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Approximation schemes for materials with discontinuities

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Abstract

Damage and fracture phenomena are related to the evolution of discontinuities both in space and in time. This contribution deals with methods from mathematical and numerical analysis to handle these: Suitable mathematical formulations and time-discrete schemes for problems with discontinuities in time are presented. For the treatment of problems with discontinuities in space, the focus lies on FE-methods for minimization problems in the space of functions of bounded variation. The developed methods are used to introduce fully discrete schemes for a rate-independent damage model and for the viscous approximation of a model for dynamic phase-field fracture. Convergence of the schemes is discussed.

1 Introduction

This contribution discusses methods from mathematical and numerical analysis developed for the numerical treatment of damage and fracture models used in engineering applications. Phenomenologically, damage and fracture processes lead to a weakening of the material's abilities to bear external loads, to a degradation of internal stresses, and ultimately result in its complete failure. Such material defects appear as discontinuities in the spatial domain, across which kinematic quantities associated with the deformable body, such as the deformation or the displacement field, may feature jumps with respect to the spatial coordinates. But often cracks also seem to form or to propagate instantaneously in previously undamaged regions, i.e., material points seem to jump in one instant from being sound to being damaged. In other words, the evolution of material defects is not only accompanied by discontinuities in space but also by discontinuities with respect to time. This effect is already reflected by Griffith' fracture criterion for brittle, quasistatic crack growth [Gri21], with states that, in a body Ω_{s_0} with a pre-existing crack of length s_0, crack growth sets in as soon as the energy released from the body by potential crack growth reaches a critical value given by the fracture toughness $\mathcal{G}_c$, i.e.,

$$-\frac{\mathrm{d}\Psi(\Omega_{s_0+s})}{\mathrm{d}s}\Big|_{s=0} = \mathcal{G}_c\,. \tag{1}$$

Here, $\Psi(\Omega_{s_0+s})$ denotes the sum of the strain energy and of the energy due to the applied forces of the body with a crack extended by the length s. The left-hand side of (1) defines the energy release rate. The growth of the crack length s is thus formally characterized by the conditions

$$\dot{s} \geq 0, \tag{2a}$$

$$\frac{\mathrm{d}\Psi(\Omega_{s_0+s})}{\mathrm{d}s}\Big|_{s=0} + \mathcal{G}_c \geq 0\,, \tag{2b}$$

$$\dot{s}\left(\frac{\mathrm{d}\Psi(\Omega_{s_0+s})}{\mathrm{d}s}\Big|_{s=0} + \mathcal{G}_c\right) = 0\,, \tag{2c}$$

where $\dot{s}$ denotes the time derivative. Above, condition (2a) states that the crack either keeps its position ($\dot{s} = 0$) or grows ($\dot{s} > 0$) with time. By condition (2b) the values of the energy release rate can never exceed the fracture toughness. Condition (2c) ensures that crack growth of any positive rate $\dot{s} > 0$ is possible if and only if (1) is satisfied. We remark here, that (2c) holds true on a formal level, only, where it is assumed that the terms involved are sufficiently regular. To simplify above explanations we have considered for (1) and (2) a two-dimensional setting, i.e. $\Omega_{s_0}, \Omega_{s_0+s} \subset \mathbb{R}^2$, so that the crack is a one-dimensional subset and s_0, resp. $s_0 + s$ indicates the position of the crack tip along this line. More general crack geometries in higher space dimensions can be described by the Francfort-Marigo model for brittle fracture, cf., e.g. [FM98], which is formulated as a minimum problem for the total energy of the body $\Omega \subset \mathbb{R}^d$

$$\mathcal{E}(\Gamma_c) = \Psi(\Omega\backslash\Gamma_c) + \int_{\Gamma_c} \mathcal{G}_c \,\mathrm{d}S \tag{3}$$

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given as the sum of the elastic bulk energy of the body and the crack surface energy along the crack Γ_c. However, the propagation of general crack geometries, which may also involve effects like crack kinking and branching, is hard to handle as sharp $(d-1)$-dimensional manifolds in a d-dimensional body from the view point of mathematical analysis and numerics. Therefore, it has become a well-established method to regularize the energy with the sharp $(d-1)$-dimensional crack surfaces by an energy functional that locates the material degradation in narrow d-dimensional volumes. In the spirit of generalized standard materials [HN75] this is done with the aid of an internal variable, which indicates the state of material degradation in each point of the domain $\Omega \subset \mathbb{R}^d$. This so-called damage or phase-field variable $z : [0,\mathsf{T}] \times \Omega \to [0,1]$ with $z(t,x) = 1$ if the material point $x \in \Omega$ is undamaged at time $t \in [0,\mathsf{T}]$ and $z(t,x) = 0$ in case of maximal damage, is a phase indicator for the damaged and undamaged phase in the body and can be understood as the volume fraction of undamaged material in each point $x \in \Omega$. In this way, a possible regularization of (3) is given by the Ambrosio-Tortorelli energy functional [Gia05]

$$\mathcal{E}(t,u,z) := \int_\Omega \Big(\tfrac{1}{2}(z+\eta)^2 \mathbb{C}e(u) : e(u) - f(t)\cdot u \Big)\,\mathrm{d}x + \int_\Omega \mathcal{G}_c \Big(\tfrac{\ell}{2}|\nabla z|^2 + \tfrac{1}{2\ell}(1-z)^2 \Big)\,\mathrm{d}x. \tag{4}$$

Herein, the second integral term can be seen as the volumetric regularization of the crack surface energy in (3). With a small parameter $\eta > 0$ the first integral term is an approximation of the elastic bulk energy $\Psi(\Omega\backslash\Gamma_c)$ in (3), which in this case takes the form $\Psi(\Omega\backslash\Gamma_c) := \int_{\Omega\backslash\Gamma_c} \Big(\frac{1}{2}\mathbb{C}e(u) : e(u) - f(t)\cdot u \Big)\,\mathrm{d}x$ with the displacements $u : [0,\mathsf{T}] \times \Omega \to \mathbb{R}^d$ and a time-dependent external volume load $f(t)$.

In this work we discuss methods from mathematical and numerical analysis that allow it to numerically handle the discontinuities in space and time exhibited by solutions of damage and fracture models. This will involve energies of the form

$$\mathcal{E}(t,u,z) := \int_\Omega \Big(\tfrac{1}{2} w_{\mathbb{C}}(z) \mathbb{C}e(u) : e(u) - f(t)\cdot u \Big)\,\mathrm{d}x + \mathcal{G}(z)\,, \tag{5}$$

where the degradation function $w_{\mathbb{C}}$ allows for generalizations of the one in (4) and $\mathcal{G}(z)$ is a gradient regularization for the damage variable. It can be a volumetric regularization of the crack surface energy as in (4) and thus our results apply to models for phase-field fracture. But we will also adress general models for volume damage with

$$\mathcal{G}(z) := \int_\Omega \tfrac{1}{r}|\nabla z|^r\,\mathrm{d}x \quad \text{for } r \in (1,\infty) \tag{6a}$$

a gradient in the sense of Sobolev spaces and, in case of $r = 1$,

$$\mathcal{G}(z) := |Dz|(\Omega) \tag{6b}$$

the total variation of z in Ω leading to a regularization in $BV(\Omega)$, the space of functions of bounded variation, see Sec. 3.1.1 for more details. While gradient regularizations of type (6a) prevent jumps of z in Ω across $(d-1)$-dimensional manifolds, such jumps are possible in the limiting case $r = 1$. Hence, a BV-regularization of type (6b) can be used to sharply distinguish between undamaged and damaged zones in a material. As already indicated along with (2), solutions of problems related to damage and fracture also exhibit discontinuities with respect to time. In particular, the time-derivate appearing in (2) cannot be understood in the classical sense, but only in the sense of measures, as solutions in general can be shown to be of bounded variation in time, only. This low regularity in time is a general feature of rate-independent evolution problems. Thus, in order to have (2c) well-defined, better regularity is required for $\frac{\mathrm{d}\Psi(\Omega_{s_0+s})}{\mathrm{d}s}$ in order to compensate for the low regularity of $\dot{s}$. However, this cannot be expected in general. Therefore, alternative formulations of the evolution problem are required, which can handle the low regularity in time. Such formulations suited for discontinuities in time and time-discrete approximation schemes thereof will be the topic of Section 2. Subsequently, Section 3 is devoted to finite-element methods for problems with discontinuities in space, in particular for the FE-approximation of minimization problems for functionals involving the BV-regularization (6b). Based on the developed FE-algorithms and on the methods of Section 2 we will present in Section 3.5 an approximation result for a rate-independent damage model and also address the challenges related to the convergence proof of the fully discrete scheme. Finally, in Section 4 we present a fully discrete scheme for the viscous approximation of dynamic phase-field fracture in visco-elastic materials and prove convergence of the method. Here, in addition to the elastic bulk energy of the type (5) also the kinetic energy of the body and further viscous potentials will play a role.

2 Mathematical formulations to handle discontinuities in time

Before introducing abstract mathematical concepts to handle discontinuities in time we once more address Griffith' model for brittle fracture (2) and motivate with this example some terms and ideas that will reappear in a more general setting lateron.

At first, we discuss that (2) is a rate-independent process. Rate-independent processes are characterized by the fact that reparametrizations in time of the given data lead to solutions of the problem, which are reparametrized in the same way. Thus, if the external loadings change twice as fast, then the solution of the problem will respond twice is fast, i.e., here the crack tip position s will move twice as fast. This effect can be described mathematically by introducing a convex, lower semicontinuous dissipation potential $\mathcal{R}_1$ that is positively 1-homogeneous to reflect rate-independence, i.e., for all admissible velocities v and all constants $\lambda > 0$ it is

$$\mathcal{R}_1(\lambda v) = \lambda \mathcal{R}_1(v) \quad \text{and } \mathcal{R}_1(0) = 0\,. \tag{7}$$

Indeed, for (2) we may set $\mathcal{R}_1(v) := \mathcal{G}_c|v| + \chi_{[0,\infty)}(v)$, where the characteristic function of the interval $[0,\infty)$ ensures that velocities are non-negative, i.e., $\chi_{[0,\infty)}(v) = 0$ if $v \geq 0$ and $\chi_{[0,\infty)}(v) = \infty$ if $v < 0$. We remark here that $\mathcal{R}_1$ is not classically differentiable in $v = 0$, but generalized derivatives can be defined in the sense of subdifferentials of convex functionals, which here takes the form

$$\partial\mathcal{R}_1(v) := \begin{cases} \{\mathcal{G}_c\} & \text{if } v > 0, \\ (-\infty, \mathcal{G}_c] & \text{if } v = 0, \\ \emptyset & \text{if } v < 0. \end{cases} \tag{8}$$

For above choice of $\mathcal{R}_1$ it can be easily checked that (7) is satisfied and from (8) one can see that $\partial\mathcal{R}_1$ is positively homogeneous of degree 0, i.e., $\partial\mathcal{R}_1(\lambda v) = \partial\mathcal{R}_1(v)$ for all admissible v and all $\lambda \geq 0$. Hence, reparametrizations in time do not change the evolution law.

We now check how (2) can be reformulated in terms of the dissipation potential $\mathcal{R}_1$. With the above choice of $\mathcal{R}_1$, by multiplication with velocities $v \geq 0$, (2b) can be formally rewritten as

$$v\left(\tfrac{\mathrm{d}\Psi(\Omega_{s_0+s})}{\mathrm{d}s}\Big|_{s=0}\right) + \mathcal{R}_1(v) \geq 0 \quad \text{for all admissible velocities } v \text{ with } v \geq 0\,. \tag{9}$$

This provides a local stability condition, restricted to test functions with $v \geq 0$. Moreover, given that the terms involved are sufficiently regular, then (2c) can be integrated over $[0,t]$ for any $t \in [0,\mathsf{T}]$. A formulation in the spirit of (9) together with a time-integrated version of (2c) adapted to the context of phase-field fracture will be obtained for limit solutions of the approximation procedure in Sec. 4.

We further observe that (9) can be assumed to hold true for all admissible velocities v such that the expression

$$v\left(\tfrac{\mathrm{d}\Psi(\Omega_{s_0+s})}{\mathrm{d}s}\Big|_{s=0}\right)$$

takes a finite value. Then the inequality is true also if $v < 0$, since then $\mathcal{R}_1(v) = \infty$. If the terms in (2c) are suitably regular, the subtraction of (2c) from (9) results in a variational inequality

$$(v - \dot{s})\left(\tfrac{\mathrm{d}\Psi(\Omega_{s_0+s})}{\mathrm{d}s}\Big|_{s=0}\right) + \mathcal{R}_1(v) - \mathcal{R}_1(\dot{s}) \geq 0 \quad \text{for all admissible velocities } v\,, \tag{10}$$

where we also used that $\mathcal{R}_1(\dot{s}) = 0$ by (2a). In view of the convexity of $\mathcal{R}_1$ and provided that all terms are sufficiently regular, the variational inequality (10) is equivalent to the subdifferential inclusion

$$-\tfrac{\mathrm{d}\Psi(\Omega_{s_0+s})}{\mathrm{d}s}\Big|_{s=0} \in \partial\mathcal{R}_1(\dot{s})\,. \tag{11}$$

We point out once more that above discussion has been carried out on a formal level in a pointwise sense in space and time always assuming that all quantities are sufficiently regular. For an elaborate mathematical analysis the reader is referred, e.g., to [KMZ08].

Solution concepts for rate-independent systems: Let us now turn to a general setting that allows us also to treat models for rate-independent damage and phase-field fracture in elastically deformable bodies. The state of the body is then characterized by a kinematic variable, such as the displacement field u and by an internal variable z responsible for the dissipative process. It is assumed that the evolution of z is rate-independent and that u evolves in a quasi-static way. The evolution of the pair (u, z) can then be given as a variational formulation in suitable Banach spaces $\mathbf{U}$ and $\mathbf{Z}$ with duals $\mathbf{U}^*$ and $\mathbf{Z}^*$ based on an energy functional $\mathcal{E} : [0,T] \times \mathbf{U} \times \mathbf{Z} \to \mathbb{R} \cup \{\infty\}$, e.g., of the form (5) and based on a convex, lower semicontinuous and positively 1-homogeneous dissipation potential $\mathcal{R}_1 : \mathbf{Z} \to [0,\infty]$:

$$\langle \mathrm{D}_u \mathcal{E}(t, u(t), z(t)), \tilde{u} \rangle_{\mathbf{U}} = 0 \quad \text{for all } \tilde{u} \in \mathbf{U}, \tag{12a}$$

$$\langle \mathrm{D}_z \mathcal{E}(t, u(t), z(t)), \tilde{z} - \dot{z}(t) \rangle_{\mathbf{Z}} + \mathcal{R}_1(\tilde{z}) - \mathcal{R}_1(\dot{z}(t)) \geq 0 \quad \text{for all } \tilde{z} \in \mathbf{Z} \tag{12b}$$

for a.e. $t \in (0,T)$. Here, $\langle \cdot, \cdot \rangle_{\mathbf{U}}$ and $\langle \cdot, \cdot \rangle_{\mathbf{Z}}$ denote the dual pairings defined for elements from the Banach spaces $\mathbf{U}$, resp. $\mathbf{Z}$ and their duals. Moreover, $\mathrm{D}_u \mathcal{E}(t, u(t), z(t)) \in \mathbf{U}^*$ and $\mathrm{D}_z \mathcal{E}(t, u(t), z(t)) \in \mathbf{Z}^*$ denote the variational derivatives of the energy functional with respect to the variables u and z. In this way, (12a) provides a weak formulation of the quasistatic momentum balance and (12b) characterizes the rate-independent evolution of z in terms of a variational inequality alike (10). Again, it has to be assumed that all quantities in (12b) are sufficiently regular in order to have the dual pairing well-defined. Since this regularity cannot be provided in general, one is interested in reformulations of (12) that avoid time-derivatives and differentials. Such reformulations are based on additional convexity assumptions for the energy functional and they can be deduced as follows:

- Multiplying (12b) by $h > 0$ and using the test function $\tilde{z} := v/h$ with $v \in \mathbf{Z}$ leads in the limit $h \to 0$ to the local stability condition

$$\langle \mathrm{D}_z \mathcal{E}(t, u(t), z(t)), v \rangle_{\mathbf{Z}} + \mathcal{R}_1(v) \geq 0 \quad \text{for all } v \in \mathbf{Z}\,, \tag{13}$$

  i.e., the analogon of (2b). In fact, with $\mathcal{R}_1(0) = 0$ one finds that (13) is equivalent to $-\mathrm{D}_z \mathcal{E}(t, u(t), z(t)) \in \partial \mathcal{R}_1(0)$. Choosing $\tilde{u} := \hat{u} - u(t)$ for $\hat{u} \in \mathbf{U}$ in (12a) and $\tilde{z} := \hat{z} - z(t)$ for $\hat{z} \in \mathbf{Z}$ in (12b), summing up, and exploiting convexity relations the energy functional, results in the global stability condition

$$\mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, \hat{u}, \hat{z}) + \mathcal{R}_1(\hat{z} - z(t)) \quad \text{for all } (\hat{u}, \hat{z}) \in \mathbf{U} \times \mathbf{Z} \tag{14}$$

  in case that $\mathcal{E}(t, \cdot, \cdot) : \mathbf{U} \times \mathbf{Z} \to \mathbb{R} \cup \{\infty\}$ is convex. Instead, if the energy is only separately convex, i.e., $\mathcal{E}(t, \cdot, \tilde{z}) : \mathbf{U} \to \mathbb{R} \cup \{\infty\}$ is convex for any fixed $\tilde{z} \in \mathbf{Z}$ and $\mathcal{E}(t, \tilde{u}, \cdot) : \mathbf{Z} \to \mathbb{R} \cup \{\infty\}$ is convex for any fixed $\tilde{u} \in \mathbf{U}$, one finds two separate stability conditions for u and z, i.e.,

$$\text{minimality:} \quad \mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, \hat{u}, z(t)) \ \text{ for all } \hat{u} \in \mathbf{U}, \tag{15a}$$

$$\text{semistability:} \quad \mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, u(t), \hat{z}) + \mathcal{R}_1(\hat{z} - z(t)) \ \text{ for all } \hat{z} \in \mathbf{Z}. \tag{15b}$$

- Testing (12b) with the test functions $\tilde{z} := 0$ and $\tilde{z} := 2\dot{z}(t)$ results in

$$\langle \mathrm{D}_z \mathcal{E}(t, u(t), z(t)), \dot{z}(t) \rangle_{\mathbf{Z}} + \mathcal{R}_1(\dot{z}(t)) = 0, \tag{16}$$

  i.e., the analogon of (2c). Testing (12a) with $\tilde{u} := \dot{u}(t)$, provided that $\dot{u}(t) \in \mathbf{U}$ is an admissible testfunction, summing the result with (16), and integrating over (t_1, t_2) for any $t_1 < t_2 \in [0, \mathrm{T}]$ results in the energy-dissipation balance

$$\mathcal{E}(t_2, u(t_2), z(t_2)) + \mathrm{Var}_{\mathcal{R}_1}(z; [t_1, t_2]) = \mathcal{E}(t_1, u(t_1), z(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{E}(t, u(t), z(t)) \, \mathrm{d}t\,. \tag{17}$$

  Here, $\partial_t \mathcal{E}(\cdot, u, z)$ denotes the partial time-derivative of the energy functional and

$$\mathrm{Var}_{\mathcal{R}_1}(z; [t_1, t_2]) := \sup_{\text{all partitions of } [t_1, t_2]} \sum_{k=1}^{N} \mathcal{R}_1(z(t_k) - z(t_{k-1}))$$

  denotes the total variation in time with respect to the potential $\mathcal{R}_1$.

The above deduction (13)–(17) motivates

Definition 1 (Notions of solution for rate-independent systems)**.** *Let $\mathbf{U}, \mathbf{Z}$ be Banach spaces, $\mathcal{E} : [0,\mathsf{T}] \times \mathbf{U} \times \mathbf{Z} \to \mathbb{R} \cup \{\infty\}$ be an energy functional and $\mathcal{R}_1 : \mathbf{Z} \to [0,\infty]$ a convex, lower semicontinuous, and positively 1-homogeneous dissipation potential. The triple $(\mathbf{U} \times \mathbf{Z}, \mathcal{R}_1, \mathcal{E})$ is called a rate-independent system.*

A pair $(u,z) : [0,\mathsf{T}] \to \mathbf{U} \times \mathbf{Z}$ is called

1. *local solution of $(\mathbf{U} \times \mathbf{Z}, \mathcal{R}_1, \mathcal{E})$, if* (12a) *and* (13) *are satisfied for almost all $t \in (0,\mathsf{T})$ and if* (17) *holds true as an upper energy-dissipation estimate, i.e., with '$\leq$' in* (17) *for all $t_1 \leq t_2 \in [0,\mathsf{T}]$;*
2. *semistable energetic solution of $(\mathbf{U} \times \mathbf{Z}, \mathcal{R}_1, \mathcal{E})$, if* (15a) *is satisfied for almost all $t \in (0,\mathsf{T})$, if* (15b) *is valid for all $t \in [0,\mathsf{T}]$, and if* (17) *holds true as an upper energy-dissipation estimate for $t_1 = 0$ and for all $t_2 \in [0,\mathsf{T}]$, i.e.:*

$$\mathcal{E}(t_2, u(t_2), z(t_2)) + \mathrm{Diss}_{\mathcal{R}_1}(z;[0,t_2]) \leq \mathcal{E}(0,u_0,z_0) + \int_0^{t_2} \partial_t \mathcal{E}(t,u(t),z(t))\,\mathrm{d}t\,; \tag{18}$$

3. *energetic solution of $(\mathbf{U} \times \mathbf{Z}, \mathcal{R}_1, \mathcal{E})$, if* (14) *and* (17) *are satisfied for all $t \in [0,T]$.*

We also refer to the monograph [MR15] for more details on the theory of rate-independent processes, cf. also [MR15, Def. 3.3.2] for further solution concepts.

Remark 1 (Approximation methods for rate-independent systems)**.** In the setting of infinite-dimensional Banach spaces $\mathbf{U} \times \mathbf{Z}$ it has been shown in [RT17a] that semistable energetic solutions for rate-independent systems (Def. 1, Item 2) can be obtained via alternate minimization: Given a partition $\Pi_\tau := \{t_\tau^k,\, k = 0,\dots N_\tau\}$ of the time interval $[0,\mathsf{T}]$ with $0 = t_\tau^0 \leq t_\tau^1 \leq \ldots \leq t_\tau^{N_\tau} = \mathsf{T}$ and starting with the given initial datum $(u_\tau^0, z_\tau^0) := (u_0,z_0) \in \mathbf{U} \times \mathbf{Z}$, for each $k \in \{0,\dots N_\tau\}$ find a pair (u_τ^k, z_τ^k) via the following staggered time-discrete scheme:

$$u_\tau^k = \mathrm{argmin}_{\tilde{u} \in \mathbf{U}}\, \mathcal{E}(t_\tau^k, \tilde{u}, z_\tau^{k-1}), \tag{19a}$$

$$z_\tau^k \in \mathrm{argmin}_{\tilde{z} \in \mathbf{Z}}\, \mathcal{E}(t_\tau^k, u_\tau^k, \tilde{z}) + \mathcal{R}_1(\tilde{z} - z_\tau^{k-1})\,. \tag{19b}$$

Convergence proofs are based on deriving a discrete version of the defining properties ((15) and (17) with $t_1 = 0$ and '$\leq$') for semistable energetic solutions, where the discrete upper energy-dissipation estimate provides suitable compactness properties, and by subsequently passing to the limit from time-discrete to time-continuous in the defining properties. Here, in particular in case of non-smooth and discontinuous dissipation potentials as in the case of damage and fracture problems, the limit passage in (15b) requires techniques for (evolutionary) Γ-convergence such as the construction of a mutual recovery sequence, cf. e.g., [MR15] and [RT17a, Hyp. 2.5] for further details. The limit passage in the upper energy-dissipation estimate is based on weak lower semicontinuity properties of the functionals and the well-preparedness of the initial data. Instead, if the upper energy-dissipation estimate was required to hold for all $t_1 \in [0,T]$ on the right-hand side of (17), then convergence of the energy along sequences of approximate solutions is needed. However, for non-smooth and non-linear energy functionals, as it is often the case for damage and fracture problems, this property is not available a priori. In some cases, as in Sec. 4, energy convergence along approximate solutions can be obtained a posteriori, e.g., if the energy dissipation estimate can be confirmed to be valid as an equality.

Energetic solutions of rate-independent systems (Def. 1, Item 3) can be obtained by the approximation with solutions of a time-discrete scheme that *simultaneously* minimizes with respect to the pair (u,z), i.e., for all $k \in \{1,\dots,N_\tau\}$ it is

$$(u_\tau^k, z_\tau^k) \in \mathrm{argmin}_{(\tilde{u},\tilde{z}) \in \mathbf{U}\times\mathbf{Z}} \big(\mathcal{E}(t_\tau^k, \tilde{u}, \tilde{z}) + \mathcal{R}_1(\tilde{z} - z_\tau^{k-1})\big)\,. \tag{20}$$

Based on this minimality property such an approximation procedure has been successfully carried out also for energy functionals with weaker convexity properties than required in the deduction (13)–(17). However, if properties (12a) and (13) are employed to determine solutions of (20), then convexity of $\mathcal{E}(t,\cdot,\cdot)$ in the pair (u,z) is needed to find energetic solutions, whereas separate convexity will in general result in semistable energetic solutions, only. Indeed, algorithms based on a FE-discretization in space use (12a) and (13), so that only the approximation of semistable energetic solutions can be expected. In fact, for many applications in damage and phase-field fracture the energy functionals are assumed to be separately convex, but in general lack the joint convexity in the pair (u,z), because of multiplicative terms of the form $w_{\mathbb{C}}(z)W(e(u))$ appearing in the bulk elastic energy, cf. (4) and (5).

Recently the concept of *Balanced Viscosity solutions* for rate-independent systems has gained attention, see, e.g., [MRS12]. This notion of solution can be obtained by introducing an additional viscous dissipation for z, weighted with a parameter

ε. As $\varepsilon \to 0$ a selection of solution for the resulting rate-independent system is made, which is characterized by a local stability condition and an energy-dissipation balance that features in comparison to (18) additional dissipative terms which become active in particular in jump regimes of the solutions, see, e.g., also [KRZ13a] in the setting of damage models. It has been shown in [KN17, AN19] that solutions of this type can be obtained for phase-field fracture problems with the aid of an alternating multi-step algorithm in time. In [ABN18] the convergence of alternating single- and multi-step algorithms in combination with FE-discretization is analyzed in the setting of L^2-gradient flows for the Ambrosio-Tortorelli phase-field fracture model with (non-vanishing) viscous regularization of the damage variable. For this model the authors show that solutions of the limit problem that satisfy the unidirectionality constraint $z(s) \geq z(t)$ for all $s \leq t \in [0,T]$ can be approximated by a posteriori truncated solutions of the discrete, unconstrained problems. Finally, we also refer to [AB19], where the P1 FE-approximation of the quasistatic evolution in terms of semistable energetic solutions is analyzed for the Ambrosio-Tortorelli functional. It is pointed out that the study of the viscous problem as an L^2-gradient flow relies on improved regularity results for elliptic systems [HMW11]. For nonlinearly coupled damage problems this restricts the results to $d = 2$ and to a quasistatic evolution of the displacements. This is why we will employ a different concept for the visco-elastodynamic problem in Sec. 4; it is rather based on the ideas below.

Concepts for rate-independent systems coupled with rate-dependent processes: While mathematical analysis of purely rate-independent systems $(\mathbf{U}\times\mathbf{Z}, \mathcal{R}_1, \mathcal{E})$ is well-established by now, results for rate-independent systems coupled with other rate-dependent processes are much less developed. These types of systems with such a mixed type of evolution arise in mechanics, e.g., if, instead of the quasistatic law (12a) the evolution of the displacements is assumed to be dynamic or subject to dissipative effects in a visco-elastic material, while the evolution of the internal variable z is still governed by a rate-independent dissipation potential $\mathcal{R}_1$. In this case, (12a) is replaced by (a weak formulation of the) momentum balance

$$\rho\ddot{u}(t) + \mathrm{D}_u\mathcal{E}(t,u(t),z(t)) + \mathrm{D}\mathcal{V}(\dot{u}) = 0 \quad \text{in } \mathbf{U}^*, \tag{21}$$

with $\rho > 0$ the mass density and $\mathcal{V} : \mathbf{U} \to [0,\infty)$ a dissipation potential of superlinear growth such that $\mathcal{V}(0) = 0$. In [Rou09] first steps towards the analysis of such coupled rate-independent/rate-dependent systems were made for the case that $\mathcal{E}$ is separately convex in u and z, and that $\mathcal{V}$ is quadratic. This type of viscous dissipation potential covers Kelvin-Voigt rheology, see also Sec. 4. Under this setting [Rou09] provides a notion of solution that consists of the weak formulation of the momentum balance (21), coupled with the semistability inequality (15b), and complemented by an upper energy-dissipation estimate in analogy to (18), see (23) below. In [RT17a] this concept was generalized to non-smooth energy functionals with (lower order) non-convexities based on the notion of Fréchet subdifferentials and also allowing for non-quadratic, convex, lower semicontinuous dissipation potentials $\mathcal{V} : \mathbf{U} \to [0,\infty)$. We denote by $\mathcal{K} : \mathbf{W} \to [0,\infty)$, $\mathcal{K}(\dot{u}) := \int_\Omega \frac{\rho}{2}|\dot{u}|^2\,\mathrm{d}x$ the kinetic energy with $\mathbf{W}$ a Hilbert space and $\mathbf{U}$ a separable Banach space such that $\mathbf{U} \subset \mathbf{W} \subset \mathbf{U}^*$ form an evolution triple. In [RT17a] two different cases are distinguished: the case $\rho \equiv 0$ in Ω where inertia is disregarded and the case $\rho > 0$ in Ω where inertia is present. In the first case, the coupled system forms a gradient system, whereas second case is a damped inertial system. The following definition is used:

Definition 2 ([RT17a, Def. 3.1 & 3.4] semistable energetic solutions for coupled rate-independent/rate-dependent systems)**.** *Let $\mathbf{U},\mathbf{Z}$ be separable Banach spaces, $\mathbf{W}$ a Hilbert space, $\mathcal{E} : [0,\mathsf{T}]\times\mathbf{U}\times\mathbf{Z} \to \mathbb{R}\cup\{\infty\}$ an energy functional, $\mathcal{K} : \mathbf{W} \to [0,\infty)$ the kinetic energy functional, $\mathcal{R}_1 : \mathbf{Z} \to [0,\infty)$ and $\mathcal{V} : \mathbf{U} \to [0,\infty)$ convex and lower semicontinuous dissipation potentials with $\mathcal{R}_1$ positively 1-homogenous and $\mathcal{V}$ of superlinear growth. A coupled rate-independent/rate-dependent system characterized by the tuple $(\mathbf{U},\mathbf{Z},\mathcal{V},\mathcal{R}_1,\mathcal{E})$ is called a gradient system and a coupled system characterized by $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_1,\mathcal{E})$ is called a damped inertial system.*

A pair $(u,z) : [0,\mathsf{T}] \to \mathbf{U}\times\mathbf{Z}$ is called a semistable energetic solution of $(\mathbf{U},\mathbf{Z},\mathcal{V},\mathcal{R}_1,\mathcal{E})$, resp. $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_1,\mathcal{E})$ if the following conditions are satisfied:

- *subdifferential inclusion for u for almost all $t \in (0,\mathsf{T})$:*

$$\rho\ddot{u}(t) + \partial_u\mathcal{E}(t,u(t),z(t)) + \partial\mathcal{V}(\dot{u}(t)) \ni 0 \text{ in } \mathbf{U}^*, \tag{22}$$

  i.e., $\rho\ddot{u}(t) + \xi(t) + \omega(t) = 0$, with $\xi(t) \in \partial_u\mathcal{E}(t,u(t),z(t))$ and $\omega(t) \in \partial\mathcal{V}(\dot{u}(t))$ for almost all $t \in (0,T)$;
- *semistability condition* (15b) *for all $t \in [0,\mathsf{T}]$;*
- *upper energy-dissipation estimate for all $t \in [0,\mathsf{T}]$:*

$$\begin{aligned}&\mathcal{K}(\dot u(t))+\int_0^t \big(\mathcal{V}(\dot u(r))+\mathcal{V}^*(-\xi(r)-\ddot u(r))\big)\,\mathrm{d}r+\mathrm{Var}_{\mathcal{R}_1}(z;[0,t])+\mathcal{E}(t,u(t),z(t))\\ &\le \mathcal{K}(\dot u(0))+\mathcal{E}(0,u(0),z(0))+\int_0^t \partial_r\mathcal{E}(r,u(r),z(r))\,\mathrm{d}r\end{aligned} \tag{23}$$

with $\xi(r)\in\partial_u\mathcal{E}(r,u(r),z(r))$ the selection from (22)*, i.e., $\xi(r)$ fulfills* (22) *for almost all $r\in(0,\mathsf{T})$.*

Moreover, a pair $(u,z):[0,\mathsf{T}]\to\mathbf{U}\times\mathbf{Z}$ is called a weak semistable energetic solution of $(\mathbf{U},\mathbf{Z},\mathcal{V},\mathcal{R}_1,\mathcal{E})$, resp. $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_1,\mathcal{E})$, if for all $t\in[0,\mathsf{T}]$ it satisfies semistability condition (15b) *and the upper energy-dissipation estimate* (23)*.*

Above in (22) the term $\partial\mathcal{V}(\dot u)$ denotes the subdifferential of the convex, lower semicontinuous potential $\mathcal{V}$. Instead $\mathcal{E}$ may feature non-convex terms of lower order, so that $\partial_u\mathcal{E}(t,u,z)$ rather is to be understood in the sense of Fréchet subdifferentials. In (23) the term $\mathcal{V}^*:\mathbf{U}^*\to[0,\infty)$ denotes the Legendre-Fenchel conjugate of the convex potential $\mathcal{V}:\mathbf{U}\to[0,\infty)$. This term in (23) stems from the DeGiorgi-principle for gradient flows, see [RT17a] for a derivation.

Remark 2. *Using suitable staggered time-discrete schemes, alike* (19)*, abstract existence results in the sense of Def. 2 were deduced in [RT17a] for four cases:*

- *semistable energetic solutions for $(\mathbf{U},\mathbf{Z},\mathcal{V},\mathcal{R}_1,\mathcal{E})$ for $\mathcal{V}$ quadratic, cf. [RT17a, Thm. 4.9];*
- *weak semistable energetic solution for $(\mathbf{U},\mathbf{Z},\mathcal{V},\mathcal{R}_1,\mathcal{E})$ for $\mathcal{V}$ with general superlinear growth, cf. [RT17a, Thm. 4.13];*
- *weak semistable energetic solution for $(\mathbf{U},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R}_1,\mathcal{E})$ for $\mathcal{V}$ with general superlinear growth, cf. [RT17a, Thm. 5.4];*
- *semistable energetic solutions for $(\mathbf{U},\mathbf{Z},\mathcal{V},\mathcal{R}_1,\mathcal{E})$ for $\mathcal{V}$ quadratic, cf. [RT17a, Thm. 5.6].*

The notion of semistable energetic solutions for coupled systems has been applied in the context of damage models [LRTT18] as well as for delamination models [RT15, RT17b, TZ17].

3 Finite element approximation for total variation regularized problems

3.1 Model problem and analytical properties

3.1.1 The space $BV(\Omega)$

Classes of weakly differentiable functions like those defined by Sobolev spaces are not suitable to describe quantities that are discontinuities. A function space that contains a large class of discontinuous functions is provided by the set of functions of bounded variation which is the subset of integrable functions on Ω whose distribtutional derivative is a bounded Radon measure, i.e.,

$$BV(\Omega) = \{v \in L^1(\Omega) : Dv \in \mathcal{M}(\Omega)\}.$$

The condition $Dv \in \mathcal{M}(\Omega)$ is specified by the requirement that Dv is of bounded total variation, i.e.,

$$|Dv|(\Omega) = \sup\Big\{ -\int_\Omega v \operatorname{div} \phi \,\mathrm{d}x : \phi \in C_c^\infty(\Omega;\mathbb{R}^d),\ |\phi(x)| \le 1\Big\} < \infty,$$

which means that the operator norm of the distributional derivative Dv is bounded as a functional on compactly supported, smooth functions. If v is weakly differentiable then we have

$$|Dv|(\Omega) = \|\nabla v\|_{L^1(\Omega)}.$$

The space $BV(\Omega)$ is larger than $L^1(\Omega)$ as, e.g., characteristic functions of sets with bounded perimeter are contained in $BV(\Omega)$. The quantity

$$\|v\|_{BV(\Omega)} = \|v\|_{L^1(\Omega)} + |Dv|(\Omega)$$

defines a norm on $BV(\Omega)$ for which it is complete. For variational problems it is important to note that the concept of weak* convergence guarantees that bounded sequences admit suitable subsequences with corresponding limits. For analyzing numerical methods an intermediate notion of convergence is needed which asserts that $v_j \to v$ *intermediately* if

$$v_j \to v \text{ in } L^1(\Omega) \quad \text{and} \quad |Dv_j|(\Omega) \to |Dv|(\Omega).$$

For this notion of convergence density of smooth functions can be established. We refer the reader to [AFP00, ABM06] for details.

3.1.2 Model problem

A model problem arising in image processing determines a regularized image $z \in BV(\Omega) \cap L^2(\Omega)$ of a noisy image $g \in L^2(\Omega)$ via minimizing

$$I(z) = |Dz|(\Omega) + \frac{\alpha}{2}\|z - g\|^2.$$

Despite the implicit definition of the total variation $|Dz|(\Omega)$, the functional has positive analytical features, cf., e.g., [ROF92, CL97, BKP10, Bar15b] for full explanations of the results summarized below.

Proposition 1 (Well posedness)**.** *(i) Given $g \in L^2(\Omega)$ there exists a unique minimizer $z \in BV(\Omega) \cap L^2(\Omega)$ for I. In particular, for every $y \in BV(\Omega) \cap L^2(\Omega)$ we have*

$$\frac{\alpha}{2}\|z - y\|^2 \le I(y) - I(z).$$

(ii) If $z, \widetilde{z} \in BV(\Omega) \cap L^2(\Omega)$ are minimizers corresponding to the data $g, \widetilde{g} \in L^2(\Omega)$ then we have that

$$\|z - \widetilde{z}\| \le \|g - \widetilde{g}\|.$$

(iii) If $g \in L^\infty(\Omega)$ then we have that $z \in L^\infty(\Omega)$ with $\|z\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)}$.

Proof (sketched). The properties are direct consequences of compactness properties of the space $BV(\Omega)$ and coercivity and strong convexity properties of the functional I. □

3.1.3 Dual problem

Important implicit properties of solutions are provided by the dual formulation of the convex minimization problem I. By using the characterization of the total variation $|Dv|(z)$ as a maximization problem we find by exchanging extrema that

$$\begin{aligned}\inf_z I(z) &= \inf_z \sup_p -\int_\Omega z \operatorname{div} p \,\mathrm{d}x - I_{K_1(0)}(p) + \frac{\alpha}{2}\|z-g\|^2 \\ &\geq \sup_p \inf_z -\int_\Omega z \operatorname{div} p \,\mathrm{d}x - I_{K_1(0)}(p) + \frac{\alpha}{2}\|z-g\|^2 \\ &= \sup_p \inf_z L(z,p).\end{aligned}$$

Here we assumed that $p \cdot n = 0$ on the boundary $\partial\Omega$. The functional $I_{K_1(0)}$ denotes the indicator functional of the subset of vector fields $p \in L^2(\Omega;\mathbb{R}^d)$ that satisfy $|p(x)| \leq 1$ almost everywhere in Ω. Given such a vector field p the optimal z in the saddle-point formulation satisfies $\partial_z L(z,p) = 0$, i.e.,

$$-\operatorname{div} p + \alpha(z-g) = 0 \quad \Longleftrightarrow \quad z = g + \alpha^{-1}\operatorname{div} p.$$

This equation complements the condition $0 \in \partial_p L(z,p)$, i.e., the subdifferential inclusion $\nabla z \in \partial I_{K_1(0)}(p)$ or equivalently $p \in \partial|\nabla z|$. Inserting the identity for z into L and using that

$$-g\operatorname{div} p - \alpha^{-1}(\operatorname{div} p)^2 + \frac{\alpha}{2}(\alpha^{-1}\operatorname{div} p)^2 = -\frac{1}{2\alpha}(\operatorname{div} p + \alpha g)^2 + \frac{\alpha}{2}g^2$$

yields the dual functional

$$\begin{aligned}D(p) &= -\frac{1}{2\alpha}\|\operatorname{div} p + \alpha g\|^2 + \frac{\alpha}{2}\|g\|^2 - I_{K_1(0)}(p) \\ &= -\frac{1}{2\alpha}\|\operatorname{div} p\|^2 - \int_\Omega \operatorname{div} p\, g \,\mathrm{d}x - I_{K_1(0)}(p).\end{aligned}$$

The derivation of the functional implies that we have the weak duality relation

$$I(z) \geq D(p)$$

for admissible functions $z \in BV(\Omega) \cap L^2(\Omega)$ and vector fields $p \in W^2_N(\operatorname{div};\Omega)$. In fact, it can be shown that strong duality applies, i.e., that equality holds at optimality, cf. [HK04].

Proposition 2 (Strong duality)**.** *The functionals I and D satisfy the strong duality relation*

$$\inf_z I(z) = \sup_p D(p).$$

Existence of a dual solution p can be established using the direct method in the calculus of variations, uniqueness cannot be expected in general. While it is difficult to establish general regularity properties for the primal problem, solutions of the dual problem may satisfy classical regularity properties such as Lipschitz continuity. The following example illustrates this aspect, cf., e.g., [Bar15b].

Example 1. *Let $r > 0$ be such that $B_r(0) \subset \Omega$ and define $g = \chi_{B_r(0)}$. Then*

$$z = \max\{0, 1 - d/(\alpha r)\}\chi_{B_r}(0)$$

is the minimizer for I subject to $z|_{\partial\Omega} = 0$. Assume that $d \leq \alpha r$ and define

$$p(x) = \begin{cases} -r^{-1}x & \text{for } |x| \leq r, \\ -rx/|x|^2 & \text{for } |x| \geq r. \end{cases}$$

Then $p \in H(\mathrm{div};\Omega)$ *with* $\mathrm{div}\, p = -(d/r)\chi_{B_r(0)}$ *and* $|p| \le 1$. *Moreover, we have* $z = (1/\alpha)\,\mathrm{div}\, p + g$. *Since* $p = -n$ *on* $\partial B_r(0)$ *we have for every* $q \in H(\mathrm{div};\Omega)$ *with* $|q| \le 1$ *that*

$$-(z, \mathrm{div}(q-p)) = -\big(1 - d/(\alpha r)\big) \int_{\partial B_r(0)} (q-p)\cdot n \,\mathrm{d}s \le 0,$$

i.e., $\nabla z \in \partial I_{K_1(0)}(p)$. *If* $d \ge \alpha r$, *we define*

$$p(x) = \begin{cases} -(\alpha/d)x & \text{for } |x| \le r, \\ -(\alpha/d) r^2 x/|x|^2 & \text{for } |x| \ge r \end{cases}$$

and verify $\mathrm{div}\, p = -\alpha\chi_{B_r(0)} = -\alpha g$, *i.e.,* $z = (1/\alpha)\,\mathrm{div}\, p + g = 0$, *and* $|p| \le \alpha r/d \le 1$. *Since* $z = 0$ *the variational inclusion* $Dz \in \partial I_{K_1(0)}(p)$ *is satisfied.*

3.2 Notation in finite element spaces

For a sequence of regular triangulations $(\mathcal{T}_h)_{h>0}$, where $h > 0$ refers to a maximal mesh-size that tends to zero, the set of elementwise polynomial functions or vector fields of maximal polynomial degree $k \ge 0$ is defined by

$$\mathcal{L}^k(\mathcal{T}_h)^\ell = \big\{ v_h \in L^1(\Omega;\mathbb{R}^\ell) : v_h|_T \in P_k(T)^\ell \text{ for all } T \in \mathcal{T}_h \big\}.$$

We let $\Pi_h : L^1(\Omega;\mathbb{R}^\ell) \to \mathcal{L}^0(\mathcal{T}_h)^\ell$ denote the L^2 projection onto elementwise constant functions or vector fields and note that Π_h is self-adjoint, i.e.,

$$\int_\Omega \Pi_h f g \,\mathrm{d}x = \int_\Omega f \Pi_h g \,\mathrm{d}x$$

for all $f, g \in L^1(\Omega)$. We let $\mathcal{S}_h$ denote the set of sides of elements and define the mesh-size function $h_{\mathcal{S}}|_S = h_S = \mathrm{diam}(S)$ for all sides $S \in \mathcal{S}_h$. We let $n_{\mathcal{S}} : \mathcal{S}_h \to \mathbb{R}^d$ denote a unit vector field given for every side $S \in \mathcal{S}_h$ by

$$n_{\mathcal{S}}|_S = n_S$$

for a fixed unit normal n_S on S which is assumed to coincide with the outer unit normal if $S \subset \partial\Omega$. The jump and average on a side S of a function $v_h \in \mathcal{L}^k(\mathcal{T}_h)^\ell$ are for $x \in S$ defined for inner sides via

$$[\![v_h]\!](x) = \lim_{\varepsilon\to 0} \big(v_h(x - \varepsilon n_S) - v_h(x + \varepsilon n_S)\big),$$
$$\{v_h\}(x) = \lim_{\varepsilon\to 0} \frac{1}{2}\big(v_h(x - \varepsilon n_S) + v_h(x + \varepsilon n_S)\big).$$

For $S \subset \partial\Omega$ we set

$$[\![v_h]\!] = \{v_h\} = v_h.$$

The integral means of jumps and averages are denoted by

$$[\![v_h]\!]_h = |S|^{-1} \int_S [\![v_h]\!] \,\mathrm{d}s, \quad \{v_h\}_h = |S|^{-1} \int_S \{v_h\} \,\mathrm{d}s,$$

which in case of elementwise affine functions coincide with the evaluation at the midpoint x_S for every $S \in \mathcal{S}_h$. We denote the space of continuous, piecewise linear functions via

$$\mathcal{S}^1(\mathcal{T}_h) = \mathcal{L}^1(\mathcal{T}_h) \cap C(\overline{\Omega}),$$

and the larger space of discontinuous, piecewise linear functions via

$$\mathcal{S}^{1,dg}(\mathcal{T}_h) = \mathcal{L}^1(\mathcal{T}_h).$$

A space of discontinuous vector fields is given by

$$\mathcal{RT}^{0,dg}(\mathcal{T}_h) = \mathcal{L}^0(\mathcal{T}_h)^d + (\mathrm{id} - x_{\mathcal{T}})\mathcal{L}^0(\mathcal{T}_h),$$

where id is the identity and $x_{\mathcal{T}} = \Pi_h \mathrm{id} \in \mathcal{L}^0(\mathcal{T}_h)^d$ the elementwise constant vector field that coincides with the midpoint x_T on every $T \in \mathcal{T}_h$. Differential operators on these spaces are defined elementwise, indicated by a subscript h, i.e., we have

$$\nabla_h v_h|_T = \nabla(v_h|_T), \quad \mathrm{div}_h\, z_h|_T = \mathrm{div}(z_h|_T)$$

for $v_h \in \mathcal{S}^{1,dg}(\mathcal{T}_h)$, $z_h \in \mathcal{RT}^{0,dg}(\mathcal{T}_h)$ and all $T \in \mathcal{T}_h$. The operators are also applied to weakly differentiable functions and vector fields in which case they coincide with the weak gradient and the weak divergence. By construction, any vector field $y_h \in \mathcal{RT}^{0,dg}(\mathcal{T}_h)$ has a piecewise constant normal component $y_h \cdot n_L$ along straight lines L with normal n_L. Subspaces of elementwise affine functions and vector fields with certain continuity properties on element sides are given by

$$\mathcal{S}_D^{1,cr}(\mathcal{T}_h) = \{v_h \in \mathcal{S}^{1,dg}(\mathcal{T}_h) : [\![v_h]\!]_h|_S = 0 \text{ for all } S \in \mathcal{S}_h \setminus \Gamma_{\mathrm{Neu}}\},$$

and

$$\mathcal{RT}_N^0(\mathcal{T}_h) = \{y_h \in \mathcal{RT}^{0,dg}(\mathcal{T}_h) : [\![y_h \cdot n_S]\!]_h|_S = 0 \text{ for all } S \in \mathcal{S}_h \setminus \Gamma_{\mathrm{Dir}}\},$$

which coincide with low order Crouzeix–Raviart and Raviart–Thomas finite element spaces introduced in [CR73, RT77]. These spaces provide quasi-interpolation operators

$$\mathcal{J}_h^{cr} : W_D^{1,p}(\Omega) \to \mathcal{S}_D^{1,cr}(\mathcal{T}_h), \quad \mathcal{J}_h^{rt} : W_N^q(\mathrm{div};\Omega) \to \mathcal{RT}_N^0(\mathcal{T}_h),$$

with the projection properties

$$\nabla_h \mathcal{J}_h^{cr} v = \Pi_h \nabla v, \quad \mathrm{div}\, \mathcal{J}_h^{rt} y = \Pi_h \mathrm{div}\, y,$$

and the interpolation estimates

$$\begin{aligned} \|v - \mathcal{J}_h^{cr} v\|_{L^p(\Omega)} &\le c_{cr,1} h \|\nabla v\|_{L^p(\Omega)}, \\ \|v - \mathcal{J}_h^{cr} v\|_{L^p(\Omega)} + h\|\nabla_h(v - \mathcal{J}_h^{cr} v)\|_{L^p(\Omega)} &\le c_{cr,2} h^2 \|D^2 v\|_{L^p(\Omega)}, \end{aligned}$$

for $v \in W_D^{2,p}(\Omega)$ with $1 \le p \le \infty$, and

$$\|y - \mathcal{J}_h^{rt} y\|_{L^q(\Omega)} \le c_{rt} h \|\nabla y\|_{L^q(\Omega)}$$

for $y \in W_N^q(\mathrm{div};\Omega)$ with $1 \le q \le \infty$. The standard nodal interpolation operator is denoted by $\mathcal{I}_h : C(\overline{\Omega}) \to \mathcal{S}^1(\mathcal{T}_h)$ and satisfies the estimate

$$\|v - \mathcal{I}_h v\|_{L^p(\Omega)} + h\|\nabla(v - \mathcal{I}_h v)\|_{L^p(\Omega)} \le c_{p1} h^2 \|D^2 v\|_{L^p(\Omega)}.$$

We refer the reader to [Cia78, BBF13, BS08, Bar16b] for details. Elementary calculations lead to the identities

$$[\![v_h y_h \cdot n_S]\!] = \begin{cases} [\![v_h]\!]\{y_h \cdot n_S\} + \{v_h\}[\![y_h \cdot n_S]\!] & \text{if } S \not\subset \partial\Omega, \\ [\![v_h]\!]\{y_h \cdot n_S\} & \text{if } S \subset \Gamma_{\mathrm{Dir}}, \\ \{v_h\}[\![y_h \cdot n_S]\!] & \text{if } S \subset \Gamma_{\mathrm{Neu}}. \end{cases}$$

By carrying out an elementwise integration by parts we thus find that for $v_h \in \mathcal{S}^{1,dg}(\mathcal{T}_h)$ and $y_h \in \mathcal{RT}^{0,dg}(\mathcal{T}_h)$ we have

$$\begin{aligned} &\int_\Omega v_h \,\mathrm{div}\, y_h \,\mathrm{d}x + \int_\Omega \nabla_h v_h \cdot y_h \,\mathrm{d}x \\ &\qquad = \int_{\mathcal{S}_h \setminus \Gamma_{\mathrm{Neu}}} [\![v_h]\!]_h \{y_h \cdot n_S\} \,\mathrm{d}s + \int_{\mathcal{S}_h \setminus \Gamma_{\mathrm{Dir}}} \{v_h\}_h [\![y_h \cdot n_S]\!] \,\mathrm{d}s. \end{aligned} \tag{24}$$

If $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ and $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$ then the terms on the right-hand side are equal to zero.

3.3 Finite element discretization

Typical finite dimensional spaces of functions on Ω such as spaces of continuous or discontinuous piecewise polynomial finite element functions define subsets of $BV(\Omega) \cap L^2(\Omega)$. We show below that their performance in discretizing the model problem can be quite different. We always consider a sequence of regular triangulations $(\mathcal{T}_h)_{h>0}$ of Ω consisting of triangles or tetrahedra for $d = 2$ and $d = 3$ respectively. We recall that associated finite element spaces are defined via sets $P_k(T)$ of polynomials of degree k on the elements $T \in \mathcal{T}_h$ and certain optional continuity conditions across interelement sides. The $P0$ finite element space of elementwise constant functions is given by

$$\mathcal{L}^0(\mathcal{T}_h) = \{v_h \in L^\infty(\Omega) : v_h|_T \in P_0(T) \text{ for all } T \in \mathcal{T}_h\}.$$

Elementwise affine, globally continuous functions are contained in the space of $P1$ finite element functions

$$\mathcal{S}^1(\mathcal{T}_h) = \{v_h \in C(\overline{\Omega}) : v_h|_T \in P_1(T) \text{ for all } T \in \mathcal{T}_h\}.$$

A space of discontinuous functions that are continuous at midpoints of element sides is the Crouzeix–Raviart finite element space

$$\mathcal{S}^{1,cr}(\mathcal{T}_h) = \{v_h \in L^\infty(\Omega) : v_h|_T \in P_1(T) \text{ for all } T \in \mathcal{T}_h, \\ v_h \text{ continuous at all } x_S \text{ for all } S \in \mathcal{S}_h\}.$$

Low order discontinuous Galerkin methods use the space of elementwise affine functions

$$\mathcal{S}^{1,dg}(\mathcal{T}_h) = \{v_h \in L^\infty(\Omega) : v_h|_T \in P_1(T) \text{ for all } T \in \mathcal{T}_h\}.$$

We discuss below the discretization of the model problem with these finite element spaces, i.e., the minimization of the functional

$$I(z) = |Dz|(\Omega) + \frac{\alpha}{2}\|z - g\|^2,$$

restricted to the spaces $\mathcal{L}^0(\mathcal{T}_h)$, $\mathcal{S}^1(\mathcal{T}_h)$, $\mathcal{S}^{1,cr}(\mathcal{T}_h)$, and $\mathcal{S}^{1,dg}(\mathcal{T}_h)$.

3.3.1 Discontinuous $P0$ elements

Using a space of discontinuous functions to discretize the model problem appears attractive as such spaces can reproduce simple discontinuities exactly. However, if the geometry of the underlying sequence of triangulations $(\mathcal{T}_h)_{h>0}$ does not approximate the discontinuity set sufficiently accurately then discrete minimizers may fail to converge to the right objects. We note that we have

$$|Dv_h|(\Omega) = \sum_{S \in \mathcal{S}_h \setminus \partial\Omega} |S| |[\![v_h]\!]_S|$$

for every $v_h \in \mathcal{L}^0(\mathcal{T}_h)$, where $[\![v_h]\!]_S$ is the jump of v_h across an inner side S of the triangulation $\mathcal{T}_h$ whose length or surface area is denoted by $|S|$.

Proposition 3 (Failure of convergence)**.** *Given $n \geq 1$ let $h = 1/n$ and $\mathcal{T}_h$ the triangulation of $\Omega = (-1, 1)^2$ as indicated in Figure 1. Then for $v(x_1, x_2) = \chi_{\{x_1>0\}}(x_1, x_2)$ we have for every sequence $(v_h)_{h>0}$ of functions $v_h \in \mathcal{L}^0(\mathcal{T}_h)$ the implication*

$$v_h \to v \text{ in } L^1(\Omega) \quad \Longrightarrow \quad |Dv_h|(\Omega) \not\to |Dv|(\Omega)$$

as $h \to 0$. In particular, the union of finite element spaces $\cup_{h>0}\mathcal{L}^0(\mathcal{T}_h)$ is not dense in $BV(\Omega)$ with respect to intermediate convergence.

The proposition implies that in general, it is not possible to correctly approximate minimizers of the model problem I via the minimization

$$I(z_h) = \sum_{S \in \mathcal{S}_h} |S| |[\![z_h]\!]_S| + \frac{\alpha}{2}\|z_h - g\|^2$$

in the set of all $z_h \in \mathcal{L}^0(\mathcal{T}_h)$ despite the consistency of the method. We refer the reader to [Bar12, Bar15b] and [BL02] for further details.

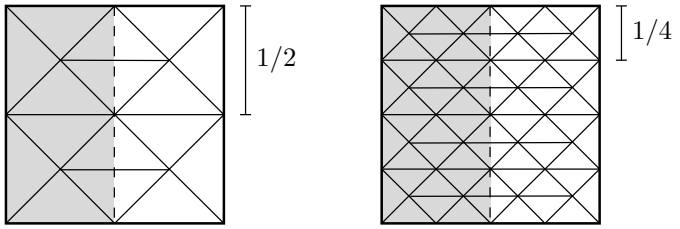

Figure 1: Triangulations $\mathcal{T}_{h_n}$ with $h_n = 1/n$ for $n = 2$ and $n = 4$ used to illustrate the failure of the $P0$ method. The length of the discontinuity set of the discontinuous function $v(x, y) = \operatorname{sign}(x)$ (indicated via gray shading) is incorrectly approximated by any sequence of piecewise constant functions $(v_h)_{h>0}$ with $v_h \to v$ in $L^1(\Omega)$.

3.3.2 Continuous $P1$ elements

The standard finite element space of piecewise affine, globally continuous functions $\mathcal{S}^1(\mathcal{T}_h)$ provides the approximation property that for all $v \in BV(\Omega)$ there exists a sequence $(v_h)_{h>0}$ with $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ for all $h > 0$ and

$$v_h \to v \text{ in } L^1(\Omega) \quad \& \quad |Dv_h|(\Omega) \to |Dv|(\Omega)$$

as $h \to 0$. This is an immediate consequence of the intermediate density of smooth functions in $BV(\Omega)$ and the density of the union of $P1$ finite element spaces in the space $W^{1,1}(\Omega)$. In fact, we have that

$$|Dv_h|(\Omega) = \int_\Omega |\nabla v_h| \,\mathrm{d}x$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$. If $z_h \in \mathcal{S}^1(\mathcal{T}_h)$ is the minimizer for

$$I(z_h) = \int_\Omega |\nabla z_h| \,\mathrm{d}x + \frac{\alpha}{2}\|z_h - g\|^2$$

in the set of all $z_h \in \mathcal{S}^1(\mathcal{T}_h)$ then it follows that

$$\frac{\alpha}{2}\|z - z_h\|^2 \le I(z_h) - I(z) \le I(v_h) - I(z) \to 0$$

as $h \to 0$ if the sequence $(v_h)_{h>0}$ is chosen such that $v_h \to z$ intermediately in $BV(\Omega)$. By an explicit construction of an approximating sequence $(v_h)_{h>0}$ for a given function $z \in BV(\Omega) \cap L^\infty(\Omega)$ it is possible to determine a convergence rate as in [WL11, Bar12, BNS14].

Proposition 4 (Suboptimal convergence)**.** *Assume that $g \in L^\infty(\Omega)$ and Ω is star shaped. Then we have that*

$$\|z - z_h\| \le ch^{1/4},$$

where $c > 0$ depends on the geometry of Ω and the triangulations, as well as α and $\|g\|_{L^\infty(\Omega)}$.

Proof (sketched). The strong convexity property of I and a binomial formula lead for arbitrary $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ to the estimate

$$\begin{aligned}
\frac{\alpha}{2}\|z - z_h\|^2 &\le I(z_h) - I(z) \\
&\le I(v_h) - I(z) \\
&= |Dv_h|(\Omega) - |Dz|(\Omega) + \frac{\alpha}{2}\int_\Omega (v_h - g)^2 - (z - g)^2 \,\mathrm{d}x \\
&\le |Dv_h|(\Omega) - |Dz|(\Omega) + \frac{\alpha}{2}\|v_h - z\|_{L^1(\Omega)}\|v_h + z + 2g\|_{L^\infty(\Omega)}.
\end{aligned}$$

By choosing a regularization $z_\varepsilon \in C^\infty(\overline{\Omega})$ of z and setting $v_{h,\varepsilon} = \mathcal{I}_h z_\varepsilon$ one derives the bounds

$$\begin{aligned}
\|z - v_{h,\varepsilon}\|_{L^1(\Omega)} &\le c\big(h^2\varepsilon^{-1} + \varepsilon\big)|Dz|(\Omega), \\
|Dv_{h,\varepsilon}|(\Omega) &\le |Dz|(\Omega) + c\big(h\varepsilon^{-1} + \varepsilon\big)|Dz|(\Omega), \\
\|v_{h,\varepsilon}\|_{L^\infty(\Omega)} &\le \|z\|_{L^\infty(\Omega)}.
\end{aligned}$$

With these estimates we deduce that

$$\frac{\alpha}{2}\|z - z_h\|^2 \leq c\big(h\varepsilon^{-1} + h^2\varepsilon^{-1} + \varepsilon\big).$$

The choice $\varepsilon = \varepsilon^{1/2}$ leads to the asserted error bound. □

The estimate can be improved if a total-variation diminishing quasi interpolation operator is available, i.e., on an intermediately dense subset $X \subset BV(\Omega)$ there exists an operator $\widetilde{\mathcal{I}}_h : X \to \mathcal{S}^1(\mathcal{T}_h)$ with the monotonicity estimate

$$\|\nabla\widetilde{\mathcal{I}}_h v\|_{L^1(\Omega)} \leq |Dv|(\Omega),$$

the approximation and stability bounds

$$\|v - \widetilde{\mathcal{I}}_h v\|_{L^1(\Omega)} \leq ch, \quad \|\widetilde{\mathcal{I}}_h v\|_{L^\infty(\Omega)} \leq c\|v\|_{L^\infty(\Omega)},$$

then by following the lines of the proof of the previous proposition one finds, cf. [BNS15], that

$$\|z - z_h\| \leq ch^{1/2}.$$

Total-variation diminishing interpolation operators can be constructed in one-dimensional settings or if the anisotropic variant of the total variation is used on regular partitions. The convergence rate $\mathcal{O}(h^{1/2})$ is optimal for the approximation of a discontinuous function by continuous finite element functions, e.g., for a generic function $v \in BV(\Omega) \cap L^\infty(\Omega)$ with discontinuity, we have for the L^2 best approximation

$$\inf_{v_h \in \mathcal{S}^1(\mathcal{T}_h)} \|v - v_h\| \geq ch^{1/2}.$$

As shown in [Bar15b] this can be verified directly in the simple setting $\Omega = (-1, 1)$, $\mathcal{T}_h$ a sequence of symmetric triangulations with respect to the origin, and the function $v(x) = \operatorname{sign}(x)$ as illustrated in Figure 2. To prove the estimate, we first note that the optimal finite element approximation satisfies $v_h(0) = 0$. This follows from the fact thatfor the unique optimal function v_h and its reflection $\widetilde{v}_h(x) = -v_h(-x)$ we have, noting that $-v(-x) = v(x)$,

$$\|v - v_h\| = \|v - \widetilde{v}_h\|.$$

Considering now the convex combination $w_h = (v_h + \widetilde{v}_h)/2$ and noting that the L^2 norm is convex we deduce that

$$\|v - w_h\| \leq \frac{1}{2}\|v - v_h\| + \frac{1}{2}\|v - \widetilde{v}_h\| = \|v - v_h\|.$$

By uniqueness we obtain that necessarily $v_h = w_h$ where w_h satisfies by construction $w_h(0) = 0$. Moreover, we obtain that v_h satisfies $v_h(-x) = -v_h(x)$. On the two elements adjacent to the origin covering the region $(-h, h)$ we have that $v_h(x) = ax$ and hence

$$\|v_h - v\|^2 \geq 2\int_0^h (ax - 1)^2 \,\mathrm{d}x = 2(a^2h^3/3 - ah^2 + h).$$

The minimal value occurs for $a = (3/2)h^{-1}$ and equals $h/2$.

3.3.3 Crouzeix–Raviart method

The error analysis of the continuous $P1$ method revealed the importance of a total-variation diminishing interpolation operator. The Crouzeix–Raviart finite element method provides an inconsistent variant of this property via the quasi-interpolation operator $\mathcal{J}_h^{cr}$ with the property

$$\nabla_h \mathcal{J}_h^{cr} v = \Pi_h \nabla v,$$

where ∇_h is the elementwise application of the gradient operator and Π_h the orthogonal projection onto elementwise constant vector fields. Jensen's inequality directly implies the monotonicity property

$$\|\nabla_h \mathcal{J}_h^{cr} v\|_{L^1(\Omega)} \leq \|\nabla v\|_{L^1(\Omega)}.$$

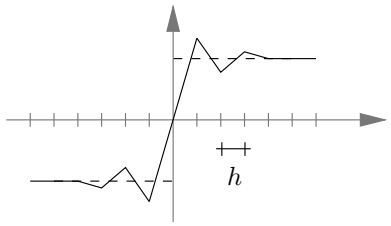

Figure 2: Oscillations in the approximation of a discontinuous function by continuous, piecewise affine functions. Oscillations occur in a neighborhood of the discontinuity and lead to suboptimal convergence behavior.

Via appropriate density arguments this property can be carried over to functions $v \in BV(\Omega)$. We follow [CP19, Bar20b] and use the discrete functional

$$I_h(z_h) = \int_\Omega |\nabla_h z_h| \,\mathrm{d}x + \frac{\alpha}{2}\|\Pi_h z_h - g_h\|^2$$

on the set $\mathcal{S}^{1,cr}(\mathcal{T}_h)$ to approximate minimizers of the model problem. The functional I_h is an inconsistent approximation of I since the first term does not coincide with the total variation $|Dz_h|(\Omega)$ of a discontinuous function $z_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$. Therefore, an error analysis has to control the effect of inconsistency of the method. This is done via a discrete duality argument. We thus consider the discrete dual problem consisting in maximizing the functional

$$D_h(p_h) = -\frac{1}{2\alpha}\|\operatorname{div} p_h + \alpha g_h\|^2 + \frac{\alpha}{2}\|g_h\|^2 - I_{K_1(0)}(\Pi_h p_h)$$

in the set of discrete vector fields $p_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$. The indicator functional $I_{K_1(0)}$ applied to the elementwise average of p_h enforces midpoint values $p_h(x_T)$ to satisfy $|p_h(x_T)| \le 1$ for all $T \in \mathcal{T}_h$. An important feature is the following discrete duality relation.

Proposition 5 (Discrete duality)**.** *Assume that $g_h \in \mathcal{L}^0(\mathcal{T}_h)$. Then, the functionals I_h defined on $\mathcal{S}^{1,cr}(\mathcal{T}_h)$ and D_h defined on $\mathcal{RT}^0_N(\mathcal{T}_h)$ are in discrete duality, i.e.,*

$$\inf_{z_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)} I_h(z_h) \ge \sup_{p_h \in \mathcal{RT}^0_N(\mathcal{T}_h)} D_h(p_h).$$

Proof. We first note that for any vector field $p_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ with $|p_h(x_T)| \le 1$ for all $T \in \mathcal{T}_h$ we have

$$\int_T p_h \cdot \nabla_h z_h \,\mathrm{d}x \le \int_T |\nabla_h z_h| \,\mathrm{d}x$$

for all $T \in \mathcal{T}_h$ since $\nabla_h z_h$ is constant on T. We use the discrete integration-by-parts (24) formula to verify

$$\begin{aligned}
&\int_\Omega |\nabla_h z_h| \,\mathrm{d}x + \frac{\alpha}{2}\|\Pi_h z_h - g_h\|^2 \\
&\qquad \ge \int_\Omega p_h \cdot \nabla_h z_h \,\mathrm{d}x - I_{K_1(0)}(\Pi_h p_h) + \frac{\alpha}{2}\|\Pi_h z_h - g_h\|^2 \\
&\qquad = -\int_\Omega \operatorname{div} p_h z_h \,\mathrm{d}x - I_{K_1(0)}(\Pi_h p_h) + \frac{\alpha}{2}\|\Pi_h z_h - g_h\|^2.
\end{aligned}$$

For the convex function $G(s) = (\alpha/2)|s-g|^2$ for $s, g \in \mathbb{R}$ we have Fenchel's inequality

$$G(s) - rs \ge -G^*(r)$$

with

$$G^*(r) = \sup_{s\in\mathbb{R}} rs - G(s) = \frac{1}{2\alpha}(r + \alpha g)^2 - \frac{\alpha}{2}g^2.$$

Hence, it follows that

$$\int_\Omega |\nabla_h z_h|\,\mathrm{d}x + \frac{\alpha}{2}\|\Pi_h z_h - g_h\|^2 \\ \geq -\frac{1}{2\alpha}\|\operatorname{div} p_h + \alpha g_h\|^2 + \frac{\alpha}{2}\|g\|^2 - I_{K_1(0)}(\Pi_h p_h).$$

Since z_h and p_h are arbitrary, this implies the asserted estimate. □

Since the modulus function and its convex conjugate can be approximated uniformly on their supports by differentiable functions, a strong duality relation can be established, i.e., that in fact equality applies in Proposition 5, cf. [CP19, Bar20b]. For the quasi-optimal error estimate stated below the weak duality result of the proposition is sufficient.

Theorem 1 (Quasi-optimality, [CP19, Bar20b]). *If $z_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$ and $z \in BV(\Omega) \cap L^\infty(\Omega)$ are the minimizers of I_h and I, respectively, for some $g \in L^\infty(\Omega)$ and with $g_h = \Pi_h g$ and if there exists a dual solution $p \in W^2_N(\operatorname{div};\Omega)$ with*

$$p \in W^{1,\infty}(\Omega;\mathbb{R}^d),$$

then we have the quasi-optimal error estimate

$$\|z - \Pi_h z_h\| \leq c h^{1/2}.$$

Proof. The discrete functional I_h satisfies the coercivity property

$$\frac{\alpha}{2}\|\Pi_h(y_h - z_h)\|^2 \leq I_h(y_h) - I_h(z_h).$$

Using the discrete duality relation $I_h(z_h) \geq D_h(p_h)$, and choosing $\widetilde{z}_h = \mathcal{J}_h^{cr} z$ and $\widetilde{p}_h = \gamma_h^{-1}\mathcal{J}_h^{rt} p$, with $\gamma_h = \max\{1, \|\mathcal{J}_h^{rt} p\|_{L^\infty(\Omega)}\}$ so that $|\widetilde{p}_h(x_T)| \leq 1$ for all $T \in \mathcal{T}_h$ and hence $\widetilde{p}_h$ is admissible in D_h, we find that

$$\frac{\alpha}{2}\|\Pi_h(\widetilde{z}_h - z_h)\|^2 \leq I_h(\widetilde{z}_h) - D_h(\widetilde{p}_h).$$

The monotonicity property $\|\nabla_h \widetilde{z}_h\|_{L^1(\Omega)} \leq |Dz|(\Omega)$ and the identity

$$\|\Pi_h(\widetilde{z}_h - g)\|^2 = \|\Pi_h \widetilde{z}_h - g\|^2 - \|g - g_h\|^2 \\ = \|z - g\|^2 + \int_\Omega \big(\Pi_h \widetilde{z}_h - z\big)\big(\Pi_h \widetilde{z}_h + z - 2g\big)\,\mathrm{d}x - \|g - g_h\|^2$$

imply that

$$\begin{aligned} I_h(\widetilde{z}_h) &= \|\nabla_h \widetilde{z}_h\|_{L^1(\Omega)} + \frac{\alpha}{2}\|\Pi_h(\widetilde{z}_h - g)\|^2 \\ &\leq |Dz|(\Omega) + \frac{\alpha}{2}\|z - g\|^2 - \frac{\alpha}{2}\|g - g_h\|^2 \\ &\quad + \frac{\alpha}{2}\|\Pi_h \widetilde{z}_h - z\|_{L^1(\Omega)}\|\Pi_h \widetilde{z}_h + z - 2g\|_{L^\infty(\Omega)} \\ &= I(z) - \frac{\alpha}{2}\|g - g_h\|^2 \\ &\quad + \frac{\alpha}{2}\|\Pi_h \widetilde{z}_h - z\|_{L^1(\Omega)}\|\Pi_h \widetilde{z}_h + z - 2g\|_{L^\infty(\Omega)}. \end{aligned}$$

Defining $\widetilde{p} = \gamma_h^{-1} p$ we have that $\widetilde{p}_h = \mathcal{J}_h^{rt}\widetilde{p}$ and

$$\operatorname{div}\widetilde{p}_h + g_h = \Pi_h(\operatorname{div}\widetilde{p} + g).$$

The identity $\|g\|^2 - \|g_h\|^2 = \|g - g_h\|^2$ shows that

$$D_h(\widetilde{p}_h) = -\frac{1}{2\alpha}\|\operatorname{div}\widetilde{p}_h + \alpha g_h\|^2 + \frac{\alpha}{2}\|g_h\|^2 \\ \geq -\frac{1}{2\alpha}\|\operatorname{div}\widetilde{p} + \alpha g\|^2 + \frac{\alpha}{2}\|g\|^2 - \frac{\alpha}{2}\|g - g_h\|^2.$$

We use that $1 \le \gamma_h \le 1 + cLh$ to deduce that

$$\begin{aligned} D_h(\widetilde{p}_h) &\ge -\frac{1}{2\alpha}\|\operatorname{div}\widetilde{p}\|^2 - \int_\Omega \operatorname{div}\widetilde{p}\, g \,\mathrm{d}x - \frac{\alpha}{2}\|g - g_h\|^2 \\ &= -\frac{1}{2\alpha}\gamma_h^{-2}\|\operatorname{div} p\|^2 - \gamma_h^{-1}\int_\Omega g \operatorname{div} p \,\mathrm{d}x - \frac{\alpha}{2}\|g - g_h\|^2 \\ &\ge -\frac{1}{2\alpha}\gamma_h^{-2}\|\operatorname{div} p\|^2 - \int \operatorname{div} p\, g \,\mathrm{d}x \\ &\quad - (1-\gamma_h^{-1})\|g\|\|\operatorname{div}\widetilde{p}\| - \frac{\alpha}{2}\|g - g_h\|^2 \\ &\ge D(p) - (1-\gamma_h^{-1})\|g\|\|\operatorname{div}\widetilde{p}\| - \frac{\alpha}{2}\|g - g_h\|^2. \end{aligned}$$

By combining the estimates, noting that $1 - \gamma_h^{-1} \le ch\|\nabla p\|_{L^\infty(\Omega)}$, and using $\|z - \Pi_h\widetilde{z}_h\|^2 \le ch|Dz|(\Omega)\|z\|_L^{\infty(\Omega)}$, we deduce the asserted error bound. □

3.3.4 Discontinuous Galerkin method

The discontinuous Galerkin finite element method generalizes the Crouzeix–Raviart method by introducing jump and average terms. It is crucial to use quadrature via midpoint evaluation to obtain a precise duality relation. We follow [Bar20a].

Definition 3 (Jumps and averages)**.** *Let $r, s \ge 1$ and let $\alpha_{\mathcal{S}}, \beta_{\mathcal{S}} : \mathcal{S}_h \to \mathbb{R}_{\ge 0}$ be piecewise constant. For $z_h \in \mathcal{S}^{1,dg}(\mathcal{T}_h)$ and $p_h \in \mathcal{RT}^{0,dg}(\mathcal{T}_h)$ define*

$$\begin{aligned} J_h(z_h) &= \frac{1}{r}\|\alpha_{\mathcal{S}}^{-1}[\![z_h]\!]_h\|_{L^r(\mathcal{S}_h\setminus\Gamma_{\mathrm{Neu}})}^r + \frac{1}{s}\|\beta_{\mathcal{S}}\{z_h\}_h\|_{L^s(\mathcal{S}_h\setminus\Gamma_{\mathrm{Dir}})}^s, \\ K_h(p_h) &= \frac{1}{r'}\|\alpha_{\mathcal{S}}\{p_h\cdot n_{\mathcal{S}}\}\|_{L^{r'}(\mathcal{S}_h\setminus\Gamma_{\mathrm{Neu}})}^{r'} + \frac{1}{s'}\|\beta_{\mathcal{S}}^{-1}[\![p_h\cdot n_{\mathcal{S}}]\!]\|_{L^{s'}(\mathcal{S}_h\setminus\Gamma_{\mathrm{Dir}})}^{s'}, \end{aligned}$$

where we require $[\![z_h]\!]_h = 0$ if $\alpha_{\mathcal{S}} = 0$ and $[\![p_h\cdot n_{\mathcal{S}}]\!] = 0$ if $\beta_{\mathcal{S}} = 0$. For $r = 1$ or $s = 1$ the functionals $(1/r')\|\cdot\|_{L^{r'}}^{r'}$ or $(1/s')\|\cdot\|_{L^{s'}}^{s'}$ are interpreted as indicator functionals $I_{K_1(0)}$ of the closed unit ball $K_1(0)$.

We have the following discrete duality result, here stated for the case of the total variation minimization problem.

Proposition 6 (dG duality)**.** *For $u_h \in \mathcal{S}^{1,dg}(\mathcal{T}_h)$ and $g_h = \Pi_h g$ let*

$$I_h(u_h) = \int_\Omega |\nabla_h z_h|\,\mathrm{d}x + \frac{\alpha}{2}\|\Pi_h z_h - g_h\|^2 + J_h(z_h),$$

Then with the discrete dual functional defined for $p_h \in \mathcal{RT}^{0,dg}(\mathcal{T}_h)$ by

$$D_h(p_h) = -I_{K_1(0)}(\Pi_h p_h) - \frac{1}{2\alpha}\|\operatorname{div}_h p_h + \alpha g_h\| + \frac{\alpha}{2}\|g_h\|^2 - K_h(z_h)$$

we have

$$I_h(z_h) \ge D_h(p_h)$$

and equality holds if and only if z_h and p_h are optimal for I_h and D_h, respectively.

By adapting the arguments that lead to the error estimate in case of the Crouzeix-Raviart method we obtain a similar estimate here.

Proposition 7 (Error estimate)**.** *Assume that $g \in L^\infty(\Omega)$ and that there exists a Lipschitz continuous solution $p \in W_N^2(\operatorname{div};\Omega)\cap W^{1,\infty}(\Omega)$ for the dual problem. Moreover, suppose that*

$$\|h_{\mathcal{S}}^{-1}\alpha_{\mathcal{S}}^{r'}\|_{L^\infty(\mathcal{S}_h)} + \|h_{\mathcal{S}}^{-1}\beta_{\mathcal{S}}^{s}\|_{L^\infty(\mathcal{S}_h)} \le ch,$$

where the first term can be omitted if $r = 1$ and $0 < \alpha_{\mathcal{S}} \le 1$. Then, for the solutions $z \in BV(\Omega)\cap L^2(\Omega)$ and $z_h \in \mathcal{S}^{1,dg}(\mathcal{T}_h)$ of the primal and discrete primal problem we have

$$\|z - \Pi_h z_h\| \le ch^{1/2}M_{z,p,g},$$

with a factor $M_{z,p,g}$ that depends on $\alpha > 0$, $\|g\|_{L^\infty(\Omega)}$, and $\|\nabla p\|_{L^\infty(\Omega)}$.

3.3.5 A posteriori error estimates

Duality relations also lead to computable error estimates for conforming discretizations.

Proposition 8 (A posteriori error estimate, [Bar15a]). *Let $z \in BV(\Omega) \cap L^2(\Omega)$ be minimal for I and $z_h \in \mathcal{S}^1(\mathcal{T}_h)$ be arbitrary. For every $p_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ with $|p_h| \leq 1$ we have that*

$$\frac{\alpha}{2}\|z - z_h\|^2 \leq \int_\Omega |\nabla z_h| - \nabla z_h \cdot p_h \,\mathrm{d}x + \frac{1}{2\alpha}\|\operatorname{div} p_h - \alpha(z_h - g)\|^2.$$

Proof. Given any $p \in W^2_N(\operatorname{div};\Omega)$ with $|p| \leq 1$ almost everywhere in Ω we have by minimality of z and conformity of the $P1$ method that

$$\frac{\alpha}{2}\|z - z_h\|^2 \leq I(z_h) - I(z).$$

The continuous duality relation yields that $I(z) \geq D(p_h)$ and hence we have that

$$\begin{aligned} I(z_h) - I(z) &\leq I(z_h) - D(p_h) \\ &= \int_\Omega |\nabla z_h| \,\mathrm{d}x + \frac{\alpha}{2}\|z_h - g\|^2 + \frac{1}{2\alpha}\|\operatorname{div} p_h + \alpha g\|^2 - \frac{\alpha}{2}\|g\|^2. \end{aligned}$$

We use that

$$-\int_\Omega \nabla z_h \cdot p_h \,\mathrm{d}x = \int_\Omega z_h \cdot \operatorname{div} p_h \,\mathrm{d}x$$

and

$$\begin{aligned} &\frac{1}{2\alpha}\|\operatorname{div} p_h - \alpha(z_h - g)\|^2 \\ &\quad = \frac{1}{2\alpha}\|\operatorname{div} p_h + \alpha g\|^2 - \int_\Omega (\operatorname{div} p + \alpha g) z_h \,\mathrm{d}x + \frac{\alpha}{2}\|z_h\|^2 \\ &\quad = \frac{1}{2\alpha}\|\operatorname{div} p_h + \alpha g\|^2 - \int_\Omega z_h \operatorname{div} p \,\mathrm{d}x + \frac{\alpha}{2}\|z_h - g\|^2 - \frac{\alpha}{2}\|g\|^2. \end{aligned}$$

A combination of the equations yields the asserted estimate. □

The a posteriori error estimate is of residual type since the optimal $z \in BV(\Omega) \cap L^2(\Omega)$ and an optimal vector field p are formally related via the identities

$$\operatorname{div} p = \alpha(z - g), \quad p = \frac{\nabla z}{|\nabla z|}.$$

The error estimate is optimal. If p_h is the solution of the dual problem restricted to the Raviart–Thomas finite element space with a relaxation of the constraint $|p_h| \leq 1$. By computing an approximation with the elementwise constraint $|\widetilde{p}_h(x_T)| \leq 1$ for all $T \in \mathcal{T}_h$ one obtains a vector field $\widetilde{p}_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ and may then define

$$p_h = \gamma_h^{-1}\widetilde{p}_h$$

where $\gamma_h = \max\{1, \|\widetilde{p}_h\|_{L^\infty(\Omega)}\}$. In fact, the elementwise quantity

$$\gamma_h(T) = \max\{0, \|\widetilde{p}_h\|_{L^\infty(T)} - 1\},$$

$T \in \mathcal{T}_h$, may be used as an additional error indicator.

3.3.6 Numerical experiments

Figures 3, 4, and 5 show the numerical results of the finite element discretization of the model problem using standard $P1$ finite elements, the Crouzeix-Raviart method, and the discretization of the dual problem using the Raviart-Thomas method. The setting was chosen as in Example 1 with $d = 2$, $\Omega = (-1,1)^2$, $r = 1/2$, and $\alpha = 10$. The advantages

of the nonconforming methods become apparent when the projections onto piecewise constant functions are plotted. The $P1$ function leads to an inaccurate approximation of the circular discontinuity set which is improved by the other methods. The discrete problems were solved with the methods described in the subsequent section and we refer the reader to [BM16, BM17, BM20] for comparisons of their performances. For iterative methods for the dual problem we refer the reader to [Cha04, HHS$^+$19].

3.4 Iterative solution methods

The nondifferentiability of the functional I and limited regularity properties of solutions lead to difficulties in the iterative solution of the discretized model problem. We discuss below possible approaches and address aspects such as choice of step sizes, monotonicity properties, and the development of stopping criteria. Throughout what follows we use for a step size $\tau > 0$ the difference quotient operator

$$d_t a^k = \tau^{-1}(a^k - a^{k-1})$$

for an arbitrary sequence $(a^k)_{k=0,1,\dots}$ in a linear space X.

3.4.1 Regularized gradient descent

A classical gradient descent approach can be used if a regularization of the functional is introduced, e.g., via a regular approximation of the modulus function or the euclidean length, e.g., for $\varepsilon > 0$ and $a \in \mathbb{R}^d$ we define

$$|a|_\varepsilon = (|a|^2 + \varepsilon^2)^{1/2}.$$

This leads to the regularized functional

$$I_\varepsilon(z) = \int_\Omega |\nabla z|_\varepsilon \,\mathrm{d}x + \frac{\alpha}{2}\|z - g\|^2.$$

The uniform estimate $0 \le |a|_\varepsilon - |a| \le \varepsilon$ for all $a \in \mathbb{R}^d$ implies that minimizers z for I and z_ε for I_ε are related via

$$\frac{\alpha}{2}\|z - z_\varepsilon\|^2 \le \varepsilon,$$

cf. [FvOP05, ES09] for related estimates. In a finite element setting this motivates using $\varepsilon = h$. With a suitable inner product $(\cdot,\cdot)_*$ and a semi-implicit treatment of the variation δI_ε we obtain the following numerical scheme.

Algorithm 1 (Regularized gradient descent)**.** *Let $z^0 \in W^{1,1}(\Omega)$ and choose $\tau, \varepsilon_{\mathrm{stop}} > 0$, set $k = 1$.*
(1) Compute $z^k \in W^{1,2}(\Omega)$ such that

$$(d_t z^k, v)_* + \int_\Omega \frac{\nabla z^k}{|\nabla z^{k-1}|_\varepsilon} \cdot \nabla v \,\mathrm{d}x + \alpha \int_\Omega (z^k - g) v \,\mathrm{d}x = 0$$

for all $v \in W^{1,2}(\Omega)$.
(2) Stop if $\|d_t z^k\|_* \le \varepsilon_{\mathrm{stop}}$; otherwise increase $k \to k+1$ and continue with (1).

The semi-implicit treatment eliminates monotonicity properties of the variation δI_ε. Remarkably, an energy decay property can be established unconditionally for $\varepsilon > 0$.

Proposition 9 (Energy decay, [BDN18])**.** *For $\varepsilon > 0$ the iterates $(z^k)_{k=0,1,\dots}$ are well defined and satisfy for every $K \ge 0$*

$$I_\varepsilon(z^K) + \tau \sum_{k=1}^{K} \|d_t z^k\|_*^2 \le I_\varepsilon(z^0).$$

In particular, we have that $d_t z^k \to 0$ as $k \to \infty$ and the sequence $(z^k)_{k\ge 0}$ converges weakly to the unique minimizer z_ε of I_ε.

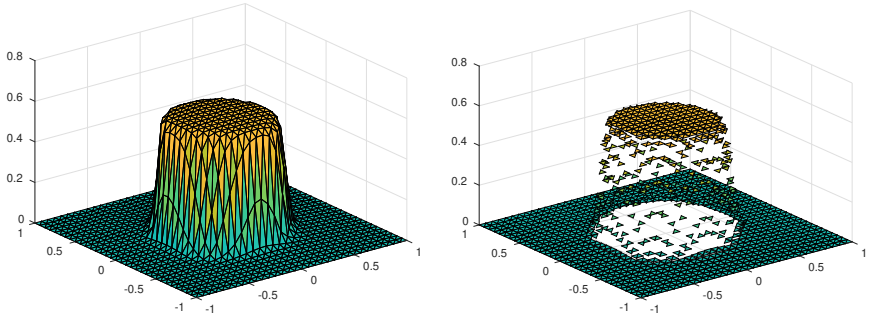

Figure 3: Numerical solution (left) obtained with the continuous $P1$ finite element method and its elementwise average (right). Although a reasonably acurate resolution of the jump is obtained its circular geometry is not well resolved.

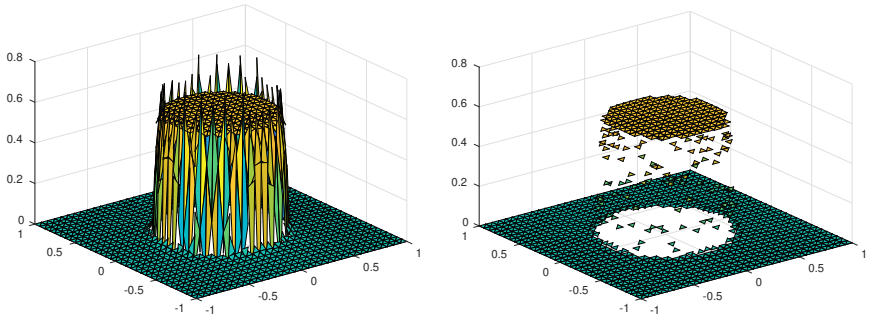

Figure 4: Numerical solution (left) obtained with the Crouzeix-Raviart finite element method and its elementwise average (right). While the approxmimation does not obey a maximum principle, the circular discontinuity set is well approximated.

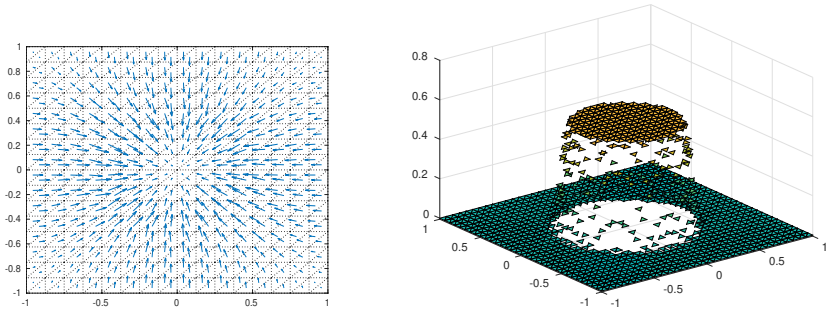

Figure 5: Numerical solution (left) obtained with the Raviart-Thomas method for the dual formulation and the resulting elementwise constant approximation $z_h = \alpha^{-1} \operatorname{div} p_h + g_h$ of the primal variable (right). The approximation is nearly identical with the averages of the Crouzeix-Raviart approximation.

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Proof (sketched). To illustrate the main idea of the proof we omit the quadratic term in I_ε, i.e., we assume for simplicity $\alpha = 0$ and note that in this case choosing $v = d_t z^k$ in Algorithm 1 shows that

$$\|d_t z^k\|_*^2 + \frac{1}{2}\int_\Omega \frac{d_t|\nabla z^k|^2 + \tau|d_t\nabla z^k|^2}{|\nabla z^{k-1}|_\varepsilon}\,\mathrm{d}x = 0.$$

To identify the regularized energy I_ε on the left-hand side we employ elementary formulas related to difference quotient calculus and derive the identity

$$\begin{aligned} d_t|a^k|_\varepsilon = d_t\frac{|a^k|_\varepsilon^2}{|a^k|_\varepsilon} &= \frac{d_t|a^k|_\varepsilon^2}{|a^{k-1}|_\varepsilon} + |a^k|_\varepsilon^2\, d_t\frac{1}{|a^k|_\varepsilon} \\ &= \frac{d_t|a^k|_\varepsilon^2}{|a^{k-1}|_\varepsilon} - |a^k|_\varepsilon^2\frac{d_t|a^k|_\varepsilon}{|a^{k-1}|_\varepsilon|a^k|_\varepsilon} \\ &= \frac{d_t|a^k|_\varepsilon^2}{|a^{k-1}|_\varepsilon} - \frac{|a^k|_\varepsilon d_t|a^k|_\varepsilon}{|a^{k-1}|_\varepsilon} \\ &= \frac{d_t|a^k|_\varepsilon^2}{|a^{k-1}|_\varepsilon} - \frac{1}{2}\frac{d_t|a^k|_\varepsilon^2 + \tau(d_t|a^k|_\varepsilon)^2}{|a^{k-1}|_\varepsilon} \\ &= \frac{1}{2}\frac{d_t|a^k|_\varepsilon^2}{|a^{k-1}|_\varepsilon} - \frac{1}{2}\frac{\tau(d_t|a^k|_\varepsilon)^2}{|a^{k-1}|_\varepsilon}. \end{aligned}$$

Using this formula with $a^k = \nabla z^k$ and noting that $d_t|a^k|_\varepsilon^2 = d_t|a^k|^2$ for the regularized euclidean length, we find that

$$\|d_t z^k\|_*^2 + d_t\int_\Omega |\nabla z^k|_\varepsilon\,\mathrm{d}x + \frac{\tau}{2}\int_\Omega \frac{|d_t\nabla z^k|^2 + (d_t|\nabla z^k|_\varepsilon)^2}{|\nabla z^{k-1}|_\varepsilon}\,\mathrm{d}x = 0.$$

This implies the asserted bound. □

While stability of the iteration is independent of ε and also of a spatial discretization, the development of an efficient stopping criterion, i.e., optimal choice of $\varepsilon_{\mathrm{stop}}$ is difficult. Related error estimates have to control the effect of the semi-implicit treatment of the operator which introduces a critical dependence on ε; we refer the reader to [BDN18, BR20] for related estimates.

3.4.2 Primal-dual iteration

The use of primal-dual methods in the context of total-variation minimization problems has been proposed in [CP11, CP16b, CP16a]. The main idea is to alternatingly update the primal and dual variables z and p in the Lagrange functional

$$L(z,p) = -\int_\Omega z\,\mathrm{div}\,p + \frac{\alpha}{2}\|z-g\|^2 - I_{K_1(0)}(p)$$

via appropriate discretizations of the dynamical system

$$\partial_t p = \delta_p L(z,p), \qquad \partial_t z = -\delta_z L(z,p).$$

When the evolution becomes stationary, a saddle-point for L has been detected. In case of a continuous $P1$ finite element discretization of the primal problem, we may carry out an integration by parts in the first term and consider the discrete Lagrange functional

$$L_h(z_h,p_h) = \int_\Omega p_h\cdot\nabla z_h\,\mathrm{d}x + \frac{\alpha}{2}\|z_h-g\|^2 - I_{K_1(0)}(p_h)$$

where we use elementwise constant vector fields $p_h \in \mathcal{L}^0(\mathcal{T}_h)^d$. An important aspect here is that the functional is quadratic in z_h and nondifferentiable but pointwise in p_h. Hence, the separate minimization and maximization in the variables can be realized efficiently. We have that a pair $(z_h,p_h) \in \mathcal{S}^1(\mathcal{T}_h)\times\mathcal{L}^0(\mathcal{T}_h)^d$ is a saddle-point for L_h if and only if $|p_h| \le 1$ in Ω and

$$(p_h,\nabla v_h) = -\alpha(z_h-g,v_h), \quad (\nabla z_h, q_h-p_h)\le 0$$

for all $(v_h, q_h) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{L}^0(\mathcal{T}_h)^d$ with $|q_h| \leq 1$ in Ω. The inequality is equivalent to the pointwise variational inclusion

$$\nabla z_h \in \partial I_{K_1(0)}(p_h).$$

The following algorithm uses appropriate implicit and explicit treatments of the discretized dynamical system to decouple the equations. The use of the extrapolated iterate

$$\widetilde{z}_h^k = z_h^{k-1} + \tau d_t z_h^{k-1}$$

is crucial to obtain moderate conditions for stability on the involved step size $\tau > 0$. Appropriate choices of the inner product $(\cdot,\cdot)_{h,s}$ to define the evolution of the primal variable will be discussed below.

Algorithm 2 (Primal-dual iteration). *Let $(\cdot,\cdot)_{h,s}$ be an inner product on $\mathcal{S}^1(\mathcal{T}_h)$, $\tau > 0$, $(z_h^0, p_h^0) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{L}^0(\mathcal{T}_h)^d$, set $d_t z_h^0 = 0$, and for $k = 1, 2, \dots$ with $\widetilde{z}_h^k = z_h^{k-1} + \tau d_t z_h^{k-1}$ solve the equations*

$$(-d_t p_h^k + \nabla \widetilde{z}_h^k, q_h - p_h^k) \leq 0,$$
$$(d_t z_h^k, v_h)_{h,s} + (p_h^k, \nabla v_h) + \alpha(z_h^k - g, v_h) = 0$$

subject to $|p_h^k| \leq 1$ in Ω for all $(v_h, q_h) \in \mathcal{S}^1(\mathcal{T}_h) \times \mathcal{L}^0(\mathcal{T}_h)^d$ with $|q_h| \leq 1$ in Ω. Stop the iteration if $\|d_t z_h^k\|_{h,s} \leq \varepsilon_{\mathrm{stop}}$.

We have that p_h^k is the unique minimizer of the nondifferentiable mapping

$$q_h \mapsto \frac{1}{2\tau}\|q_h - p_h^{k-1}\|^2 - (q_h, \nabla \widetilde{z}_h^k) + I_{K_1(0)}(q_h).$$

It is straightforward to verify that p_h^k is given by the pointwise truncation operation

$$p_h^k = \big(p_h^{k-1} + \tau \nabla \widetilde{z}_h^k\big) / \max\{1, |p_h^{k-1} + \tau \nabla \widetilde{z}_h^k|\}.$$

For this explicit formula the use of the L^2 inner product to define the evolution in the p variable is essential. The iterates of Algorithm 2 converge to a stationary point if τ is sufficiently small. The following result is obtained from arguments developed in [Roc76, EB92, Nes05, BT09, CP11, Bar12].

Proposition 10 (Convergence). *Let $z_h \in \mathcal{S}^1(\mathcal{T}_h)$ be minimal for I in $\mathcal{S}^1(\mathcal{T}_h)$ and define*

$$\theta = \sup_{v_h \in \mathcal{S}^1(\mathcal{T}_h)\setminus\{0\}} \frac{\|\nabla v_h\|}{\|v_h\|_{h,s}}.$$

If $\tau\theta \leq 1$, then the iterates of Algorithm 2 converge to z_h in the sense that they satisfy for every $K \geq 1$

$$\tau \sum_{k=1}^{K} \Big((1-\tau^2\theta^2)\frac{\tau}{2}\|d_t z_h^k\|_{h,s}^2 + \alpha\|z_h - z_h^k\|^2 \Big) \leq \frac{1}{2}\big(\|z_h - z_h^0\|_{h,s}^2 + \|p_h - p_h^0\|^2\big).$$

In general we cannot expect convergence $p_h^k \to p_h$ since p_h may fail to be unique, e.g., if $\nabla z_h|_T = 0$ for some $T \in \mathcal{T}_h$. If $(\cdot,\cdot)_{h,s}$ is the L^2 inner product then the parameter θ characterizes the constant in an inverse estimate and is given by $\theta \leq ch^{-1}$. To avoid the resulting restrictive step size condition $\tau \leq ch$ other choices of the inner product $(\cdot,\cdot)_{h,s}$ obtained as weighted combinations of the inner product in $L^2(\Omega)$ and the semi-inner product in $H^1(\Omega)$ are useful.

Proposition 11 (Discrete inner products, [Bar16a]). *For $s \in [0,1]$ and $v_h, w_h \in \mathcal{S}^1(\mathcal{T}_h)$ define*

$$(v_h, w_h)_{h,s} = (v_h, w_h) + h^{(1-s)/s}(\nabla v_h, \nabla w_h),$$

where $h^{(1-s)/s} = 0$ if $s = 0$. We then have $\|\nabla v_h\| \leq ch^{-\min\{1,(1-s)/(2s)\}}\|v_h\|_{h,s}$ for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ with $c = 1$ if $s > 0$.

A particular choice of the scalar products $(\cdot,\cdot)_{h,s}$ has to guarantee that the right-hand side in the estimate of Proposition 10 remains bounded, e.g., the choice $s = 1$ defines the H^1 norm but minimizers for the total variation minimization problem do not belong to this space, i.e., the quantity on the right-hand side will deteriorate as $h \to 0$. For $s \leq 1/2$ the upper bounded remains bounded which follows from the discrete interpolation estimate

$$h\|\nabla v_h\|^2 \leq c\|v_h\|_{L^\infty(\Omega)}\|\nabla v_h\|_{L^1(\Omega)}$$

and the fact that minimizers z_h for I_h remain bounded in the set $W^{1,1}(\Omega) \cap L^\infty(\Omega)$. To obtain robustness of the stopping criterion a smallness property of $\|d_t z_h^k\|_{h,s}$ has to be checked.

3.4.3 ADMM iteration

The idea of the *alternating direction of multiplier method* proposed in [FG83] for solving convex optimization problems of the form

$$I(z) = F(Bz) + G(z)$$

consists in introducing the variable $r = Bz$ and imposing this identity via a Lagrange multiplier λ and a stabilizing term. In the case of the total variation minimization problem the method is thus based on the augmented Lagrange functional

$$L_\tau(z, r, \lambda) = \int_\Omega |r| \,\mathrm{d}x + \frac{\alpha}{2}\|z - g\|^2 + (\lambda, \nabla z - r)_H + \frac{\tau}{2}\|\nabla z - r\|_H^2.$$

Here, a suitable Hilbert space and a parameter $\tau > 0$ have to be chosen. We have that

$$\inf_z I(z) = \inf_{z,r} \sup_\lambda L_\tau(z, r, \lambda).$$

The ADMM iteration successively minimizes L_τ with respect to z and r, and then performs an ascent step with respect to λ. Because of the splitting of the differential operator and the nonquadratic, nondifferentiable functional, the separate optimization in the different variables can be realized efficiently. To explain the algorithm and derive some features we consider the general form as stated above with convex functionals $F : X \to \mathbb{R} \cup \{+\infty\}$ and $G : Y \to \mathbb{R} \cup \{+\infty\}$ and a bounded linear operator $B : X \to Y$. Possible strong convexity of F or G is characterized by nonnegative functionals $\varrho_F : Y \times Y \to \mathbb{R}$ and $\varrho_G : X \times X \to \mathbb{R}$ in the following lemma.

Lemma 1 (Optimality conditions)**.** *A triple (z, r, λ) is a saddle point for L_τ if and only if $Bz = r$ and*

$$\begin{aligned} \big(\lambda, q - r\big)_Y + F(r) + \varrho_F(q, r) &\le F(q), \\ -\big(\lambda, B(v - z)\big)_Y + G(z) + \varrho_G(v, z) &\le G(v), \end{aligned}$$

for all $(v, q) \in X \times Y$.

We approximate a saddle-point using the following iterative scheme which coincides with the scheme introduced in [Glo84] in the case of fixed step sizes.

Algorithm 3 (Generalized ADMM)**.** *Choose $(z^0, \lambda^0) \in X \times Y$ such that $G(z^0) < \infty$. Choose $\overline{\tau} \ge \underline{\tau} > 0$ and $\overline{R} \gg 0$ and set $j = 1$.*
(1) Set $\tau_1 = \overline{\tau}$ and $R_0 = \overline{R}$.
(2) Compute a minimizer $r^j \in Y$ of the mapping

$$r \mapsto L_{\tau_j}(z^{j-1}, r; \lambda^{j-1}).$$

(3) Compute a minimizer $z^j \in X$ of the mapping

$$z \mapsto L_{\tau_j}(z, r^j; \lambda^{j-1}).$$

(4) Update $\lambda^j = \lambda^{j-1} + \tau_j(Bz^j - r^j)$.
(5) Define

$$R_j = \big(\|\lambda^j - \lambda^{j-1}\|_Y^2 + \tau_j^2\|B(z^j - z^{j-1})\|_Y^2\big)^{1/2}.$$

(6) Stop if R_j is sufficiently small.
(7) Choose step size $\tau_{j+1} \in [\underline{\tau}, \overline{\tau}]$.
(8) Set $j \to j + 1$ and continue with (2).

Further variants and related algorithms are investigated in [DR56, GM76, LM79, KM98, HY12, SX14, DY16, DHYZ17]. In [BM20] a strategy for the adjustment of τ_j based on checking certain contraction properties has been developed. Convergence of the iteration of Algorithm 3 is based on comparing the optimality conditions for L_τ to the optimality conditions arising from the iteration.

Lemma 2 (Decoupled optimality). *With $\widetilde{\lambda}^j := \lambda^{j-1} + \tau_j(Bz^{j-1} - r^j)$ the iterates $(z^j, r^j, \lambda^j)_{j=0,1,\dots}$ satisfy for $j \geq 1$ the variational inequalities*

$$\begin{aligned}\big(\widetilde{\lambda}^j, q - r^j\big)_Y + F(r^j) + \varrho_F(q, r^j) &\leq F(q),\\ -\big(\lambda^j, B(v - z^j)\big)_Y + G(z^j) + \varrho_G(v, z^j) &\leq G(v),\end{aligned}$$

for all $(v, q) \in X \times Y$. In particular, $(z^j, r^j; \lambda^j)$ is a saddle-point for L_τ if and only if $\lambda^j - \lambda^{j-1} = 0$ and $B(z^j - z^{j-1}) = 0$.

To state a convergence property of the iteration we use the symmetrized coercivity functionals

$$\widehat{\varrho}_G(z, z') = \varrho_G(z, z') + \varrho_G(z', z), \quad \widehat{\varrho}_F(r, r') = \varrho_F(r, r') + \varrho_F(r', r).$$

Typically, $\widehat{\varrho}_F$ and $\widehat{\varrho}_G$ are given by certain powers of norms of differences, e.g., $\widehat{\varrho}_G(v, w) \sim \|v - w\|^2$.

Theorem 2 (Termination). *Let $(z, r; \lambda)$ be a saddle-point for L_τ. Suppose that the step sizes satisfy the monotonicity property*

$$0 < \underline{\tau} \leq \tau_{j+1} \leq \tau_j$$

for $j \geq 1$. For the iterates $(z^j, r^j; \lambda^j)$, $j \geq 0$, of Algorithm 3, the corresponding differences $\delta_\lambda^j = \lambda - \lambda^j$, $\delta_r^j = r - r^j$ and $\delta_z^j = z - z^j$, and the distance

$$D_j^2 = \|\delta_\lambda^j\|_Y^2 + \tau_j^2 \|B\delta_z^j\|_Y^2,$$

we have for every $J \geq 1$ that

$$\frac{1}{2}D_J^2 + \sum_{j=1}^{J}\Big(\tau_j\big(\widehat{\varrho}_G(z, z^j) + \widehat{\varrho}_F(r, r^j) + \widehat{\varrho}_G(z^{j-1}, z^j)\big) + \frac{1}{2}R_j^2\Big) \leq \frac{1}{2}D_0^2.$$

In particular, $R_j \to 0$ as $j \to \infty$ and Algorithm 3 terminates.

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3.5 Fully discrete approximation of rate-independent damage processes

In [BMT18] a numerical method is developed to determine approximate solutions for a rate-independent damage model $(\mathbf{U}\times\mathbf{Z},\mathcal{R}_1,\mathcal{E})$. Here, the energy functional $\mathcal{E}:\mathbf{U}\times\mathbf{X}\to\mathbb{R}$ is of the form (5) with a gradient regularization of BV-type as in (6b), with finite sublevels on the Banach space $\mathbf{U}\times\mathbf{X}$. Here, $\mathbf{U}:=\{u\in H^1(\Omega;\mathbb{R}^2),\,u=0 \text{ on } \Gamma_D\}$, $\mathbf{X}:=BV(\Omega)$, and $\mathbf{Z}:=L^1(\Omega)$. The positively 1-homogeneous dissipation potential $\mathcal{R}_1:\mathbf{Z}\to[0,\infty]$ is given by

$$\mathcal{R}_1(v):=\int_\Omega R_1(v)\,\mathrm{d}x\,,\quad \text{with } R_1(v):=\begin{cases} a_1|v| & \text{if } v\le 0,\\ \infty & \text{otherwise.}\end{cases} \tag{25}$$

With $z=1$ for the undamaged state of the material and $z=0$ for the maximally damaged state $\mathcal{R}_1$ from (25) ensures that z has to decrease with time and thus prevents healing of the material. The non-smoothness of $\mathcal{R}_1$ together with the non-smoothness and nonlinearity of $\mathcal{E}$ impose a challenge both for numerical and mathematical analysis. To devise an iterative solution method, the staggered time-discrete scheme (19) is combined with a $P1$-FE discretization in space. To solve for the nonlinear, non-smooth discrete problem (19b) an ADMM-algorithm as described in Sec. 3.4.3 is used. It is obtained that the approximate solutions satisfy a discrete analogon of the notion of semistable energetic solutions, cf. Def. 1, Item 2, upon an error term arising from the numerical method. Thanks to a result similar to Thm. 2 it can be shown that this error is controlled and vanishes as time-step and mesh size tend to zero. This is the basis to show that the approximate solutions converge to a semistable energetic solution of the rate-independent process. The convergence of the method is shown in [BMTW20] for gradient regularizations of the type (6a) and (6b). The convergence proof is based on methods from evolutionary Γ-convergence for rate-independent systems. The interplay of the non-smooth constraint imposed by the dissipation potential with the discrete FE-spaces lead to additional error terms in the discrete semistability inequality, which are shown to vanish as $h\to 0$ if the triangulations tend to a right-angled triangulation.

4 Fully discrete approximation of dynamic phase-field fracture by viscous regularization

In this section we regularize the rate-independent damage process by a viscous damping. This means that $\mathcal{R}_1$ from (25) now is replaced by

$$\mathcal{R}_M(v)=\int_\Omega \mathrm{R}_M(v)\,\mathrm{d}x \text{ with } \quad \mathrm{R}_M(v)=\frac{M}{2}|v|^2+\chi_{(-\infty,0]}\big(v\big)\,, \tag{26}$$

with $M>0$ a viscosity parameter, and with $\chi_{(-\infty,0]}(v)=0$ if $v\in(-\infty,0]$ and $\chi_{(-\infty,0]}(v)=\infty$ if $v>0$ the characteristic function of the interval $(-\infty,0]$ to prevent healing of the material. While $\mathcal{R}_1$ from (25) allows solutions to jump in time, this is prevented by the viscous potential (26). A viscous regularization of the evolution law is often used in engineering literature, see e.g., [KM10, SWKM14, MHW10] to make numerical simulations more stable. It was used in [TBW18, TBW17], where convergence of a staggered time-discrete scheme was shown for a phase-field fracture model at finite strains. There, the focus lay on a quasistatic evolution law for the deformation ($\rho=0$ and $\mathbb{D}\equiv 0$ in (27a) below) featuring a stress tensor which takes into account the anisotropy of damage. This is achieved by applying an anisotropic split of the modified principle invariants of the right Cauchy-Green strain tensor. In this framework the existence of solutions was studied using a staggered time-discrete scheme and by showing that the time-discrete solutions converge in a weak sense to a solution of the time-continuous formulation of the model. The main challenge here comes from the non-convexity of the energy functional with respect to the deformation gradient in the finite-strain setting, where in general only polyconvexity is available, combined with the use of modified principle invariants. While complicating mathematical analysis the use of the anisotropic split and modified invariants has proved to lead to better numerical results with good qualitative agreement of simulation and experiment [HSD$^+$16].

While the viscosity $M>0$ in (26) was kept fixed in [TBW18, TBW17] and convergence was investigated for a time-discrete scheme, it is the aim of this section to prove the convergence of a discretization of a visco-elastodynamic phase-field fracture model both in *time and space* such that $M(\tau)\to 0$ as time-step size $\tau\to 0$. We will confine the analysis to the setting of small strains, but allow for a visco-elastodynamic evolution of the displacements. More precisely, the model

problem in a time interval $[0, \mathsf{T}]$ in the reference domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, $d > 1$, formally reads:

$$\rho\ddot{u} - \operatorname{div}\big(\mathbb{D}(z)e(\dot{u}) + \mathbb{C}(z)e(u)\big) = f_V \qquad \text{in } (0,\mathsf{T}) \times \Omega\,, \tag{27a}$$

$$\partial \mathrm{R}_M(\dot{z}) + \mathbb{C}'(z)e(u) : e(u) - \mathcal{G}_c\Big(\frac{1}{\ell}(1-z) - \ell \operatorname{div}\nabla z\Big) \ni 0 \qquad \text{in } (0,\mathsf{T}) \times \Omega\,. \tag{27b}$$

for the displacement $u : [0,T] \times \Omega \to \mathbb{R}^d$ and the phase-field $z : [0,T] \times \Omega \to [0,1]$ with $z = 1$ for the undamaged state and $z = 0$ for the maximally damaged state of the material. In (27a), $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ denotes the linearized strain tensor, $\rho > 0$ is the (constant) mass density, and $f_V : [0,T] \times \Omega \to \mathbb{R}^d$ a given volume force. The parameters ℓ and $\mathcal{G}_c$ are the characteristic length scale of the crack regularization and the fracture toughness appearing in the phase-field fracture energy functional $\mathcal{E}$ of the type (4). (27b) is given as a subdifferential inclusion because of the nonsmooth term $\chi_{(-\infty,0]}$. The evolution laws (27a)–(27b) are complemented by the boundary and initial conditions

$$u(t) = 0 \qquad \text{in } (0,\mathsf{T}) \times \partial_D\Omega \tag{27c}$$

$$\big(\mathbb{D}(z)e(\dot{u}) + \mathbb{C}(z)e(u)\big)\mathrm{n} = f_S \qquad \text{in } (0,\mathsf{T}) \times \partial_N\Omega, \tag{27d}$$

$$\mathcal{G}_c\ell\nabla z \cdot \mathrm{n} = 0 \qquad \text{in } (0,\mathsf{T}) \times \partial\Omega, \tag{27e}$$

$$u(0) = u_0 \qquad \text{in } \Omega, \tag{27f}$$

$$\dot{u}(0) = \dot{u}_0 \qquad \text{in } \Omega, \tag{27g}$$

$$z(0) = z_0 \qquad \text{in } \Omega, \tag{27h}$$

Above in(27e), $\partial\Omega$ denotes the boundary of Ω and n the outer unit normal vector to $\partial\Omega$. In (27c), $\partial_D\Omega$ defines the Dirichlet boundary for the displacements and $\partial_N\Omega = \partial\Omega\backslash\partial_D\Omega$ the Neumann boundary, where the surface load $f_S : [0,T] \times \partial_N\Omega \to \mathbb{R}^d$ is active. The functions $u_0, \dot{u}_0$, and z_0 are given initial data for u and z, respectively.

The phase-field energy functional $\mathcal{E} : [0,T] \times \mathbf{U} \times \mathbf{X} \to \mathbb{R}$ associated with system (27) is very similar to (4) and here takes the form

$$\begin{aligned}\mathcal{E}(t,u,z) := \int_\Omega \Big(\frac{1}{2}\mathbb{C}(z)e(u) : e(u) + \mathcal{G}_c\Big(\frac{1}{2\ell}(1-z)^2 + \frac{\ell}{2}|\nabla z|^2\Big) - f_V(t)\cdot u\Big)\,\mathrm{d}x \\ - \int_{\partial_N\Omega} f_S \cdot u\,\mathrm{d}S\,.\end{aligned} \tag{28}$$

As in Section 2 we also introduce the kinetic energy $\mathcal{K} : \mathbf{W} \to [0,\infty)$ and the viscous dissipation potential $\mathcal{V} : \mathbf{U} \to [0,\infty)$, which here take the form

$$\mathcal{K}(\dot{u}) := \int_\Omega \frac{\rho}{2}|\dot{u}|^2\,\mathrm{d}x \quad \text{and} \quad \mathcal{V}(\dot{u}) := \int_\Omega \frac{1}{2}\mathbb{D}(z)e(\dot{u}) : e(\dot{u})\,\mathrm{d}x\,. \tag{29}$$

We formally understand system (27) as a viscous approximation of a rate-independent evolution of the phase-field parameter z. We will thus investigate the limit $M \to 0$ in (26) and hence in (27b). In the limit this leads to a rate-independent, non-smooth potential $\mathcal{R} : \mathbf{Z} \to [0,\infty]$, which is here given by

$$\mathcal{R}(v) := \int_\Omega \chi_{(-\infty,0]}(v)\,\mathrm{d}x\,. \tag{30}$$

In the rate-independent limit $M \to 0$ the evolutionary inclusion (27b) for z will thus formally turn into

$$\partial\chi_{(-\infty,0]}(\dot{z}) + \mathbb{C}'(z)e(u) : e(u) - \mathcal{G}_c\Big(\frac{1}{\ell}(1-z) + \ell\operatorname{div}\nabla z\Big) \ni 0 \quad \text{in } (0,\mathsf{T}) \times \Omega\,. \tag{31}$$

While the potential $\mathcal{R}_M$ keeps rates $\dot{z}$ with values in $\mathbf{Z}_M = L^2(\Omega)$, this regularity will be lost with $M \to 0$ and one will only find that z is of bounded variation in time. For the definition of the above functionals and in the subsequent exposition we will make use of the following abbreviations for function spaces

$$\mathbf{Z} := L^1(\Omega)\,, \quad \mathbf{Z}_M := L^2(\Omega)\,, \quad \mathbf{X} := H^1(\Omega)\,, \quad \mathbf{Y} := H^1(\Omega) \cap L^\infty(\Omega)\,, \tag{32a}$$

$$\mathbf{U} := \{v \in H^1(\Omega,\mathbb{R}^d),\ v = 0 \text{ on } \partial_D\Omega\}, \quad \mathbf{W} := L^2(\Omega;\mathbb{R}^d)\,. \tag{32b}$$

Definition 4. *We denote the damped inertial system with viscous regularization from* (27) *by the tuple* $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$. *The damped inertial system obtained in the rate-independent limit* $M \to 0$ *will be denoted by* $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$.

In this section we discuss the convergence of a numerical scheme to find solutions for system

$$(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})\,.$$

Solutions of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ are defined in a weak sense in the following way:

Definition 5 (Solutions of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$)**.** *A pair* $(u, z)\colon [0, \mathsf{T}] \to \mathbf{U} \times \mathbf{X}$ *is a solution of* $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ *if it satisfies the following four conditions:*

- *one-sided variational inequality for* z*:*

$$\int_\Omega \frac{1}{2}\mathbb{C}'(z(t))e\big(u(t)\big) : e\big(u(t)\big) - \frac{\mathcal{G}_c}{\ell}\big(1 - z(t)\big)\big]\eta + \mathcal{G}_c \ell \nabla z(t) \cdot \nabla \eta \, dx \geq 0 \tag{33a}$$

  for all $t \in [0, \mathsf{T}]$ *and for all* $\eta \in \mathbf{Y}$ *such that* $\eta \leq 0$ *a.e. in* Ω*;*

- *unidirectionality: for all* $t_1 < t_2 \in [0, \mathsf{T}]$ *it is* $z(t_2) \leq z(t_1)$ *a.e. in* Ω *;* (33b)

- *weak formulation of the momentum balance for all* $t \in [0, \mathsf{T}]$ *:*

$$\begin{aligned} &\rho \int_\Omega \dot{u}(t) \cdot v(t) \,\mathrm{d}x - \rho \int_0^t \int_\Omega \dot{u}(r) \cdot \dot{v}(r) \,\mathrm{d}x \,\mathrm{d}r \\ &\qquad + \int_0^t \int_\Omega \big[\mathbb{D}(z)e(\dot{u}) + \mathbb{C}(z)e(u)\big] : e(v) \,\mathrm{d}x \,\mathrm{d}r \\ &= \rho \int_\Omega \dot{u}(0) \cdot v(0) \,\mathrm{d}x + \int_0^t \langle f(r), v(r)\rangle_{\mathbf{U}^*, \mathbf{U}} \,\mathrm{d}r \end{aligned} \tag{33c}$$

  for all $v \in L^2(0, \mathsf{T}; \mathbf{U}) \cap W^{1,1}(0, \mathsf{T}; L^2(\Omega, \mathbb{R}^d))$ *;*

- *energy-dissipation balance for all* $t \in [0, \mathsf{T}]$*:*

$$\begin{aligned} &\frac{\rho}{2} \int_\Omega |\dot{u}(t)|^2 \,\mathrm{d}x + \mathcal{E}(t, u(t), z(t)) + \int_0^t \int_\Omega \mathbb{D}(z)e(\dot{u}) : e(\dot{u}) \,\mathrm{d}x \,\mathrm{d}r \\ &\quad = \frac{\rho}{2} \int_\Omega |\dot{u}(0)|^2 \,\mathrm{d}x + \mathcal{E}(0, u(0), s(0)) + \int_0^t \partial_t \mathcal{E}\big(r, u(r), z(r)\big) \,\mathrm{d}r\,. \end{aligned} \tag{33d}$$

Remark 3 (Semistable energetic solution of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$)**.** *In fact, we obtain that solutions of*

$$(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$$

in the sense of Definition 5 also satisfy the semistability inequality for all $t \in [0, \mathsf{T}]$*:*

$$\mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, u(t), \tilde{z}) + \mathcal{R}(\tilde{z} - z(t)) \quad \text{for all } \tilde{z} \in \mathbf{X} \tag{34}$$

with $\mathcal{E}$ *from* (28) *and* $\mathcal{R}$ *from* (30)*. Thus, solutions of* $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ *are also semistable energetic solutions in the sense of Definition* (2)*.*

It is the aim of this section to show the existence of solutions for system $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ in the sense of Def. 5 by discrete approximation. For this, we will combine a staggered time-discrete scheme with a $P1$ finite-element discretization in space to find weak solutions of system $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$ corresponding to (27), see (48). While the numerical computation of solutions for the discrete version of (27a) reduces to solving a linear system of equations, solving for the discrete version of (27b) is more involved. For this we propose to regularize the non-smooth viscous dissipation potential by a smoothened version of the Yosida-regularization, cf. (42) for more details. In this way, e.g., a Newton's method will be applicable to solve the discretized version of the nonlinear problem, where the nonlinearities stem from the nonlinear dependence of the tensor $\mathbb{C}$ on z, cf. (36), and from the Yosida-regularization. We show that the approximate solutions obtained by the staggered Galerkin scheme (48) satisfy a discrete version of the notion of solution given in Def. 5. However, since the discrete nonlinear problem will only be solved approximately, error terms will appear. We derive sufficient

conditions to control the error terms, so that convergence of the approximate solutions can be shown. These sufficient conditions can serve as stopping criteria for the numerical algorithm.

The outline of this Section is as follows: After specifying the basic assumptions on the domain and given data in Section 4.1.1, we introduce the staggered Galerkin scheme (48) in Section 4.1.2 and show the existence of approximate solutions, cf. Prop. 12. This already leads to a first set of qualifying conditions, cf. (51). Subsequently, in Section 4.1.3 we deduce a second set of qualifying criteria, cf. (55), (56), and (59), that allow it to find uniform bounds for the approximate solutions and we show the convergence of (a subsequence of) approximate solutions to a solution of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ in the sense of Definition 5, cf. Theorem 3.

4.1 Basic assumptions and main result

4.1.1 Basic assumptions

Assumptions on the domain: We assume that

$$\begin{aligned}&\Omega \subset \mathbb{R}^d \text{ is a bounded domain with Lipschitz-boundary } \partial\Omega, \text{ such that}\\ &\partial_D\Omega \subset \partial\Omega \text{ is non-empty and relatively open and } \partial_N\Omega := \partial\Omega \setminus \partial_D\Omega.\end{aligned} \tag{35}$$

Assumptions on the tensors $\mathbb{C}$, $\mathbb{D}$: The tensors $\mathbb{C}, \mathbb{D} : \mathbb{R} \to \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$ depend on the phase-field parameter z through functions $w_{\mathbb{C}}$, $w_{\mathbb{D}} : \mathbb{R} \to [w_0, w^*]$ being prefactors to constant tensors $\tilde{\mathbb{C}}, \tilde{\mathbb{D}}$, i.e.,

$$\mathbb{C}(z) := w_{\mathbb{C}}(z)\tilde{\mathbb{C}} \quad \text{and} \quad \mathbb{D}(z) := w_{\mathbb{D}}(z)\tilde{\mathbb{D}} \text{ for all } z \in \mathbb{R}, \tag{36a}$$

$$\text{with constant, symmetric, and positively definite tensors } \tilde{\mathbb{C}}, \tilde{\mathbb{D}}. \tag{36b}$$

For the functions $w_{\mathbb{C}}, w_{\mathbb{D}}$ we further assume:

- Differentiability & boundedness:
$$w_{\mathbb{D}} \in C^1(\mathbb{R}, [w_0, w^*]),\ w_{\mathbb{C}} \in C^2(\mathbb{R}, [w_0, w^*]), \quad \text{with constants } 0 < w_0 < w^*, \tag{37a}$$
- Monotonicity: $w'_{\mathbb{C}}(z) \geq 0$ and $w'_{\mathbb{D}}(z) \geq 0$ for all $z \in \mathbb{R}$, (37b)
- Locally constant growth: $w'_{\mathbb{C}}(z) = 0$ and $w'_{\mathbb{D}}(z) = 0$. (37c)
$$\text{for all } z \in (-\infty, 0] \cup [z^*, \infty),$$
- Local convexity: There are $z_* \in (1, z^*)$ and $w_* \in (w_0, w^*)$ s.t.
$$w_{\mathbb{C}} : [0, z_*] \to [w_0, w_*] \text{ is convex.} \tag{37d}$$

A direct implication of (36) and (37a) is the existence of constants $0 < c^0_{\mathbb{D}} < c^*_{\mathbb{D}}$ and $0 < c^0_{\mathbb{C}} < c^*_{\mathbb{C}}$ such that for all $z \in \mathbb{R}$ and for all $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ there holds:

$$c^0_{\mathbb{D}} |A|^2 \leq \mathbb{D}(z)A : A \leq c^*_{\mathbb{D}} |A|^2 \text{ and} \tag{38a}$$

$$c^0_{\mathbb{C}} |A|^2 \leq \mathbb{C}(z)A : A \leq c^*_{\mathbb{C}} |A|^2 . \tag{38b}$$

Remark 4 (Discussion of the assumptions (37))**.** *Assumption* (37a) *on the boundedness of $w_{\mathbb{C}}$ and $w_{\mathbb{D}}$ is crucial to guarantee the existence of discrete solutions because it ensures the uniform bounds from below in* (38) *and thus the coercivity of the energy functional* (28) *and the viscous dissipation potential* (26)*. We further impose in* (37a) *the regularity $w_{\mathbb{C}} \in C^2(\mathbb{R}, [w_0, w^*])$ in order to comply with the requirements of a Newton's method to numerically solve the nonlinear equation* (48a)*. Monotonicity assumption* (37b) *reflects the physical property that an increase of damage leads to a decrease of the stresses, since it ensures*

$$w_{\mathbb{C}}(z_1) \leq w_{\mathbb{C}}(z_2) \text{ as well as } w_{\mathbb{D}}(z_1) \leq w_{\mathbb{D}}(z_2) \quad \text{for all } z_1 \leq z_2,$$

and since an increase of damage is represented by a decrease of the values of z in our model. As a direct implication of the boundedness (37a) *and the monotonicity* (37b) *the functions $w_{\mathbb{C}}$ and $w_{\mathbb{D}}$ need to be constant on subintervals of $\mathbb{R}$ as further stated in* (37c)*. It can be shown for solutions (u, z) of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ that z takes values in $[0, 1]$ a.e. in Ω*

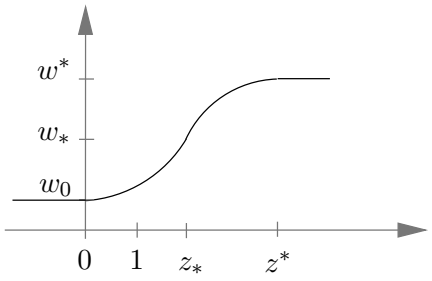

Figure 6: Qualitative shape of $w_{\mathbb{C}} : \mathbb{R} \to [w_0, w^*]$: The function is constant on $(-\infty, 0] \cup [z^*, \infty)$, monotonously increasing on $\mathbb{R}$, and convex on $(-\infty, z_*)$ with $z_* > 1$ but non-convex on $[z_*, z^*)$.

and thus can be understood as the volume fraction of undamaged material, cf. Thm. 3. To obtain this result it is important to make sure in (37c) *that the subintervals, where $w_{\mathbb{C}}$ and $w_{\mathbb{D}}$ have constant growth, do not intersect with the interval $[0, 1]$. We also refer to [KRZ13b] where similar growth assumptions and resulting observations have been made. Finally, convexity assumption* (37d) *allows it to deduce that solutions (u, z) of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ satisfy the upper energy-dissipation estimate* (18)*, which can be shown even to hold as a balance* (33d)*. This result is important from a thermodynamical point of view and from a general mathematical point of view it provides compactness properties. Alltogether, assumptions* (37) *in particular imply that $w_{\mathbb{C}}$ qualitatively is of the form indicated in Fig. 6. However, monotonicity* (37b) *together with the boundedness* (37a) *further require $w_{\mathbb{C}}$ to be non-convex on a subinterval, which is given by $[z_*, z^*]$ with $z_* > 1$ in Fig. 6. In turn, the non-convexity of $w_{\mathbb{C}}$ on the interval $[z_*, z^*]$ entails that upper energy-dissipation estimates are not yet available for approximating solutions of the fully discretized problem.*

Remark 5 (Comparison of $w_{\mathbb{C}}$ from (37) with other degradation functions from literature)**.** *We finally point out that on the interval $[0, 1]$ the degradation function $w_{\mathbb{C}}$ may take any polynomial form commonly used in literature, such as, e.g., $w_{\mathbb{C}}(z) := \eta + z^2$ with a constant $\eta > 0$ in the standard Ambrosio-Tortorelli functional, cf. [Gia05, KM10, SWKM14] or $w_{\mathbb{C}}(z) := (1 - z)^2$ in [MHW10]. Other variants like $w_{\mathbb{C}}(z) := (a_g - 2)(1 - z)^3 + (3 - a_g)(1 - z)^2$ with $a_g \in (0, 2]$ in [HGO$^+$17] or*

$$w_{\mathbb{C}}(z) := a(z^3 - z^2) + 3z^2 - 2z^3 \tag{39}$$

in [BHL$^+$16] are non-convex in the interval $[0, 1]$ for the typical choice of parameters. This is used to model a linear behaviour of the non-fractured material right before crack initiation and accomplished with a horizontal slope at the transition between sound and damaged. However, in our work convexity assumption (37d) *is a technical but crucial tool to deduce the convergence of the approximation method* (49)*. In order to comply with the requirements of mechanics and thus also to allow for non-convex degradation functions we propose here to formulate the degradation function in dependence of the mesh size h, as for example in* (39) *with $a = a(h)$ to recover convexity in the limit as $h \to 0$.*

Assumptions on the given data: For the volume force f_V in (27a) and the surface force f_S in (27d) we assume here the regularity $f_V \in C^1([0, \mathsf{T}]; \mathbf{U}^*)$ and $f_S \in C^1([0, \mathsf{T}]; L^2(\partial_N \Omega, \mathbb{R}^d))$, i.e., the external loadings are continuously differentiable in time and the time-derivative is Lipschitz-continuous with values in (a subspace of) the spatial dual. We then define the combined external loading f by

$$\langle f(t), v\rangle_{\mathbf{U}^*,\mathbf{U}} := \langle f_V(t), v\rangle_{\mathbf{U}^*,\mathbf{U}} + \int_{\partial_N \Omega} f_S(t) \cdot v \, \mathrm{d}\mathcal{H}^{d-1} \text{ for all } v \in \mathbf{U}\,. \tag{40a}$$

Above regularity assumptions on f_V and f_S imply the following properties for f:

- Regularity: $f \in C^1\big([0, T]; \mathbf{U}^*\big)\,,$ (40b)
- Boundedness of the time-derivative: $\sup\limits_{t\in[0,\mathsf{T}]} \left\|\dot f(t)\right\|_{\mathbf{U}^*} < \infty\,,$ (40c)

Additionally, we impose for the initial data in (27f)-(27h):

$$u_0 \in \mathbf{U}\,, \quad \dot u_0 \in \mathbf{U}, \tag{41a}$$

$$z_0 \in \mathbf{X} \text{ such that } z_0(x) \in [0, 1] \text{ for almost all } x \in \Omega\,. \tag{41b}$$

Yosida-regularization and mollified $\max$-function: For the numerical method we propose to replace the non-smooth dissipation potential $\mathcal{R}_M$ by a smooth approximation that allows it to compute second derivatives. This can be achieved

using a smoothened variant of the Yosida-regularization and the regularization parameter will be chosen in dependence of the time-step size τ. For this, the characteristic function $\chi_{(-\infty,0]}$ enforcing unidirectionality in (26) will be approximated by

$$r \mapsto \frac{N_\tau}{2} |m_\tau(r)|^2 \tag{42a}$$

where $m_\tau \colon \mathbb{R} \to [0,\infty)$ denotes a regularization of the function $\max\{\cdot, 0\}$, given as

$$m_\tau(r) := \begin{cases} r - \frac{\tau}{2} & r \geq \tau\,, \\ \frac{r^3}{\tau^2} - \frac{r^4}{2\tau^3} & r \in (0,\tau)\,, \\ 0 & r \leq 0. \end{cases} \tag{42b}$$

We refer to [Kop09, Sec. 4] for further details on this construction. In this way, R_M in (26) will be replaced in the discrete scheme by

$$\mathrm{R}_{M\tau}(v) := \frac{M}{2} |v|^2 + \frac{N_\tau}{2} |m_\tau(v)|^2 \tag{42c}$$

and we write $\mathcal{R}_{M\tau}$ for the corresponding integral functional.

4.1.2 Discretization of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$ in space and time

Our strategy to find weak solutions for $(\mathbf{U}, \mathbf{W}, \mathbf{Z}_M, \mathcal{V}, \mathcal{K}, \mathcal{R}_M, \mathcal{E})$ consists in a discretization in time and space using FEM. In the following we introduce the notation for the discrete setting and present the discrete scheme below in (49). Subsequently, in Section 4.1.3 we discuss the convergence of the method.

Discretization in space: For a family $(\mathcal{T}_h)_h$ of triangulations of Ω with mesh size $h = \sup_{T\in\mathcal{T}_h} \operatorname{diam} T$ let $\mathfrak{N}_h$, $\mathfrak{E}_h$ be the sets of nodal points and edges, respectively. For the infinite-dimensional Banach space $V \in \{\mathbf{X}, \mathbf{U}\}$, we consider the finite-element spaces V_h, of piecewise affine-linear functions. We assume $(\mathcal{T}_h)_h$ to be such that the finite-dimensional spaces contain each other successively as $h \to 0$, i.e., $V_{h_1} \subset V_{h_2} \subset V$ for any $h_2 < h_1$. In this way, the finite-dimensional spaces V_h are dense in V, i.e., $V = \overline{\cup_h V_h}$. Further, let N_h be the number of vertices in $\mathfrak{N}_h$. Then N_h coincides with the dimension of the FE-space in the scalar case $V_h = \mathbf{X}_h$, while for the vectorial case $V_h = \mathbf{U}_h$ the space dimension is given by $d\mathsf{N}_h$. Let $\vec{\varphi} := (\varphi_j)_{j=1}^{\mathsf{N}_h}$ denote the vector of (suitably ordered) nodal basis elements for $\mathbf{X}_h$ given by the scalar hat function with $\varphi_j(x_j) = 1$ in node x_j and $\varphi_j(x_i) = 0$ for $i \neq j$. Then the nodal basis for $\mathbf{U}_h$ given as $(\boldsymbol{\varphi}_l)_{l=1}^{d\mathsf{N}_h} = (\varphi_j \mathbf{e}_1, \dots, \varphi_j \mathbf{e}_d)_{j=1}^{\mathsf{N}_h}$, where $\mathbf{e}_i$, $i = 1, \dots, d$, are orthornormal basis vectors of $\mathbb{R}^d$. In this way, the elements $z \in \mathbf{X}_h$ and $u \in \mathbf{U}_h$ are represented by linear combinations $z = \sum_{j=1}^{\mathsf{N}_h} z_j \varphi_j$ and $u = \sum_{l=1}^{d\mathsf{N}_h} u_l \boldsymbol{\varphi}_l$ of the basis elements using coefficient vectors $\mathbf{z} = (z_j)_{j=1}^{\mathsf{N}_h}$ and $\mathbf{u} = (u_j)_{j=1}^{d\mathsf{N}_h} \in \mathbb{R}^{d\mathsf{N}_h}$. For functions $\eta \in C(\bar{\Omega})$ and $v \in C(\overline{\Omega}, \mathbb{R}^d)$ we further introduce the scalar and vectorial nodal interpolants as follows

$$P_h^{\mathbf{X}} \colon C(\overline{\Omega}) \to \mathbf{X}_h,\ P_h^{\mathbf{X}}(\eta) := \sum_{x_i \in \mathfrak{N}_h} \eta(x_i)\varphi_i, \tag{43a}$$

$$P_h^{\mathbf{U}} \colon C(\overline{\Omega}, \mathbb{R}^d) \to \mathbf{U}_h,\ P_h^{\mathbf{U}}(v) := \sum_{x_i \in \mathfrak{N}_h} \sum_{l=1}^{d} (v(x_i) \cdot \mathbf{e}_l)\varphi_i \mathbf{e}_l. \tag{43b}$$

Then, $P_h^{\mathbf{X}}(\eta) \to \eta$ strongly in $\mathbf{X}$ for any $\eta \in \mathbf{X} \cap (C^\infty(\mathbb{R}^d)|_\Omega)$ as well as $P_h^{\mathbf{U}}(v) \to v$ strongly in $\mathbf{U}$ for any $v \in \mathbf{U} \cap (C^\infty(\mathbb{R}^d)|_\Omega)^d$.

Remark 6. *By choice of piecewise affine-linear finite-element spaces, the density of $\mathbf{X}_h \subset \mathbf{X}$ can be assumed for all $h > 0$. In certain cases these approximations have to meet some additional constraints. For the inital datum $z_0 \in \mathbf{X}$ with $z_0 \in [0,1]$ a.e. in Ω it must be guaranteed that the approximations in the finite-element spaces satisfy this bound as well which can be justified as follows: First, one finds by density of smooth functions in $\mathbf{X}$ a sequence $(\eta_l)_l \subset \mathbf{X} \cap (C^\infty(\mathbb{R}^d)|_\Omega)$ such that $\eta_l \to z_0$ strongly in $\mathbf{X}$. Now, let $(\varepsilon_l)_l \subset \mathbb{R}$ be such that $\varepsilon_l \to 0$ as $l \to \infty$. Projection of η_l onto the box $[\varepsilon_l, 1-\varepsilon_l]$ defines the truncated functions $\tilde{\eta}_l := \min\{1-\varepsilon_l, \max\{\eta_l, \varepsilon_l\}\}$. Then because of $\|\tilde{\eta}_l\|_{\mathbf{X}} \leq \|\eta_l\|_{\mathbf{X}}$,*

$(\tilde\eta_l)_l$ is uniformly bounded in the separable Hilbert space $\mathbf{X}$ and thus admits passing to a (not relabeled) subsequence that

$$\tilde\eta_l \rightharpoonup \tilde z \quad \text{weakly in } \mathbf{X}. \tag{44}$$

Now, we have $\tilde\eta_l \to z$ in $L^2(\Omega)$ which can be seen by

$$\begin{aligned}\int_\Omega |\tilde\eta_l - z|^2\,dx &= \int_{[\eta_l\in[0,1]\setminus[\varepsilon_l,1-\varepsilon_l]]} |\tilde\eta_l - z|^2\,dx \\ &\quad + \int_{[\eta_l\in[\varepsilon_l,1-\varepsilon_l]]} |\tilde\eta_l - z|^2\,dx + \int_{[\eta_l\in\mathbb{R}\setminus[0,1]]} |\tilde\eta_l - z|^2\,dx \\ &\le \int_{[\eta_l\in[0,1]\setminus[\varepsilon_l,1-\varepsilon_l]]} 2\,|\tilde\eta_l - \eta_l|^2\,dx + \int_{[\eta_l\in[0,1]\setminus[\varepsilon_l,1-\varepsilon_l]]} 2\,|\eta_l - z|^2\,dx \\ &\quad + \int_{[\eta_l\in[\varepsilon_l,1-\varepsilon_l]]} |\eta_l - z|^2\,dx + \int_{[\eta_l\in\mathbb{R}\setminus[0,1]]} |\eta_l - z|^2\,dx + \varepsilon_l\mathcal{L}^d(\Omega)\,.\end{aligned}$$

The first two terms on the right-hand side converges to 0 as $l\to\infty$ because on the set $[\eta_l\in[0,1]\setminus[\varepsilon_l,1-\varepsilon_l]]\subset\Omega$ we have $|\tilde\eta_l-\eta_l|\le\varepsilon_l$. while for the second summand we already know the strong convergence in $\mathbf{U}$. For the third term notice that $\tilde\eta_l=\eta_l$ on $[\eta_l\in[\varepsilon_l,1-\varepsilon_l]]$. For the last term again one knows strong convergence on the whole domain Ω. Altogether it follows $\tilde\eta\to z$ in $L^2(\Omega)$ and thus $\tilde z = z$ with (44)*. Then, $\limsup_{l\to\infty}\|\tilde\eta_l\|_{\mathbf{X}}\le\limsup_{l\to\infty}\|\eta_l\|_{\mathbf{X}} = \|z\|_{\mathbf{X}}$, i.e. the convergence in the norms of $\mathbf{X}$, supplemented with the weak convergence $\tilde\eta_l\rightharpoonup z$ imply $\tilde\eta_l\to z$ strongly in $\mathbf{X}$. At last, mollifying $\tilde\eta_l$ one obtains functions $\hat\eta_l$ having the suitable regularity to see that for the projections $P_h^{\mathbf{X}}(\hat\eta_l)\to z$ strongly in $\mathbf{X}$ is true (see [EG13, Corollary 1.109 and 1.110, p. 61]).*

Discretization in time: Let $\Pi_\tau=\{0=t_\tau^0<t_\tau^1\ldots<t_\tau^{N_\tau}=\mathsf{T}\}$ be a uniform partition of the time interval $[0,\mathsf{T}]$ with step size $\tau=\frac{\mathsf{T}}{N_\tau}$. For a function $v\colon[0,\mathsf{T}]\to V$ we write $v_\tau^k:=v(t_\tau^k)$ for any $t_\tau^k\in\Pi_\tau$ and introduce the discrete time-derivatives

$$\mathsf{D}_\tau v_\tau^k := \tfrac{v_\tau^k - v_\tau^{k-1}}{\tau}\,, \tag{45a}$$

$$\mathsf{D}_\tau^2 v_\tau^k := \tfrac{1}{\tau}\big(\mathsf{D}_\tau v_\tau^k - \mathsf{D}_\tau v_\tau^{k-1}\big) = \tfrac{v^k - 2v^{k-1}+v^{k-2}}{\tau^2}\,. \tag{45b}$$

For the discretization of the given data and here especially for the external loading f from (40), we use an approximation

$$f_\tau^k := f(t_\tau^k) \tag{46}$$

and denote by $f_{\tau h}^k$ the restriction of $f_\tau^k\in\mathbf{U}^*$ to $\mathbf{U}_h$, where naturally

$$f_{\tau h}^k\to f_\tau^k \text{ strongly in } \mathbf{U}^* \text{ as } h\to 0 \text{ for all } k\in\{1,\ldots,N_\tau\} \text{ and } \tau>0 \text{ fixed}. \tag{47}$$

Discrete approximation scheme for $(\mathbf{U},\mathbf{W},\mathbf{Z}_M,\mathcal{V},\mathcal{K},\mathcal{R}_M,\mathcal{E})$: Keep $\tau>0$ fixed. For the initial data $(z_0,u_0,\dot u_0)$ from (41) set $z_\tau^0:=z_0$, $u_\tau^0:=u_0$, and $u_\tau^{-1}:=u_0-\tau\dot u_0$ and let $(z_{\tau h}^0)_h$, $(u_{\tau h}^0)_h$, $(u_{\tau h}^{-1})_h$ with $z_{\tau h}^0\in\mathbf{X}_h$, $u_{\tau h}^0$, $u_{\tau h}^{-1}\in\mathbf{U}_h$ for all $h>0$ be approximations of the inital data such that $z_{\tau h}^0\to z_\tau^0$, $u_{\tau h}^0\to u_\tau^0$ and $u_{\tau h}^{-1}\to u_\tau^{-1}$ as $h\to 0$. For each $\tau,h>0$ fixed, using the discrete initial data $(z_{\tau h}^0,u_{\tau h}^0,u_{\tau h}^{-1})$ our aim is to find for every time step $t_\tau^k\in\Pi_\tau$ solutions $\tilde z_{\tau h}^k\in\mathbf{X}_h$, $\tilde u_{\tau h}^k\in\mathbf{U}_h$ by solving the following staggered discrete Galerkin scheme: *For all $k\in\{1,\ldots,N_\tau\}$ find $\tilde z_{\tau h}^k\in\mathbf{X}_h$, $\tilde u_{\tau h}^k\in\mathbf{U}_h$ such that*

$$\langle \mathrm{D}_z\mathcal{E}(t_\tau^k,\tilde u_{\tau h}^{k-1},\tilde z_{\tau h}^k) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau\tilde z_{\tau h}^k),\eta_h\rangle_{\mathbf{X}^*,\mathbf{X}} = 0 \text{ for all } \eta_h\in\mathbf{Y}_h, \tag{48a}$$

$$\int_\Omega \rho\mathsf{D}_\tau^2\tilde u_{\tau h}^k\cdot v_h\,\mathrm{d}x + \langle \mathrm{D}_u\mathcal{E}(t_\tau^k,\tilde u_{\tau h}^k,\tilde z_{\tau h}^k) + \mathrm{D}\mathcal{V}(\mathsf{D}_\tau\tilde u_{\tau h}^k),v_h\rangle_{\mathbf{U}^*,\mathbf{U}} = 0 \text{ for all } v_h\in\mathbf{U}_h. \tag{48b}$$

We point out that, on an abstract level it is possible to show the existence of Galerkin solutions $(\tilde u_{\tau h}^k,\tilde z_{\tau h}^k)$ for system (48); we refer to [TT20, Prop. 3.1] for a proof. While $\tilde u_{\tau h}^k$ is obtained by solving the linear system of equations (48b), $\tilde z_{\tau h}^k$ is given by the *nonlinear* system (48a), where the nonlinearity stems from the properties (37) of the degradation function $w_{\mathbb{C}}$ and from the properties (42) of the regularized maximum-function m_τ. The abstract existence proof is based on fixed point arguments for nonlinear systems of equations and verifies that (48a) can be exactly solved. Instead, when applying

an iterative method to solve the nonlinear system (48a), it will be only solved approximately. We denote the approximate solution for (48a) obtained by the numerical method by $z^k_{\tau h}$. The approximate solution $z^k_{\tau h}$ will satisfy (48a) only up to an error, which we will indicate by $\epsilon^k_{\tau,h}$ on the right-hand side, see (49a) below. Furthermore, the approximate solution $z^k_{\tau h}$ is an input in the staggered scheme to solve for the discrete momentum balance, which, due to its linearity, can be solved exactly. With the input $z^k_{\tau h}$ this results in a solution $u^k_{\tau h}$. In conclusion, given the discrete initial data $(z^0_{\tau h}, u^0_{\tau h}, u^{-1}_{\tau h}) \in \mathbf{X}_h \times \mathbf{U}_h \times \mathbf{U}_h^*$ the numerical method provides for any choice of $h, \tau > 0$ fixed and for all $k \in \{1, \dots, N_\tau\}$ approximate solutions $(u^k_{\tau h}, z^k_{\tau h})$ satisfying

$$\langle \mathrm{D}_z \mathcal{E}(t^k_\tau, u^{k-1}_{\tau h}, z^k_{\tau h}) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z^k_{\tau h}), \eta_h \rangle_{\mathbf{X}^*,\mathbf{X}} = \epsilon^k_{\tau,h}(\eta_h) \qquad \text{for all } \eta_h \in \mathbf{Y}_h, \tag{49a}$$

$$\langle \rho \mathsf{D}^2_\tau u^k_{\tau h} + \mathrm{D}_u \mathcal{E}(t^k_\tau, u^k_{\tau h}, z^k_{\tau h}) + \mathrm{D}\mathcal{V}(\mathsf{D}_\tau u^k_{\tau h}), v_h \rangle_{\mathbf{U}^*,\mathbf{U}} = 0 \qquad \text{for all } v_h \in \mathbf{U}_h. \tag{49b}$$

Here, $\epsilon^k_{\tau,h}(\eta)$ indicates that the error induced in (49a) by the numerical method also depends on the test functions $\eta \in \mathbf{Y}_h$ and it has to be ensured by suitable stopping criteria for the numerical algorithm that this error can be controlled in such a way that $\epsilon^k_{\tau,h}(\eta) \approx 0$. The following proposition provides the existence of approximate solutions for the staggered Galerkin scheme (49) as well as uniform a priori bounds.

Proposition 12 (Existence of approximate solutions & a priori estimates)**.** *Let the assumptions* (35)–(42) *be satisfied. Keep $h, \tau > 0$, $k \in \{1, \dots, N_\tau\}$ fixed. Then there exists an approximate solution $(u^k_{\tau h}, z^k_{\tau h})$ of the staggered Galerkin scheme* (49) *for system $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}_1, \mathcal{E})$. Moreover, for all $k \in \{1, \dots, N_\tau\}$ there is a constant $\tilde{C}$ so that the approximate solutions $(u^k_{\tau h}, z^k_{\tau h})$ satisfy the a priori bounds*

$$\|u^k_{\tau h}\|_{\mathbf{U}} \leq \tilde{C}\,, \tag{50a}$$

$$\|z^k_{\tau h}\|_{\mathbf{X}} \leq \tilde{C} \tag{50b}$$

with a constant $\tilde{C} = \tilde{C}(k, \tau^{-1}) > 0$, but independent of $h > 0$.

The proof of Prop. 12 is carried out in Sec. 4.2. There it becomes apparent that a suitable stopping criterion for the algorithm to solve (49) must ensure that

$$\max\left\{|\epsilon^k_{\tau,h}(\varphi_j)|, |\epsilon^k_{\tau,h}(z^k_{\tau h})|\,,\ j = 1, \dots, \mathsf{N}_h\right\} \leq \mathsf{TOL}(h) \ll 1 \tag{51}$$

with a suitably chosen tolerance $\mathsf{TOL}(h)$ with the property

$$\mathsf{TOL}(h) \to 0 \text{ as } h \to 0. \tag{52}$$

In (51) the requirement $|\epsilon^k_{\tau,h}(\varphi_j)| \leq \mathsf{TOL}(h)$ for $j = 1, \dots, \mathsf{N}_h$ directly stems from (49a), while $|\epsilon^k_{\tau,h}(z^k_{\tau h})| \leq \mathsf{TOL}(h)$ is imposed in addition to ensure that the a priori bounds (50) are independent of $h > 0$. This is important for a limit passage $h \to 0$, while keeping τ and k fixed. For this limit passage the bounds (50) provide sufficient compactness to find suitably convergent subsequences and limit pairs (u^k_τ, z^k_τ), $k = 1, \dots, N_\tau$, that are solutions of the time-discrete, but space-continuous version of (49). Instead, due to the explicit dependence of $\tilde{C}$ on τ^{-1} estimate (50) is not sufficient to pass to the limit also with the time-step size τ or to consider the simultaneous limit $h = h(\tau)$ as $\tau \to 0$, as it will be discussed in Sec. 4.1.3 below. For this, further estimates are needed, which can be understood as discrete energy-dissipation estimates perturbed by some error terms stemming from the approximation method and from the non-convexity of the degradation function $w_{\mathbb{C}}$. As we shall see in Sec. 4.1.3, the control of these error terms will lead to additional criteria alike (51) that will impose relations between the fineness of the mesh size h and time-step size τ.

4.1.3 Convergence of the staggered Galerkin scheme (49)

In the following we discuss the convergence of the approximate solutions $(u^k_{\tau h}, z^k_{\tau h})_{k=1}^{N_\tau}$ to a pair (u, z) that provides a solution to $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ in the sense of Def. 5. For this we want to treat a simultaneous limit $h \to 0$ and $\tau \to 0$ and thus we consider mesh size h as a function of the time-step size τ, i.e., from now on we assume that

$$h = h(\tau)\,. \tag{53}$$

Yet, we will continue using the notation from Sec. 4.1.2 and only explicitely write $h(\tau)$ in sub- or superscripts when relevant. As we shall outline in what follows, the dependence of h on τ can be specified by further criteria alike (51) that are needed to verify the convergence of the approximate solutions. More precisely, the convergence is obtained from perturbed energy-dissipation estimates that need to be uniform with respect to the parameters k, τ, and h. We now discuss the main points that provide the criteria for the $h(\tau)$-dependence and refer to Sec. 4.3 for further details:

The above-mentioned perturbed energy-dissipation estimate for the approximate solutions is obtained by testing (49b) by $\tau \mathsf{D}_\tau u^k_{\tau h}$ and (49a) by $\tau \mathsf{D}_\tau z^k_{\tau h}$, summing the result and summing up over $k \in \{1, \dots, N_\tau\}$. Since the criteria for the $\tau(h)$-dependence mainly arise from contributions of (49a) we here focus on these terms. More precisely, applying above procedure to (49a) results in

$$\sum_{k=1}^{N_\tau} \langle \mathrm{D}_z \mathcal{E}(t^k_\tau, u^{k-1}_{\tau h}, z^k_{\tau h}) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z^k_{\tau h}), z^k_{\tau h} - z^{k-1}_{\tau h} \rangle_{\mathbf{X}^*, \mathbf{X}} = \sum_{k=1}^{N_\tau} \epsilon^k_{\tau,h}(z^k_{\tau h} - z^{k-1}_{\tau h}) \tag{54}$$

and the right-hand side of (54) will appear as a perturbation term in the energy-dissipation estimate. To control this perturbation one has to ensure that

$$\max\left\{ |\epsilon^k_{\tau,h}(z^k_{\tau h} - z^{k-1}_{\tau h})|\,,\ k = 1, \dots, N_\tau \right\} \le \mathsf{TOL}(h)\,. \tag{55}$$

Moreover, to make the perturbation term disappear in (54) as $h \to 0$, further requires

$$N_\tau \mathsf{TOL}(h(\tau)) = \mathsf{T} \frac{\mathsf{TOL}(h(\tau))}{\tau} \to 0 \ \text{ as } \tau \to 0\,, \tag{56}$$

which provides a first refinement of (52) and (53). In addition, a further perturbation of the energy-dissipation estimate arises on the left-hand side of (54) by the term $\frac{1}{2}\mathbb{C}'(z^k_{\tau h})(z^k_{\tau h} - z^{k-1}_{\tau h}) e(u^{k-1}_{\tau h}) : e(u^{k-1}_{\tau h})$, where $\mathbb{C}'(z^k_{\tau h}) = w'_{\mathbb{C}}(z^k_{\tau h})\tilde{\mathbb{C}}$. This error is due to non-convexity of the degradation function $w_{\mathbb{C}}$ in subsets of Ω where $z_* \le z^k_{\tau h} \le z^*$ or $z_* \le z^{k-1}_{\tau h} \le z^*$, cf. (37) and Remark 4. The treatment of this non-convex term requires the control of the integrand term

$$E_{\mathbb{C}}(z^{k-1}_{\tau h}, z^k_{\tau h}, e(u^{k-1}_{\tau h})) := \tfrac{1}{2}\Big(\mathbb{C}(z^{k-1}_{\tau h}) - \mathbb{C}(z^k_{\tau h}) + \mathbb{C}'(z^k_{\tau h})(z^k_{\tau h} - z^{k-1}_{\tau h})\Big) e(u^{k-1}_{\tau h}) : e(u^{k-1}_{\tau h}) \tag{57}$$

on the set where the non-convexity of $\mathbb{C}$ is located, which is the set

$$\begin{aligned} B^k_h := &\Big([z_* \le z^k_{\tau h} \le z^*] \cap [z_* \le z^{k-1}_{\tau h} \le z^*] \Big) \\ &\cup \Big([z^k_{\tau h} \le z_*] \cap [z^{k-1}_{\tau h} \ge z_*] \Big) \cup \Big([z^k_{\tau h} \ge z_*] \cap [z^{k-1}_{\tau h} \le z_*] \Big) \end{aligned} \tag{58}$$

with $[f \le g] := \{x \in \Omega,\ f(x) \le g(x)\}$. In other words, the energy-dissipation estimate will also feature the perturbation term $\sum_{k=1}^{N_\tau} \int_{B^k_h} E_{\mathbb{C}}(z^{k-1}_{\tau h}, z^k_{\tau h}, e(u^{k-1}_{\tau h}))\,\mathrm{d}x$ on its right-hand side. This term can be controlled by (56) if the additional condition

$$\Big| \int_{B^k_h} E_{\mathbb{C}}(z^{k-1}_{\tau h}, z^k_{\tau h}, e(u^{k-1}_{\tau h}))\,\mathrm{d}x \Big| \le \mathsf{TOL}(h) \tag{59}$$

is imposed. We will deduce in Prop. 13 of Sec. 4.3 that criterion (59) can be met.

With the stopping criteria (51) and (55) for the algorithm and with the conditions (56) and (59) on the discretization at hand we now state the convergence result:

Theorem 3 (Convergence of the staggered Galerkin scheme (49))**.** *Let the assumptions of Proposition 12 be satisfied. Further let the criteria* (51) *and* (55) *and the conditions* (56) *and* (59) *on the parameters τ and $h = h(\tau)$ be satisfied. Then the family of approximate solutions*

$$\big((u^k_{\tau h(\tau)}, z^k_{\tau h(\tau)})_{k=1}^{N_\tau}\big)_{h(\tau)}$$

obtained by the staggered Galerkin scheme (49) *provides a subsequence that suitably converges to a limit pair*

$$(u, z) \in L^\infty(0, \mathsf{T}; \mathbf{U}) \cap H^1(0, \mathsf{T}; \mathbf{U}) \times L^\infty(0, \mathsf{T}; \mathbf{X}) \cap BV(0, T; \mathbf{Z})$$

with the following properties:

1 *The pair* (u, z) *is a solution of* $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ *in the sense of Def. 5.*
2 *For all* $t \in [0, \mathsf{T}]$ *it is* $0 \le z(t, x) \le 1$ *for* $\mathcal{L}^d$*-a.e.* $x \in \Omega$.
3 *The pair* (u, z) *also satisfies the semistability inequality* (34).

The proof of Thm. 3 will be discussed in Sec. 4.3. It consists of three major steps, which treat the limit passage in h and τ separately: In a first step the limit passage $h \to 0$ from the fully discrete setting to a space-continuous but time-discrete setting is carried out, leading to the criterion (51). The second step handles the limit passage $\tau \to 0$ from the time-discrete to the time-continuous setting. In the third step, the simultaneous limit passage by suitably choosing a sequence of mesh-sizes $h = h(\tau) \to 0$ as $\tau \to 0$ is justified and in the line of this argument conditions (55), (56), and (59) are observed.

4.2 Proof of Proposition 12

In the following, the parameters $h, \tau > 0$ and $k \in \{1, \dots, N_\tau\}$ are kept fixed. Using the notation of Section 4.1.2 the finite-element scheme (49) can be rewritten as a system of (non-)linear equations for the coefficient vectors $\mathbf{z}_{\tau h}^k = (z_{\tau hi}^k)_{i=1}^{\mathsf{N}_h} \in \mathbb{R}^{\mathsf{N}_h}$ and $\mathbf{u}_{\tau h}^k \in \mathbb{R}^{d\mathsf{N}_h}$. More precisely, testing (49a) with the basis functions φ_j for $\mathbf{X}_h$, $j \in \{1, \dots, \mathsf{N}_h\}$, we find the nonlinear system of N_h equations

$$\begin{aligned}
\epsilon_{\tau,h}^k(\varphi_j) &= \langle \mathrm{D}_z \mathcal{E}(t_\tau^k, u_{\tau h}^{k-1}, z_{\tau h}^k) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_{\tau h}^k), \varphi_j \rangle_{\mathbf{X}^*, \mathbf{X}} \\
&= \int_\Omega \Big(\tfrac{1}{2} \mathbb{C}'(z_{\tau h}^k) e(u_{\tau h}^{k-1}) : e(u_{\tau h}^{k-1}) + \tfrac{N_\tau}{2} \tfrac{\mathrm{d}}{\mathrm{d}z} m_\tau^2 (\tfrac{1}{\tau}(z_{\tau h}^k - z_{\tau h}^{k-1})) \Big) \varphi_j \, \mathrm{d}x \\
&\quad + \sum_{i=1}^{\mathsf{N}_h} \Big[\int_\Omega \Big(\tfrac{M}{\tau} + \tfrac{\mathcal{G}_c}{\ell} \Big) \varphi_i \varphi_j \, \mathrm{d}x \Big] z_{\tau hi}^k + \sum_{i=1}^{\mathsf{N}_h} \Big[\int_\Omega \mathcal{G}_c \ell \nabla \varphi_i \cdot \nabla \varphi_j \, \mathrm{d}x \Big] z_{\tau hi}^k \\
&\quad - \int_\Omega \Big(\tfrac{M}{\tau} z_{\tau h}^{k-1} + \tfrac{\mathcal{G}_c}{\ell} \Big) \varphi_j \, \mathrm{d}x
\end{aligned} \tag{60}$$

for $j \in \{1, \dots, \mathsf{N}_h\}$. Recalling that $z_{\tau h}^k = \sum_{i=1}^{\mathsf{N}_h} z_{\tau hi}^k \varphi_i$ and that $\mathbb{C}'(\cdot)$ and $\frac{\mathrm{d}}{\mathrm{d}z} m_\tau^2(\cdot)$ are nonlinear functions by assumptions (36), (37), and (42) we observe that the first integral term on the right-hand side constitutes a nonlinear function $\mathbf{f}(\mathbf{z}_{\tau h}^k) = (\mathbf{f}_j(\mathbf{z}_{\tau h}^k))_{j=1}^{\mathsf{N}_h}$ of the coefficient vector $\mathbf{z}_{\tau h}^k$. Instead, the second and the third term on the right-hand side are linear in $\mathbf{z}_{\tau h}^k$ and can be rewritten as a matrix-vector multiplication with matrices $\mathbb{M}_1$ and $\mathbb{M}_2$ collecting the integrals over the basis elements and the parameters. Finally, the last term on the right-hand side is independent of $\mathbf{z}_{\tau h}^k$ and we denote it by $\mathbf{p}$. In this way, the above nonlinear system of equations rewrites as

$$\big(\epsilon_{\tau,h}^k(\varphi_j)\big)_{j=1}^{\mathsf{N}_h} = \mathbf{f}(\mathbf{z}_{\tau h}^k) + \mathbb{M}_1 \mathbf{z}_{\tau h}^k + \mathbb{M}_2 \mathbf{z}_{\tau h}^k - \mathbf{p} =: \mathbf{g}(\mathbf{z}_{\tau h}^k) \,. \tag{61}$$

Here, a suitable numerical method, such as the Newton's method, can be applied to approximate roots of the nonlinear function $\mathbf{g}$. A stopping criterion for the algorithm has to be chosen such that $\mathbf{g}(\mathbf{z}_{\tau h}^k) = \big(\epsilon_{\tau,h}^k(\varphi_j)\big)_{j=1}^{\mathsf{N}_h} \approx \mathbf{0}$. This can be ensured by enforcing in the stopping criterion that

$$\max\big\{|\epsilon_{\tau,h}^k(\varphi_j)|,\, j = 1, \dots, \mathsf{N}_h\big\} \le \mathsf{TOL}(h) \ll 1 \tag{62}$$

with a suitably chosen tolerance $\mathsf{TOL}(h)$ such that $\mathsf{TOL}(h) \to 0$ as $h \to 0$.

Similarly, testing (49b) with the nodal basis $\boldsymbol{\varphi}_j$ for $\mathbf{U}_h$, $j = 1, \dots, d\mathsf{N}_h$, we obtain the system of $d\mathsf{N}_h$ linear equations

$$\begin{aligned}
0 &= \tau^2 \langle \rho \mathsf{D}_\tau^2 u_{\tau h}^k + \mathrm{D}_u \mathcal{E}(t_\tau^k, u_{\tau h}^k, z_{\tau h}^k) + \mathrm{D}\mathcal{V}(\mathsf{D}_\tau u_{\tau h}^k), \boldsymbol{\varphi}_j \rangle_{\mathbf{U}^*, \mathbf{U}} \\
&= \sum_{i=1}^{d\mathsf{N}_h} u_{\tau hi}^k \Big(\int_\Omega \rho \boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j \, \mathrm{d}x + \int_\Omega \big(\tau^2 \mathbb{C}(z_{\tau h}^k) + \tau \mathbb{D}(z_{\tau h}^k) \big) e(\boldsymbol{\varphi}_i) : e(\boldsymbol{\varphi}_j) \, \mathrm{d}x \Big) \\
&\quad + \rho \int_\Omega (-2u_{\tau h}^{k-1} + u_{\tau h}^{k-2}) \cdot \boldsymbol{\varphi}_j - \tau \mathbb{D}(z_{\tau h}^k) e(u_{\tau h}^{k-1}) : e(\boldsymbol{\varphi}_j) \, \mathrm{d}x - \tau^2 \langle f_{\tau h}^k, \boldsymbol{\varphi}_j \rangle_{\mathbf{U}^*, \mathbf{U}}
\end{aligned}$$

for $j \in \{1, \dots, d\mathsf{N}_h\}$. We see that the first sum on the right-hand side can be rewritten as a matrix-vector multiplication of the coefficient vector $\mathbf{u}_{\tau h}^k$ with two matrices $\mathbb{M}_3$ and $\mathbb{M}_4$, which gather the integrals over the basis elements and the

material tensors $\mathbb{C}(z_{\tau h}^k)$ and $\mathbb{D}(z_{\tau h}^k)$. Moreover, the remaining terms on the right-hand side are independent of $\mathbf{u}_{\tau h}^k$ and we denote them by $\mathbf{b}$. Hence, the above system of linear equations reformulates as

$$(\mathbb{M}_3 + \mathbb{M}_4)\mathbf{u}_{\tau h}^k = \mathbf{b}\,.$$

This linear system is solvable since the matrices $\mathbb{M}_3$ and $\mathbb{M}_4$ are invertible due to the linear independence of the basis elements and thanks to the coercivity of the tensors $\mathbb{C}(z_{\tau h}^k)$ and $\mathbb{D}(z_{\tau h}^k)$ given in (38).

To verify the uniform a priori bounds (50) we argue by induction, i.e., we assume

$$\left\|u_{\tau h}^{k-1}\right\|_{\mathbf{U}} + \left\|u_{\tau h}^{k-2}\right\|_{\mathbf{U}} + \left\|z_{\tau h}^{k-1}\right\|_{\mathbf{X}} \leq C \tag{63}$$

for all $h > 0$ and show that approximate solutions $(u_{\tau h}^k, z_{\tau h}^k)$ at step k are bounded independenty of h. We note that (63) is indeed an outcome of the induction argument below starting out from uniform bounded initial data as given by (41b). For the argument, we test in (49a) and (49b) with the approximate solutions $z_{\tau h}^k$ and $u_{\tau h}^k$. Summing these two relations results in

$$\begin{aligned}\epsilon_{\tau,h}^k(z_{\tau h}^k) = \langle \mathrm{D}_z\mathcal{E}(t_\tau^k, u_{\tau h}^{k-1}, z_{\tau h}^k) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_{\tau h}^k), z_{\tau h}^k\rangle_{\mathbf{X}^*,\mathbf{X}} + \int_\Omega \rho\, \mathsf{D}_\tau^2 u_{\tau h}^k \cdot u_{\tau h}^k \\ + \int_\Omega \left[\mathbb{D}(z_{\tau h}^k)e(\mathsf{D}_\tau u_{\tau h}^k) + \mathbb{C}(z_{\tau h}^k)e(u_{\tau h}^k)\right] : e(u_{\tau h}^k)\,\mathrm{d}x - \langle f_{\tau h}^k, u_{\tau h}^k\rangle_{\mathbf{U}^*,\mathbf{U}}\,.\end{aligned} \tag{64}$$

At this point we see that, in order to obtain a uniform bound $\tilde{C}$ being independent of h as in (50), we have to make sure for the approximate solutions that also

$$|\epsilon_{\tau,h}^k(z_{\tau h}^k)| \leq \mathsf{TOL}(h) \ll 1 \tag{65}$$

in addition to (62). Now, the right-hand side of (64) can be further estimated using standard arguments; for the details, we refer to a similar estimate with a sligthly different Yosida-regularization in [TT20, Prop. 3.2]. In this way, we ultimately obtain from (64) that

$$\begin{aligned}&\int_\Omega \Big(\frac{M}{2\tau} + \frac{\mathcal{G}_c}{4\ell}\Big)\left|z_{\tau h}^k\right|^2 + \mathcal{G}_c\ell\left|\nabla z_{\tau h}^k\right|^2\,\mathrm{d}x + \frac{c_{\mathbb{C}}^0}{2c_K^2}\left\|u_{\tau h}^k\right\|_{\mathbf{U}}^2 \\ &\leq \epsilon_{\tau,h}^k(z_{\tau h}^k) + \Big(\frac{\mathcal{G}_c}{2\ell} + \frac{\mathsf{T}}{8} + \frac{c_4\mathsf{T}}{2}\Big)\mathcal{L}^d(\Omega) + \frac{c_5}{2}\left\|f_{\tau h}^k\right\|_{\mathbf{U}^*}^2 \\ &\quad + \frac{\rho}{\tau^2}\Big(2\left\|u_{\tau h}^{k-1}\right\|_{L^2}^2 + \frac{1}{2}\left\|u_{\tau h}^{k-2}\right\|_{L^2}^2\Big) + \Big(\frac{M}{2\tau} + \frac{\mathsf{T}}{2\tau^2}\Big)\left\|z_{\tau h}^{k-1}\right\|_{L^2}^2 + \frac{c_{\mathbb{D}}^{*\,2}}{2\tau c_{\mathbb{D}}^0}\left\|e(u_{\tau h}^{k-1})\right\|_{L^2}^2\end{aligned}$$

where it was used that in $[0,\tau)$, $\tau \ll 1$, m_τ, m_τ' can be estimated from above by τ and 1, respectively while $c_4, c_5 > 0$ are constants. The right-hand side indeed provides a constant $\tilde{C}$ that depends on the approximate solutions from the previous time-step and on τ^{-1}, but which is independent of h thanks to (63) and (65). This finishes the proof of the a priori estimates (50) and completes the proof of Prop. 12. □

Remark 7 (Newton's method)**.** *For h, τ, k fixed, Newton's method to find roots of the nonlinear equation* (61) *takes in every iteration step α the form*

$$\mathbf{z}_\alpha^k = \mathbf{z}_{\alpha-1}^k - \mathrm{D}\mathbf{g}(\mathbf{z}_{\alpha-1}^k)^{-1}\mathbf{g}(\mathbf{z}_{\alpha-1}^k) =: N_{\mathbf{g}}(\mathbf{z}_{\alpha-1}^k)$$

with the Newton-operator $N_{\mathbf{g}}$. Here, the nonlinear function $\mathbf{f}$ of $\mathbf{g}$ in (61) *depends on $\mathbb{C}'$. Thus, for $N_{\mathbf{g}}$ to be meaningful requires $w_{\mathbb{C}} \in C^2(\mathbb{R},[w_0,w^*])$ as demanded in* (37)*. If in addition, $w_{\mathbb{C}} \in C^3(\mathbb{R},[w_0,w^*])$, one can use a Taylor expansion near a root a of $\mathbf{g}$ to estimate*

$$\left|N_{\mathbf{g}}(\mathbf{z}_\alpha^k) - a\right| \leq \left|\mathrm{D}\mathbf{g}(\mathbf{z}_\alpha^k)^{-1}\right|\left|c\mathrm{D}^2\mathbf{g}(\xi)\right|\left|(\mathbf{z}_\alpha^k - a)\right|^2 \tag{66}$$

with ξ on a straight line between a and $\mathbf{z}_\alpha^k$. If $\left|\mathrm{D}\mathbf{g}(\cdot)^{-1}\right|$, $\left|\mathrm{D}^2\mathbf{g}(\cdot)\right|$ are uniformly bounded in a neighbourhood $U(a)$ of a and defining $K := c\inf_{\mathbf{z}\in U(a)}|\mathrm{D}\mathbf{g}(\mathbf{z})|\sup_{\mathbf{z}\in U(a)}\left|\mathrm{D}^2\mathbf{g}(\mathbf{z})\right|$, then (66) *implies with $d_n := K\left|\mathbf{z}_\alpha^k - a\right|$ that*

$$K\left|\mathbf{z}_\alpha^k - a\right| = d_n \leq d_{\alpha-1}^2 \leq \ldots \leq d_0^{2\alpha} = (K\left|\mathbf{z}_0^k - a\right|)^{2\alpha} \tag{67}$$

By induction one finds quadratic convergence to a provided the initial value z_0^k is located in a neighbourhood of a such that $\left|\mathbf{z}_0^k - a\right| < \frac{1}{K}$.

4.3 Outline of the Proof of Convergence Theorem 3

The strategy of the proof consists in three major steps:

Step 1: For $\tau > 0$ fixed, starting from approximate solutions $((u_{\tau h}^k, z_{\tau h}^k)_{k=1}^{N_\tau})_h$ given by Prop. 12 we pass to the limit $h \to 0$. By compactness arguments we find a limit pair $(u_\tau^k, z_\tau^k)_{k=1}^{N_\tau}$ for which we show that it satisfies a space-continuous but time-discrete version of (49). The results of this step are summarized in Thm. 4 below and we refer to [TT20] for the details of the proof.

Step 2: We pass to the limit $\tau \to 0$ and show that a subsequence of the time-discrete solutions $((u_\tau^k, z_\tau^k)_{k=1}^{N_\tau}))_\tau$ converges to a limit pair being a solution of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ in the sense of Def. 5. The results of this step are collected in Theorem 5 and refer to [TT20] for a proof.

Step 3: We show that the simultaneous limit in $h = h(\tau), \tau \to 0$ can be carried out by selecting a suitable diagonal sequence that complies with the constraints (51), (55), and (59). We further show that the constraint (59) on the discretization can be met. Step 3 will be carried out in Sec. 4.3.2.

4.3.1 Results of Steps 1 and 2

Theorem 4 (Existence of solutions in the space-continuous setting)**.** *Let the assumptions of Theorem 3 be satisfied. Keep $\tau > 0$ fixed. Then the following statements hold true:*

1 For each $k \in \{1, \dots, N_\tau\}$ there is a (not relabeled) subsequence $(u_{\tau h}^k, z_{\tau h}^k)_h$ and limit pairs $(u_\tau^k, z_\tau^k) \in \mathbf{U} \times \mathbf{X}$ such that

$$u_{\tau h}^k \rightharpoonup u_\tau^k \text{ weakly in } \mathbf{U}\,, \tag{68a}$$

$$z_{\tau h}^k \rightharpoonup z_\tau^k \text{ weakly in } \mathbf{X} \qquad \text{as } h \to 0. \tag{68b}$$

2 Assume that the discrete initial data satisfy

$$u_{\tau h}^0 \to u_\tau^0 \text{ in } \mathbf{U} \quad \text{and} \quad u_{\tau h}^{-1} \to u_\tau^{-1} \text{ in } \mathbf{U}, \tag{69a}$$

$$z_{\tau h}^0 \to z_\tau^0 \text{ in } \mathbf{X}. \tag{69b}$$

Then, for each $k \in \{1, \dots, N_\tau\}$ the limit pair $(u_\tau^k, z_\tau^k) \in \mathbf{U} \times \mathbf{X}$ is a solution of the time-discrete problem

$$0 = \langle \mathrm{D}_z \mathcal{E}(t_\tau^k, u_\tau^{k-1}, z_\tau^k) + \mathrm{D}\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^k), \eta \rangle_{\mathbf{X}^*, \mathbf{X}} \quad \text{for all } \eta \in \mathbf{Y}\,, \tag{70a}$$

$$0 = \int_\Omega \rho \mathsf{D}_\tau^2 u_\tau^k \cdot v + \big[\mathbb{D}(z_\tau^k) e(\mathsf{D}_\tau u_\tau^k) + \mathbb{C}(z_\tau^k) e(u_\tau^k)\big] : e(v)\,\mathrm{d}x - \big\langle f_\tau^k, v \big\rangle_{\mathbf{U}^*, \mathbf{U}}$$
$$\text{for all } v \in \mathbf{U}\,. \tag{70b}$$

3 Suppose that (69) *is satisfied. Then, in addition to* (68)*, for each $k \in \{1, \dots, N_\tau\}$ also the following improved convergence results hold true:*

$$u_{\tau h}^k \to u_\tau^k \text{ strongly in } \mathbf{U}\,, \tag{71a}$$

$$z_{\tau h}^k \to z_\tau^k \text{ strongly in } \mathbf{X}\,. \tag{71b}$$

4 Suppose that $z_{\tau h}^0 \in [0,1]$ a.e. in Ω. Then, for each $k \in \{1, \dots, N_\tau\}$ the limit function z_τ^k satisfies

$$z_\tau^k \in \mathbf{Y}, \quad \text{in particular } 0 \le z_\tau^k \le 1 \text{ a.e. in } \Omega. \tag{72}$$

5 The time-discrete solutions $(u_\tau^k, z_\tau^k)_{k=0}^{N_\tau}$ of (70) *satisfy the following upper energy-dissipation estimate for each*

$L \in \{1, \dots, N_\tau\}$:

$$\begin{aligned}
&\int_\Omega \frac{\rho}{2} \left|\mathsf{D}_\tau u_\tau^L\right|^2 \,\mathrm{d}x + \sum_{k=1}^{L} \tau \int_\Omega \mathbb{D}(z_\tau^k) e(\mathsf{D}_\tau u_\tau^k) : e(\mathsf{D}_\tau u_\tau^k) \,\mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{2} \mathbb{C}(z_\tau^L) e(u_\tau^L) : e(u_\tau^L) + \mathcal{G}_c\Big(\frac{1}{2\ell}(1 - z_\tau^L)^2 + \frac{\ell}{2} |\nabla z_\tau^L|^2\Big) \,\mathrm{d}x \\
&\quad - \langle f_\tau^L, u_\tau^L \rangle_{\mathbf{U}^*,\mathbf{U}} + \sum_{k=1}^{L} 2\tau \mathcal{R}_{M\tau}(\mathsf{D}_\tau z_\tau^k) \\
&\leq \int_\Omega \frac{\rho}{2} \left|\mathsf{D}_\tau u_\tau^0\right|^2 \,\mathrm{d}x + \int_\Omega \frac{1}{2} \mathbb{C}(z_\tau^0) e(u_\tau^0) : e(u_\tau^0) + \mathcal{G}_c\Big(\frac{1}{2\ell}(1 - z_\tau^0)^2 + \frac{\ell}{2} |\nabla z_\tau^0|^2\Big) \,\mathrm{d}x \\
&\quad - \big\langle f_\tau^0, u_\tau^0 \big\rangle_{\mathbf{U}^*,\mathbf{U}} - \tau \sum_{k=1}^{L} \big\langle \mathsf{D}_\tau f_\tau^k, u_\tau^{k-1} \big\rangle_{\mathbf{U}^*,\mathbf{U}} \,.
\end{aligned} \tag{73}$$

For the approximate solutions $(u_{\tau h}^k, z_{\tau h}^k)_{k=1}^{N_\tau}$ obtained by solving (49), piecewise constant interpolants $\bar{\mathrm{v}}_{\tau h}$, $\underline{\mathrm{v}}_{\tau h}$, and affine-linear approximations $\mathrm{v}_{\tau h}$ for $\mathrm{v} \in \{u, z\}$ are introduced, defined for $t \in (t_\tau^{k-1}, t_\tau^k]$, $k = 1, \dots N_\tau$ by

$$\bar{\mathrm{v}}_{\tau h}(t) = \mathrm{v}_{\tau h}^k,\ \underline{\mathrm{v}}_{\tau h}(t) = \mathrm{v}_{\tau h}^{k-1},\ \mathrm{v}_{\tau h}(t) = \frac{t - t_\tau^{k-1}}{\tau} \mathrm{v}_{\tau h}^k + \frac{t_\tau^k - t}{\tau} \mathrm{v}_{\tau h}^{k-1} \,. \tag{74}$$

Theorem 5 (Existence of solutions in the space- and time-continuous setting)**.** *Let the assumptions of Theorem 4 be satisfied. Further suppose that $z_\tau^0 = z_0$, $u_\tau^0 = u_0$ and $u_\tau^{-1} = u_0 - \tau \dot{u}_0$ for all $\tau > 0$. Consider the viscosity parameter M in* (26) *to depend on τ such that $M(\tau) \to 0$ as $\tau \to 0$. Then the following results hold true:*

1 *There exists a limit pair $(u, z) \colon [0, \mathsf{T}] \to \mathbf{U} \times \mathbf{X}$ and a (not relabeled) subsequence of approximate solutions $(\bar{u}_\tau, \underline{u}_\tau, u_\tau, \bar{z}_\tau, \underline{z}_\tau)_\tau$ such that*

$$\bar{u}_\tau,\ \underline{u}_\tau \overset{*}{\rightharpoonup} u \qquad \text{weakly-}* \text{ in } L^\infty(0, \mathsf{T}; \mathbf{U}) \,, \tag{75a}$$
$$u_\tau \rightharpoonup u \qquad \text{weakly in } H^1(0, \mathsf{T}; \mathbf{U}) \,, \tag{75b}$$
$$\dot{u}_\tau \overset{*}{\rightharpoonup} \dot{u} \qquad \text{weakly-}* \text{ in } L^\infty\big(0, \mathsf{T}; L^2(\Omega, \mathbb{R}^d)\big) \,, \tag{75c}$$
$$\bar{u}_\tau(t),\ \underline{u}_\tau(t) \rightharpoonup u(t) \qquad \text{weakly in } \mathbf{U} \text{ for all } t \in [0, \mathsf{T}] \,, \tag{75d}$$
$$\dot{u}_\tau(t) \rightharpoonup \dot{u}(t) \qquad \text{weakly in } L^2(\Omega, \mathbb{R}^d) \text{ for all } t \in [0, \mathsf{T}] \,, \tag{75e}$$
$$\bar{z}_\tau,\ \underline{z}_\tau \overset{*}{\rightharpoonup} z \qquad \text{weakly-}* \text{ in } L^\infty\big(0, \mathsf{T}; \mathbf{X}\big) \,, \tag{75f}$$
$$\bar{z}_\tau(t) \rightharpoonup z(t) \qquad \text{weakly in } \mathbf{X} \text{ for all } t \in [0, \mathsf{T}] \,, \tag{75g}$$
$$\bar{z}_\tau(t) \to z(t) \qquad \text{strongly in } L^2(\Omega) \text{ for all } t \in [0, \mathsf{T}] \,, \tag{75h}$$
$$\underline{z}_\tau(t) \rightharpoonup z(t) \qquad \text{weakly in } \mathbf{X} \text{ for all } t \in [0, \mathsf{T}] \,, \tag{75i}$$
$$\underline{z}_\tau(t) \to z(t) \qquad \text{strongly in } L^2(\Omega) \text{ for all } t \in [0, \mathsf{T}] \,. \tag{75j}$$

2 *The limit pair (u, z) is a solution of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ in the sense of Def. 5.*

Corollary 1. *Let the assumptions of Theorem 5 be valid. Then:*

1 *The limit pair (u, z) from Thm. 5 also satisfies semistability inequality* (34) *and thus is a semistable energetic solution in the sense of Def. 2.*
2 *As a result of the energy-dissipation* balance (33d) *there also holds*

$$\bar{z}_\tau(t) \to z(t) \ \text{ strongly in } \mathbf{X} \text{ for all } t \in [0, \mathsf{T}]. \tag{76}$$

4.3.2 Discussion of Step 3: Simultaneous limit $h = h(\tau), \tau \to 0$

In the following we verify that it is possible to select a (diagonal) subsequence of (interpolants of) approximate solutions

$$(\bar{u}_{\tau_j h_j}, \underline{\mathsf{u}}_{\tau_j h_j}, u_{\tau_j h_j}, \bar{z}_{\tau_j h_j}, \underline{\mathsf{z}}_{\tau_j h_j})_j$$

converging to a solution (u, z) of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ in the sense of Definition 5. By making use of the convergence results in Theorems 4, 5, and Cor. 1, we provide in Proposition 13 sufficient conditions (55), (56) and (59), which allow it to deduce a uniform upper energy-dissipation estimate (perturbed by error terms). Under these conditions uniform a priori bounds will be available for the approximate solutions which provide sufficient compactness so that convergence results in the topologies of (75) can be concluded. These will be sufficient to pass to the limit in the staggered Galerkin scheme (49) and to find a solution of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ in the sense of Def. 5. In addition, we will show in Lemma 3 below, that on an abstract level a better selection is possible. The idea is here that, in accordance with the necessary stopping criteria (51) and (52), h as to be chosen 'as small as possible' so that the space continuous solution $(u_\tau^k, z_\tau^k)_{k=1}^{N_\tau}$ from Theorem 4 is approximated already as 'good as possible'. We remark that solutions obtained from the uniform bound in Prop. 13 must not coincide with the one obtained in Lemma 3, because solutions of $(\mathbf{U}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ are not unique since the energy functional $\mathcal{E}(t, \cdot, \cdot)$ is non-convex.

Proposition 13 (Uniform energy-dissipation estimate and validity of fineness criterion (59))**.** *Let the assumptions of Proposition 12 be satisfied and let* $((u_{\tau h}^k, z_{\tau h}^k)_{k=1}^{N_\tau})_{\tau h}$ *be approximate solutions obtained by the staggered Galerkin scheme* (49) *such that condition* (51) *holds true.*

1 Then, $((u_{\tau h}^k, z_{\tau h}^k)_{k=1}^{N_\tau})_{\tau h}$ *comply with an upper energy-dissipation estimate up to an error:*

$$
\begin{aligned}
&\sum_{k=1}^{L} \epsilon_{\tau,h}^k (z_{\tau h}^k - z_{\tau h}^{k-1}) + \sum_{k=1}^{L} \int_{B_h^k} E_{\mathbb{C}}(z_{\tau h}^{k-1}, z_{\tau h}^k, e(u_{\tau h}^{k-1}))\,\mathrm{d}x \\
&\quad + \tau \sum_{k=1}^{L} \langle \mathsf{D}_\tau f_{\tau h}^k, u_{\tau h}^{k-1} \rangle_{\mathbf{U}^*,\mathbf{U}} + \int_\Omega \frac{\rho}{2} \left| \mathsf{D}_\tau u_{\tau h}^0 \right|^2 \mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{2} \mathbb{C}(z_{\tau h}^0) e(u_{\tau h}^0) : e(u_{\tau h}^0)\,\mathrm{d}x + \int_\Omega \mathcal{G}_c \Big(\frac{1}{2\ell}(1 - z_{\tau h}^0)^2 + \frac{\ell}{2} \left| \nabla z_{\tau h}^0 \right|^2 \Big)\,\mathrm{d}x - \langle f_{\tau h}^0, u_{\tau h}^0 \rangle_{\mathbf{U}^*,\mathbf{U}} \qquad (77) \\
&\geq \int_\Omega \frac{\rho}{2} \left| \mathsf{D}_\tau u_{\tau h}^L \right|^2 \mathrm{d}x + \sum_{k=1}^{L} \tau 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z_{\tau h}^k) + \sum_{k=1}^{L} \tau \int_\Omega \mathbb{D}(z_{\tau h}^k) e(\mathsf{D}_\tau u_{\tau h}^k) : e(\mathsf{D}_\tau u_{\tau h}^k)\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{2} \mathbb{C}(z_{\tau h}^L) e(u_{\tau h}^L) : e(u_{\tau h}^L) + \int_\Omega \mathcal{G}_c \Big(\frac{1}{2\ell}(1 - z_{\tau h}^L)^2 + \frac{\ell}{2} \left| \nabla z_{\tau h}^L \right|^2 \Big)\,\mathrm{d}x - \langle f_{\tau h}^L, u_{\tau h}^L \rangle_{\mathbf{U}^*,\mathbf{U}}
\end{aligned}
$$

for all $L \in \{1, \dots, N_\tau\}$ *and with the error terms* $\epsilon_{\tau,h}^k$, $\int_{B_h^k} E_{\mathbb{C}}(z_{\tau h}^{k-1}, z_{\tau h}^k, e(u_{\tau h}^{k-1}))\,\mathrm{d}x$ *given in* (49a) *and* (57).

2 Condition (59) *can be met.*

3 Assume in addition that (55) *and* (56) *as well as* (59) *are fulfilled.*
Then the error terms

$$\sum_{k=1}^{N_\tau} \epsilon_{\tau,h}^k (z_{\tau h}^k - z_{\tau h}^{k-1})$$

and

$$\sum_{k=1}^{N_\tau} \int_{B_h^k} E_{\mathbb{C}}(z_{\tau h}^{k-1}, z_{\tau h}^k, e(u_{\tau h}^{k-1}))\,\mathrm{d}x$$

in (77) *vanish as* $h = h(\tau) \to 0$ *and* $\tau \to 0$.
Thus, the upper energy-dissipation estimate (77) *provides a uniform a priori estimate for the approximate solutions* $((u_{\tau h}^k, z_{\tau h}^k)_{k=1}^{N_\tau})_{\tau h}$.

Proof. **of Prop. 13, Item 1:** As mentioned for (54), by testing the discrete evolution equation (48b) with $\tau \mathsf{D}_\tau u^k_{\tau h}$ and (48a) with $\tau \mathsf{D}_\tau z^k_{\tau h}$ and summing up one finds

$$
\begin{aligned}
&\sum_{k=1}^{N_\tau} \epsilon^k_{\tau,h}(z^k_{\tau h} - z^{k-1}_{\tau h}) + \sum_{k=1}^{N_\tau} \big\langle f^k_{\tau h}, u^k_{\tau h} - u^{k-1}_{\tau h} \big\rangle_{\mathbf{U}^*,\mathbf{U}} \\
&+ \int_\Omega \frac{\rho}{2} \left|\mathsf{D}_\tau u^0_{\tau h}\right|^2 \,\mathrm{d}x + \int_\Omega \mathcal{G}_c\Big(\frac{1}{2\ell}(1 - z^0_{\tau h})^2 + \frac{\ell}{2}\left|\nabla z^0_{\tau h}\right|^2\Big)\,\mathrm{d}x \\
&\geq \int_\Omega \frac{\rho}{2} \left|\mathsf{D}_\tau u^{N_\tau}_{\tau h}\right|^2 \,\mathrm{d}x + \sum_{k=1}^{N_\tau} \tau \int_\Omega \mathbb{D}(z^k_{\tau h}) e(\mathsf{D}_\tau u^k_{\tau h}) : e(\mathsf{D}_\tau u^k_{\tau h})\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{N_\tau} \tau 2\mathcal{R}_{M\tau}(\mathsf{D}_\tau z^k_{\tau h}) + \int_\Omega \mathcal{G}_c\Big(\frac{1}{2\ell}(1 - z^{N_\tau}_{\tau h})^2 + \frac{\ell}{2}\left|\nabla z^{N_\tau}_{\tau h}\right|^2\Big)\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{N_\tau} \int_\Omega \frac{1}{2}\mathbb{C}(z^k_{\tau h}) e(u^k_{\tau h}) : e(u^k_{\tau h}) - \frac{1}{2}\mathbb{C}(z^k_{\tau h}) e(u^{k-1}_{\tau h}) : e(u^{k-1}_{\tau h})\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{N_\tau} \int_\Omega \frac{1}{2}\mathbb{C}'(z^k_{\tau h})(z^k_{\tau h} - z^{k-1}_{\tau h}) e(u^{k-1}_{\tau h}) : e(u^{k-1}_{\tau h})\mathrm{d}x\,.
\end{aligned}
\tag{78}
$$

In order to make the stored elastic energy at step $k-1$ appear in (78) we would like to replace in the second last line of (78) the term $-\frac{1}{2}\mathbb{C}(z^k_{\tau h}) e(u^{k-1}_{\tau h}) : e(u^{k-1}_{\tau h})$ by $-\frac{1}{2}\mathbb{C}(z^{k-1}_{\tau h}) e(u^{k-1}_{\tau h}) : e(u^{k-1}_{\tau h})$. This has to be compensated and together with the last term in (78) we collect it in the error term $E_\mathbb{C}$ from (57), once more recalled

$$
E_\mathbb{C}\big(z^{k-1}_{\tau h}, z^k_{\tau h}, e(u^{k-1}_{\tau h})\big) := \tfrac{1}{2}\big(\mathbb{C}(z^{k-1}_{\tau h}) - \mathbb{C}(z^k_{\tau h}) + \mathbb{C}'(z^k_{\tau h})(z^k_{\tau h} - z^{k-1}_{\tau h})\big) e(u^{k-1}_{\tau h}) : e(u^{k-1}_{\tau h})\,.
$$

In this way, the last two lines in (78) can be rewritten as follows:

$$
\begin{aligned}
&\sum_{k=1}^{N_\tau} \int_\Omega \frac{1}{2}\mathbb{C}(z^k_{\tau h}) e(u^k_{\tau h}) : e(u^k_{\tau h}) - \frac{1}{2}\mathbb{C}(z^k_{\tau h}) e(u^{k-1}_{\tau h}) : e(u^{k-1}_{\tau h})\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{N_\tau} \int_\Omega \frac{1}{2}\mathbb{C}'(z^k_{\tau h})(z^k_{\tau h} - z^{k-1}_{\tau h}) e(u^{k-1}_{\tau h}) : e(u^{k-1}_{\tau h})\mathrm{d}x \\
&= \sum_{k=1}^{N_\tau} \int_\Omega \frac{1}{2}\mathbb{C}(z^k_{\tau h}) e(u^k_{\tau h}) : e(u^k_{\tau h}) - \frac{1}{2}\mathbb{C}(z^{k-1}_{\tau h}) e(u^{k-1}_{\tau h}) : e(u^{k-1}_{\tau h}) + E_\mathbb{C}\big(z^{k-1}_{\tau h}, z^k_{\tau h}, e(u^{k-1}_{\tau h})\big)\,\mathrm{d}x
\end{aligned}
\tag{79}
$$

Using (79) in (78) one arrives at the upper energy-dissipation estimate (77). □

Proof of Prop. 13, Item 2: To show that condition (59) can be met, we keep $\tau > 0$ fixed and investigate $E_\mathbb{C}$ on a partition $\Omega = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5$ with

$$
\begin{aligned}
&B_1 = [z^k_{\tau h} \leq z_*] \cap [z^{k-1}_{\tau h} \leq z_*]\,,\ B_2 = [z_* \leq z^k_{\tau h} \leq z^*] \cap [z_* \leq z^{k-1}_{\tau h} \leq z^*]\,, \\
&B_3 = [z^* \leq z^k_{\tau h}] \cap [z^* \leq z^{k-1}_{\tau h}]\,,\ B_4 = [z^k_{\tau h} \leq z_*] \cap [z_* \leq z^{k-1}_{\tau h}]\,, B_5 = [z_* \leq z^k_{\tau h}] \cap [z^{k-1}_{\tau h} \leq z_*]\,.
\end{aligned}
$$

On B_1, the error $E_\mathbb{C}\big(z^{k-1}_{\tau h}, z^k_{\tau h}, e(u^{k-1}_{\tau h})\big)$ can be estimated from below by 0 because this is a convex branch of the energy. Instead, on B_3, the error $E_\mathbb{C}\big(z^{k-1}_{\tau h}, z^k_{\tau h}, e(u^{k-1}_{\tau h})\big) = 0$ vanishes. For the remaining sets, i.e. for $B^k_h = B_2 \cup B_4 \cup B_5$, we have the inclusion

$$
B^k_h \subset \big[\big|z^k_{\tau h} - z^k_\tau\big| \geq \delta\big] \cup \big[\big|z^{k-1}_{\tau h} - z^{k-1}_\tau\big| \geq \delta\big] \tag{80}
$$

with $\delta \in (1, z_*)$. Here, clearly $\mathcal{L}^d\big([\big|z^k_{\tau h} - z^k_\tau\big| \geq \delta] \cup \big[\big|z^{k-1}_{\tau h} - z^{k-1}_\tau\big|\big) \to 0$ as $h \to 0$ in consequence of the strong convergence in $\mathbf{Z}$, cf. (71b). Hence, also

$$
\mathcal{L}^d(B^k_h) \to 0 \quad \text{as } h \to 0\,. \tag{81}
$$

Thanks to the strong convergence $u_{\tau h}^{k-1} \to u_\tau^{k-1}$ in $\mathbf{U}$ given by (71a) one finds

$$\int_\Omega E_{\mathbb{C}}(z_{\tau h}^{k-1}, z_{\tau h}^k, u_{\tau h}^{k-1})\,\mathrm{d}x \to \int_\Omega E_{\mathbb{C}}(z_\tau^{k-1}, z_\tau^k, u_\tau^{k-1})\,\mathrm{d}x \tag{82}$$

by continuity of $E_{\mathbb{C}}(\cdot,\cdot,\cdot)$ guaranteed by (36) and (37).

This provides the tools to verify that the condition (59) can be met: Using (82) the error on B_h^k can be estimated as follows:

$$\begin{aligned}
&\left|\int_{B_h^k} E_{\mathbb{C}}(z_{\tau h}^{k-1}, z_{\tau h}^k, e(u_{\tau h}^{k-1}))\right| \\
&\le \left|\int_{B_h^k} E_{\mathbb{C}}(z_{\tau h}^{k-1}, z_{\tau h}^k, e(u_{\tau h}^{k-1})) - E_{\mathbb{C}}(z_\tau^{k-1}, z_\tau^k, e(u_\tau^{k-1}))\,\mathrm{d}x\right| + \left|\int_{B_h^k} E_{\mathbb{C}}(z_\tau^{k-1}, z_\tau^k, e(u_\tau^{k-1}))\,\mathrm{d}x\right| \\
&\quad + \left|\int_{\Omega\setminus B_h^k} E_{\mathbb{C}}(z_{\tau h}^{k-1}, z_{\tau h}^k, e(u_{\tau h}^{k-1})) - E_{\mathbb{C}}(z_\tau^{k-1}, z_\tau^k, e(u_\tau^{k-1}))\,\mathrm{d}x\right| \\
&= \left|\int_\Omega E_{\mathbb{C}}(z_{\tau h}^{k-1}, z_{\tau h}^k, e(u_{\tau h}^{k-1})) - E_{\mathbb{C}}(z_\tau^{k-1}, z_\tau^k, e(u_\tau^{k-1}))\,\mathrm{d}x\right| \\
&\quad + \left|\int_{B_h^k} E_{\mathbb{C}}(z_\tau^{k-1}, z_\tau^k, e(u_\tau^{k-1}))\,\mathrm{d}x\right| \longrightarrow 0\,,
\end{aligned}$$

where the first term tends to zero by (82) and the second term by (82) and (81). Hence also

$$\left|\int_{B_h^k} E_{\mathbb{C}}(z_{\tau h}^{k-1}, z_{\tau h}^k, e(u_{\tau h}^{k-1}))\right| \to 0 \quad \text{as } h \to 0 \text{ for all } k \in \{1,\dots,N_\tau\}\,. \tag{83}$$

From this we see that the perturbation term in (77) stemming from the non-convexity of the energy is controlled if

$$\sum_{k=1}^{N_\tau} \left|\int_{B_h^k} E_{\mathbb{C}}(z_\tau^{k-1}, z_\tau^k, e(u_\tau^{k-1}))\,\mathrm{d}x\right| \to 0\,. \tag{84}$$

This can be accomplished by ensuring for all $k \in \{1,\dots,N_\tau\}$ that

$$\left|\int_{B_h^k} E_{\mathbb{C}}(z_{\tau h}^{k-1}, z_{\tau h}^k, e(u_{\tau h}^{k-1}))\right| \le \mathsf{TOL}(h) \tag{85}$$

and in addition

$$N_\tau \mathsf{TOL}(h(\tau)) = \mathsf{T}\frac{\mathsf{TOL}(h(\tau))}{\tau} \to 0 \quad \text{as } \tau \to 0\,. \tag{86}$$

This provides conditions (56) and (59). We conclude that (85), hence (59), can be met thanks to the convergence ontained in (83). □

Proof of Prop. 13, Item 3: The error term due to the non-convexity can be controlled and vanishes according to (84) if conditions (85) and (86), i.e., (59) and (56), are satisfied. Similarly, one can see that the error term $\sum_{k=1}^{N_\tau} \epsilon_{\tau,h}^k(z_{\tau h}^k - z_{\tau h}^{k-1})$ in (77) tends to zero, provided (56) and (55) are satisfied. Under these condtions the two error terms in (77) are uniformly bounded in $h = h(\tau), \tau$, so that the upper energy dissipation estimate provides uniform bounds for approximate solutions. □

Lemma 3 (Selection of a diagonal subsequence)**.** *Let the conditions* (51) *and* (52) *be satisfied. Then, it is possible to select a diagonal subsequence* $(\tau_j h_j)_{j\in\mathbb{N}}$ *of the time-step and mesh sizes, such that the corresponding approximate solutions* $(\bar u_{\tau_j h_j}, \bar z_{\tau_j h_j})_{j\in\mathbb{N}} \colon [0,\mathsf{T}] \to \mathbf{U}\times\mathbf{X}$ *converge to the limit pair* $(u,z)\colon [0,\mathsf{T}] \to \mathbf{U}\times\mathbf{X}$ *obtained in Theorem 5. More precisely, we have for all* $t \in [0,T]$*:*

$$u_{\tau_j h_j}(t) \rightharpoonup u(t) \ \textit{ weakly in } \mathbf{U}, \tag{87a}$$

$$z_{\tau_j h_j}(t) \to z(t) \ \textit{ strongly in } \mathbf{Z} \qquad \textit{as } j \to \infty. \tag{87b}$$

Proof. We first show (87b). For this, first, keep $\tau > 0$ fixed. Since there are only N_τ time-steps in the partition Π_τ of the time interval $[0, \mathsf{T}]$, the strong convergence (71b) for $h \to 0$ is *uniform* in time for the interpolants $\bar{z}_{\tau h}$. Thus, one can select $\bar{z}_{\tau h(\tau)}$ such that for all $t \in [0, \mathsf{T}]$

$$\big\|\bar{z}_{\tau h(\tau)}(t) - \bar{z}_\tau(t)\big\|_{\mathbf{X}} < \tau\,. \tag{88}$$

By strong convergence (76), $\bar{z}_\tau(t) \to z(t)$ in $\mathbf{X}$ pointwise for all $t \in [0,T]$, we find

$$\big\|\bar{z}_{\tau h(\tau)}(t) - z(t)\big\|_{\mathbf{X}} \le \big\|\bar{z}_{\tau h(\tau)}(t) - \bar{z}_\tau(t)\big\|_{\mathbf{X}} + \|\bar{z}_\tau(t) - z(t)\|_{\mathbf{X}} \le \tau + \|\bar{z}_\tau(t) - z(t)\|_{\mathbf{X}} \to 0$$

as $\tau \to 0$. We argue in a similar manner to verify (87a), based on the strong convergence (71a) and the weak convergence (75d). For each $\tau > 0$ fixed, with the same arguments as for (88), we select $\bar{u}_{\tau h(\tau)}$ such that for all $t \in [0, \mathsf{T}]$

$$\big\|\bar{u}_{\tau h(\tau)}(t) - \bar{u}_\tau(t)\big\|_{\mathbf{U}} < \tau\,.$$

For each $v \in \mathbf{U}$ with $\|v\|_{\mathbf{U}} \le 1$ we then deduce

$$\begin{aligned}\langle \bar{u}_{\tau h(\tau)}(t) - u(t), v\rangle_{\mathbf{U}^*,\mathbf{U}} &\le \big|\langle \bar{u}_{\tau h(\tau)}(t) - \bar{u}_\tau(t), v\rangle_{\mathbf{U}^*,\mathbf{U}}\big| + \big|\langle \bar{u}_\tau(t) - u(t), v\rangle_{\mathbf{U}^*,\mathbf{U}}\big| \\ &\le \|\bar{u}_{\tau h(\tau)}(t) - \bar{u}_\tau(t)\|_{\mathbf{U}}\|v\|_{\mathbf{U}} + \big|\langle \bar{u}_\tau(t) - u(t), v\rangle_{\mathbf{U}^*,\mathbf{U}}\big| \le \tau + \big|\langle \bar{u}_\tau(t) - u(t), v\rangle_{\mathbf{U}^*,\mathbf{U}}\big| \to 0\end{aligned}$$

as $\tau \to 0$ by the weak convergence (75d), $\bar{u}_\tau(t) \rightharpoonup u(t)$ in $\mathbf{U}$ pointwise for all $t \in [0,T]$ where we used the usual identification of dual and predual in Hilbert spaces. One may then choose a subsequence $\tau_j \to 0$ as $j \to \infty$ and set $h_j := h(\tau_j)$ to ultimately conclude (87). □

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References

[AB19] S. Almi and S. Belz. Consistent finite-dimensional approximation of phase-field models of fracture. *Ann. Mat. Pura Appl.*, 198(4):1191–1225, 2019.

[ABM06] H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces*, volume 6 of *MPS/SIAM Series on Optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Programming Society (MPS), Philadelphia, PA, 2006. Applications to PDEs and optimization.

[ABN18] S. Almi, S. Belz, and M. Negri. Convergence of discrete and continuous unilateral flows for Ambrosio-Tortorelli energies and application to mechanics. *ESAIM M2AN*, 53(2):659–699, 2018.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[AN19] S. Almi and M. Negri. Analysis of staggered evolutions for nonlinear energies in phase field fracture. *Arch. Ration. Mech. Anal.*, 2019.

[Bar12] S. Bartels. Total variation minimization with finite elements: convergence and iterative solution. *SIAM J. Numer. Anal.*, 50(3):1162–1180, 2012.

[Bar15a] S. Bartels. Error control and adaptivity for a variational model problem defined on functions of bounded variation. *Math. Comp.*, 84(293):1217–1240, 2015.

[Bar15b] S. Bartels. *Numerical methods for nonlinear partial differential equations*, volume 47 of *Springer Series in Computational Mathematics*. Springer, Cham, 2015.

[Bar16a] S. Bartels. Broken Sobolev space iteration for total variation regularized minimization problems. *IMA J. Numer. Anal.*, 36(2):493–502, 2016.

[Bar16b] S. Bartels. *Numerical approximation of partial differential equations*, volume 64 of *Texts in Applied Mathematics*. Springer, [Cham], 2016.

[Bar20a] S. Bartels. Error estimates for a class of discontinuous Galerkin methods for nonsmooth problems via convex duality relations, 2020.

[Bar20b] S. Bartels. Nonconforming discretizations of convex minimization problems and precise relations to mixed methods, 2020.

[BBF13] D. Boffi, F. Brezzi, and M. Fortin. *Mixed finite element methods and applications*, volume 44 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2013.

[BDN18] S. Bartels, L. Diening, and R. H. Nochetto. Unconditional stability of semi-implicit discretizations of singular flows. *SIAM J. Numer. Anal.*, 56(3):1896–1914, 2018.

[BHL$^+$16] M. J. Borden, T. J. R. Hughes, C. M. Landis, A. Anvari, and I. J. Lee. A phase-field formulation for fracture in ductile materials: Finite deformation balance law derivation, plastic degradation, and stress triaxiality effects. *Computer Methods in Applied Mechanics and Engineering*, 312:130–166, 2016.

[BKP10] K. Bredies, K. Kunisch, and T. Pock. Total generalized variation. *SIAM J. Imaging Sci.*, 3(3):492–526, 2010.

[BL02] P. Bělík and M. Luskin. A total-variation surface energy model for thin films of martensitic crystals. *Interfaces Free Bound.*, 4(1):71–88, 2002.

[BM16] S. Bartels and M. Milicevic. Stability and experimental comparison of prototypical iterative schemes for total variation regularized problems. *Comput. Methods Appl. Math.*, 16(3):361–388, 2016.

[BM17] S. Bartels and M. Milicevic. Iterative finite element solution of a constrained total variation regularized model problem. *Discrete Contin. Dyn. Syst. Ser. S*, 10(6):1207–1232, 2017.

[BM20] S. Bartels and M. Milicevic. Efficient iterative solution of finite element discretized nonsmooth minimization problems. *Computers & Mathematics with Applications*, 80(5):588 – 603, 2020.

[BMT18] S. Bartels, M. Milicevic, and M. Thomas. Numerical approach to a model for quasistatic damage with spatial BV-regularization. In E. Rocca, U. Stefanelli, and L. Truskinovsky, editors, *Proceedings of the INdAM-ISIMM Workshop on Trends on Applications of Mathematics to Mechanics, Rome, Italy, September 2016*, volume 27, pages 179–203. Springer International Publishing, Cham, 2018.

[BMTW20] S. Bartels, M. Milicevic, M. Thomas, and N. Weber. Fully discrete approximation of rate-independent damage models with gradient regularization. *WIAS-Preprint 2707*, 2020.

[BNS14] S. Bartels, R. H. Nochetto, and A. J. Salgado. Discrete total variation flows without regularization. *SIAM J. Numer. Anal.*, 52(1):363–385, 2014.

[BNS15] S. Bartels, R. H. Nochetto, and A. J. Salgado. A total variation diminishing interpolation operator and applications. *Math. Comp.*, 84(296):2569–2587, 2015.

[BR20] S. Bartels and M. Ružička. Convergence of Fully Discrete Implicit and Semi-implicit Approximations of Singular Parabolic Equations. *SIAM J. Numer. Anal.*, 58(1):811–833, 2020.

[BS08] S. C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[BT09] A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J. Imaging Sci.*, 2(1):183–202, 2009.

[Cha04] A. Chambolle. An algorithm for total variation minimization and applications. volume 20, pages 89–97. 2004. Special issue on mathematics and image analysis.

[Cia78] P. G. Ciarlet. *The finite element method for elliptic problems*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. Studies in Mathematics and its Applications, Vol. 4.

[CL97] A. Chambolle and P.-L. Lions. Image recovery via total variation minimization and related problems. *Numer. Math.*, 76(2):167–188, 1997.

[CP11] A. Chambolle and T. Pock. A first-order primal-dual algorithm for convex problems with applications to imaging. *J. Math. Imaging Vision*, 40(1):120–145, 2011.

[CP16a] A. Chambolle and T. Pock. An introduction to continuous optimization for imaging. *Acta Numer.*, 25:161–319, 2016.

[CP16b] A. Chambolle and T. Pock. On the ergodic convergence rates of a first-order primal-dual algorithm. *Math. Program.*, 159(1-2, Ser. A):253–287, 2016.

[CP19] A. Chambolle and T. Pock. Crouzeix-Raviart approximation of the total variation on simplicial meshes. HAL preprint nr. 01787012, June 2019.

[CR73] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge*, 7(no. , no. R-3):33–75, 1973.

[DHYZ17] Y.-H. Dai, D. Han, X. Yuan, and W. Zhang. A sequential updating scheme of the Lagrange multiplier for separable convex programming. *Math. Comp.*, 86(303):315–343, 2017.

[DR56] J. Douglas, Jr. and H. H. Rachford, Jr. On the numerical solution of heat conduction problems in two and three space variables. *Trans. Amer. Math. Soc.*, 82:421–439, 1956.

[DY16] W. Deng and W. Yin. On the global and linear convergence of the generalized alternating direction method of multipliers. *J. Sci. Comput.*, 66(3):889–916, 2016.

[EB92] J. Eckstein and D. P. Bertsekas. On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators. *Math. Programming*, 55(3, Ser. A):293–318, 1992.

[EG13] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159. Springer Science & Business Media, 2013.

[ES09] C. M. Elliott and S. A. Smitheman. Numerical analysis of the TV regularization and H^{-1} fidelity model for decomposing an image into cartoon plus texture. *IMA J. Numer. Anal.*, 29(3):651–689, 2009.

[FG83] M. Fortin and R. Glowinski. *Augmented Lagrangian methods*, volume 15 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1983. Applications to the numerical solution of boundary value problems, Translated from the French by B. Hunt and D. C. Spicer.

[FM98] G. A. Francfort and J-J Marigo. Revisiting brittle fracture as an energy minimization problem. *Journal of the Mechanics and Physics of Solids*, 46(8):1319–1342, 1998.

[FvOP05] X. Feng, M. von Oehsen, and A. Prohl. Rate of convergence of regularization procedures and finite element approximations for the total variation flow. *Numer. Math.*, 100(3):441–456, 2005.

[Gia05] A. Giacomini. Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fractures. *Calculus of Variations and Partial Differential Equations*, 22(2):129–172, 2005.

[Glo84] R. Glowinski. *Numerical methods for nonlinear variational problems*. Springer Series in Computational Physics. Springer-Verlag, New York, 1984.

[GM76] D. Gabay and B. Mercier. A dual algorithm for the solution of nonlinear variational problems via finite element approximation. *Computers & Mathematics with Applications*, 2(1):17 – 40, 1976.

[Gri21] A. A. Griffith. Vi. the phenomena of rupture and flow in solids. *Philosophical transactions of the royal society of london. Series A, containing papers of a mathematical or physical character*, 221(582-593):163–198, 1921.

[HGO+17] C. Hesch, A.J. Gil, R. Ortigosa, M. Dittmann, C. Bilgen, P. Betsch, M. Franke, A. Janz, and K. Weinberg. A framework for polyconvex large strain phase-field methods to fracture. *Computer Methods in Applied Mechanics and Engineering*, 317:649–683, 2017.

[HHS+19] M. Herrmann, R. Herzog, S. Schmidt, J. Vidal-Núñez, and G. Wachsmuth. Discrete total variation with finite elements and applications to imaging. *J. Math. Imaging Vision*, 61(4):411–431, 2019.

[HK04] M. Hintermüller and K. Kunisch. Total bounded variation regularization as a bilaterally constrained optimization problem. *SIAM J. Appl. Math.*, 64(4):1311–1333, 2004.

[HMW11] R. Herzog, C. Meyer, and G. Wachsmuth. Integrability of displacement and stresses in linear and nonlinear elasticity with mixed boundary conditions. *Journal of Mathematical Analysis and Applications*, 382(2):802 – 813, 2011.

[HN75] B. Halphen and Q.S. Nguyen. Sur les matériaux standards généralisés. *J. Mécanique*, 14:39–63, 1975.

[HSD+16] C. Hesch, S. Schuß, M. Dittmann, M. Franke, and K. Weinberg. Isogeometric analysis and hierarchical refinement for higher-order phase-field models. *Computer Methods in Applied Mechanics and Engineering*, 303:185–207, 2016.

[HY12] B. He and X. Yuan. On the $O(1/n)$ convergence rate of the Douglas-Rachford alternating direction method. *SIAM J. Numer. Anal.*, 50(2):700–709, 2012.

[KM98] S. Kontogiorgis and R. R. Meyer. A variable-penalty alternating directions method for convex optimization. *Math. Programming*, 83(1, Ser. A):29–53, 1998.

[KM10] C. Kuhn and R. Müller. A continuum phase field model for fracture. *Engineering Fracture Mechanics*, 77(18):3625–3634, 2010.

[KMZ08] D. Knees, A. Mielke, and C. Zanini. On the inviscid limit of a model for crack propagation. *Math. Models Methods Appl. Sci.*, 18:1529–1569, 2008.

[KN17] D. Knees and M. Negri. Convergence of alternate minimization schemes for phase-field fracture and damage. *Math. Models Methods Appl. Sci.*, 27(9):1743–1794, 2017.

[Kop09] I. Kopacka. *MPECs/MPCCs in function space: first order optimality concepts, path-following, and multilevel algorithms*. na, 2009.

[KRZ13a] D. Knees, R. Rossi, and C. Zanini. A vanishing viscosity approach to a rate-independent damage model. *Math. Models Methods Appl. Sci.*, 23(04):565–616, 2013.

[KRZ13b] D. Knees, R. Rossi, and C. Zanini. A vanishing viscosity approach to a rate-independent damage model. *Mathematical Models and Methods in Applied Sciences*, 23(04):565–616, 2013.

[LM79] P.-L. Lions and B. Mercier. Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.*, 16(6):964–979, 1979.

[LRTT18] G. Lazzaroni, R. Rossi, M. Thomas, and R. Toader. Rate-independent damage in thermo-viscoelastic materials with inertia. *Journal of Dynamics and Differential Equations*, 30:1311–1364, 2018.

[MHW10] C. Miehe, M. Hofacker, and F. Welschinger. A phase field model for rate-independent crack propagation: Robust algorithmic implementation based on operator splits. *Computer Methods in Applied Mechanics and Engineering*, 199(45-48):2765–2778, 2010.

[MR15] A. Mielke and T. Roubíček. *Rate-independent Systems: Theory and Application*, volume 193 of *Applied Mathematical Sciences*. Springer, 2015.

[MRS12] A. Mielke, R. Rossi, and G. Savaré. BV solutions and viscosity approximations of rate-independent systems. *ESAIM Control Optim. Calc. Var.*, 18(1):36–80, 2012.

[Nes05] Y. Nesterov. Smooth minimization of non-smooth functions. *Math. Program.*, 103(1, Ser. A):127–152, 2005.

[Roc76] R. T. Rockafellar. Monotone operators and the proximal point algorithm. *SIAM J. Control Optim.*, 14(5):877–898, 1976.

[ROF92] L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. volume 60, pages 259–268. 1992. Experimental mathematics: computational issues in nonlinear science (Los Alamos, NM, 1991).

[Rou09] T. Roubíček. Rate-independent processes in viscous solids at small strains. *Math. Methods Appl. Sci.*, 32(7):825–862, 2009.

[RT77] P.-A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*, pages 292–315. Lecture Notes in Math., Vol. 606, 1977.

[RT15] R. Rossi and M. Thomas. From an adhesive to a brittle delamination model in thermo-visco-elasticity. *ESAIM Control Optim. Calc. Var.*, 21:1–59, 2015.

[RT17a] R. Rossi and M. Thomas. Coupling rate-independent and rate-dependent processes: Existence results. *SIAM Journal on Mathematical Analysis*, 49(2):1419–1494, 2017.

[RT17b] R. Rossi and M. Thomas. From adhesive to brittle delamination in visco-elastodynamics. *Mathematical Models and Methods in Applied Sciences*, 27(08):1489–1546, 2017.

[SWKM14] A. Schlüter, A. Willenbücher, C. Kuhn, and R. Müller. Phase field approximation of dynamic brittle fracture. *Computational Mechanics*, 54(5):1141–1161, 2014.

[SX14] Y. Shen and M. Xu. On the $O(1/t)$ convergence rate of Ye-Yuan's modified alternating direction method of multipliers. *Appl. Math. Comput.*, 226:367–373, 2014.

[TBW17] M. Thomas, C. Bilgen, and K. Weinberg. Analysis and simulations for a phase-field fracture model at finite strains based on modified invariants. *WIAS-Preprint 2456*, 2017.

[TBW18] M. Thomas, C. Bilgen, and K. Weinberg. Phase-field fracture at finite strains based on modified invariants: A note on its analysis and simulations. *GAMM-Mitteilungen*, 40(3):207–237, 2018.

[TT20] M. Thomas and S. Tornquist. Discrete approximation of dynamic phase-field fracture in visco-elastic materials. *WIAS Preprint*, No.0000:–, 2020.

[TZ17] M. Thomas and C. Zanini. Cohesive zone-type delamination in visco-elasticity. *Discrete & Continuous Dynamical Systems - S*, 10(6):1487–1517, 2017.

[WL11] J. Wang and B. J. Lucier. Error bounds for finite-difference methods for Rudin-Osher-Fatemi image smoothing. *SIAM J. Numer. Anal.*, 49(2):845–868, 2011.