{|C^{k_q}|} u(x)} \times \frac{S_{|C^{k_q}|} u(x)}{S_q u(x)} \\ &\geq \dim_u \mu_2 - \delta', \end{aligned} \tag{20}$$

for all $q \geq q_1$, some integer $q_1 \geq q_0$, and some $\delta' = \delta'(q) \in (\delta, 2\delta)$. Hence, when (19) holds, the desired result follows from (20).

Assume now that (19) does not hold. In this case, we have

$$\mu(C_q(x)) = \mu(C^{k_q}) \nu_{p_{k_q}}(\widetilde{C}) \leq \mu(C^{k_q}) \mu_{p_{k_q}}(\widetilde{C}),$$

where $C_q(x) = C^{k_q} C \widetilde{C}$ and the cylinder set $\widetilde{C}$ contains an element of $\mathfrak{C}_{k_q+1}$; moreover, $|C| < \psi_{|C^{k_q}|}$ and $|\widetilde{C}| > \ell_{k_q+1}$. Thus

$$q \leq |C^{k_q}| + \psi_{|C^{k_q}|} + |\widetilde{C}|.$$

Therefore, by induction, the definition of $\Gamma^{\ell_{k_q+1}}_{p_{k_q+1}}$, and the positivity of u,

$$\begin{aligned}-\frac{\log\mu(C_q(x))}{S_q u(x)} &\geq \frac{1}{S_q u(x)}\left(-\log\mu(C^{k_q}) - \log\mu_{p_{k_q+1}}(\widetilde{C})\right)\\ &\geq \frac{S_{|C^{k_q}|}u(x) + S_{|\widetilde{C}|}u(x)}{S_q u(x)}(\dim_u\mu_2 - \delta_1)\\ &\geq \dim_u\mu_2 - \delta_2,\end{aligned}$$

for all $q \geq q_2$, some integer $q_2 \geq q_1$, and some $\delta_1 = \delta_1(q)$, $\delta_2 = \delta_2(q) \in (\delta, 2\delta)$. This completes the proof of the lemma. □

By Theorem 3.1 in [11], and Lemma 3.3, we obtain

$$\dim_u\Lambda \geq \dim_u(\mu|\Lambda) \geq \dim_u\mu_2 - 2\delta.$$

Since δ is arbitrary,

$$\dim_u\Lambda \geq \dim_u\mu_2.$$

Thus, by (18), $\dim_u\widehat{\mathcal{F}}(F) \geq \dim_u\mu_2$. Since $\dim_u\mu_1 \geq \dim_u\mu_2$, this completes the proof of the theorem in the case $m = 1$.

We now briefly discuss how to deal with the case $m > 1$. We consider the sequence $\Psi = \{\max\{\psi_n^1, \ldots, \psi_n^k\}\}_{n\in\mathbb{N}}$, where $\Psi^i = \{\psi_n^i\}_{n\in\mathbb{N}}$ for $i = 1, \ldots, k$. For each integer $s \geq 1$, we set $p_s = s \pmod k + 1$.

Without loss of generality, we may assume that $\dim_u\mu_{j_1(i)} \leq \dim_u\mu_{j_2(i)}$ for all $1 \leq i \leq m$, and $\dim_u\mu_j \geq \dim_u\mu_k$ for all $1 \leq j \leq k$. For each integer $\ell \geq 1$, let $\widehat{\Gamma}^\ell_{j_1(i)} \subset \Sigma_\Psi$ be the set of points $x \in \Sigma_\Psi$ such that for all $m \geq \ell$ and $t = 1$, $j_2(i)$, we have

$$|f_m^i(x) - a_1| < \delta \quad\text{and}\quad -\frac{\log\mu_t(C_m(x))}{S_m u(x)} > \dim_u\mu_t - \delta.$$

For each $\ell \geq 1$, let $\widehat{\Gamma}^\ell_{j_2(i)} \subset \Sigma_\Psi$ be the set of points $x \in \Sigma_\Psi$ such that for all $m \geq \ell$,

$$|f_m^i(x) - a_2| < \delta \quad\text{and}\quad \left|-\frac{\log\mu_{j_1(i)}(C_m(x))}{S_m u(x)} - \dim_u\mu_{j_1(i)}\right| < \delta.$$

Clearly $\widehat{\Gamma}_i^{\ell+1} \supset \widehat{\Gamma}_i^\ell$ for each $\ell \geq 1$, and $i = 1, 2$. □

The following result is a simple application of Theorem 3.2, illustrating the power of this theorem.

Corollary 3.4. *Let $\mu_1, \ldots, \mu_\ell$ be ergodic measures such that the condition* (11) *holds for each measure μ_i with respect to some sequence Ψ^i. If not all the numbers $\dim_u\mu_1$, $\ldots$, $\dim_u\mu_\ell$ are equal, then, for any Hölder continuous function $u\colon \Sigma \to \mathbb{R}^+$ we have*

$$\dim_u\widehat{\mathcal{H}}(\mu_1, \ldots, \mu_\ell; u) \geq \min\{\dim_u\mu_1, \ldots, \dim_u\mu_\ell\}.$$

Proof. □

In fact, one can proof the following apparently stronger statement.

Corollary 3.5. *Under the assumption of Theorem 3.2 and Corollary 3.4, we have*

$$\dim_u \left(\widehat{\mathcal{H}}(\mu_1 \dots, \mu_\ell; u) \cap \widehat{\mathcal{F}}(F^1, \dots, F^m)\right) \geq \min\{\dim_u \mu_1, \dots, \dim_u \mu_\ell\}.$$

Proof. □

We now obtain entropy lower bounds for the sets of irregular points $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{H}}(\mu)$. Since these sets may not be compact we need the notion of topological entropy for non-compact sets introduced independently by Bowen in [6], and by Pesin and Pitskel' in [12].

Corollary 3.6. *Let μ_1 and μ_2 be two distinct ergodic σ-invariant probability measures on Σ for which* (11) *holds for some sequences Ψ^1 and Ψ^2, respectively. Then:*

1. $h(\sigma|\widehat{\mathcal{B}}) \geq \min\{h_{\mu_1}(\sigma), h_{\mu_2}(\sigma)\}$;
2. *if* $h_{\mu_1}(\sigma) > h_{\mu_2}(\sigma)$, *then* $h(\sigma|\widehat{\mathcal{H}}(\mu_2) \cap \widehat{\mathcal{B}}) \geq h_{\mu_2}(\sigma)$.

Proof. Since μ_1 and μ_2 are distinct, there is a continuous function g on Σ such that

$$\int_\Sigma g\, d\mu_1 \neq \int_\Sigma g\, d\mu_2.$$

By Birkhoff Ergodic Theorem, the measures μ_1, μ_2 compose a distinguishing system for $\frac{1}{n}\{S_n g\}_{n\in\mathbb{N}}$. Hence, setting $u \equiv 1$, Theorem 3.2 implies the first inequality.

If, in addition, $h_{\mu_1}(\sigma) > h_{\mu_2}(\sigma)$ then, by Proposition 2.1, the same system μ_1, μ_2 is also distinguishing for $\frac{1}{n}\{S_n g\}_{n\in\mathbb{N}}$, $\{f_n^2\}_{n\in\mathbb{N}}$, where $f_n^2 = -\frac{1}{n}\log \mu_2(C_n(x))$ for each $n \in \mathbb{N}$. Theorem 3.2 implies the second statement of the corollary. □

By the statement 1 in Corollary 3.6, we have

$$\begin{aligned} h(\sigma|\widehat{\mathcal{B}}) &\geq \sup\{h_\mu(f) : h_\mu(f) \neq h(\sigma),\ \mu \text{ is ergodic, and (11) holds}\} \\ &= \sup\{h_\mu(f) : h_\mu(f) \neq h(\sigma) \text{ and (11) holds}\}, \end{aligned} \tag{21}$$

with the convention that $h(\sigma|\varnothing) = \sup \varnothing = 0$.

4. Applications to Subshifts of Finite Type

In this and the following sections we present several applications of Theorem 3.2. Among others, we consider the sets of irregular points where the local entropy, the pointwise dimension, and the Birkhoff average do not exist, both for subshifts of finite type, for repellers of conformal expanding maps, and for horseshoes, and show that these carries carry full topological entropy (with some natural exceptions). In each case, we choose appropriate sequences of functions F^i and u, and a system of distinguishing measures, and apply Theorem 3.2. The choice of the system of measures is based on the theory multifractal analysis. To our best knowledge this constitutes the first mathematical application of that theory.

4.1. **Multifractal analysis and sets of irregular points.** We now present effect a complete multifractal analysis for the u-dimension. For every real number α such that $K_\alpha = \{x : d_{\mu,u}(x) = \alpha\} \neq \varnothing$, we write

$$\mathcal{D}_u(\alpha) = \dim_u K_\alpha.$$

The function $\alpha \mapsto \mathcal{D}_u(\alpha)$ is called the *u-dimension spectrum for u-pointwise dimensions* (with respect to the measure μ). Let φ be a continuous function on Σ. For every real number q, we define the function

$$\varphi_q = -T_u(q)u + q\varphi,$$

where the number $T_u(q)$ is chosen such that $P(\varphi_q) = 0$. We denote by ν_q and m_u, respectively, the equilibrium measures of φ_q and $-\dim_u \Sigma \cdot u$ with respect to σ.

The following is a complete multifractal analysis of the spectrum $\mathcal{D}_u$ for subshifts of finite type. It follows from a combination of the results in [13, 4, 16] for subshifts of finite and repellers of conformal expanding maps (see Section 5 for the definition; some care is necessary when we transfer results originally formulated for repellers to results formulated in terms of the underlying symbolic dynamics).

Theorem 4.1. *Let $\sigma\colon \Sigma \to \Sigma$ be a subshift of finite type, u and φ Hölder continuous functions on Σ, such that u is positive and $P(\varphi) = 0$, and μ the equilibrium measure of φ with respect to σ. Then, the following properties hold:*

1. *For μ-almost every $x \in \Sigma$, the u-pointwise dimension of μ at x exists and*
$$d_{\mu,u}(x) = \frac{\int_\Sigma \varphi \, d\mu}{\int_\Sigma u \, d\mu}.$$
2. *The function $q \mapsto T_u(q)$ is real analytic on $\mathbb{R}$, and satisfies $T_u'(q) \leq 0$ and $T_u''(q) \geq 0$ for every $q \in \mathbb{R}$. Moreover, $T_u(0) = \dim_u \Sigma$ and $T_u(1) = 0$.*
3. *The domain of the function $\alpha \mapsto \mathcal{D}_u(\alpha)$ is a closed interval in $[0, +\infty)$ and coincides with the range of the function $\alpha_u(q) = -T_u'(q)$. For every $q \in \mathbb{R}$, we have*
$$\mathcal{D}_u(\alpha_u(q)) = T_u(q) + q\alpha_u(q),$$
and
$$\alpha_u(q) = \frac{\int_\Sigma \varphi \, d\nu_q}{\int_\Sigma u \, d\nu_q}.$$
4. *If $\mu \neq m_u$, then $\mathcal{D}_u$ and T_u are analytic strictly convex functions, and hence, $(\mathcal{D}_u, T_u)$ is a Legendre pair with respect to the variables α, q.*
5. *If $\mu = m_u$, then $\mathcal{D}_u$ is the delta function*
$$\mathcal{D}_D(\alpha) = \begin{cases} \dim_u \Sigma & \text{if } \alpha = \dim_u \Sigma \\ 0 & \text{if } \alpha \neq \dim_u \Sigma \end{cases}.$$
6. *For every $q \in \mathbb{R}$, we have $\nu_q(K_\alpha) = 1$ and*
$$d_{\nu_q,u}(x) = T_u(q) + q\alpha_u(q)$$

for ν_q-almost all $x \in K_\alpha$. Moreover, $\mathcal{D}_u(\alpha_u(q)) = \dim_u \nu_q$ for every $q \in \mathbb{R}$.

We call m_u the *measure of maximal u-dimension*, and ν_q the *full measure* for the spectrum $\mathcal{D}_u$ at the point $\alpha_u(q)$. We note that when $u = 1$, the spectrum $\mathcal{D}_u$ coincides with the entropy spectrum for local entropies introduced in [4], and that when $u = \log a$ for some Hölder continuous function a we recover the dimension spectrum for pointwise dimensions on a repeller of a $C^{1+\varepsilon}$ conformal expanding map f such that $a(x) = \|d_x f\|$ (expressed in terms of its underlying symbolic representation by a subshift of finite type). See [4] for details.

Theorem 4.2. *Let $\varphi_1, \dots, \varphi_n$ be Hölder continuous potentials on Σ, $g \equiv 1$ or $g = \log a$, and $u \equiv 1$ or $u = \log a$ with φ_i not cohomologous to g for $i = 1, \dots, n$. Then, for any $\varepsilon > 0$ there is a distinguishing system of measures $\mu_1, \dots, \mu_{2n}$ for the sequences of functions $\{\frac{S_k\varphi_1}{S_kg}\}_{k\in\mathbb{N}}, \dots, \{\frac{S_k\varphi_n}{S_kg}\}_{k\in\mathbb{N}}$ such that*

$$\min\{\dim_u \mu_1, \dots, \dim_u \mu_{2n}\} > \dim_u \Sigma - \varepsilon.$$

Proof. Without loss of generality we may assume that $P(\varphi_i) = 0$; $i = 0, \dots, n$. The proof consists of three cases.

Case 1. $g = u$. Fix $\varepsilon > 0$. Since no function φ_i is cohomologous to $g = u$ their corresponding equilibrium states $\nu_i = \nu_{\varphi_i}$ have a non-trivial spectrum with respect to the function g. In the case of $g = u \equiv 1$ this spectrum is the entropy spectrum of the local entropies and in case of $g = u = \log a$ this spectrum is the dimension spectrum of the pointwise dimension (see [4]). The non-triviality of these spectra implies that for each $1 \le i \le n$ we can find two measures $\nu_i^1 = \nu_{q_1}$ and $\nu_i^2 = \nu_{q_2}$ for some q_1, $q_2 > 1$ such that

$$\min\{\dim_u \nu_i^1, \dim_u \nu_i^1\} > \dim_u \Sigma$$

and it exist

$$\lim_{n\to\infty} \frac{S_n\varphi_i(x)}{S_ng(x)} = a_i^j \quad \text{for } \nu_i^j\text{-almost every } x$$

and $a_i^1 \ne a_i^2$. We the set $\mu_{2i-1} = \nu_i^1$ and $\mu_{2i} = \nu_i^2$. This system is a distinguishing one.

Case 2. $u \equiv 1$ and $g = \log a$. This case is different from the previous one since we do not have a sufficient description of the entropy spectrum of pointwise dimensions. But as in the above case it is enough to find for each i two measures of different Hausdorff dimension but large enough metric entropy. Let i and ε be fixed, μ_E be the measure of maximal entropy which is a Gibbs measure for a constant potential.

We consider a neighborhood of $\varphi \equiv -h(\sigma|\Sigma)$ in the space $C_\theta(\Sigma)$ of Hölder continuous functions on Σ with Hölder exponent θ. For each function $\varphi \in C_\theta(\Sigma)$ we define its norm $\|\varphi\|_\theta$ by

$$\|\varphi\|_\theta = \sup\{|\varphi(x)| : x \in \Sigma\} + \inf\{K : \varphi \in C_{\theta,K}(\Sigma)\},$$

where

$$C_{\theta,K}(\Sigma) = \left\{\varphi \in C(\Sigma) : |\varphi(x) - \varphi(y)| \le Kd(x,y)^\theta \text{ for every } x, y \in \Sigma\right\}.$$

The entropy of a Gibbs measure with potential ψ with $P(\psi) = 0$ is given by

$$h_{\mu_\psi}(\sigma) = \frac{d}{dt}P(t\psi)|_{t=1} = \int_\Sigma \psi\, d\mu_\psi.$$

This implies the analytic dependence of the entropy on the potentials. Let us now assume that we cannot find a potential close enough to a constant such that the pointwise dimension of the corresponding Gibbs measure is different from that of the measure of maximal entropy. This means that for all potentials η in a certain small $C_\theta(\Sigma)$-neighborhood U of the constant potential $-h(\sigma|\Sigma)$ the equality

$$\frac{d}{dt}P(-T_\eta(t)\log a + \varphi_i + t\eta)|_{t=1} = -\frac{\int_\Sigma \varphi_i\, d\mu_\eta}{\int_\Sigma \log a\, d\mu_\eta} = -\frac{\int_\Sigma \varphi_i\, d\mu_E}{\int_\Sigma \log a\, d\mu_E} = C$$

holds, where $T_\eta(t)$ is chosen such that $P(-T_\eta(t)\log a + \varphi_i + t\eta) = 0$. By the analitycity of the pressure functional we have for all Gibbs states μ on Σ

$$\int_\Sigma \varphi_i\, d\mu = C\int_\Sigma \log a\, d\mu.$$

But this implies that $\varphi_i \sim \log a$ what contradicts the assumptions. Hence in every neighborhood, and in particular in a neighborhood small enough to have entropy of all its Gibbs states larger than $h(\sigma|\Sigma) - \varepsilon$, of the potential $-h(\sigma|\Sigma)$ there is a potential η such that

$$\lim_{n\to\infty} \frac{S_n\varphi_i(x)}{S_n g(x)} = a_i^1 \quad \text{for } \mu_\eta\text{-almost every } x$$

and

$$\lim_{n\to\infty} \frac{S_n\varphi_i(x)}{S_n g(x)} = a_i^2 \neq a_i^1 \quad \text{for } \mu_E\text{-almost every } x.$$

We set $\mu_{2i-1} = \mu_\eta$ and $\mu_{2i} = \mu_E$. By the above arguments this gives rise to a system of distinguishing measures.

Case 3. $g \equiv 1$ and $u = \log a$. The proof in this case is analog to the proof in case 2. The only change is to consider the measure of maximal dimension μ_D which is the Gibbs measure for $-\dim_{\log a}\Sigma \log a$ instead of the measure of maximal entropy. □

4.2. **Sets of irregular points for local entropies and Birkhoff averages.** We start with a preliminary result.

Proposition 4.3. *If $\sigma|\Sigma$ is a topologically mixing subshift of finite type, then*

$$\widehat{\mathcal{B}} = \{x \in \widehat{\mathcal{H}}(\mu) : \mu \textit{ is a Gibbs measure}\}. \tag{22}$$

Proof. Observe that $\widehat{\mathcal{H}}(\mu) \subset \widehat{\mathcal{B}}$ whenever μ is a Gibbs measure (if μ is a Gibbs measure for the continuous potential φ, one considers the function $g = P_{\sigma|\Sigma}(\varphi) - \varphi$ in (6)). Hence, if $\widehat{\mathcal{A}} \subset \Sigma$ is the set defined by the right-hand side of (22), then $\widehat{\mathcal{A}} \subset \widehat{\mathcal{B}}$.

Let $x \in \widehat{\mathcal{A}}$. Then the limit

$$\lim_{n\to\infty} \frac{1}{n} S_n\varphi(x)$$

exists for every Hölder continuous function φ on Σ. For a given continuous function g on Σ let $\{\varphi_m\}_{m\in\mathbb{N}}$ be a sequence of Hölder continuous functions on Σ such that $\sup_{x\in\Sigma}|g(x)-\varphi_m(x)| \to 0$ as $m\to\infty$. This implies that

$$\begin{aligned}0 &\le \overline{\lim_{n\to\infty}} \frac{1}{n} S_n g(x) - \underline{\lim_{n\to\infty}} \frac{1}{n} S_n g(x)\\ &\le \overline{\lim_{n\to\infty}} \frac{1}{n} S_n \varphi_m(x) - \underline{\lim_{n\to\infty}} \frac{1}{n} S_n \varphi_m(x) + 2\sup_{x\in\Sigma}|g(x)-\varphi_m(x)|\\ &= 2\sup_{x\in\Sigma}|g(x)-\varphi_m(x)| \to 0\end{aligned}$$

as $m\to\infty$, and hence, $x\in\widehat{\mathcal{B}}$. This implies that $\widehat{\mathcal{B}}\subset\widehat{\mathcal{A}}$, and hence, $\widehat{\mathcal{A}}=\widehat{\mathcal{B}}$. □

Theorem 4.4. *Let $\sigma|\Sigma$ be a mixing subshift of finite type. If $g_1, \ldots, g_k$ are Hölder continuous functions on Σ non-cohomologous to 0, then*

$$h(\sigma|\widehat{\mathcal{B}}(g_1)\cap\cdots\cap\widehat{\mathcal{B}}(g_k)) = h(\sigma).$$

Proof. Choose $2k$ measures μ_i^1, μ_i^2 for $i=1,\ldots,k$ such that $\int_\Sigma g_i\,d\mu_i^1 \ne \int_\Sigma g_i\,d\mu_i^2$ for each i. Their existence follows easily from the multifractal analysis of the entropy spectrum for pointwise entropies, with respect to the Gibbs measures with potentials $g_1, \ldots, g_k$ (see Theorem 4.1). These measures are distinguishing for the sequences $\{\frac{1}{n}S_n g_1\}_{n\in\mathbb{N}}, \ldots, \{\frac{1}{n}S_n g_k\}_{n\in\mathbb{N}}$, and the desired result follows from Theorems 3.2 and 3.4. □

For a mixing subshift of finite type, it follows from Theorem 4.4 that

$$h(\sigma|\widehat{\mathcal{B}}) = h(\sigma).$$

The following result shows that, in the case of subshifts of finite type, the set $\widehat{\mathcal{B}}$ as well as the sets $\widehat{\mathcal{H}}(\mu)$ carry all the topological entropy.

Theorem 4.5. *If $\sigma|\Sigma$ is a subshift of finite type, then the following properties hold:*

1. *$h(\sigma|\widehat{\mathcal{B}}) = h(\sigma)$;*
2. *if μ is a Gibbs measure on Σ which is not a measure of maximal entropy, then $\widehat{\mathcal{H}}(\mu)\subset\widehat{\mathcal{B}}$ and $h(\sigma|\widehat{\mathcal{H}}(\mu)) = h(\sigma)$.*

Proof. Both statements are trivial when $h(\sigma)=0$, and hence, we may assume that $h(\sigma)>0$ without loss of generality.

Since Σ is of finite type there are always two distinct Gibbs measures with entropy arbitrary close to the topological entropy of Σ. Namely, by Theorems 4.4 and 5.4 in [4] (the entropy spectrum for local entropies is analytic and has a family of full measures, and hence), for each $\varepsilon>0$ there exist Gibbs measures μ_1 and μ_2 such that:

1. $h_{\mu_1}(\sigma) > h_{\mu_2}(\sigma) > h(\sigma)-\varepsilon$;
2. for μ_1-almost every $x\in\Sigma$,

$$\lim_{n\to\infty} -\frac{1}{n}\log\mu(C_n(x)) = h_{\mu_1}(\sigma);$$

3. for μ_2-almost every $x \in \Sigma$,
$$\lim_{n\to\infty} -\frac{1}{n} \log \mu(C_n(x)) = h_{\mu_2}(\sigma).$$

Moreover, we have that $\widehat{\mathcal{H}}(\mu) \subset \widehat{\mathcal{B}}$ since $h_{\mu_\varphi}(\sigma) = \lim_{n\to\infty} -\frac{1}{n} S_n\varphi(x)$ for μ_φ-almost every x for any Gibbs measure μ_φ corresponding to a Hölder continuous potential φ and Theorem 4.2 immediately implies the remainder of the statements. □

Statement 1 in Theorem 4.5 was first established by Pesin and Pitskel' in [12] in the special case of the Bernoulli shift on two symbols (their proof immediately generalizes to any Bernoulli shift).

Remarks.

1. All the statements in the former section remain true when we substitute $S^{\mathbb{N}}$ by the two-sided space $S^{\mathbb{Z}}$.
2. It follows from the special type of metric introduced in (4) that for any subset $Z \subset \Sigma$, we have $h(\sigma|Z) = \dim_H Z \cdot \log \beta$. Hence, by Theorem 4.5, for a subshift of finite type $\sigma|\Sigma$, and a Gibbs measure μ on Σ which is not the measure of maximal entropy, we have
$$\dim_H \widehat{\mathcal{H}}(\mu) = \dim_H \Sigma.$$

5. Applications to Repellers

Let $f\colon M \to M$ be a C^1 map of a smooth manifold, and J a compact subset of M. We say that f is *expanding* and J is a *repeller* of f if:

1. There are constants $C > 0$ and $\beta > 1$ such that $\|d_x f^n V\| \ge C\beta^n \|V\|$ for all $x \in J$, $V \in T_x M$, and $n \ge 1$;
2. $J = \bigcap_{n\ge 0} f^{-n} V$ for some open neighborhood V of J.

One can easily show that $fJ = J$.

We recall that a finite cover $\{R_1, \ldots, R_p\}$ of M by closed sets is called a *Markov partition* of J (with respect to f) if:

1. $\overline{\operatorname{int} R_i} = R_i$ for each $i = 1, \ldots, p$;
2. $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ if $i \ne j$;
3. each fR_i is a union of sets R_j.

It is well known that repellers admit Markov partitions of arbitrarily small diameter.

Let J be a repeller of an expanding map f, and $\{R_1, \ldots, R_p\}$ a Markov partition of J. We define a $p \times p$ *transfer* matrix $A = (a_{ij})$ by
$$a_{ij} = \begin{cases} 1 & \text{if } R_i \cap f^{-1}R_j \ne \varnothing \\ 0 & \text{otherwise} \end{cases}. \tag{23}$$

setting Consider the associated one-sided subshift of finite type $\sigma|\Sigma$. For each $\omega = (i_0 i_1 \cdots) \in \Sigma$, the set
$$\chi(\omega) = \left\{x \in X : f^k x \in R_{i_k} \text{ for every } k \ge 0\right\}$$

consists of a single point in J. We obtain a *coding map* $\chi\colon \Sigma \to J$ for the repeller, which is continuous, onto, and satisfies $f \circ \chi = \chi \circ \sigma$.

A smooth map $f\colon M \to M$ is called *conformal* if $d_x f$ is a multiple of an isometry at every point $x \in M$. Well-known examples of conformal expanding maps include one-dimensional Markov maps and holomorphic maps. We write $a(x) = \|d_x f\|$ for each $x \in M$. The equilibrium measure m_D of $-\dim_H J \cdot \log a$ on J is called the *measure of maximal dimension* (for a conformal $C^{1+\varepsilon}$ expanding map, it is the unique f-invariant measure μ such that $\dim_H \mu = \dim_H J$). We denote by m_E the *measure of maximal entropy*.

Set $\mathcal{B} = \chi(\widehat{\mathcal{B}})$ and $\mathcal{H}(\mu) = \chi(\widehat{\mathcal{H}}(\mu))$. We observe that

$$\mathcal{B} = \left\{ x \in J : \lim_{n\to\infty} \frac{1}{n} S_n g(x) \text{ does not exist for some } g \in C(J) \right\}.$$

We also define the sets

$$\mathcal{L} = \left\{ x \in J : \lim_{n\to\infty} \frac{1}{n} \log\|d_x f^n\| \text{ does not exist} \right\}$$

and, for each probability measure μ on J,

$$\mathcal{D}(\mu) = \left\{ x \in J : \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ does not exist} \right\}.$$

By Kingman's Subadditive Ergodic Theorem, we have $\mu(\mathcal{L}) = 0$ for any f-invariant probability measure μ on J.

We now enumerate several sets of irregular points which carry full topological entropy and full Hausdorff dimension. The proofs are based on Theorems 3.2 and 4.2.

Theorem 5.1. *If J is a repeller of a conformal $C^{1+\varepsilon}$ expanding map f, for some $\varepsilon > 0$, then the following properties hold:*

1. $h(f|\mathcal{B}) = h(f|J)$ *and* $\dim_H \mathcal{B} = \dim_H J$;
2. *if* $m_D \neq m_E$, *then* $h(f|\mathcal{L} \cap \mathcal{B}) = h(f|J)$ *and* $\dim_H(\mathcal{L} \cap \mathcal{B}) = \dim_H J$;
3. *if μ is an equilibrium measure on J and $\mu \neq m_D$, then* $h(f|\mathcal{D}(\mu) \cap \mathcal{B}) = h(f|J)$ *and* $\dim_H(\mathcal{D}(\mu) \cap \mathcal{B}) = \dim_H J$;
4. *if μ is an equilibrium measure on J and $\mu \neq m_E$, then* $h(f|\mathcal{H}(\mu) \cap \mathcal{B}) = h(f|J)$ *and* $\dim_H(\mathcal{H}(\mu) \cap \mathcal{B}) = \dim_H J$;
5. *if μ is an equilibrium measure on J, $\mu \neq m_D$, and $\mu \neq m_E$, then* $h(f|\mathcal{D}(\mu) \cap \mathcal{H}(\mu) \cap \mathcal{B}) = h(f|J)$ *and* $\dim_H(\mathcal{D}(\mu) \cap \mathcal{H}(\mu) \cap \mathcal{B}) = \dim_H J$;
6. *if μ is an equilibrium measure on J, and the three measures μ, m_D, and m_E are distinct, then* $h(f|\mathcal{D}(\mu) \cap \mathcal{H}(\mu) \cap \mathcal{L} \cap \mathcal{B}) = h(f|J)$ *and* $\dim_H(\mathcal{D}(\mu) \cap \mathcal{H}(\mu) \cap \mathcal{L} \cap \mathcal{B}) = \dim_H J$.

Proof. We proceed the proof in pointing out the appropriate sets $(f_k^1, \dots, f_k^n)$ of functions, the function u and corresponding sets of distinguishing measure in order to apply Theorem 3.2. Theorem 4.2 will help us to find these measures.

1. The first equality is contained in Theorem 4.5 by observing that the repeller J is coded by Σ. The second is a corollary of the main Theorem 3.2 by setting $u = \log a$

and choosing two distinct measures with Hausdorff dimension arbitrary close to the Hausdorff dimension of the repeller.

2. Choose for $\varepsilon > 0$ two distinct Gibbs measures μ_φ and μ_ψ from the entropy (dimension spectrum) of the Lyapunov exponents having metric entropy (respectively, Hausdorff dimension) not less than $h(J) - \varepsilon$ (respectively, $\dim_h J - \varepsilon$). This can be done since $m_D \neq m_E$. Then these measures are distinguishing for the set of functions $(\{\frac{1}{n}S_n \log a\}_{n\in\mathbb{N}}, \{S_n f\}_{n\in\mathbb{N}})$ where f is some function with $\int f\, d\mu_\varphi \neq \int f\, d\mu_\psi$. Then Theorem 3.2 concludes the proof with $u \equiv 1$ ($u = \log a$) and ε tending to zero.

3. Both equalities follow immediately from Theorems 3.2 and 4.2 by setting $f_k^1 = \frac{S_k\varphi}{S_k \log a}$ and $u \equiv 1$ or $u = \log a$, respectively, where φ is a potential with zero pressure for μ.

4. Both equalities follow immediately from Theorems 3.2 and 4.2 by setting $f_k^1 = \frac{1}{k}S_k\varphi$ and $u \equiv 1$ or $u = \log a$, respectively, where φ is a potential with zero pressure for μ.

5. The proof is the same as for statements 3 and 4 by setting $f_k^1 = \frac{1}{k}S_k\varphi$, $f_k^2 = \frac{S_k\varphi}{S_k \log a}$ and $u \equiv 1$ or $u = \log a$, respectively, where φ is a potential with zero pressure for μ.

6. The proof follows by setting $f_k^1 = \frac{1}{k}S_k\varphi$, $f_k^2 = \frac{S_k\varphi}{S_k \log a}$, $f_k^3 = \frac{1}{n}S_k \log a$ and $u \equiv 1$ or $u = \log a$. □

We emphasize that the sets of irregular points enumerated in Theorem 5.1 are important but nevertheless only special cases of many examples that one can obtain using Theorems 3.2 and 4.2.

6. Applications to Horseshoes

Let $f\colon M \to M$ be a C^1 diffeomorphism of a smooth manifold M, and $\Lambda \subset M$ a compact locally maximal hyperbolic set for f. Then, there is a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u$, and constants $C > 0$ and $\lambda \in (0,1)$ such that for each $x \in \Lambda$:

1. $d_x f\, E_x^s = E_{fx}^s$ and $d_x f\, E_x^u = E_{fx}^u$;
2. $\|d_x f^n v\| \le C\lambda^n \|v\|$ for all $v \in E_x^s$ and $n \ge 0$;
3. $\|d_x f^{-n} v\| \le C\lambda^n \|v\|$ for all $v \in E_x^u$ and $n \ge 0$.

For each point $x \in \Lambda$ there exist *local stable* and *unstable manifolds* $W^s(x)$ and $W^u(x)$, with $T_x W^s(x) = E_x^s$ and $T_x W^u(x) = E_x^u$. Moreover, there exists $\delta > 0$ such that for all x, $y \in \Lambda$ with $\rho(x,y) < \delta$, the set $W^s(x) \cap W^u(y)$ consists of a single point, which we denote by $[x,y]$, and the map

$$[\cdot,\cdot]\colon \{(x,y) \in \Lambda \times \Lambda : \rho(x,y) < \delta\} \to \Lambda$$

is continuous. For each $x \in M$, we write

$$a^u(x) = \|df|E^u(x)\| \quad \text{and} \quad a^s(x) = \|df|E^s(x)\|.$$

The functions a^s and a^u satisfy $a^u(x) > 1$ and $a^s(x) < 1$ for every $x \in \Lambda$, and if f is of class $C^{1+\varepsilon}$, then they are Hölder continuous.

A non-empty closed set $R \subset M$ is called a *rectangle* if $\operatorname{diam} R < \delta$, $R = \overline{\operatorname{int} R}$, and $[x, y] \in R$ whenever $x, y \in R$. For each $x \in R$, we write $W^s(x, R) = W^s(x) \cap R$ and $W^u(x, R) = W^u(x) \cap R$. A finite cover $\{R_1, \ldots, R_p\}$ of Λ by rectangles is called a *Markov partition* of Λ (with respect to f) if:

1. $R_i \cap R_j \subset \partial R_i \cap \partial R_j$ for any $i \neq j$;
2. for each $x \in \operatorname{int} R_i \cap f^{-1} \operatorname{int} R_j$, we have

$$fW^u(x, R_i) \supset W^u(fx, R_j) \quad \text{and} \quad fW^s(x, R_i) \subset W^s(fx, R_j).$$

Locally maximal hyperbolic sets have Markov partitions of arbitrarily small diameter.

Let Λ be a compact locally maximal hyperbolic set for f, and $\{R_1, \ldots, R_p\}$ a Markov partition of J. We define a $p \times p$ *transfer* matrix $A = (a_{ij})$ by (23). Consider the associated two-sided subshift of finite type $\sigma|\Sigma$. For each $\omega = (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma$, the set

$$\chi(\omega) = \{x \in X : f^k x \in R_{i_k} \text{ for every } k \in \mathbb{Z}\}$$

consists of a single point in Λ. We obtain the *coding map* $\chi\colon \Sigma \to \Lambda$ for the hyperbolic set, which is continuous, onto, and satisfies $f \circ \chi = \chi \circ \sigma$. For each point $\omega \in \Sigma$, and each non-negative integers n, m, we define the cylinder set

$$C_m^n = C_m^n(\omega) = \{(\cdots j_{-1} j_0 j_1 \cdots) \in \Sigma : j_k = i_k \text{ for } i = -m, \ldots, n\}.$$

When we pass from a one-sided to a two-sided shift (coding a hyperbolic set), there is an asymmetry which apparently was never mentioned in the literature. Thsi problem occurs only for non-compact or non-invariant subsets of Σ. Namely, with the definition of topological entropy introduced in [6] and [12], $h(\sigma|Z)$ and $h(\sigma^{-1}|\Sigma)$ may not coincide; however if Z is compact and σ-invariant, then $h(\sigma|Z) = h(\sigma^{-1}|\Sigma)$. This asymmetry is due to the fact that $h(\sigma|Z)$ takes only into account the complexity in the "future".

We introfude a new notion of topological entropy which takes into account the "complexity" both in the "future" and in the "past". For every set $Z \subset \Sigma$ and real number α, we set

$$m(Z, \alpha) = \varliminf_{n+m \to \infty} \inf_{\mathcal{U}} \sum_i \exp(-(n_i + m_i)\alpha),$$

where the infimum is taken over all finite or countable covers $\mathcal{U}$ of Z by cylinder sets $C_{m_i}^{n_i}$ such that $m_i \geq m$ and $n_i \geq n$ for each i. Then there is a unique critical value $\alpha = h^*(\sigma|Z)$ at which $m(Z, \cdot)$ jumps from $+\infty$ to 0. It is easy to see that for every subset $Z \in \Sigma$, we have

$$h^*(\sigma|Z) = \min\{h(\sigma|Z), h(\sigma^{-1}|Z)\},$$

the minimum of the contributions from the "future" and from the "past", respectively. Clearly, $h^*(\sigma|Z) = h^*(\sigma^{-1}|Z)$. We propose to call $h^*(\sigma|Z)$ (instead of $h(\sigma|Z)$) the *topological entropy* of σ on the set Z. See [11] for a similar discussions see also .

In order to obtain results for horseshoes similar to those for repellers in the previous section, our strategy is to deal separately with the stable and unstable manifolds, and deduce the corresponding statement using Theorem 5.1.

The following result enumerates sets of irregular points which carry full topological entropy and full Hausdorff dimension. Let $\mathcal{M}_D$ be the set of f-invariant measures μ such that $\dim_H \mu = \dim_H \Lambda$. Note that this set may be empty.

Theorem 6.1. *Let f be a topologically mixing $C^{1+\varepsilon}$ surface diffeomorphism, for some $\varepsilon > 0$, and Λ a compact locally maximal saddle-type hyperbolic set of f. Then the following properties hold:*

1. $h(f|\mathcal{B}) = h(f|\Lambda)$ *and* $\dim_H \mathcal{B} = \dim_H \Lambda$*;*
2. *if* $m_E \notin \mathcal{M}_D$*, then* $h(f|\mathcal{L} \cap \mathcal{B}) = h(f|\Lambda)$ *and* $\dim_H(\mathcal{L} \cap \mathcal{B}) = \dim_H \Lambda$*;*
3. *if* μ *is an equilibrium measure on* Λ *and* $\mu \notin \mathcal{M}_D$*, then* $h(f|\mathcal{D}(\mu) \cap \mathcal{B}) = h(f|\Lambda)$ *and* $\dim_H(\mathcal{D}(\mu) \cap \mathcal{B}) = \dim_H \Lambda$*;*
4. *if* μ *is an equilibrium measure on* Λ *and* $\mu \neq m_E$*, then* $h(f|\mathcal{H}(\mu) \cap \mathcal{B}) = h(f|\Lambda)$ *and* $\dim_H(\mathcal{H}(\mu) \cap \mathcal{B}) = \dim_H \Lambda$*;*
5. *if* μ *is an equilibrium measure on* Λ*,* $\mu \notin \mathcal{M}_D$*, and* $\mu \neq m_E$*, then* $h(f|\mathcal{D}(\mu) \cap \mathcal{H}(\mu) \cap \mathcal{B}) = h(f|\Lambda)$ *and* $\dim_H(\mathcal{D}(\mu) \cap \mathcal{H}(\mu) \cap \mathcal{B}) = \dim_H \Lambda$*;*
6. *if* $\mu \neq m_E$ *is an equilibrium measure on* Λ*, and* μ*,* $m_E \notin \mathcal{M}_D$*, then* $h(f|\mathcal{D}(\mu) \cap \mathcal{H}(\mu) \cap \mathcal{L} \cap \mathcal{B}) = h(f|\Lambda)$ *and* $\dim_H(\mathcal{D}(\mu) \cap \mathcal{H}(\mu) \cap \mathcal{L} \cap \mathcal{B}) = \dim_H \Lambda$.

Proof. We present the proof of statement 3. The other statements of the above theorem are similar with the obvious changes of the set of functions and the distinguishing system of measures. The idea is to decompose the invariant hyperbolic set into local stable and unstable manifolds. For this we use the following fact. For a given Gibbs measure μ on Σ and any point $x \in \Lambda$ we can define two conditional measures μ_x^s and μ_x^u on the local stable and unstable manifold of x, respectively. There is a constant C such that $C^{-1}\mu < \mu_x^s \circ \chi^{-1} \times \mu_x^u \circ \chi^{-1} < C\mu$ on the rectangle $R(x)$ and the measure μ_x^u (μ_x^s) is a Gibbs measure with some potential φ_x^u (φ_x^s) on Σ. Since $\mu \neq \mu_D$ the conditional measures μ_x^u and μ_x^s cannot both be equivalent to the measures of maximal dimension on the unstable or stable manifold of x. Without loss of generality let us assume that $\mu_x^u \neq \mu_{D,x}^u$.

Let $\mathcal{D}_x^u(\mu)$ be the set of points in $W^u(x)$ such that the pointwise dimension of μ_x^u does not exist. This set is the image under χ of the set of points $\underline{y} \in \Sigma$ for whose the limit of $\frac{S_n \varphi_x^u}{S_n \log a^u}$ does not exist.

If f is a Hölder continuous functions on the two-sided subshift Σ, then there is a function f' cohomologous to f and depending only on the "future" coordinates (see [7]):

$$f'(\ldots, x_{-1}, x_0, x_1, \ldots) = f(\ldots, x'_{-1}, x'_0, x'_1, \ldots)$$

if $x_i = x'_i$ for $i \geq 0$. This allows us to identify f' with a function on Σ which has the same irregular set $\widehat{\mathcal{B}}(f)$ on Σ_A^+ as well as $\mathcal{B}(f)$ on Λ as the function f itself. We set $\mathcal{B}_x^u = \chi(\widehat{\mathcal{B}})$. As in the proof of Theorem 5.1 we have

$$\begin{aligned} h^+(f|\mathcal{D}_x^u(\mu) \cap \mathcal{B}_x^u) &= h^+(f|\Lambda \cap W^u(x)) \\ &= h\left(\widehat{\mathcal{F}}\left(\left\{\frac{S_n \varphi_x^u}{S_n \log a^u}\right\}_{n\in\mathbb{N}}, \left\{\frac{1}{n} S_n f\right\}_{n\in\mathbb{N}}\right)\right) \end{aligned}$$

and

$$\begin{aligned}\dim_H(\mathcal{D}^u_x(\mu)\cap\mathcal{B}^u_x) &= \dim_H(\Lambda\cap W^u(x))\\ &= \dim_{\log a^u}\widehat{\mathcal{F}}\left(\left\{\frac{S_n\varphi^u_x}{S_n\log a^u}\right\}_{n\in\mathbb{N}},\left\{\frac{1}{n}S_nf\right\}_{n\in\mathbb{N}}\right).\end{aligned}$$

Now we consider the measure of maximal stable dimension μ^s_x on $W^s(x)$. We have that μ^s_x-almost every $y\in W^s(x)$ does not belong to $\mathcal{D}^s_x(\mu)$ and $\mathcal{B}^s_x$ (where $\mathcal{D}^s_x(\mu)$ and $\mathcal{B}^s_x$ are the corresponding sets defined for the restriction of f to $W^s(x)$ and the measure μ^s_x.). Hence for $y\in W^s(x)$ out of a set $G^s_{D,x}$ of full $\mu^s_{D,x}$ measure the sets $\mathcal{D}^u_y(\mu)\cap\mathcal{B}^u_y$ are contained in $\mathcal{D}\cap\mathcal{B}$. Let us now consider the set $\bigcup_{y\in G^s_{D,x}}\mathcal{D}^u_y(\mu)\cap\mathcal{B}^u_y$. Then this set has full stable and unstable dimension and hence the dimension assertion is proved. To show the assertion about the topological entropy we observe that the set $\bigcup_{y\in G^s_{E,x}}\mathcal{D}^u_y(\mu)\cap\mathcal{B}^u_y$ – which is defined by means of $\mu^s_{E,x}$ instead of $\mu^s_{D,x}$ – has full topological entropy for f and f^{-1} at the same time. Hence,

$$\begin{aligned}h(\mathcal{D}\cap\mathcal{B}) &= \min\left\{h^+(\mathcal{D}\cap\mathcal{B}),h^-(\mathcal{D}\cap\mathcal{B})\right\}\\ &\geq \min\left\{h^+\Big(\bigcup_{y\in G^s_x}\mathcal{D}^u_y(\mu)\cap\mathcal{B}^u_y\Big),h^-\Big(\bigcup_{y\in G^s_x}\mathcal{D}^u_y(\mu)\cap\mathcal{B}^u_y\Big)\right\}\\ &= h(\Lambda).\end{aligned}$$

This completes the proof of the theorem. □

Remark. The conclusions of Theorem 6.1 also hold for diffeomorphisms on manifolds of arbitrary dimension provided that $df|E^u(x)$ and $df|E^s(x)$ are multiples of isometries for each $x\in\Lambda$, i.e., when the dynamics in the stable and unstable manifolds is conformal.

In [3] (see also [2]), we proved that $\mu(\mathcal{D}(\mu))=0$ for any hyperbolic measure μ invariant under a $C^{1+\varepsilon}$ diffeomorphism.

It was established by Shereshevsky in [17] that $\dim_H\mathcal{D}(\mu)>0$, and $\overline{\mathcal{D}(\mu)}\supset\Lambda$ for a generic C^2 surface diffeomorphism possessing a locally maximal hyperbolic set Λ, and a generic Hölder continuous potential, with respect to the C^0 topology, with Gibbs measure μ.

7. Further Applications

We have seen in the previous sections that sets of irregular points are quite "large" in numerous situations. Namely, they have the same topological entropy and Hausdorff dimension as the set of all typical points. In the case of two-dimensional horseshoes this is even worse. It is well-known that "typical" two-dimensional horseshoes (for maps out of an open dense set in the space of all horseshoe maps) do not have an invariant measure of maximal dimension (see [10]). Moreover the dimension of any invariant measure is uniformly bounded away from the Hausdorff dimension of the horseshoe.

For each measure ν on a manifold M, let $\mathcal{G}(\nu)$ be the set of points $x \in M$ such that $\lim_{n\to\infty} \frac{1}{n} S_n g(x) = \int_M g\, d\nu$ for every continuous function g on M. The following is a simple consequence of the above discussion, and Proposition 2.5 and Theorem 6.1.

Theorem 7.1. *For a generic C^2 surface diffeomorphism possessing a locally maximal hyperbolic set, and any Gibbs measure $\mu \neq m_E$ which is not a measure of maximal dimension, we have*

$$\dim_H(\mathcal{D}(\mu) \cap \mathcal{H}(\mu) \cap \mathcal{L} \cap \mathcal{B}) > \dim_H \bigcup_\nu \mathcal{G}(\nu).$$

Since a locally maximal hyperbolic set admits a coding by a subshift of finite type, metric properties such as topological entropy are preserved under the coding map. Together with Theorem 4.5, this immediately implies the following statement.

Theorem 7.2. *If Λ is a locally maximal hyperbolic set of a C^1 diffeomorphism f of a compact manifold, then the following properties hold:*

1. *$h(f|\mathcal{B}) = h(f)$;*
2. *if μ is a Gibbs measure on Λ which is not a measure of maximal entropy, then $\mathcal{H}(\mu) \subset \mathcal{B}$ and $h(f|\mathcal{H}(\mu)) = h(f)$.*

We note that this result is also valid for Axiom $\sharp$ homeomorphisms (see [1]).

In [9], Katok proved that for an ergodic hyperbolic measure ν (i.e., an ergodic measure with non-zero Lyapunov exponents), invariant under a $C^{1+\varepsilon}$ diffeomorphism $f\colon M \to M$, given $\delta > 0$ there exists a closed f-invariant hyperbolic set $\Gamma \subset M$ such that the restriction of f to Γ is topologically conjugate to a topological Markov chain with topological entropy $h(f|\Gamma) \geq h_\nu(f) - \delta$. In other words, the entropy of a hyperbolic measure can be approximated by the topological entropies of invariant hyperbolic sets. Using this approximation result one can prove the following.

Theorem 7.3. *If f is a $C^{1+\varepsilon}$ surface diffeomorphism, for some $\varepsilon > 0$, and the Lyapunov exponents are not constant everywhere, then*

$$\dim_H(\mathcal{L} \cap \mathcal{B}) \geq \sup_\mu \dim_H \mu_x^u + \sup_\mu \dim_H \mu_x^s$$

and

$$h(\mathcal{B} \cap \mathcal{L}) = h(f).$$

Let M be a d-dimensional Riemannian manifold. Then there are d matrix-cocycles on the exterior forms of M whose determinants generate the Lyapunov exponents on a set of total measure one [14]. Again one can approximate the set of typical points of any hyperbolic measure together with its topological entropy by measures on hyperbolic horseshoes. In this way we get a sequence of subshifts of finite type coding those horseshoes approximating the sets of typical points. It is not hard to see that the assumptions of Theorem 6.1 are fulfilled for all approximation steps when none of the cocycles above are cohomologous to a constant one. In this situation we say that the Lyapunov spectrum is not constant everywhere.

Theorem 7.4. *If f is a $C^{1+\varepsilon}$ diffeomorphism, for some $\varepsilon > 0$, of a compact manifold with not everywhere constant Lyapunov spectrum, then*

$$h(\mathcal{L} \cap \mathcal{B}) \geq \sup\{h_\mu(f) : \mu \in \mathcal{M}_{\text{ergodic}}\}.$$

We emphasize that the results in Theorem 7.2 and 7.4 hold for manifolds of arbitrary (finite) dimension.

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