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A wavelet algorithm for the boundary element solution of a geodetic boundary value problem

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Abstract

In this paper we consider a piecewise bilinear collocation method for the solution of a singular integral equation over a part of the surface of the earth. This singular equation is the boundary integral equation corresponding to the oblique derivative boundary problem for Laplace's equation. We introduce special wavelet bases for the spaces of test and trial functions. Analogously to well-known results on wavelet algorithms, the stiffness matrices with respect to these bases can be reduced to sparse matrices such that the assembling of the matrices and the iterative solution of the matrix equations become fast. Though the theoretical results apply only to integral equations with "smooth" solutions over "smooth" manifolds, we present numerical tests for a geometry as difficult as the surface of the earth.

0 Introduction

It is a well-known fact that usual finite element discretisations of linear integral equations (e.g. of boundary integral equations) lead to systems of linear equations with fully populated matrices. Thus, even an iterative solution method requires a huge number of arithmetic operations and a large storage capacity. In order to improve these finite element approaches, several new algorithms have been developed. For a relatively wide class of boundary integral equations, Rokhlin and Greengard [29, 15] have introduced their methods of multipole expansion, Hackbusch and Nowak [16] (cf. also [30]) have considered panel clustering algorithms, and Brandt and Lubrecht [3] have set up multilevel schemes. Another approach for saving storage and computation time consists in employing wavelet bases of the finite element spaces. This idea goes back to Beylkin, Coifman, and Rokhlin [2], and has been thoroughly investigated by Dahmen, v.Petersdorff, Prößdorf, Schneider, and Schwab [11, 12, 10, 13, 26, 25, 24, 31] (cf. also the contributions by Alpert, Harten, Yad-Shalom, Dorobantu, Kleemann, and the author [1, 17, 14, 7, 8, 28]). Note that all the different algorithms from multipole expansion to wavelets seem to have a common multilevel background.

The subject of the present paper is to apply the wavelet technique from [2] to the solution of a two-dimensional singular integral equation which corresponds to the oblique derivative problem for Laplace's equation. For the computation of the gravity field of the earth in geodesy, this equation has to be considered over the surface of the earth or part of it (cf. Klees [19, 18] or cf. the similar equation for the Molodensky problem in Moritz [22], Section 43). If the underlying surface is smooth (piecewise continuously differentiable up to a certain order), then it is clear that the wavelet algorithms (cf. [10, 25]) admit high order compressions. In addition, if the surface is piecewise analytic, then the algorithms are fast (cf. [31, 25]). The surface of the earth, however, cannot supposed to be smooth. Thus, we will apply a wavelet algorithm for a complicated real-life geometry and report on numerical tests for a situation where no proofs are available yet.

Following Klees [19, 18], we will describe the problem of oblique derivative, the corresponding boundary integral equation, and its discretisation via piecewise bilinear collocation over uniform grids in Section 1. In Section 2 we will introduce the wavelet algorithm based on biorthogonal wavelets for the bilinear trial space (cf. Cohen, Daubechies, and Feauveau [5] as well as their application in [26, 28]) and on wavelets defined by Harten and Yad-Shalom [17] for the space of test functionals (cf. also the earlier paper by Brandt and Lubrecht [3]). We will discuss two different compression schemes. The first one will be similar to that of Dahmen, Prößdorf, Schneider, v.Petersdorff, and Schwab [10, 25]. For the number of degrees of freedom n, it admits a reduction from n^2 entries in the full matrix to $n^{4/3}$ entries in the compressed sparse matrix. Though this compression is not asymptotically optimal, we expect shorter computation times for $n \leq 10\,000$. The second compression algorithm is a heuristic scheme including the determination of approximate values for some entries of the stiffness matrix. Section 3 is devoted to the quadrature algorithm for the computation of the stiffness matrix implemented in the numerical tests. The rules are based on singularity subtraction and graded meshes defined with the help of Duffy's transformation. We present further details of the algorithm and the results of the numerical tests in Section 4. For a $9\,025 \times 9\,025$ matrix, it turns out that, in order to keep the relative compression error less than 10^{-5}, it is sufficient to compute only 5% of the stiffness matrix. The computing time for a $2\,025 \times 2\,025$ system can be reduced from 1005s (seconds) in case of the conventional finite element code to 405s for the wavelet algorithm. The saving of computing time is likely to be much higher for the $9\,025 \times 9\,025$ system. Unfortunately, the main memory of 96 MB was not enough to perform the test for conventional finite elements. Finally, in order to understand the features of the matrix assembling, we present a modified quadrature algorithm for a model problem in Section 5. For this, we estimate the quadrature error and the number of arithmetic operations. We will show that the $n^{4/3}$ entries of the compressed stiffness matrix can be computed with $O(n^{5/3}\{\log n\}^2)$ operations. A further reduction of the number of operations up to, roughly speaking, the number of entries in the compressed matrix will be indicated. Note that, in contrast to the quadrature algorithms in [25, 31], we do not use any analyticity assumption on the kernel or on the underlying geometry.

In summary, the wavelet algorithm presented here provides a good tool for an improvement in computing time and storage requirements even for complicated geometries. However, the question remains open which of the fast methods from multipole expansion to wavelets is the best. For an answer, a lot of further numerical test would be necessary. From the theoretical point of view, panel clustering seems to be more suitable for the test example in consideration since the kernel function of the integral operator is, roughly speaking, the restriction of an analytic function in space to the complicated surface (The kernel function is the sum of products of such analytic kernels with non-smooth functions depending on one variable only!). If a Galerkin discretisation is chosen, then even the sparse grid methods of Zenger [32] can be applied successfully. This approach is closely related to a wavelet method where the higher dimensional wavelets are replaced by tensor products of univariate wavelets.

1 The Boundary Element Method for the Solution of a Geodetic Boundary Value Problem

1.1. The Singular Integral Equation

In this section we recall the boundary element approach of Klees [19, 18] (cf. also [21], Section 23). The potential w of the gravity field in the external space Ω_a of the earth's body can be represented as a sum of a known reference potential w_0 and an unknown difference $\delta w = w - w_0$. If $g = |\nabla w|$ is given by measurements on the earth's surface Γ and if a small order term in the non-linear boundary condition $|\nabla(\delta w + w_0)| = g$ is neglected, then δw is the solution of the boundary value problem

$$\Delta \delta w = 0, \text{ in } \Omega_a, \tag{1.1}$$

$$\langle \tilde{\tau}, \nabla \delta w \rangle = \tilde{v}, \text{ on } \Gamma, \tag{1.2}$$

where

$$\tilde{v} := \frac{g^2 - |\nabla w_0|^2}{2}, \quad \tilde{\tau} := \nabla w_0.$$

We seek a solution δw in form of a single layer potential

$$\delta w(z) := \frac{1}{4\pi} \int_\Gamma \frac{\tilde{u}(y)}{|y-z|} d_y\Gamma \tag{1.3}$$

with a yet unknown density function $\tilde{u}$ on Γ. Substituting (1.3) into the boundary condition (1.2) and applying the jump relations for the derivative of the single layer potential, we arrive at to the singular integral equation

$$-\frac{1}{2}\cos[\tilde{n}(x), \tilde{\tau}(x)]\tilde{u}(x) + \frac{1}{4\pi} p.v. \int_\Gamma \frac{\cos[\tilde{\tau}(x), y-x]}{|y-x|^2} \tilde{u}(y) d_y\Gamma = \tilde{v}(x), \quad x \in \Gamma, \tag{1.4}$$

where $\tilde{n}(x)$ denotes the unit normal of the surface Γ at x pointing into Ω_a, and the integral over Γ is to be understood in the principal value sense (cf. [20]). By $\cos[\tilde{n}(x), \tilde{\tau}(x)]$ we denote the cosine of the angle between the vectors $\tilde{n}(x)$ and $\tilde{\tau}(x)$ and $\cos[\tilde{\tau}(x), y-x]$ is defined analogously. Throughout this paper we will assume $\cos[\tilde{n}(x), \tilde{\tau}(x)] > 0$ for any $x \in \Gamma$. This guarantees that the operator on the left-hand side of (1.4) is strongly elliptic (cf. [19, 20]). Moreover, we will assume that the null space of the operator is trivial. This together with the strong ellipticity implies the operator on the left-hand side of (1.4) to be invertible in the space $L^2(\Gamma)$.

If we wanted to discretise the whole surface of the earth, any work station would be likely to collapse because of the enormous number of grid points we would have to take into account. Therefore, we restrict the domain to a rectangular part of Γ which we denote by the same letter. More precisely, Γ is the set of points with latitude between 48.60^0 and 56.65^0 and with longitude between 5.35^0 and 13.40^0. The surface is given by a C^1 parametrisation F over the square $S := [0,1] \times [0,1]$. This parametrisation is the tensor product Overhauser interpolation (cf. [23]) over a uniform rectangular grid. Using F, we arrive at the equivalent singular integral equation $Au = v$ of the form

$$-\frac{1}{2}\cos[n(P), \tau(P)]u(P) + \frac{1}{4\pi} p.v. \int_S \frac{\cos[\tau(P), F(Q) - F(P)]}{|F(Q) - F(P)|^2} F'(Q)u(Q) d_Q S = v(P), \tag{1.5}$$

where P runs over S, $n(P) := \tilde{n}(F(P))$, $\tau(P) := \tilde{\tau}(F(P))$, $u(P) := \tilde{u}(F(P))$, $v(P) := \tilde{v}(F(P))$, and

$$F'(Q) := |\partial_x F((x,y)) \times \partial_y F((x,y))| \tag{1.6}$$

for $Q = (x, y) \in S$. Equation (1.5) is uniquely solvable in $L^2(S)$, and we are going to solve it numerically by bilinear collocation. If u_h denotes the approximate solution of u, then, to get an approximate value $\delta w_h(z)$ to the difference potential $\delta w(z)$, we set $\tilde{u}_h(F(P)) := u_h(P)$ and

$$\delta w_h(z) := \frac{1}{4\pi} \int_\Gamma \frac{\tilde{u}_h(y)}{|y - z|} d_y \Gamma = \frac{1}{4\pi} \int_S \frac{u_h(P)}{|F(P) - z|} F'(P) d_P S. \tag{1.7}$$

1.2. The Collocation Method

In order to set up the collocation system, we introduce a basis of trial functions $\{\varphi_I^N\}$ and the collocation points $\{P_I^N\}$. We choose an integer N and set $h := \frac{1}{N-1}$, $x_i := ih$, $i = 0, 1, \ldots, N-1$ as well as $P_I := P_I^N := (x_i, x_j)$ for $I \in \mathcal{M} := \{(i, j) : i, j = 1, \ldots, N-2\}$. Over the uniform grid $\{x_i\}$ on $[0, 1]$, we define the scaled hat functions

$$\varphi_i^N(x) := \frac{1}{\sqrt{h}} \begin{cases} \frac{x - x_{i-1}}{h} & \text{if } x_{i-1} < x < x_i \\ \frac{x_{i+1} - x}{h} & \text{if } x_i < x < x_{i+1} \\ 0 & \text{else} \end{cases} \tag{1.8}$$

for $i = 1, \ldots, N-2$. By taking tensor products, we obtain $\varphi_I^N(Q) := \varphi_i^N(x) \cdot \varphi_j^N(y)$ for $I \in \mathcal{M}$ and $Q = (x, y) \in S$. Note that the trial functions from the space $\mathcal{X}_h :=$span$\{\varphi_I^N : I \in \mathcal{M}\}$ vanish on the boundary of S.

Now with the collocation method one seeks an approximate solution $u_h = \sum_{I \in \mathcal{M}} \xi_I \varphi_I^N$, $\xi_I \in \mathbb{R}$, which fulfills Equation (1.5) at least at the collocation points P_I, $I \in \mathcal{M}$, i.e.,

$$A u_h(P_I) = v(P_I), \ I \in \mathcal{M}. \tag{1.9}$$

Obviously, (1.9) is equivalent to the matrix equation $A_h \xi = \eta$ with $\xi := \{\xi_I\}_{I \in \mathcal{M}}$, $\eta := \{\eta_I\} := \{h v(P_I)\}_{I \in \mathcal{M}}$ and $A_h := (h(A\varphi_I^N)(P_J))_{J,I \in \mathcal{M}}$. The matrix A_h can be identified with the operator in the space $\mathcal{X}_h$ whose matrix with respect to the basis $\{\varphi_I^N : I \in \mathcal{M}\}$ coincides with A_h. The space $\mathcal{X}_h$ is a subspace of $L^2(S)$. Now the collocation method is called stable if, for h sufficiently small, the approximate operators A_h are invertible and the norms $\|A_h^{-1}\|_{\mathcal{L}(\mathcal{X}_h)}$ are uniformly bounded. In other words, the collocation is stable if and only if the l^2 condition numbers of the matrices A_h are uniformly bounded with respect to h. The following theorem is due to Prößdorf and Schneider [27].

Theorem 1.1 *If the surface Γ is sufficiently smooth, then the piecewise bilinear collocation method is stable. Moreover, if the exact solution u belongs to the Sobolev space H^2 over S and vanishes at the boundary of S, then there exists a constant C such that $\|u - u_h\|_{L^2} \le C\, h^2$ holds for the approximate solutions u_h of the collocation method. Furthermore, for a fixed $z \in \Omega_a$ and δw_h from (1.7), the estimate $|D^\beta \delta w(z) - D^\beta \delta w_h(z)| \le C\, h^2$ is valid where D^β denotes the partial derivative of multiindex β and the constant C depends on β, z, and u but not on h.*

Remark 1.1 *Obviously, the error for the numerical approximations to the solution u of (1.5) and to the potential δw is affected by the restriction of the boundary integral equation to a part of the earth's surface and by employing the Overhauser parametrisation defined over the same grid as the trial functions. Though a rigorous analysis of the collocation method would require the investigation of these influences, we will ignore them throughout the present paper.*

2 The Wavelet Algorithm

2.1. Basis Transformations and the Wavelet Transform of the Stiffness Matrix

The wavelet algorithm for the bilinear collocation method relies on the introduction of two new wavelet bases which will be defined in the multilevel settings of the next subsections. The first is the basis $\{\psi_I,\, I \in \mathcal{N}\}$ of wavelet basis functions in the space $\mathcal{X}_h$. The set $\mathcal{N}$ is a new index set with the same cardinality as $\mathcal{M}$. This basis $\{\psi_I,\, I \in \mathcal{N}\}$ is chosen such that the functions ψ_I have a possibly small support and such that certain moment conditions are satisfied, i.e., the ψ_I are orthogonal to polynomials of an order less than a certain prescribed number. Similarly, we chose a wavelet basis $\{\vartheta_I,\, I \in \mathcal{N}\}$ in the space of test functionals spanned by the functionals of function evaluation at the collocation points. In fact, the ϑ_I will be linear combinations of a small number of function evaluations. Analogously to the moment conditions for the ψ_I, we require that these functionals vanish at polynomials of low order. We arrive at

$$\mathcal{X}_h = \operatorname{span}\{\varphi_I,\, I \in \mathcal{M}\} = \operatorname{span}\{\psi_I,\, I \in \mathcal{N}\}, \tag{2.1}$$

$$\operatorname{span}\{h\delta_{P_I},\, I \in \mathcal{M}\} = \operatorname{span}\{\vartheta_I,\, I \in \mathcal{N}\}. \tag{2.2}$$

In view of these new bases, the collocation equation (1.9) is equivalent to

$$\langle Au_h, \vartheta_J\rangle = \langle v, \vartheta_J\rangle,\ J \in \mathcal{N}, \qquad u_h = \sum_{I\in\mathcal{N}} \mu_I \psi_I. \tag{2.3}$$

The matrix equation $A_h\xi = \eta$ can be replaced by $B_h\mu = \nu$, where $\nu := \{\nu_J\}_{J\in\mathcal{N}} := \{\langle v, \vartheta_J\rangle\}_{J\in\mathcal{N}}$ and $\mu := \{\mu_J\}_{J\in\mathcal{N}}$. The matrix $B_h := (\langle A\psi_I, \vartheta_J\rangle)_{J,I\in\mathcal{N}}$ is called the wavelet transform of A_h. We define the basis transform E_h by $E_h\{\xi_I\}_{I\in\mathcal{M}} = \{\mu_I\}_{I\in\mathcal{N}}$ for $\sum_{I\in\mathcal{M}} \xi_I\varphi_I = \sum_{J\in\mathcal{N}} \mu_J\psi_J$ and the basis transform R_h by $R_h\{\nu_I\}_{I\in\mathcal{N}} = \{\eta_I\}_{I\in\mathcal{M}}$, where $\eta_I := hv_h(P_I)$ and v_h is the unique function in $\mathcal{X}_h$ with $\langle v_h, \vartheta_J\rangle = \nu_J$. Then we get $A_h = R_hB_hE_h$.

Now the wavelet algorithm looks as follows. We solve the equation $A_h\xi = \eta$ iteratively (e.g. by classical iteration, Jacobi's iteration or by GMRes). The main part of the computation is spent for the multiplication of iterative solutions $\tilde{\xi}$ or residual vectors $\tilde{\xi}$ by the matrix A_h. In the wavelet algorithm, this step is done by first multiplying $\tilde{\xi}$ by E_h, then by B_h, and finally by R_h. As we will see in the next subsections, the basis transforms $\tilde{\xi} \mapsto E_h\tilde{\xi}$ and $[B_hE_h\tilde{\xi}] \mapsto R_h[B_hE_h\tilde{\xi}]$ can be realized via fast pyramid type algorithms. For the multiplication by B_h, we will prove that, due to the moment conditions and the smallness of the supports of the bases $\{\vartheta_I,\, I \in \mathcal{N}\}$ and $\{\psi_I,\, I \in \mathcal{N}\}$, the majority of entries in B_h is very small (cf. Lemma 2.3). Thus, setting these entries equal to zero, we end up with a compressed matrix C_h and the multiplication by B_h can be replaced by the multiplication with C_h. The additional error due to the compression will be less than the discretisation error of the collocation (cf. Theorem 2.1). Since the matrix C_h is sparse, the multiplication by C_h is fast.

Remark 2.1 *In some situations it is possible to solve $B_h\mu = \nu$ directly. For instance, if the transforms E_h, R_h are uniformly bounded in suitable Sobolev spaces and if the* *norm of a trial function u_h can easily be expressed in terms of its wavelet coefficients, then B_h admits a diagonal preconditioning and $B_h\mu = \nu$ can be solved with a number of iterations independent of the mesh size h. For details we refer to the papers by Dahmen, Kunoth, Prößdorf, and Schneider [9, 12]. In the situation considered in the present paper, however, R_h will be unbounded and no diagonal preconditioner is known. In general, if the condition number of the original matrix A_h is uniformly bounded, then the actual value of the condition number of the wavelet transform B_h is often much worse even if it is uniformly bounded. For this case, the iterative solution of the original matrix equation is faster.*

In any case, the main part of the computing time for boundary element methods is spent for the calculation of the stiffness matrix. For the wavelet algorithm, we do not need the whole matrices A_h or B_h but only the compressed matrix C_h which saves a lot of computing time. However, this reduction in computing time is not so easy to achieve as it might seem at first glance. In fact, a sophisticated algorithm of quadrature is needed to guarantee small quadrature errors and to reduce the amount of work. We will discuss this issue in the Sections 3 and 5.

2.2. The Bilinear Wavelets for the Trial Space

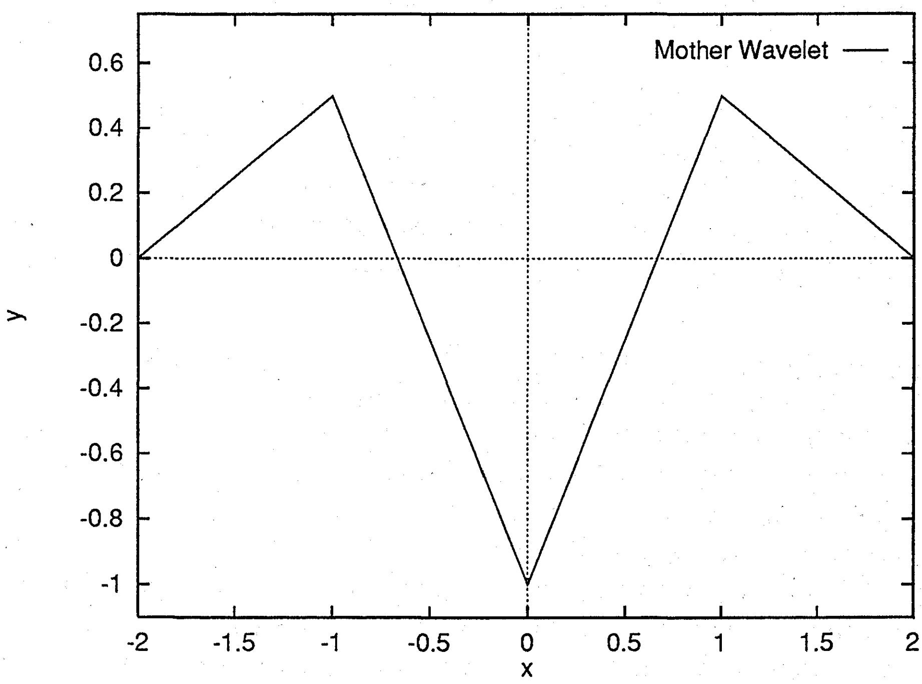

Figure 1: Mother wavelet ψ.

Set $N_l = 3 \cdot 2^l + 1,\ l = 0, 1, \ldots$ and suppose the number N in Section 1.2 is equal to N_L with $L \geq 1$. Recall that $h = \frac{1}{N_L - 1}$. Similarly to the functions φ_i^N and $\varphi_{(i,j)}^N$ of Section 1.2, we define $\varphi_i^{N_l}$ and $\varphi_{(i,j)}^{N_l}$, respectively. We begin with <u>univariate wavelets</u> and introduce

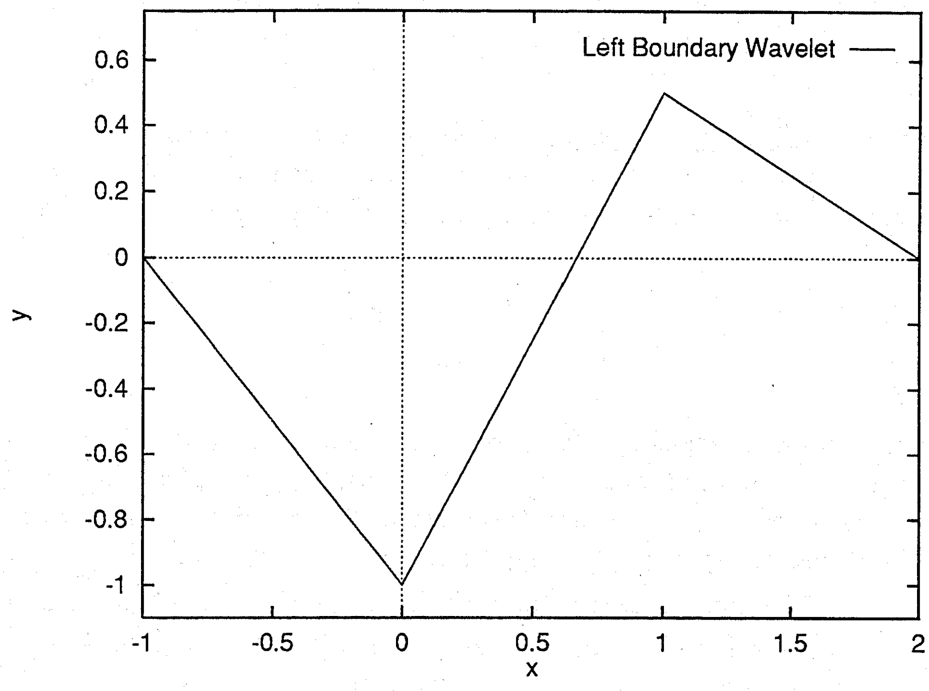

Figure 2: Boundary wavelet ψ_a.

the shape function ψ together with its boundary modifications ψ_a and ψ_b by (cf. the Figures 1 and 2)

$$\psi(x) := \begin{cases} \frac{3}{2}|x| - 1 & \text{if } |x| \leq 1 \\ 1 - \frac{1}{2}|x| & \text{if } 1 < |x| \leq 2 \\ 0 & \text{else ,} \end{cases} \tag{2.4}$$

$$\psi_a(x) := \begin{cases} \psi(x) & \text{if } 0 \leq x \\ -1 - x & \text{if } -1 < x \leq 0 \\ 0 & \text{else ,} \end{cases} \tag{2.5}$$

$$\psi_b(x) := \psi_a(-x). \tag{2.6}$$

Then a multilevel basis of $\text{span}\{\varphi_j^{N_L},\ j = 1, \ldots N_L - 2\}$ is given as follows. We define basis functions on each level $l = 0, \ldots, L$. For $l = 0$, we easily take the hat functions $\{\varphi_i^{N_0} : i = 1, 2\}$ on the coarsest grid. For $l > 0$, we take the dilated shape function $x \mapsto \psi([3 \cdot 2^l]x)$ and shift it to each point of the difference grid $\{\frac{i}{N_l - 1} : i = 1, \ldots, N_l - 2\} \setminus \{\frac{i}{N_{l-1} - 1} : i = 1, \ldots, N_{l-1} - 2\}$. In other words, we set

$$\psi_{0,i}(x) := \varphi_i^{N_0}(x), \quad i = 1, 2 = N_0 - 2, \tag{2.7}$$

$$\psi_{l,1}(x) := \psi_a\left([3 \cdot 2^l] \cdot x - 1\right) \sqrt{3 \cdot 2^l}, \quad l = 1, \ldots L,$$

$$\psi_{l,j}(x) := \psi\left([3 \cdot 2^l] \cdot x - (2j - 1)\right) \sqrt{3 \cdot 2^l}, \quad j = 2, \ldots, N_{l-1} - 2, \ l = 1, \ldots L,$$

$$\psi_{l,N_{l-1}-1}(x) := \psi_b\left([3 \cdot 2^l] \cdot x - (2[N_{l-1} - 1] - 1)\right) \sqrt{3 \cdot 2^l}, \quad l = 1, \ldots L. \tag{2.8}$$

Now we set

$$\mathcal{N} := \{(l, t, i, j) : \ l = 0, \ t = 1, \ i, j = 1, 2, \tag{2.9}$$

$$l = 1, \ldots, L,\ t = 1,\ i = 1, \ldots, N_{l-1} - 1,\ j = 1, \ldots, N_{l-1} - 2,$$
$$l = 1, \ldots, L,\ t = 2,\ i = 1, \ldots, N_{l-1} - 2,\ j = 1, \ldots, N_{l-1} - 1,$$
$$l = 1, \ldots, L,\ t = 3,\ i, j = 1, \ldots, N_{l-1} - 1\}$$

and introduce the two-dimensional wavelets as follows. We define basis functions for each level $l \in \{0, 1, \ldots, L\}$. For $l = 0$, we take the tensor product hat functions $(x, y) \mapsto \varphi_i^{N_0}(x)\varphi_j^{N_0}(y)$ with $i, j = 1, 2$ on the coarsest grid. Now let φ stand for the univariate hat function

$$\varphi(x) := \begin{cases} 1 - \frac{|x|}{2} & \text{if } |x| \leq 2 \\ 0 & \text{else} . \end{cases} \tag{2.10}$$

Then, instead of the univariate shape function ψ in the definition of univariate wavelets, we consider the three shape functions $(x, y) \mapsto \psi(x)\varphi(y)$, $(x, y) \mapsto \varphi(x)\psi(y)$, and $(x, y) \mapsto \psi(x)\psi(y)$. For each level $l > 0$, we take the dilated shape functions and shift them to the points of the difference grid $\{(\frac{i}{N_l - 1}, \frac{j}{N_l - 1}) : i, j = 1, \ldots, N_l - 2\} \setminus \{(\frac{i}{N_{l-1} - 1}, \frac{j}{N_{l-1} - 1}) : i, j = 1, \ldots, N_{l-1} - 2\}$. More exactly, we shift $(x, y) \mapsto \psi([3 \cdot 2^l]x)\ \varphi([3 \cdot 2^l]y)$ to the points $\left\{\left(\frac{2i-1}{N_l - 1}, \frac{2j}{N_l - 1}\right)\right\}$, the function $(x, y) \mapsto \varphi([3 \cdot 2^l]x)\psi([3 \cdot 2^l]y)$ to $\left\{\left(\frac{2i}{N_l - 1}, \frac{2j-1}{N_l - 1}\right)\right\}$, and $(x, y) \mapsto \psi([3 \cdot 2^l]x)\psi([3 \cdot 2^l]y)$ to the points $\left\{\left(\frac{2i-1}{N_l - 1}, \frac{2j-1}{N_l - 1}\right)\right\}$. In other words, we set (cf. the Figures 3 and 4)

$$\psi_{(0,1,i,j)} := \varphi_i^{N_0}(x)\varphi_j^{N_0}(y), \tag{2.11}$$
$$\psi_{(l,1,i,j)} := \psi_{l,i}(x)\varphi_j^{N_{l-1}}(y), \tag{2.12}$$
$$\psi_{(l,2,i,j)} := \varphi_i^{N_{l-1}}(x)\psi_{l,j}(y), \tag{2.13}$$
$$\psi_{(l,3,i,j)} := \psi_{l,i}(x)\psi_{l,j}(y). \tag{2.14}$$

It is easy to see that the functions $\psi_{l,j}$, $l \geq 1$, $j = 2, \ldots, N_{l-1} - 2$ are orthogonal to linear functions. Hence, the functions $\psi_I = \psi_{(l,t,i,j)}$ are orthogonal to bilinear functions if $l \geq 1$, $t = 1$ and $i \neq 1, N_{l-1} - 1$ or if $l \geq 1$, $t = 2$ and $j \neq 1, N_{l-1} - 1$ or if $l \geq 1$, $t = 3$ and $\{i, j\} \not\subseteq \{1, N_{l-1} - 1\}$. The functions $\{\psi_I,\ I \in \mathcal{N}\}$ belong to the class of biorthogonal wavelets in the sense of [5]. More exactly, the functions without the boundary modification including ψ_a, ψ_b belong to this class. However, the boundary modification at the end points 0 and 1 of the interval is well known, too. Namely, the functions from $\text{span}\{\varphi_j^{N_L},\ j = 1, \ldots, N_L - 2\}$ can be extended to odd functions over $[-1, 1]$ and further to 2-periodic functions over the real axis $\mathbb{R}$. The boundary function $x \mapsto \psi_a(3 \cdot 2^l \cdot x - 1)$ is the difference of the basis functions $x \mapsto \psi(3 \cdot 2^l \cdot x - 1)$ and $x \mapsto \psi(3 \cdot 2^l \cdot x + 1)$ over $\mathbb{R}$ and, thus, the restriction of a natural basis function from the space of extended functions. This fact guarantees the boundedness of the wavelet basis transform also for the case of the interval. Note that the univariate function $\psi_{l,j}$ have been used also in [26, 28]. All properties of the functions $\{\psi_I,\ I \in \mathcal{N}\}$ mentioned in this subsection are well known from the theory of wavelets (cf. [6, 4, 5]).

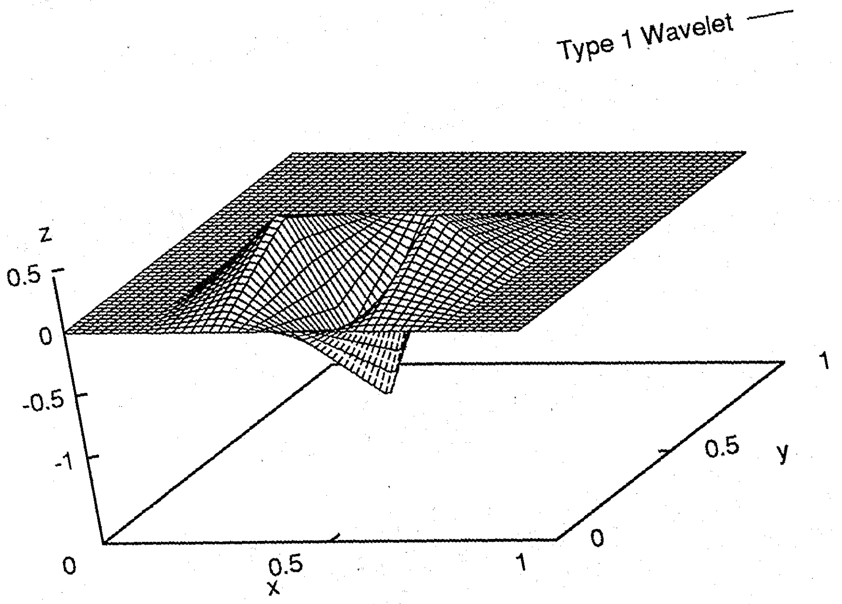

Figure 3: Wavelet $\psi_{(1,1,2,1)}$.

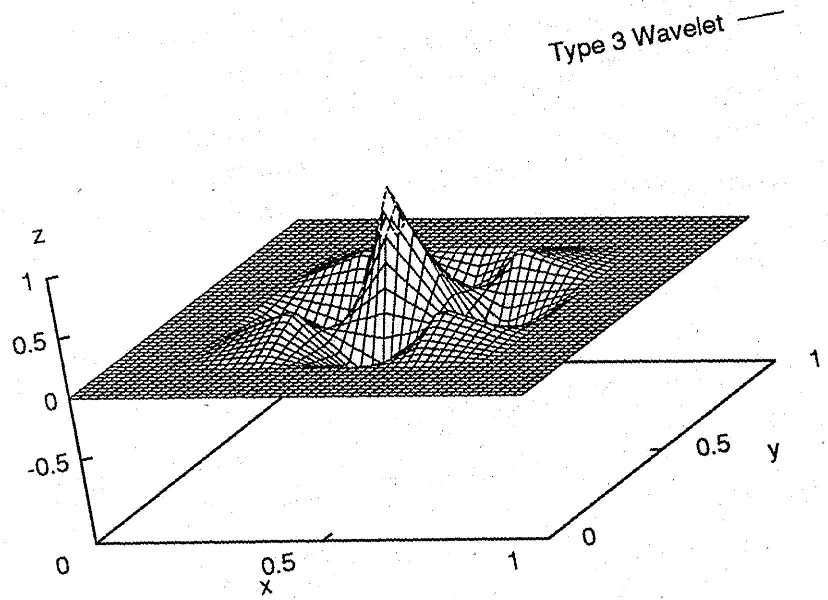

Figure 4: Wavelet $\psi_{(1,3,2,2)}$.

Lemma 2.1 *a) The wavelet transform E_h introduced in Subsection 2.1 and its inverse E_h^{-1} are bounded in the l^2 operator norm and the bounds are uniform with respect to L (recall that $h = \frac{1}{N-1}$, $N = 3 \cdot 2^L + 1$).*
b) The basis $\{\psi_I, I \in \mathcal{N}\}$ represents a hierarchical basis of $\mathcal{X}_h$, i.e., for $l = 1, \ldots, L$ one has

$$\operatorname{span}\left\{\varphi_{i,j}^{N_l} : i,j = 1,\ldots,N_l - 2\right\} = \operatorname{span}\left\{\varphi_{i,j}^{N_{l-1}} : i,j = 1,\ldots,N_{l-1} - 2\right\} \bigoplus \operatorname{span}\left\{\psi_I : I = (l', t, i, j) \in \mathcal{N}, l' = l\right\}. \tag{2.15}$$

This is a consequence of so called two-scale relations including the relations

$$\psi_{l,j} = \frac{1}{2}\varphi_{2j-2}^{N_l} - \varphi_{2j-1}^{N_l} + \frac{1}{2}\varphi_{2j}^{N_l}, \qquad j = 1,\ldots,N_{l-1} - 1, \tag{2.16}$$

$$\varphi_j^{N_{l-1}} = \sqrt{2}\left\{\frac{1}{2}\varphi_{2j-1}^{N_l} + \varphi_{2j}^{N_l} + \frac{1}{2}\varphi_{2j+1}^{N_l}\right\}, \quad j = 1,\ldots,N_{l-1} - 2, \tag{2.17}$$

where we have to set $\varphi_0^{N_l} := 0$ and $\varphi_{N_l-1}^{N_l} := 0$ in (2.16) for $j = 1$ or $j = N_{l-1} - 1$. As a consequence of Lemma 2.1, b) we obtain the following fast pyramid algorithm for the application of E_h. Suppose we are given the vector of coefficients $\{\xi_{(i,j)}^L, i,j = 1,\ldots N_L - 2\}$ of the function $u^L = \sum \xi_{(i,j)}^L \varphi_{(i,j)}^{N_L}$. In view of (2.15) with $l = L$, we split $u^L = u^{L-1} + v^L$ such that $u^{L-1} = \sum \xi_{(i,j)}^{L-1} \varphi_{(i,j)}^{N_{L-1}}$ and $v^L = \sum \eta_{(L,t,i,j)} \psi_{(L,t,i,j)}$. We have obtained the first wavelet coefficients $\eta_{(l,t,i,j)}$ for $l = L$. In the next step we use (2.15) with $l = L-1$ and split $u^{L-1} = u^{L-2} + v^{L-1}$ such that $u^{L-2} = \sum \xi_{(i,j)}^{L-2} \varphi_{(i,j)}^{N_{L-2}}$ and $v^{L-1} = \sum \eta_{(L-1,t,i,j)} \psi_{(L-1,t,i,j)}$. This gives us the values $\eta_{(l,t,i,j)}$ for $l = L-1$. Proceeding in the same fashion for $l = L-2, \ldots, 1$, we finally get all the coefficients $\eta_{(l,t,i,j)}$ for $l = L, \ldots, 1$ and $\eta_{(0,1,i,j)} = \xi_{(i,j)}^0$.

It remains to show that the splitting $u^l = u^{l-1} + v^l$ with $u^l = \sum \xi_{(i,j)}^l \varphi_{(i,j)}^{N_l}$, $u^{l-1} = \sum \xi_{(i,j)}^{l-1} \varphi_{(i,j)}^{N_{l-1}}$, and $v_l = \sum \eta_{(l,t,i,j)} \psi_{(l,t,i,j)}$ can be done by a fast algorithm. To this end, we observe that the bases $\{\varphi_{(i,j)}^{N_l}\}$, $\{\varphi_{(i,j)}^{N_{l-1}}\}$, and $\{\psi_{(l,t,i,j)}\}$ are tensor products of the one-dimensional bases $\{\varphi_i^{N_l}\}$, $\{\varphi_j^{N_{l-1}}\}$, and $\{\psi_{l,i}\}$, respectively. Consequently, we only need to apply one-dimensional splittings. Indeed, for every i, we split $\sum_j \xi_{(i,j)}^l \varphi_j^{N_l}(y)$ into $\sum_j \alpha_{(i,j)}^l \psi_{l,j}(y) + \sum_j \beta_{(i,j)}^l \varphi_j^{N_{l-1}}(y)$, i.e.,

$$\sum_{i,j} \xi_{(i,j)}^l \varphi_i^{N_l}(x) \varphi_j^{N_l}(y) = \sum_{i,j} \alpha_{(i,j)}^l \varphi_i^{N_l}(x) \psi_{l,j}(y) + \sum_{i,j} \beta_{(i,j)}^l \varphi_i^{N_l}(x) \varphi_j^{N_{l-1}}(y).$$

Then, for any j, we split $\sum_i \alpha_{(i,j)}^l \varphi_i^{N_l}(x)$ and $\sum_i \beta_{(i,j)}^l \varphi_i^{N_l}(x)$ into the sums $\sum_i \gamma_{(i,j)}^l \psi_{l,i}(x) + \sum_i \delta_{(i,j)}^l \varphi_i^{N_{l-1}}(x)$ and $\sum_i \epsilon_{(i,j)}^l \psi_{l,i}(x) + \sum_i \zeta_{(i,j)}^l \varphi_i^{N_{l-1}}(x)$, respectively. In other words

$$\begin{aligned}\sum_{i,j} \xi_{(i,j)}^l \varphi_i^{N_l}(x) \varphi_j^{N_l}(y) &= \sum_{i,j} \gamma_{(i,j)}^l \psi_{l,i}(x) \psi_{l,j}(y) + \sum_{i,j} \delta_{(i,j)}^l \varphi_i^{N_{l-1}}(x) \psi_{l,j}(y) \\ &\quad + \sum_{i,j} \epsilon_{(i,j)}^l \psi_{l,i}(x) \varphi_j^{N_{l-1}}(y) + \sum_{i,j} \zeta_{(i,j)}^l \varphi_i^{N_{l-1}}(x) \varphi_j^{N_{l-1}}(y).\end{aligned}$$

Setting $\xi_{(i,j)}^{l-1} = \zeta_{(i,j)}^l$, $\eta_{(l,1,i,j)} = \epsilon_{(i,j)}^l$, $\eta_{(l,2,i,j)} = \delta_{(i,j)}^l$, and $\eta_{(l,3,i,j)} = \gamma_{(i,j)}^l$, we arrive at the two-dimensional splitting.

For the one-dimensional splitting, we have to determine $\{\alpha_i\}$, $\{\beta_i\}$ from $\{\xi_i\}$ such that

$$\sum \xi_i \varphi_i^{N_l} = \sum \alpha_i \psi_{l,i} + \sum \beta_i \varphi_i^{N_{l-1}}. \tag{2.18}$$

If we substitute (2.16), (2.17) into the right-hand side of (2.18) and compare the coefficients of the $\varphi_i^{N_l}$, we arrive at a tridiagonal linear system for the unknowns α_i, β_i. This, however, can be solved with $O(N_l)$ arithmetic operations. The whole algorithm for the multiplication by E_h requires no more than $O(2^{2L}) = O((N_L-2)^2)$ operations, where $(N_L-2)^2$ is the dimension of the trial space.

Remark 2.2 *For the numerical tests, we modify the boundary wavelets ψ_a and ψ_b. We set*

$$\psi_a(x) := \begin{cases} 1-\frac{1}{2}x & \text{if } 1<x<2 \\ x-\frac{1}{2} & \text{if } 0<x<1 \\ -\frac{1}{2}(x+1) & \text{if } -1<x<0 \\ 0 & \text{else} \end{cases} \tag{2.19}$$

and $\psi_b(x) := \psi_a(-x)$. Though we do not know whether Lemma 2.1 a) holds for this modification, we have not observed any instability in the computations. The advantage of the modification is that we obtain the moment condition of order zero, i.e., the new boundary wavelets are orthogonal to constant functions.

2.3. The Wavelets for the Test Space

Retain the notation of $\mathcal{N}$ and N_l from the previous subsection and, in accordance with Section 1.2, set $P_{(i,j)}^{N_l} := (\frac{i}{N_l-1}, \frac{j}{N_l-1})$. We denote the functionals of function evaluation at $P_{(i,j)}^{N_l}$ by $\delta_{P_{(i,j)}^{N_l}}$, i.e., we set $\langle f, \delta_{P_{(i,j)}^{N_l}}\rangle = f(P_{(i,j)}^{N_l})$. Now we introduce the wavelet functionals as follows. We define the basis functionals for each level $l \in \{0,1,\ldots,L\}$. If $l=0$, then we choose the $\delta_{P_{(i,j)}^{N_0}}$ with $i,j=1,2$. For $l>0$, we define the basis functionals as differences $\delta_P - \delta_Q$, where P is a point of the difference grid $\{P_{(i,j)}^{N_l} : i,j=1,\ldots,N_l-2\} \setminus \{P_{(i,j)}^{N_{l-1}} : i,j=1,\ldots,N_{l-1}-2\}$ and Q is a neighbor point of P on the grid $\{P_{(i,j)}^{N_l} : i,j = 1,\ldots,N_l-2\}$. If $P = \left(\frac{2i-1}{N_l-1}, \frac{2j}{N_l-1}\right)$ with $2i < N_l-1$ then we choose the right neighbor $Q := \left(\frac{2i}{N_l-1}, \frac{2j}{N_l-1}\right)$. For $P = \left(\frac{2i-1}{N_l-1}, \frac{2j}{N_l-1}\right)$ with $2i = N_l-1$, we take the left neighbor $Q := \left(\frac{2i-2}{N_l-1}, \frac{2j}{N_l-1}\right)$ since $\left(\frac{2i}{N_l-1}, \frac{2j}{N_l-1}\right)$ is no more in the grid $\{P_{(i,j)}^{N_l} : i,j=1,\ldots,N_l-2\}$. Similarly, we choose the upper neighbor $Q := \left(\frac{i}{N_l-1}, \frac{2j}{N_l-1}\right)$ for $P = \left(\frac{i}{N_l-1}, \frac{2j-1}{N_l-1}\right)$ with $2j < N_l-1$ as well as the lower neighbor $Q := \left(\frac{i}{N_l-1}, \frac{2j-2}{N_l-1}\right)$ for $P = \left(\frac{i}{N_l-1}, \frac{2j-1}{N_l-1}\right)$ with $2j = N_l-1$. In other words, for $J=(l,t,i,j) \in \mathcal{N}$, we introduce the wavelet functionals

$$
\vartheta_J := [3\cdot 2^l]^{-1}\begin{cases}
\delta_{P^{N_0}_{(i,j)}} & \text{if } l=0,\ t=1,\ i,j=1,2\\
\delta_{P^{N_l}_{(2i-1,2j)}} - \delta_{P^{N_l}_{(2i,2j)}} & \text{if } l\ge 1,\ t=1,\ i,j=1,\dots,N_{l-1}-2\\
\delta_{P^{N_l}_{(2i-1,2j)}} - \delta_{P^{N_l}_{(2i-2,2j)}} & \text{if } l\ge 1,\ t=1,\\
& i=N_{l-1}-1,\ j=1,\dots,N_{l-1}-2\\
\delta_{P^{N_l}_{(2i,2j-1)}} - \delta_{P^{N_l}_{(2i,2j)}} & \text{if } l\ge 1,\ t=2,\\
& i,j=1,\dots,N_{l-1}-2\\
\delta_{P^{N_l}_{(2i,2j-1)}} - \delta_{P^{N_l}_{(2i,2j-2)}} & \text{if } l\ge 1,\ t=2,\\
& i=1,\dots,N_{l-1}-2,\ j=N_{l-1}-1\\
\delta_{P^{N_l}_{(2i-1,2j-1)}} - \delta_{P^{N_l}_{(2i-1,2j)}} & \text{if } l\ge 1,\ t=3,\\
& i=1,\dots,N_{l-1}-1,\ j=1,\dots,N_{l-1}-2\\
\delta_{P^{N_l}_{(2i-1,2j-1)}} - \delta_{P^{N_l}_{(2i-1,2j-2)}} & \text{if } l\ge 1,\ t=3,\\
& i=1,\dots,N_{l-1}-1,\ j=N_{l-1}-1.
\end{cases} \tag{2.20}
$$

It is easy to see that $\text{span}\{\vartheta_J,\ J\in\mathcal{N}\} = \text{span}\{h\delta_{P_I^N},\ I\in\mathcal{M}\}$.

In order to analyse the wavelet transform R_h, we introduce the bidual wavelet system. First, for each level $l\in\{0,1,\dots,L\}$, we consider the uniform partition $(0,1]\times(0,1] = \cup_{i,j=1}^{N_l-1}(\frac{i-1}{N_l-1},\frac{i}{N_l-1}]\times(\frac{j-1}{N_l-1},\frac{j}{N_l-1}]$. The space of piecewise constant functions over this partition is $(N_l-1)^2$ dimensional. To get an $(N_l-2)^2$ dimensional space we enlarge the subdomains intersecting the right and the upper part of the boundary ∂S. Thus, we consider the partition

$$
\begin{aligned}
(0,1)\times(0,1) &= \cup_{i,j=1}^{N_l-2} S_{i,j}^{N_l}, \qquad (2.21)\\
S_{i,j}^{N_l} &:= \begin{cases}
(\frac{i-1}{N_l-1},\frac{i}{N_l-1}]\times(\frac{j-1}{N_l-1},\frac{j}{N_l-1}] & \text{if } i,j=1,\dots,N_l-3\\
(\frac{i-1}{N_l-1},\frac{i+1}{N_l-1}]\times(\frac{j-1}{N_l-1},\frac{j}{N_l-1}] & \text{if } i=N_l-2,\ j=1,\dots,N_l-3\\
(\frac{i-1}{N_l-1},\frac{i}{N_l-1}]\times(\frac{j-1}{N_l-1},\frac{j+1}{N_l-1}] & \text{if } i=1,\dots,N_l-3,\ j=N_l-2\\
(\frac{i-1}{N_l-1},\frac{i+1}{N_l-1}]\times(\frac{j-1}{N_l-1},\frac{j+1}{N_l-1}] & \text{if } i=j=N_l-2.
\end{cases}
\end{aligned}
$$

The set of piecewise constant functions over this partition of level l is spanned by the functions $\chi_{(i,j)}^{N_l}$, $i,j,=1,\dots,N_l-2$, where

$$
\chi_{(i,j)}^{N_l}(x) := \begin{cases} [3\cdot 2^l] & \text{if } x\in S_{i,j}^{N_l}\\ 0 & \text{else .}\end{cases} \tag{2.22}
$$

Now we introduce a simple hierarchical basis for the space $\mathcal{Y}_h := \text{span}\{\chi_I^{N_L} : I\in\mathcal{M}\}$ of piecewise constant functions over the finest level. For the coarsest level $l=0$, we take the $\chi_{(i,j)}^{N_0}$ with $i,j=1,2$. For $l>0$, we observe that the piecewise constant functions of level l over the square $S_{i,j}^{N_{l-1}}$ are spanned by the level $(l-1)$ function $\chi_{i,j}^{N_{l-1}}$ and the three level l functions $\chi_{2i-1,2j}^{N_l}+\chi_{2i-1,2j-1}^{N_l}$, $\chi_{2i,2j-1}^{N_l}$, and $\chi_{2i-1,2j-1}^{N_l}$. Thus, for each i,j, we take these three level l functions as basis functions. In other words, the hierarchical basis in $\mathcal{Y}_h$ is given by

$$
\omega_{(0,1,i,j)} := \chi_{(i,j)}^{N_0},\ i,j=1,2 \tag{2.23}
$$

$$\begin{aligned}
\omega_{(l,1,i,j)} &:= \chi^{N_l}_{(2i-1,2j)} + \chi^{N_l}_{(2i-1,2j-1)},\ i = 1,\dots,N_{l-1}-1,\ j = 1,\dots,N_{l-1}-2 \\
\omega_{(l,2,i,j)} &:= \chi^{N_l}_{(2i,2j-1)},\ i = 1,\dots,N_{l-1}-2,\ j = 1,\dots,N_{l-1}-1 \\
\omega_{(l,3,i,j)} &:= \chi^{N_l}_{(2i-1,2j-1)},\ i,j = 1,\dots,N_{l-1}-1, \\
& \quad l = 1,\dots,L.
\end{aligned}$$

For this basis it is not hard to verify that, for any $I, J \in \mathcal{N}$, there holds

$$\langle \omega_I, \vartheta_J \rangle = \delta_{I,J} \tag{2.24}$$

and the projection P_h onto the space $\mathcal{Y}_h$ interpolating at the points $\{P_I,\ I \in \mathcal{M}\}$ can be represented as

$$P_h f = \sum_{I\in\mathcal{M}} h f(P_I)\chi^{N_L}_I = \sum_{J\in\mathcal{N}} \langle f, \vartheta_J\rangle \omega_J. \tag{2.25}$$

Now the matrix R_h of Subsection 2.1 is nothing else than the basis transform from $\{\omega_J\}$ to $\{\chi^{N_L}_I\}$, i.e., $R_h\nu = \mu$ if $\sum_{J\in\mathcal{N}} \nu_J\omega_J = \sum_{I\in\mathcal{M}} \mu_I\chi^{N_L}_I$. In view of the uniform norm equivalence

$$\| \sum_{I\in\mathcal{M}} \mu_I \chi^{N_L}_I \|_{L^2(S)} \sim \sqrt{\sum_{I\in\mathcal{M}} |\mu_I|^2}, \tag{2.26}$$

it is natural to supply the image space of R_h with the l^2 norm. Unfortunately, the analogous norm equivalence for the system $\{\omega_J\}$ is not valid. We get

Lemma 2.2 *a) For the basis transform $R_h \in \mathcal{L}(l^2(\mathcal{N}), l^2(\mathcal{M}))$, there exists a constant C independent of L (recall that $h = \frac{1}{N-1}$, $N = 3\cdot 2^L + 1$) such that*

$$C^{-1}\sqrt{L} \le \|R_h\| \le C\sqrt{L}, \tag{2.27}$$
$$C^{-1}2^L \le \|R_h^{-1}\| \le C2^L. \tag{2.28}$$

b) The basis $\{\omega_J,\ J \in \mathcal{N}\}$ represents a hierarchical basis of $\mathcal{Y}_h$, i.e., for $l = 1,\dots,L$, we get

$$\begin{aligned}
\operatorname{span}\left\{\chi^{N_l}_{i,j} : i,j = 1,\dots,N_l-2\right\} &= \operatorname{span}\left\{\chi^{N_{l-1}}_{i,j} : i,j = 1,\dots,N_{l-1}-2\right\} \\
&\bigoplus \operatorname{span}\left\{\omega_I :\ I = (l',t,i,j) \in \mathcal{N}, l' = l\right\}.
\end{aligned} \tag{2.29}$$

This fact is a consequence of the two-scale relations which include the relations (2.23) and

$$\chi^{N_{l-1}}_{(i,j)} = 2\left\{\chi^{N_l}_{(2i-1,2j-1)} + \chi^{N_l}_{(2i-1,2j)} + \chi^{N_l}_{(2i,2j-1)} + \chi^{N_l}_{(2i,2j)}\right\}. \tag{2.30}$$

Proof. Since b) is easy to check, we only show a). Now and in the following, let C stand for a general constant the value of which varies from instance to instance. Setting $v = \sum_{J\in\mathcal{N}} \nu_J\omega_J = \sum_{I\in\mathcal{M}} \mu_I\chi^{N_L}_I$, we get $R_h\nu = \mu$. From (2.25) we infer

$$\mu_I = hv(P_I) = \sum_{J\in\mathcal{N}} \nu_J h\omega_J(P_I). \tag{2.31}$$

The last sum contains no more than $C \cdot L$ terms different from zero and, for $J = (l_J, t_J, i_J, j_J)$, each term can be estimated by

$$|\nu_J| \cdot h \cdot \sup_P |\omega_J(P)| \leq C|\nu_J| 2^{l_J - L}. \tag{2.32}$$

By the Cauchy Schwarz inequality, we conclude

$$\begin{aligned} |\mu_I|^2 &\leq CL \sum_{J \in \mathcal{N}: \omega_J(P_I) \neq 0} 2^{2(l_J - L)} |\nu_J|^2, \\ \sum_{I \in \mathcal{M}} |\mu_I|^2 &\leq CL \sum_{J \in \mathcal{N}} 2^{2(l_J - L)} |\nu_J|^2 \sum_{I \in \mathcal{M}: \omega_J(P_I) \neq 0} 1. \end{aligned} \tag{2.33}$$

Taking into account that the support of ω_J contains no more than $C 2^{2(L - l_J)}$ grid points P_I, we continue

$$\sum_{I \in \mathcal{M}} |\mu_I|^2 \leq CL \sum_{J \in \mathcal{N}} |\nu_J|^2. \tag{2.34}$$

This proves $\|R_h\| \leq C\sqrt{L}$. For the converse estimate, we choose $\nu_J := 2^{-l_J}$. A simple calculation yields $\|\nu\| \leq C\sqrt{L}$ and $|\mu_I| \geq C 2^{-L} L$ for P_I on the difference grid $\{P_I, I \in \mathcal{M}\} \setminus \{(\frac{i}{N_{L-1}-1}, \frac{j}{N_{L-1}-1}), i,j = 1, \ldots, N_{L-1} - 2\}$. Hence, we conclude $\|\mu\| \geq CL$ and $\|R_h\| \geq C\sqrt{L}$.

Now we turn to R_h^{-1}. Analogously to (2.31), we arrive at

$$\nu_J = \sum_{I \in \mathcal{M}} \mu_I \langle \chi_I^{N_L}, \vartheta_J \rangle. \tag{2.35}$$

In this sum the number of terms different from 0 is bounded by a constant. Each term can be estimated by $|\mu_I| 2^{(L - l_J)}$, and the Cauchy Schwarz inequality yields

$$\begin{aligned} |\nu_J|^2 &\leq C 2^{2(L - l_J)} \sum_{I \in \mathcal{M}: \langle \chi_I, \vartheta_J \rangle \neq 0} |\mu_I|^2, \\ \sum_{J \in \mathcal{N}} |\nu_J|^2 &\leq C \sum_{I \in \mathcal{M}} |\mu_I|^2 \sum_{J \in \mathcal{N}: \langle \chi_I, \vartheta_J \rangle \neq 0} 2^{2(L - l_J)}. \end{aligned} \tag{2.36}$$

For fixed $I \in \mathcal{M}$ and fixed l_J, $0 \leq l_J \leq L$, the number of $J = (l_J, t_J, i_J, j_J) \in \mathcal{N}$ with $\langle \chi_I, \vartheta_J \rangle \neq 0$ is bounded by a constant. Consequently, we obtain

$$\begin{aligned} \sum_{J \in \mathcal{N}} |\nu_J|^2 &\leq C \sum_{I \in \mathcal{M}} |\mu_I|^2 \sum_{l_j = 0}^{L} 2^{2(L - l_J)}, \\ \|\nu\|_{l^2} &\leq C 2^L \|\mu\|_{l^2} \end{aligned} \tag{2.37}$$

and $\|R_h^{-1}\| \leq C 2^L$. On the other hand, choosing $\mu_I := 2^{-L}$ for the coarse grid point $P_I = (\frac{1}{3}, \frac{1}{3})$ and $\mu_I := 0$ else, we arrive at $\|\mu\| \leq C 2^{-L}$ and $|\nu_{(0,1,1,1)}| \geq C$. In other words, $\|\nu\| \geq C$ and $\|R_h^{-1}\| \geq C 2^{-L}$.

$\diamond$

Similarly to E_h, the basis transform R_h can be performed with the help of a fast pyramid scheme. Indeed, suppose we are given the vector of wavelet coefficients $\nu = \{\nu_J\}_{J \in \mathcal{N}}$ of a function $v^L = \sum_{J\in\mathcal{N}} \nu_J \omega_J$. In view of $\omega_{(0,1,i,j)} = \chi^{N_0}_{(i,j)}$ and the relation (2.29) with $l = 1$, we compute the coefficients $\mu^1_{(i,j)}$ of the representation $v^1 = \sum_{i,j=1,\dots,N_1-2} \mu^1_{(i,j)}\chi^{N_1}_{(i,j)}$ for the function $v^1 := \sum_{l=0}^{1}\sum_{t,i,j} \nu_{(l,t,i,j)}\omega_{(l,t,i,j)}$. Next, in view of (2.29) with $l = 2$, we compute the coefficients $\mu^2_{(i,j)}$ of the representation $v^2 = \sum_{i,j=1,\dots,N_2-2} \mu^2_{(i,j)}\chi^{N_2}_{(i,j)}$ for the function $v^2 := \sum_{i,j=1,\dots,N_1-2} \mu^1_{(i,j)}\chi^{N_1}_{(i,j)} + \sum_{t,i,j} \nu_{(2,t,i,j)}\omega_{(2,t,i,j)}$. Proceeding in this manner, we finally determine the coefficients $\mu_{(i,j)} = \mu^L_{(i,j)}$ of the representation $\sum_{i,j=1,\dots,N_L-2} \mu^L_{(i,j)}\chi^{N_L}_{(i,j)}$ for the function $v^L := \sum_{i,j=1,\dots,N_{L-1}-2} \mu^{L-1}_{(i,j)}\chi^{N_{L-1}}_{(i,j)} + \sum_{t,i,j} \nu_{(L,t,i,j)}\omega_{(L,t,i,j)}$. Remark that, for each level l, the computation of the $\mu^l_{(i,j)}$ from the $\mu^{l-1}_{(i,j)}$ and the $\nu_{(l,t,i,j)}$ can be realised by substituting the two scale relations (2.23) and (2.30) into $\sum_{i,j} \mu^l_{(i,j)}\chi^{N_l}_{(i,j)} = \sum_{i,j} \mu^{l-1}_{(i,j)}\chi^{N_{l-1}}_{(i,j)} + \sum_{t,i,j} \nu_{(l,t,i,j)}\omega_{(l,t,i,j)}$ and by comparing the coefficients of the functions $\chi^{N_l}_{(i,j)}$. In other words, on each level l, the vectors $\{\mu^{l-1}_{(i,j)}\}_{(i,j)}$ and $\{\nu_{(l,t,i,j)}\}_{(t,i,j)}$ are to be multiplied by a matrix containing only a small number of non-zero entries in each row. By this way, the number of all arithmetic operations for the multiplication by R_h is less than $O(2^{2L}) = O((N_L-2)^2)$, where $(N_L-2)^2$ is the dimension of the test space.

2.4. The Compression Schemes

In this section we describe two compression algorithms. For the first, the results and proofs are completely analogous to those given by Dahmen, Prößdorf, Schneider, v.Petersdorff, and Schwab [12, 25]. Hence, we present the results and only those parts of the proofs which are necessary for the analysis of the quadrature algorithm in Subsection 5.2. We begin with an estimate for the entries of B_h.

Lemma 2.3 *Suppose the parametrisation F of the surface Γ is smooth. Then, for $J = (l_J, t_J, i_J, j_J)$ and $I = (l_I, t_I, i_I, j_I)$ from $\mathcal{N}$ such that ψ_I is orthogonal to bilinear functions, the entry $b_{J,I} := \langle A\psi_I, \vartheta_J\rangle$ of the wavelet transform B_h can be estimated as*

$$|b_{J,I}| \le 2^{-3l_I-2l_J}\left[\operatorname{dist}(\operatorname{supp}\psi_I, \operatorname{supp}\vartheta_J)\right]^{-5}, \tag{2.38}$$

where $\operatorname{supp}\psi_I$ and $\operatorname{supp}\vartheta_J$ denote the supports of the function ψ_I and the functional ϑ_J, respectively. By $\operatorname{dist}(\operatorname{supp}\psi_I, \operatorname{supp}\vartheta_J)$ we have denoted the distance between the sets $\operatorname{supp}\psi_I$ and $\operatorname{supp}\vartheta_J$.

Proof. Instead of repeating the rigorous proof of [12, 25], let us only explain, where the different factors in (2.38) come from. One factor 2^{-l_J} is from the scaling factor $[3\cdot 2^{l_J}]^{-1}$ in the definition of (2.20). The second factor 2^{-l_J} is due to the second term in the Taylor series expansion of the kernel function at a point P of $\operatorname{supp}\vartheta_J$. Indeed, applying ϑ_J to $f := A\psi_I$ and using that ϑ_J vanishes over constant functions, we get

$$\begin{aligned} f(Q) &= f(P) + \nabla f(P')\cdot(P-Q), \\ |\langle f, [3\cdot 2^{l_J}]\vartheta_J\rangle| &\le C\sup|\nabla f(P')| \sup_{Q\in\operatorname{supp}\vartheta_J} |P-Q| \le C\sup|\nabla f(P')|\, 2^{-l_J}. \end{aligned} \tag{2.39}$$

Similarly, writing $\langle A\psi_I, \vartheta_J\rangle = \langle \psi_I, A^*\vartheta_J\rangle = \int f\psi_I$ with $f := A^*\vartheta_J$, using the moment conditions of order two for the trial wavelet, and choosing $P \in \operatorname{supp}\psi_I$, we conclude

$$f(Q) = f(P) + \nabla f(P)\cdot(P-Q) + \frac{1}{2}\nabla^2 f(P')\cdot(P-Q)\cdot(P-Q),$$

$$\begin{aligned}
\int f\psi_I &= \int \frac{1}{2}\nabla^2 f(P')\cdot(P-Q)\cdot(P-Q)\psi_I(Q)dQ, \\
\left|\int f\psi_I\right| &\le C\sup|\nabla^2 f(P')|\int_{\operatorname{supp}\psi_I}|P-Q|^2|\psi_I(Q)|dQ \\
&\le C\sup|\nabla^2 f(P')|2^{-2l_I}\int_{\operatorname{supp}\psi_I}|\psi_I(Q)|dQ.
\end{aligned} \tag{2.40}$$

Thus, a factor 2^{-2l_I} in (2.38) is due to the second order moment conditions of the wavelet in the trial space and an additional 2^{-l_I} arises from the scaling factor $[3\cdot 2^{l_I}] = \sqrt{3\cdot 2^{l_I}}^2$ in (2.8), (2.12)-(2.14) and from the measure $meas(\operatorname{supp}\psi_I) \sim 2^{-2l_I}$. Applying these Taylor series arguments to the integrand in $\langle A\psi_I, \vartheta_J\rangle$, it remains to estimate the third order derivatives of the kernel function $K(P,Q)$ of the operator A for P between the points of $\operatorname{supp}\vartheta_J$ and $Q \in \operatorname{supp}\psi_I$. It is not hard to show that

$$|\partial_P^{\alpha_1}\partial_Q^{\alpha_2}K(P,Q)| \le C|P-Q|^{-2-|\alpha_1|-|\alpha_2|}. \tag{2.41}$$

Hence, the estimate of the kernel function leads to the factor $[\operatorname{dist}(\operatorname{supp}\psi_I, \operatorname{supp}\vartheta_J)]^{-5}$ in (2.38).

◇

Theorem 2.1 *Suppose that the parametrisation F of the surface Γ is smooth and that the solution u and the right-hand side v of (1.5) belong to the Sobolev space H^2 and vanish on the boundary of the domain S. Choose β, γ, and α such that $\frac{1}{3} \le \beta \le 1$, $1 \le \gamma < \frac{4}{3}$, and $\alpha = \frac{1}{3} + \beta$. Let the compressed matrix $C_h = (c_{J,I})_{J,I\in\mathcal{N}}$ be defined by*

$$c_{J,I} := \begin{cases} b_{J,I} & \text{if } \operatorname{dist}(\operatorname{supp}\psi_I, \operatorname{supp}\vartheta_J) \le \max\{2^{-l_I},\, 2^{-l_J},\, (a2^{\alpha L})2^{-\beta l_J - \gamma l_I}\} \\ 0 & \text{else} \end{cases} \tag{2.42}$$

with a suitable constant a. If a is large enough, then there is an $h_0 > 0$ such that, for any $h = \frac{1}{N_L - 1} < h_0$, the operator $\tilde{A}_h := [R_h C_h E_h] \in \mathcal{L}(\mathcal{X}_h)$ is invertible and its inverse $\tilde{A}_h^{-1}$ is uniformly bounded. Additionally, if u_h denotes the solution of $\tilde{A}_h u_h = v_h$ with $v_h = \sum_{I\in\mathcal{M}} h v_h(P_I)\varphi_I$, then

$$\|u - u_h\|_{L^2} \le Ca^{-3}h^2 \begin{cases} \{\log h^{-1}\}^{3/2} & \text{if } \beta = \frac{1}{3} \\ \sqrt{\log h^{-1}} & \text{else} \end{cases} \tag{2.43}$$

and the number of non-zero entries in the $(N_L-2)^2 \times (N_L-2)^2$ matrix C_h is less than $Ca^2[(N_L-2)^2]^{4/3}[\log(N_L-2)]^b = Ca^2[h^{-2}]^{4/3}[\log h^{-1}]^b$, where

$$b := \begin{cases} 2 & \text{if } \beta = \gamma = 1 \\ 1 & \text{if } \beta = 1 \ne \gamma \text{ or if } \gamma = 1 \ne \beta \\ 0 & \text{else}\ . \end{cases} \tag{2.44}$$

In other words, if the matrix C_h is given and if the equation $\tilde{A}_h u_h = v_h$ is solved by an optimal iterative algorithm, then an approximate solution with an error less than $Ch^2\sqrt{\log h^{-1}}$ can be computed with no more than $C[h^{-2}]^{4/3}[\log h^{-1}]^b$ arithmetic operations.

Proof. For a proof we again refer to [12, 25]. We only present that part of the proof which will be important for the treatment of the quadrature algorithm in Subsection 5.2. More precisely, for the stability and for the convergence estimate, we will prove

$$\|(A_h - \tilde{A}_h)u_h\|_{L^2} \le Ca^{-3}h^{\min\{2,4-3\gamma+s\}}\|u_h\|_{H^s} \begin{cases} \{\log h^{-1}\}^{3/2} & \text{if } \beta = \frac{1}{3} \\ & \text{or if } s = 3\gamma - 2 \\ \sqrt{\log h^{-1}} & \text{else} \end{cases} \quad (2.45)$$

with $0 \le s \le 2$. After this, we will count the non-zero entries of C_h.

To prove (2.45), we set $D_h := B_h - C_h = (d_{J,I})_{J,I\in\mathcal{N}}$ and get $A_h - \tilde{A}_h = R_h D_h E_h$. In view of Lemma 2.2 and the estimate (cf. e.g. [9])

$$\sqrt{\sum_{I=(l_I,t_I,i_I,j_I)\in\mathcal{N}} |\mu_I|^2 2^{2sl_I}} \le C\|\sum_{I\in\mathcal{N}} \mu_I \psi_I\|_{H^s}, \quad (2.46)$$

we have to estimate the matrix $D_h^s := (d_{J,I}^s)_{J,I\in\mathcal{N}} \in \mathcal{L}(l^2(\mathcal{N}))$ with $d_{J,I}^s := d_{J,I}2^{-sl_I}$. By Schur's lemma the norm can be bounded as follows.

$$\begin{aligned} \|D_h^s\|_{\mathcal{L}(l^2(\mathcal{N}))} &\le \max\{\sigma_1, \sigma_2\}, \quad (2.47) \\ \sigma_1 &:= \sup_{J\in\mathcal{N}} \left[2^{l_J} \sum_{I\in\mathcal{N}} \left|d_{J,I}^s\right| 2^{-l_I}\right], \quad \sigma_2 := \sup_{I\in\mathcal{N}} \left[2^{l_I} \sum_{J\in\mathcal{N}} \left|d_{J,I}^s\right| 2^{-l_J}\right]. \end{aligned}$$

Set $a_* := \max\{2^{-l_I}, 2^{-l_J}, (a2^{\alpha L})2^{-\beta l_J - \gamma l_I}\}$ and $\text{dist} := \text{dist}(\text{supp}\,\psi_I, \text{supp}\,\vartheta_J)$. Using (2.38) and (2.42), we get

$$\begin{aligned} \sigma_1 &\le C \sup_{J\in\mathcal{N}} \left[2^{l_J} \sum_{I\in\mathcal{N}:\,\text{dist}>a_*} 2^{-3l_I - 2l_J}\text{dist}^{-5}2^{-sl_I}2^{-l_I}\right] \quad (2.48) \\ &\le C \sup_{J\in\mathcal{N}} \left[2^{-l_J} \sum_{l_I,t_I} 2^{-l_I(2+s)} \sum_{(i_I,j_I):\,\text{dist}>a_*} \text{dist}^{-5}2^{-2l_I}\right]. \end{aligned}$$

Applying

$$\sum_{(i_I,j_I):\,\text{dist}>a_*} \text{dist}^{-5}2^{-2l_I} \le C\int_{|P|>a_*} |P|^{-5}dP \le Ca_*^{-3}, \quad (2.49)$$

we arrive at

$$\begin{aligned} \sigma_1 &\le C \sup_{J\in\mathcal{N}} \left[2^{-l_J} \sum_{l_I,t_I} 2^{-l_I(2+s)} a_*^{-3}\right] \quad (2.50) \\ &\le C \sup_{J\in\mathcal{N}} \left[2^{-l_J} \sum_{l_I,t_I} 2^{-l_I(2+s)} \left((a2^{\alpha L})2^{-\beta l_J - \gamma l_I}\right)^{-3}\right] \\ &\le C \sup_{J\in\mathcal{N}} \left[a^{-3}2^{-3\alpha L}2^{(3\beta-1)l_J} \sum_{l_I=0}^{L} 2^{l_I(3\gamma-2-s)}\right] \\ &\le Ca^{-3}2^{\{-3\alpha+(3\beta-1)_+ + (3\gamma-2-s)_+\}L} \begin{cases} L & \text{if } (3\gamma-2-s)=0 \\ 1 & \text{else}. \end{cases} \end{aligned}$$

Here $(\ldots)_+$ denotes the non-negative part of the expression $(\ldots)$. Analogously we obtain

$$\begin{aligned}
\sigma_2 &\leq C \sup_{I\in\mathcal{N}} \left[2^{l_I} \sum_{J\in\mathcal{N}:\,\text{dist}>a_*} 2^{-3l_I-2l_J}\text{dist}^{-5}2^{-sl_I}2^{-l_J} \right] \\
&\leq C \sup_{I\in\mathcal{N}} \left[2^{-(2+s)l_I} \sum_{l_J,t_J} 2^{-l_J} \sum_{(i_J,j_J):\,\text{dist}>a_*} \text{dist}^{-5}2^{-2l_J} \right] \\
&\leq C \sup_{I\in\mathcal{N}} \left[2^{-(2+s)l_I} \sum_{l_J} 2^{-l_J} \left((a2^{\alpha L})2^{-\beta l_J-\gamma l_I} \right)^{-3} \right] \\
&\leq C \sup_{I\in\mathcal{N}} \left[a^{-3}2^{-3\alpha L}2^{(3\gamma-2-s)l_I} \sum_{l_J=0}^{L} 2^{l_J(3\beta-1)} \right] \\
&\leq C a^{-3} 2^{\{-3\alpha+(3\beta-1)_++(3\gamma-2-s)_+\}L} \begin{cases} L & \text{if } (3\beta-1)=0 \\ 1 & \text{else .} \end{cases}
\end{aligned} \tag{2.51}$$

The choice $\alpha=\beta+\frac{1}{3}$ and the bounds (2.50) and (2.51) yield that $\|D_h^s\|_{\mathcal{L}(l^2(\mathcal{N}))}$ is less than $Ch^{\min\{2,4-3\gamma+s\}}$ times maybe a logarithmic factor $L\sim\log h^{-1}$. This together with (2.46) and Lemma 2.2 implies (2.45).

Now we count the number of non-zero entries in C_h. For a fixed row corresponding to a fixed ϑ_J with $J=(l_J,t_J,i_J,j_J)$, the number of non-zero entries is bounded by

$$\begin{aligned}
\sum_{I=(l_I,t_I,i_I,j_I):\,\text{dist}\leq a_*} 1 &\leq C\sum_{l_I=0}^{L} 2^{2l_I} \sum_{(i_I,j_I):\,dist\leq a_*} 2^{-2l_I} \\
&\leq C\sum_{l_I=0}^{L} 2^{2l_I} \int_{|P|\leq a_*} dP \leq C\sum_{l_I=0}^{L} 2^{2l_I}a_*^2 .
\end{aligned} \tag{2.52}$$

Applying $a_*^2 \leq 2^{-2l_I}+2^{-2l_J}+(a^22^{2\alpha L})2^{-2\beta l_J-2\gamma l_I}$, we continue

$$\sum_{I:\,\text{dist}\leq a_*} 1 \leq C\sum_{l_I=0}^{L} 1 + C2^{-2l_J}\sum_{l_I=0}^{L} 2^{2l_I} + C(a^22^{2\alpha L})2^{-2\beta l_J}\sum_{l_I=0}^{L} 2^{2(1-\gamma)l_I}. \tag{2.53}$$

Now we observe that there exist no more than $C2^{2l_J}$ test functionals at level l_J. Summing up over all rows, i.e., over all J leads to the bound

$$\begin{aligned}
C\sum_{l_J=0}^{L} 2^{2l_J} \sum_{I:\,\text{dist}\leq a_*} 1 &\leq CL2^{2L} + Ca^22^{\{2\alpha+2(1-\gamma)_++2(1-\beta)_+\}L} \begin{cases} L^2 & \text{if } \beta=\gamma=1 \\ L & \text{if } \beta=1\neq\gamma \\ & \text{or if } \beta\neq 1=\gamma \\ 1 & \text{else} \end{cases} \\
&\leq Ca^2[h^{-2}]^{4/3}\{\log h^{-1}\}^b .
\end{aligned} \tag{2.54}$$

Thus, the estimate for the number of non-zero entries in C_h is proved.
◇

Remark 2.3 *An optimal compression C_h of B_h to a number of non-zero entries less than $C(N_L-2)^2$ like for the Galerkin scheme in [31] (cf. also the almost optimal compression in [12, 26]) seems to be impossible since the moment conditions for the wavelets are not strong* *enough. However, we have chosen the test wavelets not to achieve optimal asymptotic orders of complexity but to develop a fast algorithm for the case $(N_L-2)^2 \leq 10\,000$. In certain parts (cf. Algorithm 3 in Subsection 4.1) of the matrix assembling the number of points in the test functionals appears as a factor for the computing time. This is the reason why we have chosen the particular wavelets presented in Subsections 2.2 and 2.3.*

Remark 2.4 *From the Lemmata 2.2 and 2.1 we get $\|C_h\| = \|R_h^{-1}\tilde{A}_h E_h^{-1}\| \sim 2^L$ and $\|R_h\| \sim L$. Thus, the multiplication of a certain vector $\tilde{\xi}_h$ by $R_h C_h E_h$ can lead to an additional error of $C2^L L$ times the numerical error of $\tilde{\xi}_h$.*

In the numerical computations we have tested several choices of β and γ. Choosing $\beta = \frac{1}{3}$ and $\gamma = \frac{4}{3}$ has turned out to be a little bit worse than the others. However, the compression rates (compression rate = number of all entries divided by the number of non-zero entries in C_h) have not differed too much. The numerical results presented in Section 4 are obtained with $\beta = 1$ and $\gamma = 1$. The constant a has been determined by tests such that the additional error caused by the compression is less than a prescribed error. We will call the just defined compression (2.42) an a priori compression since the pattern of the compressed matrix is given a priori.

In addition to the scheme (2.42), we will consider a second adaptive compression algorithm which we will call oracle scheme. For large systems, this scheme performs better than (2.42). However, we have not tried to prove asymptotic orders like in Theorem 2.1. To motivate the oracle scheme, we apply B_h to the yet unknown solution $u_h = \sum_{I\in\mathcal{N}} \eta_I \psi_I$. The resulting coefficients are given by $[B_h u_h]_J = \sum_{I\in\mathcal{N}} b_{J,I}\eta_I$. Analogously to the Galerkin method, we seek $B_h u_h$ in a space with a norm "equivalent" to a negative Sobolev norm. In other words, for $J = (l_J, t_J, i_J, j_J)$, we consider

$$2^{-l_J}[B_h u_h]_J = \sum_{I\in\mathcal{N}} 2^{-l_J} b_{J,I}\eta_I. \tag{2.55}$$

Now the compression step from B_h to C_h restricts the last summation to a fewer number of terms. This should be accomplished in such a manner that the restricted sum is close to the complete sum in (2.55), i.e., such that the additional compression error leads to a small perturbation in the consistency estimate. In view of (2.55) we choose a positive threshold ε and neglect those $b_{J,I}$ for which $|2^{-l_J} b_{J,I}\eta_I| < \varepsilon$, i.e., we set

$$c_{J,I} := \begin{cases} b_{J,I} & \text{if } |2^{-l_J} b_{J,I}\eta_I| \geq \varepsilon \\ 0 & \text{else.} \end{cases} \tag{2.56}$$

For a realisation of this scheme, we need an "oracle" to tell us the values of $b_{J,I}$ and η_I. Clearly, it is sufficient to take "approximate" values which are of the same size. We replace the coefficient η_I of $u_h = \sum_{I\in\mathcal{N}} \eta_I\psi_I$ by the coefficients $\tilde{\eta}_I$ of the right-hand side $\tilde{v}_h = \sum_{I\in\mathcal{M}} hv(P_I)\varphi_I = \sum_{I\in\mathcal{N}} \tilde{\eta}_I\psi_I$. The entries $b_{J,I}$ are replaced by an approximate value $\tilde{b}_{J,I} := \langle A\tilde{\psi}_I, \vartheta_J\rangle$, where $A\psi_I$ in $b_{J,I} := \langle A\psi_I, \vartheta_J\rangle$ is replaced by a crude quadrature approximation $A\tilde{\psi}_I$. More precisely, we set

$$\begin{aligned} \tilde{\psi}_{(0,1,i,j)} &:= [3\cdot 2^0]^{-1}\delta_{P^{N_0}_{(i,j)}}, \\ \tilde{\psi}_{(l,1,i,j)} &:= [3\cdot 2^l]^{-1}\left\{\frac{1}{2}\delta_{P^{N_l}_{(2i-2,2j)}} - \delta_{P^{N_l}_{(2i-1,2j)}} + \frac{1}{2}\delta_{P^{N_l}_{(2i,2j)}}\right\} \end{aligned} \tag{2.57}$$

$$\tilde{\psi}_{(l,2,i,j)} := [3 \cdot 2^l]^{-1} \left\{ \frac{1}{2}\delta_{P^{N_l}_{(2i,2j-2)}} - \delta_{P^{N_l}_{(2i,2j-1)}} + \frac{1}{2}\delta_{P^{N_l}_{(2i,2j)}} \right\}$$

$$\tilde{\psi}_{(l,3,i,j)} := [3 \cdot 2^l]^{-1} \left\{ \frac{1}{4}\delta_{P^{N_l}_{(2i-2,2j-2)}} + \frac{1}{4}\delta_{P^{N_l}_{(2i-1,2j)}} - \delta_{P^{N_l}_{(2i-1,2j-1)}} + \frac{1}{4}\delta_{P^{N_l}_{(2i,2j-2)}} + \frac{1}{4}\delta_{P^{N_l}_{(2i,2j)}} \right\}$$

where $l = 1, \ldots, L$. However, $\tilde{b}_{J,I}$ is an acceptable approximate value for $b_{J,I}$ only if the support of ψ_I does not intersect the boundary ∂S of domain S and if $\operatorname{dist}(\operatorname{supp}\psi_I, \operatorname{supp}\vartheta_J) > \min\{\operatorname{diam}(\operatorname{supp}\psi_I), \operatorname{diam}(\operatorname{supp}\vartheta_J)\}$. Here, by $\operatorname{diam}(\operatorname{supp}\psi_I)$ and $\operatorname{diam}(\operatorname{supp}\vartheta_J)$ we have denoted the diameters of the sets $\operatorname{supp}\psi_I$ and $\operatorname{supp}\vartheta_J$, respectively. We arrive at the compression scheme

$$c_{J,I} := \begin{cases} b_{J,I} & \text{if } \operatorname{dist}(\operatorname{supp}\psi_I, \operatorname{supp}\vartheta_J) \le \min\{\operatorname{diam}(\operatorname{supp}\psi_I), \operatorname{diam}(\operatorname{supp}\vartheta_J)\} \\ & \text{or if } \operatorname{supp}\psi_I \cap \partial S \neq \emptyset \text{ or if } \left|2^{-l_J}\left\langle A\tilde{\psi}_I, \vartheta_J\right\rangle \tilde{\eta}_I\right| \ge \varepsilon \\ 0 & \text{else.} \end{cases} \tag{2.58}$$

Of course, this scheme and also (2.42) lead to $O((N_L-2)^4)$ arithmetic operations for the assembling of the $(N_L-2)^2 \times (N_L-2)^2$ matrix C_h if we check for each entry whether it is to be computed or not. For a fast algorithm, we will use the tree structure of the hierarchical wavelet basis. Namely, to any wavelet ψ_I with $I = (l,t,i,j)$ on level l with $1 \le l \le L$, there correspond four sons ψ_{I_1}, ψ_{I_2}, ψ_{I_3}, and ψ_{I_4} on level $l+1$, where $I_1 = (l+1, t, 2i-1, 2j-1)$, $I_2 = (l+1, t, 2i, 2j-1)$, $I_3 = (l+1, t, 2i-1, 2j)$, and $I_4 = (l+1, t, 2i, 2j)$. We call ψ_I the father of ψ_{I_k}, $k = 1, \ldots, 4$. For the scheme (2.42), we observe that, if $b_{J,I}$ is neglected, then also the entries b_{J,I_k}, $k = 1, \ldots, 4$ are neglected, i.e., $c_{J,I} = 0$ implies $c_{J,I_k} = 0$, $k = 1, \ldots, 4$. Consequently, if an entry is neglected, then all the entries corresponding to the sons, the sons of the sons and etc. can be neglected without any test whether the compression criterion is fulfilled or not. For the oracle scheme (2.58), we simply assume that this property is satisfied. We arrive at the algorithm for the determination of the matrix pattern of C_h presented as Algorithm 1. This is the algorithm which has shown better results in the numerical computations than the a priori scheme (2.42), i.e., for $N_5 = 97$ we have obtained a higher compression rate with Algorithm 1. However, to check the compression criterion, a lot of kernel functions have to be evaluated. In the end, the computing times for both schemes are comparable.

Remark 2.5 *If the boundary wavelets are as in Remark 2.2, then the criterion* $\operatorname{supp}\psi_I \cap \partial S \neq \emptyset$ *in Algorithm 1 can be dropped. However, corresponding to the zero order moment condition, the definition of* $\tilde{\psi}_I$ *in (2.57) is to be modified if* $\operatorname{supp}\psi_I \cap \partial S \neq \emptyset$. *For* $l \ge 1$ *and* $t = 1, 2$, *we have to set*

$$\begin{aligned} \tilde{\psi}_{(l,1,1,j)} &:= [3 \cdot 2^l]^{-1} \left\{ \frac{1}{2}\delta_{P^{N_l}_{(1,2j)}} - \frac{1}{2}\delta_{P^{N_l}_{(2,2j)}} \right\} \\ \tilde{\psi}_{(l,1,N_{l-1}-1,j)} &:= [3 \cdot 2^l]^{-1} \left\{ -\frac{1}{2}\delta_{P^{N_l}_{(2N_{l-1}-3,2j)}} + \frac{1}{2}\delta_{P^{N_l}_{(2N_{l-1}-2,2j)}} \right\} \\ \tilde{\psi}_{(l,2,i,1)} &:= [3 \cdot 2^l]^{-1} \left\{ \frac{1}{2}\delta_{P^{N_l}_{(2i,1)}} - \frac{1}{2}\delta_{P^{N_l}_{(2i,2)}} \right\} \\ \tilde{\psi}_{(l,2,i,N_{l-1}-1)} &:= [3 \cdot 2^l]^{-1} \left\{ -\frac{1}{2}\delta_{P^{N_l}_{(2i,2N_{l-1}-3)}} + \frac{1}{2}\delta_{P^{N_l}_{(2i,2N_{l-1}-2)}} \right\}. \end{aligned} \tag{2.59}$$

```
do for any index J ∈ 𝒩 of test functionals
  do for l_I = 0, ..., L
    if l_I = 0 then
      do for any index I = (0, t, i, j) ∈ 𝒩 of trial functions
        Compute c_{J,I} = b_{J,I}.
      end do
    else if l_I = 1 then
      do for any index I = (1, t, i, j) ∈ 𝒩 of trial functions
        if dist(supp ψ_I, supp ϑ_J) ≤ min{diam(supp ψ_I), diam(supp ϑ_J)}
          or supp ψ_I ∩ ∂S ≠ ∅ or |2^{-l_J} ⟨A ψ̃_I, ϑ_J⟩ η̃_I| ≥ ε then
          Compute c_{J,I} = b_{J,I}.
        else
          Set c_{J,I} = 0.
        end if
      end do
    else if l_I > 1 then
      do for any index I = (l_I, t, i, j) ∈ 𝒩 of trial functions
        Determine I' the index of the father of index I.
        if c_{J,I'} ≠ 0 and
        if dist(supp ψ_I, supp ϑ_J) ≤ min{diam(supp ψ_I), diam(supp ϑ_J)}
          or supp ψ_I ∩ ∂S ≠ ∅ or |2^{-l_J} ⟨A ψ̃_I, ϑ_J⟩ η̃_I| ≥ ε then
          Compute c_{J,I} = b_{J,I}.
        else
          Set c_{J,I} = 0.
        end if
      end do
    end if
  end do
end do
```

Algorithm 1: Algorithm of the oracle scheme.

3 The Quadrature Algorithm for the Computation of the Stiffness Matrix

In the present section, we describe the quadrature rules which we have used for the numerical tests. A different quadrature algorithm has been introduced and analysed by Schneider, v.Petersdorff, and Schwab [31, 25]. This algorithm for the Galerkin method should apply also to collocation. However, it relies on Gauß rules with variable degrees and requires the analyticity of the kernel and the underlying geometry. The quadrature rules we have implemented in the numerical tests of the present paper (cf. also the similar quadrature algorithm for one-dimensional double layer equations in [28]) are easier than those of [31, 25]. Instead of composite Gauß rules of variable order we apply composite two point rules. A proof for a similar quadrature scheme applied to a simple model problem will be presented in Section 5 (cf. Theorem 5.3). Due to the use of mesh grading, the quadrature procedures are not asymptotically optimal. However, at the end of Section 5 we will give a remark on how to improve them.

3.1. Quadrature Rules for the Collocation without Wavelets

We have to define quadrature rules for the integral

$$Int := \int_S K(P,Q)u_h(Q)d_QS, \tag{3.1}$$

where P is a collocation point, u_h is a bilinear function of $\mathcal{X}_h$, and the kernel (cf. (1.5))

$$K(P,Q) := \frac{1}{4\pi}\frac{\cos[\tau(P), F(Q)-F(P)]}{|F(Q)-F(P)|^2}F'(Q) \tag{3.2}$$

is singular for $Q = P$. We denote the Fréchet derivative of the parametrisation F by DF and introduce the main part $H(P,Q)$ of the kernel $K(P,Q)$ by

$$H(P,Q) := \frac{1}{4\pi}\frac{\cos[\tau(P), DF(P)\cdot(Q-P)]}{|DF(P)\cdot(Q-P)|^2}F'(P). \tag{3.3}$$

Substituting $K = H + (K - H)$ into (3.1) and applying singularity subtraction to the integral with the main part H, we arrive at

$$\begin{aligned} Int &= \int_S H(P,Q)d_QSu_h(P) + \int_S H(P,Q)[u_h(Q)-u_h(P)]d_QS \\ &\quad + \int_S [K(P,Q)-H(P,Q)]u_h(Q)d_QS. \end{aligned} \tag{3.4}$$

In the last two integrals we employ a quadrature rule of the form

$$\int_S f(Q)d_QS \sim \sum_{I\in\mathcal{O}} f(Q_I)w_I \tag{3.5}$$

to get

$$\begin{aligned} Int &\sim \int_S H(P,Q)d_QSu_h(P) + \sum_{K\in\mathcal{O}} H(P,Q_K)[u_h(Q_K)-u_h(P)]w_K \\ &\quad + \sum_{K\in\mathcal{O}}[K(P,Q_K)-H(P,Q_K)]u_h(Q_K)w_K, \end{aligned} \tag{3.6}$$

$$Int \sim \left\{\int_S H(P,Q)d_QS - \sum_{K\in\mathcal{O}} H(P,Q_K)w_K\right\}u_h(P) + \sum_{K\in\mathcal{O}} K(P,Q_K)u_h(Q_K)w_K, \tag{3.7}$$

where the integral $\int_S H(P,Q)d_QS$ can be computed with the help of the formula

$$\begin{aligned} &\int_{a-1}^{a}\int_{b-1}^{b}\frac{cx+dy}{\{ex^2+fxy+gy^2\}^{3/2}}dydx = \\ &-2\frac{2gc-fd}{\sqrt{g}[4eg+f^2]}\log\left\{\frac{[2gb+fa+2\sqrt{g}\sqrt{ea^2+fab+gb^2}](1-a)^2}{[2gb-f(1-a)+2\sqrt{g}\sqrt{e(1-a)^2-f(1-a)b+gb^2}]a^2}\right. \end{aligned} \tag{3.8}$$

$$\cdot \frac{2g(1-b) - fa + 2\sqrt{g}\sqrt{ea^2 - fa(1-b) + g(1-b)^2}}{2g(1-b) + f(1-a) + 2\sqrt{g}\sqrt{e(1-a)^2 + f(1-a)(1-b) + g(1-b)^2}} \Bigg\}$$

$$-2\frac{2ed - fc}{\sqrt{e}[4eg + f^2]} \log \Bigg\{ \frac{[2ea + fb + 2\sqrt{e}\sqrt{ea^2 + fab + gb^2}](1-b)^2}{[2ea - f(1-b) + 2\sqrt{e}\sqrt{ea^2 - fa(1-b) + g(1-b)^2}]b^2}$$

$$\cdot \frac{2e(1-a) - fb + 2\sqrt{e}\sqrt{e(1-a)^2 - fb(1-a) + gb^2}}{2e(1-a) + f(1-b) + 2\sqrt{e}\sqrt{e(1-a)^2 + f(1-a)(1-b) + g(1-b)^2}} \Bigg\}$$

The integrands $Q \mapsto H(P,Q)[u_h(Q) - u_h(P)]$ and $Q \mapsto [K(P,Q) - H(P,Q)]u_h(Q)$ are weakly singular at the point $Q = P$. Note that

$$\begin{aligned} |H(P,Q)[u_h(Q) - u_h(P)]| &\leq C|Q-P|^{-1}, \\ |[K(P,Q) - H(P,Q)]u_h(Q)| &\leq C|Q-P|^{-1}. \end{aligned} \tag{3.9}$$

Therefore, we have to choose a rule (3.5) which depends on the singularity point P and is accurate for integrals of weakly singular functions.

To set up (3.5), we will give a rectangular partition of S which is adapted to the singular behaviour of the integrand and to the uniform grid which has been used for the definition of the trial space. From this partition, we obtain (3.5) by taking the tensor product of the two point Gauß rule over each subdomain. Thus, to define the underlying partition in dependence on the collocation point, we fix $P = (x_P, y_P) = (i_P h, j_P h)$ and consider the eight parts of the domain S obtained by cuts along the straight lines through P parallel to the sides and diagonals of S. For symmetry reasons, it is sufficient to present the partition over one of these eight parts. In other words, we restrict the consideration to $S' := S \cap \{(x,y) : x_P \leq x,\ 0 \leq y - y_P \leq x - x_P\}$ and define those rectangular subdomains of the partition which intersect S'. The first term of the asymptotics at P for the singular integrands in (3.4) takes the form

$$\Phi(\frac{y_Q - y_P}{x_Q - x_P})\frac{1}{x_Q - x_P}, \tag{3.10}$$

where $Q = (x_Q, y_Q) \in S'$ and Φ stands for a continuous function. For this kind of integrands, we apply two concepts:

i) An appropriate quadrature partition for singular functions containing factors like $\Phi(\frac{y_Q - y_P}{x_Q - x_P})$ is generated by Duffy's transformation. Roughly speaking, this means that, to a step size h_x in x direction at $x = x_Q$, there should correspond a step size $h_y \leq h_x(x_Q - x_P)$ in y direction.

ii) Since the derivatives of the trial functions have jump discontinuities along the lines of the uniform rectangular partition with mesh size h, the quadrature partition should be a refinement of this uniform grid.

Combining these ideas, we get the partition as follows. We consider the uniform partition

$$\begin{aligned} S' &\subseteq \bigcup_{i=0}^{N_L-2-i_P} \bigcup_{j=0}^{\min\{N_L-2-j_P,i\}} S_{i,j}, \\ S_{i,j} &:= \{(x_Q, y_Q) : ih < x_Q - x_P < (i+1)h,\ jh < y_Q - y_P < (j+1)h\} \end{aligned} \tag{3.11}$$

and divide each square $S_{i,j}$. For the subdomain $S_{0,0}$ adjacent to the singularity point P, we apply Duffy's transformation to get

$$\begin{aligned}
&\int_{x_P}^{x_P+h}\int_{y_P}^{y_P+h} f(x,y)dydx \qquad (3.12)\\
=& \int_0^h\int_0^x f(x_P+x,y_P+y)dydx + \int_0^h\int_0^y f(x_P+x,y_P+y)dxdy\\
=& \int_0^h\int_0^1 f(x_P+x,y_P+zx)xdzdx + \int_0^h\int_0^1 f(x_P+zy,y_P+y)ydzdy\\
=& \sum_{i=1}^{N_L-1}\int_0^h\int_{(i-1)h}^{ih} f(x_P+x,y_P+zx)xdzdx + \sum_{i=1}^{N_L-1}\int_0^h\int_{(i-1)h}^{ih} f(x_P+zy,y_P+y)ydzdy.
\end{aligned}$$

In accordance with this splitting, we set $k_0^{0,0} := 2(N_L-1)$ and introduce

$$\begin{aligned}
S_{0,0,k} &:= \{(x_P+x,y_P+zx) : 0\le x\le h,\ (k-1)h<z<kh\}, \qquad (3.13)\\
S_{0,0,N_L-1+k} &:= \{(x_P+zy,y_P+y) : 0\le y\le h,\ (k-1)h<z<kh\}, k=1,\dots,N_L-1.
\end{aligned}$$

In the case of $S_{i,i}$ with $i>0$, we leave the subdomain unchanged and set $k_0^{i,i} := 1$ and $S_{i,i,1} = S_{i,i}$. For $S_{i,j}$ and $i>j$, we do not change the step size $h_x = h$ in x-direction but reduce the step size in y-direction. We define $k_0^{i,j}$ to be the smallest integer greater or equal to $\frac{N_L-1}{i+1}$ and introduce

$$S_{i,j,k} := \left\{(x,y)\in S_{i,j} : (k-1)\frac{h}{k_0^{i,j}} < y-y_P-jh < k\frac{h}{k_0^{i,j}}\right\},\ k=1,\dots,k_0^{i,j}. \qquad (3.14)$$

Note that the step size $h_y = h/k_0^{i,j}$ is chosen in accordance with i), i.e., $h_y \le h[(i+1)h]$. Finally, we arrive at the partition

$$S' \subseteq \bigcup_{i=0}^{N_L-2-i_P}\ \bigcup_{j=0}^{\min\{N_L-2-j_P,i\}}\ \bigcup_{k=1}^{k_0^{i,j}} S_{i,j,k}. \qquad (3.15)$$

The rule (3.5) is the composite quadrature rule with a tensor product two point Gauß rule over each subdomain $S_{i,j,k}$ of S' and over each similar subdomain defined for the seven other parts of S. Of course, for the square $S_{0,0}$ and its subdomains $S_{0,0,k}$, the Gauß rule is meant to be applied to the $2(N_L-1)$ integrals on the right-hand side of (3.12).

Remark 3.1 *Note that the quadrature rule can be improved, i.e., the same accuracy can be achieved with fewer quadrature knots. For details we refer to Section 5 and, especially, to Remark 5.1.*

3.2. Quadrature Rules for the Wavelet Algorithm

The quadrature rule for the wavelet algorithm is very similar to that of Subsection 3.1. Now the quadrature rules are to be applied to the last two integrals in (3.4), where $u_h \in \mathcal{X}_h$ is spanned by some but not all of the wavelet basis functions ψ_I. Since we will use a composite rule of tensor product two point Gauß rules, we only have to define the partition.

Consider a fixed collocation point P. This point belongs to the support of at most $2L+3$ test functionals ϑ_J. Let us denote them by $\vartheta_{J_1}, \vartheta_{J_2}, \ldots, \vartheta_{J_m}$. In view of the compression, we have to compute (3.4) for u_h in the span $\mathcal{Z}_h$ of all ψ_I, $I \in \mathcal{N}$ such that there exists a non-zero entry $c_{J_i,I}$, $1 \le i \le m$ of C_h. Thus, analogously to point ii) of the previous subsection the underlying partition of the quadrature should be a refinement of the partition $Part_L$ for which the functions of $\mathcal{Z}_h$ are smooth on each subdomain. More precisely, the partition $Part_L$ is defined recursively in the following way. For $L=1$, we set $Part_1 = \{T_{i,j} : i,j = 1,\ldots,N_1-1\}$ with $T_{i,j} := \{(x,y) : \frac{i-1}{N_1-1} < x < \frac{i}{N_1-1}, \frac{j-1}{N_1-1} < y < \frac{j}{N_1-1}\}$. If $Part_{l-1}$ is given, then $Part_l$ is the refinement of $Part_{l-1}$, where only those subdomains are divided into four equal squares for which there exists a non-differentiable function in $\mathcal{Z}_h$ or which are adjacent to P.

To get the refined partition of the quadrature, we leave the squares of $Part_L$ with side length greater than h unchanged and divide the squares with side length h by the same manner as in Subsection 3.1. Finally, we define the composite quadrature rule over this partition by taking the tensor product two point Gauß rule over each subdomain.

Since we have started from the partition $Part_L$ which is much coarser than the uniform partition (3.11), the number of quadrature knots will be smaller than that for the rule in Subsection 3.1. This is the essential point which leads to a reduction in computing time for the assembling of the stiffness matrix.

4 The Numerical Tests

4.1. The Algorithm for the Assembling of the Stiffness Matrix

Since the time for the set up of the matrix is much longer than that for the iterative solution of the matrix equation, we consider the assembling of the matrix in more detail. Instead of the scaled stiffness matrix $A_h = (hA\varphi_J(P_I))_{I,J\in\mathcal{M}}$ we compute $A_h^u := (A\varphi_J(P_I))_{I,J\in\mathcal{M}}$, and instead of the compressed matrix C_h of the scaled wavelet transform $B_h = (\langle\vartheta_I, A\psi_J\rangle)_{I,J\in\mathcal{N}}$ we set up the compressed matrix C_h^u of the matrix $B_h^u := (\langle[3\cdot 2^{l_I}]\vartheta_I, A\psi_J\rangle)_{I,J\in\mathcal{N}}$. In other words, we have to compute the entries $a_{I,J}^u \approx A\varphi_J(P_I)$ with $I,J \in \mathcal{M}$ of the collocation matrix without wavelets by (cf. (3.7))

$$\begin{aligned} a_{I,J}^u &= \sum_{K\in\mathcal{O}} K(P_I,Q_K)\varphi_J(Q_K)w_K + \\ &\quad \left[-\frac{1}{2}\cos[n(P_I),\tau(P_I)] + \int_S H(P_I,Q)d_QS - \sum_{K\in\mathcal{O}} H(P_I,Q_K)w_K\right]\delta_{I,J} \end{aligned} \tag{4.1}$$

and the non-zero entries $c_{I,J}^u \approx \langle[3\cdot 2^{l_I}]\vartheta_I, A\psi_J\rangle$, $I,J\in\mathcal{N}$ of the wavelet algorithm by

$$\begin{aligned} c_{I,J}^u &= \sum_{K\in\mathcal{O}} K(P_I^1,Q_K)\psi_J(Q_K)w_K - \varrho\sum_{K\in\mathcal{O}} K(P_I^2,Q_K)\psi_J(Q_K)w_K \\ &\quad + \left[-\frac{1}{2}\cos[n(P_I^1),\tau(P_I^1)] + \int_S H(P_I^1,Q)d_QS - \sum_{K\in\mathcal{O}} H(P_I^1,Q_K)w_K\right]\psi_J(P_I^1) \end{aligned} \tag{4.2}$$

$$-\varrho\left[-\frac{1}{2}\cos[n(P_I^2),\tau(P_I^2)]+\int_S H(P_I^2,Q)d_QS-\sum_{K\in\mathcal{O}} H(P_I^2,Q_K)w_K\right]\psi_J(P_I^2),$$

where P_I^1, P_I^2 are defined such that $\langle[3\cdot 2^{l_I}]\vartheta_I,f\rangle = f(P_I^1)-\varrho f(P_I^2)$ with $\varrho=0$ for $I=(0,1,i,j)$ and with $\varrho=1$ else (cf. (2.20)). The most time consuming elements of the algorithm are the evaluations of the kernel function which include several Overhauser interpolations and transitions from spherical coordinates to the parameter coordinates over the square S. One computation of a kernel function requires more than twenty times the time of a single evaluation of a wavelet function. Therefore, we have to arrange the do loops in such an order that the number of kernel evaluations is minimised. For the algorithm without wavelets, this leads to the scheme of Algorithm 2. Proceeding analogously, the wavelet algorithm looks as in Algorithm 3. The essential parts in the Algorithms 2 and 3 are the do loops over the quadrature knots Q_k. Comparing these, we observe that the wavelet algorithm contains an additional do loop over the functionals $\vartheta_{\tilde I}$ with $P_I\in\operatorname{supp}\vartheta_{\tilde I}$. In view of this loop, we have chosen the test wavelets $\vartheta_{\tilde I}$ as linear combinations of only two Dirac delta functionals. Hence, for this choice, the loop over the $\vartheta_{\tilde I}$ is in the average over two items. On the other hand, to the do loop over two functions φ_J with $\varphi_J(Q_k)\neq 0$ in Algorithm 2, there corresponds a do loop over more than two functions ψ_J with $\psi_J(Q_k)\neq 0$ in Algorithm 3. The greater the number of vanishing moments, the larger is the support of the wavelets and the larger is the number of ψ_J not vanishing at Q_k. Though the higher compression for wavelets with more vanishing moments reduces the number of ψ_J with $\psi_J(Q_k)\neq 0$ and $J\in\mathcal{N}_U(P_I)$, we believe that the number of vanishing moments should not be too large, and we have chosen two vanishing moments in Subsection 2.2. Nevertheless, the additional do loop over the $\vartheta_{\tilde I}$ and the

do for any collocation point P_I, $I\in\mathcal{M}$
  do for any φ_J with $J\in\mathcal{M}$
    Set the initial value for the matrix entry $a^u_{I,J}$ equal to zero.
  end do
  Compute $d_I := -\frac{1}{2}\cos[n(P_I),\tau(P_I)]+\int_S H(P_I,Q)d_QS$ by (3.8).
  Determine the knots Q_K and the weights w_K, $K\in\mathcal{O}$ of the quadrature
  rule (3.5) in dependence on P_I.
* *
  do for any quadrature knot Q_K with $K\in\mathcal{O}$
    Compute $[H(P_I,Q_K)w_K]$ and $[K(P_I,Q_K)w_K]$.
    Subtract $[H(P_I,Q_K)w_K]$ from d_I.
    Determine the set $\mathcal{M}(Q_K)$ of all indices $J\in\mathcal{M}$ with $\varphi_J(Q_K)\neq 0$.
    do for any φ_J with $J\in\mathcal{M}(Q_K)$
      Compute $\varphi_J(Q_K)$ and $[K(P_I,Q_K)w_K]\varphi_J(Q_K)$.
      Add $[K(P_I,Q_K)w_K]\varphi_J(Q_K)$ to $a^u_{I,J}$.
    end do
  end do
* *
  Add d_I to $a^u_{I,I}$.
end do

Algorithm 2: Matrix computation for the collocation.

Compute the matrix pattern according to (2.42) or Algorithm 1.
do for any collocation point P_I, $I \in \mathcal{M}$
    Determine the set $\mathcal{N}(P_I)$ of $\tilde{I} \in \mathcal{N}$ with $P_I \in \operatorname{supp} \vartheta_{\tilde{I}}$.
    do for any $\vartheta_{\tilde{I}}$ with $\tilde{I} \in \mathcal{N}(P_I)$
        Determine the set $\mathcal{N}(P_I, \tilde{I})$ of $J \in \mathcal{N}$ such that, according to the just determined matrix pattern, there exists a non-zero matrix entry $c^{u}_{\tilde{I},J}$ with $J \in \mathcal{N}$.
        do for any ψ_J with $J \in \mathcal{N}(P_I, \tilde{I})$
            Set the initial value for the matrix entry $c^{u}_{\tilde{I},J}$ equal to zero.
        end do
    end do
    Determine the union $\mathcal{N}_U(P_I)$ of the sets $\mathcal{N}(P_I, \tilde{I})$.
    Compute $d_I := -\frac{1}{2}\cos[n(P_I), \tau(P_I)] + \int_S H(P_I, Q) d_Q S$ by (3.8).
    Determine the knots Q_K and weights w_K, $K \in \mathcal{O}$ of the quadrature rule (3.5) in dependence on P_I and $\mathcal{N}_U(P_I)$.
* *
    do for any quadrature knot Q_K with $K \in \mathcal{O}$
        Compute $[H(P_I, Q_K) w_K]$ and $[K(P_I, Q_K) w_K]$.
        Subtract $[H(P_I, Q_K) w_K]$ from d_I.
        Determine the set $\mathcal{N}_Q(Q_K)$ of all indices $J \in \mathcal{N}_U(P_I)$ with $\psi_J(Q_K) \neq 0$.
        do for any ψ_J with $J \in \mathcal{N}_Q(Q_K)$
            Compute $\psi_J(Q_K)$ and $[K(P_I, Q_K) w_K] \psi_J(Q_K)$.
            do for any $\vartheta_{\tilde{I}}$ with $\tilde{I} \in \mathcal{N}(P_I)$
                if $J \in \mathcal{N}(P_I, \tilde{I})$ *then*
                    Add $\pm [K(P_I, Q_K) w_K] \psi_J(Q_K)$ to $c^{u}_{\tilde{I},J}$.
                end if
            end do
        end do
    end do
* *
    Determine the set $\mathcal{N}_A(P_I)$ of $J \in \mathcal{N}$ with $\psi_J(P_I) \neq 0$.
    do for any ψ_J with $J \in \mathcal{N}_A(P_I)$
        Compute $[d_I \psi_J(P_I)]$.
        do for any $\vartheta_{\tilde{I}}$ with $\tilde{I} \in \mathcal{N}(P_I)$
            Add $\pm [d_I \psi_J(P_I)]$ to $c^{u}_{\tilde{I},J}$.
        end do
    end do
end do

Algorithm 3: Matrix computation for the wavelet approach.

longer do loop over the ψ_J suggest longer computing times for the wavelet algorithm. The only way for Algorithm 3 to be faster than Algorithm 2 is that the number of quadrature knots is less for Algorithm 3. Indeed, for high compression rates, the number of knots will be essentially reduced for the wavelet algorithm and, hence, a fewer number of time consuming kernel evaluations will reduce the overall computing time.

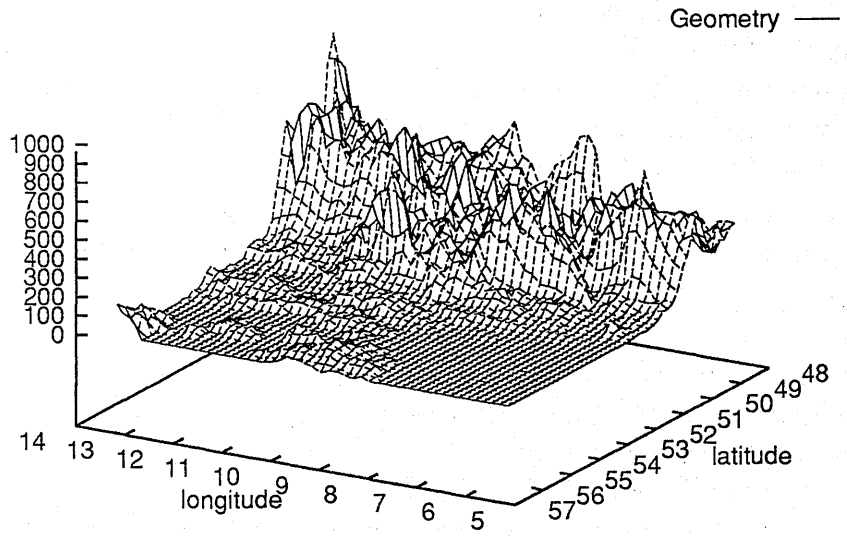

Figure 5: Height depending on latitude and longitude.

Now we make a remark on how to store the matrices. For a computation on a work station, it is important to take into account the number of I/O operations between CPU and main memory. In case of Algorithm 2, the fully populated matrix A_h can be stored as a two dimensional field. For Algorithm 3, the non-zero entries of C_h are stored in a one-dimensional field and their indices in extra fields. Doing so, we frequently have to access to entries from C_h stored at different places of the one-dimensional field and, consequently, the ratio of I/O time per time for arithmetic operations is much worse than for Algorithm 2. To improve this ratio, we recommend to introduce an additional two-dimensional field which, for a fixed collocation point P_I, stores the values for the entries of C_h depending on $\tilde{I} \in \mathcal{N}(P_I)$ and on $J \in \mathcal{N}$ (cf. Algorithm 3 and recall that $\mathcal{N}(P_I)$ contains no more than $2L+3$ indices.). In the do loop over the quadrature knots, we sum up the values of the entries and put them into the two-dimensional field. After the loop, we write the values into the one-dimensional array and store the indices in extra fields.

Finally, let us note that, to reduce the computing time for the evaluation of wavelet functions, the scaling factors $\sqrt{3 \cdot 2^l}$ (cf. (2.7)-(2.8)) should be dropped and the recursive structure of the function system should be applied. Namely, knowing the values of the two univariate scaling functions $\varphi_j^{N_l}$ and $\varphi_{j+1}^{N_l}$ and of the two univariate wavelet functions $\psi_{l,j'}$ and $\psi_{l,j'+1}$ of level l not vanishing at a certain point Q_k, it takes only a few number of operations to compute the values of the level $l-1$ scaling and wavelet functions not vanishing at Q_k.

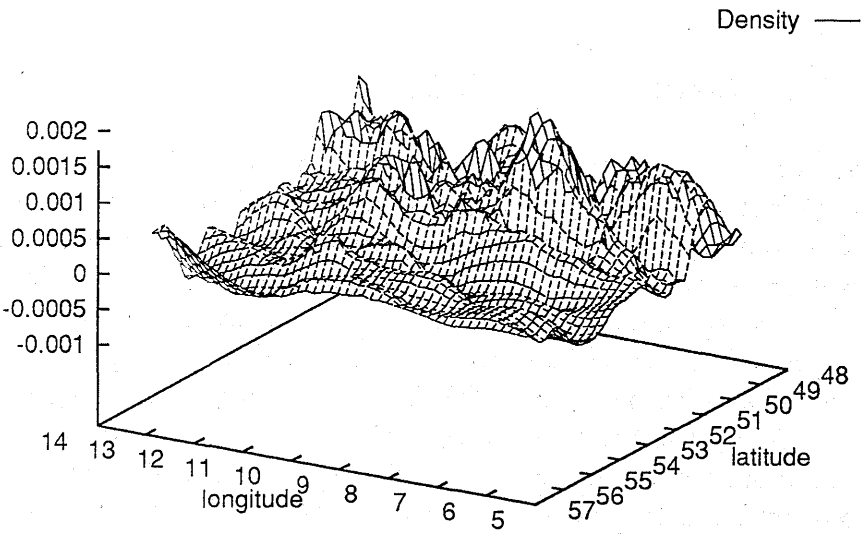

Figure 6: Density depending on latitude and longitude.

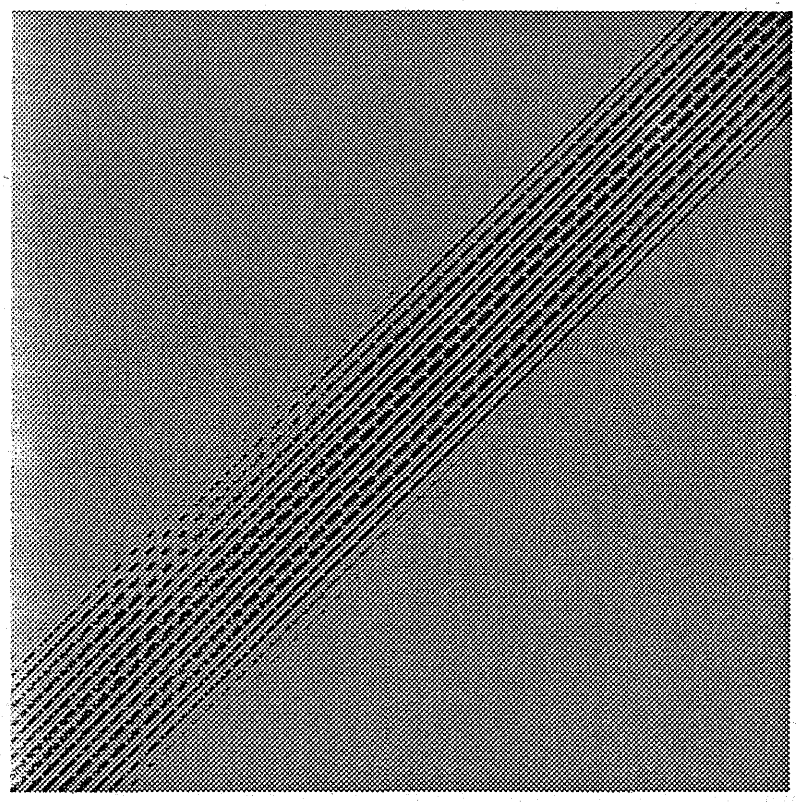

Figure 7: Matrix pattern of the stiffness matrix A_h for the collocation without wavelets.

L	1	2	3	4
err_Q	$8.8 \cdot 10^{-8}$	$1.2 \cdot 10^{-8}$	$4.0 \cdot 10^{-9}$	$1.4 \cdot 10^{-9}$

Table 1: Quadrature error for the collocation without wavelets.

4.2. Numerical Results of the Tests

For the numerical tests, we have taken a rectangular part Γ of the earth's surface containing Northern Germany (cf. Figure 5). The reference potential including the centrifugal potential is computed in a preprocessing step from an expansion into spherical harmonics up to degree/order 10. We have solved equation (1.5) by collocation with $N = N_L = 3 \cdot 2^L + 1$ and $L = 0, \ldots, 5$. The approximate density function $\tilde{u}_h$ (cf. (1.7)) is given in Figure 6. In order to check the quality of the quadrature rules from Subsection 3.1, we have computed another solution u_h^+ by halving the size of the underlying quadrature mesh and by refining the subdomains $S_{i,j}$, $i, j \geq 0$ in x direction with additional grid points of a polynomially graded mesh (cf. Subsection 5.1). We have obtained the estimates for the quadrature errors $err_Q := \sup |u_h(P_I) - u_h^+(P_I)|$ given in Table 1. The supremum in the definition of err_Q is taken over four different points, two fixed in the interior of S and two as close as possible to corners of ∂S. Note that we are dealing with errors for values with a size of about $5 \cdot 10^{-4}$. These estimates show that the quadrature rules of Subsection 3.1 are satisfactory or even too good. The quadrature rules of Subsection 3.2 for the wavelet algorithm are also quite accurate since replacing them by those of Subsection 3.1 has not led to a reduction of the quadrature error.

Unfortunately, we have no good estimate for the discretisation errors of the collocation method (1.9). We believe them to be much larger than the quadrature errors. Since discretisation errors play the role of thresholds for the compression errors, we somehow have to choose an estimate. In view of a comment in [19] on the Galerkin discretisation error over a different domain, we assume a relative discretisation error in the supremum norm of $4^{5-L} \cdot 10^{-5}$ for the collocation solution u_h with $h = \frac{1}{N_L - 1}$, $N_L = 3 \cdot 2^L + 1$. In other words, we suppose an error 10^{-5} for $L = 5$ and determine the errors for the lower levels by assuming a second order convergence rate. Now let us have a look at the size of the entries in the stiffness matrices A_h^u and B_h^u (cf. Subsection 4.1), respectively. In the Figures 7 and 8 we have marked the entries greater than 10^{-4} by black pixels and those less than 10^{-4} but greater than 10^{-5} by grey ones. These pictures obtained for $N = N_4$ clearly demonstrate that the wavelet transform B_h^u contains much more small entries than A_h^u. In other words, B_h^u seems to be appropriate for a compression.

We have implemented the a priori compression (2.42), and the constant a has been chosen by experiment such that the additional error due to the compression is less than the estimate for the discretisation error. However, instead of the supremum norm we have taken the maximum over the four points used for the computation of err_Q. The values for the optimal constants a and the compression rates rat (rat = number of all entries divided by the number of non-zero entries in C_h) are presented in Table 2 (cf. Figure 9). By $knot_C$ we have denoted the sum $\sum_P \#\mathcal{O}$ of the numbers $\#\mathcal{O}$ of all quadrature knots in the rule of Subsection 3.1 depending on the collocation point P. Similarly, by $knot_W$ we have denoted the number of all quadrature knots in all the rules of Subsection 3.2. These numbers are also the numbers of evaluations of kernel functions in the Algorithms

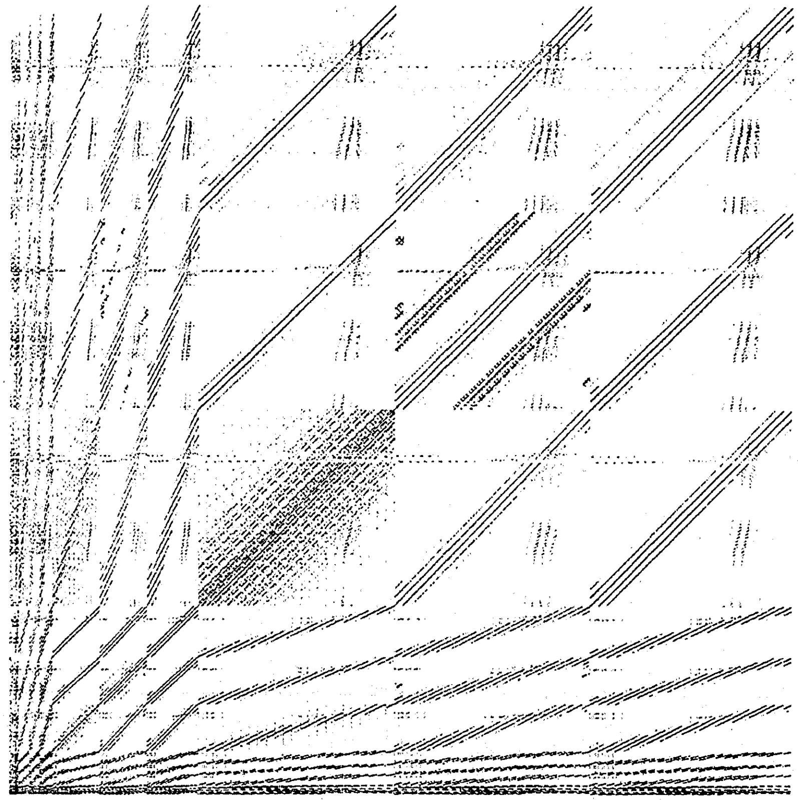

Figure 8: Matrix pattern of the stiffness matrix B_h, i.e., of the wavelet transform of A_h.

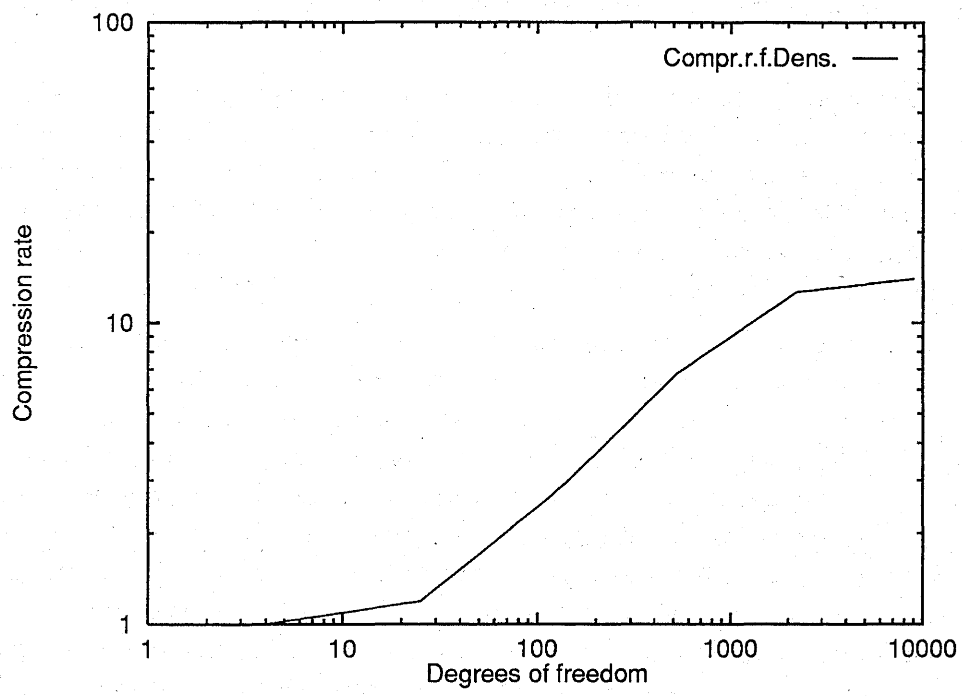

Figure 9: Compression rates of the wavelet algorithm depending on the degrees of freedom $(N_L - 2)^2$ for the approximation of the density function.

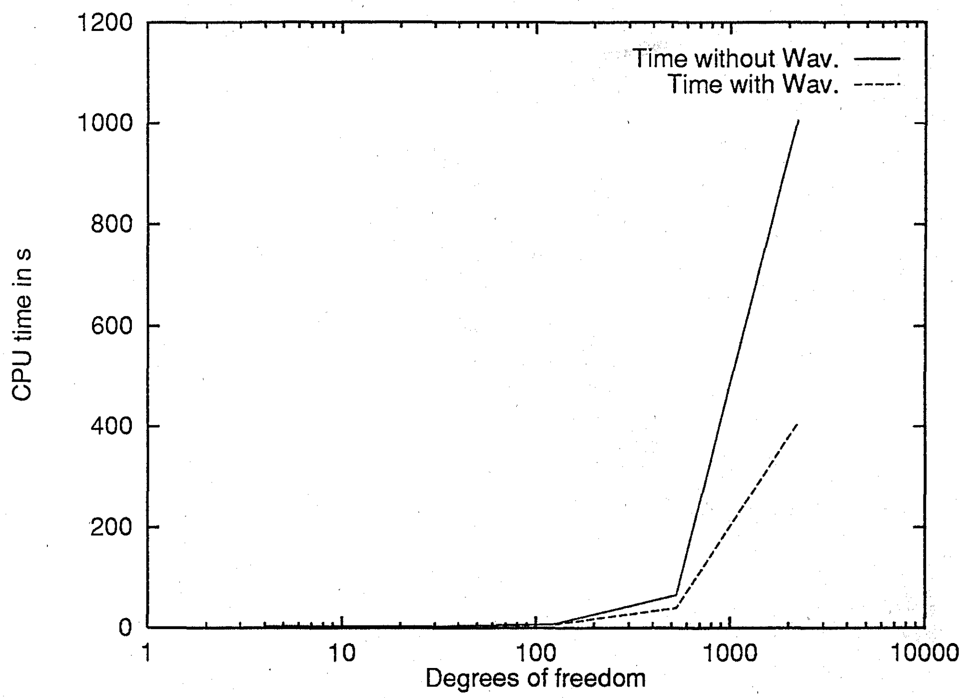

Figure 10: Computing times depending on the degrees of freedom $(N_L - 2)^2$ for the approximation of the density function.

2 and 3, respectively. The last two rows in Table 2 contain the computing times tim_C for Algorithm 2 without wavelets and tim_W for the wavelet algorithm Algorithm 3 (cf. Figure 10).

We note that the storage capacity of 96 MB of our work station was not sufficient for solving (1.5) on level $L = 5$ without wavelets. The introduction of wavelets admits compressions of the matrix up to 7% for $N = N_5$. Of course, the saving of storage is a little bit less since additional working space is required. The smaller numbers of kernel evaluations lead to a reduction in the computing time from $tim_C = 1\,005s$ to $tim_W = 405s$ for $N = N_4$ and from estimated $tim_C = 15\,447s$ to $tim_W = 5\,580s$ for $N = N_5$.

The compression rates and computing times depend strongly on the threshold for the compression. In a second experiment we have computed a directional derivative of the potential δw_h from (1.7) over eight points placed in a height of 0.01m, 0.1m, 1m, 10m, 100m, 1 000m, 10 000m, and 100 000m over the surface point with longitude 9.333 and latitude 52.666. The parameter a in the compression has been chosen such that the relative compression errors for the resulting eight derivatives of the potential are less than $4^{5-L} \cdot 10^{-5}$. The values a, the compression rates rat, the numbers of quadrature knots $knot_C$, $knot_W$, as well as the computation times tim_C, tim_W are presented in Table 3 (cf. also Figures 11 and 12). It has turned out that, for the computation of the potential far from the points of the boundary ∂S, much higher compression rates are possible. The same is likely to be true for the determination of the density u_h over a part of S which has a positive distance to ∂S.

Finally, we remark that, for $N = N_5$ and a compression not affecting the supremum norm

L	0	1	2	3	4	5
N_L	4	7	13	25	49	97
$(N_L-2)^2$	4	25	121	529	2209	9025
$a \cdot 2^L$	0	0	0	0.1	0.5	1.4
rat	1	1.19	2.69	6.82	12.64	13.99
$knot_C$	464	8000	125232	1948096	31064672	
$knot_W$	464	8000	100104	1054664	10694100	134108232
tim_C	2.80s	2.92s	6.64s	65.49s	1005.81s	
tim_W	2.90s	3.08s	6.06s	39.55s	405.17s	5580.07s

Table 2: Compression rates, number of quadrature knots, and computation time for the algorithms without and with wavelets. Approximation of the density function.

error of the density function, we have obtained the higher compression rate 23.03 using the oracle scheme (cf. Algorithm 1 of Subsection 2.4). The computation took a time of 6981s from which 1090s were spent only for the determination of the matrix pattern. In other words, we recommend to take the a priori scheme. The oracle scheme is preferable only if the saving of storage is more important than the saving of time and if a high accuracy is required.

All the computations have been performed on a DEC 3000 AXP 400 α- processor work station.

L	0	1	2	3	4	5
N_L	4	7	13	25	49	97
$(N_L-2)^2$	4	25	121	529	2209	9025
$a \cdot 2^L$	0	0	0	0	0.2	0.5
rat	1	1.19	2.69	6.83	19.44	36.90
$knot_C$	464	8000	125232	1948096	31064672	
$knot_W$	464	8000	100104	1054664	9435704	90949376
tim_C	2.90s	3.13s	7.41s	68.33s	1017.03s	
tim_W	2.90s	3.09s	6.06s	39.55s	351.63s	3641.05s

Table 3: Compression rates, number of quadrature knots, and computation time for the algorithms without and with wavelets. Approximation of the potential.

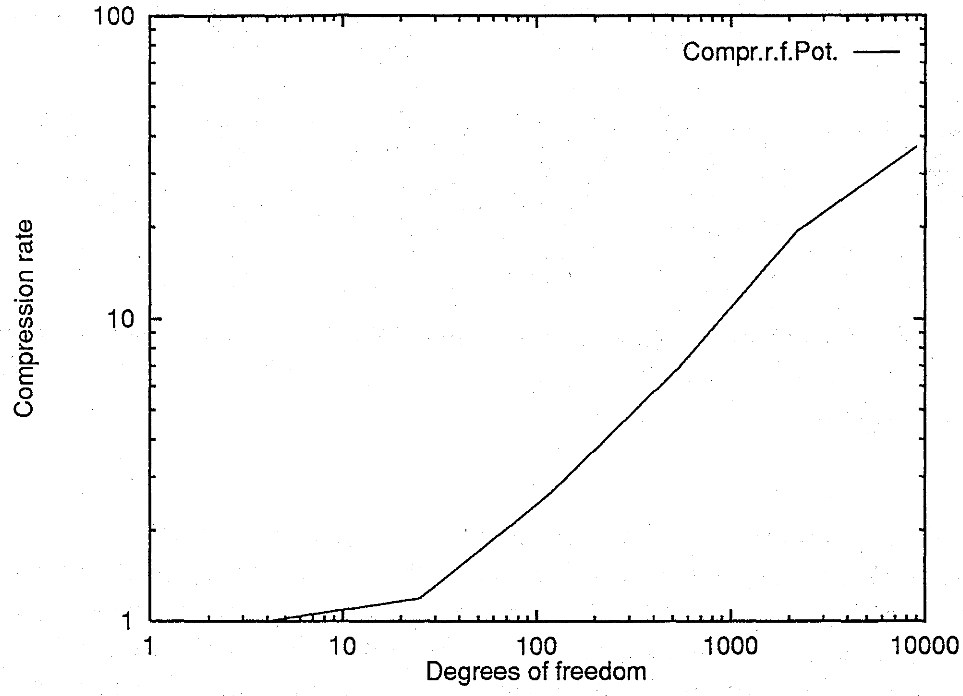

Figure 11: Compression rates of the wavelet algorithm depending on the degrees of freedom $(N_L - 2)^2$ for the approximation of the potential.

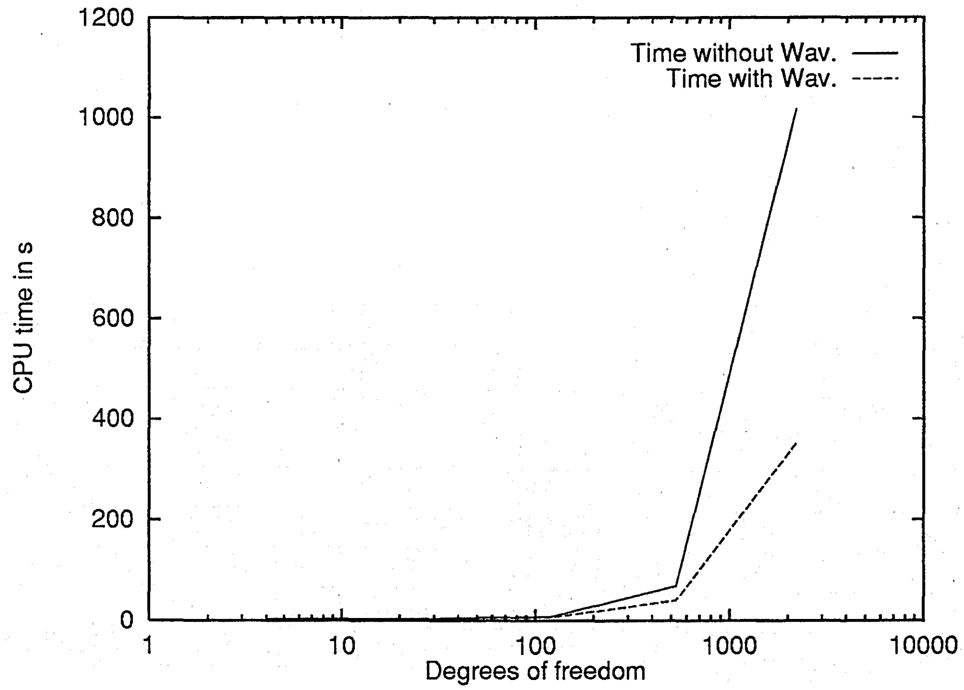

Figure 12: Computing times depending on the degrees of freedom $(N_L - 2)^2$ for the approximation of the potential.

5 Asymptotic Quadrature Estimates

5.1. An Asymptotic Estimate for a Quadrature Algorithm Applied to Collocation without Wavelets in Case of a Model Operator

In order to simplify formulae and proofs, let us consider a model operator M instead of A. Since we do not want to treat singularities at the boundary of the domain S, we consider the torus $\mathbb{T}^2$ instead of S, i.e., we set $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ and identify $\mathbb{T}^2$ with the product of the 1-periodic interval $[-\frac{1}{2}, \frac{1}{2}]$ by itself. We introduce M as the sum of a multiplication operator and a convolution operator, i.e., we set

$$\begin{aligned} Mu(P) &:= a_M(P)u(P) + p.v. \int_{\mathbb{T}^2} K_M(P,Q)u(Q) d_Q\mathbb{T}^2 \\ &:= a_M(P)u(P) + p.v. \int_{x_P-\frac{1}{2}}^{x_P+\frac{1}{2}} \int_{y_P-\frac{1}{2}}^{y_P+\frac{1}{2}} K_M(P,(x_Q,y_Q))u((x_Q,y_Q)) dy_Q dx_Q, \end{aligned} \tag{5.1}$$

where $P = (x_P, y_P) \in \mathbb{T}^2$, a_M is at least twice continuously differentiable, and the convolution kernel K_M takes the form

$$K_M(P,Q) := \Phi\left(\frac{Q-P}{|Q-P|}\right)|Q-P|^{-2}, \quad P,Q \in \mathbb{R}^2. \tag{5.2}$$

The function Φ defined on the unit circle S^1 in the complex plane is supposed to be twice continuously differentiable and to admit an expansion into a trigonometric series of the form

$$\Phi(e^{i\lambda}) = \sum_{k=1}^{\infty} a_k \sin(k\lambda) + \sum_{k=1}^{2} b_k \cos(k\lambda). \tag{5.3}$$

Note that $\Phi(\theta) = -\Phi(-\theta)$ is sufficient for (5.3). The condition (5.3) implies

$$\int_{S^1} \Phi(\theta) d_\theta S^1 = 0 \tag{5.4}$$

and, hence, the existence of the principal value integral in (5.1) (cf. [20]). Moreover, (5.3) implies

$$\int_{-a}^{a}\int_{-a}^{a} \Phi\left(\frac{(x,y)}{\sqrt{x^2+y^2}}\right)\frac{1}{x^2+y^2} dy dx = 0, \quad a > 0 \tag{5.5}$$

which will be important for the next proof. In addition, we assume that the model operator M is invertible in $L^2(\mathbb{T}^2)$ and strongly elliptic.

Analogously to the collocation for operator A, we can introduce the bilinear collocation to the numerical solution of equation $Mu = v$. The necessary modification due to the change of the underlying domain is straightforward. For instance, we have to set $\mathcal{M} := \{(i,j) : i,j = 0,\dots,N_L - 2\}$ and $P_{(i,j)} := \left(\frac{i}{N_L-1}, \frac{j}{N_L-1}\right) \in \mathbb{T}^2$ for $(i,j) \in \mathcal{M}$. The trial functions

$\varphi_{i,j}$ are defined by formula (1.8) over $\left(\frac{i-1}{N_L-1}, \frac{i+1}{N_L-1}\right) \times \left(\frac{j-1}{N_L-1}, \frac{j+1}{N_L-1}\right)$, extended by zero to $\left(\frac{i}{N_L-1} - \frac{1}{2}, \frac{i}{N_L-1} + \frac{1}{2}\right) \times \left(\frac{j}{N_L-1} - \frac{1}{2}, \frac{j}{N_L-1} + \frac{1}{2}\right)$, and, from this, extended to a 1-periodic function. We arrive at a collocation method seeking $u_h \in \mathcal{X}_h := \text{span}\{\varphi_I : I \in \mathcal{M}\}$ which satisfies $Mu_h(P_J) = v(P_J)$ for $J \in \mathcal{M}$. Analogously to A_h, we denote the corresponding stiffness matrix $(M\varphi_I(P_J))_{J,I\in\mathcal{M}}$ by M_h and identify it with the corresponding operator in $\mathcal{L}(\mathcal{X}_h)$.

Next we introduce a quadrature approximation of M_h. Following Subsection 3.1 and using (5.5), we write

$$\begin{aligned} & p.v. \int_{x_P-\frac{1}{2}}^{x_P+\frac{1}{2}} \int_{y_P-\frac{1}{2}}^{y_P+\frac{1}{2}} K_M(P,(x_Q,y_Q))u_h(Q)dy_Q dx_Q \qquad (5.6) \\ = & \int_{x_P-\frac{1}{2}}^{x_P+\frac{1}{2}} \int_{y_P-\frac{1}{2}}^{y_P+\frac{1}{2}} K_M(P,(x_Q,y_Q))[u_h(Q)-u_h(P)]dy_Q dx_Q \\ \sim & \sum_{K\in\mathcal{O}} K_M(P,Q_K)[u_h(Q_K)-u_h(P)]w_K \\ = & -\left\{\sum_{K\in\mathcal{O}} K_M(P,Q_K)w_k\right\} u_h(P) + \sum_{K\in\mathcal{O}} K_M(P,Q_K)u_h(Q_K)w_k, \end{aligned}$$

where the rule

$$\int_{x_P-\frac{1}{2}}^{x_P+\frac{1}{2}} \int_{y_P-\frac{1}{2}}^{y_P+\frac{1}{2}} f((x,y))dydx \sim \sum_{K\in\mathcal{O}} f(Q_K)w_K \qquad (5.7)$$

depending on the collocation point P will be specified later. In any case the quadrature rule will be symmetric with respect to P in the sense that, if Q_K is a quadrature knot with weight w_K, then also the points defined by the operations $P-(Q_K-P)$, $P-\sqrt{-1}(Q_K-P)$, and $P+\sqrt{-1}(Q_K-P)$ in the field of complex numbers are knot points with the same weight. This kind of symmetry together with assumption (5.3) implies

$$\sum_{K\in\mathcal{O}} K_M(P,Q_K)w_K = 0. \qquad (5.8)$$

The quadrature step (5.6) leads us to an approximate matrix $M_h' := (hM'\varphi_I(P_J))_{J,I\in\mathcal{M}}$ for the stiffness matrix M_h, where

$$M'\varphi_I(P_J) := a_M(P_J)\delta_{J,I} + \sum_{K\in\mathcal{O}} K_M(P_J,Q_K)\varphi_I(Q_K)w_K. \qquad (5.9)$$

The quadrature rule (5.7) employed in (5.9) will be defined as follows. We start with the underlying partition and, similarly to Subsection 3.1, we give this partition only over

$$S' = [x_P - \frac{1}{2}, x_P + \frac{1}{2}] \times [y_P - \frac{1}{2}, y_P + \frac{1}{2}] \cap \{(x,y) : x_P \le x,\, 0 \le y - y_P \le x - x_P\}.$$

Without loss of generality we may suppose $P = (x_P, y_P) = (0,0)$. Now we choose a parameter $q > 1$ and start with the partitioning in x-direction. We would like to take a graded partition of the type $\{[ih]^q : i = 1, \ldots\}$. However, in accordance with point ii) of

Subsection 3.1, we need a refinement of the uniform partition $\{ih : i = 1, \ldots\}$. Thus, we introduce the partition $0 = z_0 < z_1 < \ldots < z_m = \frac{1}{2}$ of $[0, \frac{1}{2}]$ by

$$\{z_i : i = 0, \ldots, m\} := \left\{ih : i = 0, 1, \ldots, \frac{N_L - 1}{2}\right\} \cap \{[ih]^q : i = 0, 1, \ldots, [2^{-1/q}h^{-1}]\}$$

and arrive at a partition into m strips.

$$S' \subseteq \cup_{i=1}^{m} \left\{(x, y) \in [-\frac{1}{2}, \frac{1}{2}]^2 : z_{i-1} \leq x \leq z_i,\ 0 \leq y \leq z_i\right\}.$$

Now we turn to the subdivision in y-direction. In the strip $\{(x, y) \in [-\frac{1}{2}, \frac{1}{2}]^2 : z_{i-1} \leq x \leq z_i,\ 0 \leq y \leq z_i\}$ the partitioning should be uniform with step size less than $z_i - z_{i-1}$. Again, we have to take care of point ii) of Subsection 3.1. Consequently, we introduce the partition $0 = \zeta_{i,0} < \zeta_{i,1} < \ldots < \zeta_{i,m_i-1} = z_{i-1} < \zeta_{i,m_i} = z_i$, where

$$\begin{aligned} \{\zeta_{i,j} : j = 1, \ldots, m_i - 1\} &= \{jh \in [0, z_{i-1}] : j = 1, \ldots\} \\ &\quad \cup \{j[z_i - z_{i-1}] \in [0, z_{i-1}] : j = 1, \ldots\}. \end{aligned} \tag{5.10}$$

Finally, we arrive at the partition

$$\begin{aligned} S' &\subseteq \cup_{i=1}^{m} \cup_{j=1}^{m_i} S^{i,j}, \\ S^{i,j} &:= \left\{(x, y) \in [-\frac{1}{2}, \frac{1}{2}]^2 : z_{i-1} \leq x \leq z_i,\ \zeta_{i,j-1} \leq y \leq \zeta_{i,j}\right\}. \end{aligned} \tag{5.11}$$

Now the rule (5.7) is the composite quadrature rule, where the mid-point rule is taken over each subdomain $S^{i,j}$ and over the corresponding subdomains of the parts of $\mathbb{T}^2$ defined similarly to S'.

Theorem 5.1 *The quadrature approximation M_h' of M_h is stable. If the solution u of $Mu = v$ is twice continuously differentiable and if $u_h' \in \mathcal{X}_h$ is the approximate solution defined by $M_h' u_h' = \{hv(P_I)\}_{I \in \mathcal{M}}$, then there exists a constant C independent of h such that*

$$\|u - u_h'\|_{L^2} \leq C \begin{cases} h^2 & \text{if } q > 2 \\ h^2 \log h^{-1} & \text{if } q = 2 \\ h^q & \text{if } q < 2. \end{cases} \tag{5.12}$$

Proof. a) First we suppose M_h' is stable, i.e., $\|[M_h']^{-1}\|_{\mathcal{L}(\mathcal{X}_h)}$ is uniformly bounded. Let Q_h stand for the interpolation projection onto $\mathcal{X}_h \subseteq L^2(\mathbb{T}^2)$ defined by $Q_h f := \sum_{I \in \mathcal{M}} hf(P_I)\varphi_I$ and set $\tilde{u}_h := Q_h u$, $v_h := Q_h v = Q_h M u$. Then we have

$$\begin{aligned} u - u_h' &= u - Q_h u + \\ &\quad [M_h']^{-1}\{M_h'\tilde{u}_h - Q_h M \tilde{u}_h + (Q_h - I)M(Q_h - I)u + M(Q_h - I)u\}, \\ \|u - u_h'\|_{L^2} &\leq C\|u - Q_h u\|_{L^2} + C\|M_h'\tilde{u}_h - Q_h M\tilde{u}_h\|_{L^2} + \|(Q_h - I)M(Q_h - I)u\|_{L^2}. \end{aligned} \tag{5.13}$$

Using the second order estimate $\|f - Q_h f\|_{L^2(\mathbb{T}^2)} \leq Ch^2\|f\|_{H^2(\mathbb{T}^2)}$ and the uniform boundedness of the projections Q_h in $H^2(\mathbb{T}^2)$, we arrive at

$$\begin{aligned} \|u - u_h'\|_{L^2} &\leq Ch^2\|u\|_{H^2} + Ch^2\|M(Q_h - I)u\|_{H^2} + C\|M_h'\tilde{u}_h - Q_h M\tilde{u}_h\|_{L^2}. \\ &\leq Ch^2 + C\|M_h'\tilde{u}_h - Q_h M\tilde{u}_h\|_{L^2}. \end{aligned} \tag{5.14}$$

It suffices to estimate $\|M'_h\tilde{u}_h - Q_hM\tilde{u}_h\|_{L^\infty}$, i.e., to estimate $|M'_h\tilde{u}_h(P) - M\tilde{u}_h(P)|$ for $P \in \{P_J,\ J \in \mathcal{M}\}$. Without loss of generality, we suppose $P = (0,0)$ and, in view of (5.6), we get

$$\begin{aligned} M\tilde{u}_h(P) - M'_h\tilde{u}_h(P) &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} K_M(P,Q)[\tilde{u}_h(Q) - \tilde{u}_h(P)]dy_Q dx_Q \\ &\quad - \sum_{K\in\mathcal{M}} K_M(P,Q_K)[\tilde{u}_h(Q_K) - \tilde{u}_h(P)]w_K. \end{aligned} \tag{5.15}$$

To estimate this quadrature error, we use the formula

$$\begin{aligned} &\left|\int_a^b\int_c^d f(x,y)dydx - f\left(\frac{a+b}{2},\frac{c+d}{2}\right)[b-a][d-c]\right| \leq \\ &C[b-a]^3[d-c]\sup_{\substack{a<x<b\\c<y<d}}\left|\frac{\partial^2}{\partial x^2}f(x,y)\right| + C[b-a][d-c]^3\sup_{\substack{a<x<b\\c<y<d}}\left|\frac{\partial^2}{\partial y^2}f(x,y)\right| \end{aligned} \tag{5.16}$$

on each subdomain $S^{i,j}$ of the partition (5.11). For the derivatives of the integrand $f(Q) = K_M(P,Q)[\tilde{u}_h(Q) - \tilde{u}_h(P)]$, we apply the simple estimates

$$\begin{aligned} &\left|\left(\frac{\partial}{\partial x}\right)^{\alpha_1}\left(\frac{\partial}{\partial y}\right)^{\alpha_2} K_M(P,(x,y))\right| = \\ &\left|\left(\frac{\partial}{\partial x}\right)^{\alpha_1}\left(\frac{\partial}{\partial y}\right)^{\alpha_2}\left\{\Phi_M\left(\frac{(x,y)-P}{|(x,y)-P|}\right)|(x,y)-P|^{-2}\right\}\right| \leq C|(x,y)-P|^{-2-\alpha_1-\alpha_2}, \\ &\qquad\qquad \text{if } 0 \leq \alpha_1+\alpha_2 \leq 2, \end{aligned} \tag{5.17}$$

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha_1}\left(\frac{\partial}{\partial y}\right)^{\alpha_2}[\tilde{u}_h((x,y)) - \tilde{u}_h(P)]\right| \leq C\begin{cases} |(x,y)-P| & \text{if } \alpha_1=\alpha_2=0 \\ 1 & \text{if } \alpha_1<2,\ \alpha_2<2, \\ & \text{and } 0<\alpha_1+\alpha_2 \\ 0 & \text{else.}\end{cases} \tag{5.18}$$

Thus, the quadrature error of (5.15) over $S^{i,j}$ with $i > 1$ is less than

$$\begin{aligned} &C(z_i - z_{i-1})^3(\zeta_{i,j} - \zeta_{i,j-1})|P - (z_{i-1},\zeta_{i,j-1})|^{-3} \\ \leq\ &C(z_i - z_{i-1})^3|z_{i-1} + \zeta_{i,j-1}|^{-3}(\zeta_{i,j} - \zeta_{i,j-1}). \end{aligned} \tag{5.19}$$

Summing up this estimate for $j = 1,\ldots,m_i$ and using

$$\sum_{j=1}^{m_i}|z_{i-1} + \zeta_{j-1}|^{-3}(\zeta_{i,j} - \zeta_{i,j-1}) \leq C\int_0^\infty [z_{i-1}+y]^{-3}dy \leq Cz_{i-1}^{-2}, \tag{5.20}$$

we arrive at the estimate $C(z_i - z_{i-1})^3z_{i-1}^{-2}$ for the quadrature error over $\cup_{j=1}^{m_i}S^{i,j}$. For the error over $\cup_{i=2}^{m}\cup_{j=1}^{m_i}S^{i,j}$, we get the bound

$$\begin{aligned} C\sum_{i=2}^{m}(z_i - z_{i-1})^3z_{i-1}^{-2} &\leq C\sum_{i=2}^{\frac{N_L-1}{2}}\left([ih]^q - [(i-1)h]^q\right)^3\left([(i-1)h]^q\right)^{-2} \\ &\leq C\begin{cases} h^2 & \text{if } q>2 \\ h^2\log h^{-1} & \text{if } q=2 \\ h^q & \text{if } q<2. \end{cases} \end{aligned} \tag{5.21}$$

The quadrature error over $S^{1,1}$ is less than

$$C\int_{S^{1,1}} |Q|^{-1} dy_Q dx_Q + Ch^q h^q \left[\frac{h^q}{2}\right]^{-1} \le Ch^q. \tag{5.22}$$

Formulae (5.21) and (5.22) prove the convergence estimate (5.12).

b) Now we turn to the question of stability. We estimate the quadrature error for the matrix entries separately, i.e., we set $u_h = \varphi_I$ in (5.6). First consider $a_{J,I} = hM\varphi_I(P_J)$ with $P = P_J = (0,0)$ not in the support of φ_I. We have to estimate

$$h\int K_M(P,Q)\varphi_I(Q)d_Q\mathbb{T}^2 - h\sum_{K\in O} K_M(P,Q_K)\varphi_I(Q_K)w_K. \tag{5.23}$$

Using the exactness of the quadrature formula on the set of trial functions, we can write the last expression as

$$h\sum_i\sum_j\int_{S^{i,j}}[K_M(P,Q) - K_M(P,Q_K)]\varphi_I(Q)d_Q\mathbb{T}^2. \tag{5.24}$$

Note that we may suppose $i > 1$ since $P = (0,0)$ is not in the support of φ_I. The expression (5.24) is less than

$$C\sup_{i>1,j\ge1}\sup_{Q\in S^{i,j}}\frac{|K_M(P,Q) - K_M(P,Q_K)|}{|P-Q|^{-2}}\, h\int_{\mathbb{T}^2}|P-Q|^{-2}\varphi_I(Q)d_Q\mathbb{T}^2. \tag{5.25}$$

Now it is not hard to see that $h\int|P-\cdot|^{-2}\varphi_I$ is less than $|(i_I,j_I)-(i_J,j_J)|^{-2} = \{(i_I - i_J)^2 + (j_I - j_J)^2\}^{-1}$ for $I=(i_I,j_I)$ and $P = P_J$, $J=(i_J,j_J)$. Hence, by Young's inequality for convolution operators, the norm of the matrix $((1-\delta_{J,I})h\int|P-\cdot|^{-2}\varphi_I)_{J,I}$ is less than

$$C\sum_{|i|\le N_L}\sum_{|j|\le N_L}\frac{1}{i^2+j^2} \le C\log N_L \le C\log h^{-1}. \tag{5.26}$$

On the other hand, from (5.17) we infer

$$\begin{aligned}\sup_{Q\in S^{i,j}}\frac{|K_M(P,Q) - K_M(P,Q_K)|}{|P-Q|^{-2}} &\le C\sup_{Q\in S^{i,j}}\frac{|\nabla K_M(P,Q)|\cdot|Q-Q_K|}{|P-Q|^{-2}}\\ &\le C\sup_{Q\in S^{i,j}}\frac{|Q-Q_K|}{|P-Q|}.\end{aligned} \tag{5.27}$$

Now we observe that $P=(0,0)\notin \operatorname{supp}\varphi_I$ and that we can restrict the consideration to $S^{i,j}\subseteq\operatorname{supp}\varphi_I$. Consequently, we may suppose $[0,h]^2\cap S^{i,j}=\emptyset$ and, hence (cf. (5.11)), $S^{i,j}\subseteq\{(x,y): z_{i-1}\le x\le z_i\}$ such that $[z_{i-1},z_i]\subseteq[[(k-1)h]^q,[kh]^q]$ and $[(k-1)h]^q\ge h$. We get $k\ge h^{1/q-1}$ and

$$\sup_{Q\in S^{i,j}}\frac{|K_M(P,Q) - K_M(P,Q_K)|}{|P-Q|^{-2}} \le C\frac{[kh]^q - [(k-1)h]^q}{[(k-1)h]^q} \le C\frac{1}{k} \le Ch^{1-\frac{1}{q}}. \tag{5.28}$$

In other words, by the estimates (5.25), (5.26), and (5.28), the matrix of the quadrature errors corresponding to the entries $a_{J,I}$ satisfying $P_J\notin\operatorname{supp}\varphi_I$ has norm less than $\log h^{-1}\, h^{1-1/q}$ and tends to 0 for $h\longrightarrow 0$.

Now consider the case of entries $a_{J,I}$ for which $P_J \in \operatorname{supp} \varphi_I$. On $S^0 := [0,h]^2$ we have to estimate

$$h \int_{S^0} K_M(P,Q)[\varphi_I(Q) - \varphi_I(P)] d_Q \mathbb{T}^2 - h \sum_{K \in \mathcal{O}: Q_K \in S^0} K_M(P,Q_K)[\varphi_I(Q_K) - \varphi_I(P)] w_K \tag{5.29}$$

instead of (5.23). In view of

$$\left| \left(\frac{\partial}{\partial x}\right)^{\alpha_1} \left(\frac{\partial}{\partial y}\right)^{\alpha_2} [\varphi_I((x,y)) - \varphi_I(P)] \right| \leq Ch^2 \begin{cases} |(x,y) - P| & \text{if } \alpha_1 = \alpha_2 = 0 \\ 1 & \text{if } \alpha_1 < 2,\ \alpha_2 < 2, \\ & \quad \text{and } 0 < \alpha_1 + \alpha_2 \\ 0 & \text{else,} \end{cases}$$

the difference (5.29) can be shown to be less than $Ch^{q-1} \log h^{-1}$ analogously to part a) of this proof. For the analogous estimation over $\mathbb{T}^2 \setminus S^0$, we have to consider (5.23) over $\mathbb{T}^2 \setminus S^0$ and

$$\left\{ h \int_{\mathbb{T}^2 \setminus S^0} K_M(P,Q) d_Q \mathbb{T}^2 - h \sum_{K \in \mathcal{O}: Q_K \notin S^0} K_M(P,Q_K) w_K \right\} \varphi_I(P) \tag{5.30}$$

separately. The term (5.23) can be treated like the entries $a_{J,I}$ with $P_J \notin \operatorname{supp} \varphi_I$ and, analogously to (5.8) and (5.5), the term (5.30) vanishes due to (5.3).
◇

5.2. An Asymptotic Estimate for a Quadrature Algorithm Applied to the Wavelet Approach

We consider the equation $Mu = v$ and the collocation of the previous subsection. Analogously to Section 2, we apply the wavelet algorithm including (2.42) and the parameters $\alpha = \frac{4}{3}$, $\beta = \gamma = 1$. Again, the modification due to the replacement of S by $\mathbb{T}^2$ is straightforward. Indeed, we set

$$\begin{aligned} \mathcal{N} \ :=\ & \{(l,t,i,j):\ l = 0,\ t = 1,\ i,j = 0,1,2, \\ & l = 1,\dots,L,\ t = 1,\ i = 1,\dots,N_{l-1} - 1,\ j = 1,\dots,N_{l-1} - 2, \\ & l = 1,\dots,L,\ t = 2,\ i = 1,\dots,N_{l-1} - 2,\ j = 1,\dots,N_{l-1} - 1, \\ & l = 1,\dots,L,\ t = 3,\ i,j = 1,\dots,N_{l-1} - 1\} \end{aligned} \tag{5.31}$$

and replace the boundary wavelets $\psi_{l,1}$ and $\psi_{l,N_{l-1}-1}$ by

$$\psi_{l,j}(x) := \psi([3 \cdot 2^l]x - (2j-1))\sqrt{3 \cdot 2^l}, \quad l = 1,\dots,L,\ j = 1, N_{l-1} - 1. \tag{5.32}$$

These functions and the functions $\psi_{l,j}$, $1 < j < N_{l-1} - 1$ of (2.8) are to be defined over $\left(\frac{2j-3}{N_l - 1}, \frac{2j+1}{N_l - 1}\right)$ and to be extended to a 1-periodic function. Using these univariate wavelets, we define the two-dimensional wavelet functions ψ_I, $I \in \mathcal{N}$ by (2.11)-(2.14). The test functionals ϑ_J are given by

$$\vartheta_J := [3 \cdot 2^l]^{-1} \begin{cases} \delta_{P^{N_0}_{(i,j)}} & \text{if } l = 0,\ t = 1,\ i,j = 0,1,2 \\ \delta_{P^{N_l}_{(2i-1,2j)}} - \delta_{P^{N_l}_{(2i,2j)}} & \text{if } l \geq 1,\ t = 1,\ i = 1,\dots,N_{l-1}-1 \\ & \quad j = 1,\dots,N_{l-1}-2 \\ \delta_{P^{N_l}_{(2i,2j-1)}} - \delta_{P^{N_l}_{(2i,2j)}} & \text{if } l \geq 1,\ t = 2,\ i = 1,\dots,N_{l-1}-2 \\ & \quad j = 1,\dots,N_{l-1}-1 \\ \delta_{P^{N_l}_{(2i-1,2j-1)}} - \delta_{P^{N_l}_{(2i-1,2j)}} & \text{if } l \geq 1,\ t = 3, i,j = 1,\dots,N_{l-1}-1, \end{cases} \tag{5.33}$$

We arrive at a wavelet transform $B_h := (\langle M\psi_I, \vartheta_J\rangle)_{J,I\in\mathcal{N}}$ and denote the compressed matrix of B_h by C_h. The wavelet algorithm is the same as that in Subsection 2.1.

For the numerical computation of C_h, we introduce a quadrature approximation C_h'. To this end we apply a new rule of the form (5.7). The underlying partition of this rule will be a refinement of $Part_L$ (cf. Subsection 3.2). This refinement is obtained by leaving unchanged those squares of $Part_L$ which belong to a coarser grid, i.e., for which the side length is greater than $\frac{1}{N_{L-1}}$. The squares with side length $\frac{1}{N_{L-1}}$ are divided by the same procedure as in the previous subsection using a parameter $q > 2$. We denote by $Part_L^c$ the set of all squares in $Part_L$ with side length greater than $\frac{1}{N_{L-1}}$ and by $Part_L^f$ the set of the subdomains resulting from the subdivision of the squares with side length $\frac{1}{N_{L-1}}$. For this refinement $Part_L^c \cup Part_L^f$ of $Part_L$, the new rule (5.7) is just the following composite rule. Over the subdomains of $Part_L^f$ we take the mid-point rule and over those of $Part_L^c$ we take the tensor product two point Gauß rule. Note that the tensor product two point Gauß rule is exact for polynomials of degree less than three, i.e., less than the degree of functions in the trial space two plus the order of moment conditions one. Applying this new rule to (5.6) with u_h from the span of certain wavelet basis functions in $\mathcal{X}_h$ and using (5.5) as well as (5.8), we arrive at the approximate value (cf. (4.2))

$$\begin{aligned} c'_{J,I} &= \sum_{K\in\mathcal{O}} K_M(P_J^1, Q_K)\psi_I(Q_K)w_K - \varrho \sum_{K\in\mathcal{O}} K_M(P_J^2, Q_K)\psi_I(Q_K)w_K \\ &\quad + a_M(P_J^1)\psi_I(P_J^1) - \varrho a_M(P_J^2)\psi_I(P_J^2) \end{aligned} \tag{5.34}$$

for the non-zero entry $\langle M\psi_I, \vartheta_J\rangle$ of the compressed stiffness matrix C_h corresponding to M. We denote the compressed approximate matrix by $C_h' := (c'_{J,I})_{J,I\in\mathcal{N}}$.

Theorem 5.2 *i) The quadrature approximation* $\tilde{M}_h' := [R_h C_h' E_H]$ *is stable.*
ii) If the solution u *of* $Mu = v$ *is twice continuously differentiable and if* $\tilde{u}_h' \in \mathcal{X}_h$ *is the approximate solution defined by* $\tilde{M}_h'\tilde{u}_h' = v_h$, *then there exists a constant* C *independent of* h *such that*

$$\|u - \tilde{u}_h'\|_{L^2(\mathbb{T}^2)} \leq C\{\log h^{-1}\}^2 h^2. \tag{5.35}$$

iii) The sum $\sum_{J\in\mathcal{M}} \#\mathcal{O}$ *of the numbers* $\#\mathcal{O}$ *of all quadrature knots for the rules (5.7) depending on* P_J *is less than* $C[h^{-2}]^{2-2/(3q)}$. *Hence, the number of arithmetic operations in the wavelet algorithm (cf. Algorithm 3 of Subsection 4.1) applied to* M *instead of* A *is less than* $C[h^{-2}]^{2-2/(3q)}[\log h^{-1}]^2$.

Proof. i) Similarly to Theorem 2.1, the sequence $\tilde{M}_h := [R_h C_h E_h]$ defined for the operator M instead of A is stable. Thus, it suffices to show that $\tilde{M}'_h - \tilde{M}_h = R_h[C'_h - C_h]E_h$ tends to zero for $h \longrightarrow 0$. The matrix $C'_h - C_h$ of quadrature errors splits into the sum $C^c_h + C^f_h$, where C^c_h denotes the matrix of quadrature errors over the domains of $Part^c_L$ and C^f_h that of the errors over the squares of $Part^f_L$. We get $\tilde{M}'_h - \tilde{M}_h = \tilde{M}^c_h + \tilde{M}^f_h$ with $\tilde{M}^c_h := R_h C^c_h E_h$ and $\tilde{M}^f_h := R_h C^f_h E_h$.

First we show that $\|\tilde{M}^f_h\| \longrightarrow 0$ for $h \longrightarrow 0$. Analogously to the matrix A_h for A in Subsection 1.2, we denote by M_h the collocation matrix without wavelets for operator M. We let M'_h stand for its quadrature approximation (cf.(5.9)) and $\tilde{M}^Q_h$ for the compressed matrix M'_h obtained by applying algorithm (2.42) to its wavelet transform. We observe that the quadrature rules of the wavelet algorithm depend on the collocation point but not on the wavelet functional ϑ_J containing the collocation point in its support. Consequently, the product $\tilde{M}^f_h = R_h C^f_h E_h$ is equal to the matrix of quadrature errors in $\tilde{M}_h$ if the same quadrature rules are applied over the $Part^f_L$ domains. Over these domains the quadrature rules of the wavelet algorithm are the same as those of the algorithm of the previous subsection. Thus, the entries of $\tilde{M}^f_h$ are bounded by the estimates for the entries in $\tilde{M}_h - \tilde{M}^Q_h$. Hence, it remains to estimate $\|\tilde{M}_h - \tilde{M}^Q_h\|$ by giving a bound for the norm of the matrix of absolute values. Now, analogously to the stability result in Theorem 2.1, we conclude $\|\tilde{M}_h - M_h\| \longrightarrow 0$ and part b) of the proof to Theorem 5.1 implies $\|M_h - M'_h\| \longrightarrow 0$. Consequently, we only have to estimate the norm of $M'_h - \tilde{M}^Q_h$, i.e., the compression error for the quadrature approximation M'_h. To this end we fix a ϑ_J and a ψ_I and consider the entry $b'_{J,I}$ of the wavelet transform to M'_h. Analogously to Lemma 2.3, we arrive at

$$|b'_{J,I}| \leq 2^{-2l_I - 2l_J}[\mathrm{dist}(\mathrm{supp}\,\psi_I, \mathrm{supp}\,\vartheta_J)]^{-4}. \tag{5.36}$$

Note that the fourth order exponent in (5.36) instead of the fifth order exponent in Lemma 2.3 is due to the fact that the integral over the kernel function multiplied by a wavelet with two vanishing moments is replaced by a quadrature for this product. The functional, mapping a function to the quadrature value for the integral of the function multiplied by a wavelet, has one vanishing moment only. Now we take into account the condition (2.42) of the compression algorithm as well as the estimate (5.36) and apply a Schur lemma argument similar to that proving Theorem 2.1. Finally, we arrive at $\|M'_h - \tilde{M}^Q_h\| \leq C \log h^{-1}\, h^{2/3}$.

Now we turn to $\tilde{M}^c_h$. We fix a collocation point P from the difference grid

$$\begin{array}{l} \left\{ \left(\frac{i}{N_l - 1}, \frac{j}{N_l - 1} \right),\ i,j = 1, \dots, N_l - 2 \right\} \setminus \\ \left\{ \left(\frac{i}{N_{l-1} - 1}, \frac{j}{N_{l-1} - 1} \right),\ i,j = 1, \dots, N_{l-1} - 2 \right\} \end{array} \tag{5.37}$$

with $l = l_P$ and consider the quadrature error of the two point Gauß rule over a subdomain $D \in Part^c_L$. Then $P \in \mathrm{supp}\,\vartheta_J$, $J = (l_J, t_J, i_J, j_J)$ is possible only if $l_J > l_P$. Let us consider a $\varphi^{N_l}_I$ with $l = l_D$ and $I = (i,j) \in \{(i,j) : i,j = 0, 1, , \dots, N_l - 2\}$. Since D belongs to the partition $Part^c_L$ and since any $\psi_{I'}$ of level $l_{I'} > l_D$ with $\mathrm{supp}\,\psi_{I'} \cap D \neq \emptyset$ has a discontinuous derivative over D, we conclude that these $\psi_{I'}$ are discarded, i.e., they satisfy (cf. (2.42))

$$\mathrm{dist}(\mathrm{supp}\,\psi_{I'}, \mathrm{supp}\,\vartheta_J) > \max\{2^{-l_{I'}}, 2^{-l_J}, (a2^{\frac{4}{3}L})2^{-l_{I'} - l_J}\}. \tag{5.38}$$

Consequently, for the $\varphi_I^{N_l}$ with $\operatorname{supp}\varphi_I^{N_l} \cap D \neq \emptyset$, there holds

$$C\operatorname{dist}(\operatorname{supp}\varphi_I^{N_l}, \operatorname{supp}\vartheta_J) > \max\{2^{-l},\, 2^{-l_J},\, (a2^{\frac{4}{3}L})2^{-l-l_J}\}. \tag{5.39}$$

Now we estimate the quadrature error of the tensor product two point Gauß rule over D for the integral in $\langle \vartheta_J, M\varphi_I^{N_l}\rangle$ using a third order variant of the second order estimate (5.16) and we apply

$$|\vartheta_J(f)| = \frac{|f(P_J^1) - f(P_J^2)|}{3 \cdot 2^{l_J}} \leq \sup|\nabla f| \frac{|P_J^1 - P_J^2|}{3 \cdot 2^{l_J}} \leq C 2^{-2l_J} \sup|\nabla f|. \tag{5.40}$$

The derivatives arising from (5.40) and the formula for the quadrature error are estimated by (5.17) and by

$$\left| \left(\frac{\partial}{\partial x}\right)^{\alpha_1} \left(\frac{\partial}{\partial y}\right)^{\alpha_2} \varphi_I^{N_l}((x,y)) \right| \leq C \begin{cases} \left[2^l\right]^{1+\alpha_1+\alpha_2} & \text{if } 0 \leq \alpha_1, \alpha_2 \leq 1 \\ 0 & \text{else.} \end{cases} \tag{5.41}$$

We arrive at the bound

$$2^{-5l_D} 2^{-2l_J} \left\{ \operatorname{dist}(\operatorname{supp}\varphi_I^{N_l}, \operatorname{supp}\vartheta_J)^{-6} 2^{l_D} + \operatorname{dist}(\operatorname{supp}\varphi_I^{N_l}, \operatorname{supp}\vartheta_J)^{-5} 2^{2l_D} \right\} \tag{5.42}$$

for the quadrature error of $\langle \vartheta_J, M\varphi_I^{N_l}\rangle$ over D. Summing up over all D with $\operatorname{supp}\varphi_I^{N_l} \cap D \neq \emptyset$ and applying $\operatorname{dist}(\operatorname{supp}\varphi_I^{N_l}, \operatorname{supp}\vartheta_J) \geq C2^{-l_D}$ (cf. (5.39)), we estimate the quadrature error of $\langle \vartheta_J, M\varphi_I^{N_l}\rangle$ by

$$C 2^{-2l_J} 2^{-3l_D} \operatorname{dist}(\operatorname{supp}\varphi_I^{N_l}, \operatorname{supp}\vartheta_J)^{-5}. \tag{5.43}$$

This is the same estimate as that of Lemma 2.3. Now the operator of quadrature errors $\tilde{M}_h^c$ is determined by the entries $\langle \vartheta_J, M\varphi_I^{N_l}\rangle$, where $\varphi_I^{N_l}$ is a function on level l such that there exists a subdomain $D \in Part_L^c$ with $l_D = l$ and $\operatorname{supp}\varphi_I^{N_l} \cap D \neq \emptyset$. To estimate such an operator, we supply $\mathcal{X}_h$ with the norm

$$\|w_h\|_* := \sum_{l=0}^{L} \sqrt{\sum_{i,j} |\xi_{i,j}^l|^2}, \tag{5.44}$$

where $w_h = \sum_{l,t,i,j} \mu_{(l,t,i,j)} \psi_{(l,t,i,j)} \in \mathcal{X}_h$ and $\xi_{i,j}^l$ is defined by $\sum_{l'=0}^{l} \sum_{t,i,j} \mu_{(l',t,i,j)} \psi_{(l',t,i,j)} = \sum_{i,j} \xi_{i,j}^l \varphi_{i,j}^{N_l}$. Then, using (5.43) and (5.39) and repeating the Schur lemma argument leading to Theorem 2.1, we conclude

$$\|\tilde{M}_h^c\|_* \leq C\sqrt{\log h^{-1}}\, h. \tag{5.45}$$

Here $\|\tilde{M}_h^c\|_*$ is the operator norm of $\tilde{M}_h^c$ acting from $\mathcal{X}_h$ supplied with $\|\cdot\|_*$ to $\mathcal{X}_h$ supplied with the norm $\|w_h\| = \sqrt{\sum_{l,t,i,j} |\mu_{(l,t,i,j)}|^2}$ which is equivalent to the L^2 norm restricted to $\mathcal{X}_h$. Indeed, the norm equivalence $\|\sum_{I\in\mathcal{M}} \xi_I^L \varphi_I^{N_L}\|_{L^2} \sim \sqrt{\sum_{I\in\mathcal{M}} |\xi_I^L|^2}$ is not hard to verify and $\sqrt{\sum_{I\in\mathcal{M}} |\xi_I^L|^2} \sim \sqrt{\sum_{I\in\mathcal{N}} |\mu_I|^2}$ is a consequence of Lemma 2.1. From Lemma 2.1 a), we also infer that $\|w_h\|_* \leq CL\|w_h\|$. Consequently, the operator norm $\|\tilde{M}_h^c\|$ induced by the norm $\|\cdot\|$ is less than $C\{\log h^{-1}\}^{3/2}h$. Hence, $\|\tilde{M}_h^c\| \longrightarrow 0$ for $h \longrightarrow 0$ and $\tilde{M}_h'$ is stable.

ii) In view of (5.13), we have to estimate $\tilde{M}_h'\tilde{u}_h - M_h\tilde{u}_h$ for $\tilde{u}_h := Q_h u$. We observe that the convergence estimate of Theorem 2.1 is valid also for A replaced by M. The proof of this results relies upon an argument as in (5.13) and the estimate $\|\tilde{M}_h\tilde{u}_h - M_h\tilde{u}_h\|_{L^2} \le C\sqrt{\log h^{-1}}h^2$. In view of the last inequality it remains to consider $\tilde{M}_h'\tilde{u}_h - \tilde{M}_h\tilde{u}_h$, i.e, $\tilde{M}_h^c\tilde{u}_h$ and $\tilde{M}_h^f\tilde{u}_h$. For $\tilde{M}_h^c\tilde{u}_h$ we remark that the arguments leading to (5.45) imply

$$\|\tilde{M}_h^c\tilde{u}_h\|_{L^2} \le C\{\log h^{-1}\}^{3/2}h^2\|\tilde{u}_h\|_{H^2}. \tag{5.46}$$

Thus, we turn to $\tilde{M}_h^f\tilde{u}_h$ and fix a test functional ϑ_J as well as a collocation point P with $P \in \operatorname{supp}\vartheta_J$. Following part a) of the proof to Theorem 5.1, we first will estimate $\langle\tilde{M}_h^f\tilde{u}_h, \vartheta_J\rangle$. For the fixed ϑ_J, we denote by $\tilde{u}_h^*$ the compressed sum of $\tilde{u}_h$, i.e., the sum over all $\eta_I\psi_I$ in the representation of $\tilde{u}_h$ for which the entry $c_{J,I}$ of the wavelet transform C_h of $\tilde{M}_h$ is different from zero. We observe that the quadrature error $\langle\tilde{M}_h^f\tilde{u}_h, \vartheta_J\rangle$ is nothing else than the quadrature error $\langle M_h'\tilde{u}_h^*, \vartheta_J\rangle$ restricted to the domains of $Part_L^f$. Hence, following part a) of the proof to Theorem 5.1, we obtain

$$|\langle M_h'\tilde{u}_h^*, \vartheta_J\rangle| \le C \max_{P\in\operatorname{supp}\vartheta_J} 2^{-l_J}|M_h'\tilde{u}_h^*(P)| \le Ch^2\log h^{-1}\, 2^{-l_J}. \tag{5.47}$$

Here the factor $\log h^{-1}$ is due to the fact that instead of (5.18) we have used

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha_1}\left(\frac{\partial}{\partial y}\right)^{\alpha_2}[\tilde{u}_h^*((x,y)) - \tilde{u}_h^*(P)]\right| \tag{5.48}$$

$$\le C\log h^{-1}\begin{cases} |(x,y)-P| & \text{if } \alpha_1=\alpha_2=0\\ 1 & \text{if } \alpha_1<2,\ \alpha_2<2,\\ & \quad\text{and } 0<\alpha_1+\alpha_2\\ 0 & \text{else}\end{cases}$$

which will be shown in a moment. We finally get (cf. Lemma 2.2 a))

$$\begin{aligned}\|\tilde{M}_h^f\tilde{u}_h\|_{L^2} &\le C\|\{\langle\tilde{M}_h^f\tilde{u}_h,\vartheta_J\rangle\}_{J\in\mathcal{N}}\|_{l^2}\sqrt{L}\\ &\le C\{\log h^{-1}\}^{3/2}h^2\|\{2^{-l_J}\}_{J\in\mathcal{N}}\|_{l^2} \le Ch^2\{\log h^{-1}\}^2.\end{aligned} \tag{5.49}$$

It remains to prove (5.48). For a subset $\mathcal{N}_U(P_J)$ of $\mathcal{N}$, we get

$$\begin{aligned}\tilde{u}_h^* &= \sum_{I\in\mathcal{N}_U(P_J)} \eta_I\psi_I\,,\\ \|\nabla\tilde{u}_h^*\|_{L^\infty} &\le \sum_{l=1}^{L}\left\|\sum_{t,i,j:\,I=(l,t,i,j)\in\mathcal{N}_U(P_J)} \eta_I\nabla\psi_I\right\|_{L^\infty}\\ &\le \sum_{l_I=1}^{L}\sup_{t,i,j:\,I=(l_I,t,i,j)\in\mathcal{N}_U(P_J)}\left|\eta_I 2^{2l_I}\right| \quad \le CL\sup_{I\in\mathcal{N}}\left|\eta_I 2^{2l_I}\right|.\end{aligned} \tag{5.50}$$

Since the trial functions are biorthogonal wavelets, there exist bidual mother wavelet functions ψ_t^D, $t=1,2,3$ of exponential decay such that

$$\begin{aligned}\eta_{(l,t,i,j)} &= \int \tilde{u}_h(P)[3\cdot 2^l]\psi_t^D([3\cdot 2^l]P-(2i-1,2j-1))dP\\ &= [3\cdot 2^l]^{-1}\int \tilde{u}_h([3\cdot 2^l]^{-1}[P+(2i-1,2j-1)])\psi_t^D(P)dP.\end{aligned} \tag{5.51}$$

If we expand $\tilde{u}_h$ at $[3\cdot 2^l]^{-1}(2i-1,2j-1)$ into the Taylor series

$$\tilde{u}_h\left([3\cdot 2^l]^{-1}[P+(2i-1,2j-1)]\right) = \tilde{u}_h\left([3\cdot 2^l]^{-1}(2i-1,2j-1)\right) + \nabla\tilde{u}_h\left([3\cdot 2^l]^{-1}[P'+(2i-1,2j-1)]\right)\cdot[3\cdot 2^l]^{-1}P \quad (5.52)$$

and take into account that ψ_t^D is orthogonal to the constant functions, we may continue as

$$|\eta_{(l,t,i,j)}| \le [3\cdot 2^l]^{-1}\int \sup|\nabla\tilde{u}_h|\,[3\cdot 2^l]^{-1}|P|\,|\psi_t^D(P)|dP \le C2^{-2l}. \quad (5.53)$$

This together with (5.50) shows (5.48) for the case of first order derivatives. The other cases follow in a similar way.

iii) Now we count the quadrature knots. For domains $D\in Part_L^c$, we have four knots for every square D. Let D' stand for the square on the next coarser level (l_D-1) containing D. If the collocation point is fixed, then to each $D\in Part_L^c$ there corresponds a wavelet basis function ψ_I on the same level as D such that $\operatorname{supp}\psi_I\cap D'\neq\emptyset$ and that the entry $c_{J,I}$ is non-zero for a J with $P\in\operatorname{supp}\vartheta_J$ (cf. the definition of $Part_L$ in Subsection 3.2). Hence, the numbers of knots over the domains $D\in Part_L^c$ is less than constant times the number of non-zero entries in the compressed matrix C_h, i.e., less than $C[h^{-2}]^{4/3}[\log h^{-1}]^2$.

It remains to count the knots defined by the algorithm of Subsection 5.1 over the domains $D\in Part_L^f$. We fix a collocation point P from the difference grid (5.37). In view of (2.42), there is a square shaped neighborhood of width $C2^{\min\{L/3-l_P,0\}}$ which is divided by the algorithm of Subsection 5.1. The subdomains of $Part_L$ which are adjacent to this neighborhood have already a size of $\frac{1}{N_{L-1}-1}$, i.e., the number of knots inside of these subdomain is already covered by the estimate $C[h^{-2}]^{4/3}[\log h^{-1}]^2$. Thus, only the knots in the neighborhood of size $C2^{\min\{L/3-l_P,0\}}$ are to be counted. This neighborhood is split into i_0 strips $\{(x,y): z_{i-1}<x<z_i\}$, where i_0 is defined by $z_{i_0}\sim[i_0h]^q\sim C2^{\min\{L/3-l_P,0\}}$. In other words, $i_0\le C2^{\min\{L/3-l_P,0\}/q}h^{-1}$. Each strip is divided into $[z_{i-1}/(z_i-z_{i-1})]+1$ subdomains, i.e., in about $[ih]^q/\{[ih]^q-[(i-1)h]^q\}\sim i$ squares. The whole number of squares is bounded by

$$C\sum_{i=1}^{i_0} i \le Ci_0^2 \le C2^{2\{L/3-l_P\}/q+2L}. \quad (5.54)$$

if $l_P>L/3$ and by 2^{2L} if $l_P\le L/3$. Summing up over all collocation points, we arrive at the estimate

$$C\sum_{l_P=0}^{L/3} 2^{2l_P}2^{2L} + C\sum_{l_P=L/3+1}^{L} 2^{2l_P}2^{2\{L/3-l_P\}/q+2L} \le C\left[2^{2L}\right]^{2-2/(3q)}. \quad (5.55)$$

which proves iii).

◇

Remark 5.1 *The previous proof shows that the non optimal estimate for the number of quadrature knots is caused by the quadrature rule applied to the singular and almost* *singular elements which are handled as in the algorithm without wavelets. If optimal rules for these elements are applied, then the reduction in computing time will be similar to the reduction in storage. Such optimal rules leading to $O(h^{-2}\{\log h^{-1}\}^b)$, $b > 0$, quadrature knots over the domains which are treated as in the non wavelet algorithm are well known. The singular elements are to be simplified by Duffy's transformation and Gauß quadrature rules with degrees depending on the logarithmic distance to the collocation points are to be exploited. For the estimate of the quadrature errors, the piecewise analyticity of the trial functions and of the parametrisation by Overhauser interpolation is essential. The analyticity of the geometry is not required. Note, however, that such an approach requires estimates for the second order derivatives of the coordinate mappings of Γ in order to control the maximal domains of analyticity for the approximate kernels which arise after the introduction of the parametrisation in the integral equation. The quadrature rules we have applied should be more robust with respect to the geometry.*

At the end of the present section, we consider a quadrature algorithm defined by Duffy's transformation leading to $O([h^{-2}]^{5/3})$ knot points. This algorithm is close to that which we have implemented in the numerical tests (cf. Section 3). For the new quadrature algorithm, we have to define a new quadrature rule depending on P. Without loss of generality, we suppose $P = (0,0)$. Moreover, for symmetry reasons, it is sufficient to define the quadrature rule over $S' := \{(x,y) : 0 \le x \le \frac{1}{2}, 0 \le y \le x\}$. We will set up the rule separately over $S' \cap [0,h]^2$, over $S' \cap [(i-1)h, ih]^2$, $i = 2, \dots, (N_L - 1)/2$, and over the strips $[(i-1)h, ih] \times [0, (i-1)h]$, $i = 2, \dots, (N_L-1)/2$. For the subdomain $S' \cap [0,h]^2$, we apply Duffy's transformation and obtain

$$\int_0^h \int_0^x f(x,y)dydx = \int_0^h \int_0^1 f(x,zx)xdzdx = \sum_{j=1}^{N_L-1} \int_0^h \int_{(j-1)h}^{jh} f(x,zx)xdzdx. \tag{5.56}$$

To these integrals we apply the tensor product of the two point Gauß rule

$$\int_0^1 g(s)ds \sim \frac{1}{2}\sum_{\kappa=1}^{2} g(\tau_\kappa), \qquad \tau_1 := \frac{1}{2}(1 - \frac{1}{\sqrt{3}}),\ \tau_2 := \frac{1}{2}(1 + \frac{1}{\sqrt{3}}) \tag{5.57}$$

to get

$$\int_0^h \int_0^x f(x,y)dydx \sim \sum_{j=1}^{N_L-1} \frac{h^2}{4} \sum_{\kappa=1}^{2}\sum_{\iota=1}^{2} f\left([h\tau_\kappa], [(j-1)h + h\tau_\iota][h\tau_\kappa]\right)[h\tau_\kappa]. \tag{5.58}$$

If the subdomain is $S' \cap [(i-1)h, ih]^2$, then we write

$$\int_{(i-1)h}^{ih} \int_{(i-1)h}^{x} f(x,y)dydx \sim \frac{h}{2}\sum_{\kappa=1}^{2} \int_{(i-1)h}^{(i-1)h+\tau_\kappa h} f([(i-1)h + \tau_\kappa h], y)dy. \tag{5.59}$$

For $i > 1$, we divide $[(i-1)h, (i-1)h + \tau_\kappa h]$ into $n_{i,\kappa}$ equal parts so that each subinterval is less than $[(i-1)h + \tau_\kappa h]/(N_L - 1) = [(i-1) + \tau_\kappa]h^2$. Thus, we denote by $n_{i,\kappa}$ the smallest integer greater than $\{[(i-1)h + \tau_\kappa h] - [(i-1)h]\}/\{[(i-1) + \tau_\kappa]h^2\}$ and arrive at

$$\begin{aligned}\int_{(i-1)h}^{ih} \int_{(i-1)h}^{x} f(x,y)dydx &\sim \frac{h}{2}\sum_{\kappa=1}^{2}\sum_{j=1}^{n_{i,\kappa}} \int_{(i-1)h+(j-1)\tau_\kappa h/n_{i,\kappa}}^{(i-1)h+j\tau_\kappa h/n_{i,\kappa}} f\left([(i-1)h + \tau_\kappa h], y\right)dy \\ &\sim \frac{h}{2}\sum_{\kappa=1}^{2}\sum_{j=1}^{n_{i,\kappa}} \frac{\tau_\kappa h}{2n_{i,\kappa}} \sum_{\iota=1}^{2} f\left([(i-1)h + \tau_\kappa h], [(i-1)h + (j-1)\frac{\tau_\kappa h}{n_{i,\kappa}} + \tau_\iota \frac{\tau_\kappa h}{n_{i,\kappa}}]\right).\end{aligned} \tag{5.60}$$

Finally, we consider $\{(x,y) : (i-1)h \leq x \leq ih, 0 \leq y \leq (i-1)h\}$. Introducing the partition $0 = z_0 < z_1 < \ldots < z_{m_i} = (i-1)h$ by

$$\begin{aligned} \{z_j : j = 0,\ldots,m_i\} &:= \{jh : j = 0,1,\ldots,i-1\} \cap \\ &\quad \{j[i-1/2]h^2 \in [0,(i-1)h] : j = 1,\ldots\}, \end{aligned} \tag{5.61}$$

we obtain the composite tensor product Gauß rule

$$\begin{aligned} \int_{(i-1)h}^{ih} \int_0^{(i-1)h} f(x,y)dydx &\sim \sum_{j=1}^{m_i} \int_{(i-1)h}^{ih} \int_{z_{j-1}}^{z_j} f(x,y)dydx \\ &\sim \sum_{j=1}^{m_i} \frac{h}{2} \frac{z_j - z_{j-1}}{2} \sum_{\kappa=1}^{2} \sum_{\iota=1}^{2} f([(i-1)h + \tau_\kappa h], [z_{j-1} + \tau_\iota(z_j - z_{j-1})]). \end{aligned} \tag{5.62}$$

Putting formulae (5.58), (5.60), and (5.62) together, we get a new rule of the form (5.7).

Theorem 5.3 *Applying the just defined quadrature rules (5.58), (5.60), and (5.62) over the subdomains of $Part_L^f$ and combining it with the tensor product two point Gauß rules over the $Part_L^c$ subdomains, we arrive at a wavelet algorithm (cf. Algorithm 3 of Subsection 4.1) with no more than $O([h^{-2}]^{5/3}\{\log h^{-1}\}^2)$ arithmetic operations and an error estimate of $O(h^2\{\log h^{-1}\}^{3/2})$ provided the discretised wavelet method is stable and the exact solution u of $Mu = v$ is three times continuously differentiable.*

Proof. We set

$$f(x,y) := \Phi\left(\frac{(x,y)-(0,0)}{|(x,y)-(0,0)|}\right) \frac{1}{|(x,y)-(0,0)|^2} [\tilde{u}_h(x,y) - \tilde{u}_h(0,0)], \tag{5.63}$$

where $\tilde{u}_h \in \mathcal{X}_h$ is the bilinear interpolation of the solution u. For this f, we will show that the quadrature error for the integral of f over S' is less than $O(h^2 \log h^{-1})$. Starting from this estimate instead of (5.12), the proof of the theorem is analogous to that of Theorem 5.2. We only remark that, this time, the quadrature estimate over the $Part_L^f$ domains requires that $\tilde{u}_h^*$ is the interpolant of a three times continuously differentiable function. Since this is not true, only the quadrature error over the $Part_L^f$ squares with $\tilde{u}_h^* = \tilde{u}_h$ can be estimated. However, due to the application of two point Gauß rules, the squares with $\tilde{u}_h^* \neq \tilde{u}_h$ can be treated like the $Part_L^c$ domains. Arguing this way, one gets the $O(h^2\{\log h^{-1}\}^{3/2})$ estimate instead of the $O(h^2\{\log h^{-1}\}^2)$ in Theorem 5.2.

First we consider the quadrature error for f over $S' \cap [0,h]^2$. Note that the integrand takes the form

$$f(x,y) = \Psi\left(\frac{y}{x}\right) \frac{1}{x^2} [\tilde{u}_h(x,y) - \tilde{u}_h(0,0)] \tag{5.64}$$

where Ψ is at least twice continuously differentiable. Duffy's transformation changes the integrand into

$$f(x,zx)x = \Psi(z) \frac{[\tilde{u}_h(x,xz) - \tilde{u}_h(0,0)]}{x} = \Psi(z) \int_0^1 \nabla\tilde{u}_h(\lambda x, \lambda xz) \cdot (1,z)\, d\lambda. \tag{5.65}$$

Since the last expression is smooth, we arrive at an $O(h^2)$ estimate for the quadrature error over $S' \cap [0,h]^2$.

Next we consider the integral and the quadrature rule over $S' \setminus [0,h]^2$. Over this domain, we can replace $\tilde{u}_h$ in the definition of f by the exact solution u. Indeed, this leads to an error

$$\begin{aligned} & \left| \int_h^{1/2} \int_0^x \Psi\left(\frac{y}{x}\right) \frac{1}{x^2} [\tilde{u}_h(x,y) - u(x,y)] dy dx \right| \qquad (5.66) \\ \leq\ & \sup |\tilde{u}_h(x,y) - u(x,y)| \int_h^{1/2} \int_0^x \left| \Psi\left(\frac{y}{x}\right) \right| \frac{1}{x^2} dy dx \\ \leq\ & Ch^2 \int_h^{1/2} \frac{1}{x} dx \ \leq\ Ch^2 \log h^{-1} \end{aligned}$$

for the integral and, analogously, to an error less than $Ch^2 \log h^{-1}$ for the quadrature rule. Hence, we have to estimate the quadrature error for the integrand

$$f(x,y) = \Psi\left(\frac{y}{x}\right) \frac{1}{x^2} [u(x,y) - u(0,0)]. \qquad (5.67)$$

We observe that the function

$$\begin{aligned} x \mapsto \int_0^x f(x,y) dy &= \int_0^1 \Psi(z) \frac{[u(x,zx) - u(0,0)]}{x} dz \qquad (5.68) \\ &= \int_0^1 \Psi(z) \int_0^1 \nabla u(\lambda x, \lambda z x) \cdot (1,z)\, d\lambda dz \end{aligned}$$

is twice continuously differentiable. Consequently,

$$\left| \int_h^{1/2} \left[\int_0^x f(x,y) dy \right] dx - \sum_{i=2}^{\frac{N_L-1}{2}} \frac{h}{2} \sum_{\kappa=1}^{2} \left[\int_0^{(i-1)h + \tau_\kappa h} f([(i-1)h + \tau_\kappa h], y) dy \right] \right| \leq Ch^2.$$

Hence, it remains to estimate the difference of $\frac{h}{2} \sum_{\kappa=1}^2 \int_0^{(i-1)h+\tau_\kappa h} f([(i-1)h + \tau_\kappa h], y) dy$ and the quadrature rule over $\{(x,y) \in S' : (i-1)h \leq x \leq ih\}$. Using $|(\partial/\partial y)^2 f(x,y)| \leq C[ih]^{-3}$ and the fact that the step size in y-direction is less than Cih^2, the sum of the considered differences is less than

$$\begin{aligned} & C \sum_{i=2}^{\frac{N_L-1}{2}} \frac{h}{2} \sum_{\kappa=1}^{2} \left\{ \sum_{j=1}^{m_i} (ih)^{-3} (z_j - z_{j-1})^3 + \sum_{j=1}^{n_{\kappa,i}} (ih)^{-3} \left(\frac{h}{2n_{i,\kappa}} \right)^3 \right\} \qquad (5.69) \\ \leq\ & C \sum_{i=2}^{\frac{N_L-1}{2}} h\, (ih)^{-3} (ih^2)^3 \{ m_i + n_{i,1} + n_{i,2} \} \leq Ch^4 \sum_{i=2}^{\frac{N_L-1}{2}} (N_L - 1) \ \leq\ Ch^2. \end{aligned}$$

◇

Remark 5.2 *Note that a stability proof for the quadrature discretisation of the wavelet algorithm in Theorem 5.3 seems to be very hard. However, using the ideas of the proof to Theorem 5.2, it is not hard to define a slight modification which can be proved to be stable.*

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