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Abstract

We study the existence of uniformly bounded extension and trace operators for $W^{1,p}$-functions on randomly perforated domains, where the geometry is assumed to be stationary ergodic. Such extension and trace operators are important for compactness in stochastic homogenization. In contrast to former approaches and results, we use very weak assumptions on the geometry which we call local (δ, M)-regularity, isotropic cone mixing and bounded average connectivity. The first concept measures local Lipschitz regularity of the domain while the second measures the mesoscopic distribution of void space. The third is the most tricky part and measures the "mesoscopic" connectivity of the geometry.

In contrast to former approaches we do not require a minimal distance between the inclusions and we allow for globally unbounded Lipschitz constants and percolating holes. We will illustrate our method by applying it to the Boolean model based on a Poisson point process and to a Delaunay pipe process.

We finally introduce suitable Sobolev spaces on $\mathbb{R}^d$ and Ω in order to construct a stochastic two-scale convergence method and apply the resulting theory to the homogenization of a p-Laplace problem on a randomly perforated domain.

Contents

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1 Introduction

In 1979 Papanicolaou and Varadhan [31] and Kozlov [24] for the first time independently introduced concepts for the averaging of random elliptic operators. At that time, the periodic homogenization theory had already advanced to some extend (as can be seen in the book [32] that had appeared one year before) dealing also with non-uniformly elliptic operators [26] and domains with periodic holes [7].

Even though the works [24, 31] clearly guide the way to a stochastic homogenization theory, this theory advanced quite slowly over the past 4 decades. Compared to the stochastic case, periodic homogenization developed very strong with methods that are now well developed and broadly used. The most popular methods today seem to be the two-scale convergence method by Allaire and Nguetseng [2, 30] in 1989/1992 and the periodic unfolding method [6] by Cioranescu, Damlamian and Griso in 2002. Both methods are conceptually related to asymptotic expansion and very intuitive to handle. It is interesting to observe that the stochastic counterpart, the stochastic two-scale convergence, was developed only in 2006 by Zhikov and Piatnitsky [39], with the stochastic unfolding developed only recently in [29, 18].

A further work by Bourgeat, Mikelic and Wright [5] introduced two-scale convergence in the mean. This sense of two-scale convergence is indeed a special case of the stochastic unfolding, which can only be applied in an averaged sense, too. This leads us to a fundamental difference between the periodic and the stochastic homogenization. In stochastic homogenization we distinguish between quenched convergence, i.e. for almost every realization one can prove homogenization, and homogenization in the mean, which means that homogenization takes place in expectation.

In particular in nonlinear non-convex problems (that is: we cannot rely on weak convergence methods) the quenched convergence is of uttermost importance, as this sense of convergence allows to use - for each fixed ω - compactness in the spaces $H^1(\mathbf{Q})$. On the other hand, convergence in the mean deals with convergence in $L^2(\Omega; H^1(\mathbf{Q}))$, which goes in hand with a loss of compactness.

The results presented below are meant for application in quenched convergence. The estimates for the extension and trace operators which are derived strongly depends on the realization of the geometry - thus on ω. Nevertheless, if the geometry is stationary, a corresponding estimate can be achieved for almost every ω.

The Problem

The discrepancy in the speed of progress between periodic and stochastic homogenization is due to technical problems that arise from the randomness of parameters. In this work we will consider uniform extension operators for randomly perforated stationary domains. We use stationarity (see Def. 2.16) as this is the standard way to cope with the lack of periodicity. Let us first have a look at a typical application to illustrate the need of the extension operators that we construct below.

Let $\mathbf{P}(\omega) \subset \mathbb{R}^d$ be a stationary random open set and let $\varepsilon > 0$ be the smallness parameter and let $\tilde{\mathbf{P}}(\omega)$ be a connected component of $\mathbf{P}(\omega)$. For a bounded open domain, we consider $\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}(\omega) := \mathbf{Q} \cap \varepsilon\tilde{\mathbf{P}}(\omega)$ and $\Gamma^\varepsilon(\omega) := \mathbf{Q} \cap \varepsilon\partial\tilde{\mathbf{P}}(\omega)$ with outer normal $\nu_{\Gamma^\varepsilon(\omega)}$. We study the following problem in

Section 10.6:

$$
\begin{aligned}
-\operatorname{div}\left(a\left|\nabla u^{\varepsilon}\right|^{p-2}\nabla u^{\varepsilon}\right) &= g && \text{on } \mathbf{Q}^{\varepsilon}_{\tilde{\mathbf{P}}}(\omega)\,,\\
u &= 0 && \text{on } \partial\mathbf{Q}\,,\\
\left|\nabla u^{\varepsilon}\right|^{p-2}\nabla u^{\varepsilon}\cdot\nu_{\Gamma^{\varepsilon}(\omega)} &= f(u^{\varepsilon}) && \text{on } \Gamma^{\varepsilon}(\omega)\,.
\end{aligned}
\tag{1.1}
$$

Note that for simplicity of illustration, the only randomness that we consider in this problem is due to $\mathbf{P}(\omega)$, i.e. we assume $a\equiv const$.

Problem (1.1) can be recast into a variational problem, i.e. solutions of (1.1) are local minimizers of the energy functional

$$
\mathcal{E}_{\varepsilon,\omega}(u)=\int_{\mathbf{Q}^{\varepsilon}_{\tilde{\mathbf{P}}}(\omega)}\left(\frac{1}{p}\left|\nabla u\right|^{p}-gu\right)+\int_{\Gamma^{\varepsilon}(\omega)}\int_{0}^{u}F(s)\mathrm{d}s\,,
$$

where F is convex with $\partial F=f$. This problem will be treated in Theorem 10.20 and the final Remark 10.22.

One way to prove homogenization of (1.1) is to prove Γ-convergence of $\mathcal{E}_{\varepsilon,\omega}$. Conceptually, this implies convergence of the minimizers u^{ε} to a minimizer of the limit functional. However, the minimizers are elements of $W^{1,p}(\mathbf{Q}^{\varepsilon}_{\tilde{\mathbf{P}}})$ and since this space changes with ε, we lack compactness in order to pass to the limit in the nonlinearity. The canonical path to circumvent this issue in *periodic* homogenization is via uniformly bounded extension operators $\mathcal{U}_{\varepsilon}:\,W^{1,p}(\mathbf{Q}^{\varepsilon}_{\tilde{\mathbf{P}}})\to W^{1,p}(\mathbf{Q})$, see [20, 22], combined with uniformly bounded trace operators, see [12, 13].

The first proof for the existence of periodic extension operators was due to Cioranescu and Paulin [7] in 1979, while the proof in its full generality was provided only recently by Höpker and Böhm [22] and Hp̈ker [21]. In this work we will generalize parts of the results of [21] to a stochastic setting. A modified version of the original proof of [21] is provided in Section 3. It relies on three ingredients: the local Lipschitz regularity of the surface, the periodicity of the geometry and the connectedness. Local Lipschitz regularity together with periodicity imply global Lipschitz regularity of the surface. In particular, one can construct a local extension operator on every cell $z+(-\delta,1+\delta)^{d}$, $z\in\mathbb{Z}^{d}$ which might then be glued together using a periodic partition of unity of $\mathbb{R}^{d}$. The connectedness of the geometry assures that the difference of the average of a function u on two different cells z_1 and z_2 can be computed from the gradient along a path connecting the two cells and being fully comprised in $z_1+(-1,2)^{d}$.

In the stochastic case the proof of existence of suitable extension operators is much more involved and not every geometry will eventually allow us to be successful. In fact, we will not be able - in general - to even provide extension operators $\mathcal{U}_{\varepsilon}:\,W^{1,p}\left(\mathbf{Q}^{\varepsilon}_{\tilde{\mathbf{P}}}(\omega)\right)\to W^{1,p}(\mathbf{Q})$ but rather obtain $\mathcal{U}_{\varepsilon}:\,W^{1,p}\left(\mathbf{Q}^{\varepsilon}_{\tilde{\mathbf{P}}}(\omega)\right)\to W^{1,r}(\mathbf{Q})$, where $r<p$ depends (among others) on the dimension and on the distribution of the Lipschitz constant of $\partial\tilde{\mathbf{P}}(\omega)$. This is due to the presence of arbitrarily "bad" local behavior of the geometry.

The theory developed below also allows to provide estimates on the trace operator

$$
\mathcal{T}_{\omega}:\,C^{1}\big(\overline{\mathbf{P}}(\omega)\big)\to C(\partial\mathbf{P}(\omega))
$$

when seen as an operator $\mathcal{T}_{\omega}:\,W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega))\to L^{r}_{\mathrm{loc}}(\partial\mathbf{P}(\omega))$, where again $1\leq r<p$ in general.

We summarize the above discussion in the following.

Problem 1.1. Find (computationally or rigorously) verifiable conditions on stationary random geometries that allow to prove existence of extension operators

$$\mathcal{U}_\varepsilon:\,W^{1,p}_{0,\partial\mathbf{Q}}\left(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}(\omega)\right)\to W^{1,r}(\mathbf{Q})\quad\text{s.t.}\quad\|\nabla\mathcal{U}_\varepsilon u\|_{L^r(\mathbf{Q})}\leq C\,\|\nabla u\|_{L^p\left(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}(\omega)\right)}\,,$$

where $r\geq 1$ and $C>0$ are independent of ε and where

$$W^{1,p}_{0,\partial\mathbf{Q}}\left(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}(\omega)\right)=\left\{u\in W^{1,p}\left(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}(\omega)\right)\,:\,u|_{\partial\mathbf{Q}}\equiv 0\right\}\,.$$

Problem 1.2. Find (computationally or rigorously) verifiable conditions on stationary random geometries that allow to prove an estimate

$$\varepsilon\,\|\mathcal{T}_\varepsilon u\|^r_{L^r(\mathbf{Q}\cap\varepsilon\partial\mathbf{P})}\leq C\left(\|u\|^r_{L^p(\mathbf{Q}\cap\varepsilon\mathbf{P}(\omega))}+\varepsilon^r\,\|\nabla u\|^r_{L^p(\mathbf{Q}\cap\varepsilon\mathbf{P}(\omega))}\right)\,,$$

where $r\geq 1$ and $C>0$ are independent of ε.

Let us mention at this place existing results in literature. In recent years, Guillen and Kim [13] have proved existence of uniformly bounded extension operators $\mathcal{U}_\varepsilon\,:\,W^{1,p}\left(\mathbf{Q}^\varepsilon_{\tilde{\mathbf{P}}}(\omega)\right)\to W^{1,p}(\mathbf{Q})$ in the context of minimally smooth surfaces, i.e. uniformly Lipschitz and uniformly bounded inclusions with uniform minimal distance. A homogenization result of integral functionals on randomly perforated domains with uniformly bounded inclusions was provided by Piat and Piatnitsky [33]. Concerning unbounded inclusions and non-uniformly Lipschitz geometries, the present work seems to be the first approach. Since Problem 1.2 is easier to handle, we first explain our concept of microscopic regularity in view of $\mathcal{T}_\omega$ and then go on to extension operators.

(δ,M)-Regularity and the Trace Operator

We introduce two concepts which are suited for the current and potentially also for further studies. The first of these two concepts is inspired by the concept of minimal smoothness [35] and accounts for the local regularity of $\partial\mathbf{P}$. Deviating from [35] we will call it *local (δ,M)-regularity* (see Definition 4.2). Although this assumption is very weak, its consequences concerning local coverings of $\partial\mathbf{P}$ are powerful. Based on this concept, we introduce the functions δ, $\hat\rho$ and ρ on $\partial\mathbf{P}$ as well as $M_{[\eta]}$ and $M_{[\eta],\mathbb{R}^d}$ for $\eta\in\{\delta,\hat\rho,\rho\}$ in Lemmas 4.4, 4.6, 4.8 and 4.12 and make the following assumptions:

Assumption 1.3. *Let $\mathbf{P}(\omega)$ be a random open set such that for $1\leq r<p_0<p$ and $\eta\in\{\rho,\hat\rho,\delta\}$ it holds either*

$$\int_\Omega\eta^{-\frac{1}{p_0-r}}\mathrm{d}\mu_{\Gamma,\mathcal{P}}+\mathbb{E}\left(M_{[\frac{1}{8}\eta],\mathbb{R}^d}^{\left(\frac{1}{p_0}+1\right)\frac{p}{p-p_0}}\right)<\infty\,,$$

$$\text{or}\int_\Omega\left(\eta M_{[\frac{1}{16}\eta],\mathbb{R}^d}\right)^{-\frac{1}{p-r}}\mathrm{d}\mu_{\Gamma,\mathcal{P}}<\infty\,.$$

Having studied the properties of (δ,M)-regular sets in detail in Sections 4.1 and 2.5 it is very easy to prove the following trace theorem (for notations we refer to Section 2 and Section 4.1). Note that via a simple rescaling, this provides a solution to Problem 1.1.

Theorem 1.4 (Solution of Problem 1.1)**.** *Let $\mathbf{P}(\omega)$ be a stationary and ergodic random open set which is almost surely locally (δ,M) regular and let Assumption 1.3 hold. For given ω let $\mathcal{T}_\omega\,:\,C^1\left(\overline{\mathbf{P}}(\omega)\right)\to C(\partial\mathbf{P}(\omega))$ be the trace operator. Then for almost every ω the extension $\mathcal{T}_\omega\,:\,W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega))\to L^r_{\mathrm{loc}}(\partial\mathbf{P}(\omega))$ is continuous and there exists a constant $C_\omega>0$ s.t. it holds for every bounded Lipschitz domain $\mathbf{Q}\supset\mathbb{B}_1(0)$ and every $n\in\mathbb{N}$*

$$\|\mathcal{T}_\omega u\|_{L^r(\partial\mathbf{P}\cap n\mathbf{Q})}\leq C_\omega\,\|u\|_{W^{1,p}(\mathbb{B}_{\mathfrak{r}}(n\mathbf{Q})\cap\mathbf{P})}\,.$$

Proof. This is a consequence of Theorem 5.9, stationarity and ergodicity and the ergodic theorem. □

Construction of Extension Operators

The main results of this work is on extension operators on randomly perforated domains. In order to construct a suitable extension operator, we use

Step 1: (δ, M)**-regularity** Concerning extension results, the concept of (δ, M)-regularity suggests the naive approach to use a local open covering of $\partial\mathbf{P}$ and to add the local extension operators via a partition of unity in order to construct a global extension operator. We call this ansatz naive since one would not chose this approach even in the periodic setting, as it is known to lead to unbounded gradients. Nevertheless, this ansatz is followed in Section 5.2 for two reasons. The first reason is illustration of an important principle: The extension operator $\mathcal{U} = \tilde{\mathcal{U}} + \hat{\mathcal{U}}$ can be split up into a *local* part $\tilde{\mathcal{U}}$, whose norm can be estimated by local properties of $\partial\mathbf{P}$, and a *global* part $\hat{\mathcal{U}}$ whose norm is determined by *connectivity*, an issue which has to be resolved afterwards, and corresponds to Step 2 in the proof of Theorem 3.2 below (periodic case), where one glues together the local extension operators on the periodic cells. The second reason is that this first estimate, although it cannot be applied globally, is very well suited for constructing a local extension operator. Lemma 5.6 hence provides estimates of a certain extension operator which has the property that the constant in the estimate tends to $+\infty$ as the domain grows.

However, this first ansatz grants some insight into the structure of the extension problem. In particular, we find the following result which will provide a better understanding of the Sobolev spaces $W^{1,p}(\Omega)$ and $W^{1,r,p}(\Omega, \mathbf{P})$ on the probability space Ω.

Assumption 1.5. *Let $\mathbf{P}(\omega)$ be a random open set such that Assumption 5.5 hold and let $\hat{d}$ be the constant from (5.8).*

1 *Assume for $r < p$ that*
$$\mathbb{E}\left(\tilde{M}_{[\frac{1}{8}\hat{\rho}]}^{\frac{p(\hat{d}+1)}{p-r}}\right) + \mathbb{E}\left(\tilde{M}_{[\frac{1}{8}\hat{\rho}]}^{\frac{p(\hat{d}+\alpha)}{p-r}}\right) < \infty \tag{1.2}$$

2 *Assume for $r < p_0 < p_1 < p$ that (1.2) and either*
$$\mathbb{E}\left(\tilde{M}_{[\frac{1}{8}\hat{\rho}]}^{\frac{p_1(d-2)(p_0-r)}{r(p_1-p_0)}}\right) + \mathbb{E}\left(\tilde{M}_{[\frac{1}{8}\hat{\rho}]}^{\frac{\alpha p_1 p}{p-p_1}}\right) + \mathbb{E}\left(\rho^{1-\frac{rp_0}{p_0-r}}\right) < \infty$$
or
$$\mathbb{E}\left(\tilde{M}_{[\frac{1}{8}\hat{\rho}]}^{\alpha p_0 p}\right) + \mathbb{E}\left(\rho_{\mathrm{bulk}}^{-\frac{rp_0}{p_0-r}}\right) < \infty\,,$$
where
$$\rho_{\mathrm{bulk}}(x) := \inf\left\{\rho(\tilde{x}) \;:\; \tilde{x} \in \partial\mathbf{P} \text{ s.t. } x \in \mathbb{B}_{\frac{1}{8}\rho(\tilde{x})}(\tilde{x})\right\}\,,$$

Theorem 1.6. *Let Assumption 1.5 hold and let τ be ergodic. Then for almost every ω the extension operator $\mathcal{U} : W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega)) \to W^{1,r}_{\mathrm{loc}}(\mathbb{R}^d)$ provided in (5.14) is well defined and for $\mathbf{Q} \subset \mathbb{R}^d$ a bounded domain with Lipschitz boundary there exists a constant $C(\omega)$ such that for every positive $n \geq 1$ and every $u \in W^{1,p}(\mathbf{P}(\omega) \cap \mathbb{B}_{\mathfrak{r}}(n\mathbf{Q}))$*
$$\frac{1}{n^d\,|\mathbf{Q}|}\int_{n\mathbf{Q}}\left(|\mathcal{U}u|^r + |\nabla\mathcal{U}u|^r\right) \leq C(\omega)\left(\frac{1}{n^d\,|\mathbf{Q}|}\int_{\mathbf{P}(\omega)\cap\mathbb{B}_{\mathfrak{r}}(n\mathbf{Q})}|u|^p + |\nabla u|^p\right)^{\frac{r}{p}}\,.$$

Proof. This follows from Lemmas 5.6, 5.8 and 4.13 on noting that $\nabla\phi_0 \leq C\rho_{\mathrm{bulk}}^{-1}$. □

Theorem 1.6, though useful, is not satisfactory for homogenization, as $\nabla\mathcal{U}u$ is bounded by u and not solely ∇u. Therefore, some more work is needed.

Step 2: isotropic cone mixing In order to account for the issue of connectedness in a proper way on the macroscopic level, we propose our second fundamental concept of *isotropic cone mixing geometries* (see Definition 4.17), which allow to construct a global Voronoi tessellation of $\mathbb{R}^d$ with good local covering properties. This definition, though being rather technical, can be verified rather easily using Criterion 4.18.

In short, isotropic cone mixing allows to distribute balls $B_i = \mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_i)$ of a uniform minimal radius $\frac{1}{2}\mathfrak{r}$ within $\mathbf{P}$ such that the centers x_i of the balls B_i generate a Voronoi mesh of cells G_i with diameter d_i, distributed according to a function $f(d)$ (see Lemma 4.20). These Voronoi cells in general might be of arbitrary large diameter d_i, although they are bounded in the statistical average. Due to this lack of a uniform bound, we call the distribution of Voronoi cells the *mesoscopic regularity* of the geometry.

Step 3: gluing The Voronoi cells resulting from an isotropic cone mixing geometry are well suited for the gluing of local extension operators. We will construct the macroscopic extension operator in an analogue way to [21], replacing the periodic cells by the Voronoi cells (see Figure 5). In Theorem 6.3 we provide a first abstract result how the norm of the glued operator can be estimated from the distribution of M, the geometry of the Voronoi mesh and the connectivity, even though the last two properties enter rather indirectly. To make this more clear, we note at this points that the extension operator depends on two types of local averages: To each Voronoi cell G_i we take the average $\mathcal{M}_i u$ over B_i. Furthermore, to every local microscopic extension operator chosen in Section 5 there corresponds a local average $\tau_j u$ close to the boundary. We will see that the norm of the extension operator strongly depends on the differences $|\mathcal{M}_i u - \mathcal{M}_j u|$ and $|\mathcal{M}_j u - \tau_k u|$.

In Theorem 6.7 we will see that the dependence on $|\mathcal{M}_i u - \mathcal{M}_j u|$ can be eliminated with the price to increase the cost of "unfortunate distributions" of G_i and of the local (δ, M) regularity. The remaining dependence which we leave unresolved is the dependence on $|\mathcal{M}_j u - \tau_k u|$. This dependence is linked to quantitative connectedness properties of the geometry. By this we mean more than the topological question of connectedness. In particular, we need an estimate of the type $|\mathcal{M}_j u - \tau_k u|^r \leq \int_{G_i} C(x)\,|\nabla u(x)|^r\,\mathrm{d}x$ which will finally allow us an estimate of $\sum_{j,k} |\mathcal{M}_j u - \tau_k u|^r$ in terms of ∇u. Unfortunately, the classical percolation theory, which deals with connectedness of random geometries, is not developed to answer this question. In this paper, we will use two workarounds which we call "statistically harmonic" and "statistically connected". However, further research has to be conducted. We state our first main theorem.

Theorem 1.7. *Let $\mathbf{P}(\omega)$ be a stationary ergodic random open set which is almost surely (δ, M)-regular (Def. 4.2) and isotropic cone mixing for $\mathfrak{r} > 0$ and $f(R)$ (Def. 4.17) and statistically harmonic (Def. 6.9) and let $1 \leq r < p \leq \infty$. Let $\mathbf{Q} \subset \mathbb{R}^d$ be bounded open with Lipschitz boundary as well as $s \in (r, p)$ such that*

$$\mathbb{E}\left(\tilde{M}^{\frac{2pd}{p-r}}\right) < +\infty\,,$$

$$\sum_{k=1}^{\infty} (k+1)^{d\left(\frac{2p-s}{p-s}\right)+(d+1)(2r+2)\frac{p}{p-s}+r\left(\frac{p}{p-s}-1\right)} f(k) < +\infty\,,$$

$$\mathbb{E}\left(\sup_R \frac{1}{R^d}\int_{\mathbb{B}_R(0)}\left(\sum_k P(d_k)\chi_{\mathfrak{A}_{4,k}} C_k\right)^{\frac{p}{p-s}}\right) < +\infty\,.$$

Then for almost every ω the extension operator $\mathcal{U}\,:\, W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega)) \to W^{1,r}_{\mathrm{loc}}(\mathbb{R}^d)$ provided in (6.6) is *well defined with a constant $C(\omega)$ such that for every positive $n\geq 1$*

$$\frac{1}{n^d\left|\mathbf{Q}\right|}\int_{n\mathbf{Q}}\left|\mathcal{U}u\right|^r\leq C(\omega)\left(\frac{1}{n^d\left|\mathbf{Q}\right|}\int_{\mathbf{P}(\omega)\cap n\mathbf{Q}}\left|u\right|^p\right)^{\frac{r}{p}}$$
$$\frac{1}{n^d\left|\mathbf{Q}\right|}\int_{n\mathbf{Q}}\left|\nabla\mathcal{U}u\right|^r\leq C(\omega)\left(\frac{1}{n^d\left|\mathbf{Q}\right|}\int_{\mathbf{P}(\omega)\cap n\mathbf{Q}}\left|\nabla u\right|^p\right)^{\frac{r}{p}}.$$

Proof. This follows from Theorems 6.3, 6.7 and 6.10 on noting that in the general case we have to assume $\alpha=\hat{d}=d$. Furthermore, we need Lemma 4.21. □

In practical applications, one would need to verify whether $\mathbf{P}$ is statistically harmonic via numerical simulations. The problem particularly results in the numerical evaluation of a Laplace operator.

Based on this insight, we develop an alternative approach: The connectedness of $\mathbf{P}$ is quantified by introducing directly a discrete graph on $\mathbf{P}$ and a discrete Poisson equation on this graph. The construction of the graph and the evaluation of the Poisson equation can be done numerically, but with the advantage that the discrete quantities are now directly connected to the analytical theory. Additionally to the (δ,M)-regularity we have to deal with the average diameter d_j of the cells of a the global Voronoi tessellation and the local stretch factor S_j. We impose the following assumptions:

Assumption 1.8. *Let $\mathbf{P}(\omega)$ be a random open set such that Assumption 5.5 hold and let $\hat{d}$ be the constant from (5.8). Let (1.2) and for $r<\tilde{s}<s<p$ let either*

$$\mathbb{E}\left(\tilde{M}_{[\frac{1}{8}\hat{\rho}]}^{\frac{p_1(d-2)(\tilde{s}-r)}{r(s-\tilde{s})}}\right)+\mathbb{E}\left(\rho^{1-\frac{\tilde{s}r}{\tilde{s}-r}}\right)<\infty$$

or

$$\mathbb{E}\left(\rho_{\mathrm{bulk}}^{-\frac{sr}{s-r}}\right)<\infty\,.$$

Furthermore, let $\mathbf{P}(\omega)$ be almost surely isotropic cone mixing for $\mathfrak{r}>0$ and $f(R)$ (Def. 4.17) as well as locally connected and let the local stretch factor (see Definition Theorem 7.7 and Definition 7.8) satisfy $\mathbb{P}(\mathsf{S}>S_0)\leq f_s(S_0)$ such that

$$\sum_{k=1}^{\infty}(k+1)^{2d+\frac{r(d-1)+drs}{s-r}}f(k)<+\infty\,,$$
$$\sum_{k,N=1}^{\infty}\left[(N+1)\,(k+1)\right]^{d\frac{2p-s}{p-s}+\frac{s-1}{s}\frac{p}{p-s}+r\frac{s}{p-s}}(k+1)^{d\frac{p}{p-s}}f(k)f_S(N)<+\infty\,.$$

The second main theorem can be formulated as follows:

Theorem 1.9. *Let $\mathbf{P}(\omega)$ be a stationary ergodic random open set which is almost surely (δ,M)-regular (Def. 4.2) and isotropic cone mixing for $\mathfrak{r}>0$ and $f(R)$ (Def. 4.17) as well as locally connected and satisfy $\mathbb{P}(\mathsf{S}>S_0)\leq f_s(S_0)$ such that Assumption 1.8 holds. For $1\leq r<\tilde{s}<s<p\leq\infty$ and $\mathbf{Q}\subset\mathbb{R}^d$ a bounded domain with Lipschitz boundary. Then for almost every ω the extension operator $\mathcal{U}:\,W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega))\to W^{1,r}_{\mathrm{loc}}\left(\mathbb{R}^d\right)$ provided in (6.6) is well defined with a constant $C(\omega)$ such that for every positive $n\geq 1$ and every $u\in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{P}(\omega)\cap n\mathbf{Q})$*

$$\frac{1}{n^d\left|\mathbf{Q}\right|}\int_{n\mathbf{Q}}\left|\mathcal{U}u\right|^r\leq C(\omega)\left(\frac{1}{n^d\left|\mathbf{Q}\right|}\int_{\mathbf{P}(\omega)\cap n\mathbf{Q}}\left|u\right|^p\right)^{\frac{r}{p}},\tag{1.3}$$
$$\frac{1}{n^d\left|\mathbf{Q}\right|}\int_{n\mathbf{Q}}\left|\nabla\mathcal{U}u\right|^r\leq C(\omega)\left(\frac{1}{n^d\left|\mathbf{Q}\right|}\int_{\mathbf{P}(\omega)\cap n\mathbf{Q}}\left|\nabla u\right|^p\right)^{\frac{r}{p}}.\tag{1.4}$$

Proof. We combine Theorem 6.3 with Lemmas 6.4 and 6.5 as well as Lemmas 4.13 and 4.21 to obtain the first and second condition. The remaining condition is inferred from Theorem 7.7 and Lemma 4.21. □

Sobolev Spaces on Ω

Besides the evident benefit of the above extension and trace theorems, let us note that these theorems are also needed for the construction of the suitable Sobolev spaces on Ω. In Section 9 we recall some standard construction of Sobolev spaces on the probability space Ω and provide some links between two major approaches which seem to be hard to find in one place. We will need this summing up in order to better illustrate the generalization to perforated domains.

To understand our ansatz, we recall a result from [14] that there exist $\mathbf{P} \subset \Omega$ and $\Gamma \subset \Omega$ such that for almost every $\omega \in \Omega$ $\chi_{\mathbf{P}(\omega)}(x) = \chi_{\mathbf{P}}(\tau_x\omega)$ and $\chi_{\Gamma(\omega)}(x) = \chi_{\Gamma}(\tau_x\omega)$, where $\Gamma(\omega) := \partial\mathbf{P}(\omega)$. The random set $\mathbf{P}(\omega)$ leads to Sobolev spaces $W^{1,p}(\mathbf{P}(\omega))$, e.g. by defining $W^{1,p}(\mathbf{P}(\omega)) := \left\{\chi_{\mathbf{P}(\omega)}u : u \in W^{1,p}(\mathbb{R}^d)\right\}$. We will see that we can introduce spaces $W^{1,p}(\mathbf{P})$, but this construction is more involved than in $\mathbb{R}^d$ and heavily relies on the almost sure extension property guarantied by Theorem 1.6. Once we have introduced the spaces $W^{1,p}(\mathbf{P})$ we can also introduce "trace"-operators $\mathcal{T}_\Omega : W^{1,p}(\mathbf{P}) \to L^r(\Gamma)$, where $\Gamma \subset \Omega$ with $\chi_{\Gamma(\omega)}(x) = \chi_{\Gamma}(\tau_x\omega)$, and $L^r(\Gamma)$ is to be understood w.r.t. the Palm measure on Γ. This construction will rely on Theorems 1.6 and 1.4. In all our results, we only provide sufficient conditions for the existence of the respective spaces and operators. Necessary conditions are left for future studies.

Discussion: Random Geometries and Applicability of the Method

In Section 10 we will discuss how the present results can be applied in the framework of the stochastic two-scale convergence method. However, this concerns only the analytic aspect of applicability.

The more important question is the applicability of the presented theory from the point of view of random geometries. Of course our result can be applied to periodic geometries and hence also to stochastic geometries which originate from random perturbations of periodic geometries as long as these perturbations are - in the statistical average - "not to large". However, it is a well justified question if the estimates presented here are applicable also for other models.

In Section 8 we discuss three standard models from the theory of stochastic geometries. The first one is the Boolean model based on a Poisson point process. Here we can show that the micro- and mesoscopic assumptions are fulfilled, at least in case $\mathbf{P}$ is given as the union of balls. If we choose $\mathbf{P}$ as the complement of the balls, we currently seem to run into difficulties. However, this problem might be overcome using a Matern modification of the Poisson process. We deal with such Matern modifications in Section 8.2. What remains challenging in both settings are the proofs of statistical harmony or statistical connectivity. However, if the Matern process strongly excludes points that are to close to each other, the connectivity issue can be resolved.

A further class which will be discussed are a system of Delaunay pipes based on a Matern process. In this case, even though the geometry might locally become very irregular, all properties can be verified. Hence, we identified at least one non-trivial, non-quasi-periodic geometry to which our approach can be applied for sure.

The above mentioned construction of Sobolev spaces and the application in the homogenization result of Theorem 10.20 clearly demonstrate the benefits of the new methodology.

Notes

Structure of the article

We close the introduction by providing an overview over the article and its main contributions. In Section 2 we collect some basic concepts and inequalities from the theory of Sobolev spaces, random geometries and discrete and continuous ergodic theory. We furthermore establish local regularity properties for what we call η-regular sets, as well as a related covering theorem in Section 2.5. In Section 2.11 we will demonstrate that stationary ergodic random open sets induce stationary processes on $\mathbb{Z}^d$, a fact which is used later in the construction of the mesoscopic Voronoi tessellation in Section 4.2.

In Section 3 we provide a proof of the periodic extension result in a simplified setting. This is for completeness and self-containedness of the paper, in order to make a comparison between stochastic and periodic approach easily accessible to the reader.

In Section 4 we introduce the regularity concepts of this work. More precisely, in Section 4.1 we introduce the concept of local (δ, M)-regularity and use the theory of Section 2.5 in order to establish a local covering result for $\partial\mathbf{P}$, which will allow us to infer most of our extension and trace results. In Section 4.2 we show how isotropic cone mixing geometries allow us to construct a stationary Voronoi tessellation of $\mathbb{R}^d$ such that all related quantities like “diameter” of the cells are stationary variables whose expectation can be expressed in terms of the isotropic cone mixing function f. Moreover we prove the important integration Lemma 4.21.

In Sections 5–7 we finally provide the aforementioned extension operators and prove estimates for these extension operators and for the trace operator.

In Section 8 we study some sample geometries and in Section 10 we discuss the homogenization problem.

A Remark on Notation

This article uses concepts from partial differential equations, measure theory, probability theory and random geometry. Additionally, we introduce concepts which we believe have not been introduced before. This makes it difficult to introduce readable self contained notation (the most important aspect being symbols used with different meaning) and enforces the use of various different mathematical fonts. Therefore, we provide an index of notation at the end of this work. As a rough orientation, the reader may keep the following in mind:

We use the standard notation $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$ for natural (>0), rational, real and integer numbers. $\mathbb{P}$ denotes a probability measure, $\mathbb{E}$ the expectation. Furthermore, we use special notation for some geometrical objects, i.e. $\mathbb{T}^d = [0,1)^d$ for the torus ($\mathbb{T}$ equipped with the topology of the torus), $\mathbb{I}^d = (0,1)^d$ the open interval as a subset of $\mathbb{R}^d$ (we often omit the index d), $\mathbb{B}$ a ball, $\mathbb{C}$ a cone and $\mathbb{X}$ a set of points. In the context of finite sets A, we write $\#A$ for the number of elements.

Bold large symbols ($\mathbf{U}$, $\mathbf{Q}$, $\mathbf{P}$,...) refer to open subsets of $\mathbb{R}^d$ or to closed subsets with $\partial\mathbf{P} = \partial\mathring{\mathbf{P}}$. The Greek letter Γ refers to a $d-1$ dimensional manifold (aside from the notion of Γ-convergence).

Calligraphic symbols ($\mathcal{A}$, $\mathcal{U}$, ...) usually refer to operators and large Gothic symbols ($\mathfrak{B}, \mathfrak{C}, \dots$) indicate topological spaces, except for $\mathfrak{A}$.

2 Preliminaries

We first collect some notation and mathematical concepts which will be frequently used throughout this paper. We first start with the standard geometric objects, which will be labeled by bold letters.

2.1 Fundamental Geometric Objects

Unit cube The torus $\mathbb{T} = [0,1)^d$ has the topology of the metric $d(x,y) = \min_{z\in\mathbb{Z}^d} |x-y+z|$. In contrast, the open interval $\mathbb{I}^d := (0,1)^d$ is considered as a subset of $\mathbb{R}^d$. We often omit the index d if this does not provoke confusion.

Balls Given a metric space (M,d) we denote $\mathbb{B}_r(x)$ the open ball around $x \in M$ with radius $r > 0$. The surface of the unit ball in $\mathbb{R}^d$ is $\mathbb{S}^{d-1}$.

Points A sequence of points will be labeled by $\mathbb{X} := (x_i)_{i\in\mathbb{N}}$.

A cone in $\mathbb{R}^d$ is usually labeled by $\mathbb{C}$. In particular, we define for a vector ν of unit length, $0 < \alpha < \frac{\pi}{2}$ and $R > 0$ the cone

$$\mathbb{C}_{\nu,\alpha,R}(x) := \{z \in \mathbb{B}_R(x) \;:\; z\cdot\nu > |z|\cos\alpha\} \quad \text{and} \quad \mathbb{C}_{\nu,\alpha}(x) := \mathbb{C}_{\nu,\alpha,\infty}(x)\,.$$

Inner and outer hull We use balls of radius $r > 0$ to define for a closed set $\mathbf{P} \subset \mathbb{R}^d$ the sets

$$\begin{aligned} \mathbf{P}_r &:= \overline{\mathbb{B}_r(\mathbf{P})} := \left\{x \in \mathbb{R}^d \;:\; \operatorname{dist}(x,\mathbf{P}) \le r\right\}\,, \\ \mathbf{P}_{-r} &:= \mathbb{R}^d \backslash \left[\mathbb{B}_r\big(\mathbb{R}^d \setminus \mathbf{P}\big)\right] := \left\{x \in \mathbb{R}^d \;:\; \operatorname{dist}\big(x, \mathbb{R}^d\setminus\mathbf{P}\big) \ge r\right\}\,. \end{aligned} \tag{2.1}$$

One can consider these sets as inner and outer hulls of $\mathbf{P}$. The last definition resembles a concept of “negative distance” of $x \in \mathbf{P}$ to $\partial\mathbf{P}$ and “positive distance” of $x \notin \mathbf{P}$ to $\partial\mathbf{P}$. For $A \subset \mathbb{R}^d$ we denote $\operatorname{conv}(A)$ the closed convex hull of A.

The natural geometric measures we use in this work are the Lebesgue measure on $\mathbb{R}^d$, written $|A|$ for $A \subset \mathbb{R}^d$, and the k-dimensional Hausdorff measure, denoted by $\mathcal{H}^k$ on k-dimensional submanifolds of $\mathbb{R}^d$ (for $k \le d$).

2.2 Local Extensions and Traces

Let $\mathbf{P} \subset \mathbb{R}^d$ be an open set and let $p \in \partial\mathbf{P}$ and $\delta > 0$ be a constant such that $\mathbb{B}_\delta(p) \cap \partial\mathbf{P}$ is graph of a Lipschitz function. We denote

$$\begin{aligned} M(p,\delta) := \inf \big\{ M \;:\; &\exists\phi : U \subset \mathbb{R}^{d-1} \to \mathbb{R} \\ &\phi \text{ Lipschitz, with constant } M \text{ s.t. } \mathbb{B}_\delta(p) \cap \partial\mathbf{P} \text{ is graph of } \phi \big\}\,. \end{aligned} \tag{2.2}$$

Remark 2.1. For every p, the function $M(p,\cdot)$ is monotone increasing in δ.

In the following, we formulate some extension and trace results. Although it is well known how such results are proved and the proofs are standard, we include them for completeness.

Lemma 2.2 (Uniform Extension for Balls)**.** *Let $\mathbf{P} \subset \mathbb{R}^d$ be an open set, $0 \in \partial\mathbf{P}$ and assume there exists $\delta > 0$, $M > 0$ and an open domain $U \subset \mathbb{B}_\delta(0) \subset \mathbb{R}^{d-1}$ such that $\partial\mathbf{P} \cap \mathbb{B}_\delta(0)$ is graph of a Lipschitz function $\varphi : U \subset \mathbb{R}^{d-1} \to \mathbb{R}^d$ of the form $\varphi(\tilde{x}) = (\tilde{x}, \phi(\tilde{x}))$ in $\mathbb{B}_\delta(0)$ with Lipschitz* *constant M and $\varphi(0)=0$. Writing $x=(\tilde{x},x_d)$ and defining $\rho=\delta\sqrt{4M^2+2}^{-1}$ there exist an extension operator*

$$\left(\mathcal{U}u\right)(x)=\begin{cases}u(x) & \textit{if } x_d<\phi(\tilde{x})\\ 4u\left(\tilde{x},-\frac{x_d}{2}+\frac{3}{2}\phi(\tilde{x})\right)-3u\left(\tilde{x},-x_d+2\phi(\tilde{x})\right) & \textit{if } x_d>\phi(\tilde{x})\end{cases}, \tag{2.3}$$

such that for

$$\mathcal{A}\left(0,\mathbf{P},\rho\right):=\left\{(\tilde{x},-x_d+2\phi(\tilde{x}))\,:\,(\tilde{x},x_d)\in\mathbb{B}_\rho(0)\backslash\mathbf{P}\right\}, \tag{2.4}$$

and for every $p\in[1,\infty]$ the operator

$$\mathcal{U}:\,W^{1,p}(\mathcal{A}\left(0,\mathbf{P},\rho\right))\to W^{1,p}(\mathbb{B}_\rho(0))\,,$$

is continuous with

$$\left\|\mathcal{U}u\right\|_{L^p(\mathbb{B}_\rho(0)\backslash\mathbf{P})}\leq 7\left\|u\right\|_{L^p(\mathcal{A}(0,\mathbf{P},\rho))}\,,\qquad\left\|\nabla\mathcal{U}u\right\|_{L^p(\mathbb{B}_\rho(0)\backslash\mathbf{P})}\leq 14M\left\|\nabla u\right\|_{L^p(\mathcal{A}(0,\mathbf{P},\rho))}\,. \tag{2.5}$$

Remark 2.3. It is well known ([10, chapter 5]) that for every bounded domain $\mathbf{U}\subset\mathbb{R}^d$ with $C^{0,1}$-boundary there exists a continuous extension operator $\mathcal{U}:\,W^{1,p}(\mathbf{U})\to W^{1,p}(\mathbb{R}^d)$.

Proof of Lemma 2.2. The extended function $\varphi:\,U\times\mathbb{R}\to U\times\mathbb{R}$, $\varphi(x)=(\tilde{x},\phi(\tilde{x})+x_d)$ is bijective with $\varphi^{-1}(x)=(\tilde{x},x_d-\phi(\tilde{x}))$. In particular, both φ and φ^{-1} are Lipschitz continuous with Lipschitz constant $M+1$.

W.l.o.g. we assume that

$$\varphi\left(U\times(-\infty,0)\right)\cap\mathbb{B}_\delta(0)=\mathbf{P}\cap\mathbb{B}_\delta(0)\cap\left(U\times\mathbb{R}\right)$$

implying $\varphi\left(U\times(0,\infty)\right)\cap\mathbf{P}=\emptyset$.

Step 1: We consider the extension operator $\mathcal{U}_+:\,W^{1,p}(\mathbb{R}^{d-1}\times(-\infty,0))\to W^{1,p}(\mathbb{R}^d)$ having the form [10, chapter 5], [1]

$$\left(\mathcal{U}_+u\right)(x)=\begin{cases}u(x) & \text{if } x_d<0\\ 4u\left(\tilde{x},-\frac{x_d}{2}\right)-3u\left(\tilde{x},-x_d\right) & \text{if } x_d>0\end{cases}.$$

We make use of this operator and define

$$\mathcal{U}u(x):=\left(\mathcal{U}_+\left(u\circ\varphi\right)\right)\circ\varphi^{-1}(x)\,.$$

Note that all three operators $u\mapsto u\circ\varphi$, $\mathcal{U}_+$ and $v\mapsto v\circ\varphi^{-1}$ map $W^{1,p}$-functions to $W^{1,p}$-functions. By the definition of $\mathcal{U}_+$ we may explicitly calculate (2.3). In particular, $\mathcal{U}u(x)$ is well defined for $x\in\mathbb{B}_\delta(0)\backslash\mathbf{P}$ whenever

$$(\tilde{x},-x_d+2\phi(\tilde{x}))\in\mathbb{B}_\delta(0)\,. \tag{2.6}$$

Step 2: We seek for $\rho>0$ such that (2.6) is satisfied for every $x\in\mathbb{B}_\rho(0)\backslash\mathbf{P}$ and such that $\mathcal{A}\left(0,\mathbf{P},\rho\right)\subset\mathbb{B}_\delta(0)$. For $\rho<\delta$ and $x=(\tilde{x},x_d)\in\mathbb{B}_\rho(0)$, we find with $\varphi(0)=0$ and $|x_d|\leq\sqrt{\rho^2-|\tilde{x}|^2}$ that

$$\begin{aligned}-x_d+2\phi(\tilde{x})&\in\left(x_d-2M|\tilde{x}|\,,\,x_d+2M|\tilde{x}|\right)\\ &\subset\left(-\sqrt{\rho^2-|\tilde{x}|^2}-2M|\tilde{x}|\,,\,\sqrt{\rho^2-|\tilde{x}|^2}+2M|\tilde{x}|\right)\,.\end{aligned}$$

In particular,

$$\max_{(\tilde{x},x_d)\in\mathbb{B}_\rho(0)\backslash\mathbf{P}} |-x_d+2\phi(\tilde{x})| \leq \rho\sqrt{4M^2+1}$$

and (2.6) holds if

$$|-x_d+2\phi(\tilde{x})|^2+|\tilde{x}|^2 \leq \rho^2\left(4M^2+1\right)+\rho^2 \leq \delta^2\,.$$

Hence we require $\rho=\delta\sqrt{4M^2+2}^{-1}$. It is now easy to verify (2.5) from the definition of $\mathcal{U}$ and the chain rule. □

Lemma 2.4. *Let $\mathbf{P}\subset\mathbb{R}^d$ be an open set, $0\in\partial\mathbf{P}$ and assume there exists $\delta>0$, $M>0$ and an open domain $U\subset\mathbb{B}_\delta(0)\subset\mathbb{R}^{d-1}$ such that $\partial\mathbf{P}\cap\mathbb{B}_\delta(0)$ is graph of a Lipschitz function $\varphi:\,U\subset\mathbb{R}^{d-1}\to\mathbb{R}^d$ of the form $\varphi(\tilde{x})=(\tilde{x},\phi(\tilde{x}))$ in $\mathbb{B}_\delta(0)$ with Lipschitz constant M and $\varphi(0)=0$. Writing $x=(\tilde{x},x_d)$ we consider the trace operator $\mathcal{T}:\,C^1\left(\mathbf{P}\cap\mathbb{B}_{2\delta}(0)\right)\to C\left(\partial\mathbf{P}\cap\mathbb{B}_\delta(0)\right)$. For every $p\in[1,\infty]$ and every $r<\frac{p(1-d)}{(p-d)}$ the operator $\mathcal{T}$ can be continuously extended to*

$$\mathcal{T}:\,W^{1,p}\left(\mathbf{P}\cap\mathbb{B}_{2\delta}(0)\right)\to L^r(\partial\mathbf{P}\cap\mathbb{B}_\delta(0))\,,$$

such that

$$\|\mathcal{T}u\|_{L^r(\partial\mathbf{P}\cap\mathbb{B}_\delta(0))} \leq C_{r,p}\delta^{\frac{d(p-r)}{rp}-\frac{1}{r}}\sqrt{4M^2+2}^{\frac{1}{r}+1}\|u\|_{W^{1,p}(\mathbf{P}\cap\mathbb{B}_{2\delta}(0))}\,. \tag{2.7}$$

Proof. We proceed similar to the proof of Lemma 2.2.

Step 1: Writing $B_\delta=\mathbb{B}_\delta(0)$ together with $B_\delta^-=\{x\in B_\delta:\,x_d<0\}$ and $\Sigma_\delta:=\{x\in B_\delta:\,x_d=0\}$ we recall the standard estimate

$$\left(\int_{\Sigma_1}|u|^r\right)^{\frac{1}{r}} \leq C_{r,p}\left(\left(\int_{B_1^-}|\nabla u|^p\right)^{\frac{1}{p}}+\left(\int_{B_1^-}|u|^p\right)^{\frac{1}{p}}\right)\,,$$

which leads to

$$\left(\int_{\Sigma_\delta}|u|^r\right)^{\frac{1}{r}} \leq C_{r,p}\delta^{\frac{d(p-r)}{rp}-\frac{1}{r}}\left(\left(\int_{B_\delta^-}|\nabla u|^p\right)^{\frac{1}{p}}+\left(\int_{B_\delta^-}|u|^p\right)^{\frac{1}{p}}\right)\,.$$

Step 2: Using the transformation rule and the fact that $1\leq|\det D\varphi|\leq\sqrt{4M^2+2}$ we infer (2.7) similar to Step 2 in the proof of Lemma 2.2.

$$\begin{aligned}
\left(\int_{\partial\mathbf{P}\cap\mathbb{B}_\delta(0)}|u|^r\right)^{\frac{1}{r}} &\leq \sqrt{4M^2+2}^{\frac{1}{r}}\left(\int_{\Sigma_\delta}|u\circ\varphi|^r\right)^{\frac{1}{r}}\\
&\leq C_{r,p}\delta^{\frac{d(p-r)}{rp}-\frac{1}{r}}\sqrt{4M^2+2}^{\frac{1}{r}}\left(\left(\int_{B_\delta^-}|\nabla\left(u\circ\varphi\right)|^p\right)^{\frac{1}{p}}+\left(\int_{B_\delta^-}|u\circ\varphi|^p\right)^{\frac{1}{p}}\right)\\
&\leq C_{r,p}\delta^{\frac{d(p-r)}{rp}-\frac{1}{r}}\sqrt{4M^2+2}^{\frac{1}{r}+1}\cdot\\
&\quad\cdot\left(\left(\int_{B_\delta^-}|(\nabla u)\circ\varphi|^p\det D\varphi\right)^{\frac{1}{p}}+\left(\int_{B_\delta^-}|u\circ\varphi|^p\det D\varphi\right)^{\frac{1}{p}}\right)
\end{aligned}$$

and from this we conclude the Lemma. □

2.3 Poincaré Inequalities

We denote

$$W^{1,p}_{(0),r}(\mathbb{B}_r(0)) := \left\{ u \in W^{1,p}(\mathbb{B}_r(0)) \,:\, \exists x:\, B_r(x) \subset \mathbb{B}_r(0) \,\vee\, \fint_{B_r(x)} u = 0 \right\}.$$

Note that this is not a linear vector space.

Lemma 2.5. *For every $p \in (1,\infty)$ there exists $C_p > 0$ such that the following holds: Let $r < 1$ and $x \in \mathbb{B}_1(0)$ such that $B_r(x) \subset \mathbb{B}_1(0)$ then for every $u \in W^{1,p}(\mathbb{B}_1(0))$*

$$\|u\|^p_{L^p(\mathbb{B}_1(0))} \leq C_p \left(\|\nabla u\|^p_{L^p(\mathbb{B}_1(0))} + \frac{1}{r^d} \|u\|^p_{L^p(B_r(x))} \right), \tag{2.8}$$

and for every $u \in W^{1,p}_{(0),r}(\mathbb{B}_1(0))$ it holds

$$\|u\|^p_{L^p(\mathbb{B}_1(0))} \leq C_p \left(1 + r^{p-d}\right) \|\nabla u\|^p_{L^p(\mathbb{B}_1(0))} . \tag{2.9}$$

Remark. In case $p \geq d$ we find that (2.9) holds iff $u(x) = 0$ for some $x \in \mathbb{B}_1(0)$.

Proof. In a first step, we assume $x = 0$. The underlying idea of the proof is to compare every $u(y)$, $y \in \mathbb{B}_1(0)\backslash\mathbb{B}_r(0)$ with $u(rx)$. In particular, we obtain for $y \in \mathbb{B}_1(0)\backslash\mathbb{B}_r(0)$ that

$$u(y) = u(ry) + \int_0^1 \nabla u(ry + t(1-r)y) \cdot (1-r)y \,\mathrm{d}t$$

and hence by Jensen's inequality

$$|u(y)|^p \leq C \left(\int_0^1 |\nabla u(ry + t(1-r)y)|^p \,(1-r)^p \,|y|^p \,\mathrm{d}t + |u(ry)|^p \right).$$

We integrate the last expression over $\mathbb{B}_1(0)\backslash\mathbb{B}_r(0)$ and find

$$\begin{aligned}
\int_{\mathbb{B}_1(0)\backslash\mathbb{B}_r(0)} |u(y)|^p \,\mathrm{d}y &\leq \int_{S^{d-1}} \int_r^1 C \left(\int_0^1 |\nabla u(rs\nu + t(1-r)s\nu)|^p \,(1-r)^p s^p \,\mathrm{d}t \right) s^{d-1} \mathrm{d}s\mathrm{d}\nu \\
&\quad + \int_{\mathbb{B}_1(0)\backslash\mathbb{B}_r(0)} |u(ry)|^p \,\mathrm{d}y \\
&\leq \int_{S^{d-1}} \int_r^1 C \left(\int_{rs}^s |\nabla u(t\nu)|^p \,(1-r)^{p-1} s^{p-1} \,\mathrm{d}t \right) s^{d-1} \mathrm{d}s \\
&\quad + \int_{\mathbb{B}_1(0)\backslash\mathbb{B}_r(0)} |u(ry)|^p \,\mathrm{d}y \\
&\leq C \|\nabla u\|^p_{L^p(\mathbb{B}_1(0))} + \frac{1}{r^d} \|u\|^p_{L^p(\mathbb{B}_r(0))} .
\end{aligned}$$

For general $x \in \mathbb{B}_1(0)$, use the extension operator $\mathcal{U} :\, W^{1,p}(\mathbb{B}_1(0)) \to W^{1,p}(B_4(0))$ (see Remark 2.3) such that $\|\mathcal{U}u\|_{W^{1,p}(B_4(0))} \leq C \|u\|_{W^{1,p}(\mathbb{B}_1(0))}$ and $\|\nabla\mathcal{U}u\|_{W^{1,p}(B_4(0))} \leq C \|\nabla u\|_{W^{1,p}(\mathbb{B}_1(0))}$. Since $\mathbb{B}_1(0) \subset B_2(x) \subset B_4(0)$ we infer

$$\|u\|^p_{L^p(\mathbb{B}_1(0))} \leq \|\mathcal{U}u\|^p_{L^p(B_2(x))} \leq C \left(\|\nabla\mathcal{U}u\|^p_{L^p(B_2(x))} + \frac{1}{r^d} \|\mathcal{U}u\|^p_{L^p(B_r(x))} \right).$$

and hence (2.8). Furthermore, since there holds $\|u\|^p_{L^p(\mathbb{B}_1(0))} \leq C \|\nabla u\|^p_{L^p(\mathbb{B}_1(0))}$ for every $u \in W^{1,p}_{(0)}(\mathbb{B}_1(0))$, a scaling argument shows $\|u\|^p_{L^p(\mathbb{B}_r(0))} \leq C r^p \|\nabla u\|^p_{L^p(\mathbb{B}_r(0))}$ for every $u \in W^{1,p}_{(0),r}(\mathbb{B}_1(0))$ and hence (2.9). □

Lemma 2.6. *Let $0 < r < R < \infty$ and $p \in (1,\infty)$ and $q \le pd/(d-p)$ (if $p < d$) or $q = \infty$ (if $p \ge d$). Then there exists $C_{p,q}$ such that for every convex set $\mathbf{P}$ with polytope boundary $\partial\mathbf{P} \subset \mathbb{B}_R(0)\backslash\overline{\mathbb{B}_r(0)}$*

$$\|u\|^p_{L^q(\mathbf{P})} \le C_{p,q}R^{-d\left(1-\frac{p}{q}\right)}\left(\int_{\mathbf{P}}\left(R^p\left(\frac{R}{r}\right)^{p+1}|\nabla u|^p + \frac{R^{d+1}}{r^{d+1}}|u|^p\right)\right)\,, \tag{2.10}$$

and for every $u \in W^{1,p}_{(0),r}\left(\mathbb{B}_R(0)\right)$

$$\|u\|^p_{L^q(\mathbb{B}_R(0))} \le \mathfrak{C}_{p,q}(R,r)\,\|\nabla u\|^p_{L^p(\mathbb{B}_R(0))}\,, \tag{2.11}$$

where

$$\mathfrak{C}_{p,q}(R,r) := C_{p,q}R^{-d\left(1-\frac{p}{q}\right)+p}\left(\left(\frac{R}{r}\right)^{p+1} + \left(\frac{R}{r}\right)^{d+1}\right) \tag{2.12}$$

Remark 2.7. For the critical Sobolev index $q = \frac{pd}{d-p}$ we infer $d\left(1-\frac{p}{q}\right) = p$.

Proof. First note that by a simple scaling argument based on the integral transformation rule the equations (2.8) yields for every $u \in W^{1,p}(\mathbb{B}_r(0))$

$$\|u\|^p_{L^q(\mathbb{B}_R(0))} \le C_{p,q}R^{-d\left(1-\frac{p}{q}\right)}\left(R^p\,\|\nabla u\|^p_{L^p(\mathbb{B}_R(0))} + \frac{R^d}{r^d}\,\|u\|^p_{L^p(\mathbb{B}_r(0))}\right) \tag{2.13}$$

and (2.9) yields for every $u \in W^{1,p}_{(0),r}(\mathbb{B}_r(0))$

$$\|u\|^p_{L^q(\mathbb{B}_R(0))} \le C_{p,q}R^pR^{-d\left(1-\frac{p}{q}\right)}\left(1+\left(\frac{r}{R}\right)^{p-d}\right)\|\nabla u\|^p_{L^p(\mathbb{B}_R(0))}\,. \tag{2.14}$$

Now, for $\nu \in \mathbb{S}^{d-1}$ we denote $P(\nu)$ as the unique $p \in \partial\mathbf{P}\cap(0,\infty)\nu$ and for $x \in \mathbb{R}^d\backslash\{0\}$ we denote $\nu_x := \frac{x}{\|x\|}$ and consider the bijective Lipschitz map

$$\varphi_P:\,\mathbf{P}\to\mathbb{B}_r(0)\,,\qquad x\mapsto R\frac{x}{\|P(\nu_x)\|}\,.$$

Then we infer from (2.13)

$$\left\|u\circ\tilde\varphi_P^{-1}\right\|^p_{L^q(\mathbb{B}_R(0))} \le CR^{-d\left(1-\frac{p}{q}\right)}\left(R^p\left\|\nabla\left(u\circ\tilde\varphi_P^{-1}\right)\right\|^p_{L^p(\mathbb{B}_R(0))} + \frac{R^d}{r^d}\left\|u\circ\tilde\varphi_P^{-1}\right\|^p_{L^p(\mathbb{B}_r(0))}\right)$$

or, after transformation of integrals,

$$\left(\int_{\mathbf{P}}|u|^q\left|\det\mathrm{D}\tilde\varphi_P\right|\right)^{\frac{p}{q}} \le CR^{-d\left(1-\frac{p}{q}\right)}\left(\int_{\mathbf{P}}\left(R^p\left|(\nabla u)\left(\mathrm{D}\tilde\varphi_P\right)^{-1}\right|^p + \frac{R^d}{r^d}\chi_{\tilde\varphi_P^{-1}\mathbb{B}_r(0)}|u|^p\right)\left|\det\mathrm{D}\tilde\varphi_P\right|\right)\,.$$

It remains to estimate the derivatives of φ_P. In polar coordinates, the radial derivative is $\partial_r\varphi_P(x) = \frac{R}{\|P(\nu_x)\|}$, while the tangential derivative is more complicated to calculate. However, in case $\nu\perp\mathrm{T}_{P(\nu)}$ we obtain $\partial_{\mathbb{S}^{d-1}}\varphi_P(x) = \mathbb{I}_{\mathbb{R}^{d-1}}$, which is by the same time the minimal absolute value for each tangential derivative, and $\partial_{\mathbb{S}^{d-1}}\varphi_P(x)$ becomes maximal in edges where $2\tan\alpha = r^{-1}\sqrt{R^2-r^2}$ and $\|\partial\varphi_P\|(x_0) = \left\|\frac{R}{\|x_0\|}\mathrm{id} - \frac{Rx_0}{\|x_0\|^3}\otimes x\right\| \le 2\frac{R}{r}$ (see Figure ......).Now we make use of the fact that $\tilde\varphi_P$ increases the volume locally with a rate smaller than $\|\partial\varphi_P\|$and hence $\left|\det\mathrm{D}\tilde\varphi_P\right| \ge 1$. On the other hand, we have $\left|\left(\mathrm{D}\tilde\varphi_P\right)^{-1}\right| < \frac{R}{r}$ and hence (2.10). In a similar way we infer (2.11) from (2.14). □

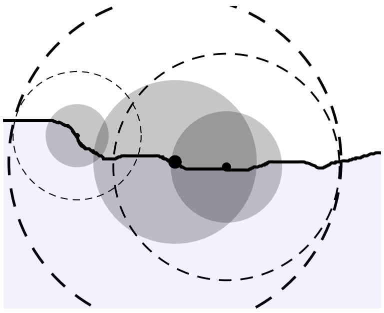

Figure 1: An illustration of η-regularity. In Theorem 2.13 we will rely on a “gray” region like in this picture.

2.4 Voronoi Tessellations and Delaunay Triangulation

Definition 2.8 (Voronoi Tessellation)**.** Let $\mathbb{X} = (x_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$ with $x_i \neq x_k$ if $i \neq k$. For each $x \in \mathbb{X}$ let

$$G(x) := \left\{ y \in \mathbb{R}^d \,:\, \forall \tilde{x} \in \mathbb{X}\backslash \{x\} : \, |x-y| < |\tilde{x}-y| \right\} \,.$$

Then $(G(x_i))_{i\in\mathbb{N}}$ is called the *Voronoi tessellation* of $\mathbb{R}^d$ w.r.t. $\mathbb{X}$. For each $x \in \mathbb{X}$ we define $d(x) := \mathrm{diam} G(x)$.

We will need the following result on Voronoi tessellation of a minimal diameter.

Lemma 2.9. *Let $\mathfrak{r} > 0$ and let $\mathbb{X} = (x_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$ with $|x_i - x_k| > 2\mathfrak{r}$ if $i \neq k$. For $x \in \mathbb{X}$ let $\mathcal{I}(x) := \{y \in \mathbb{X} \,:\, G(y) \cap \mathbb{B}_{\mathfrak{r}}(G(x)) \neq \emptyset\}$. Then*

$$\#\mathcal{I}(x) \leq \left(\frac{4d(x)}{\mathfrak{r}}\right)^d \,. \tag{2.15}$$

Proof. Let $\mathbb{X}_k = \left\{ x_j \in \mathbb{X} \,:\, \mathcal{H}^{d-1}(\partial G_k \cap \partial G_j) \geq 0 \right\}$ the neighbors of x_k and $d_k := d(x_k)$. Then all $x_j \in \mathbb{X}$ satisfy $|x_k - x_j| \leq 2d_k$. Moreover, every $\tilde{x} \in \mathbb{X}$ with $|\tilde{x} - x_k| > 4d_k$ has the property that $\mathrm{dist}(\,\partial G\,(\tilde{x})\,,\, x_k\,) > 2d_k > d_k + \mathfrak{r}$ and $\tilde{x} \notin \mathcal{I}_k$. Since every Voronoi cell contains a ball of radius $\mathfrak{r}$, this implies that $\#\mathcal{I}_k \leq |\mathbb{B}_{4d_k}(x_k)| \,/\, |\mathbb{B}_{\mathfrak{r}}(0)| = \left(\frac{4d_k}{\mathfrak{r}}\right)^d$. □

Definition 2.10 (Delaunay Triangulation)**.** Let $\mathbb{X} = (x_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$ with $x_i \neq x_k$ if $i \neq k$. The Delaunay triangulation is the dual graph of the Voronoi tessellation, i.e. we say $\mathbb{D}(\mathbb{X}) := \left\{ (x,y) : \, \mathcal{H}^{d-1}(\partial G(x) \cap \partial G(y)) \neq 0 \right\}$.

2.5 Local η-Regularity

We say that a function $F : A \to \{0,1\}$ holds “true” in $a \in A$ if $F(a) = 1$ and “false” if $F(a) = 0$.

Definition 2.11 (η-regularity)**.** A set $\mathbf{P} \subset \mathbb{R}^d$ is called locally η-regular with $f : \, \mathbf{P} \times (0, r] \to \{0,1\}$ and $\mathfrak{r} > 0$ if $f(p,\cdot)$ is decreasing and

$$f(p,\eta) = 1 \quad \Rightarrow \quad \forall\, \varepsilon \in \left(0, \frac{1}{2}\right),\, \tilde{p} \in \mathbb{B}_{\varepsilon\eta}(p) \cap \mathbf{P}\,,\, \tilde{\eta} \in (0,\, (1-\varepsilon)\eta) \,:\, f(\tilde{p},\tilde{\eta}) = 1\,. \tag{2.16}$$

For $p \in \mathbf{P}$ we write $\eta(p) := \sup\{\eta \in (0,\mathfrak{r}) \,:\, f(p,\eta) = 1\}$.

Lemma 2.12. *Let $\mathbf{P}$ be a locally η-regular set with f and $\mathfrak{r}$ and $\eta(p)$. Then $\eta : \mathbf{P} \to \mathbb{R}$ is locally Lipschitz continuous with Lipschitz constant 4 and for every $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $\tilde{p} \in \mathbb{B}_{\varepsilon\eta}(p) \cap \mathbf{P}$ it holds*

$$\frac{1-\varepsilon}{1-2\varepsilon}\eta(p) > \eta(\tilde{p}) > \eta(p) - |p-\tilde{p}| > (1-\varepsilon)\,\eta(p)\,. \tag{2.17}$$

Furthermore,

$$|p-\tilde{p}| \leq \varepsilon \max\{\eta(p), \eta(\tilde{p})\} \quad \Rightarrow \quad |p-\tilde{p}| \leq \frac{\varepsilon}{1-\varepsilon}\min\{\eta(p), \eta(\tilde{p})\} \tag{2.18}$$

Proof. We infer from (2.16) for every $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $\tilde{p}$ such that $|\tilde{p}-p| < \varepsilon\eta(p)$ let $\tilde{\eta} < \eta(p)$ such that also $|\tilde{p}-p| < \varepsilon\tilde{\eta}$. It then holds $f(\tilde{p}, (1-\varepsilon)\,\tilde{\eta}) = 1$ and hence $\eta(\tilde{p}) \geq (1-\varepsilon)\,\tilde{\eta}$. Taking the supremum over $\sup\{\tilde{\eta} : \tilde{\eta} < \eta(p)\}$ we find $\eta(\tilde{p}) \geq (1-\varepsilon)\,\eta(p)$ i.e.

$$\begin{aligned}\eta(\tilde{p}) &\geq \sup_{\hat{p}}\{(1-\varepsilon)\,\eta(\hat{p}) \;:\; |\tilde{p}-\hat{p}| < \varepsilon\eta(\hat{p})\} \\ &\geq \eta(p) - |p-\tilde{p}| > (1-\varepsilon)\,\eta(p)\end{aligned}$$

which implies $|\tilde{p}-p| < \frac{\varepsilon}{1-\varepsilon}\eta(\tilde{p})$. This in turn leads to $\eta(p) > \left(1 - \frac{\varepsilon}{1-\varepsilon}\right)\eta(\tilde{p})$ or

$$\eta(p) = \frac{1-\varepsilon}{1-\varepsilon}\eta(p) < \frac{1}{1-\varepsilon}\left(\eta(p) - |p-\tilde{p}|\right) < \frac{1}{1-\varepsilon}\eta(\tilde{p}) \leq \frac{1}{1-2\varepsilon}\eta(p)\,,$$

implying (2.17) and continuity of η.

Let $|p-\tilde{p}| = \varepsilon\eta(p) \leq 2\varepsilon\eta(\tilde{p})$, the last inequality particularly implies also $\eta(p) \geq (1-2\varepsilon)\,\eta(\tilde{p})$. Together with $|p-\tilde{p}| \leq 2\varepsilon\eta(\tilde{p}) \leq 4\varepsilon\eta(p) = 4\,|p-\tilde{p}|$ we have

$$4\,|p-\tilde{p}| \geq 2\varepsilon\eta(\tilde{p}) \geq \eta(\tilde{p}) - \eta(p) \geq -\varepsilon\eta(p) = -\,|p-\tilde{p}|\;.$$

Finally, in order to prove (2.18), w.l.o.g. let $\eta(\tilde{p}) \leq \eta(p)$. Then

$$|p-\tilde{p}| \leq \varepsilon\eta(p) \leq \frac{\varepsilon}{1-\varepsilon}\eta(\tilde{p})\,.$$

□

We make use of the latter Lemmas in order to prove the following covering-regularity of $\partial\mathbf{P}$.

Theorem 2.13. *Let $\Gamma \subset \mathbb{R}^d$ be a closed set and let $\eta(\cdot) \in C(\Gamma)$ be bounded and satisfy for every $\varepsilon \in \left(0, \frac{1}{2}\right)$ and for $|p-\tilde{p}| < \varepsilon\eta(p)$*

$$\frac{1-\varepsilon}{1-2\varepsilon}\,\eta(p) > \eta(\tilde{p}) > \eta(p) - |p-\tilde{p}| > (1-\varepsilon)\,\eta(p)\,. \tag{2.19}$$

and define $\tilde{\eta}(p) = 2^{-K}\eta(p)$, $K \geq 2$. Then for every $C \in (0,1)$ there exists a locally finite covering of Γ with balls $\mathbb{B}_{\tilde{\eta}(p_k)}(p_k)$ for a countable number of points $(p_k)_{k\in\mathbb{N}} \subset \Gamma$ such that for every $i \neq k$ with $\mathbb{B}_{\tilde{\eta}(p_i)}(p_i) \cap \mathbb{B}_{\tilde{\eta}(p_k)}(p_k) \neq \emptyset$ it holds

$$\begin{aligned}&\frac{2^{K-1}-1}{2^{K-1}}\tilde{\eta}(p_i) \leq \tilde{\eta}(p_k) \leq \frac{2^{K-1}}{2^{K-1}-1}\tilde{\eta}(p_i) \\ \text{and}\quad &\frac{2^{K}-1}{2^{K-1}-1}\min\{\tilde{\eta}(p_i), \tilde{\eta}(p_k)\} \geq |p_i-p_k| \geq C\max\{\tilde{\eta}(p_i), \tilde{\eta}(p_k)\}\end{aligned} \tag{2.20}$$

Proof. W.o.l.g. assume $\tilde{\eta} < (1-\delta)$. Consider $\tilde{Q} := \left[0, \frac{1}{n}\right]^d$, let $q_{1,\dots,n^d}$ denote the n^d elements of $[0,1)^d \cap \frac{\mathbb{Q}^d}{n}$ and let $\tilde{Q}_{z,i} = \tilde{Q} + z + q_i$. We set $B_{(0)} := \emptyset$, $\Gamma_1 = \Gamma$, $\eta_k := (1-\delta)^k$ and for $k \geq 1$ we construct the covering using inductively defined open sets $B_{(k)}$ and closed set Γ_k as follows:

1 Define $\Gamma_{k,1} = \Gamma_k$. For $i = 1, \dots, n^d$ do the following:

1.1 For every $z \in \mathbb{Z}^d$ do

$$\begin{array}{ll} \text{if } \exists p \in \left(\eta_k \tilde{Q}_{z,i}\right) \cap \Gamma_{k,i},\ \tilde{\eta}(p) \in (\eta_k, \eta_{k-1}] & \text{then set } b_{z,i} = \mathbb{B}_{\tilde{\eta}(p)}(p)\,,\ \mathbb{X}_{z,i} = \{p\} \\ \text{otherwise} & \text{set } b_{z,i} = \emptyset\,,\ \mathbb{X}_{z,i} = \emptyset\,. \end{array}$$

1.2 Define $B_{(k),i} := \bigcup_{z\in\mathbb{Z}^d} b_{z,i}$ and $\Gamma_{k,i+1} = \Gamma_k \backslash B_{(k),i}$ and $\mathbb{X}_{(k),i} := \bigcup_{z\in\mathbb{Z}^d} \mathbb{X}_{z,i}$. Observe: $p_1, p_2 \in \mathbb{X}_{(k),i}$ implies $|p_1 - p_2| > \left(1 - \frac{1}{n}\right)\eta_k$ and $p_3 \in \mathbb{X}_{(k),j}$, $j < i$ implies $p_1 \notin \mathbb{B}_{\eta_k}(p_3)$ and hence $|p_1 - p_3| > \eta_k$. Similar, $p_3 \in \mathbb{X}_l$, $l < k$, implies $|p_1 - p_3| > \eta_l > \eta_k$.

2 Define $\Gamma_{k+1} := \Gamma_{k,2^d+1}$, $\mathbb{X}_k := \bigcup_i \mathbb{X}_{(k),i}$.

The above covering of Γ is complete in the sense that every $x \in \Gamma$ lies into one of the balls (by contradiction). We denote $\mathbb{X} := \bigcup_k \mathbb{X}_k = (p_i)_{i\in\mathbb{N}}$ the family of centers of the above constructed covering of Γ and find the following properties: Let $p_1, p_2 \in \mathbb{X}$ be such that $\mathbb{B}_{\tilde{\eta}(p_1)}(p_1) \cap \mathbb{B}_{\tilde{\eta}(p_2)}(p_2) \neq \emptyset$. W.l.o.g. let $\tilde{\eta}(p_1) \geq \tilde{\eta}(p_2)$. Then the following two properties are satisfied due to (2.19)

1 It holds $|p_1 - p_2| \leq 2\tilde{\eta}(p_1) \leq \frac{1}{2^{K-1}}\eta(p_1)$ and hence $\mathbb{B}_{\tilde{\eta}(p_2)}(p_2) \subset \mathbb{B}_{2^{2-K}\eta(p_1)}(p_1)$ and $\eta(p_2) \geq \frac{2^{K-1}-1}{2^{K-1}}\eta(p_1)$. Furthermore $\tilde{\eta}(p_1) \geq \tilde{\eta}(p_2) \geq \frac{2^{K-1}-1}{2^{K-1}}\tilde{\eta}(p_1)$.

2 Let k such that $\tilde{\eta}(p_1) \in (\eta_k, \eta_{k+1}]$. If also $\tilde{\eta}(p_2) \in (\eta_k, \eta_{k+1}]$ then observation 1.(b) implies $|p_1 - p_2| \geq \left(1 - \frac{1}{n}\right)\eta_k \geq \left(1 - \frac{1}{n}\right)(1-\delta)\,\tilde{\eta}(p_1)$. If $\tilde{\eta}(p_2) \notin [\eta_k, \eta_{k+1})$ then $\tilde{\eta}(p_2) < \eta_k$ and hence $p_2 \notin \mathbb{B}_{\tilde{\eta}(p_1)}(p_1)$, implying $|p_1 - p_2| > \tilde{\eta}(p_1)$.

Choosing n and δ appropriately, this concludes the proof. □

2.6 Dynamical Systems

Assumption 2.14. *Throughout this work we assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space with countably generated σ-algebra $\mathscr{F}$.*

Due to the insight in [14], shortly sketched in the next two subsections, after a measurable transformation the probability space Ω can be assumed to be metric and separable, which always ensures Assumption 2.14.

Definition 2.15 (Dynamical system)**.** A dynamical system on Ω is a family $(\tau_x)_{x\in\mathbb{R}^d}$ of measurable bijective mappings $\tau_x : \Omega \mapsto \Omega$ satisfying (i)-(iii):

(i) $\tau_x \circ \tau_y = \tau_{x+y}$, $\tau_0 = id$ (Group property)

(ii) $\mathbb{P}(\tau_{-x}B) = \mathbb{P}(B) \quad \forall x \in \mathbb{R}^d,\ B \in \mathscr{F}$ (Measure preserving)

(iii) $A :\ \mathbb{R}^d \times \Omega \to \Omega \qquad (x,\omega) \mapsto \tau_x\omega$ is measurable (Measurability of evaluation)

A set $A \subset \Omega$ is almost invariant if $\mathbb{P}\left((A \cup \tau_x A) \setminus (A \cap \tau_x A)\right) = 0$. The family

$$\mathscr{I} = \left\{A \in \mathscr{F} \,:\, \forall x \in \mathbb{R}^d\, \mathbb{P}\left((A \cup \tau_x A) \setminus (A \cap \tau_x A)\right) = 0\right\} \tag{2.21}$$

of almost invariant sets is σ-algebra and

$$\mathbb{E}\left(f|\mathscr{I}\right) \text{ denotes the expectation of } f\,:\,\Omega \to \mathbb{R} \text{ w.r.t. } \mathscr{I}\,. \tag{2.22}$$

A concept linked to dynamical systems is the concept of stationarity.

Definition 2.16 (Stationary)**.** Let X be a measurable space and let $f\,:\,\Omega \times \mathbb{R}^d \to X$. Then f is called (weakly) stationary if $f(\omega, x) = f(\tau_x\omega, 0)$ for (almost) every x.

Definition 2.17. A family $(A_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ is called convex averaging sequence if

(i) each A_n is convex

(ii) for every $n \in \mathbb{N}$ holds $A_n \subset A_{n+1}$

(iii) there exists a sequence r_n with $r_n \to \infty$ as $n \to \infty$ such that $B_{r_n}(0) \subseteq A_n$.

We sometimes may take the following stronger assumption.

Definition 2.18. A convex averaging sequence A_n is called regular if

$$|A_n|^{-1}\,\#\left\{z \in \mathbb{Z}^d \,:\, (z + \mathbb{T}) \cap \partial A_n \neq \emptyset\right\} \to 0\,.$$

The latter condition is evidently fulfilled for sequences of cones or balls. Convex averaging sequences are important in the context of ergodic theorems.

Theorem 2.19 (Ergodic Theorem [8] Theorems 10.2.II and also [36])**.** *Let $(A_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a convex averaging sequence, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f\,:\,\Omega \to \mathbb{R}$ be measurable with $|\mathbb{E}(f)| < \infty$. Then for almost all $\omega \in \Omega$*

$$|A_n|^{-1} \int_{A_n} f(\tau_x\omega)\,\mathrm{d}x \to \mathbb{E}(f|\mathscr{I})\,. \tag{2.23}$$

We observe that $\mathbb{E}\left(f|\mathscr{I}\right)$ is of particular importance. For the calculations in this work, we will particularly focus on the case of trivial $\mathscr{I}$. This is called ergodicity, as we will explain in the following.

Definition 2.20 (Ergodicity and Mixing)**.** A dynamical system $(\tau_x)_{x\in\mathbb{R}^d}$ which is given on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called *mixing* if for every measurable $A, B \subset \Omega$ it holds

$$\lim_{\|x\|\to\infty} \mathbb{P}(A \cap \tau_x B) = \mathbb{P}(A)\,\mathbb{P}(B)\,. \tag{2.24}$$

A dynamical system is called *ergodic* if

$$\lim_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} \mathbb{P}(A \cap \tau_x B)\mathrm{d}x = \mathbb{P}(A)\,\mathbb{P}(B)\,. \tag{2.25}$$

Remark 2.21. a) Let $\Omega = \{\omega_0 = 0\}$ with the trivial σ-algebra and $\tau_x\omega_0 = \omega_0$. Then τ is evidently mixing. However, the realizations are constant functions $f_\omega(x) = c$ on $\mathbb{R}^d$ for some constant c.

b) A typical ergodic system is given by $\Omega = \mathbb{T}$ with the Lebesgue σ-algebra and $\mathbb{P} = \mathcal{L}$ the Lebesgue measure. The dynamical system is given by $\tau_x y := (x+y) \mod \mathbb{T}$.

c) It is known that $(\tau_x)_{x\in\mathbb{R}^d}$ is ergodic if and only if every almost invariant set $A \in \mathscr{I}$ has probability $\mathbb{P}(A) \in \{0,1\}$ (see [8] Proposition 10.3.III) i.e.

$$[\ \forall x\, \mathbb{P}((\tau_x A \cup A) \setminus (\tau_x A \cap A)) = 0\ ] \ \Rightarrow\ \mathbb{P}(A) \in \{0,1\}\ . \tag{2.26}$$

d) It is sufficient to show (2.24) or (2.25) for A and B in a ring that generates the σ-algebra $\mathscr{F}$. We refer to [8], Section 10.2, for the later results.

A further useful property of ergodic dynamical systems, which we will use below, is the following:

Lemma 2.22 (Ergodic times mixing is ergodic)**.** *Let $(\tilde\Omega, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and $(\hat\Omega, \hat{\mathscr{F}}, \hat{\mathbb{P}})$ be probability spaces with dynamical systems $(\tilde\tau_x)_{x\in\mathbb{R}^d}$ and $(\hat\tau_x)_{x\in\mathbb{R}^d}$ respectively. Let $\Omega := \tilde\Omega \times \hat\Omega$ be the usual product measure space with the notation $\omega = (\tilde\omega, \hat\omega) \in \Omega$ for $\tilde\omega \in \tilde\Omega$ and $\hat\omega \in \hat\Omega$. If $\tilde\tau$ is ergodic and $\hat\tau$ is mixing, then $\tau_x(\tilde\omega, \hat\omega) := (\tilde\tau_x\tilde\omega, \hat\tau_x\hat\omega)$ is ergodic.*

Proof. Relying on Remark 2.21.c) we verify (2.25) by proving it for sets $A = \tilde A \times \hat A$ and $B = \tilde B \times \hat B$ which generate $\mathscr{F} := \tilde{\mathscr{F}} \otimes \hat{\mathscr{F}}$. We make use of $A \cap B = \left(\tilde A \cap \tilde B\right) \times \left(\hat A \cap \hat B\right)$ and observe that

$$\begin{aligned}
\mathbb{P}(A \cap \tau_x B) &= \mathbb{P}\Big(\Big(\tilde A \cap \tilde\tau_x \tilde B\Big) \times \Big(\hat A \cap \hat\tau_x \hat B\Big)\Big) = \hat{\mathbb{P}}\Big(\hat A \cap \hat\tau_x \hat B\Big)\, \tilde{\mathbb{P}}\Big(\tilde A \cap \tilde\tau_x \tilde B\Big) \\
&= \hat{\mathbb{P}}\Big(\hat A \cap \hat B\Big)\, \tilde{\mathbb{P}}\Big(\tilde A \cap \tilde\tau_x \tilde B\Big) + \Big[\hat{\mathbb{P}}\Big(\hat A \cap \hat\tau_x \hat B\Big) - \hat{\mathbb{P}}\Big(\hat A \cap \hat B\Big)\Big]\, \tilde{\mathbb{P}}\Big(\tilde A \cap \tilde\tau_x \tilde B\Big)\,.
\end{aligned}$$

Using ergodicity, we find that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} \hat{\mathbb{P}}\Big(\hat A \cap \hat B\Big)\, \tilde{\mathbb{P}}\Big(\tilde A \cap \tilde\tau_x \tilde B\Big) \mathrm{d}x &= \hat{\mathbb{P}}\Big(\Big(\hat A \cap \hat B\Big)\Big)\, \tilde{\mathbb{P}}\Big(\tilde A \cap \tilde B\Big) \\
&= \mathbb{P}(A \cap B)\,.
\end{aligned} \tag{2.27}$$

Since $\hat\tau$ is mixing, we find for every $\varepsilon > 0$ some $R > 0$ such that $\|x\| > R$ implies

$$\left|\hat{\mathbb{P}}\Big(\hat A \cap \hat\tau_x \hat B\Big) - \hat{\mathbb{P}}\Big(\hat A \cap \hat B\Big)\right| < \varepsilon\,.$$

For $n > R$ we find

$$\begin{aligned}
&\frac{1}{(2n)^d} \int_{[-n,n]^d} \left|\hat{\mathbb{P}}\Big(\hat A \cap \hat\tau_x \hat B\Big) - \hat{\mathbb{P}}\Big(\hat A \cap \hat B\Big)\right|\, \tilde{\mathbb{P}}\Big(\tilde A \cap \tilde\tau_x \tilde B\Big) \\
&\qquad\qquad \le \frac{1}{(2n)^d} \int_{[-n,n]^d} \varepsilon + \frac{1}{(2n)^d} \int_{[-R,R]^d} 2 \to \varepsilon \quad \text{as } n \to \infty\,.
\end{aligned} \tag{2.28}$$

The last two limits (2.27) and (2.28) imply (2.25). □

Remark 2.23. The above proof heavily relies on the mixing property of $\hat\tau$. Note that for $\hat\tau$ being only ergodic, the statement is wrong, as can be seen from the product of two periodic processes in $\mathbb{T} \times \mathbb{T}$ (see Remark 2.21). Here, the invariant sets are given by $I_A := \{((y+x) \mod \mathbb{T}\,,\, x)\ :\ y \in A\}$ for arbitrary measurable $A \subset \mathbb{T}$.

2.7 Random Measures and Palm Theory

We recall some facts from random measure theory (see [8]) which will be needed for homogenization. Let $\mathfrak{M}(\mathbb{R}^d)$ denote the space of locally bounded Borel measures on $\mathbb{R}^d$ (i.e. bounded on every bounded Borel-measurable set) equipped with the Vague topology, which is generated by the sets

$$\left\{\mu\,:\,\int f\,\mathrm{d}\mu\in A\right\}\text{ for every open }A\subset\mathbb{R}^d\text{ and }f\in C_c\big(\mathbb{R}^d\big)\,.$$

This topology is metrizable, complete and countably generated. However, note that it is not locally compact, which implies that the Alexandroff compactification cannot be applied. A random measure is a measurable mapping

$$\mu_\bullet\,:\,\Omega\to\mathfrak{M}(\mathbb{R}^d)\,,\qquad\omega\mapsto\mu_\omega$$

which is equivalent to both of the following conditions

1 For every bounded Borel set $A\subset\mathbb{R}^d$ the map $\omega\mapsto\mu_\omega(A)$ is measurable

2 For every $\omega\mapsto\int f\mathrm{d}\mu_\omega$ the map $\omega\mapsto\int f\,\mathrm{d}\mu_\omega$ is measurable.

A random measure is stationary if the distribution of $\mu_\omega(A)$ is invariant under translations of A that is $\mu_\omega(A)$ and $\mu_\omega(A+x)$ share the same distribution. From stationarity of μ_ω one concludes the existence ([14, 31] and references therein) of a dynamical system $(\tau_x)_{x\in\mathbb{R}^d}$ on Ω such that $\mu_\omega\left(A+x\right)=\mu_{\tau_x\omega}\left(A\right)$. By a deep theorem due to Mecke (see [28, 8]) the measure

$$\mu_{\mathcal{P}}(A)=\int_\Omega\int_{\mathbb{R}^d}g(s)\,\chi_A(\tau_s\omega)\,\mathrm{d}\mu_\omega(s)\,\mathrm{d}\mathbb{P}(\omega)$$

can be defined on Ω for every positive $g\in L^1(\mathbb{R}^d)$ with compact support. $\mu_{\mathcal{P}}$ is independent from g and in case $\mu_\omega=\mathcal{L}$ we find $\mu_{\mathcal{P}}=\mathbb{P}$. Furthermore, for every $\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\Omega)$-measurable non negative or $\mu_{\mathcal{P}}\times\mathcal{L}$- integrable functions f the Campbell formula

$$\int_\Omega\int_{\mathbb{R}^d}f(x,\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega)=\int_{\mathbb{R}^d}\int_\Omega f(x,\omega)\,\mathrm{d}\mu_{\mathcal{P}}(\omega)\,\mathrm{d}x$$

holds. The measure μ_ω has finite intensity if $\mu_{\mathcal{P}}(\Omega)<+\infty$.

We denote by

$$\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I}):=\int_\Omega f\text{ the expectation of }f\text{ w.r.t. the }\sigma\text{-algebra }\mathscr{I}\text{ and }\mu_{\mathcal{P}}\,.\tag{2.29}$$

For random measures we find a more general version of Theorem 2.19.

Theorem 2.24 (Ergodic Theorem [8] 12.2.VIII)**.** *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, $(A_n)_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be a convex averaging sequence, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f\,:\,\Omega\to\mathbb{R}$ be measurable with $\int_\Omega|f|\,\mathrm{d}\mu_{\mathcal{P}}<\infty$. Then for $\mathbb{P}$-almost all $\omega\in\Omega$*

$$|A_n|^{-1}\int_{A_n}f(\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\to\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I})\,.\tag{2.30}$$

Given a bounded open (and convex) set $\mathbf{Q}\subset\Omega$, it is not hard to see that the following generalization holds:

Theorem 2.25 (General Ergodic Theorem)**.** *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, $\mathbf{Q}\subset\mathbb{R}^d$ be a convex bounded open set with $0\in\mathbf{Q}$, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f:\,\Omega\to\mathbb{R}$ be measurable with $\int_\Omega |f|\,\mathrm{d}\mu_{\mathcal{P}}<\infty$. Then for $\mathbb{P}$-almost all $\omega\in\Omega$ it holds*

$$\forall\varphi\in C(\overline{\mathbf{Q}}):\quad n^{-d}\int_{n\mathbf{Q}}\varphi(\frac{x}{n})f(\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\to\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I})\int_{\mathbf{Q}}\varphi\,. \tag{2.31}$$

Sketch of proof. Chose a countable family of characteristic functions that spans $L^1(\mathbf{Q})$. Use a Cantor argument and Theorem 2.24 to prove the statement for a countable dense family of $C(\overline{\mathbf{Q}})$. From here, we conclude by density.

The last result can be used to prove the most general ergodic theorem which we will use in this work: □

Theorem 2.26 (General Ergodic Theorem for the Lebesgue measure)**.** *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, $\mathbf{Q}\subset\mathbb{R}^d$ be a convex bounded open set with $0\in\mathbf{Q}$, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f\in L^p(\Omega;\mu_{\mathcal{P}})$ and $\varphi\in L^q(\mathbf{Q})$, where $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then for $\mathbb{P}$-almost all $\omega\in\Omega$ it holds*

$$n^{-d}\int_{n\mathbf{Q}}\varphi(\frac{x}{n})f(\tau_x\omega)\,\mathrm{d}x\to\mathbb{E}(f)\int_{\mathbf{Q}}\varphi\,.$$

Proof. Let $\varphi_\delta\in C(\overline{\mathbf{Q}})$ with $\|\varphi-\varphi_\delta\|_{L^q(\mathbf{Q})}<\delta$. Then

$$\begin{aligned}&\left|n^{-d}\int_{n\mathbf{Q}}\varphi(\frac{x}{n})f(\tau_x\omega)\,\mathrm{d}x-\mathbb{E}(f)\int_{\mathbf{Q}}\varphi\right|\\&\qquad\leq\|\varphi-\varphi_\delta\|_{L^q(\mathbf{Q})}\left(n^{-d}\int_{n\mathbf{Q}}|f(\tau_x\omega)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\&\qquad\quad+\left|n^{-d}\int_{n\mathbf{Q}}\varphi_\delta(x)f(\tau_x\omega)\,\mathrm{d}x-\mathbb{E}(f)\int_{\mathbf{Q}}\varphi_\delta\right|+\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I})\int_{\mathbf{Q}}|\varphi-\varphi_\delta|\,,\end{aligned}$$

which implies the claim. □

2.8 Random Sets

The theory of random measures and the theory of random geometry are closely related. In what follows, we recapitulate those results that are important in the context of the theory developed below and shed some light on the correlations between random sets and random measures.

Let $\mathfrak{F}(\mathbb{R}^d)$ denote the set of all closed sets in $\mathbb{R}^d$. We write

$$\mathfrak{F}_V:=\ \left\{F\in\mathfrak{F}(\mathbb{R}^d)\ :\ F\cap V\neq\emptyset\right\}\quad\text{if } V\subset\mathbb{R}^d\quad\text{is an open set}\,, \tag{2.32}$$

$$\mathfrak{F}^K:=\ \left\{F\in\mathfrak{F}(\mathbb{R}^d)\ :\ F\cap K=\emptyset\right\}\quad\text{if } K\subset\mathbb{R}^d\quad\text{is a compact set}\,. \tag{2.33}$$

The *Fell-topology* $\mathscr{T}_F$ is created by all sets $\mathfrak{F}_V$ and $\mathfrak{F}^K$ and the topological space $(\mathfrak{F}(\mathbb{R}^d),\mathscr{T}_F)$ is compact, Hausdorff and separable[27].

Remark 2.27. We find for closed sets F_n,F in $\mathbb{R}^d$ that $F_n\to F$ if and only if [27]

1 for every $x\in F$ there exists $x_n\in F_n$ such that $x=\lim_{n\to\infty}x_n$ and

2 if F_{n_k} is a subsequence, then every convergent sequence x_{n_k} with $x_{n_k} \in F_{n_k}$ satisfies $\lim_{k\to\infty} x_{n_k} \in F$.

If we restrict the Fell-topology to the compact sets $\mathfrak{K}(\mathbb{R}^d)$ it is equivalent with the Hausdorff topology given by the Hausdorff distance

$$\mathrm{d}(A,B) = \max\left\{\sup_{y\in B}\inf_{x\in A}|x-y|\ ,\ \sup_{x\in A}\inf_{y\in B}|x-y|\right\}\,.$$

Remark 2.28. For $A\subset\mathbb{R}^d$ closed, the set

$$\mathfrak{F}(A) := \left\{F\in\mathfrak{F}(\mathbb{R}^d)\ :\ F\subset A\right\}$$

is a closed subspace of $\mathfrak{F}\big(\mathbb{R}^d\big)$. This holds since

$$\mathfrak{F}\big(\mathbb{R}^d\big)\backslash\mathfrak{F}(A) = \left\{B\in\mathfrak{F}\big(\mathbb{R}^d\big)\ :\ B\cap\big(\mathbb{R}^d\backslash A\big)\neq\emptyset\right\} = \mathfrak{F}_{\mathbb{R}^d\backslash A}\quad\text{is open.}$$

.

Lemma 2.29 (Continuity of geometric operations)**.** *The maps* $\tau_x\,:\ A\mapsto A+x$ *and* $b_\delta\,:\ A\mapsto\overline{\mathbb{B}_\delta(A)}$ *are continuous in* $\mathfrak{F}\left(\mathbb{R}^d\right)$*.*

Proof. We show that preimages of open sets are open. For open sets V we find

$$\begin{aligned}\tau_x^{-1}\left(\mathfrak{F}_V\right) &= \left\{F\in\mathfrak{F}(\mathbb{R}^d)\ :\ \tau_xF\cap V\neq\emptyset\right\} = \left\{F\in\mathfrak{F}(\mathbb{R}^d)\ :\ F\cap\tau_{-x}V\neq\emptyset\right\} = \mathfrak{F}_{\tau_{-x}V}\,,\\ b_\delta^{-1}\left(\mathfrak{F}_V\right) &= \left\{F\in\mathfrak{F}(\mathbb{R}^d)\ :\ \overline{\mathbb{B}_\delta(F)}\cap V\neq\emptyset\right\} = \left\{F\in\mathfrak{F}(\mathbb{R}^d)\ :\ F\cap\mathbb{B}_\delta(V)\neq\emptyset\right\} = \mathfrak{F}_{(b_\delta V)^\circ}\,.\end{aligned}$$

The calculations for $\tau_x^{-1}\big(\mathfrak{F}^K\big) = \mathfrak{F}^{\tau_{-x}K}$ and $b_\delta^{-1}\big(\mathfrak{F}^K\big) = \mathfrak{F}^{b_\delta K}$ are analogue. □

Remark 2.30. The *Matheron-σ-field* $\sigma_{\mathfrak{F}}$ is the Borel-σ-algebra of the Fell-topology and is fully characterized either by the class $\mathfrak{F}_V$ of $\mathfrak{F}^K$.

Definition 2.31 (Random closed / open set according to Choquet (see [27] for more details))**.**

a) Let $(\Omega,\sigma,\mathbb{P})$ be a probability space. Then a *Random Closed Set (RACS)* is a measurable mapping
$$A:(\Omega,\sigma,\mathbb{P})\longrightarrow(\mathfrak{F},\sigma_{\mathfrak{F}})$$

b) Let τ_x be a dynamical system on Ω. A random closed set is called stationary if its characteristic functions $\chi_{A(\omega)}$ are stationary, i.e. they satisfy $\chi_{A(\omega)}(x) = \chi_{A(\tau_x\omega)}(0)$ for almost every $\omega\in\Omega$ for almost all $x\in\mathbb{R}^d$. Two random sets are jointly stationary if they can be parameterized by the same probability space such that they are both stationary.

c) A random closed set $\Gamma\,:\,(\Omega,\sigma,P)\longrightarrow(\mathfrak{F},\sigma_{\mathfrak{F}})\quad\omega\mapsto\Gamma(\omega)$ is called a *Random closed C^k-Manifold* if $\Gamma(\omega)$ is a piece-wise C^k-manifold for P almost every ω.

d) A measurable mapping
$$A:(\Omega,\sigma,\mathbb{P})\longrightarrow(\mathfrak{F},\sigma_{\mathfrak{F}})$$
is called *Random Open Set (RAOS)* if $\omega\mapsto\mathbb{R}^d\backslash A(\omega)$ is a RACS.

The importance of the concept of random geometries for stochastic homogenization stems from the following Lemma by Zähle. It states that every random closed set induces a random measure. Thus, every stationary RACS induces a stationary random measure.

Lemma 2.32 ([38] Theorem 2.1.3 resp. Corollary 2.1.5)**.** *Let $\mathfrak{F}_m \subset \mathfrak{F}$ be the space of closed m-dimensional sub manifolds of $\mathbb{R}^d$ such that the corresponding Hausdorff measure is locally finite. Then, the σ-algebra $\sigma_{\mathfrak{F}} \cap \mathfrak{F}_m$ is the smallest such that*

$$M_B : \mathfrak{F}_m \to \mathbb{R} \quad M \mapsto \mathcal{H}^m(M \cap B)$$

is measurable for every measurable and bounded $B \subset \mathbb{R}^d$.

This means that

$$M_{\mathbb{R}^d} : \mathfrak{F}_m \to \mathfrak{M}(\mathbb{R}^d) \quad M \mapsto \mathcal{H}^m(M \cap \cdot)$$

is measurable with respect to the σ-algebra created by the Vague topology on $\mathfrak{M}(\mathbb{R}^d)$. Hence a random closed set always induces a random measure. Based on Lemma 2.32 and on Palm-theory, the following useful result was obtained in [14] (See Lemma 2.14 and Section 3.1 therein).

Theorem 2.33. *Let (Ω, σ, P) be a probability space with an ergodic dynamical system τ. Let $A : (\Omega, \sigma, P) \longrightarrow (\mathfrak{F}, \sigma_{\mathfrak{F}})$ be a stationary random closed m-dimensional C^k-Manifold.*

a) There exists a separable metric space $\tilde{\Omega} \subset \mathfrak{M}(\mathbb{R}^d)$ with an ergodic dynamical system $\tilde{\tau}$ and a mapping $\tilde{A} : (\tilde{\Omega}, \mathcal{B}_{\tilde{\Omega}}, \mathbb{P}) \to (\mathfrak{F}, \sigma_{\mathfrak{F}})$ such that A and $\tilde{A}$ have the same law and such that $\tilde{A}$ still is stationary. Furthermore, $(x, \omega) \mapsto \tau_x\omega$ is continuous. We identify $\tilde{\Omega} = \Omega$, $\tilde{A} = A$ and $\tilde{\tau} = \tau$.

b) The mapping

$$\mu_\bullet : \Omega \to \mathfrak{M}(\mathbb{R}^d), \quad \omega \mapsto \mu_\omega(\cdot) := \mathcal{H}^m(M \cap \cdot)$$

is a stationary random measure on $\mathbb{R}^d$ and there exists a corresponding Palm-measure $\mu_{\mathcal{P}}$ if and only if $\mu_\bullet$ has finite intensity.

c) There exists a measurable set $\hat{A} \subset \Omega$, called the prototype *of A, such that $\chi_{A(\omega)}(x) = \chi_{\hat{A}}(\tau_x\omega)$ for $\mathcal{L} + \mu_\omega$-almost every x and $\mathbb{P}$-almost surely. The Palm-measure $\mu_{\mathcal{P}}$ of μ_ω concentrates on $\hat{A}$, i.e. $\mu_{\mathcal{P}}(\Omega \backslash \hat{A}) = 0$.*

d) If A is a random closed m-dimensional C^k-manifold, then $\mathbb{P}(\hat{A}) = 0$.

Also the following result will be useful below.

Lemma 2.34. *Let μ be a Radon measure on $\mathbb{R}^d$ and let $\mathbf{Q} \subset \mathbb{R}^d$ be a bounded open set. Let $\mathfrak{F}_0 \subset \mathfrak{F}\left(\overline{\mathbf{Q}}\right)$ be such that $\mathfrak{F}_0 \to \mathbb{R}$, $A \mapsto \mu(A)$ is continuous. Then*

$$m : \mathfrak{F} \times \mathfrak{F}_0 \to \mathfrak{M}(\mathbb{R}^d), \quad (P, B) \mapsto \begin{cases} A \mapsto \mu(A \cap B) & B \subset P \\ 0 & \textit{else} \end{cases}$$

is measurable.

Proof. For $f \in C_c(\mathbb{R}^d)$ we introduce m_f through

$$m_f : (P, B) \mapsto \begin{cases} \int_B f \,\mathrm{d}\mu & B \subset P \\ 0 & \text{else} \end{cases}$$

and observe that m is measurable if and only if for every $f \in C_c(\mathbb{R}^d)$ the map m_f is measurable (see Section 2.7). Hence, if we prove the latter property, the lemma is proved.

We assume $f \geq 0$ and we show that the mapping m_f is even upper continuous. In particular, let $(P_n, B_n) \to (P, B)$ in $\mathfrak{F} \times \mathfrak{F}_0$ and assume that $B_n \subset P_n$ for all $n > N_0$. Since $\overline{\mathbf{Q}}$ is compact, Remark 2.27. 2. implies that $B \subset P \cap \overline{\mathbf{Q}}$. Furthermore, since f has compact support, we find $\left|\int_{B_n} f \,\mathrm{d}\mu - \int_B f \,\mathrm{d}\mu\right| \leq \|f\|_\infty |\mu(B_n) - \mu(B)| \to 0$. On the other hand, if there exists a subsequence such that $B_n \not\subset P_n$ for all n, then either $B \not\subset P$ and $m_f(P_n, B_n) = 0 \to m_f(P, B) = 0$ or $B \subset P$ and $0 = \lim_{n\to\infty} m_f(P_n, B_n) \leq \int_B f \mathrm{d}\mu = m_f(P, B)$. For $f \leq 0$ we obtain lower semicontinuity and for general f the map m_f is the sum of an upper and a lower semicontinuous map, hence measurable. □

2.9 Point Processes

Definition 2.35 ((Simple) point processes)**.** A $\mathbb{Z}$-valued random measure μ_ω is called point process. In what follows, we consider the particular case that for almost every ω there exist points $(x_k(\omega))_{k\in\mathbb{N}}$ and values $(a_k(\omega))_{k\in\mathbb{N}}$ in $\mathbb{Z}$ such that

$$\mu_\omega = \sum_{k\in\mathbb{N}} a_k \delta_{x_k(\omega)} \,.$$

The point process μ_ω is called simple if almost surely for all $k \in \mathbb{N}$ it holds $a_k \in \{0, 1\}$.

Example 2.36 (Poisson process)**.** A particular example for a stationary point process is the Poisson point process $\mu_\omega = \mathbb{X}_\omega$ with intensity λ. Here, the probability $\mathbb{P}(\mathbb{X}(A) = n)$ to find n points in a Borel-set A with finite measure is given by a Poisson distribution

$$\mathbb{P}(\mathbb{X}(A) = n) = e^{-\lambda|A|} \frac{\lambda^n |A|^n}{n!} \tag{2.34}$$

with expectation $\mathbb{E}(\mathbb{X}(A)) = \lambda |A|$. The last formula implies that the Poisson point process is stationary.

We can use a given random point process to construct further processes.

Example 2.37 (Hard core Matern process)**.** The hard core Matern process is constructed from a given point process $\mathbb{X}_\omega$ by mutually erasing all points with the distance to the nearest neighbor smaller than a given constant r. If the original process $\mathbb{X}_\omega$ is stationary (ergodic), the resulting hard core process is stationary (ergodic) respectively.

Example 2.38 (Hard core Poisson–Matern process)**.** If a Matern process is constructed from a Poisson point process, we call it a Poisson–Matern point process.

Lemma 2.39. *Let μ_ω be a simple point process with $a_k = 1$ almost surely for all $k \in \mathbb{N}$. Then $\mathbb{X}_\omega = (x_k(\omega))_{k\in\mathbb{N}}$ is a random closed set. On the other hand, if $\mathbb{X}_\omega = (x_k(\omega))_{k\in\mathbb{N}}$ is a random closed set that almost surely has no limit points then μ_ω is a point process.*

Proof. Let μ_ω be a point process. For open $V \subset \mathbb{R}^d$ and compact $K \subset \mathbb{R}^d$ let

$$f_{V,R}(x) = \mathrm{dist}\big(x, \mathbb{R}^d \backslash (V \cap \mathbb{B}_R(0))\big) , \qquad f_\delta^K(x) = \max\left\{1 - \frac{1}{\delta}\mathrm{dist}(x, K) , 0\right\} .$$

Then $f_{V,R}$ is Lipschitz with constant 1 and f_δ^K is Lipschitz with constant $\frac{1}{\delta}$ and support in $\mathbb{B}_\delta(K)$. Moreover, since μ_ω is locally bounded, the number of points x_k that lie within $\mathbb{B}_1(K)$ is bounded. In particular, we obtain

$$\begin{aligned}\mathbb{X}^{-1}(\mathfrak{F}_V) &= \bigcup_{R>0}\left\{\omega\,:\,\int_{\mathbb{R}^d} f_{V,R}\,\mathrm{d}\mu_\omega > 0\right\}\,,\\ \mathbb{X}^{-1}\big(\mathfrak{F}^K\big) &= \bigcap_{\delta>0}\left\{\omega\,:\,\int_{\mathbb{R}^d} f_\delta^K\,\mathrm{d}\mu_\omega > 0\right\}\,,\end{aligned}$$

are measurable. Since $\mathfrak{F}_V$ and $\mathfrak{F}^K$ generate the σ-algebra on $\mathfrak{F}\left(\mathbb{R}^d\right)$, it follows that $\omega \to \mathbb{X}_\omega$ is measurable.

In order to prove the opposite direction, let $\mathbb{X}_\omega = (x_k(\omega))_{k\in\mathbb{N}}$ be a random closed set of points. Since $\mathbb{X}_\omega$ has almost surely no limit points the measure μ_ω is locally bounded almost surely. We prove that μ_ω is a random measure by showing that

$$\forall f\in C_c\big(\mathbb{R}^d\big)\;:\qquad F\,:\,\omega\mapsto\int_{\mathbb{R}^d} f\,\mathrm{d}\mu_\omega \text{ is measurable.}$$

For $\delta>0$ let $\mu_\omega^\delta(A) := \left(\left|\mathbb{S}^{d-1}\right|\delta^d\right)^{-1}\mathcal{L}(A\cap\mathbb{B}_\delta(\mathbb{X}_\omega))$. By Lemmas 2.29 and 2.34 we obtain that $F_\delta\,:\,\omega\mapsto\int_{\mathbb{R}^d} f\,\mathrm{d}\mu_\omega^\delta$ are measurable. Moreover, for almost every ω we find $F_\delta(\omega)\to F(\omega)$ uniformly and hence F is measurable. ☐

Corollary 2.40. *A random simple point process μ_ω is stationary iff $\mathbb{X}_\omega$ is stationary.*

Hence we can provide the following definition based on Definition 2.31.

Definition 2.41. A point process μ_ω and a random set $\mathbf{P}$ are jointly stationary if $\mathbf{P}$ and $\mathbb{X}$ are jointly stationary.

Lemma 2.42. *Let $\mathbb{X}_\omega = (x_i)_{i\in\mathbb{N}}$ be a Matern point process from Example 2.37 with distance r and let for $\delta<\frac{r}{2}$ be $\mathbf{B}(\omega) := \bigcup_i \overline{\mathbb{B}\delta x_i}$. Then $\mathbf{B}(\omega)$ is a random closed set.*

Proof. This follows from Lemma 2.29: $\mathbb{X}_\omega$ is measurable and $\mathbb{X}\mapsto\overline{\mathbb{B}_\delta(\mathbb{X})}$ is continuous. Hence $\mathbf{B}(\omega)$ is measurable. ☐

2.10 Unoriented Graphs on Point Processes

Definition 2.43 ((Unoriented) Graph)**.** Let $\mathbb{X} = (x_i)_{i\in\mathbb{N}}\subset\mathbb{R}^d$ be a countable set of points. A graph $(\mathbb{G},\mathbb{X})$ on $\mathbb{X}$ (or simply $\mathbb{G}$ on $\mathbb{X}$) is a subset $\mathbb{G}\subset\mathbb{X}^2$. The graph $\mathbb{G}$ is unoriented if $(x,y)\in\mathbb{G}$ implies $(y,x)\in\mathbb{G}$. For $(x,y)\in\mathbb{G}$ we write $x\sim y$.

Elements of $\mathbb{G}$ are usually referred to as *edges*. Classically, a graph consists of vertices $\mathbb{X}$ and edges $\mathbb{G}$, so the graph is given through $(\mathbb{G},\mathbb{X})$. However, in this work the set of points $\mathbb{X}$ will usually be given and we will mostly discuss the properties of $\mathbb{G}$. This is why we adopt standard notations.

Definition 2.44 (Paths and connected graphs)**.** Let $\mathbb{X} = (x_i)_{i\in\mathbb{N}}\subset\mathbb{R}^d$ be a countable set of points with a graph $\mathbb{G}\subset\mathbb{X}^2$. A *path* in $\mathbb{X}$ is a sorted family of points $(y_1,\dots,y_N)\in\mathbb{X}^N$, $N\in\mathbb{N}$, such that for every $k\in\{1,\dots,N-1\}$ it holds $y_k\sim y_{k+1}$. The family of all paths in $\mathbb{X}$ is hence a subset of $\bigcup_{N\in\mathbb{N}}\mathbb{X}^N$. The graph $\mathbb{G}$ is said to be *connected* if for every $x,y\in\mathbb{G}$, $x\neq y$, there exists $N>2$ and a path $(y_1,\dots,y_N)\in\mathbb{X}^N$ such that $y_1=x$ and $y_N=y$.

Remark 2.45. Let $(y_1,\dots,y_k)$ with be a path from y_1 to y_k . A path from y_k to y_1 is given by reversing the order, i.e. by $(y_k,\dots,y_1)$.

Definition 2.46 (Local extrema on graphs)**.** Let $\mathbb{X}\subset\mathbb{R}^d$ be a countable set of points with a graph $\mathbb{G}$. A function $u:\ A\subset\mathbb{X}\to\mathbb{R}$ has a *local maximum* resp. *minimum* in $y\in A$ if for all $\tilde y\in A$ with $\tilde y\sim y$ it holds $u(y)\geq u(\tilde y)$ resp. $u(y)\leq u(\tilde y)$

2.11 Dynamical Systems on $\mathbb{Z}^d$

Definition 2.47. Let $\left(\hat\Omega,\hat{\mathscr{F}},\hat{\mathbb{P}}\right)$ be a probability space. A discrete dynamical system on $\hat\Omega$ is a family $(\hat\tau_z)_{z\in r\mathbb{Z}^d}$ of measurable bijective mappings $\hat\tau_z:\hat\Omega\mapsto\hat\Omega$ satisfying (i)-(iii) of Definition 2.15. A set $A\subset\hat\Omega$ is almost invariant if for every $z\in r\mathbb{Z}^d$ it holds $\mathbb{P}\left((A\cup\hat\tau_zA)\setminus(A\cap\hat\tau_zA)\right)=0$ and $\hat\tau$ is called ergodic w.r.t. $r\mathbb{Z}^d$ if every almost invariant set has measure 0 or 1.

Similar to the continuous dynamical systems, also in this discrete setting an ergodic theorem can be proved.

Theorem 2.48 (See Krengel and Tempel'man [25, 36])**.** *Let $(A_n)_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be a convex averaging sequence, let $(\hat\tau_z)_{z\in r\mathbb{Z}^d}$ be a dynamical system on $\hat\Omega$ with invariant σ-algebra $\mathscr{I}$ and let $f:\hat\Omega\to\mathbb{R}$ be measurable with $|\mathbb{E}(f)|<\infty$. Then for almost all $\hat\omega\in\hat\Omega$*

$$|A_n|^{-1}\sum_{z\in A_n\cap r\mathbb{Z}^d}f(\hat\tau_z\hat\omega)\to r^{-d}\mathbb{E}(f|\mathscr{I})\,. \tag{2.35}$$

In the following, we restrict to $r=1$ for simplicity of notation.

Let $\Omega_0\subset\mathbb{R}^d$. We consider an enumeration $(\xi_i)_{i\in\mathbb{N}}$ of $\mathbb{Z}^d$ such that $\hat\Omega:=\Omega_0^{\mathbb{Z}^d}=\Omega_0^{\mathbb{N}}$ and write $\hat\omega=(\hat\omega_{\xi_1},\hat\omega_{\xi_2},\dots)=(\hat\omega_1,\hat\omega_2,\dots)$ for all $\hat\omega\in\hat\Omega$. We define a metric on $\hat\Omega$ through

$$d(\hat\omega_1,\hat\omega_2)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{|\hat\omega_{1,\xi_k}-\hat\omega_{2,\xi_k}|}{1+|\hat\omega_{1,\xi_k}-\hat\omega_{2,\xi_k}|}\,.$$

We write $\Omega_n:=\Omega_0^n$ and $\mathbb{N}_n:=\{k\in\mathbb{N}:\ k\geq n+1\}$. The topology of $\hat\Omega$ is generated by the open sets $A\times\Omega_0^{\mathbb{N}_n}$, where for some $n>0$, $A\subset\Omega_n$ is an open set. In case Ω_0 is compact, the space $\hat\Omega$ is compact. Further, $\hat\Omega$ is separable in any case since Ω_0 is separable (see [23]).

We consider the ring

$$\mathcal{R}=\bigcup_{n\in\mathbb{N}}\left\{A\times\Omega_0^{\mathbb{N}_n}\ :\ A\subset\Omega_n\text{ is measurable}\right\}$$

and suppose for every $n\in\mathbb{N}$ that there exists a probability measure $\mathbb{P}_n$ on Ω_n such that for every measurable $A_n\subset\Omega_n$ it holds $\mathbb{P}_{n+k}\big(A_n\times\Omega^k\big)=\mathbb{P}_n(A_n)$. Then we define

$$\mathbb{P}\big(A_n\times\Omega_0^{\mathbb{N}_n}\big):=\mathbb{P}_n(A_n)\,.$$

We make the observation that $\mathbb{P}$ is additive and positive on $\mathcal{R}$ and $\mathbb{P}(\emptyset)=0$. Next, let $(A_j)_{j\in\mathbb{N}}$ be an increasing sequence of sets in $\mathcal{R}$ such that $A:=\bigcup_j A_j\in\mathcal{R}$. Then, there exists $\tilde A_1\subset\Omega_0^n$ such that $A_1=\tilde A_1\times\Omega_0^{\mathbb{N}_n}$ and since $A_1\subset A_2\subset\dots\subset A$, for every $j>1$, we conclude $A_j=\tilde A_j\times\Omega_0^{\mathbb{N}_n}$ for some $\tilde A_j\subset\Omega_n$. Therefore, $\mathbb{P}(A_j)=\mathbb{P}_n(\tilde A_j)\to\mathbb{P}_n(\tilde A)=\mathbb{P}(A)$ where $A=\tilde A\times\Omega_0^{\mathbb{N}_n}$. We have thus proved that $\mathbb{P}:\mathcal{R}\to[0,1]$ can be extended to a measure on the Borel-σ-Algebra on Ω (See [3, Theorem 6-2]).

We define for $z\in\mathbb{Z}^d$ the mapping

$$\hat{\tau}_z:\,\hat{\Omega}\to\hat{\Omega}\,,\qquad \hat{\omega}\mapsto\hat{\tau}_z\hat{\omega}\,,\quad\text{where }\left(\hat{\tau}_z\hat{\omega}\right)_{\xi_i}=\hat{\omega}_{\xi_i+z}\text{ component wise}\,.$$

Remark 2.49. In this paper, we consider particularly $\Omega_0=\{0,1\}$. Then $\hat{\Omega}:=\Omega_0^{\mathbb{Z}^d}$ is equivalent to the power set of $\mathbb{Z}^d$ and every $\hat{\omega}\in\hat{\Omega}$ is a sequence of 0 and 1 corresponding to a subset of $\mathbb{Z}^d$. Shifting the set $\hat{\omega}\subset\mathbb{Z}^d$ by $z\in\mathbb{Z}^d$ corresponds to an application of $\hat{\tau}_z$ to $\hat{\omega}\in\hat{\Omega}$.

Now, let $\mathbf{P}(\omega)$ be a stationary ergodic random open set and let $r>0$. Recalling (2.1) the map $\omega\mapsto\mathbf{P}_{-r}(\omega)$ is measurable due to Lemma 2.29 and we can define $\mathbb{X}_r(\mathbf{P}(\omega)):=2r\mathbb{Z}^d\cap\mathbf{P}_{-\frac{r}{2}}(\omega)$.

Lemma 2.50. *If $\mathbf{P}$ is a stationary ergodic random open set then the set*

$$\mathbb{X}=\mathbb{X}_r(\omega):=\mathbb{X}_r(\mathbf{P}(\omega)):=2r\mathbb{Z}^d\cap\mathbf{P}_{-r}(\omega)\tag{2.36}$$

is a stationary random point process w.r.t. $2r\mathbb{Z}^d$.

Proof. By a simple scaling we can w.l.o.g. assume $2r=1$ and write $\mathbb{X}=\mathbb{X}_r$. Evidently, $\mathbb{X}$ corresponds to a process on $\mathbb{Z}^d$ with values in $\Omega_0=\{0,1\}$ writing $\mathbb{X}(z)=1$ if $z\in\mathbb{X}$ and $\mathbb{X}(z)=0$ if $z\notin\mathbb{X}$. In particular, we write $(\omega,z)\mapsto\mathbb{X}(\omega,z)$. This process is stationary as the shift invariance of $\mathbf{P}$ induces a shift-invariance of $\hat{\mathbb{P}}$ with respect to $\hat{\tau}_z$. It remains to observe that the probabilities $\mathbb{P}(\mathbb{X}(z)=1)$ and $\mathbb{P}(\mathbb{X}(z)=0)$ induce a random measure on $\hat{\Omega}$ in the way described in Remark 2.49. □

Remark 2.51. If $\mathbf{P}$ is mixing one can follow the lines of the proof of Lemma 2.22 to find that $\mathbb{X}_r(\mathbf{P}(\omega))$ is ergodic. However, in the general case $\mathbb{X}_r(\mathbf{P}(\omega))$ is not ergodic. This is due to the fact that by nature $(\tau_z)_{z\in\mathbb{Z}^d}$ on Ω has more invariant sets than$(\tau_x)_{x\in\mathbb{R}^d}$. For sufficiently complex geometries the map $\Omega\to\hat{\Omega}$ is onto.

Definition 2.52 (Jointly stationary)**.** We call a point process $\mathbb{X}$ with values in $2r\mathbb{Z}^d$ to be strongly jointly stationary with a random set $\mathbf{P}$ if the functions $\chi_{\mathbf{P}(\omega)}$, $\chi_{\mathbb{X}(\omega)}$ are strongly jointly stationary w.r.t. the dynamical system $(\tau_{2rx})_{x\in\mathbb{Z}^d}$ on Ω.

3 Periodic Extension Theorem

We study extension theorems on periodic geometries. In what follows, we assume that the torus is split into $\mathbb{T}=\mathbb{T}_1\cup\mathbb{T}_2$ and we denote $\mathbf{T}_1$ and $\mathbf{T}_2$ the periodic extensions of $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively. In order to get familiar with our approach, we first prove the following standard result, which was already obtained in [7] and generalized to $\mathbb{R}^d$ and $W^{1,p}(\mathbf{T}_1)$ in [20] (see also [22]).

Theorem 3.1 (Extension Theorem)**.** *Let $\mathbb{T}=\mathbb{T}_1\cup\mathbb{T}_2$ with $\mathbb{T}_2\subset\subset(0,1)^d$ compactly and such that $\partial\mathbb{T}_2$ is Lipschitz. Then, for every $p\in[1,\infty)$ there exists C depending only on $\mathbb{T}_2$, p and d such that for every $u\in W^{1,p}(Y_1)$:*

$$\int_{\mathbb{R}^d}\left|\tilde{\mathcal{U}}u\right|^p\leq C\int_{Y_1}|u|^p\,,\tag{3.1}$$

$$\int_{\mathbb{R}^d}\left|\nabla\left(\tilde{\mathcal{U}}u\right)\right|^p\leq C\int_{Y_1}|\nabla u|^p\,.\tag{3.2}$$

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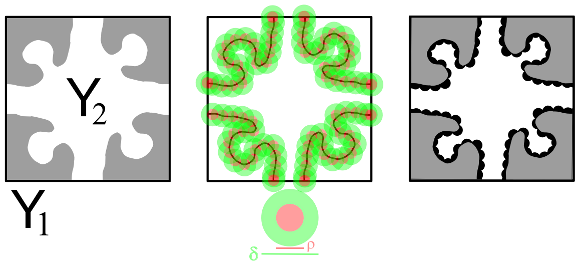

Figure 2: Left: The periodic geometry $\mathbf{T}_1$ and $\mathbf{T}_2$. Middle: The boarder $\partial\mathbf{T}_1$ is covered by balls of a uniform size such that on each center x_i there exists an extension operator from $\mathbf{T}_1 \cap \mathbb{B}_\delta(x_i)$ to $\mathbf{T}_2 \cap \mathbb{B}_\rho(x_i)$. Right: The microscopically glued extension operator maps functions with support $\mathbf{T}_1$ onto functions with support in the black and gray domain.

Proof. Since $\mathbb{T}_2 \subset\subset (0,1)^d$ one proves by contradiction the existence of $C > 0$ such that

$$\forall \varphi \in W^{1,p}((0,1)^d \backslash \mathbb{T}_2) \, : \qquad \int_{\mathbb{T}_1} |\varphi|^p \leq C \left(\int_{\mathbb{T}_1} |\nabla\varphi|^p + \left| \fint_{\mathbb{T}_1} \varphi \right| \right) . \tag{3.3}$$

In what follows we write $\overline{\varphi} = \fint_{\mathbb{T}_1} \varphi$. Since $\partial\mathbb{T}_2$ is Lipschitz, there exists a continuous operator $\tilde{\mathcal{U}}$: $W^{1,p}((0,1)^d \backslash \mathbb{T}_2) \to W^{1,p}((0,1)^d)$. Due to (3.3) it holds

$$\begin{aligned}
\int_{\mathbb{T}} |\mathcal{U}\,(u - \overline{u}) + \overline{u}|^p &\leq C \int_{\mathbb{T}_1} |u|^p \,, \\
\int_{\mathbb{T}_2} |\nabla\,(\mathcal{U}\,(u - \overline{u}) + \overline{u})|^p &= \int_{\mathbb{T}_2} |\nabla\mathcal{U}\,(u - \overline{u})|^p \\
&\leq C \left(\int_{\mathbb{T}_1} |u - \overline{u}|^p + \int_{\mathbb{T}_1} |\nabla\,(u - \overline{u})|^p \right) \\
&\leq C \int_{\mathbb{T}_1} |\nabla u|^p \,.
\end{aligned}$$

For $u \in W^{1,p}(\mathbf{T}_1)$ and $k \in \mathbb{Z}^d$, we define $\mathcal{U}$ on $\mathbb{R}^d$ by applying it locally on every cell $I_k := k + [0,1)^d$. Hence $\mathcal{U}$ satisfies (3.1)–(3.2). □

The last proof heavily relied on the disconnectedness of $\mathbf{T}_2$. In case $\mathbf{T}_2$ is connected, the "gluing" of the local extensions is more delicate.

Theorem 3.2. *Let $\mathbb{T} = \mathbb{T}_1 \cup \mathbb{T}_2$ such that ,$\partial\mathbf{T}_1$ is locally Lipschitz. Then there exist an extension operator*

$$\mathcal{U} : \, W^{1,p}(\mathbf{T}_1) \to W^{1,p}(\mathbb{R}^d)$$

such that for some $C > 0$ depending only on δ and p it holds

$$\int_{\mathbb{R}^d} |\mathcal{U}u|^p \leq C \int_{\mathbf{T}_1} |u|^p \,, \tag{3.4}$$

$$\int_{\mathbb{R}^d} |\nabla\,(\mathcal{U}u)|^p \leq C \int_{\mathbf{T}_1} |\nabla u|^p \,. \tag{3.5}$$

Idea of Proof: In order to highlight the structure of the following proof, let us explain how the extension operator is constructed. In Figure 2 we see on the left a Lipschitz surface $\partial\mathbf{T}_1$ with maximal Lipschitz constant M, which can be locally covered by balls of radius $\rho = \delta\sqrt{4M^2+2}^{-1}$ (middle). Using the extension operators given by Lemma 2.2, we can extend u to the red balls that intersect $\mathbf{T}_2$. The extension operators on the various red balls are then glued together using a suitable partition of unity. However, this leads to steep gradients in the black region on the right hand side, while $\mathcal{U}u \equiv 0$ in the white region. In particular, if $u(x) \equiv c$ is locally constant, these gradients are of order $\frac{c}{\rho}$. Hence, proceeding globally in this way, the gradient $\nabla\left(\tilde{\mathcal{U}}u\right)$ cannot be bounded by ∇u.

To avoid this problem, in Step 2 we use a mesoscopic correction: Writing $K_\alpha := (-\alpha, 1+\alpha)^d$, and $K_\alpha(z) = z + K_\alpha$ for $z \in \mathbb{Z}^d$ with a partition of unity $\tilde{\eta}_z$ and the local extension operator $\mathcal{U}_z$ on $K_\alpha(z)$, we define the global extension operator through:

$$\mathcal{U}u := \sum_{z\in\mathbb{Z}^d} \tilde{\eta}_z \Big(\tilde{\mathcal{U}}_z(u - \tau_z u) + \tau_z u\Big) \tag{3.6}$$

where $\tau_z u = ⨍_{B(z)} u$ for some suitable ball $B(z)$. By this, we assign to the void space an averaged value of the surrounding matrix. In Step 2 we heavily rely on the periodicity, which allows to apply a $\mathbb{T}$-periodic partitioning to $\mathbb{R}^d$.

Proof. **Step 1 (Local extension operator on $(0,1)^d$):** W.l.o.g. we can assume that $\delta \ll 1$. Writing $K_\alpha := (-\alpha, 1+\alpha)^d$ the set $\partial\mathbf{T}_1 \cap K_\delta$ is precompact and can be covered by a finite number of balls $B_{\rho/2}(x_k)$, where $\rho = \delta\sqrt{4M^2+2}^{-1}$ and $(x_k)_{k=1,\dots,K} \subset \partial\mathbf{T}_1 \cap K_\delta$.

In what follows, let $\eta \in C_0^\infty(-1,1)$ be a positive symmetric smooth function with $0 < \eta(x) \le 1$ on $(-1,1)$, $\eta(0) = 1$ and monotone on $(0,1)$. We denote $\eta_0 := \eta \circ \mathrm{dist}(\,\cdot\,, \partial\mathbf{T}_1\,)$ and $\eta_k(x) := \eta(\rho^{-1}|x - x_k|)$ for $k \ge 1$. In what follows we identify η_k with their periodized versions. For every $k \ge 0$ let $\tilde{\eta}_k = \left(\sum_{j=0}^\infty \eta_j\right)^{-1} \eta_k$ and note that $\tilde{\eta}_k$ defines a partition of unity on $\partial\mathbf{T}_1 \cap K_\delta$. Writing $\mathcal{U}_i$ for the corresponding extension operator from Lemma 2.2 on $B_\rho(x_i)$, we extend u by 0 to $\mathbb{R}^d \backslash \mathbf{T}_1$ and consider

$$\begin{aligned} \tilde{\mathcal{U}} :\ & W^{1,p}(K_{2\delta} \cap \mathbf{T}_1) \to W^{1,p}(K_\delta) \\ & \tilde{\mathcal{U}}u := \sum_{i\in\mathbb{N}} \tilde{\eta}_i \mathcal{U}_i u + \eta_0 u\,. \end{aligned} \tag{3.7}$$

For the following calculation, we further note that

$$\nabla\tilde{\eta}_k = \left(\sum_{j=0}^\infty \eta_j\right)^{-1} \nabla\eta_k - \left(\sum_{j=0}^\infty \eta_j\right)^{-2} \eta_k \sum_{j=0}^\infty \nabla\eta_j$$

$$\text{and } 1 \le \sum_{j=0}^\infty \eta_j \le \hat{N} \text{ as well as } \left|\sum_{j=0}^\infty \nabla\eta_j\right| \le \hat{N}\,\|\nabla\eta\|_\infty\,,$$

for some $\hat{N}$ depending only on the dimension d. Let $\tilde{\boldsymbol{B}} := \{B_\rho(x_k)\}$. For every $i \in \{1,\dots,k\}$, the number $\#\left\{\tilde{B}_j \in \tilde{\boldsymbol{B}} \mid \tilde{B}_j \cap \tilde{B}_i \neq \emptyset\right\}$ of balls in $\tilde{\boldsymbol{B}}$ intersecting with $\tilde{B}_i$ is bounded by $\hat{N}$. On each ball we infer from Lemma 2.2

$$\int_{B_i} |\tilde{\eta}_i \mathcal{U}_i u|^p \le 7 \int_{B_\delta(x_i)\cap\mathbf{T}_1} |u|^p\,,$$

$$\int_{B_i} |\nabla(\tilde{\eta}_i \mathcal{U}_i u)|^p \le 7\,\|\nabla\tilde{\eta}\|_\infty^p \int_{B_\delta(x_i)\cap P} |u|^p + 14M \int_{B_\delta(x_i)\cap\mathbf{T}_1} |\nabla u|^p\,.$$

Similar estimates also hold for $\eta_0 u$ and summing over i, we obtain

$$\int_{K_\delta} \left|\tilde{\mathcal{U}}u\right|^p \leq 7\hat{N} \int_{K_{2\delta}\cap\mathbf{T}_1} |u|^p \,, \tag{3.8}$$

$$\int_{K_\delta} \left|\nabla\left(\tilde{\mathcal{U}}u\right)\right|^p \leq 7\hat{N}\frac{1}{\rho^p} \int_{K_{2\delta}\cap\mathbf{T}_1} |u|^p + 14M\hat{N} \int_{K_{2\delta}\cap\mathbf{T}_1} |\nabla u|^p \,. \tag{3.9}$$

Now let $B \subset (2\delta, 1-2\delta)^d \cap T_1$ be a ball with positive radius. By a contradiction argument, we obtain

$$\int_{K_{2\delta}\cap\mathbf{T}_1} |u|^p \leq C\left(\int_{K_2\cap\mathbf{T}_1} |\nabla u|^p + \left|\fint_B u\right|^p\right) \tag{3.10}$$

and hence defining $\tau u := \fint_B u$ we find

$$\int_{K_\delta} \left|\nabla\left(\tilde{\mathcal{U}}\left(u-\tau u\right)\right)\right|^p \leq 28M\hat{N} \int_{K_2\cap\mathbf{T}_1} |\nabla u|^p \,. \tag{3.11}$$

Step 2 (gluing together the local extension operators): In what follows, for every $z \in \mathbb{Z}^d$ let $\left(\tilde{\mathcal{U}}_z u\right)(\cdot) := \tilde{\mathcal{U}}(u(\cdot+z))(\cdot-z)$ the operator $\tilde{\mathcal{U}}$ shifted onto the cell $z+K_{2\delta}$. Given some positive $\overline{\eta} \in C_c(K_\delta)$ with $\overline{\eta}|_{(0,1)^d} \equiv 1$ and symmetric w.r.t. the center of $(0,1)^d$ we write $\eta_z := \overline{\eta}(\cdot - z)$ such that $\eta_z|_{z+(0,1)^d} \equiv 1$ and introduce $\tilde{\eta}_z = \eta_z / \left(\sum_{x\in\mathbb{Z}^d} \eta_x\right)$ which provide a $(0,1)^d$-periodic partition of unity. Note that at each $x \in \mathbb{R}^d$ at most 2^d functions $\tilde{\eta}_z$ are different from 0. We now define the operator $\mathcal{U}$ according to (3.6) with $\tau_z u := \fint_{B+z} u$ and $\mathcal{U}_z$ from Step 1 to find

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus\mathbf{T}_1} |\nabla\mathcal{U}u|^p &= \int_{\mathbb{R}^d\setminus\mathbf{T}_1} \left|\nabla\sum_{z\in\mathbb{Z}^d} \tilde{\eta}_z\left(\tilde{\mathcal{U}}_z(u-\tau_z u)+\tau_z u\right)\right|^p \\
&= \int_{\mathbb{R}^d\setminus\mathbf{T}_1} \left|\sum_{z\in\mathbb{Z}^d}\left[\nabla\tilde{\eta}_z\left(\tilde{\mathcal{U}}_z(u-\tau_z u)+\tau_z u\right)+\tilde{\eta}_z\nabla\left(\tilde{\mathcal{U}}_z(u-\tau_z u)\right)\right]\right|^p \\
&\leq C\left\|\nabla\tilde{\eta}\right\|_\infty^p \sum_{z\in\mathbb{Z}^d}\left\|\tilde{\mathcal{U}}_z(u-\tau_z u)\right\|^p_{L^p(z+K_\delta\setminus\mathbf{T}_1)} + C\int_{\mathbb{R}^d}\left|\sum_{z\in\mathbb{Z}^d}\tau_z u\nabla\tilde{\eta}_z\right|^p \\
&\quad + C\sum_{z\in\mathbb{Z}^d}\int_{z+K_\delta}\left|\nabla\left(\tilde{\mathcal{U}}_z(u-\tau_z u)\right)\right|^p ,
\end{aligned} \tag{3.12}$$

In order to derive an estimate on $\int_{\mathbb{R}^d}\left|\sum_{z\in\mathbb{Z}^d}\tau_z u\nabla\tilde{\eta}_z\right|^p$, note that for $z_1, z_2 \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$ for all $i = 1,\dots,d$ it holds $\partial_i\tilde{\eta}_{z_1} = -\partial_i\tilde{\eta}_{z_2}$ by symmetry and hence (writing $K_\delta(z) = z + K_\delta$

$$\int_{\mathbb{R}^d}\left|\sum_{z\in\mathbb{Z}^d}\tau_z u\nabla\tilde{\eta}_z\right|^p \leq \sum_{z_1\in\mathbb{Z}^d}\sum_{z_2\in\mathbb{Z}^d}\int_{K_\delta(z_1)\cap K_\delta(z_2)} |\nabla\tilde{\eta}_z|^p\,|\tau_{z_1}u-\tau_{z_2}u|^p \,.$$

Thus, let $z_1, z_2 \in \mathbb{Z}^d$ such that $(z_1+K_{2\delta})\cap(z_2+K_{2\delta}) \neq \emptyset$. Since $\mathbf{T}_1$ is open and connected, one can prove

$$|\tau_{z_1}u-\tau_{z_2}u|^p \leq C\int_{\mathbf{T}_1\cap[(z_1+K_2)\cup(z_2+K_2)]} |\nabla u|^p \,, \tag{3.13}$$

where C depends on d, p and $\mathbf{T}_1$. Together with (3.9)–(3.11) we infer (3.5). Estimate (3.4) can be proved in an analogue way. □

4 Quantifying Nonlocal Regularity Properties of the Geometry

We have to account for three types of randomness. One is local, namely the local Lipschitz regularity. The other is of global nature: We have to find a partition of $\mathbb{R}^d$ such that on each partition cell the extension can be explicitly constructed in a well defined way. In the case of periodicity this is evidently trivial. However, since we lack periodicity, we have to replace the periodic construction of the extension operator in Section 3 by something similar, but of stochastic nature. The key to this will be the local (δ, M)-regularity

The second problem will be overcome using a random distribution of balls within $\mathbf{P}(\omega)$ and a Voronoi tessellation which is such that every Ball is contained in exactly one Voronoi cell. This construction is based on the following observation.

Lemma 4.1. *Let $\mathbf{P}(\omega)$ be a stationary and ergodic random open set such that*

$$\mathbb{P}(\mathbf{P} \cap \mathbb{I} = \emptyset) < 1 \,.$$

Then there exists $\mathfrak{r} > 0$ such that with positive probability $p_{\mathfrak{r}} > 0$ the set $(0,1)^d \cap \mathbf{P}$ contains a ball with radius $4\sqrt{d}\mathfrak{r}$.

Proof. Assume that the lemma was wrong. Then for every $r > 0$ the set $(0,1)^d \cap \mathbf{P}$ almost surely does not contain an open ball with radius r. In particular with probability 1 the set $(0,1)^d \cap \mathbf{P}$ does not contain any ball. Hence $(0,1)^d \cap \mathbf{P} = \emptyset$ almost surely, contradicting the assumptions. □

The numbers $\mathfrak{r}$ and $p_{\mathfrak{r}}$ from Lemma 4.1 will finally lead to the concept of *mesoscopic regularity* of the geometry $\mathbf{P}(\omega)$, see Definition 4.19. Particularly the number $\mathfrak{r}$ is important, as it affects also the construction of the extension operator on the very microscopic level.

The third problem is the hardest: It is the necessity to quantify connectedness of a domain geometrically and analytically.

4.1 Microscopic Regularity

Definition 4.2 ((δ, M)-Regularity)**.** Let $\mathbf{P} \subset \mathbb{R}^d$ be an open set.

1 $\mathbf{P}$ is called (δ, M)*-regular* in $p_0 \in \partial\mathbf{P}$ if $M(p,\delta) < \infty$ and $M > M(p,\delta)$, i.e. there exists an open set $U \subset \mathbb{R}^{d-1}$ and a Lipschitz continuous function $\phi : U \to \mathbb{R}$ with Lipschitz constant M such that $\partial\mathbf{P} \cap \mathbb{B}_\delta(p_0)$ is graph of the function $\varphi : U \to \mathbb{R}^d \,, \; \tilde{x} \mapsto (\tilde{x}, \phi(\tilde{x}))$ in some suitable coordinate system.

2 $\mathbf{P}$ is called *locally* (δ, M)*-regular* if for every $p_0 \in \partial\mathbf{P}$ there exists $\delta(p_0) > 0$ and $M(p_0) > 0$ such that $\mathbf{P}$ is $(\delta(p_0), M(p_0))$-regular in p_0.

3 $\mathbf{P}$ is called *(globally)* (δ, M)*-regular* or *minimally smooth* if there exist constants $\delta, M > 0$ s.t. $\mathbf{P}$ is (δ, M)-regular in every $p_0 \in \partial\mathbf{P}$.

The concept of (global) (δ, M)-regularity or minimally smoothness can be found in the book [35]. The theory of [35] was recently used in [13] to derive extension theorems for minimally smooth stochastic geometries. A first application of the concept of (δ, M)-regularity is the following Lemma, which is important for the application of the PoincarÃ© inequalities proved in Section 2 during the construction of the local extension operators in Section 5.

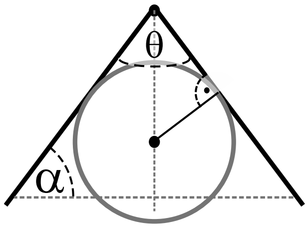

Figure 3: How to fit a ball into a cone.

Lemma 4.3. *Let $\mathbf{P}$ be locally (δ, M)-regular. Then for every $p_0 \in \partial\mathbf{P}$ with $\delta(p_0) > 0$ the following holds: For every $\delta < \delta(p_0)$ let $M := M(p_0, \delta) > 0$ such that $\partial\mathbf{P} \cap \mathbb{B}_\delta(p_0)$ is a $M(p_0, \delta)$ Lipschitz manifold. Then there exists $y \in \mathbf{P}$ with $|p_0 - y| = \frac{\delta}{4}$ such that with $r(p_0) := \frac{\delta}{4(1+M)}$ it holds $\mathbb{B}_{r(p_0)}(y) \subset \mathbb{B}_{\delta/2}(p_0)$.*

Proof. We can assume that $\partial\mathbf{P}$ is locally a cone as in Figure 3. With regard to Figure 3, for $p_0 \in \partial\mathbf{P}$ with δ and M as in the statement we can place a right circular cone with vertex (apex) p_0 and axis ν and an aperture $\theta = \pi - 2\arctan M$ inside $\mathbb{B}_\delta(p_0)$, where $\alpha = \arctan M(p_0)$. In other words, it holds $\tan(\alpha) = \tan\left(\frac{\pi-\theta}{2}\right) = M$. Along the axis we may select y with $|p_0 - y| = \frac{\delta}{4}$. Then the distance R of y to the cone is given through

$$|y - p_0|^2 = R^2 + R^2\tan^2\left(\frac{\pi-\theta}{2}\right) \quad \Rightarrow \quad R = \frac{|y-p_0|}{\sqrt{1+M^2}}\,.$$

In particular $r(p_0)$ as defined above satisfies the claim. □

Continuity properties of δ, M and ϱ

Our main extension and trace theorems will be proved for *locally* (δ, M)-regular sets $\mathbf{P}$ and is based on some simple properties of such sets which we summarize in this section. Additionally we introduce the quantity ρ.

Lemma 4.4. *Let $\mathfrak{r} > 0$, $\mathbf{P}$ be a locally (δ, M)-regular open set and let $M_0 \in (0, +\infty]$ such that for every $p \in \partial\mathbf{P}$ there exists $\delta > 0$, $M < M_0$ such that $\partial\mathbf{P}$ is (δ, M)-regular in p. Define for every $p \in \partial\mathbf{P}$*

$$\Delta(p) := \sup_{\delta<\mathfrak{r}}\left\{\exists M \in (0, M_0):\ \mathbf{P}\text{ is }(\delta, M)\text{-regular in }p\right\}\,,\quad \delta_\Delta(p) := \frac{\Delta(p)}{2}$$

Then $\partial\mathbf{P}$ is δ_Δ-regular in the sense of Definition 2.11 with

$$f(p,\delta) := \left(\exists M \in (0, M_0):\ \mathbf{P}\text{ is }(\delta, M)\text{-regular in }p\right)\,.$$

In particular, $\delta_\Delta : \partial\mathbf{P} \to \mathbb{R}$ is locally Lipschitz continuous with Lipschitz constant 4 and for every $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $\tilde{p} \in \mathbb{B}_{\varepsilon\delta}(p) \cap \partial\mathbf{P}$ it holds

$$\frac{1-\varepsilon}{1-2\varepsilon}\delta_\Delta(p) > \delta_\Delta(\tilde{p}) > \delta_\Delta(p) - |p - \tilde{p}| > (1-\varepsilon)\,\delta_\Delta(p)\,. \tag{4.1}$$

Remark 4.5. The latter lemma does ***not*** imply global Lipschitz regularity of δ_Δ. It could be that $2\delta_\Delta(p) < |p - \tilde{p}| < 3\delta_\Delta(p)$ and p and $\tilde{p}$ are connected by a path inside $\partial\mathbf{P}$ with the shortest path of length $10\delta_\Delta(p)$. Then Lemma 4.4 would have to be applied successively along this path yielding an estimate of $|\delta_\Delta(p) - \delta_\Delta(\tilde{p})| \leq 40\,|p - \tilde{p}|$.

Proof of Lemma 4.4. It is straight forward to verify that f and $\delta_\triangle$ satisfy the conditions of Lemma 2.12. $\square$

With regard to Lemma 2.2, the relevant quantity for local extension operators is related to the variable $\delta(p)/\sqrt{4M(p)^2+2}$, where $M(p)$ is the related Lipschitz constant. While we can quantify $\delta(p)$ in terms of $\delta(\tilde{p})$ and $|p-\tilde{p}|$, this does not work for $M(p)$. Hence we cannot quantify $\delta(p)/\sqrt{4M(p)^2+2}$ in terms of its neighbors. This drawback is compensated by a variational trick in the following statement.

Lemma 4.6. *Let $\mathbf{P}$ be locally (δ, M)-regular and let $\delta \leq \delta_\triangle$ satisfy (4.1) such that $\partial\mathbf{P}$ is δ-regular. For $p \in \partial\mathbf{P}$ and $r < \delta(p)$ let $M_r(p)$ be the Lipschitz constant of $\partial\mathbf{P}$ in $\mathbb{B}_r(p)$ and define*

$$\rho(p) := \sup_{r<\delta(p)} r\sqrt{4M_r(p)^2+2}^{\,-1}, \tag{4.2}$$

$$\hat{\rho}(p) := \inf\left\{\delta \leq \delta(p) \,:\, \sup_{r<\delta} r\sqrt{4M_r(p)^2+2}^{\,-1} = \rho(p)\right\}. \tag{4.3}$$

Then, ρ and $\hat{\rho}$ are positive and locally Lipschitz continuous on $\partial\mathbf{P}$ with Lipschitz constant 4 and $\partial\mathbf{P}$ is ρ and $\hat{\rho}$-regular in the sense of Definition 2.11. In particular, for $|p-\tilde{p}| < \varepsilon\rho(p)$ or $|p-\tilde{p}| < \varepsilon\hat{\rho}(p)$ it holds respectively

$$\frac{1-\varepsilon}{1-2\varepsilon}\rho(p) > \rho(\tilde{p}) > \rho(p) - |p-\tilde{p}| > (1-\varepsilon)\,\rho(p)\,,$$
$$\frac{1-\varepsilon}{1-2\varepsilon}\hat{\rho}(p) > \hat{\rho}(\tilde{p}) > \hat{\rho}(p) - |p-\tilde{p}| > (1-\varepsilon)\,\hat{\rho}(p)\,.$$

Remark 4.7*.* For the same reason as in Remark 4.5. The latter lemma does ***not*** imply global Lipschitz regularity of ρ or $\hat{\rho}$.

Proof. Positivity is given by $\rho(p) \geq \delta(p)\sqrt{4M(p)^2+2}^{\,-1}$. Let $\varepsilon > 0$ and $|p-\tilde{p}| < \varepsilon\hat{\rho}(p)$. For $r < \hat{\rho}(p)$ sufficiently large it holds $|p-\tilde{p}| < \varepsilon r$ implying $\tilde{p}$ is $((1-\varepsilon)r, M_r(p))$-regular. From here we conclude that $\partial\mathbf{P}$ is $\hat{\rho}$-regular and the above chain of inequalities follows from Lemma 2.12.

Now let $|p-\tilde{p}| < \varepsilon\rho(p) < \varepsilon\delta(p)$ implying $\delta(\tilde{p}) \geq (1-\varepsilon)\,\delta(p)$ by Lemma 4.4. For every $\eta > 0$ let $r_\eta < \delta(p)$ such that $\rho(p) \leq (1+\eta)\, r_\eta \sqrt{4M_{r_\eta}(p)^2+2}^{\,-1}$. Since $r_\eta > \rho(p)$ and $|p-\tilde{p}| < \varepsilon\rho(p)$ we find $\mathbb{B}_{r_\eta}(p) \supset \mathbb{B}_{(1-\varepsilon)r_\eta}(\tilde{p})$ and hence $M_{(1-\varepsilon)r_\eta}(\tilde{p}) \leq M_{r_\eta}(p)$. This implies at the same time that $\partial\mathbf{P}$ is ρ-regular and that

$$\rho(\tilde{p}) \geq \frac{(1-\varepsilon)\,r_\eta}{\sqrt{4M_{(1-\varepsilon)r_\eta}(\tilde{p})^2+2}} \geq \frac{(1-\varepsilon)\,r_\eta}{\sqrt{4M_{r_\eta}(p)^2+2}} \geq \frac{(1-\varepsilon)}{(1+\eta)}\rho(p)\,.$$

Since η was arbitrary, we conclude $\rho(\tilde{p}) \geq (1-\varepsilon)\,\rho(p)$. Moreover, we find $|p-\tilde{p}| < \frac{\varepsilon}{1-\varepsilon}\rho(\tilde{p})$. From here, we conclude with Lemma 2.12. $\square$

Lemma 4.8. *Let $\mathfrak{r} > 0$, $\mathbf{P} \subset \mathbb{R}^d$ be a locally (δ, M)-regular open set and let $M_0 \in (0, +\infty]$ such that for every $p \in \partial\mathbf{P}$ there exists $\delta > 0$, $M < M_0$ such that $\partial\mathbf{P}$ is (δ, M)-regular in p. For*

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$\alpha \in (0,1]$ let $\eta(p) = \alpha\delta(p)$ from Lemma 4.4 or $\eta(p) = \alpha\rho(p)$ or $\eta(p) = \alpha\hat\rho(p)$ from Lemma 4.6 and define

$$\mathrm{M}_{[\eta]}(p) := \inf_{\delta>\eta(p)} \inf_{M} \left\{ \mathbf{P} \text{ is } (\delta, M)\text{-regular in } p \right\} . \tag{4.4}$$

$$\mathfrak{m}_{[\eta]}(p,\xi) := \inf_{\delta>\min\{\delta(p),\xi\}} \inf_{M} \left\{ \mathbf{P} \text{ is } (\delta, M)\text{-regular in } p \right\} , \tag{4.5}$$

Then, for fixed ξ, $\mathrm{M}_{[\eta]}(\cdot), \mathfrak{m}_{[\eta]}(p,\xi) : \partial\mathbf{P} \to \mathbb{R}$ are upper semicontinuous and on each bounded measurable set $A \subset \mathbb{R}^d$ the quantity

$$\mathrm{M}_{[\eta],A} := \sup_{p\in\overline{A}\cap\partial\mathbf{P}} M_{[\eta]}(p) \tag{4.6}$$

with $\mathrm{M}_{[\eta],A} = 0$ if $\overline{A} \cap \partial\mathbf{P} = \emptyset$ is well defined. The functions

$$\mathrm{M}_{[\eta],A} : \mathbb{R}^d \to \mathbb{R}\,, \quad \mathrm{M}_{[\eta],A}(x) := \mathrm{M}_{[\eta],A+x} \quad \textit{with } \mathrm{M}_{[\eta],A}(0) = \mathrm{M}_{[\eta],A}$$

are upper semicontinuous.

Remark 4.9. In order to prevent confusion, let us note at this point that M_η defined in (4.9) is different from $M_{[\eta]}$. In particular, M_η is a quantity on $\mathbb{R}^d$, while $M_{[\eta]}$ is a quantity on $\partial\mathbf{P}$. Furthermore, as the last lemma shows, $M_{[\eta]}$ is upper semi continuous, while M_η is only measurable.

Notation 4.10. The infimum in (4.4) is a $\liminf$ for $\delta \searrow \eta(p)$. We sometimes use the special notation

$$M_{[\eta],\mathfrak{r}}(x) := M_{[\eta],\mathbb{B}_{\mathfrak{r}}(0)}(x) \,.$$

Proof of Lemma 4.8. Let $p, \tilde p \in \partial\mathbf{P}$ with $|p-\tilde p| < \varepsilon\eta(p)$. Writing $\tilde\varepsilon := \frac{\varepsilon}{1-\varepsilon}$ and $r(p,\varepsilon) := \left(\frac{1}{1-2\varepsilon} + \varepsilon\right)\eta(p)$ and

$$M(p,\varepsilon) := \inf_M \left\{ \mathbb{B}_{r(p,\varepsilon)}(p) \cap \partial\mathbf{P} \text{ is } M\text{-Lipschitz graph} \right\}$$

as well as we observe from η-regularity that $\mathbb{B}_{\eta(\tilde p)}(\tilde p) \subset \mathbb{B}_{r(p,\varepsilon)}(p)$ and $\mathbb{B}_{\eta(p)}(p) \subset \mathbb{B}_{r(\tilde p,\tilde\varepsilon)}(\tilde p)$. Hence we find

$$\mathrm{M}_{[\eta]}(\tilde p) \le M(p,\varepsilon)\,.$$

Observing that $M(p,\varepsilon) \searrow \mathrm{M}_{[\eta]}(p)$ as $\varepsilon \to 0$ we find $\limsup_{\tilde p\to p} \mathrm{M}_{[\eta]}(\tilde p) \le \mathrm{M}_{[\eta]}(p)$ and M is u.s.c.

Let $x \to 0$. First observe that $\mathrm{M}_{[\eta],A} = \max_{y\in\overline{A}} \mathrm{M}_{[\eta]}(y)$. The set $\overline{A}$ is compact and hence $\overline{A}+x \to \overline{A}$ in the Hausdorff metric as $x \to 0$. Let $y_x \in \overline{A}+x$ such that $\mathrm{M}_{[\eta]}(y_x) = \mathrm{M}_{[\eta],A}(x)$. Since $\overline{A}+x \to \overline{A}$ w.l.o.g. we find $y_x \to y$ converges and $y \in \overline{A}$. Hence

$$\mathrm{M}_{[\eta]}(y) \ge \limsup_{x\to 0} \mathrm{M}_{[\eta]}(y_x) = \limsup_{x\to 0} \mathrm{M}_{[\eta],A}(x)\,.$$

In particular, $M_{[\eta],A}(\cdot)$ is u.s.c. The u.s.c of $\mathfrak{m}_{[\eta]}(p,\xi)$ can be proved similarly. □

Corollary 4.11. *Let $\mathfrak{r} > 0$ and let $\mathbf{P} \subset \mathbb{R}^d$ be a locally (δ, M)-regular open set, where we restrict δ by $\delta(\cdot) \le \frac{\mathfrak{r}}{4}$. Then there exists a countable number of points $(p_k)_{k\in\mathbb{N}} \subset \partial\mathbf{P}$ such that $\partial\mathbf{P}$ is completely covered by balls $\mathbb{B}_{\tilde\rho(p_k)}(p_k)$ where $\tilde\rho(p) := 2^{-5}\rho(p)$. Writing*

$$\tilde\rho_k := \tilde\rho(p_k)\,, \qquad \delta_k := \delta(p_k)\,.$$

For two such balls with $\mathbb{B}_{\tilde{\rho}_k}(p_k) \cap \mathbb{B}_{\tilde{\rho}_i}(p_i) \neq \emptyset$ *it holds*

$$\begin{aligned} & \frac{15}{16}\tilde{\rho}_i \le \tilde{\rho}_k \le \frac{16}{15}\tilde{\rho}_i \\ \text{and} \quad & \frac{31}{15}\min\{\tilde{\rho}_i,\tilde{\rho}_k\} \ge |p_i - p_k| \ge \frac{1}{2}\max\{\tilde{\rho}_i,\tilde{\rho}_k\}\ . \end{aligned} \tag{4.7}$$

Furthermore, there exists $\mathfrak{r}_k \ge \frac{\tilde{\rho}_k}{32\left(1+\mathfrak{m}_{[\tilde{\rho}]}(p_k,\tilde{\rho}_k/4)\right)}$ *and* y_k *such that* $\mathbb{B}_{\mathfrak{r}_k}(y_k) \subset \mathbb{B}_{\tilde{\rho}_k/8}(p_k) \cap \mathbf{P}$ *and* $\mathbb{B}_{2\mathfrak{r}_k}(y_k) \cap \mathbb{B}_{2\mathfrak{r}_j}(y_j) = \emptyset$ *for* $k \neq j$*.*

Proof. The existence of the points and Balls satisfying (4.7) follows from Theorem 2.13, in particular (2.20). It holds for $\mathbb{B}_{\tilde{\rho}_k}(p_k) \cap \mathbb{B}_{\tilde{\rho}_i}(p_i) \neq \emptyset$

$$|p_i - p_k| \le \tilde{\rho}_i + \tilde{\rho}_k \le \left(\frac{16}{15}+1\right)\tilde{\rho}_i\,.$$

Lemma 4.3 yields existence of y_k such that $\mathbb{B}_{\mathfrak{r}_k}(y_k) \subset \mathbb{B}_{\tilde{\rho}_k/8}(p_k) \cap \mathbf{P}$. The latter implies $\mathbb{B}_{\mathfrak{r}_k}(y_k) \cap \mathbb{B}_{\mathfrak{r}_j}(y_j) = \emptyset$ for $k \neq j$. □

Measurability and Integrability of Extended Variables

Lemma 4.12. *Let* $\mathfrak{r} > 0$*,* $\mathbf{P} \subset \mathbb{R}^d$ *be a locally* (δ, M)*-regular open set and let* $M_0 \in (0,+\infty]$ *such that for every* $p \in \partial\mathbf{P}$ *there exists* $\delta > 0$*,* $M < M_0$ *such that* $\partial\mathbf{P}$ *is* (δ, M)*-regular in* p*. For* $\alpha \in (0,1]$ *let* $\eta(p) = \alpha\delta(p)$ *from Lemma 4.4 or* $\eta(p) = \alpha\rho(p)$ *or* $\eta(p) = \alpha\hat{\rho}(p)$ *from Lemma 4.6 and define*

$$\tilde{\eta}(x) := \inf\left\{\eta(\tilde{x}) \;:\; \tilde{x} \in \partial\mathbf{P} \text{ s.t. } x \in \mathbb{B}_{\frac{1}{8}\eta(\tilde{x})}(\tilde{x})\right\}\,, \tag{4.8}$$

$$M_{[\eta],\mathbb{R}^d}(x) := \sup\left\{M_{[\eta]}(\tilde{x}) \;:\; \tilde{x} \in \partial\mathbf{P} \text{ s.t. } x \in \overline{\mathbb{B}_{\eta(\tilde{x})}(\tilde{x})}\right\}\,, \tag{4.9}$$

where $\inf\emptyset = \sup\emptyset := 0$ *for notational convenience. Furthermore, write* $A := F^{-1}\big((0,\frac{3}{2}\mathfrak{r})\big)$ *for*

$$F := \inf_{p\in\partial\mathbf{P}} f_p\,, \qquad f_p(x) := \begin{cases} \eta(p) & \text{if } x \in \mathbb{B}_{\frac{\eta(p)}{4}}(p) \\ 2\mathfrak{r} & \text{else} \end{cases}\,.$$

then $\tilde{\eta}$ *is measurable and* $M_{[\eta]}$ *is upper semicontinuous.*

Proof. Step 1: Let $(p_i)_{i\in\mathbb{N}} \subset \partial\mathbf{P}$ be a dense subset. If $x \in \mathbb{B}_{\frac{1}{8}\eta(p)}(p)$ for some $p \in \partial\mathbf{P}$ then also $x \in \mathbb{B}_{\frac{1}{8}\eta(\tilde{p})}(\tilde{p})$ for $|p-\tilde{p}|$ sufficiently small, by continuity of η. For every $p \in \partial\mathbf{P}$ consider the function $f_p(x)$ as introduced above. Then every f_p is upper semicontinuous and $F := \inf_{i\in\mathbb{N}} f_{p_i}$ is measurable. In particular, the set A is measurable and thus $\tilde{\eta} = \chi_A F$ is measurable.

Step 2: We show that for every $a \in \mathbb{R}$ the preimage $M_{[\eta],\mathbb{R}^d}^{-1}([a,+\infty))$ is closed. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence with $M_{[\eta],\mathbb{R}^d}(x_k) \in [a,+\infty)$. Let $(p_k) \subset \partial\mathbf{P}$ be a sequence with $|x_k - p_k| \le \eta(p_k)$. W.l.o.g. assume $p_k \to p \in \partial\mathbf{P}$ and $x_k \to x \in \mathbb{R}^d$. Since η is continuous, it follows $|x-p| \le \eta(p)$. On the other hand $M_{[\eta]}(p) \ge \limsup_{k\to\infty} M_{[\eta]}(p_k)$ and thus $M_{[\eta],\mathbb{R}^d}(x) \ge M_{[\eta]}(p) \ge a$. □

Lemma 4.13. *Under the assumptions of Lemma 4.12 there exists a constant $C > 0$ only depending on the dimension d such that for every bounded open domain $\mathbf{Q}$ it holds*

$$\int_{A\cap\mathbf{Q}} \chi_{\tilde\eta>0}\tilde\eta^{-\alpha} \leq C\int_{A\cap\mathbb{B}_{\frac{\mathfrak{r}}{4}}(\mathbf{Q})\cap\partial\mathbf{P}} \eta^{1-\alpha} M^{d-2}_{[\frac{\eta}{4}],\mathbb{R}^d}\,, \tag{4.10}$$

$$\int_{A\cap\mathbf{Q}} \tilde\eta^{-\alpha} M^{r}_{[\frac{\eta}{8}]} \leq C\int_{A\cap\mathbb{B}_{\frac{\mathfrak{r}}{4}}(\mathbf{Q})\cap\partial\mathbf{P}} \eta^{1-\alpha} M^{r+d-2}_{[\frac{\eta}{4}],,\mathbb{R}^d}\,. \tag{4.11}$$

Finally, it holds

$$x\in\mathbb{B}_{\frac18\eta(p)}(p) \quad\Rightarrow\quad \eta(p)>\tilde\eta(x)>\frac34\eta(p)\,. \tag{4.12}$$

Remark 4.14. Estimates (4.10)–(4.11) are only rough estimates and better results could be obtained via more sophisticated calculations that make use of particular features of given geometries.

Proof. Step 1: Given $x\in\mathbb{R}^d$ with $\tilde\eta(x)>0$ let

$$p_x\in\operatorname{argmin}\left\{\eta(\tilde x)\;:\;\tilde x\in\partial\mathbf{P}\text{ s.t. }x\in\overline{\mathbb{B}_{\frac18\eta(\tilde x)}(\tilde x)}\right\}\,. \tag{4.13}$$

Such p_x exists because $\partial\mathbf{P}$ is locally compact. We observe with help of the definition of p_x, the triangle inequality and (2.19)

$$x\in\mathbb{B}_{\frac18\eta(p)}(p) \quad\Rightarrow\quad \eta(p_x)\leq\eta(p) \quad\Rightarrow\quad |p-p_x|<\frac{\eta(p)}{4} \quad\Rightarrow\quad \eta(p_x)>\frac34\eta(p)\,.$$

The last line particularly implies (4.12) and

$$\forall p\in\partial\mathbf{P}\;\forall x\in\mathbb{B}_{\frac{\eta(p)}{8}}(p):\quad \tilde\eta(x)>\frac{3\eta(p)}{4}\,.$$

Step 2: By Theorem 2.13 we can chose a countable number of points $(p_k)_{k\in\mathbb{N}}\subset\partial\mathbf{P}$ such that $\Gamma=\partial\mathbf{P}$ is completely covered by balls $B_k:=\mathbb{B}_{\xi(p_k)}(p_k)$ where $\xi(p):=2^{-4}\eta(p)$. For simplicity of notation we write $\eta_k:=\eta(p_k)$ and $\xi_k:=\xi(p_k)$. Assume $x\in A$ with $p_x\in\Gamma$ given by (4.13). Since the balls B_k cover Γ, there exists p_k with $|p_x-p_k|<\xi_k=2^{-4}\eta_k$, implying $\eta(p_x)<\frac{2^4}{2^4-1}\eta_k$ and hence $|x-p_k|\leq\left(2^{-4}+\frac{2^{-3}2^4}{2^4-1}\right)\eta_k<\frac{3}{16}\eta_k$. Hence we find

$$\forall x\in A\;\exists p_k\;:\quad x\in\mathbb{B}_{\frac{3}{16}\eta_k}(p_k)\,.$$

Step 3: For $p\in\Gamma$ with $x\in\mathbb{B}_{\frac14\eta(p)}(p)\cap\mathbb{B}_{\frac18\eta(p_x)}(p_x)$ we can distinguish two cases:

1 $\eta(p)\geq\eta(p_x)$: Then $p_x\in\mathbb{B}_{\frac38\eta(p)}(p)$ and hence $\eta(p_x)\geq\frac58\eta(p)$ by (2.19).

2 $\eta(p)<\eta(p_x)$: Then $p\in\mathbb{B}_{\frac38\eta(p_x)}(p_x)$ and hence$\eta(p_x)>\frac{1-\frac38}{1-\frac68}\eta(p)=\frac52\eta(p)$ by (2.19).

and hence

$$x\in\mathbb{B}_{\frac14\eta(p)}(p) \qquad\Rightarrow\qquad \tilde\eta(x)=\eta(p_x)>\frac58\eta(p)\,.$$

Step 4: Let $k\in\mathbb{N}$ be fixed and define $B_k=\mathbb{B}_{\frac14\eta_k}(p_k)$, $M_k:=M(p_k,\frac14\eta_k)$. By construction, every B_j with $B_j\cap B_k\neq\emptyset$ satisfies $\eta_j\geq\frac12\eta_k$ and hence if $B_j\cap B_k\neq\emptyset$ and $B_i\cap B_j\neq\emptyset$ we find $|p_j-p_i|\geq\frac14\eta_k$ and $|p_j-p_k|\leq3\eta_k$. This implies that

$$\exists C>0:\;\forall k\quad\#\left\{j\;:\;B_j\cap B_k\neq\emptyset\right\}\leq C\,.$$

We further observe that the minimal surface of $B_k \cap \partial\mathbf{P}$ is given in case when $B_k \cap \partial\mathbf{P}$ is a cone with opening angle $\frac{\pi}{2} - \arctan M(p_k)$. The surface area of $B_k \cap \partial\mathbf{P}$ in this case is bounded by $\frac{1}{d-1}\left|\mathbb{S}^{d-2}\right| \eta_k^{d-1} \left(M_k+1\right)^{2-d}$. This particularly implies up to a constant independent from k:

$$\begin{aligned}\int_{A\cap\mathbf{Q}\cap\mathbf{P}} \tilde{\eta}^{-\alpha} &\lesssim \sum_{k:\, B_k\cap\mathbf{Q}\neq\emptyset} \int_{A\cap B_k\cap\mathbf{P}} \eta_k^{-\alpha} \\ &\lesssim \sum_{k:\, B_k\cap\mathbf{Q}\neq\emptyset} \int_{A\cap B_k\cap\partial\mathbf{P}} \eta^{1-\alpha} M_{\left[\frac{\eta}{4}\right]}^{d-2} \\ &\lesssim \int_{A\cap\mathbb{B}_{\frac{\mathfrak{r}}{4}}(\mathbf{Q})\cap\partial\mathbf{P}} \eta^{1-\alpha} M_{\left[\frac{\eta}{4}\right]}^{d-2} .\end{aligned}$$

The second integral formula follows in a similar way. □

4.2 Mesoscopic Regularity and Isotropic Cone Mixing

In what follows, we built upon Lemma 4.1 to motivate our definition of mesoscopic regularity (Definition 4.17 by the following two Lemmas.

Lemma 4.15. *Recall $\mathbb{X}_r(\mathbf{P}(\omega)) := 2r\mathbb{Z}^d \cap \mathbf{P}_{-r}(\omega) = \left\{x \in 2r\mathbb{Z}^d \,:\, \mathbb{B}_{\frac{r}{2}}(x) \subset \mathbf{P}\right\}$ from Lemma 2.50 and assume $\mathfrak{r} < \frac{1}{8}$. Let*

$$\mu_{\omega,\mathfrak{r}}(\,\cdot\,) := \mathcal{L}\Big(\,\cdot\, \cap \mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbb{X}_{\mathfrak{r}}(\mathbf{P}(\omega)))\Big) ,$$

then there exists a constant $\lambda_0 > 0$ such that for almost every $\omega \in \Omega$ it holds for all regular convex averaging sequences A_n

$$\liminf_{n\to\infty} |A_n|^{-1} \mu_{\omega,\mathfrak{r}}(A_n) \geq \lambda_0 . \tag{4.14}$$

Remark. Note that $\mu_{\omega,\mathfrak{r}}$ is stationary with respect to shifts in $2\mathfrak{r}\mathbb{Z}^d$ but not ergodic in general. It corresponds to the function $\left|\mathbb{S}^{d-1}\right| \left(\frac{\mathfrak{r}}{2}\right)^d \mathbb{X}_{\mathfrak{r}}(\mathbf{P}(\omega))$ on $2\mathfrak{r}\mathbb{Z}^d$ and by stationarity, Theorem 2.48 yields convergence

$$|A_n|^{-1} \sum_{z\in A_n\cap\mathbb{X}_r(\mathbf{P}(\omega))} \left|\mathbb{S}^{d-1}\right| \left(\frac{\mathfrak{r}}{2}\right)^d \mathbb{X}_{\mathfrak{r}}(\mathbf{P}(\omega)) \to \mathbb{E}\left(\mu_{\omega,\mathfrak{r}}|\mathscr{I}\right) .$$

Inequality (4.14) implies $\mathbb{E}(\mu_{\omega,\mathfrak{r}}|\mathscr{I}) \geq \lambda_0$ a.s.

Proof of Lemma 4.15. Due to Lemma 4.1, with probability $p_{\mathfrak{r}} > 0$ the set $\mathbb{I} \cap \mathbf{P}$ contains a ball $\mathbb{B}_{4\sqrt{d}\mathfrak{r}}(x)$ and thus the set $(\mathbb{I} \cap \mathbf{P})_{-3\sqrt{d}\mathfrak{r}}$ contains a ball $\mathbb{B}_{\sqrt{d}\mathfrak{r}}(x)$. In particular, the stationary ergodic random measure $\tilde{\mu}_\omega(\,\cdot\,) := \mathcal{L}\big(\,\cdot\, \cap \mathbf{P}_{-3\sqrt{d}\mathfrak{r}}(\omega)\big)$ has positive intensity $\tilde{\lambda}_0 > p_{\mathfrak{r}} \left|\mathbb{S}^{d-1}\left(\sqrt{d}\mathfrak{r}\right)^d\right|$. Let $\tilde{\mu}_\omega\big(\mathbb{I}_{-3\sqrt{d}\mathfrak{r}}\big) > 0$. Then there exists $x \in (\mathbb{I} \cap \mathbf{P})_{-3\sqrt{d}\mathfrak{r}}$ and thus there exists $x \in \mathbb{X}_{\mathfrak{r}}(\mathbf{P}(\omega)) \cap \mathbb{I}$ with $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x) \subset \mathbb{I}$. It follows

$$\tilde{\mu}_\omega\big(\mathbb{I}_{-3\sqrt{d}\mathfrak{r}}\big) \leq \tilde{\mu}_\omega\left(\mathbb{I}\right) \leq 1 = \frac{2^d}{\mathfrak{r}^d\left|\mathbb{S}^{d-1}\right|} \left|\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x)\right| \leq \frac{2^d}{\mathfrak{r}^d\left|\mathbb{S}^{d-1}\right|} \mu_{\omega,\mathfrak{r}}(\mathbb{I}) .$$

Since $\tilde{\mu}_\omega$ is stationary ergodic and A_n is regular we find

$$p_{\mathfrak{r}}\left|\mathbb{S}^{d-1}\left(\sqrt{d}\mathfrak{r}\right)^d\right| < \tilde{\mu}_\omega\left(\mathbb{I}_{-3\sqrt{d}\mathfrak{r}}\right) < \tilde{\lambda}_0 \leq \mathbb{E}(\tilde{\mu}_\omega\left(\mathbb{I}\right)) = \liminf_{n\to\infty}|A_n|^{-1}\tilde{\mu}_\omega(A_n)$$
$$\leq \frac{2^d}{\mathfrak{r}^d\left|\mathbb{S}^{d-1}\right|}\liminf_{n\to\infty}|A_n|^{-1}\mu_{\omega,\mathfrak{r}}(A_n)\,.$$

□

Lemma 4.1 suggests that starting at the origin and walking into an arbitrary direction, it is almost impossible to not meet a ball of radius $\mathfrak{r}$ that fully lies within $\mathbf{P}(\omega)$. However, this is in general wrong, as for a given fixed direction one may already find periodic counter examples. In what follows, we will therefore use the weaker concept of isotropic cone mixing (Definition 4.17) which is based on the following observation:

Lemma 4.16. *Let $((\nu_j,\alpha_j))_{j\in\mathbb{N}}\subset\mathbb{S}^{d-1}\times\left(0,\frac{\pi}{2}\right)$ be countable. Then for every $x\in\mathbb{R}^d$ and each $j\in\mathbb{N}$ there holds*

$$\lim_{R\to\infty}\mathbb{P}\left(\mathbb{X}_{\mathfrak{r}}\left(\mathbf{P}\right)\cap\mathbb{C}_{\nu_j,\alpha_j,R}(x)\neq\emptyset\right)=1\,.$$

Proof. By stationarity, we can assume $x=0$ and by Lemma 4.15 the random measure $\mu_{\omega,\mathfrak{r}}$ has strictly positive intensity.

We write $\mathbb{C}_R:=\mathbb{C}_{\nu_j,\alpha_j,e^R}(0)$ and denote by $\tilde{\mathbb{C}}_R$ the cone with the same base as $\mathbb{C}_R$ but with apex $-\nu_jR$. Then $\tilde{\mathbb{C}}_R$ is a regular convex averaging sequence. Furthermore, it holds $\mathcal{L}\left(\tilde{\mathbb{C}}_R\right)/\mathcal{L}((\mathbb{C}_R))=\frac{e^R+R}{e^R}\to1$ implying $\mathcal{L}\left(\tilde{\mathbb{C}}_R\backslash\mathbb{C}_R\right)\mathcal{L}\left(\tilde{\mathbb{C}}_R\right)^{-1}\to0$ as $R\to\infty$. Thus

$$\mu_0\leq\liminf_{R\to\infty}\mathcal{L}\left(\tilde{\mathbb{C}}_R\right)^{-1}\mu_{\omega,\mathfrak{r}}\left(\tilde{\mathbb{C}}_R\right)$$
$$=\liminf_{R\to\infty}\mathcal{L}\left(\tilde{\mathbb{C}}_R\right)^{-1}\left(\mu_{\omega,\mathfrak{r}}(\mathbb{C}_R)+\mu_{\omega,\mathfrak{r}}\left(\tilde{\mathbb{C}}_R\backslash\mathbb{C}_R\right)\right)=\liminf_{R\to\infty}\mathcal{L}\left(\tilde{\mathbb{C}}_R\right)^{-1}\mu_{\omega,\mathfrak{r}}(\mathbb{C}_R)\,.$$

where we use $0\leq\mu_{\omega,\mathfrak{r}}\left(\tilde{\mathbb{C}}_R\backslash\mathbb{C}_R\right)\leq\mathcal{L}\left(\tilde{\mathbb{C}}_R\backslash\mathbb{C}_R\right)\to0$ as $R\to\infty$. We infer that $\mu_{\omega,\mathfrak{r}}((\mathbb{C}_R))=O\left(R^d\right)$ and hence the statement ($\mathbb{C}_R$ has to contain infinitely many balls $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_l)$). □

The following definition is a quantification of Lemma 4.16.

Definition 4.17 (Isotropic cone mixing)**.** A random set $\mathbf{P}(\omega)$ is isotropic cone mixing if there exists a jointly stationary point process $\mathbb{X}$ in $\mathbb{R}^d$ or $2\mathfrak{r}\mathbb{Z}^d$, $\mathfrak{r}>0$, such that almost surely two points $x,y\in\mathbb{X}$ have mutual minimal distance $2\mathfrak{r}$ and such that $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbb{X}(\omega))\subset\mathbf{P}(\omega)$. Further there exists a function $f(R)$ with $f(R)\to0$ as $\mathbb{R}\to\infty$ and $\alpha\in\left(0,\frac{\pi}{2}\right)$ such that with $\mathbf{E}:=\{e_1,\dots e_d\}\cup\{-e_1,\dots-e_d\}$ ($\{e_1,\dots e_d\}$ being the canonical basis of $\mathbb{R}^d$)

$$\mathbb{P}(\forall e\in\mathbf{E}:\ \mathbb{X}\cap\mathbb{C}_{e,\alpha,R}(0)\neq\emptyset)\geq1-f(R)\,. \tag{4.15}$$

Criterion 4.18 (A simple sufficient criterion for (4.15))**.** *Let $\mathbf{P}$ be a stationary ergodic random open set, let $\tilde{f}$ be a positive, monotonically decreasing function with $\tilde{f}(R)\to0$ as $R\to\infty$ and let $\mathfrak{r}>0$ s.t.*

$$\mathbb{P}\left(\exists x\in\mathbb{B}_R(0):\ \mathbb{B}_{4\sqrt{d}\mathfrak{r}}(x)\subset\mathbb{B}_R(0)\cap\mathbf{P}\right)\geq1-\tilde{f}(R)\,. \tag{4.16}$$

Then $\mathbf{P}$ is isotropic cone mixing with $f(R)=2d\tilde{f}\left((a+1)^{-1}R\right)$ and with $\mathbb{X}=\mathbb{X}_{\mathfrak{r}}(\mathbf{P})$. Vice versa, if $\mathbf{P}$ is isotropic cone mixing for f then $\mathbf{P}$ satisfies (4.16) with $\tilde{f}=f$.

Definition 4.19 (Mesoscopic regularity)**.** A random set $\mathbf{P}$ satisfying Criterion 4.18 is also called mesoscopically regular and $\tilde{f}$ is the regularity. $\mathbf{P}$ is called polynomially (exponentially) regular if $1/\tilde{f}$ grows polynomially (exponentially).

Proof of Criterion 4.18. Because of $\mathbb{P}(A\cup B)\leq\mathbb{P}(A)+\mathbb{P}(B)$ it holds for $a>1$

$$\mathbb{P}\big(\exists e\in\mathbf{E}\,:\,\nexists x\in\mathbb{B}_R(aRe)\,:\,\mathbb{B}_{4\sqrt{d}\mathfrak{r}}(x)\subset\mathbb{B}_R(aRe)\cap\mathbf{P}\big)\leq 2d\tilde{f}(R)\,.$$

The existence of $\mathbb{B}_{4\sqrt{d}\mathfrak{r}}(x)\subset\mathbb{B}_R(aRe)\cap\mathbf{P}(\omega)$ implies that there exists at least one $x\in\mathbb{X}_{\mathfrak{r}}\left(\mathbf{P}\left(\omega\right)\right)$ such that $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x)\subset\mathbb{B}_R(aRe)\cap\mathbf{P}(\omega)$ and we find

$$\mathbb{P}\Big(\exists e\in\mathbf{E}\,:\,\nexists x\in\mathbb{X}_{\mathfrak{r}}(\mathbf{P})\,:\,\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x)\subset\mathbb{B}_R(aRe)\cap\mathbf{P}\Big)\leq 2d\tilde{f}(R)\,.$$

In particular, for $\alpha=\arccos a$ and R large enough we discover

$$\mathbb{P}\left(\exists e\in\mathbf{E}\,:\,\mathbb{X}_{\mathfrak{r}}(\mathbf{P})\cap\mathbb{C}_{e,\alpha,(a+1)R}\left(0\right)=\emptyset\right)\leq 2d\tilde{f}(R)\,.$$

The relation (4.15) holds with $f(R)=2d\tilde{f}\big((a+1)^{-1}R\big)$.

The other direction is evident. □

Note that Criterion 4.18 is much easier to verify than Definition 4.17. However, Definition 4.17 is formulated more generally and is easier to handle in the proofs below, that are all built on properties of Voronoi meshes.

The formulation of Definition 4.17 is particularly useful for the following statement.

Lemma 4.20 (Size distribution of cells)**.** *Let $\mathbf{P}(\omega)$ be a stationary and ergodic random open set that is isotropic cone mixing for $\mathbb{X}(\omega)$, $\mathfrak{r}>0$, $f\,:\,(0,\infty)\to\mathbb{R}$ and $\alpha\in\left(0,\frac{\pi}{2}\right)$. Then $\mathbb{X}$ and its Voronoi tessellation have the following properties:*

1 *If $G(x)$ is the open Voronoi cell of $x\in\mathbb{X}(\omega)$ with diameter $d(x)$ then d is jointly stationary with $\mathbb{X}$ and for some constant $C_\alpha>0$ depending only on α*

$$\mathbb{P}(d(x)>D)<f\left(C_\alpha^{-1}\frac{D}{2}\right)\,. \tag{4.17}$$

2 *For $x\in\mathbb{X}(\omega)$ let $\mathcal{I}(x):=\{y\in\mathbb{X}\,:\,G(y)\cap\mathbb{B}_{\mathfrak{r}}(G(x))\neq\emptyset\}$. Then*

$$\#\mathcal{I}(x)\leq\left(\frac{4d(x)}{\mathfrak{r}}\right)^d\,. \tag{4.18}$$

Proof. 1. W.l.o.g. let $x_k=0$. The first part follows from the definition of isotropic cone mixing: We take arbitrary points $x_{\pm j}\in C_{\pm\mathrm{e}_j,\alpha,R}(0)\cap\mathbb{X}$. Then the planes given by the respective equations $\left(x-\frac{1}{2}x_{\pm j}\right)\cdot x_{\pm j}=0$ define a bounded cell around 0, with a maximal diameter $D(\alpha,R)=2C_\alpha R$ which is proportional to R. The constant C_α depends nonlinearly on α with $C_\alpha\to\infty$ as $\alpha\to\frac{\pi}{2}$. Estimate (4.17) can now be concluded from the relation between R and $D(\alpha,R)$ and from (4.15).

2. This follows from Lemma 2.15. □

Lemma 4.21. *Let $\mathbb{X}_\mathfrak{r}$ be a stationary and ergodic random random points process with minimal mutual distance $2\mathfrak{r}$ for $\mathfrak{r}>0$ and let $f\,:\,(0,\infty)\to\mathbb{R}$ be such that the Voronoi tessellation of $\mathbb{X}$ has the property*

$$\forall x\in\mathfrak{r}\mathbb{Z}^d\,:\quad \mathbb{P}(d(x)>D)=f(D)\,.$$

Furthermore, let $n,s\,:\,\mathbb{X}_\mathfrak{r}\to[1,\infty)$ be measurable and i.i.d. among $\mathbb{X}_\mathfrak{r}$ and let n,s,d be independent from each other. Let

$$G_{n(x)}(x)=n(x)\left(G(x)-x\right)+x$$

be the cell $G(x)$ enlarged by the factor $n(x)$, let $d(x)=\operatorname{diam}G(x)$ and let

$$\mathfrak{b}_n(y):=\sum_{x\in\mathbb{X}_\mathfrak{r}}\chi_{G_n(x)}d(x)^\eta s(x)^\xi n(x)^\zeta\,,$$

where $\eta,\xi,\zeta>0$ is a constant. Then $\mathfrak{b}_n$ is jointly stationary with $\mathbb{X}_\mathfrak{r}$ and for every $r>1$ there exists $C\in(0,+\infty)$ such that

$$\begin{aligned}&\mathbb{E}(\mathfrak{b}_n^p)\\ &\le C\left(\sum_{k,N=1}^\infty (k+1)^{d(p+1)+\eta p+r(p-1)}\,(S+1)^{\xi p+r(p-1)}\,(N+1)^{d(p+1)+\zeta p+r(p-1)}\,\mathbb{P}_{d,k}\mathbb{P}_{n,N}\mathbb{P}_{s,S}\right)\,.\end{aligned}\tag{4.19}$$

where

$$\begin{aligned}\mathbb{P}_{d,k}&:=\mathbb{P}(d(x)\in[k,k+1))=f(k)-f(k+1)\,,\\ \mathbb{P}_{n,N}&:=\mathbb{P}(n(x)\in[N,N+1))\,,\\ \mathbb{P}_{s,S}&:=\mathbb{P}(s(x)\in[S,S+1))\,.\end{aligned}$$

Proof. We write $\mathbb{X}_\mathfrak{r}=(x_i)_{i\in\mathbb{N}}$, $d_i=d(x_i)$, $n_i=n(x_i)$, $s_i:=s(x_i)$. Let

$$X_{k,N,S}(\omega):=\{x_i\in\mathbb{X}_\mathfrak{r}\;:\;d_i\in[k,k+1),\;n_i\in[N,N+1),\;s_i\in[S,S+1)\}$$

with $A_{k,N,S}:=\bigcup_{x\in X_{k,N,S}}G_{n(x)}(x)$. We observe that

$$\forall x\in\mathbb{R}^d:\quad\#\left\{x_i\in X_{k,N,S}:\,x\in G_{n(x_i)}(x_i)\right\}\le\mathbb{S}^{d-1}\,(N+1)^d\,(k+1)^d\,\mathfrak{r}^{-d}\,,\tag{4.20}$$

which follows from the uniform boundedness of cells $G_{n(x)}(x)$, $x\in X_{k,N}$ and the minimal distance of $|x_i-x_j|>2\mathfrak{r}$. Then, writing $B_R:=\mathbb{B}_R(0)$ for every $y\in\mathbb{R}^d$ it holds by stationarity and the ergodic theorem

$$\begin{aligned}\mathbb{P}(y\in G_{n_i}(x_i)\;:\;x_i\in X_{k,N})&=\lim_{R\to\infty}|B_R|^{-1}\,|A_{k,N}\cap B_R|\\ &\le\lim_{R\to\infty}|B_R|^{-1}\left|B_R\cap\bigcup_{x_i\in X_{k,N}}G_{n_i}(x_i)\right|\\ &\le\lim_{R\to\infty}|B_R|^{-1}\sum_{x_i\in X_{k,N}\cap B_R}\left|\mathbb{S}^{d-1}\right|(N+1)^d\,(k+1)^d\,\mathfrak{r}^{-d}\\ &\to\mathbb{P}_{d,k}\mathbb{P}_{n,N}\,(N+1)^d\left|\mathbb{S}^{d-1}\right|(k+1)^d\,\mathfrak{r}^{-d}\,.\end{aligned}$$

In the last inequality we made use of the fact that every cell $G_{n(x)}(x)$, $x\in X_{k,N}$, has volume smaller than $\mathbb{S}^{d-1}\left(N+1\right)^{d}\left(k+1\right)^{d}$. We note that for $\frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned}
&\int_{\mathbf{Q}}\left(\sum_{x\in\mathbb{X}_{\mathfrak{r}}}\chi_{G_{n}(x)}d(x)^{\eta}s(x)^{\xi}n(x)^{\zeta}\right)^{p}\\
&\quad\leq\int_{\mathbf{Q}}\left(\sum_{k=1}^{\infty}\sum_{N=1}^{\infty}\sum_{S=1}^{\infty}\left(\sum_{x\in X_{k,N,S}}\chi_{G_{n(x)}(x)}\left(k+1\right)^{\eta}\left(N+1\right)^{\xi}(S+1)^{\zeta}\right)\right)^{p}\\
&\quad\leq\int_{\mathbf{Q}}\left(\sum_{k,N,S=1}^{\infty}\alpha_{k,N,S}^{q}\right)^{\frac{p}{q}}\left(\sum_{k,N,S=1}^{\infty}\alpha_{k,N,S}^{-p}\left(\sum_{x\in X_{k,N,S}}\chi_{G_{n(x)}(x)}\left(k+1\right)^{\eta}\left(N+1\right)^{\xi}(S+1)^{\zeta}\right)^{p}\right).
\end{aligned}$$

Due to (4.20) we find

$$\sum_{x\in X_{k,N}}\chi_{G_{n(x)}(x)}\leq\chi_{A_{k,N}}\left(N+1\right)^{d}\left(k+1\right)^{d}\left|\mathbb{S}^{d-1}\right|$$

and obtain for $q=\frac{p}{p-1}$ and $C_q:=\left(\sum_{k,N,S=1}^{\infty}\alpha_{k,N,S}^{q}\right)^{\frac{p}{q}}\left|\mathbb{S}^{d-1}\right|^{p}$:

$$\begin{aligned}
&\frac{1}{|B_R|}\int_{B_R}\left(\sum_{x\in\mathbb{X}_{\mathfrak{r}}}\chi_{G_{n}(x)}d(x)^{\eta}s(x)^{\xi}n(x)^{\zeta}\right)^{p}\\
&\qquad\leq C_q\frac{1}{|B_R|}\int_{B_R}\left(\sum_{k,N,S=1}^{\infty}\alpha_{k,N,S}^{-p}\chi_{A_{k,N,S}}\left(N+1\right)^{dp+\zeta p}\left(k+1\right)^{dp+\eta p}\left(S+1\right)^{\xi p}\right)\\
&\qquad\to C_q\left(\sum_{k,N,S=1}^{\infty}\alpha_{k,N,S}^{-p}\left(k+1\right)^{d(p+1)+\eta p}\left(N+1\right)^{d(p+1)+\zeta p}\left(S+1\right)^{\xi p}\mathbb{P}_{s,S}\mathbb{P}_{d,k}\mathbb{P}_{n,N}\right)
\end{aligned}$$

For the sum $\sum_{k,N,S=1}^{\infty}\alpha_{k,N,S}^{q}$ to converge, it is sufficient that $\alpha_{k,N,S}^{q}=\left(k+1\right)^{-r}\left(N+1\right)^{-r}\left(S+1\right)^{-r}$ for some $r>1$. Hence, for such r it holds $\alpha_{k,N,S}=\left(k+1\right)^{-r/q}\left(N+1\right)^{-r/q}\left(S+1\right)^{-r/q}$ and thus (4.19).

□

4.3 Discretizing the Connectedness of (δ,M)-Regular Sets

Let $\mathbf{P}(\omega)$ be a stationary ergodic random open set which is isotropic cone mixing for $\mathfrak{r}>0$, $f:(0,\infty)\to\mathbb{R}$ and $\alpha\in\left(0,\frac{\pi}{2}\right)$. Then $\mathbb{X}_{\mathfrak{r}}(\mathbf{P}(\omega))=\left(x_k\right)_{k\in\mathbb{N}}$ generates a Voronoi tessellation according to Lemma 4.20 with cells G_k and balls $B_{k,\mathfrak{r}}=\mathbb{B}_{\mathfrak{r}/2}(x_k)$. While the (δ,M)-regularity of $\mathbf{P}$ is a strictly local property with a radius of influence of δ, the isotropic cone mixing is a mesoscopic property, with the influence ranging from $\mathfrak{r}$ to ∞.

In this part, we close the gap by introducing graphs on $\mathbf{P}$ that connect the small local balls covering $\partial\mathbf{P}$ with $\mathbb{X}_{\mathfrak{r}}$ in $\mathbf{P}$. The resulting family of graphs and paths on these graphs will be essential for the last step in Section 7.

Definition 4.22 (Admissible and simple graphs)**.** Let $\partial\mathbb{X}:=\left(p_k\right)_{k\in\mathbb{N}}\subset\partial\mathbf{P}$ with corresponding $\mathbb{Y}_{\partial\mathbb{X}}:=\left(y_k\right)_{k\in\mathbb{N}}$ like in Corollary 4.11 and let $\mathbb{Y}\subset\mathbf{P}$ be a countable set of points with $\partial\mathbb{X}\cup\mathbb{Y}_{\partial\mathbb{X}}\cup$

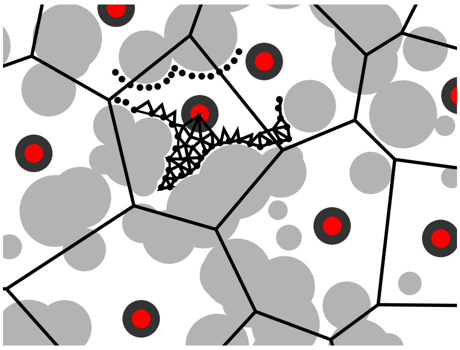

Figure 4: In order to treat the differences $|\tau_i u - \mathcal{M}_j u|^s$ appearing in Theorem 6.3 below, it is necessary to construct a graph that connects the boundary with the centers of the Voronoi tessellation.

$\mathbb{X}_{\mathfrak{r}} \subset \mathbb{Y}$ and let $(\mathbb{Y}, \mathbb{G}_*(\mathbf{P}))$ be a graph. Then the graph $\mathbb{G}_*(\mathbf{P})$ on $\mathbb{Y}$ is *admissible* if it is connected and every $p_k \in \partial\mathbb{X}$ has exactly one neighbor $y = y_k \in \mathbb{Y}_{\partial\mathbb{X}}$. An admissible graph is called *simple* if every $y_k \in \mathbb{Y}_{\partial\mathbb{X}}$ has - besides p_k - only neighbors in $\mathbb{Y}\backslash\mathbb{Y}_{\partial\mathbb{X}}$.

The following concept will become important later in Section . For reasons of self-containedness, we introduce it already at this point.

Definition 4.23 (Locally connected $\mathbf{P}$ and $\mathbb{G}_{\text{flat}}$)**.** Assume that $(\mathbb{Y}, \mathbb{G}(\mathbf{P}))$ is an admissible graph on $\mathbf{P}$ with the property that for $y_1, y_2 \in \mathbb{Y}_{\partial\mathbb{X}}$ with corresponding $p_1, p_2 \in \partial\mathbb{X}$ it holds $y_1 \sim y_2$ iff $\mathbb{B}_{\tilde\rho_1}(p_1) \cap \mathbb{B}_{\tilde\rho_2}(p_2) \neq \emptyset$. The graph $\mathbb{G}_{\text{flat}}(\mathbf{P})$ consists of all elements of $\mathbb{G}(\mathbf{P})$, except those $(y_1, y_2) \in \mathbb{Y}^2_{\partial\mathbb{X}}$ for which there is no path in $\mathbb{B}_{2\tilde\rho_1}(p_1) \cap \mathbf{P}$ or in $\mathbb{B}_{2\tilde\rho_2}(p_2) \cap \mathbf{P}$ connecting y_1 with y_2. If $\mathbb{G}_{\text{flat}}(\mathbf{P})$ is connected, the set $\mathbf{P}$ is called *locally connected*.

Locally flat geometries will turn out to be particularly useful as they allow to construct tubes around paths that fully lie within $\mathbf{P}$ and connect the local with the mesoscopic balls.

Definition 4.24 (Admissible paths)**.** Let $(\mathbb{Y}, \mathbb{G}(\mathbf{P}))$ be an admissible graph on $\mathbf{P}$ and let $\mathbb{AX}(y, x)$ be a family of paths from $y \in \mathbb{Y}_{\partial\mathbb{X}}$ to $x \in \mathbb{X}_{\mathfrak{r}}$ which are constructed from a deterministic algorithm that terminates after finitely many steps. Assume that for every $Y = (y_1, \dots, y_k) \in \mathbb{AX}(y, x)$. If $\mathfrak{r}_1 = \mathfrak{r}(y)$ is the radius of y from Corollary 4.11 assume there exists

$$Y_0 \in C([0,1] \times \mathbb{B}_{\mathfrak{r}_1}(y); \mathbf{P}) \quad \text{with} \quad Y_0(t, \mathbb{B}_{\mathfrak{r}_1}(y)) = \mathbb{B}_{\frac{\mathfrak{r}}{16}}(x)\,,$$

such that $Y_0(t, \mathbb{B}_{\mathfrak{r}_1}(y))$ is invertible for every t and $Y_0(0, x) = x$. Then the family $\mathbb{AX}(y, x)$ is called admissible.

A general approach to construct admissible graphs and paths on locally connected $\mathbf{P}$

For a particular family of random geometries, there might be sophisticated ways to construct $\mathbb{Y}$ and the families $\mathbb{AX}(\cdot, \cdot)$. However, it is interesting to know that such a graph can be constructed very generally for every locally connected geometry. In this section, we will thus introduce a concept how to transform the domain $\mathbf{P}$ into such a graph, thereby bridging the gap between the local regularity of $\partial\mathbf{P}$ and the mesoscopic regularity. The basic Idea is sketched in Figure 4.

The grid Let $\mathbf{P} \subset \mathbb{R}^d$ be open and $\mathfrak{r} > 0$. For $x \notin \partial\mathbf{P}$ let

$$\eta(x) := \min\{\operatorname{dist}(x, \partial\mathbf{P})\,,\ 2\mathfrak{r}\} \tag{4.21}$$

and $\tilde\eta = \frac{1}{4}\eta$. Then we find the following:

Lemma 4.25. *Let $\mathbf{P}$ be a connected open set which is locally (δ, M)-regular. For $\mathfrak{r} > 0$ let $\mathbb{X}_\mathfrak{r} = (x_k)_{k\in\mathbb{N}}$ be a family of points with a mutual distance of at least $2\mathfrak{r}$ satisfying $\operatorname{dist}(x_k, \partial\mathbf{P}) > 2\mathfrak{r}$ and let $\partial\mathbb{X} := (p_k)_{k\in\mathbb{N}} \subset \partial\mathbf{P}$ with corresponding $(\tilde\rho_k)_{k\in\mathbb{N}}, (\mathfrak{r}_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ and $\mathbb{Y}_{\partial\mathbb{X}} := (y_k)_{k\in\mathbb{N}}$ like in Corollary 4.11. Then there exists a family of points $\mathring{\mathbb{X}} = (\hat p_j)_{j\in\mathbb{N}} \subset \mathbf{P}$ with $\mathbb{X}_\mathfrak{r} \subset \mathring{\mathbb{X}}$ such that with $\tilde\eta_k := \tilde\eta(\hat p_k)$, $\hat B_k := \mathbb{B}_{\tilde\eta_k}(\hat p_k)$ and $B_k := \mathbb{B}_{\tilde\rho_k}(p_k)$ the family $(B_k)_{k\in\mathbb{N}} \cup \left(\hat B_k\right)_{k\in\mathbb{N}}$ covers $\mathbf{P}$ and*

$$\hat B_k \cap \hat B_i \neq \emptyset \quad \Rightarrow \quad \begin{cases} & \frac{1}{2}\tilde\eta_i \le \tilde\eta_k \le 2\tilde\eta_i \\ \textit{and} & 3\min\{\tilde\eta_i, \tilde\eta_k\} \ge |\hat p_i - \hat p_k| \ge \frac{1}{2}\max\{\tilde\eta_i, \tilde\eta_k\} \,. \end{cases} \tag{4.22}$$

Furthermore, $B_k \cap \hat B_j \neq \emptyset$ implies

$$\frac{3}{14}\tilde\rho_k \le \tilde\eta_j \le \frac{1}{3}\tilde\rho_k \,, \quad 4\tilde\eta_j \le |\hat p_j - p_k| \le \frac{4}{3}\tilde\rho_k \,, \tag{4.23}$$

i.e. $\mathbb{B}_{\mathfrak{r}_k}(y_k) \cap \mathbb{B}_{\frac18\tilde\eta_j}(\hat p_j) = \emptyset$. Finally, there exists $C > 0$ such that for every $x \in \mathbf{P}$

$$\#\left\{j \in \mathbb{N} :\; x \in \mathbb{B}_{\frac18\tilde\eta_j}(\hat p_j)\right\} + \#\left\{k \in \mathbb{N} :\; x \in \mathbb{B}_{\mathfrak{r}_k}(y_k)\right\} \le C \,. \tag{4.24}$$

Notation 4.26. Summing up and extending the notation of Lemma 4.25 we write

$$\begin{aligned} \partial\mathbb{Y} := \partial\mathbb{X} := (p_k)_{k\in\mathbb{N}} \subset \partial\mathbf{P}\,, \quad & \mathbb{X}_\mathfrak{r} \subset \mathring{\mathbb{X}} := (\hat p_j)_{j\in\mathbb{N}} \subset \mathbf{P}\,, \quad \mathbb{X} := \partial\mathbb{X} \cup \mathring{\mathbb{X}}\,, \\ \mathbb{Y}_{\partial\mathbb{X}} := (y_k)_{k\in\mathbb{N}}\,, \quad & \mathring{\mathbb{Y}} := (y_k)_{k\in\mathbb{N}} \cup \mathring{\mathbb{X}} \quad\; \mathbb{Y} := \mathring{\mathbb{Y}} \cup \partial\mathbb{Y}\,. \end{aligned} \tag{4.25}$$

The meaning of introducing the symbol $\mathbb{Y}$ will be clarified below.

For $p \in \partial\mathbb{X}$ we write $\tilde\eta(p) := \tilde\rho(p)$ and for $p \in \mathring{\mathbb{X}}$ we use the above notation (4.21) and further define

$$\mathfrak{r}(y) := \mathfrak{r}_j \text{ for } y = y_j \in \mathbb{Y}_{\partial\mathbb{X}}\,, \qquad \mathfrak{r}(y) := \frac18\tilde\eta(y) \text{ for } y \in (\hat p_k)_k \,. \tag{4.26}$$

We finally introduce the following bijective mappings

$$x(y) = \begin{cases} p_k & \text{if } y = y_k \in \mathbb{Y}_{\partial\mathbb{X}} \\ \hat p_j & \text{if } y = \hat p_j \in \mathring{\mathbb{X}} \end{cases}, \qquad y(x) = \begin{cases} y_k & \text{if } x = p_k \in \partial\mathbb{X} \\ \hat p_j & \text{if } x = \hat p_j \in \mathring{\mathbb{X}} \end{cases}. \tag{4.27}$$

Proof of Lemma 4.25. We recall $\tilde\rho_k := \tilde\rho(p_k) := 2^{-5}\rho(p_k)$ and $\mathfrak{r}_k = \frac{\tilde\rho_k}{32(1+M_k)}$ and that (4.7) holds. Furthermore, $\mathbb{B}_{\mathfrak{r}_k}(y_k) \subset \mathbb{B}_{\tilde\rho_k/8}(p_k) \cap \mathbf{P}$ and hence $\mathbb{B}_{\mathfrak{r}_k}(y_k) \cap \mathbb{B}_{\mathfrak{r}_j}(y_j) = \emptyset$ for $k \neq j$.

If we define $\mathbf{P}_B := \overline{\mathbf{P}\backslash\bigcup_k B_k}$ and observe that $\mathbf{P}_B$ is η-regular (for η defined in (4.21)). Then Lemma 2.12 and Theorem 2.13 yield a cover of $\mathbf{P}_B$ by a locally finite family of balls $\hat B_k = \mathbb{B}_{\tilde\eta_k}(\hat p_k)$, where $(\hat p_k)_{k\in\mathbb{N}} \subset \mathbf{P}_B$, and where (4.22) holds. Looking into the proof of Theorem 2.13 we can assume w.l.o.g. that $(x_k)_{k\in\mathbb{N}} \subset (\hat p_k)_{k\in\mathbb{N}}$ by suitably bounding η.

Furthermore, we find for $B_k \cap \hat B_j \neq \emptyset$ that

$$\tilde\eta_j + \tilde\rho_k \ge |\hat p_j - p_k| > 4\tilde\eta_j \quad \Rightarrow \quad \tilde\eta_j \le \frac13\tilde\rho_k \text{ and } |\hat p_j - p_k| \le \frac43\tilde\rho_k \,.$$

Next, for such p_k we consider all B_i such that $p_i \in \mathbb{B}_{4\tilde\rho_k}(p_k)$ and since $\hat p_j \notin B_i$ for all such i, we infer $\operatorname{dist}(\hat p_j, \partial\mathbf{P}) \ge \tilde\rho(p_i)$ and hence by Lemma 4.6

$$\tilde\eta_j \ge \frac14\tilde\rho_i = 2^{-7}\rho_i \ge 2^{-7}\frac{1-2\frac18}{1-\frac18}\rho_k > \frac{3}{14}\tilde\rho_k \,.$$

Finally, $\mathbb{B}_{\mathfrak{r}_k}(y_k)\cap\mathbb{B}_{\frac{1}{8}\tilde\eta_j}(\hat p_j)=\emptyset$ follows from $\frac{12}{14}\tilde\rho_k\le 4\tilde\eta_j\le|\hat p_j-p_k|$.

To see (4.24) let $x\in\mathbf{P}$ and let $\hat p_j$ such that $\tilde\eta_j$ is maximal among all $\hat B_j$ with $x\in\hat B_j$. Let $\hat p_i$ with $x\in\hat B_i\cap\hat B_j$ and observe that both $|\hat p_i-\hat p_j|$ and $\tilde\eta_i$ are bounded from below and above by a multiple of $\tilde\eta_j$. If $x\in\hat B_i\cap\hat B_k\cap\hat B_j$, $|\hat p_i-\hat p_k|$ is bounded from above and below by $\tilde\eta_i$, hence by $\tilde\eta_j$. This provides a uniform bound on $\#\left\{j\in\mathbb{N}:\ x\in\mathbb{B}_{\tilde\eta_j}(\hat p_j)\right\}$. The second part of (4.24) follows in an analogue way. □

Definition 4.27 (Neighbors)**.** Under the assumptions and notations of Lemma 4.25, for two points $y_1,y_2\in\mathring{\mathbb{X}}\cup\mathbb{Y}_{\partial\mathbb{X}}$ let $x_1=x(y_1)$, $x_2=x(y_2)$. We say that y_1 and y_2 are neighbors, written $y\sim y_2$, if $\mathbb{B}_{\tilde\eta(x_1)}(x_1)\cap\mathbb{B}_{\tilde\eta(x_2)}(x_2)\neq\emptyset$. This implies a definition of "neighbor" for $x_1,x_2\in\mathring{\mathbb{X}}$. For $x\in\partial\mathbb{X}$ and $y\in\mathbb{Y}_{\partial\mathbb{X}}$ we write $x\sim y$ if $x=x(y)$. We denote by $\mathbb{G}_0(\mathbf{P},\mathbb{X})$, $\mathbb{G}_0(\mathbf{P})$ or simply $\mathbb{G}(\mathbf{P})$ the graph on $\mathring{\mathbb{X}}\cup\mathbb{Y}_{\partial X}\cup\partial\mathbb{X}$ generated by $\sim$.

Remark 4.28*.* a) Every $y\in\mathbb{Y}_{\partial\mathbb{X}}$ has a neighbor $x\in\partial\mathbb{X}$.

b) Besides $y(x)$, points $x\in\partial\mathbb{X}$ have no other neighbors.

The admissible paths We will see below that $\mathbb{G}_0(\mathbf{P})$ is admissible if $\mathbf{P}$ is connected. Besides $\mathbb{G}_0(\mathbf{P})$ we introduce further (reduced) graphs on $\mathbb{X}$, which are based on continuous paths. For two points $x,y\in\mathbf{P}$ we denote

$$\mathbf{P}_0(x,y):=\{f\in C([0,1];\mathbf{P}):\ f(0)=x,\ f(1)=y\}\ .$$

Definition 4.29. Using the notation of Lemma 4.25, the graph

$$\mathbb{G}_{\mathrm{simple}}(\mathbf{P}):=\left\{(y_1,y_2)\in\mathbb{G}_{\mathrm{flat}}(\mathbf{P})\ :\ (y_1,y_2)\not\in\mathbb{Y}^2_{\partial\mathbb{X}}\right\}$$

is the subset of $\mathbb{G}(\mathbf{P})$ where all elements (y_1,y_2) and $(x(y_1),x(y_2))$ are removed for which $y_1,y_2\in\mathbb{Y}_{\partial\mathbb{X}}$. Furthermore, if $y_k\in\mathbb{Y}_{\partial\mathbb{X}}$ with $p_k=x(y_k)\in\partial\mathbb{X}$ has a neighbor $\hat p_j\in\mathring{\mathbb{X}}$ such that y_k and $\hat p_j$ are *not* connected through a path which lies in $\mathbb{B}_{3\tilde\rho(p_k)}(p_k)\cap\mathbf{P}$, then $(y(p_k),\hat p_j)$, $(\hat p_j,y(p_k))$ are removed.

We write $\mathbb{G}_*(\mathbf{P})$ for either $\mathbb{G}_0(\mathbf{P})$, $\mathbb{G}_{\mathrm{flat}}(\mathbf{P})$, $\mathbb{G}_{\mathrm{simple}}(\mathbf{P})$ or any other subset of $\mathbb{G}_0(\mathbf{P})$ which is connected.

Lemma 4.30. *Assume $(\mathbb{X},\mathbb{G}_{\mathrm{flat}}(\mathbf{P}))$ is connected, assume $y\in\mathbb{Y}_{\partial\mathbb{X}}$ and $y_1\sim y$. Then there exists $\gamma\in C\Big([0,1]\times\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0);\,\mathbf{P}\cap\mathbb{B}_{3\tilde\rho(x(y))}(x(y))\Big)$ such that $\gamma(\cdot,x)$ is a path from $y+\frac{16}{\mathfrak{r}}\mathfrak{r}(y)x$ to $y_1+\frac{16}{\mathfrak{r}}\mathfrak{r}(y_1)x$, for two points $x_1,x_2\in\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)$ it holds either $\gamma(\cdot,x_1)\cap\gamma(\cdot,x_2)=\emptyset$ or $\gamma(\cdot,x_1)\subset\gamma(\cdot,x_2)$ or $\gamma(\cdot,x_1)\supset\gamma(\cdot,x_2)$ and there exist constants c_1,c_2,c_3 depending only on the dimension but not on y or y_1 such that*

$$\begin{aligned}&\forall t\in[0,1]\ \mathbb{B}_{c_1\min\{\mathfrak{r}(y),\mathfrak{r}(y_1)\}}(\gamma(t,0))\subset\gamma\Big([0,1]\times\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\Big)\\&\forall x\in\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\ \mathrm{Length}\gamma(t,x)\le c_2\,|y-y_1|\ .\end{aligned}$$

We denote γ as $\gamma[y,y_1]$.

Proof. Let $\tilde\gamma\in\mathbf{P}_0(y,y_1)$. If $\tilde y\in\mathbb{Y}_{\partial\mathbb{X}}$ we infer from Lemma 5.2 below that $\mathbb{B}_{\frac12\tilde\rho(\tilde y)}(\tilde y)\subset\mathbb{B}_{3\tilde\rho(x(y))}(x(y))$. We recall that $\partial\mathbf{P}\cap\mathbb{B}_{3\tilde\rho(x(y))}(x(y))$ is a graph $(\cdot,\phi(\cdot))$ of a Lipschitz continuous function $\phi:\mathbb{R}^{d-1}\to\mathbb{R}$ and that both $\mathbb{B}_{\mathfrak{r}(y)}(y)$ and $\mathbb{B}_{\mathfrak{r}(y_1)}(y_1)$ as well as $\tilde\gamma([0,1])$ lie below that graph. We project $\mathbb{B}_{\mathfrak{r}(y)}(y)$

and $\mathbb{B}_{\mathfrak{r}(y_1)}(y_1)$ as well as $\tilde{\gamma}([0,1])$ onto the sphere $x(y)+2\tilde{\rho}(x(y))\mathbb{S}^{d-1}$, which still do not intersect with the graph of ϕ. From here we may construct γ satisfying the claimed estimates. Since $\mathbb{B}_{\mathfrak{r}(y)}(y)\subset\mathbb{B}_{\frac{1}{2}\tilde{\rho}(x(y))}(x(y))$ and $\mathbb{B}_{\mathfrak{r}(y_1)}(y_1)\subset\mathbb{B}_{\frac{1}{2}\tilde{\rho}(x(y_1))}(x(y_1))$ and $|y-y_1|>\frac{15}{16}\min\left\{\tilde{\rho}(x(y_1)),\tilde{\rho}(x(y))\right\}$, we conclude that the constants can be chosen independently from y.

If $y\in\mathring{\mathbb{Y}}$ we can proceed analogously. □

Lemma 4.31 ($\mathbb{G}_0(\mathbf{P})$ is admissible)**.** *Under the assumptions and notations of Lemma 4.25 for every $y_0,y_1\in\mathbb{Y}$ there exists a discrete path from y_0 to y_1 in $(\mathbb{X},\mathbb{G}_0(\mathbf{P}))$.*

Proof. Since $\mathbf{P}$ is connected, there exists a continuous path $\gamma:[0,1]\to\mathbf{P}$ with $\gamma(0)=y_0$, $\gamma(1)=y_1$. Since $\gamma([0,1])$ is compact, it is covered by a finite family of balls $\mathbb{B}_{\tilde{\eta}(y)}(y)$, $y\in\mathbb{Y}$. If $\gamma([0,1])\subset\mathbb{B}_{\tilde{\eta}(y_0)}(y_0)$ the statement is obvious. Otherwise there exists a maximal interval $[0,a)$, $a<1$, such that $\gamma([0,a))\subset\mathbb{B}_{\tilde{\eta}(y_0)}(y_0)$, $\gamma(a)\not\in\mathbb{B}_{\tilde{\eta}(y_0)}(y_0)$and there exists $y\neq y_0$ such that for some $\varepsilon>0$ $\gamma((a-\varepsilon,a+\varepsilon))\subset\mathbb{B}_{\tilde{\eta}(y_0)}(y_0)\cap\mathbb{B}_{\tilde{\eta}(y)}(y)$. One may hence iteratively continue with $y_0':=y$ on the interval $[a,1]$. □

Hence, every two points in $\mathbb{Y}$ can be connected by a discrete path. However, the choice of the path is not unique, there might be even infinitely many with arbitrary large deviation from the “shortest” path. Luckily, it turns out that it suffices to provide a deterministically constructed finite family of paths.

Definition 4.32 (Admissible paths on $\mathbb{G}_*(\mathbf{P})$)**.** Let $\mathbf{P}\subset\mathbb{R}^d$ be open, connected and locally connected with $\mathbb{G}_*(\mathbf{P})$ such that the assumptions of Lemma 4.25 are satisfied. Let $x\in\mathbb{X}_r$. We call any family of paths which connect $y\in\mathbb{Y}\backslash\{x\}$ to x *admissible*, if it is generated by a deterministic algorithm that terminates after a finite number of steps. Hence, an *admissible path* from y to x in $\mathbb{G}_*(\mathbf{P})$ is a path $(x_1,\dots,x_k)$ with $x_1=y$, $x_k=x$ generated according to this algorithm. We denote the set of admissible paths from y to x by $\mathbb{AX}_*(y,x)$.

Notation 4.33. Let $x_j\in\mathbb{X}_{\mathfrak{r}}$, $p_i\in\mathbb{Y}_{\partial\mathbb{X}}$ and $Y=(y_1,\dots,y_N)\in\mathbb{AX}(p_i,x_j)$. Recalling (4.26), for $x\in\mathbb{B}_{\mathfrak{r}_1}(0)$ we define $Y_0(x)$ the set of paths connecting y_1+x, $y_2+\frac{\mathfrak{r}(y_2)}{\mathfrak{r}_1}x$, ... $y_N+\frac{\mathfrak{r}(y_N)}{\mathfrak{r}_1}x$ chosen as straight line if $y_i,y_{i+1}\in\mathbb{Y}_0$ and $\gamma(\cdot,x)$ from Lemma 4.30 else and

$$Y_0(\mathbb{B}_{\mathfrak{r}_1}(0)):=\bigcup_{x\in\mathbb{B}_{\mathfrak{r}_1}(0)}Y(x)\,.$$

In what follows, we are usually working with the latter expression and hence introduce for simplicity of notation the identification $Y\equiv Y_0(\mathbb{B}_{\mathfrak{r}_i}(0))$. In this way, Y is an open set and the characteristic function $\chi_Y\in L^1(\mathbb{R}^d)$ is integrable as Lemma the next Lemma 4.38 will reveal. Finally, by Lemma 4.25 there exists $C>1$ such that independent from x_j, p_i and $x\in\mathbb{B}_{\mathfrak{r}_i}(0)$ it holds

$$\frac{1}{C}\mathrm{Length}(Y(x))\leq\mathrm{Length}(Y):=\mathrm{Length}(Y(0))\leq C\mathrm{Length}(Y(x))\,. \tag{4.28}$$

Remark 4.34. 1. Every path admissible in the sense of 4.32 is admissible in the sense of 4.24. This follows from Lemma 4.30 and the fact that for $y,\tilde{y}\in\mathbb{Y}_0$ with $y\sim\tilde{y}$ it holds $\mathbb{B}_{\frac{1}{4}\tilde{\eta}(y)}(y)\subset\mathbb{B}_{4\tilde{\eta}(\tilde{y})}(\tilde{y})\subset\mathbf{P}$.

2. A particular family of admissible paths is given by the shortest distance. In particular, if $x\in\mathbb{X}_r$ and $y\in\mathbb{Y}\backslash\{x\}$ we define the shortest paths as

$$\mathbb{AX}_{\mathrm{short}}(y,x):=\arg\min\left\{\sum_{i=1}^k|x_{i+1}-x_i|\,:\,(x_1,\dots,x_k)\text{ path in }\mathbb{G}_*(\mathbf{P})\,,\right.$$
$$\left.k\in\mathbb{N},\,x_1=y,\,x_k=x\right\}$$

Construct a finite family In what follows, we will construct a class of admissible paths on $\mathbb{G}_*(\mathbf{P})$ which does not rely on the metric graph distance. We study the discrete Laplacian $\mathcal{L}_* : L^2(\mathbb{Y}) \to L^2(\mathbb{Y})$ on an admissible graph $\mathbb{G}_*(\mathbf{P})$ given by

$$(\mathcal{L}_* u)(x) := -\sum_{(y,x)\in\mathbb{G}_*(\mathbf{P})} \frac{1}{|x-y|^2}\left(u(y)-u(x)\right) .$$

It is well known that $\mathcal{L}$ is a discrete version of an elliptic second order operator, see [4, 11, 16] and references therein. This may be quickly verified for the "classical" choice $\mathbb{Y} = h\mathbb{Z}^d$ with $x \sim y$ iff $|x-y| = h$ (using Taylor expansion and the limit $h \to 0$).

The discrete Laplacian is connected to the following discrete Poincaré inequality.

Lemma 4.35. *Let $\mathbf{P} \subset \mathbb{R}^d$ be open, connected and satisfy the assumptions of Lemma 4.25, let $(\mathbb{X}, \mathbb{G}_*(\mathbf{P}))$ be admissible and let $0 \in \mathring{\mathbb{Y}}$. Writing*

$$H_0(\mathbb{Y}) := \{u : \mathbb{Y} \to \mathbb{R} \,:\, \forall y \in \partial\mathbb{Y} :\; u(y) = 0\} ,$$
$$H_0(\mathbb{Y}\cap\mathbb{B}_R(0)) := \{u \in H_0(\mathbb{Y}) \,:\, \forall y \notin \mathbb{B}_R(0) :\; u(y) = 0\} .$$

There exists $R_0 > 0$ and $C_{R_0} > 0$ such that for every $R > R_0$ the following discrete PoincarÃ© estimate holds:

$$\forall u \in H_0(\mathbb{Y}) : \quad u(0)^2 \le C_{R_0} \sum_{\substack{y_1,y_2\in\mathbb{Y}\cap\mathbb{B}_R(0)\\ y_1\sim y_2}} \frac{\left(u(y_1)-u(y_2)\right)^2}{|y_1-y_2|^2} . \tag{4.29}$$

Proof. This is straight forward from a contradiction argument (using connectedness of $(\mathbb{X}, \mathbb{G}_*(\mathbf{P}))$). □

For the following result we introduce the notation:

$$\text{For } x \in \mathring{\mathbb{Y}} \text{ define } \delta_x(y) := \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases} .$$

Lemma 4.36 (A discrete maximum principle)**.** *Let $\mathbf{P} \subset \mathbb{R}^d$ be open, connected and satisfy the assumptions of Lemma 4.25, let $(\mathbb{X}, \mathbb{G}_*(\mathbf{P}))$ be admissible and let $x \in \mathring{\mathbb{Y}}$. Then the equation*

$$\begin{aligned} (\mathcal{L}_* u)(y) + |y-x|\, u(y) &= \delta_x(y) && \text{for } y \in \mathring{\mathbb{Y}} \\ u(y) &= 0 && \text{for } y \in \partial\mathbb{Y} \end{aligned} \tag{4.30}$$

has a unique solution which satisfies $u(y) > 0$ for all $y \in \mathring{\mathbb{Y}}$ and attains its unique local (and thus global) maximum in x. Furthermore, $u(y) \to 0$ as $|y| \to \infty$ and for $C_{R_0} > 0$ from Lemma 4.35 it holds

$$u(x) + \sum_{(y_1,y_2)\in\mathbb{G}_*(\mathbf{P})} \frac{1}{|y_1-y_2|^2}\left(u(y_1)-u(y_2)\right)^2 + \sum_{y\in\mathbb{Y}} |x-y|\, u(y)^2 \le 5C_{R_0} . \tag{4.31}$$

Proof. W.l.o.g. let $x = 0$ and write $y_1 \sim y_2$ iff $(y_1, y_2) \in \mathbb{G}_*(\mathbf{P})$. Using the notation of Lemma 4.35 and $B_R := \mathbb{B}_R(0)$ and $B_R^{\complement} := \mathbb{R}^d \setminus \mathbb{B}_R(0)$ we divide the proof in three parts.

Approximation: We consider the problem

$$\mathcal{L}_* u_R + |\cdot|\, u = \delta_0 , \qquad u_R(y) = 0 \text{ for } y \in \partial\mathbb{Y}, \text{ and } y \in \mathbb{Y}\cap B_R^{\complement} . \tag{4.32}$$

Putting $v(x)=0$ for $v\in H_0(\mathbb{Y}\cap B_R)$ and all $x\notin B_R$, we find

$$\sum_{y\in\mathring{\mathbb{Y}}\cap B_R} v(y)\,\mathcal{L}_*u(y)=\sum_{z\sim y}\frac{1}{|y-z|^2}\left(u(y)-u(z)\right)\left(v(y)-v(z)\right)\,, \tag{4.33}$$

which is a strictly positive definite bilinear symmetric form on $\mathbb{R}^{\mathring{\mathbb{Y}}\cap B_R}$. Hence, there exists a unique solution u_R to (4.32).

Since $\mathbb{Y}\cap B_R$ is finite, u_R attains a maximum and a minimum. If u_R attains a local maximum in y, it holds $\mathcal{L}_*u_R(y)\geq 0$ and if u_R attains a local miminum in y it holds $\mathcal{L}_*u_R(y)\leq 0$. If u_R attains negative values, it has a negative minimum in $y_0\in\mathring{\mathbb{Y}}$ and hence $(\mathcal{L}_*u)(y_0)+|y_0-x|\,u(y_0)<0$, a contradiction. Thus, $u_R>0$ in every $y\notin\partial\mathbb{Y}$. Furthermore, because of (4.32) u_R can attain a local maximum only in 0.

Passage $R\to\infty$: Using Lemma 4.35, for some large enough $R_0\in\mathbb{R}$ we find the following estimate, which holds for every $R>R_0$ due to (4.32) and (4.29) applied to R_0

$$\begin{aligned}\sum_{y\in\mathring{\mathbb{X}}\cap B_R}\left(u_R(y)\,\mathcal{L}u_R(y)+|y|\,u(y)^2\right)&=\sum_{z\sim y}\frac{\left(u_R(y)-u_R(z)\right)^2}{|y-z|^2}+\sum_{y\in\mathring{\mathbb{X}}\cap B_R}|y|\,u(y)^2\\&=\sum_{y\in\mathring{\mathbb{X}}\cap B_R}u_R(y)\,\delta_0(y)\leq u(0)\leq 2C_{R_0}+\frac{1}{2C_{R_0}}u_R(0)^2\\&\leq 2C_{R_0}+\frac{1}{2}\sum_{\substack{y_1,y_2\in\mathbb{Y}\cap B_{R_0}\\ y_1\sim y_2}}\frac{\left(u_R(y_1)-u_R(y_2)\right)^2}{|y_1-y_2|^2}\end{aligned} \tag{4.34}$$

Together with (4.33), the latter yields a uniform estimate for all $R>R_0$. In particular (due to a Cantor argument), there exists a subsequence $u_{R'}$ such that $u_{R'}(y)\to u(y)$ converges for every $y\in\mathbb{Y}$ as $R'\to\infty$. Evidently, u solves (4.30), is non-negative, attains its maximum in 0 and satisfies the estimate (4.31). The limit $u(y)\to 0$ as $y\to\infty$ follows from (4.31) and (4.34). u has a unique local maximum in 0 for the same reason as for u_R.

Uniqueness of u: Finally, let u and $\tilde{u}$ be two solutions such that $v=u-\tilde{u}$ satisfies

$$\begin{aligned}(\mathcal{L}_*v)(y)+|y-x|\,v(y)&=0\quad\text{for }y\in\mathbb{Y}\setminus\mathbb{Y}_\partial\\ v(y)&=0\quad\text{for }y\in\mathbb{Y}_\partial\end{aligned}\,.$$

Multiplying the above equation with v and summing over all y, we find

$$\sum_{y\in\mathring{\mathbb{X}}}\left(v(y)\,\mathcal{L}_*v(y)+|y|\,v(y)^2\right)=0\,,$$

which implies $v=0$. □

Definition 4.37. Let $x\in\mathbb{X}_r$, let u_x be the solution of (4.30) and $y\in\mathbb{Y}\backslash\{x\}$. An *admissible harmonic path* from y to x in $\mathbb{G}_*(\mathbf{P})$ is a path $(x_1,\dots,x_k)$ with $x_1=y$, $x_k=x$ such that $u_x(x_{i+1})\geq u_x(x_i)$. We denote the set of admissible harmonic paths from y to x by $\mathbb{AX}_*(y,x)$. If $\mathbb{G}_*(\mathbf{P})=\mathbb{G}_0(\mathbf{P})=\mathbb{G}(\mathbf{P})$ we simply write $\mathbb{AX}(y,x)$. Note that

$$\mathbb{AX}(y,x)\supseteq\mathbb{AX}_*(y,x)\,.$$

Lemma 4.38. *Let $\mathbf{P} \subset \mathbb{R}^d$ be open, connected and satisfy the assumptions of Lemma 4.25. Let $(\mathbb{Y}, \mathbb{G}_*(\mathbf{P}))$ be admissible and let $x \in \mathring{\mathbb{Y}}$ and $y \in \mathbb{Y}$. There exists $R > 0$, depending on $\mathbf{P}$, x and y such that every admissible harmonic path $(x_1, \dots, x_k) \in \mathbb{AX}_*(y, x)$ from y to x lies in $\mathbb{B}_R(x)$. If $C_0, C > 0$ are the natural numbers such that for every $y \in \mathbb{Y}$ it holds $C_0 \leq \#\{z \sim y : z \in \mathbb{Y}\} \leq C$ (which exist due to Lemma 4.25) then we can choose*

$$\forall y \in \mathbb{Y} : \quad R \leq \mathsf{R}_0(x, y) := C\frac{u(x)}{u(y)} \tag{4.35}$$

Proof. Let us recall that $u(z) > 0$ for every $z \notin \partial\mathbb{Y}$ by Lemma 4.36. Again we write $x \sim y$ if $(x, y) \in \mathbb{G}_*(\mathbf{P})$.

For an admissible path $(x_1, \dots, x_k)$ from $y = x_1$ to $x = x_k$ it follows $u(x_j) \geq u(y) > 0$ for every $j > 1$. On the other hand

$$(\mathcal{L}_* u)(x_j) + |x_j - x|\, u(x_j) = 0$$

Let us further recall, that with C_0 and C independent from y. Given $u(y)$ we can therefore conclude the necessary condition

$$(C_0 + |x_j - x|)\, u(x_j) - \sum_{z \sim x_j} u(z) \leq 0\,.$$

On the other hand, it holds $u(z) \leq u(x)$. This implies that the left hand side of the last inequality is bounded from below by

$$(C_0 + |x_j - x|)\, u(x_j) - Cu(x).$$

Hence we conclude (4.35) from

$$|x_j - x| \leq C\frac{u(x)}{u(y)} - C_0\,.$$

□

The most important and concluding result in this context is the following, which states that the set of admissible paths is not empty and the $\mathbb{G}(\mathbf{P})$ is connected:

Theorem 4.39 (Admissible $\mathbb{G}_*(\mathbf{P})$ are connected through admissible harmonic paths)**.** *Let $\mathbf{P} \subset \mathbb{R}^d$ be open, connected and let $\mathbf{P}$ as well as $(\mathbb{Y}, \mathbb{G}_*(\mathbf{P}))$ satisfy the assumptions of Lemma 4.36. Then for $x \in \mathring{\mathbb{Y}}$ let u_x be the solution of (4.30) and for $y \in \mathbb{Y}$ let $x_1 := y$. As long as $x_i \neq x$ select iteratively $x_{i+1} \in \{z \in \mathbb{Y} : z \sim x_i,\, u_x(z) > u_x(x_i)\}$. Then this algorithm terminates after finite steps, i.e. there exists $i \in \mathbb{N}$ such that $x_i = x$. In particular $\mathbb{G}_*(\mathbf{P})$ is connected via admissible paths.*

Proof. According to Lemma 4.38, the number of points that can be reached by the iterative process is finite, i.e. the algorithm will stop when x_i is a local maximum of u_x. But this is given by $x_i = x$ according to Lemma 4.36. □

5 Extension and Trace Properties from (δ, M)-Regularity

5.1 Preliminaries

For this whole section, let $\mathbf{P}$ be a locally (δ, M)-regular open set and let δ be bounded by $\mathfrak{r} > 0$ and satisfy (4.1). In view of Corollary 4.11, there exists a complete covering of $\partial\mathbf{P}$ by balls $\mathbb{B}_{\tilde{\rho}(p_k)}(p_k)$, $(p_k)_{k\in\mathbb{N}}$, where $\tilde{\rho}(p) := 2^{-5}\rho(p)$. We define with $\tilde{\rho}_k := \tilde{\rho}(p_k)$, $\hat{\rho}_k := \hat{\rho}(p_k)$ given in Lemma 4.6

$$A_{1,k} := \mathbb{B}_{\tilde{\rho}_k}(p_k)\,, \quad A_{2,k} := \mathbb{B}_{3\tilde{\rho}_k}(p_k)\,, \quad A_{3,k} := \mathbb{B}_{\frac{\hat{\rho}_k}{8}}(p_k) \tag{5.1}$$

and recall (4.8), which we apply to δ in order to obtain the measurable function

$$\tilde{\delta}(x) := \tilde{\hat{\rho}}(x) = \min\left\{\hat{\rho}(\tilde{x}) \ : \ \tilde{x} \in \partial\mathbf{P} \text{ s.t. } x \in \mathbb{B}_{\frac{1}{8}\hat{\rho}(\tilde{x})}(\tilde{x})\right\} . \tag{5.2}$$

Similarly, in view of (4.9), we define the measurable function

$$\tilde{M}(x) := M_{[\frac{1}{8}\hat{\rho}],\mathbb{R}^d}(x) + 1 = \max\left\{M_{[\frac{1}{8}\hat{\rho}]}(\tilde{x}) + 1 \ : \ \tilde{x} \in \partial\mathbf{P} \text{ s.t. } x \in \overline{\mathbb{B}_{\frac{1}{8}\hat{\rho}(\tilde{x})}(\tilde{x})}\right\} , \tag{5.3}$$

Here we have used the convention $\max\emptyset = \min\emptyset = 0$.

Remark 5.1. a) In view of Lemma 4.8 we recall Remark 4.9 on the difference between $M_{[\eta]}$ and M_η and additionally remark that $\mathrm{M}_{[\frac{\hat{\rho}}{8}]}(x) + 1 \leq \tilde{M}_{\hat{\rho}}(x)$ for every $x \in \partial\mathbf{P}$.

b) We could equally work with δ replacing $\hat{\rho}$. However, Lemma 4.6 suggests that the natural choice is $\hat{\rho}$.

Additionally introduce (recalling (4.6))

$$\mathfrak{m}_k := \mathfrak{m}_{[\frac{1}{8}\hat{\rho}]}(p_k, \frac{1}{4}\tilde{\rho}) , \qquad \tilde{M}_k := \tilde{M}(p_k) , \qquad M_k := M\left(p_k, \frac{1}{8}\hat{\rho}(p_k)\right) \tag{5.4}$$

We further recall that there exists $\mathfrak{r}_k = \frac{\tilde{\rho}_k}{32(1+\mathfrak{m}_k)}$, and y_k such that

$$B_k := \mathbb{B}_{\mathfrak{r}_k}(y_k) \subset \mathbf{P} \cap \mathbb{B}_{\frac{1}{8}\tilde{\rho}_k}(p_k)$$

Lemma 5.2. *For two balls $A_{1,k} \cap A_{1,j} \neq \emptyset$ either $A_{1,k} \subset A_{2,j}$ or $A_{1,j} \subset A_{2,k}$ and*

$$A_{1,k} \cap A_{1,j} \neq \emptyset \quad \Rightarrow \quad \mathbb{B}_{\frac{1}{2}\tilde{\rho}_k}(p_k) \subset A_{2,j} \text{ and } \mathbb{B}_{\frac{1}{2}\tilde{\rho}_j}(p_j) \subset A_{2,k} . \tag{5.5}$$

Furthermore, there exists a constant C depending only on the dimension d and some $\hat{d} \in [0, d]$ such that

$$\begin{aligned}
&\forall k && \#\{j \ : \ A_{1,j} \cap A_{1,k} \neq \emptyset\} + \#\{j \ : \ A_{2,j} \cap A_{2,k} \neq \emptyset\} \leq C , && (5.6)\\
&\forall x && \#\{j \ : \ x \in A_{1,j}\} + \#\{j \ : \ x \in A_{2,j}\} \leq C + 1 , && (5.7)\\
&\forall x && \#\left\{j : x \in \overline{\mathbb{B}_{\frac{1}{8}\hat{\rho}_j}(p_j)}\right\} < C\tilde{M}(x)^{\hat{d}} . && (5.8)
\end{aligned}$$

Finally, there exist non-negative functions ϕ_0 and $(\phi_k)_{k\in\mathbb{N}}$ such that for $k \geq 1$: $\mathrm{supp}\phi_k \subset A_{1,k}$, $\phi_k|_{B_j} \equiv 0$ for $k \neq j$. Further, $\phi_0 \equiv 0$ on all B_k and on $\partial\mathbf{P}$ and $\sum_{k=0}^{\infty}\phi_k \equiv 1$ and there exists C depending only on d such that for all $j \in \mathbb{N}\cup\{0\}$, $k \in \mathbb{N}$ it holds and

$$x \in A_{1,k} \quad \Rightarrow \quad |\nabla\phi_j(x)| \leq C\tilde{\rho}_k^{-1} . \tag{5.9}$$

Remark 5.3. We usually can improve $\hat{d}$ to at least $\hat{d} = d - 1$. To see this assume $\partial\mathbf{P}$ is locally connected. Then all points p_i lie on a $d-1$-dimensional plane and we can thus improve the argument in the following proof to $\hat{d} = d - 1$.

Proof. (5.5) follows from $(4.7)_2$.

Let $k \in \mathbb{N}$ be fixed. By construction in Corollary 4.11, every $A_{1,j}$ with $A_{1,j} \cap A_{1,k} \neq \emptyset$ satisfies $\tilde{\rho}_j \geq \frac{1}{2}\tilde{\rho}_k$ and hence if $A_{1,j} \cap A_{1,k} \neq \emptyset$ and $A_{1,i} \cap A_{1,k} \neq \emptyset$ we find $|p_j - p_i| \geq \frac{1}{4}\tilde{\rho}_k$ and $|p_j - p_k| \leq 3\tilde{\rho}_k$. This implies (5.6)–(5.7) for $A_{1,j}$ and the statement for $A_{2,j}$ follows analogously.

For two points p_i, p_j such that $x \in A_{3,i} \cap A_{3,j}$ it holds due to the triangle inequality $|p_i - p_j| \leq \max\left\{\frac{1}{4}\hat\rho_i, \frac{1}{4}\hat\rho_j\right\}$. Let $\mathbb{X}(x) := \left\{p_i \in \mathbb{X} :\, x \in \overline{\mathbb{B}_{\frac{1}{8}\hat\rho_i}(p_i)}\right\}$ and choose $\tilde p(x) = \tilde p \in \mathbb{X}(x)$ such that $\hat\rho_{\mathsf{m}} := \hat\rho(\tilde p)$ is maximal. Then $\mathbb{X}(x) \subset \mathbb{B}_{\frac{1}{4}\hat\rho_{\mathsf{m}}}(\tilde p)$ and every $p_i \in \mathbb{X}(x)$ satisfies $\hat\rho_{\mathsf{m}} > \hat\rho_i > \frac{1}{3}\hat\rho_{\mathsf{m}}$. Correspondingly, $\tilde\rho_i > \frac{1}{3}\hat\rho_{\mathsf{m}} 2^{-5}\tilde M_i^{-1}$ for all such p_i. In view of (4.7) this lower local bound of $\tilde\rho_i$ implies a lower local bound on the mutual distance of the p_i. Since this distance is proportional to $\hat\rho_{\mathsf{m}}\tilde M_i^{-1}$, and since $\hat\rho_{\mathsf{m}} > \hat\rho_i > \frac{1}{3}\hat\rho_{\mathsf{m}}$, this implies (5.8) with $\hat d = d$. This is by the same time the upper estimate on $\hat d$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be symmetric, smooth, monotone on $(0,\infty)$ with $\phi' \leq 2$ and $\phi = 0$ on $(1,\infty)$. For each k we consider a radially symmetric smooth function $\tilde\phi_k(x) := \phi\left(\frac{|x-p_k|^2}{\tilde\rho_k}\right)$ and an additional function $\tilde\phi_0\,(x) = \mathrm{dist}(\,x,\, \partial\mathbf{P} \cup \bigcup_k \mathbb{B}_{\mathfrak{r}_k}(y_k)\,)$. In a similar way we may modify $\tilde\phi_k$ such that $\tilde\phi_k|_{B_j} \equiv 0$ for $j \neq k$. Then we define $\phi_k := \tilde\phi / \left(\tilde\phi_0 + \sum_j \tilde\phi_j\right)$. Note that by construction of $\mathfrak{r}_k$ and y_k we find $\phi_k|_{B_k} \equiv 1$ and $\sum_{k\geq 1}\phi_k \equiv 1$ on $\partial\mathbf{P}$.

Estimate (5.9) follows from (5.6). □

5.2 Extension Estimate Through (δ, M)-Regularity of $\partial\mathrm{P}$

By Lemmas 4.6 and 2.2 the local extension operator

$$\mathcal{U}_k :\; W^{1,p}(\mathbf{P} \cap A_{3,k}) \;\to\; W^{1,p}\Big(\mathbb{B}_{\frac{1}{8}\rho_k}(p_k)\backslash\mathbf{P}\Big) \;\hookrightarrow\; W^{1,p}(A_{2,k}\backslash\mathbf{P}) \tag{5.10}$$

is linear continuous with bounds

$$\left\|\nabla\mathcal{U}_k u\right\|_{L^p(A_{2,k}\backslash\mathbf{P})} \leq 14 M_k \left\|\nabla u\right\|_{L^p\left(A_{3,k}\cap\mathbf{P}\right)}\,, \tag{5.11}$$

$$\left\|\mathcal{U}_k u\right\|_{L^p(A_{2,k}\backslash\mathbf{P})} \leq 7 \left\|u\right\|_{L^p\left(A_{3,k}\cap\mathbf{P}\right)}\,, \tag{5.12}$$

and for constants c we find

$$\left\|c - \mathcal{U}_k c\right\|_{L^p(A_{2,k}\backslash\mathbf{P})} = 0\,. \tag{5.13}$$

Definition 5.4. For every $\mathbf{Q} \subset \mathbb{R}^d$ let $\tau_i u := \frac{1}{\left|\mathbb{B}_{\mathfrak{r}_i}(y_i)\right|}\int_{\mathbb{B}_{\mathfrak{r}_i}(y_i)} u$ and

$$\begin{aligned}\mathcal{U}_{\mathbf{Q}} :\; C^1\Big(\overline{\mathbf{P} \cap \mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})}\Big) &\to C^1\Big(\overline{\mathbf{Q}\backslash\mathbf{P}}\Big)\,,\\ u &\mapsto \chi_{\mathbf{Q}\backslash\mathbf{P}} \sum_k \phi_k\left(\mathcal{U}_k(u - \tau_k u) + \tau_k u\right)\end{aligned}$$

where $\mathcal{U}_k$ are the extension operators on $A_{3,k}$ given by Lemma 2.2, respectively (5.10)–(5.13). Furthermore, we observe

$$\mathcal{U}_{\mathbf{Q}} = \tilde{\mathcal{U}}_{\mathbf{Q}} + \hat{\mathcal{U}}_{\mathbf{Q}}\,, \qquad \text{with} \qquad \tilde{\mathcal{U}}_{\mathbf{Q}} u := \chi_{\mathbf{Q}\backslash\mathbf{P}} \sum_k \phi_k\, \mathcal{U}_k(u - \tau_k u)\,, \quad \hat{\mathcal{U}}_{\mathbf{Q}} u := \chi_{\mathbf{Q}\backslash\mathbf{P}} \sum_k \phi_k\, \tau_k u \tag{5.14}$$

For two points p_i and p_j such that $A_{1,i}\cap A_{1,j}\neq\emptyset$ we find

$$\begin{aligned}|\tau_i u-\tau_j u|^r &= \left(\left|\mathbb{B}_{\frac{\mathfrak{r}_j}{2}}(0)\right|^{-1}\left|\int_{\mathbb{B}_{\frac{\mathfrak{r}_j}{2}}(x_j)}(u(\cdot)-\tau_i u)\right|\right)^r\\ &\leq \left|\mathbb{B}_{\frac{\mathfrak{r}_j}{2}}(0)\right|^{-1}\int_{\mathbb{B}_{\frac{\mathfrak{r}_j}{2}}(x_j)}|u(\cdot)-\tau_i u|^r\leq \left|\mathbb{B}_{\frac{\mathfrak{r}_j}{2}}(0)\right|^{-1}\int_{A_{1,i}}|u(\cdot)-\tau_i u|^r\\ &\leq \left|\mathbb{B}_{\frac{\mathfrak{r}_j}{2}}(0)\right|^{-1}\rho_i^r\int_{\mathrm{conv}(A_{1,i}\cup A_{1,j})}|\nabla\mathcal{U}_i u|^r\leq \left|\mathbb{B}_{\frac{\mathfrak{r}_j}{2}}(0)\right|^{-1}\rho_i^r\int_{A_{2,i}}|\nabla\mathcal{U}_i u|^r\,. \end{aligned}\tag{5.15}$$

The latter expression is not symmetric in i,j. Hence we can play a bit with the indices in order to optimize our estimates below. We have seen that $\mathfrak{r}_j\simeq\rho_j M_j^{-1}$, and hence we expect in view of (5.11)

$$|\tau_i u-\tau_j u|^r\leq CM_j^d\tilde\rho_j^{-d}\tilde\rho_i^r\int_{A_{3,i}\cap\mathbf{P}}|\nabla u|^r\,.\tag{5.16}$$

However, this needs not to be the optimal estimate. Instead of the general and restrictive estimate (5.16), we make the following Assumption:

Assumption 5.5. *There exists $\alpha\in[0,d]$ and $C>0$ such that for every k it holds $\mathfrak{r}_k\geq C\hat\rho_k M_k^{-\frac{\alpha}{d}}$. In particular, for two points $p_i,p_j\in\partial\mathbb{Y}$ with $p_i\sim p_j$ it holds*

$$|\tau_i u-\tau_j u|^r\leq C\tilde\rho_j^{-d}M_j^\alpha\tilde\rho_i^r\int_{A_{3,i}\cap\mathbf{P}}|\nabla u|^r\,.\tag{5.17}$$

In order to formulate our main results we define the general sets

$$\mathbb{R}^d_1:=\bigcup_k A_{1,k}\,,\qquad \mathbb{R}^d_3:=\bigcup_k A_{3,k}\tag{5.18}$$

and for every bounded set $\mathbf{Q}\subset\mathbb{R}^d$ we define

$$\mathbf{Q}_1:=\mathbf{Q}\cap\mathbb{R}^d_1\,,\qquad \mathbf{Q}_3:=\mathbf{Q}\cap\mathbb{R}^d_3\,.\tag{5.19}$$

Lemma 5.6. *Let $\mathbf{P}\subset\mathbb{R}^d$ be a locally (δ,M)-regular open set with delta bounded by $\mathfrak{r}>0$ and let Assumption 5.5 hold and let $\hat d$ be the constant from (5.8). Then for every bounded open $\mathbf{Q}\subset\mathbb{R}^d$, $1\leq r<p$ the operators*

$$\tilde{\mathcal{U}}_{\mathbf{Q}},\hat{\mathcal{U}}_{\mathbf{Q}}:\,W^{1,p}\left(\mathbf{P}\cap\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\right)\to W^{1,r}\left(\mathbf{Q}\backslash\mathbf{P}\right)$$

are linear, well defined and satisfy

$$\left\|\nabla\tilde{\mathcal{U}}_{\mathbf{Q}}u\right\|^r_{L^r(\mathbf{Q}\backslash\mathbf{P})}\leq C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\tilde M^{\frac{p(\hat d+1)}{p-r}}\right)^{\frac{p-r}{p}}\|\nabla u\|^{\frac{r}{p}}_{L^p\left(\mathbf{P}\cap\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\right)}\tag{5.20}$$

$$\left\|\nabla\hat{\mathcal{U}}_{\mathbf{Q}}u\right\|^r_{L^r(\mathbf{Q}\backslash\mathbf{P})}\leq C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\tilde M^{\frac{p(\hat d+\alpha)}{p-r}}\right)^{\frac{p-r}{p}}\|\nabla u\|^{\frac{r}{p}}_{L^p\left(\mathbf{P}\cap\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\right)}\tag{5.21}$$

$$+C_0\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\backslash\mathbf{P}}|\nabla\phi_0|^r\sum_{j\neq0:\,\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}|\tau_j u|^r\,,\tag{5.22}$$

$$\|\mathcal{U}_{\mathbf{Q}}u\|^r_{L^r(\mathbf{Q}\backslash\mathbf{P})}\leq C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\tilde M^{\frac{p\hat d}{p-r}}\right)^{\frac{p-r}{p}}\|u\|^{\frac{r}{p}}_{L^p\left(\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\right)}\tag{5.23}$$

where $D_{l+} := \sum_{j\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0}|\partial_l\phi_j|$*. Furthermore, for constant functions* $x\mapsto c\in\mathbb{R}$ *it holds*

$$\|c-\mathcal{U}_{\mathbf{Q}}c\|_{L^r(\mathbf{Q}\backslash\mathbf{P})}\leq |c|\,|\mathbf{Q}\backslash\mathbf{P}|^{\frac{1}{r}}\ . \tag{5.24}$$

The second term in (5.21) imposes severe problems, as we will see in Sections 7, 6.2 or even in Lemma 5.8 below.

Lemma 5.7. *Let* α_i*,* u_i*,* $i=1\dots n$*, be a family of real numbers such that* $\sum_i\alpha_i=0$ *and let* $\alpha_+ := \sum_{i:\,\alpha_i>0}\alpha_i$*. Then*

$$\sum_i\alpha_iu_i=\sum_{i:\,\alpha_i>0}\sum_{j:\,\alpha_j<0}\frac{\alpha_i\,|\alpha_j|}{\alpha_+}\left(u_i-u_j\right)\ .$$

Proof.

$$\begin{aligned}\sum_i\alpha_iu_i&=\sum_{i:\,\alpha_i>0}\alpha_iu_i+\sum_{j:\,\alpha_j<0}\alpha_ju_j\\&=\sum_{i:\,\alpha_i>0}\alpha_i\sum_{j:\,\alpha_j<0}\frac{-\alpha_j}{\alpha_+}u_i+\sum_{j:\,\alpha_j<0}\alpha_j\sum_{i:\,\alpha_i>0}\frac{\alpha_i}{\alpha_+}u_j\\&=\sum_{i:\,\alpha_i>0}\sum_{j:\,\alpha_j<0}\frac{\alpha_i\,|\alpha_j|}{\alpha_+}\left(u_i-u_j\right)\ .\end{aligned}$$

□

Proof of Lemma 5.6. For shortness of notation (and by abuse of notation) we write

$$⨍_{\mathbf{P}\cap\mathbf{Q}}g:=\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}g\,,\qquad ⨍_{\mathbf{Q}\backslash\mathbf{P}}g:=\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\backslash\mathbf{P}}g$$

and similar for integrals over $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}$ and $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\backslash\mathbf{P}$.

Step 1: We note that $\tilde\rho_k\leq\frac{1}{8}\delta_k$ as well as $\sqrt{4M_k^2+2}\leq 2\tilde M_k$. The integral over $\nabla\left(\tilde{\mathcal{U}}_{\mathbf{Q}}u\right)$ can be estimated via

$$⨍_{\mathbf{Q}\backslash\mathbf{P}}\left|\nabla\sum_{i\neq 0}\phi_i\mathcal{U}_i\left(u-\tau_iu\right)\right|^r\leq C_r\left(I_1+I_2\right) \tag{5.25}$$

$$I_1=⨍_{\mathbf{Q}\backslash\mathbf{P}}\left|\sum_{i\neq 0}\mathcal{U}_i\left(u-\tau_iu\right)\nabla\phi_i\right|^r\,,\qquad I_2:=⨍_{\mathbf{Q}\backslash\mathbf{P}}\left|\sum_{i\neq 0}\phi_i\nabla\mathcal{U}_i\left(u-\tau_iu\right)\right|^r\ .$$

(5.9) together with Jensen's yields

$$\begin{aligned}⨍_{\mathbf{Q}\backslash\mathbf{P}}\left|\sum_{i\neq 0}\mathcal{U}_i\left(u-\tau_iu\right)\nabla\phi_i\right|^r&\leq C\sum_{i\neq 0}⨍_{\mathbf{Q}\backslash\mathbf{P}}\left|\mathcal{U}_i\left(u-\tau_iu\right)\right|^r\delta_i^{-r}\chi_{A_{1,i}}\\&\leq C\sum_{i\neq 0}⨍_{\mathbf{Q}}\chi_{A_{2,i}}\left|\nabla\mathcal{U}_i\left(u-\tau_iu\right)\right|^r\\&\leq C\sum_{i\neq 0}\tilde M_i⨍_{\mathbf{Q}\cap\mathbf{P}}\chi_{A_{3,i}}\left|\nabla u\right|^r\end{aligned}$$

where we used Lemma 2.6 with $\frac{R}{r}=3$ and inequality (5.11). In a similar way, we conclude

$$\begin{aligned}
&\fint_{\mathbf{Q}\setminus\mathbf{P}}\left|\sum_{i\neq 0}\phi_i\nabla\mathcal{U}_i\left(u-\tau_i u\right)\right|^r\\
&\leq\fint_{\mathbf{Q}\setminus\mathbf{P}}\sum_{i\neq 0}\phi_i\left|\nabla\mathcal{U}_i\left(u-\tau_i u\right)\right|^r\leq\fint_{\mathbf{Q}\setminus\mathbf{P}}\sum_{i\neq 0}\chi_{A_{1,i}}\left|\nabla\mathcal{U}_i\left(u-\tau_i u\right)\right|^r\\
&\leq C\sum_{i\neq 0}\fint_{\mathbf{Q}}\chi_{A_{2,i}}\left|\nabla\mathcal{U}_i\left(u-\tau_i u\right)\right|^r\leq C\sum_{i\neq 0}\tilde{M}_i\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\chi_{A_{3,i}}\left|\nabla u\right|^r\,.
\end{aligned}$$

It only remains to estimate $\sum_i\chi_{A_{3,i}}(x)$. Inequality (5.8) yields

$$\begin{aligned}
\sum_{i\neq 0}\tilde{M}_i\fint_{\mathbf{Q}\cap\mathbf{P}}\chi_{A_{3,i}}\left|\nabla u\right|^r&\leq\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\sum_{i\neq 0}\chi_{A_{3,i}}\tilde{M}\left|\nabla u\right|^r\\
&\leq\left(\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\left(\sum_{i\neq 0}\chi_{A_{3,i}}\right)^{\frac{p}{p-r}}\tilde{M}^{\frac{p}{p-r}}\right)^{\frac{p-r}{p}}\left(\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\left|\nabla u\right|^p\right)^{\frac{r}{p}}\\
&\leq\left(\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\tilde{M}^{\frac{p(\hat{d}+1)}{p-r}}\right)^{\frac{p-r}{p}}\left(\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}}\left|\nabla u\right|^p\right)^{\frac{r}{p}}\,.
\end{aligned}\tag{5.26}$$

Step2: We now study $\hat{\mathcal{U}}_{\mathbf{Q}}$ and use Lemma 5.7 which yields

$$\sum_j\partial_l\phi_j=0\quad\Rightarrow\quad D_{l+}:=\sum_{j\colon\partial_l\phi_j>0}\partial_l\phi_j=-\sum_{j\colon\partial_l\phi_j<0}\partial_l\phi_j\tag{5.27}$$

that

$$\begin{aligned}
\fint_{\mathbf{Q}\setminus\mathbf{P}}\left|\nabla\sum_j\phi_j\tau_j u\right|^r&\leq C\sum_{l=1}^d\fint_{\mathbf{Q}\setminus\mathbf{P}}\left|\sum_j\partial_l\phi_j\tau_j u\right|^r\\
&\leq C\sum_{l=1}^d\fint_{\mathbf{Q}\setminus\mathbf{P}}\left|\sum_{i\neq 0\colon\partial_l\phi_i>0}\ \sum_{j\neq 0\colon\partial_l\phi_j<0}\frac{\partial_l\phi_i\left|\partial_l\phi_j\right|}{D_{l+}}\left|\tau_i u-\tau_j u\right|\right|^r+I_3\,,
\end{aligned}\tag{5.28}$$

where

$$I_3=C\sum_{l=1}^d\fint_{\mathbf{Q}\setminus\mathbf{P}}\left|\sum_{j\neq 0\colon\partial_l\phi_j\partial_l\phi_0<0}\frac{\left|\partial_l\phi_0\right|\left|\partial_l\phi_j\right|}{D_{l+}}\left|\tau_j u\right|\right|^r\,.\tag{5.29}$$

Since in (5.29) $\sum_{j\neq 0\colon\partial_l\phi_j\partial_l\phi_0<0}\left|\partial_l\phi_j\right|=D_{l+}$ we obtain

$$I_3=C\sum_{l=1}^d\fint_{\mathbf{Q}\setminus\mathbf{P}}\left|\partial_l\phi_0\right|^r\sum_{j\neq 0\colon\partial_l\phi_j\partial_l\phi_0<0}\frac{\left|\partial_l\phi_j\right|}{D_{l+}}\left|\tau_j u\right|^r\,.$$

We will now derive an estimate on $\left|\tau_i u-\tau_j u\right|$. For this reason, denote l_{ij} the line from x_i to x_j and by $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(l_{ij})$ the set of all points with distance less than $\frac{\mathfrak{r}}{2}$ to l_{ij}. We exploit the fact that every term in the sum on the right hand side of (5.28) appears only once and introduce

$$E_l(x)=\left\{(i,j)\ :\ \partial_l\phi_i\partial_l\phi_j<0\text{ and }\mathfrak{r}_i<\mathfrak{r}_j\text{ or }\left(\mathfrak{r}_i=\mathfrak{r}_j\text{ and }i<j\right)\right\}\,.$$

We make use of (5.17) and successively apply Jensen's inequality, $|\nabla\phi_i| \leq C\tilde{\rho}_i^{-1}$, $\frac{1}{C}\rho_i \leq \rho_j \leq C\rho_i$ and $\left|\mathbb{B}_{\frac{\mathfrak{r}_j}{2}}(0)\right|^{-1} \leq \tilde{M}_i^d \left|A_{1,i}\right|^{-1}$ to obtain

$$\begin{aligned}
S := &\left|\sum_{i:\,\partial_l\phi_i>0}\ \sum_{j:\,\partial_l\phi_j<0} \frac{\partial_l\phi_i\left|\partial_l\phi_j\right|}{D_{l+}}\left|\tau_i u-\tau_j u\right|\right|^r = \left|\sum_{(i,j)\in E_l} \frac{\left|\partial_l\phi_i\right|\left|\partial_l\phi_j\right|}{D_{l+}}\left|\tau_i u-\tau_j u\right|\right|^r \\
&\leq \sum_{(i,j)\in E_l} \frac{\tilde{\rho}_i^{-r}\left|\partial_l\phi_j\right|}{D_{l+}} C\tilde{\rho}_j^{\,-d} M_j^\alpha \tilde{\rho}_i^r \int_{A_{3,i}\cap\mathbf{P}} \left|\nabla u\right|^r .
\end{aligned}$$

Hence we find

$$\begin{aligned}
\fint_{\mathbf{Q}\setminus\mathbf{P}} \left|\nabla\sum_j \phi_j\tau_j u\right|^r &\lesssim C\sum_{l=1}^d \sum_{(i,j)\in E_l} \frac{\tilde{\rho}_i^{-r}\left|\partial_l\phi_j\right|}{D_{l+}} C\tilde{\rho}_j^{\,-d} M_j^\alpha \tilde{\rho}_i^r \int_{A_{3,i}\cap\mathbf{P}} \left|\nabla u\right|^r . \\
&\lesssim C\frac{1}{|\mathbf{Q}|}\sum_i \tilde{M}_i^\alpha \int_{A_{3,i}\cap\mathbf{P}} \left|\nabla u\right|^r
\end{aligned} \tag{5.30}$$

Similar to (5.26) we may conclude (5.21).

Step 3: We observe with Jensen's inequality and the fact that $\mathcal{U}_i$ are linear with $\mathcal{U}_i c = c$ for constants c that

$$\begin{aligned}
\fint_{\mathbf{Q}\setminus\mathbf{P}} \left|\sum_i \phi_i\mathcal{U}_i\left(u-\tau_i u\right)+\phi_i\tau_i u\right|^r &\leq \fint_{\mathbf{Q}\setminus\mathbf{P}} \sum_i \phi_i \left(\mathcal{U}_i u\right)^r \leq \fint_{\mathbf{Q}\setminus\mathbf{P}} \sum_i \phi_i \left(\mathcal{U}_i u\right)^r \\
&\leq \fint_{\mathbf{Q}\setminus\mathbf{P}} \sum_i \chi_{A_{1,i}} \left(\mathcal{U}_i u\right)^r \\
&\leq 7\fint_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\cap\mathbf{P}} \sum_i \chi_{A_{3,i}}\, u^r
\end{aligned}$$

From here we may proceed as in (5.26) to conclude . □

Lemma 5.8. *Let $\mathbf{P}\subset\mathbb{R}^d$ be a locally (δ,M)-regular open set with delta bounded by $\mathfrak{r}>0$ and let Assumption 5.5 hold and let $\hat{d}$ be the constant from (5.8). Then for every bounded open $\mathbf{Q}\subset\mathbb{R}^d$, $1\leq r<p_0<p_1<p$*

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}} \left|\nabla\phi_0\right|^r \sum_{j\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0} \frac{\left|\partial_l\phi_j\right|}{D_{l+}}\left|\tau_j u\right|^r \\
&\quad\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}} \left|\nabla\phi_0\right|^{\frac{rp_0}{p_0-r}} \tilde{M}^{2-d}\right)^{\frac{p_0-r}{p_0}} \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}} \tilde{M}^{\frac{p_1(d-2)(p_0-r)}{r(p_1-p_0)}}\right)^{r\frac{p_1-p_0}{p_1p_0}} \\
&\qquad\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})} \tilde{M}^{\frac{\alpha p_1 p}{p-p_1}}\right)^{r\frac{p-p_1}{pp_1}} \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})} |u|^p\right)^{\frac{r}{p}}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}} \left|\nabla\phi_0\right|^r \sum_{j\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0} \frac{\left|\partial_l\phi_j\right|}{D_{l+}}\left|\tau_j u\right|^r \\
&\quad\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}} \left|\nabla\phi_0\right|^{\frac{rp_0}{p_0-r}}\right)^{\frac{p_0-r}{p_0}} \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})} \tilde{M}^{\frac{\alpha p_0 p}{p-p_0}}\right)^{r\frac{p-p_0}{pp_0}} \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})} |u|^p\right)^{\frac{r}{p}}
\end{aligned}$$

Proof. We observe with Hölder and Jensens inequality on $\mathbb{R}^d$ and in the sum $\sum_{j\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}=1$ respectively that

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}}|\nabla\phi_0|^r\sum_{j\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}|\tau_j u|^r\leq\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}}|\nabla\phi_0|^{\frac{rp_0}{p_0-r}}\tilde{M}^{2-d}\right)^{\frac{p_0-r}{p_0}}$$
$$\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}}\tilde{M}^{\frac{1}{r}(d-2)(s-r)}\sum_{j\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}|\tau_j u|^{p_0}\right)^{\frac{r}{p_0}}.$$

Applying the same trick again we find

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}}\tilde{M}^{\frac{1}{r}(d-2)(p_0-r)}\sum_{j\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}|\tau_j u|^{p_0}$$
$$\leq\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}}\tilde{M}^{\frac{p_1(d-2)(p_0-r)}{r(p_1-p_0)}}\right)^{\frac{p_1-p_0}{p_1}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}}\sum_{j\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}|\tau_j u|^{p_1}\right)^{\frac{p_0}{p_1}}$$

From the definition of τ_j and (5.7) we find

$$\begin{aligned}\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})\setminus\mathbf{P}}&\sum_{j\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}|\tau_j u|^{p_1}\\
&\leq\frac{1}{|\mathbf{Q}|}\sum_{p_j\in\mathbb{B}_{\frac{\mathfrak{r}}{4}}(\mathbf{Q})}\tilde{\rho}_j^d\frac{M_j^{\alpha p_1}}{\tilde{\rho}_j^d}\int_{\mathbb{B}_{\tilde{\rho}_j}(p_j)}|u|^{p_1}\\
&\leq\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})}\tilde{M}^{\alpha p_1}|u|^{p_1}\\
&\leq\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})}\tilde{M}^{\frac{\alpha p_1 p}{p-p_1}}\right)^{\frac{p-p_1}{p}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q})}|u|^p\right)^{\frac{p_1}{p}}.\end{aligned}$$

□

5.3 Traces on (δ, M)-Regular Sets

Theorem 5.9. *Let $\mathbf{P}\subset\mathbb{R}^d$ be a locally (δ, M)-regular open set, $\frac{1}{8}>\mathfrak{r}>0$ and let $\mathbf{Q}\subset\mathbb{R}^d$ be a bounded open set and let $1\leq r<p_0<p$. Then the trace operator $\mathcal{T}$ satisfies for every $u\in W^{1,p}_{\mathrm{loc}}(\mathbf{P})$*

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\partial\mathbf{P}}|\mathcal{T}u|^r\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac{1}{4}}(\mathbf{Q})\cap\mathbf{P}}|u|^p+|\nabla u|^p\right)^{\frac{r}{p}}$$

where for some constant C_0 depending only on p_0, p and r and d and for $\eta\in\{\rho,\hat\rho,\delta\}$ one may chose between

$$C=C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac14\mathfrak{r}}(\mathbf{Q})\cap\partial\mathbf{P}}\eta^{-\frac{1}{p_0-r}}\right)^{\frac{p_0-r}{p_0}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac14\mathfrak{r}}(\mathbf{Q})\cap\mathbf{P}}\tilde M_{[\frac18\eta],\mathbb{R}^d}^{\left(\frac{1}{p_0}+1\right)\frac{p}{p-p_0}}\right)^{\frac{p-p_0}{p_0p}}\,, \tag{5.31}$$

$$C=C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac14\mathfrak{r}}(\mathbf{Q})\cap\partial\mathbf{P}}\left(\eta M_{[\frac{1}{16}\eta],\mathbb{R}^d}\right)^{-\frac{1}{p-r}}\right)^{\frac{p-r}{p}}\,. \tag{5.32}$$

Proof. Using Theorem 2.13, we cover $\partial\mathbf{P}$ by balls $B_k=\mathbb{B}_{\frac{1}{16}\eta(p_k)}(p_k)$ with $(p_k)_{k\in\mathbb{N}}\subset\partial\mathbf{P}$ and define $\hat B_k=\mathbb{B}_{\frac18\eta(p_k)}(p_k)$ and $M_k=M_{[\frac{1}{16}\eta]}(p_k)$. Like for (5.7) we can show that the covering with both B_k and $\hat B_k$ is locally uniformly bounded by a constant C. Due to Lemma 2.4 we find locally

$$\|\mathcal{T}u\|_{L^{p_0}(\partial\mathbf{P}\cap B_k)}\le C_{p_0,p_0}\eta^{-\frac{1}{p_0}}\sqrt{4M_k^2+2}^{\frac{1}{p_0}+1}\|u\|_{W^{1,p_0}\left(\hat B_k\right)}\,. \tag{5.33}$$

If ϕ_k is a partition of 1 on $\partial\mathbf{P}$ with respective support B_k we obtain

$$\begin{aligned}&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\partial\mathbf{P}}\left|\sum_k\phi_k\mathcal{T}_ku\right|^r\\&\qquad\le\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{B}_{\frac14}(\mathbf{Q})\cap\partial\mathbf{P}}\sum_k\chi_{B_k}\eta_k^{-\frac{1}{p_0-r}}\right)^{\frac{p_0-r}{p_0}}\left(\frac{1}{|\mathbf{Q}|}\sum_k\int_{\mathbb{B}_{\frac14}(\mathbf{Q})\cap\partial\mathbf{P}}\chi_{B_k}\eta_k\left|\mathcal{T}_ku\right|^{p_0}\right)^{\frac{r}{p_0}}\end{aligned}$$

which yields by the uniform local bound of the covering, $\tilde\eta$ defined in Lemma 4.13, twice the application of (4.12) and (5.33)

$$\begin{aligned}\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\partial\mathbf{P}}\left|\sum_k\phi_k\mathcal{T}_ku\right|^r&\le\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\partial\mathbf{P}}\eta^{-\frac{1}{p_0-r}}\right)^{\frac{p_0-r}{p_0}}\cdot\\&\quad\cdot\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbf{P}}\sum_k\chi_{\hat B_k}\sqrt{4M_k^2+2}^{\frac{1}{p_0}+1}\left(|\nabla u|^{p_0}+|u|^{p_0}\right)\right)^{\frac{r}{p_0}}\,.\end{aligned}$$

With Hölders inequality, the last estimate leads to (5.31). The second estimate goes analogue since the local covering by $A_{2,k}$ is finite. □

6 Construction of Macroscopic Extension Operators I: General Considerations

In this section, we provide the extension results which answer the question of the existence of such uniformly bounded families of operators up to the issue of quantifying *connectedness*. We will discuss what we mean by that in Section 6.2. In Section 6.4 we provide a first attempt to from the point of view of continuous PDE, which is - in some sense - a tautology. However, verifying the conditions of Theorem 6.10 in a computer based approach (for real life geometries) leads to a discretization of an elliptic second order operator. Therefore, in Section 7 we use the construction of Section 4.3 to introduce a quantity which can be directly calculated from a numerical algorithm.

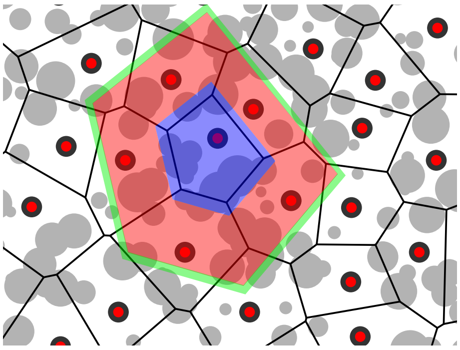

Figure 5: Gray: a Poisson ball process. Black balls: balls of radius $\mathfrak{r} > 0$. Red Balls: radius $\frac{\mathfrak{r}}{2}$. The Voronoi tessellation is generated from the centers of the red balls. The existence of such tessellations is discussed in Section 4.2. Blue region: $\mathfrak{A}_{1,k}$ according to Assumption 6.6. Red region: $\mathfrak{A}_{2,k}$. Green region: an alternative choice of $\mathfrak{A}_{3,k}$.

6.1 Extension for Voronoi Tessellations

Assumption 6.1. *Let $\mathbf{P}$ be an open set and let $\mathbb{X}_\mathfrak{r} = (x_i)_{i=\in\mathbb{N}}$ have mutual distance $|x_i - x_k| > 2\mathfrak{r}$ if $i \neq k$ and with $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_i) \subset \mathbf{P}$ for every $i \in \mathbb{N}$ (e.g. $\mathbb{X}_\mathfrak{r}(\mathbf{P})$, see (2.36)). We construct from $\mathbb{X}_\mathfrak{r}$ a Voronoi tessellation and denote by $G_i := G(x_i)$ the Voronoi cell corresponding to x_i with diameter d_i. We denote $\mathfrak{A}_{1,i} := \mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_i)$ and*

$$\mathcal{M}_i u := \left|\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\right|^{-1} \int_{\mathbb{B}_{\frac{\mathfrak{r}}{16}}(x_i)} u\,. \tag{6.1}$$

Let $\tilde\Phi_0 \in C^\infty(\mathbb{R};[0,1])$ be monotone decreasing with $\tilde\Phi_0' > -\frac{4}{\mathfrak{r}}$, $\tilde\Phi_0(x) = 1$ if $x \le 0$ and $\tilde\Phi_0(x) = 0$ for $x \ge \frac{\mathfrak{r}}{2}$. We define on $\mathbb{R}^d$ the functions

$$\tilde\Phi_i(x) := \tilde\Phi_0\left(\operatorname{dist}(x, G_i)\right) \quad \textit{and} \quad \Phi_i(x) := \tilde\Phi_i(x)\left(\sum_j \tilde\Phi_j(x)\right)^{-1}. \tag{6.2}$$

Lemma 4.20.2) implies

$$\forall x \in \mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_i) : \quad \#\left\{k \,:\, x \in \mathfrak{A}_{1,k}\right\} \le \left(\frac{4d_i}{\mathfrak{r}}\right)^d \tag{6.3}$$

and thus (6.2) yields for some C depending only on $\tilde\Phi_0$ that

$$|\nabla\Phi_i| \le C d_i^d \quad \text{and} \quad \forall k:\ |\nabla\Phi_k|\,\chi_{\mathfrak{A}_{1,i}} \le C d_i^d\,. \tag{6.4}$$

Definition 6.2 (Weak Neighbors)**.** Under the Assumption 6.1, two points x_i and x_j are called to be weakly connected (or weak neighbors), written $i \sim\sim j$ or $x_i \sim\sim x_j$ if $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_i) \cap \mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_j) \neq \emptyset$. For $\mathbf{Q} \subset \mathbb{R}^d$ open we say $\mathfrak{A}_{1,j} \sim\sim \mathbf{Q}$ if $\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathfrak{A}_{1,j}) \cap \mathbf{Q} \neq \emptyset$. We then define

$$\mathbb{X}_\mathfrak{r}(\mathbf{Q}) := \left\{x_j \in \mathbb{X}_\mathfrak{r} :\ \mathfrak{A}_{1,j} \sim\sim \mathbf{Q} \neq \emptyset\right\}, \quad \mathbf{Q}^{\sim\sim} := \bigcup_{\mathfrak{A}_{1,j}\sim\sim\mathbf{Q}} \mathfrak{A}_{1,j}\,. \tag{6.5}$$

Let $\mathbf{P}$ be locally (δ, M)-regular and satisfy Assumption 6.1. Then we can construct continuous local extension operators $\mathcal{U}_{G_j} : W^{1,p}(\mathbb{B}_\mathfrak{r}(G_i)) \to W^{1,r}\left(\mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_i)\right)$ from Lemma 5.6. These can be glued together via

$$\mathcal{U}_\mathbf{Q} u := \sum_j \Phi_j\left(\mathcal{U}_{G_j}\left(u - \mathcal{M}_j u\right) + \mathcal{M}_j u\right)\,.$$

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However, using the partition of unity from Lemma 5.2 and the definition of U_{G_i} from (5.14) we obtain

$$\mathcal{U}_{\mathbf{Q}}u=\sum_j\Phi_j\left(\sum_i\phi_i\left[\mathcal{U}_i\left(u-\mathcal{M}_ju-\tau_i\left(u-\mathcal{M}_ju\right)\right)+\tau_i\left(u-\mathcal{M}_ju\right)\right]+\mathcal{M}_ju\right)\,.$$

Using $\tau_i\mathcal{M}_ju=\mathcal{M}_ju$ the latter yields

$$\begin{aligned}\mathcal{U}_{\mathbf{Q}}u&=\sum_j\Phi_j\left(\sum_i\phi_i\left[\mathcal{U}_i\left(u-\tau_iu\right)+\tau_iu-\mathcal{M}_ju\right]+\mathcal{M}_ju\right)\\&=\sum_i\sum_j\Phi_j\left(\phi_i\,\left(\mathcal{U}_i(u-\tau_iu)+\tau_iu-\mathcal{M}_ju\right)+\mathcal{M}_ju\right)\,,\end{aligned}$$

where we used that $\mathcal{U}_i$ maps constants onto constants via the identity. Note that

$$\mathcal{U}_{\mathbf{Q}}u\neq\sum_i\phi_i\left[\mathcal{U}_i\left(u-\tau_iu\right)+\tau_iu\right]\,,$$

as $\sum_{i\neq0}\phi_i\neq1$ in most points.

Theorem 6.3 (Extensions for locally regular, isotropic cone mixing geometries)**.** *Let the open set $\mathbf{P}$ be locally (δ,M)-regular, δ bounded by $\frac{\mathfrak{r}}{2}>0$, and satisfy Assumptions 5.5, 6.1 and $\hat{d}$ be the constant from (5.8),. Let $1<r<s<t<p<+\infty$ and $s<p_0\leq p$ with $1-\frac{\hat{d}}{r}\geq\frac{\hat{d}}{s}$.*

Recalling (5.4) and defining $\mathbf{Q}_{\mathfrak{r}}:=\mathbb{B}_{\mathfrak{r}}(\mathbf{Q})$ as well as

$$\mathcal{U}u:=\sum_i\sum_j\Phi_j\left(\phi_i\,\left(\mathcal{U}_i(u-\tau_iu)+\tau_iu-\mathcal{M}_ju\right)+\mathcal{M}_ju\right)\tag{6.6}$$

the following estimates hold:

$$\begin{aligned}\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\backslash\mathbf{P}}|\nabla\mathcal{U}u|^r&\leq C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}_{\mathfrak{r}}\cap\mathbf{P}}\tilde{M}^{\frac{p(\hat{d}+\alpha)}{p-r}}\right)^{\frac{p-r}{p}}\|\nabla u\|^{\frac{r}{p}}_{L^p\left(\mathbf{P}\cap\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q}_{\mathfrak{r}})\right)}\\&\quad+\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}^{\sim\sim}}|f(u)|^r\end{aligned}\tag{6.7}$$

$$\begin{aligned}\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\backslash\mathbf{P}}|\mathcal{U}u|^r&\leq C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}_{\mathfrak{r}}\cap\mathbf{P}}\tilde{M}^{\frac{p\hat{d}}{p-r}}\right)^{\frac{p-r}{p}}\cdot\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}}|u|^p\right)^{\frac{r}{p}}\\&\quad+C_0\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbf{P}}\left(\sum_{x_j\in\mathbb{X}_{\mathfrak{r}}(\mathbf{Q})}\chi_{G_j}\left|\mathfrak{A}_{1,j}\right|\right)^{\frac{p}{p-r}}\right)^{\frac{p-r}{p}}\cdot\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}}|u|^p\right)^{\frac{r}{p}}\,,\end{aligned}\tag{6.8}$$

where

$$\begin{aligned}f(u)=&\sum_{l=1}^d\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{\partial_l\Phi_k\left|\partial_l\Phi_j\right|}{D^{\Phi}_{l+}}\left(2-\phi_0\right)\left(\mathcal{M}_ku-\mathcal{M}_ju\right)\\&-\sum_{l=1}^d\sum_{i\neq0:\,\partial_l\phi_i\partial_l\phi_0<0}\sum_j\frac{\partial_l\phi_0\left|\partial_l\phi_i\right|}{D_{l+}}\Phi_j\left(\tau_iu-\mathcal{M}_ju\right)\,.\end{aligned}$$

with functions

$$D_{l+}:=\sum_{j\neq0:\,\partial_l\phi_j\partial_l\phi_0<0}\left|\partial_l\phi_j\right|\,,\qquad D^{\Phi}_{l+}:=\sum_{j\neq0:\,\partial_l\Phi_j<0}\left|\partial_l\Phi_j\right|\,.$$

Proof. Let us note that on $\mathbf{Q}$ it holds

$$\mathcal{U}u := \sum_i \phi_i \mathcal{U}_i(u-\tau_i u) + \sum_{x_j\in\mathbb{X}_\mathfrak{r}(\mathbf{Q})} \sum_i \Phi_j\phi_i\left(\tau_i u - \mathcal{M}_j u\right) + \sum_{x_j\in\mathbb{X}_\mathfrak{r}(\mathbf{Q})} \Phi_j\mathcal{M}_j u \tag{6.9}$$

$$= \sum_i \phi_i\,\left(\mathcal{U}_i(u-\tau_i u)+\tau_i u\right) + \sum_{x_j\in\mathbb{X}_\mathfrak{r}(\mathbf{Q})} \Phi_j\phi_0\mathcal{M}_j u\,. \tag{6.10}$$

We first observe in (6.10) that

$$\begin{aligned}\frac{1}{|\mathbf{Q}|}\int_\mathbf{Q}\left|\sum_{x_j\in\mathbb{X}_\mathfrak{r}(\mathbf{Q})}\Phi_j\phi_0\mathcal{M}_j u\right|^r &\leq \frac{1}{|\mathbf{Q}|}\int_\mathbf{Q}\sum_{x_j\in\mathbb{X}_\mathfrak{r}(\mathbf{Q})}\Phi_j\phi_0\left|\mathcal{M}_j u\right|^r \leq \frac{1}{|\mathbf{Q}|}\int_\mathbf{Q}\sum_{x_j\in\mathbb{X}_\mathfrak{r}(\mathbf{Q})}\chi_{\mathfrak{A}_{1,j}}\left|\mathcal{M}_j u\right|^r\\ &\leq \frac{1}{|\mathbf{Q}|}\sum_{x_j\in\mathbb{X}_\mathfrak{r}(\mathbf{Q})}\left|\mathfrak{A}_{1,j}\right|\left|\mathbb{S}^{d-1}\right|\left(\frac{\mathfrak{r}}{2}\right)^d\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_i)}|u|^r\\ &\leq \frac{1}{|\mathbf{Q}|}\int_\mathbf{Q}|u|^r\sum_{x_j\in\mathbb{X}_\mathfrak{r}(\mathbf{Q})}\chi_{G_j\cap\mathbf{P}}\left|\mathfrak{A}_{1,j}\right|\left|\mathbb{S}^{d-1}\right|\left(\frac{\mathfrak{r}}{2}\right)^d\end{aligned}$$

From the last inequality and Lemma 5.6 we obtain (6.8). Furthermore, the first term on the right hand side of (6.9) with Lemma 5.6 provides the first line of (6.7).

In what follows, we write for simplicity $\sum_{x_j\in\mathbb{X}_\mathfrak{r}(\mathbf{Q})} = \sum_j$ but have in mind the respective meaning. The same holds for $\sum_{k:\,\partial_l\Phi_k>0}$.

Concerning the second term in (6.9), we observe

$$\begin{aligned}&\nabla\sum_{x_j\in\mathbb{X}_\mathfrak{r}(\mathbf{Q})}\sum_{i\in\mathbb{N}}\Phi_j\phi_i\left(\tau_i u-\mathcal{M}_j u\right)\\ &\qquad = \sum_j\sum_{i\in\mathbb{N}}\phi_i\left(\tau_i u-\mathcal{M}_j u\right)\nabla\Phi_j + \sum_j\sum_{i\in\mathbb{N}}\Phi_j\left(\tau_i u-\mathcal{M}_j u\right)\nabla\phi_i\,,\end{aligned}$$

and obtain with help of Lemma 5.7 and $\sum_j\nabla\Phi_j(x)=0$ for $x\in\mathbf{Q}$

$$\begin{aligned}&\sum_j\sum_{i\in\mathbb{N}}\phi_i\left(\tau_i u-\mathcal{M}_j u\right)\nabla\Phi_j\\ &\quad=\sum_{l=1}^d\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{\partial_l\Phi_k\left|\partial_l\Phi_j\right|}{D^\Phi_{l+}}\left(\sum_{i\in\mathbb{N}}\phi_i\left(\tau_i u-\mathcal{M}_k u\right)-\sum_{i\in\mathbb{N}}\phi_i\left(\tau_i u-\mathcal{M}_j u\right)\right)\\ &\quad=\sum_{l=1}^d\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{\partial_l\Phi_k\left|\partial_l\Phi_j\right|}{D^\Phi_{l+}}\left(1-\phi_0\right)\left(\mathcal{M}_k u-\mathcal{M}_j u\right)\,.\end{aligned}$$

Similarly, we use Lemma 5.7 together with $\sum_i \nabla\phi_i = -\nabla\phi_0$ and find

$$\begin{aligned}
&\sum_j \sum_{i\in\mathbb{N}} \Phi_j\left(\tau_i u - \mathcal{M}_j u\right)\nabla\phi_i\,,\\
&\quad= \sum_{l=1}^d \sum_{k:\,\partial_l\phi_k>0}\sum_{i:\,\partial_l\phi_i<0} \frac{\partial_l\phi_k\left|\partial_l\phi_i\right|}{D_{l+}}\left(\sum_j \Phi_j\left(\tau_k u - \mathcal{M}_j u\right) - \sum_j \Phi_j\left(\tau_i u - \mathcal{M}_j u\right)\right)\\
&\qquad - \sum_{l=1}^d \sum_{i\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0} \frac{\partial_l\phi_0\left|\partial_l\phi_i\right|}{D_{l+}}\Phi_j\left(\tau_i u - \mathcal{M}_j u\right)\\
&\quad= \sum_{l=1}^d \sum_{k:\,\partial_l\phi_k>0}\sum_{j:\,\partial_l\phi_i<0} \frac{\partial_l\phi_k\left|\partial_l\phi_i\right|}{D_{l+}}\left(\tau_k u - \tau_i u\right)\\
&\qquad - \sum_{l=1}^d \sum_{i\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0} \frac{\partial_l\phi_0\left|\partial_l\phi_i\right|}{D_{l+}}\Phi_j\left(\tau_i u - \mathcal{M}_j u\right)\,,
\end{aligned}$$

where the first term on the right hand side can be estimated like in Lemma 5.6. Finally, from a similar calculation using Lemma 5.7 it is now obvious for the third term in (6.9) that

$$\sum_j \mathcal{M}_j u\nabla\Phi_j \leq \sum_{l=1}^d \sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0} \frac{\partial_l\Phi_k\left|\partial_l\Phi_j\right|}{D^\Phi_{l+}}\left(\mathcal{M}_k u - \mathcal{M}_j u\right)\,.$$

□

6.2 The Issue of Connectedness

In Theorem 6.3 we discovered the integral $\int_{\mathbf{Q}} |f(u)|^r$ as part of the estimate for $\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\setminus\mathbf{P}} |\nabla\mathcal{U}u|^r$, where we recall that f was given through

$$\begin{aligned}
f(u) = &\sum_{l=1}^d \sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0} \frac{\partial_l\Phi_k\left|\partial_l\Phi_j\right|}{D^\Phi_{l+}}\left(2-\phi_0\right)\left(\mathcal{M}_k u - \mathcal{M}_j u\right)\\
&- \sum_{l=1}^d \sum_{i\neq 0:\,\partial_l\phi_i\partial_l\phi_0<0}\sum_j \frac{\partial_l\phi_0\left|\partial_l\phi_i\right|}{D_{l+}}\Phi_j\left(\tau_i u - \mathcal{M}_j u\right)\,.
\end{aligned}$$

We seek for an interpretation of the two sums appearing on the right hand side. The first one is related to the difference of mean values around x_k and x_j in case they are weak neighbors, i.e. $x_k \sim\!\sim x_j$. In Theorem 6.7 below we provide a rough estimate on this part in terms of $\tau_i u - \mathcal{M}_j u$ but on a larger area. In the present section, we first want to "isolate" $|\tau_i u - \mathcal{M}_j u|$ and $|\mathcal{M}_k u - \mathcal{M}_j u|$ from the other geometric properties of $\mathbf{P}$. In Section 7 we will see how these quantities are related to the connectivity of $\mathbf{P}$.

Lemma 6.4. *Under Assumptions 5.5, 6.1 and using the notation of Theorem 6.3 let $(f_j)_{j\in\mathbb{N}}$ be non-negative and have support $\operatorname{supp} f_j \supset \mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_j)$ and let $\sum_{j\in\mathbb{N}} f_j \equiv 1$.*

Writing $\mathbb{X}(\mathbf{Q}) := \{x_j :\ \mathrm{supp} f_j \cap \mathbf{Q} \neq \emptyset\}$, *for every* $l = 1, \dots d$ *and* $r < \tilde{s} < s < p$ *it holds*

$$\left| \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}_\mathfrak{r}} \sum_{i \neq 0:\, \partial_l \phi_i \partial_l \phi_0 < 0} \sum_j \frac{\partial_l \phi_0 \, |\partial_l \phi_i|}{D_{l+}} f_j \left(\tau_i u - \mathcal{M}_j u\right) \right|$$

$$\leq \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{P} \cap \mathbf{Q}_\mathfrak{r} \cap \mathbb{R}^d_3} |\partial_l \phi_0|^{\frac{sr}{s-r}} \right)^{\frac{s-r}{s}} \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{P} \cap \mathbf{Q}_\mathfrak{r}} \sum_{i \neq 0:\, \partial_l \phi_i \partial_l \phi_0 < 0} \sum_{x_j \in \mathbb{X}(\mathbf{Q})} f_j \frac{|\partial_l \phi_i|}{D_{l+}} |\tau_i u - \mathcal{M}_j u|^s \right)^{\frac{r}{s}}$$

and

$$\left| \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}_\mathfrak{r}} \sum_{i \neq 0:\, \partial_l \phi_i \partial_l \phi_0 < 0} \sum_j \frac{\partial_l \phi_0 \, |\partial_l \phi_i|}{D_{l+}} f_j \left(\tau_i u - \mathcal{M}_j u\right) \right|$$

$$\leq \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{P} \cap \mathbf{Q}_\mathfrak{r} \cap \mathbb{R}^d_3} |\partial_l \phi_0|^{\frac{\tilde{s}r}{\tilde{s}-r}} \tilde{M}^{2-d} \right)^{\frac{\tilde{s}-r}{\tilde{s}}} \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathbf{Q}) \setminus \mathbf{P}} \tilde{M}^{\frac{p_1 (d-2)(\tilde{s}-r)}{r(s-\tilde{s})}} \right)^{r \frac{s-\tilde{s}}{\tilde{s}s}}$$
$$\left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{P} \cap \mathbf{Q}_\mathfrak{r}} \sum_{i \neq 0:\, \partial_l \phi_i \partial_l \phi_0 < 0} \sum_{x_j \in \mathbb{X}(\mathbf{Q})} f_j \frac{|\partial_l \phi_i|}{D_{l+}} |\tau_i u - \mathcal{M}_j u|^s \right)^{\frac{r}{s}} .$$

Proof. We find from Hölder's and Jensen's inequality

$$\frac{1}{|\mathbf{Q}|} \int_{\mathbf{P} \cap \mathbf{Q}} \left| \sum_{i \neq 0:\, \partial_l \phi_i \partial_l \phi_0 < 0} \sum_j \frac{\partial_l \phi_0 \, |\partial_l \phi_i|}{D_{l+}} f_j \left(\tau_i u - \mathcal{M}_j u\right) \right|^r$$
$$\leq \frac{1}{|\mathbf{Q}|} \int_{\mathbf{P} \cap \mathbf{Q}} |\partial_l \phi_0|^r \sum_{i \neq 0:\, \partial_l \phi_i \partial_l \phi_0 < 0} \sum_j \frac{|\partial_l \phi_i|}{D_{l+}} f_j |\tau_i u - \mathcal{M}_j u|^r$$
$$\leq \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{P} \cap \mathbf{Q} \cap \mathbb{R}^d_3} |\partial_l \phi_0|^{\frac{sr}{s-r}} \right)^{\frac{s-r}{s}} \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{P} \cap \mathbf{Q}} \sum_{i \neq 0:\, \partial_l \phi_i \partial_l \phi_0 < 0} \sum_j f_j \frac{|\partial_l \phi_i|}{D_{l+}} |\tau_i u - \mathcal{M}_j u|^s \right)^{\frac{r}{s}} .$$

The other inequality can be derived similarly, see also the proof of Lemma 5.8. □

Lemma 6.5. *Under Assumptions 5.5, 6.1 for every* $l = 1, \dots d$ *it holds*

$$\frac{1}{|\mathbf{Q}|} \int_{\mathbf{P} \cap \mathbf{Q}} \left| \sum_{k:\, \partial_l \Phi_k > 0} \sum_{j:\, \partial_l \Phi_j < 0} \frac{\partial_l \Phi_k \, |\partial_l \Phi_j|}{D^\Phi_{l+}} \left(2 - \phi_0\right) \left(\mathcal{M}_k u - \mathcal{M}_j u\right) \right|^r$$
$$\leq \left(\frac{1}{|\mathbf{Q}|} \int_{\mathbf{P} \cap \mathbf{Q}} \left(\sum_{j:\, \partial_l \Phi_j < 0} d_j^{\frac{r(d-1)+drs}{s-r}} \chi_{\nabla \Phi_j \neq 0} \right)^{\frac{s}{s-r}} \right)^{\frac{s-r}{s}} \frac{1}{|\mathbf{Q}|} \sum_{\substack{x_k \sim\sim x_j \\ x_k, x_j \in \mathbb{X}_\mathfrak{r}(\mathbf{Q})}} |\mathcal{M}_k u - \mathcal{M}_j u|^s$$

Proof. For this we observe with help of (6.4) and with Lemma 4.20.2)

$$\begin{aligned} \forall x :\quad \sup_k |\partial_l \Phi_k|(x) &\leq \sup \left\{ |\nabla \Phi_k(x)| \ :\ x \in \mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_k) \right\} \\ &\leq C \sup \left\{ d_k^d \ :\ x \in G_k \right\} , \end{aligned} \tag{6.11}$$

$$\sup_{x \in \mathbb{B}_{\frac{\mathfrak{r}}{2}}(G_j)} |\partial_l \Phi_j|(x) \leq C d_j^d . \tag{6.12}$$

We write

$$I := \frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\left|\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{\partial_l\Phi_k\,|\partial_l\Phi_j|}{D^{\Phi}_{l+}}\left(2-\phi_0\right)\left(\mathcal{M}_k u-\mathcal{M}_j u\right)\right|^r$$

and find

$$\begin{aligned}I &\leq C\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{|\partial_l\Phi_k|^r\,|\partial_l\Phi_j|}{D^{\Phi}_{l+}}\left|\mathcal{M}_k u-\mathcal{M}_j u\right|^r\\
&\leq CC\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\left(\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{d_j^{\alpha\frac{s}{s-r}}\,|\partial_l\Phi_k|^{\frac{sr}{s-r}}\,|\partial_l\Phi_j|}{D^{\Phi}_{l+}}\right)^{\frac{s-r}{s}}\dots\\
&\qquad\dots\left(\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{d_j^{-\alpha\frac{s}{r}}\,|\partial_l\Phi_j|}{D^{\Phi}_{l+}}\left|\mathcal{M}_k u-\mathcal{M}_j u\right|^s\right)^{\frac{r}{s}}.\end{aligned}$$

Now we make use of (6.11) and once more of Lemma 4.20.2) to obtain for the first bracket on the right hand side an estimate of the form

$$|\partial_l\Phi_k|^{\frac{sr}{s-r}}\,|\partial_l\Phi_j|\leq|\partial_l\Phi_k|\,|\partial_l\Phi_k|^{\frac{sr}{s-r}-1}\,|\partial_l\Phi_j|\leq C\,|\partial_l\Phi_k|\,d_j^{d\frac{sr-s+r}{s-r}}\,d_j^d\leq C\,|\partial_l\Phi_k|\,d_j^{d\frac{sr}{s-r}}\,,$$

which implies

$$\begin{aligned}\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{d_j^{\alpha\frac{s}{s-r}}\,|\partial_l\Phi_k|^{\frac{sr}{s-r}}\,|\partial_l\Phi_j|}{D^{\Phi}_{l+}} &\leq C\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{d_j^{\alpha\frac{s}{s-r}}\,d_j^{\frac{dsr}{s-r}}\,|\partial_l\Phi_k|}{D^{\Phi}_{l+}}\\
&\leq C\sum_{j:\,\partial_l\Phi_j<0}d_j^{\alpha\frac{s}{s-r}}\,d_j^{\frac{dsr}{s-r}}\,\chi_{\nabla\Phi_j\neq0}\,,\end{aligned}$$

where we used $\sum|\partial_l\Phi_k| = D^{\Phi}_{l+}$. We make use of $\alpha = rs^{-1}(d-1)$ in the above estimates and Hölder's inequality to find

$$\begin{aligned}&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\left|\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{\partial_l\Phi_k\,|\partial_l\Phi_j|}{D^{\Phi}_{l+}}\left(2-\phi_0\right)\left(\mathcal{M}_k u-\mathcal{M}_j u\right)\right|^r\\
&\quad\leq C\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\left(\sum_{j:\,\partial_l\Phi_j<0}d_j^{\frac{r(d-1)+drs}{s-r}}\chi_{\nabla\Phi_j\neq0}\right)^{\frac{s}{s-r}}\right)^{\frac{s-r}{s}}\dots\\
&\qquad\dots\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{d_j^{1-d}\,|\partial_l\Phi_j|}{D^{\Phi}_{l+}}\left|\mathcal{M}_k u-\mathcal{M}_j u\right|^s\right)^{\frac{r}{s}}.\end{aligned}$$

Since $\int_{\mathbf{P}\cap\mathbf{Q}}\frac{d_j^{1-d}|\partial_l\Phi_j|}{D^{\Phi}_{l+}}\leq C$ for some $C>0$ independent from j, we obtain

$$\begin{aligned}&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\sum_{k:\,\partial_l\Phi_k>0}\sum_{j:\,\partial_l\Phi_j<0}\frac{d_j^{1-d}\,|\partial_l\Phi_j|}{D^{\Phi}_{l+}}\left|\mathcal{M}_k u-\mathcal{M}_j u\right|^s\\
&\quad\leq\frac{1}{|\mathbf{Q}|}\sum_{x_k\sim\sim x_j}\left|\mathcal{M}_k u-\mathcal{M}_j u\right|^s\,.\end{aligned}$$

□

6.3 Estimates Related to Mesoscopic Regularity of the Geometry

Assumption 6.6 (Mesoscopic Regularity). *Under the Assumption 6.1 and introducing the notation* $\mathcal{I}_i := \left\{x_j \in \mathbb{X}_\mathfrak{r}\,:\,\mathcal{H}^{d-1}\left(\partial G_i \cap \partial G_j\right) \geq 0\right\}$ *we construct* $\mathfrak{A}_{2,i}$ *and* $\mathfrak{A}_{3,i}$ *from* $\mathfrak{A}_{1,i}$ *by*

$$\mathfrak{A}_{2,i} := \mathbb{B}_{2d_i}(\mathfrak{A}_{1,i})\,, \qquad \mathfrak{A}_{3,i} := \mathbb{B}_{2d_i+\mathfrak{r}}(\mathfrak{A}_{2,i})\,. \tag{6.13}$$

We infer from Lemma 5.6 that $\mathcal{U}\,:\,W^{1,p}(\mathfrak{A}_{3,i}) \to W^{1,r}(\mathfrak{A}_{2,i})$ *is continuous with the estimate and constants given by Lemma 5.6.*

Theorem 6.7 (Extensions for mesoscopic regular, isotropic cone mixing geometries). *Let* $\mathbf{P}(\omega)$ *be an open connected set and let Assumption 6.6 hold. Let* $\mathbf{P}$ *be locally* (δ, M)*-regular and satisfy Assumptions 5.5, 6.1 and* $\hat{d}$ *be the constant from (5.8). Then for almost every* ω *it holds: for every* $l = 1, \dots, d$ *and* $1 \leq r < s, \tilde{s} < p$*:*

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_\mathbf{Q}\left|\sum_{k:\,\partial_l\Phi_k>0}\ \sum_{j:\,\partial_l\Phi_j<0}\frac{\partial_l\Phi_k\left|\partial_l\Phi_j\right|}{D^\Phi_{l+}}\left(2-\phi_0\right)\left|\mathcal{M}_j u - \mathcal{M}_k u\right|\right|^r\\
&\qquad\leq C(\mathbf{P}(\omega))\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\cap\mathbf{P}}|\nabla u|^p\right)^{\frac{r}{p}}\\
&\qquad\quad + C_P(\mathbf{P}(\omega))\sum_l\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\backslash\mathbf{P}}\sum_k\frac{\chi_{\mathfrak{A}_{3,k}}}{\mathfrak{a}}\left|\nabla\phi_0\right|^{\tilde{s}}\sum_{j\neq 0:\,\partial_l\phi_j\partial_l\phi_0<0}\frac{\left|\partial_l\phi_j\right|}{D_{l+}}\left|\tau_j u - \mathcal{M}_k u\right|^{\tilde{s}}\right)^{\frac{r}{\tilde{s}}}
\end{aligned}\tag{6.14}$$

where with $P(x) = x^{d(2r-1)+r}(x^{r+1}+x^{d+1})$, $\mathfrak{a} := \sum_k \chi_{\mathfrak{A}_{3,k}}$ *and it holds*

$$\begin{aligned}
C(\mathbf{P}(\omega)) &= \left(\frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\left(\sum_k P(d_k)\chi_{\mathfrak{A}_{3,k}}\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}}\left(\frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\tilde{M}^{\frac{2p\hat{d}}{s-r}}\right)^{\frac{s-r}{p}}\\
C_P(\mathbf{P}(\omega)) &= \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\backslash\mathbf{P}}\sum_k P(d_k)^{\frac{\tilde{s}}{\tilde{s}-r}}\mathfrak{a}^{\frac{\tilde{s}}{\tilde{s}-r}}\frac{\chi_{\mathfrak{A}_{3,k}}}{\mathfrak{a}}\right)^{\frac{\tilde{s}-r}{\tilde{s}}}
\end{aligned}$$

Remark 6.8. A combination with Lemma 6.4 is possible.

Proof. We make use of (6.3) as well as the following observation: for each $k = 1, \dots K$ let $\alpha_k \geq K$. Then

$$\left(\sum_{k=1}^K f_k\right)^r \leq K^{r-1}\sum_{k=1}^K f_k^r \leq \sum_{k=1}^K \alpha_k^{r-1} f_k^r\,.$$

Hence

$$\begin{aligned}
&\left|\sum_{k:\,\partial_l\Phi_k>0}\ \sum_{j:\,\partial_l\Phi_j<0}\frac{\partial_l\Phi_k\left|\partial_l\Phi_j\right|}{D^\Phi_{l+}}\left(2-\phi_0\right)\left|\mathcal{M}_j u - \mathcal{M}_k u\right|\right|^r\\
&\qquad\leq \sum_{k:\,\partial_l\Phi_k>0}\ \sum_{j:\,\partial_l\Phi_j<0}\left(\frac{4d_k}{\mathfrak{r}}\right)^{d(r-1)}\frac{\left|\partial_l\Phi_k\right|^r\left|\partial_l\Phi_j\right|}{D^\Phi_{l+}}2\left|\mathcal{M}_j u - \mathcal{M}_k u\right|^r
\end{aligned}$$

Given $|\nabla\Phi_k| \leq \chi_{\mathfrak{A}_{1,k}}\left(\frac{4d_k}{\mathfrak{r}}\right)^d$ we hence find an estimate by

$$\sum_{k:\,\partial_l\Phi_k>0}\;\sum_{j:\,\partial_l\Phi_j<0}\left(\frac{4d_k}{\mathfrak{r}}\right)^{d(2r-1)}\chi_{\mathfrak{A}_{1,k}}\frac{|\partial_l\Phi_j|}{D^\Phi_{l+}}2\left|\mathcal{M}_j u-\mathcal{M}_k u\right|^r\,.$$

Next, we obtain

$$|\mathcal{M}_j u-\mathcal{M}_k u|^r \leq \left|\mathfrak{r}^d\mathbb{S}^{d-1}\right|^{-1}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_j)}|u-\mathcal{M}_k u|^r$$

and thus

$$\begin{aligned}
&\left|\sum_{k:\,\partial_l\Phi_k>0}\;\sum_{j:\,\partial_l\Phi_j<0}\frac{\partial_l\Phi_k\,|\partial_l\Phi_j|}{D^\Phi_{l+}}\left(2-\phi_0\right)|\mathcal{M}_j u-\mathcal{M}_k u|\right|^r\\
&\quad\leq 2\left|\mathfrak{r}^d\mathbb{S}^{d-1}\right|^{-1}\sum_k\;\sum_{j:\,\partial_l\Phi_j<0}\left(\frac{4d_k}{\mathfrak{r}}\right)^{d(2r-1)}\chi_{\mathfrak{A}_{1,k}}\frac{|\partial_l\Phi_j|}{D^\Phi_{l+}}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_j)}|u-\mathcal{M}_k u|^r\\
&\quad\leq 2\left|\mathfrak{r}^d\mathbb{S}^{d-1}\right|^{-1}\sum_k\left(\frac{4d_k}{\mathfrak{r}}\right)^{d(2r-1)}\chi_{\mathfrak{A}_{1,k}}\int_{\mathfrak{A}_{2,k}}|u-\mathcal{M}_k u|^r\\
&\quad\leq 2\left|\mathfrak{r}^d\mathbb{S}^{d-1}\right|^{-1}\sum_k\left(\frac{4d_k}{\mathfrak{r}}\right)^{d(2r-1)}\chi_{\mathfrak{A}_{1,k}}C_k\int_{\mathfrak{A}_{2,k}}|\nabla\mathcal{U}u|^r\\
&\quad\leq C\sum_k\left(\frac{4d_k}{\mathfrak{r}}\right)^{d(2r-1)}\chi_{\mathfrak{A}_{1,k}}C_k\hat{C}_{k,r,s}\left(\frac{1}{|\mathfrak{A}_{3,k}|}\int_{\mathfrak{A}_{3,k}\cap\mathbf{P}}|\nabla u|^s\right)^{\frac{r}{s}}\\
&\quad\quad+C\sum_k\left(\frac{4d_k}{\mathfrak{r}}\right)^{d(2r-1)}\chi_{\mathfrak{A}_{1,k}}C_k\frac{C_0}{|\mathfrak{A}_{3,k}|}\int_{\mathfrak{A}_{3,k}\setminus\mathbf{P}}|\nabla\phi_0|^r\sum_{j\neq0:\,\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}|\tau_j u-\mathcal{M}_k u|^r
\end{aligned}$$

where according to Lemmas 2.6 and 5.6 for some C depending only on r and $\mathfrak{r}$:

$$\begin{aligned}
C_k&=Cd_k^r\left(d_k^{r+1}+d_k^{d+1}\right)\,,\\
\hat{C}_{k,r,s}&=\left(\fint_{\mathfrak{A}_{3,k}\cap\mathbf{P}}\tilde{M}^{\frac{2s\hat{d}}{s-r}}\right)^{\frac{s-r}{s}}\,.
\end{aligned}$$

We integrate with respect to $\mathbf{Q}$ and obtain with $P(x)=x^{d(2r-1)+r}(x^{r+1}+x^{d+1})$

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\sum_k\left(\frac{4d_k}{\mathfrak{r}}\right)^{d(2r-1)}\chi_{\mathfrak{A}_{1,k}}C_k\hat{C}_{k,r,s}\left(\frac{1}{|\mathfrak{A}_{3,k}|}\int_{\mathfrak{A}_{3,k}\cap\mathbf{P}}|\nabla u|^s\right)^{\frac{r}{s}}\\
&\quad\leq\sum_k\left(\frac{4d_k}{\mathfrak{r}}\right)^{d(2r-1)}|\mathfrak{A}_{1,k}|\,C_k\hat{C}_{k,r,s}\left(\frac{1}{|\mathfrak{A}_{3,k}|}\int_{\mathfrak{A}_{3,k}\cap\mathbf{P}}|\nabla u|^s\right)^{\frac{r}{s}}\\
&\quad\leq C\left(\frac{1}{|\mathbf{Q}|}\sum_k P(d_k)\frac{|\mathfrak{A}_{1,k}|}{|\mathfrak{A}_{3,k}|}\int_{\mathfrak{A}_{3,k}\cap\mathbf{P}}|\nabla u|^s\right)^{\frac{r}{s}}\left(\frac{1}{|\mathbf{Q}|}\sum_k P(d_k)\frac{|\mathfrak{A}_{1,k}|}{|\mathfrak{A}_{3,k}|}\int_{\mathfrak{A}_{3,k}\cap\mathbf{P}}\tilde{M}^{\frac{2s\hat{d}}{s-r}}\right)^{\frac{s-r}{s}}\,.
\end{aligned}\tag{6.15}$$

For the measurable function $g=\tilde{M}^{\frac{2s\hat{d}}{s-r}}$ on $\mathbb{R}^d$ we find for every $\frac{1}{\tilde{p}}+\frac{1}{\tilde{q}}=1$

$$\begin{aligned}
&\frac{C}{|\mathbf{Q}|}\sum_k P(d_k)\frac{|\mathfrak{A}_{1,k}|}{|\mathfrak{A}_{3,k}|}\int_{\mathfrak{A}_{3,k}\cap\mathbf{P}} g(x)\,\mathrm{d}x\\
&\quad\leq \frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}} g(x)\sum_k P(d_k)\chi_{\mathfrak{A}_{3,k}}(x)\,\mathrm{d}x\\
&\quad\leq \left(\frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}} g^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}}\left(\frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\left(\sum_k P(d_k)\chi_{\mathfrak{A}_{3,k}}\right)^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}}.
\end{aligned} \tag{6.16}$$

For the remaining expression note that

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\sum_k\left(\frac{4d_k}{\mathfrak{r}}\right)^{d(2r-1)}\chi_{\mathfrak{A}_{1,k}}C_k\frac{C_0}{|\mathfrak{A}_{3,k}|}\int_{\mathfrak{A}_{3,k}\setminus\mathbf{P}}|\nabla\phi_0|^r\sum_{j\neq 0\colon\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}|\tau_j u-\mathcal{M}_k u|^r\\
&\quad\leq\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\setminus\mathbf{P}}\sum_k P(d_k)\chi_{\mathfrak{A}_{3,k}}|\nabla\phi_0|^r\sum_{j\neq 0\colon\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}|\tau_j u-\mathcal{M}_k u|^r
\end{aligned} \tag{6.17}$$

We denote the right hand side of (6.17) by I_1. Using $\mathfrak{a}$ we obtain from Hölder's inequality together with Jensen's inequality

$$I_l\leq C_P\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\setminus\mathbf{P}}\sum_k\frac{\chi_{\mathfrak{A}_{3,k}}}{\mathfrak{a}}|\nabla\phi_0|^{\tilde{s}}\sum_{j\neq 0\colon\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}|\tau_j u-\mathcal{M}_k u|^{\tilde{s}}\right)^{\frac{r}{\tilde{s}}}, \tag{6.18}$$

where C_P and $\mathfrak{a}$ are defined in the statement and where we used $\sum_k\frac{\chi_{\mathfrak{A}_{3,k}}}{\mathfrak{a}}\equiv 1$ and $\sum\frac{|\partial_l\phi_j|}{D_{l+}}\equiv 1$.

Taking together (6.15)–(6.18) we conclude for $\tilde{p}=\frac{p}{s}$ and with boundedness $0<c<\frac{|\mathfrak{A}_{1,k}|}{|\mathfrak{A}_{3,k}|}<C<\infty$. □

6.4 Extension for Statistically Harmonic Domains

Definition 6.9. A random geometry $\mathbf{P}(\omega)$ is statistically s-harmonic if there exist constants $C_k>0$, $k\in\mathbb{N}$ and sets $\mathfrak{A}_{4,k}\supset\mathfrak{A}_{3,k}$ such that for every $x_k\in\mathbb{X}_\omega$

$$\int_{\mathfrak{A}_{3,k}\cap\mathbb{R}^d_3\cap\mathbf{P}}|u-\mathcal{M}_k u|^s\leq\int_{\mathfrak{A}_{4,k}\cap\mathbf{P}}C_k|\nabla u|^s\,.$$

Theorem 6.10. *Let $\mathbf{P}(\omega)$ be a stationary ergodic random open set which is (δ,M)-regular, isotropic cone mixing for $\mathfrak{r}>0$ and $f(R)$, statistically s-harmonic and let Assumption 6.6 hold. Then for every*

$l=1,\dots,d$ *and* $1\leq r<s<p$ *and every* $1<\alpha,\tilde{p}<\infty$ *it holds*

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\setminus\mathbf{P}}\sum_k P(d_k)\chi_{\mathfrak{A}_{3,k}}\left|\nabla\phi_0\right|^r\sum_{j\neq 0\colon\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}\left|\tau_j u-\mathcal{M}_k u\right|^r\\
&\quad\leq\left(\frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}\cap\mathbb{R}^d_3}\left(\sum_k P(d_k)\chi_{\mathfrak{A}_{3,k}}\right)^{\frac{\tilde{p}}{\tilde{p}-1}}\right)^{\frac{(s-1)(\tilde{p}-1)}{\tilde{p}s}}\left(\frac{C}{|\mathbf{Q}|}\int_{\partial\mathbf{P}\cap\mathbf{Q}}\tilde{\delta}^{1-\alpha r\tilde{p}\frac{s}{s-r}}\right)^{\frac{1}{\alpha\tilde{p}}}\cdot\\
&\quad\left(\frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}\cap\mathbb{R}^d_3}\left(\left(\tilde{M}^{d+r}\right)^{\tilde{p}\frac{s}{s-r}}\tilde{M}^{\frac{(d-2)}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha\tilde{p}}}\cdot\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}|\nabla u|^p\right)^{\frac{s}{p}}\cdot\\
&\quad\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\sum_k P(d_k)\chi_{\mathfrak{A}_{4,k}}C_k\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{ps}r}
\end{aligned}$$

Proof. We make use of $|\nabla\phi_0|\leq C\rho_j$ on $A_{1,j}$ as well as the definition of $\tau_j u$ to obtain that the latter expression is bounded by (compare also with the calculation leading to (5.16))

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\setminus\mathbf{P}}\sum_k P(d_k)\chi_{\mathfrak{A}_{3,k}}\left|\nabla\phi_0\right|^r\sum_{j\neq 0\colon\partial_l\phi_j\partial_l\phi_0<0}\frac{|\partial_l\phi_j|}{D_{l+}}\left|\tau_j u-\mathcal{M}_k u\right|^r\\
&\quad\leq\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}\setminus\mathbf{P}}\sum_k P(d_k)\chi_{\mathfrak{A}_{3,k}}\sum_{j\neq 0}\rho_j^{-r}\rho_j^{-d}M_j^d\chi_{A_{1,j}}\int_{\mathbb{B}_{\mathfrak{r}_j}(y_j)\cap\mathfrak{A}_{3,k}}|u-\mathcal{M}_k u|^r\\
&\quad\leq\frac{1}{|\mathbf{Q}|}\sum_k P(d_k)\sum_{j\neq 0}\rho_j^{-r}M_j^d\int_{\mathbb{B}_{\mathfrak{r}_j}(y_j)\cap\mathfrak{A}_{3,k}}|u-\mathcal{M}_k u|^r\\
&\quad\leq\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\sum_k P(d_k)\sum_{j\neq 0}\rho_j^{-r}M_j^d\chi_{\mathbb{B}_{\mathfrak{r}_j}(y_j)\cap\mathfrak{A}_{3,k}}|u-\mathcal{M}_k u|^r\\
&\quad\leq\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\sum_k P(d_k)\sum_{j\neq 0}\left(\rho_j^{-r}M_j^d\right)^{\frac{s}{s-r}}\chi_{\mathbb{B}_{\mathfrak{r}_j}(y_j)\cap\mathfrak{A}_{3,k}}\right)^{\frac{s-r}{s}}\\
&\qquad\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\sum_k P(d_k)\sum_{j\neq 0}\chi_{\mathbb{B}_{\mathfrak{r}_j}(y_j)\cap\mathfrak{A}_{3,k}}|u-\mathcal{M}_k u|^s\right)^{\frac{r}{s}}.
\end{aligned}$$

We use that $\mathbb{B}_{\mathfrak{r}_j}(y_j)$ are mutually disjoint and $\mathbb{B}_{\mathfrak{r}_j}(y_j)\subset\mathbb{R}^d_3$ to find

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\sum_k P(d_k)\sum_{j\neq 0}\chi_{\mathbb{B}_{\mathfrak{r}_j}(y_j)\cap\mathfrak{A}_{3,k}}|u-\mathcal{M}_k u|^s\\
&\quad\leq\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\sum_k P(d_k)\chi_{\mathfrak{A}_{3,k}\cap\mathbb{R}^d_3}|u-\mathcal{M}_k u|^s\\
&\quad\leq\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\sum_k P(d_k)\chi_{\mathfrak{A}_{4,k}}C_k\,|\nabla u|^s
\end{aligned}$$

where we have used the statistical s-connectedness. Similar to the proof of Theorem 6.7 we observe that

$$\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\sum_k P(d_k)\sum_{j\neq 0}\left(\rho_j^{-r}M_j^d\right)^{\frac{s}{s-r}}\chi_{\mathbb{B}_{\mathfrak{r}_j}(y_j)\cap\mathfrak{A}_{3,k}}\mathrm{d}x\right)$$
$$\leq\left(\frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}\cap\mathbb{R}^d_3}\left(\tilde\rho^{-r}M_j^d\right)^{\tilde p\frac{s}{s-r}}\right)^{\frac{1}{\tilde p}}\left(\frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}\cap\mathbb{R}^d_3}\left(\sum_k P(d_k)\chi_{\mathfrak{A}_{3,k}}\right)^{\tilde q}\right)^{\frac{1}{\tilde q}}$$

and

$$\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\sum_k P(d_k)\chi_{\mathfrak{A}_{4,k}}C_k\left|\nabla u\right|^s$$
$$\leq\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\sum_k P(d_k)\chi_{\mathfrak{A}_{4,k}}C_k\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}}\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left|\nabla u\right|^p\right)^{\frac{s}{p}}.$$

Finally, Lemma 4.13 yields with $\tilde\rho\geq C\tilde\delta/\tilde M$

$$\frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}\cap\mathbb{R}^d_3}\left(\tilde\rho^{-r}\tilde M^d\right)^{\tilde p\frac{s}{s-r}}$$
$$\leq\left(\frac{C}{|\mathbf{Q}|}\int_{\partial\mathbf{P}\cap\mathbf{Q}}\tilde\delta^{1-\alpha r\tilde p\frac{s}{s-r}}\right)^{\frac{1}{\alpha}}\left(\frac{C}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}\cap\mathbb{R}^d_3}\left(\left(\tilde M^{d+r}\right)^{\tilde p\frac{s}{s-r}}\tilde M^{\frac{(d-2)}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}}$$

□

7 Construction of Macroscopic Extension Operators II: Admissible Paths

In this section, we will use admissible paths on connected sets in order to estimate the (so far uncontrolled) terms $|\tau_i u-\mathcal{M}_j u|$ in Theorems 6.3 and 6.7 in terms of ∇u.

Knowing there exists an admissible path (by Theorem 4.39), it remains to deal with the non-uniqueness of the path. Note there is no clear distinction which puts one path in favor of others. While this could be seen as a drawback, it can also be considered as an opportunity, since it allows to distribute the "weight" of integration along the paths more uniformly among the total volume. This is the basic idea of this section.

7.1 Preliminaries

Given an open connected set $\mathbf{P}$ and a countable family of points $\mathbb{X}_\mathfrak{r}$ satisfying Assumption 6.1 we extend the covering $A_{1,j}$ resp. $A_{2,j}$ of $\partial\mathbf{P}$ from Section 5.1 (e.g. (5.1)) to the inner of $\mathbf{P}$ using Lemma 4.25. In this context, we remind the reader of (4.25) and Definition 4.27 and introduce the notation

$$A_1(y)=\begin{cases}A_{1,k} & \text{if } y=p_k\in\mathbb{X}_\partial\\ \mathbb{B}_{\tilde\eta(y)}(y) & \text{if } y\in\mathring{\mathbb{Y}}\end{cases},\quad A_2(y)=\begin{cases}A_{2,k} & \text{if } y=p_k\in\mathbb{X}_\partial\\ \mathbb{B}_{3\frac{1}{2}\tilde\eta(y)}(y) & \text{if } y\in\mathring{\mathbb{Y}}\end{cases}.$$

We find the following

Lemma 7.1. *There exists $C>0$ independent from $\mathbf{P}$ such that for every $x\in\mathbf{P}$*

$$\#\left\{y\in\mathring{\mathbb{Y}}:\ x\in A_2(y)\right\}\leq C\,.$$

Proof. For two points $p_i,p_j\in\partial\mathbb{X}$ such that $x\in A_{2,i}\cap A_{2,j}$ it holds due to the triangle inequality

$$|p_i-p_j|\leq|x-p_j|+|p_i-x|\leq 3\left(\tilde\rho_i+\tilde\rho_j\right)\leq\max\left\{6\tilde\rho_i,6\tilde\rho_j\right\}\,. \tag{7.1}$$

Let $\mathbb{X}_\partial(x):=\left\{p_i\in\partial\mathbb{X}:\ x\in\overline{\mathbb{B}_{3\tilde\rho_i}(p_i)}\right\}$ and choose $\tilde p\in\mathbb{X}_\partial(x)$ such that $\tilde\rho_{\mathsf{m}}:=\tilde\rho(\tilde p)$ is maximal. Then $\mathbb{X}_\partial(x)\subset\mathbb{B}_{6\tilde\rho_{\mathsf{m}}}(\tilde p)$ by (7.1) and every $p_i\in\mathbb{X}_\partial(x)$ satisfies $\tilde\rho_{\mathsf{m}}>\tilde\rho_i>\frac13\tilde\rho_{\mathsf{m}}$ (Lemma 2.12). In view of (4.7) this lower local bound of $\tilde\rho_i$ implies a lower local bound on the mutual distance of the p_i. Since this distance is proportional to $\tilde\rho_{\mathsf{m}}$, and since $\tilde\rho_{\mathsf{m}}>\tilde\rho_i>\frac13\tilde\rho_{\mathsf{m}}$, this implies for some constant $C>0$ independent of x or $\mathbf{P}$ that

$$\#\left\{y\in\partial\mathbb{X}:\ x\in A_2(y)\right\}\leq C\,.$$

Now let $y\in\mathring{\mathbb{Y}}\backslash\partial\mathbb{X}$ and $x\in A_2(y)=\mathbb{B}_{\frac78\eta}(y)$. We show

$$\eta(y)<8\eta(x)<16\eta(y)\,.$$

For the first inequality, observe that $\eta(x)\leq\frac18\eta(y)$ is equivalent with $\mathrm{dist}(x,\partial\mathbf{P})\leq\frac18\mathrm{dist}(y,\partial\mathbf{P})$ and hence

$$\begin{aligned}
&& \mathrm{dist}(y,\partial\mathbf{P}) &\leq\mathrm{dist}(x,\partial\mathbf{P})+|x-y|\\
&& &\leq\frac18\mathrm{dist}(y,\partial\mathbf{P})+|x-y|\\
\Rightarrow && |x-y| &\geq\frac78\mathrm{dist}(y,\partial\mathbf{P})\,.
\end{aligned}$$

For the second inequality, assume $\tilde\eta(y)<\tilde\eta(x)$. Then y lies closer to the boundary than x and $x\in A_2(y)$ implies

$$\eta(x)=\mathrm{dist}\left(x,\partial\mathbf{P}\right)\leq\mathrm{dist}(y,\partial\mathbf{P})+|x-y|\leq\eta(y)+\frac72\tilde\eta(y)\leq2\eta(y)\,.$$

The mutual minimal distance of neighboring points in terms of $\tilde\eta$ now implies for some C independent from x and $\mathbf{P}$

$$\#\left\{y\in\mathring{\mathbb{Y}}\backslash\partial\mathbb{X}:\ x\in A_2(y)\right\}\leq C\,.$$

□

Definition 7.2. Let $\mathbb{G}_*(\mathbf{P})$ be a connected sub-graph of $\mathbb{G}_0(\mathbf{P})$. Let $x_i\in\mathbb{X}_{\mathfrak{r}}$ and $u_i:=u_{x_i}$ be the solution of the discrete Laplace equation (4.30) for $x=x_i$ on the graph $\mathbb{G}_*(\mathbf{P})$. For every $z\in\mathring{\mathbb{Y}}\backslash\{x_i\}$ let

$$\mathbb{O}_{*,i}(z):=\left\{\tilde y\in\mathring{\mathbb{Y}}:\ u_i(\tilde y)>u_i(z)\right\}$$

the neighbors corresponding the *outgoing branches of admissible paths through* y, and we assign to each $\tilde y\in\mathbb{O}_{*,i}(z)$ the weight $w_*(z,\tilde y)=w_{*,1,2}(z,\tilde y)$ of the branch $(z,\tilde y)$ where either

$$\begin{aligned}
w_{*,1}(z,\tilde y)&=\left(u_i(\tilde y)-u_i(z)\right)/\left(\sum_{y\in\mathbb{O}_{*,i}(z)}\left(u_i(y)-u_i(z)\right)\right)\,,\\
w_{*,2}(z,\tilde y)&=\#\mathbb{O}_{*,i}(z)^{-1}\,.
\end{aligned}$$

For $Y=(y_1,\dots y_N)\in\mathbb{AX}_*(p_j,x_i)$ we define the weight of the path Y by

$$W_*(Y):=W_*(y_1,\dots y_N):=\prod_{i=1}^{N-1}w_*(y_i,y_{i+1})\,.$$

Remark 7.3. We observe

$$\sum_{Y\in\mathbb{AX}_*(p_j,x_i)}W_*(Y)=1\,.$$

This holds by induction along the path and different branches since in every $z\in\mathring{\mathbb{Y}}\backslash\{x_i\}$ it holds $\sum_{y\in\mathbb{O}_{*,i}(z)}w_*(z,y)=1.$

7.2 Extension for Connected Domains

In this section, we discuss how the graphs built in Section 4.3 can be used to derive estimates on $f(u)$ given in Theorem 6.3. The remaining constant on the right hand side is given in terms of the balls $\mathbb{B}_{\mathfrak{r}_i}(p_i)$ and length of the paths between p_i and x_j or x_j and x_k respectively. Although one could go even more into details and try to generally decouple these effects, this is not helpful for our examples in Section 8 below. Hence we leave the results of this section as they are but encourage further investigation in the future.

The idea

We first consider the case of a general graph $(\mathbb{Y},\mathbb{G}(\mathbf{P}))$ on $\mathbf{P}$ and do not claim that paths in the classes $\mathbb{AX}$ are fully embedded into $\mathbf{P}$. In particular, we drop for a moment the concept of local connectivity and we allow paths to intersect with $\mathbb{R}^d\backslash\mathbf{P}$. Let $x_j\in\mathbb{X}_{\mathfrak{r}}$, $p_i\in\mathbb{Y}_{\mathbb{X}\partial}$ and $Y=(y_1,\dots,y_N)\in\mathbb{AX}(p_i,x_j)$. In the following short calculation, one may think of $\widetilde{\nabla u}$ as a function related to $\nabla\left(\mathcal{U}u\right)$, though the following calculations will reveal that it is not exactly what we mean. Nevertheless, recalling Notation 4.33 for $Y(x)$ and $Y=\bigcup_x Y(x)$ it holds

$$\begin{aligned}
\left|\tau_i u-\mathcal{M}_j u\right|^s &= \left|\frac{1}{\left|\mathbb{B}_{\mathfrak{r}_i}(0)\right|}\int_{\mathbb{B}_{\mathfrak{r}_i}(0)}u(x+p_i)-\frac{1}{\left|\mathbb{B}_{\frac{\mathfrak{r}}{16}}(x_j)\right|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)}u(x+x_j)\right|^s\\
&=\left|\frac{1}{\left|\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\right|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)}\left(u\!\left(\frac{16}{\mathfrak{r}}\mathfrak{r}_i(x+p_i)\right)-u(x+x_j)\right)\right|^s\\
&\leq\sum_{Y\in\mathbb{AX}(p_i,x_j)}W(Y)\left|\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\right|^{-1}\int_{\mathbb{B}_{\mathfrak{r}_i}(0)}\left|\int_{Y(x)}\left|\widetilde{\nabla u}\right|\right|^s\mathrm{d}x\\
&\leq C\sum_{Y\in\mathbb{AX}(p_i,x_j)}W(Y)\left|\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\right|^{-1}\int_Y\left|\widetilde{\nabla u}\right|^s\mathrm{Length}(Y)^{\frac{s-1}{s}}\,.
\end{aligned}\tag{7.2}$$

Since $\widetilde{\nabla u}$ is related to $\nabla\mathcal{U}u$, the latter formula reveals that the terms $\left|\tau_i u-\mathcal{M}_j u\right|^s$ may lead to an "entanglement" of $\widetilde{M}_{\hat{\rho}}$ and the properties of the paths $\mathbb{AX}$. In what follows, we will resolve the latter calculation in more details to prepare this discussion.

In what follows, we will make use of $Y=(y_1=p_i,\dots y_N=x_j)$ and

$$u\left(\frac{16}{\mathfrak{r}}\mathfrak{r}_i(x+p_i)\right)-u(x+x_j)=\sum_{k=1}^{N-1}u\biggl(\frac{16}{\mathfrak{r}}\mathfrak{r}(y_k)x+y_k\biggr)-u\biggl(\frac{16}{\mathfrak{r}}\mathfrak{r}(y_{k+1})x+y_{k+1}\biggr),$$

and we write $Y(y_k,y_{k+1},x)$ for the straight line segment connecting $\frac{16}{\mathfrak{r}}\mathfrak{r}(y_k)x+y_k$ with $\frac{16}{\mathfrak{r}}\mathfrak{r}(y_{k+1})x+y_{k+1}$. We distinguish 4 cases:

Case $y_k,y_{k+1}\in\mathbb{Y}_{\partial\mathbb{X}}$: According to Lemma 5.2 it holds $\mathbb{B}_{\mathfrak{r}(y_{k+1})}(y_{k+1})\subset A_2(y_k)$ and if $\mathcal{U}_k:W^{1,p}(A_{3,k})\to W^{1,r}(A_{2,k})$ is the corresponding local extension operator it holds

$$u\biggl(\frac{16}{\mathfrak{r}}\mathfrak{r}(y_k)x+y_k\biggr)-u\biggl(\frac{16}{\mathfrak{r}}\mathfrak{r}(y_{k+1})x+y_{k+1}\biggr)\leq\int_{Y(y_k,y_{k+1},x)}\nabla\mathcal{U}_k u\,.$$

Case $y_k\in\mathbb{Y}_{\partial\mathbb{X}}$, $y_{k+1}\in\mathring{\mathbb{Y}}$: According to Lemma 4.25 it holds $\mathbb{B}_{\mathfrak{r}(y_{k+1})}(y_{k+1})\subset A_2(y_k)$ and if $\mathcal{U}_k:W^{1,p}(A_{3,k})\to W^{1,r}(A_{2,k})$ is the corresponding local extension operator it holds

$$u\biggl(\frac{16}{\mathfrak{r}}\mathfrak{r}(y_k)x+y_k\biggr)-u\biggl(\frac{16}{\mathfrak{r}}\mathfrak{r}(y_{k+1})x+y_{k+1}\biggr)\leq\int_{Y(y_k,y_{k+1},x)}\nabla\mathcal{U}_k u\,.$$

Case $y_{k+1}\in\mathbb{Y}_{\partial\mathbb{X}}$, $y_k\in\mathring{\mathbb{Y}}$: According to Lemma 4.25 it holds $\mathbb{B}_{\mathfrak{r}(y_k)}(y_k)\subset A_2(y_{k+1})$ and if $\mathcal{U}_{k+1}:W^{1,p}(A_{3,k+1})\to W^{1,r}(A_{2,k+1})$ is the corresponding local extension operator it holds

$$u\biggl(\frac{16}{\mathfrak{r}}\mathfrak{r}(y_k)x+y_k\biggr)-u\biggl(\frac{16}{\mathfrak{r}}\mathfrak{r}(y_{k+1})x+y_{k+1}\biggr)\leq\int_{Y(y_k,y_{k+1},x)}\nabla\mathcal{U}_{k+1} u\,.$$

Case $y_k,y_{k+1}\in\mathring{\mathbb{Y}}$: According to Lemma 4.25 it holds $\mathbb{B}_{\mathfrak{r}(y_k)}(y_k)\subset A_2(y_{k+1})\subset\mathbf{P}$ and

$$u\biggl(\frac{16}{\mathfrak{r}}\mathfrak{r}(y_k)x+y_k\biggr)-u\biggl(\frac{16}{\mathfrak{r}}\mathfrak{r}(y_{k+1})x+y_{k+1}\biggr)\leq\int_{Y(y_k,y_{k+1},x)}\nabla u\,.$$

However, in case of local connectivity, we face a simpler situation. In case $y_k,y_{k+1}\in\mathring{\mathbb{Y}}$ we can use the above estimates while in the other cases, we can use the Lemma 4.30.

Locally connected $\mathbf{P}$

In what follows, we consider $\mathbb{G}_*(\mathbf{P})=\mathbb{G}_{\text{flat}}(\mathbf{P})$ (see Definition 4.29) with a suitable family of admissible paths $\mathbb{AX}_{\text{flat}}$, and we also recall $Y(x)$ from Notation 4.33. We repeat the calculations of (7.2) in view of Lemma 4.30. In particular, if $\tilde{y}\sim y$ are connected via a path γ in $\mathbb{B}_{3\tilde{\rho}(x(y))}(x(y))$, which additionally has the property that the corresponding tube exists, then the length of γ is bounded by

$C\,|y-\tilde{y}|$, where C is determined by the dimension. Hence we have

$$\begin{aligned}
|\tau_i u-\mathcal{M}_j u|^s &= \left|\frac{1}{|\mathbb{B}_{\mathfrak{r}_i}(0)|}\int_{\mathbb{B}_{\mathfrak{r}_i}(0)} u(x+p_i)-\frac{1}{\left|\mathbb{B}_{\frac{\mathfrak{r}}{16}}(x_j)\right|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)} u(x+x_j)\right|^s\\
&= \left|\frac{1}{\left|\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\right|}\int_{\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)}\left(u\Big(\frac{16}{\mathfrak{r}}\mathfrak{r}_i(x+p_i)\Big)-u(x+x_j)\right)\right|^s\\
&\le \sum_{Y\in\mathbb{AX}_{\mathrm{flat}}(p_i,x_j)} W(Y)\frac{1}{\left|\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\right|}\int_{\mathbb{B}_{\mathfrak{r}_i}(0)}\left|\int_{Y(x)}|\nabla u|\right|^s\mathrm{d}x\\
&\le C\sum_{Y\in\mathbb{AX}_{\mathrm{flat}}(p_i,x_j)} W(Y)\frac{1}{\left|\mathbb{B}_{\frac{\mathfrak{r}}{16}}(0)\right|}\int_Y|\nabla u|^s\,\mathrm{Length}(Y)^{\frac{s-1}{s}}\,. \qquad (7.3)
\end{aligned}$$

The last calculation is at the heart of the results in this section. In what follows, we adopt the situation of Lemma 6.4:

Lemma 7.4. *Let $\mathbf{P}$ be locally connected. Under Assumptions 5.5, 6.1 and using the notation of Theorem 6.3 let $(f_j)_{j\in\mathbb{N}}$ be non-negative and have support $\mathrm{supp}f_j\supset\mathbb{B}_{\frac{\mathfrak{r}}{2}}(x_j)$ and let $\sum_{j\in\mathbb{N}}f_j\equiv 1$. Let $\mathbb{G}_*(\mathbf{P})=\mathbb{G}_{\mathrm{flat}}(\mathbf{P})$ (see Definition 4.29) with a suitable family of admissible paths $\mathbb{AX}_{\mathrm{flat}}$. Writing $\mathbb{X}(\mathbf{Q}):=\{x_j:\ \mathrm{supp}f_j\cap\mathbf{Q}\neq\emptyset\}$*

$$Y^{\mathrm{local}}_{\mathrm{all\,paths}}(\mathbf{Q}):=\bigcup_{x_j\in\mathbb{X}(\mathbf{Q})}\ \bigcup_{p_i\in\mathrm{supp}f_j\cap\mathbb{Y}_{\partial\mathbb{X}}}\ \bigcup_{Y\in\mathbb{AX}_{\mathrm{flat}}(p_i,x_j)}\chi_Y$$

$\chi_{f_j}(x):=(x\in\mathrm{supp}f_j)$ and for every $l=1,\dots d$ it holds

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}}\sum_{i\neq0:\,\partial_l\phi_i\partial_l\phi_0<0}\ \sum_{x_j\in\mathbb{X}(\mathbf{Q})} f_j\frac{|\partial_l\phi_i|}{D_{l+}}|\tau_i u-\mathcal{M}_j u|^s\\
&\quad\le C\left(\frac{1}{|\mathbf{Q}|}\int_{Y^{\mathrm{local}}_{\mathrm{all\,paths}}(\mathbf{Q})}|\nabla u|^p\right)^{\frac{s}{p}}\\
&\qquad\left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{R}^d}\left(\sum_{x_j\in\mathbb{X}(\mathbf{Q})}\sum_i\chi_{f_j}(p_i)\tilde{\rho}_i^d\sum_{Y\in\mathbb{AX}_{\mathrm{flat}}(p_i,x_j)}\chi_Y W(Y)\,\mathrm{Length}(Y)^{\frac{s-1}{s}}\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}}
\end{aligned}$$

Proof. We find

$$\begin{aligned}
&\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}}\sum_{i\neq0:\,\partial_l\phi_i\partial_l\phi_0<0}\ \sum_{x_j\in\mathbb{X}(\mathbf{Q})} f_j\frac{|\partial_l\phi_i|}{D_{l+}}|\tau_i u-\mathcal{M}_j u|^s\\
&\quad\le\frac{C}{|\mathbf{Q}|}\sum_{i\neq0:\,\partial_l\phi_i\partial_l\phi_0<0}\ \sum_{x_j\in\mathbb{X}(\mathbf{Q})}\chi_{f_j}(p_i)\tilde{\rho}_i^d\sum_{Y\in\mathbb{AX}_{\mathrm{flat}}(p_i,x_j)}W(Y)\int_Y|\nabla u|^s\,\mathrm{Length}(Y)^{\frac{s-1}{s}}
\end{aligned}$$

which leads to the result. □

And finally, we provide an estimate for the remaining term in Lemma 6.5. The proof is similar to the last Lemma.

Lemma 7.5. *Let $\mathbf{P}$ be locally connected. Under Assumptions 5.5, 6.1 it holds for*

$$Y_{\text{all paths}}^{\text{global}}(\mathbf{Q}) := \bigcup_{\substack{x_k \sim\sim x_j \\ x_k, x_j \in \mathbb{X}_{\mathfrak{r}}(\mathbf{Q})}} \bigcup_{Y \in \mathbb{AX}_{\text{flat}}(x_k, x_j)} \chi_Y$$

that

$$\frac{1}{|\mathbf{Q}|} \sum_{\substack{x_k \sim\sim x_j \\ x_k, x_j \in \mathbb{X}_{\mathfrak{r}}(\mathbf{Q})}} |\mathcal{M}_k u - \mathcal{M}_j u|^s \leq C \left(\frac{1}{|\mathbf{Q}|} \int_{Y_{\text{all paths}}^{\text{global}}(\mathbf{Q})} |\nabla u|^p \right)^{\frac{s}{p}}$$
$$\left(\frac{1}{|\mathbf{Q}|} \int_{\mathbb{R}^d} \left(\sum_{\substack{x_k \sim\sim x_j \\ x_k, x_j \in \mathbb{X}_{\mathfrak{r}}(\mathbf{Q})}} \sum_{Y \in \mathbb{AX}_{\text{flat}}(x_k, x_j)} W(Y) \text{Length}(Y)^{\frac{s-1}{s}} \right)^{\frac{p}{p-s}} \right)^{\frac{p-s}{p}} .$$

7.3 Statistical Stretch Factor for Locally Connected Geometries

Definition 7.6. Let $\mathbf{P} \subset \mathbb{R}^d$ be an open set with $\mathbb{X}_{\mathfrak{r}}$ satisfying Assumption 6.1. Generalizing the notation of Lemma 4.38 and recalling the Notation 4.33 let for $x \in \mathbb{X}_{\mathfrak{r}}$ and $y \in \mathbb{Y}$ and a family of admissible paths $\mathbb{AX}(y, x)$

$$\mathsf{R}_0(x, y) := \inf \left\{ R > 0 : \bigcup_{Y \in \mathbb{AX}(y,x)} Y \subset \mathbb{B}_R(x) \right\} .$$

For an open set $\mathfrak{A}$ with $x \in \mathfrak{A}$ we denote

$$\mathsf{R}_0(x, \mathfrak{A}) := \sup_{y \in \mathbb{Y} \cap \mathfrak{A}} \mathsf{R}_0(x, y) .$$

Theorem 7.7. *Let the Assumptions of Theorem 6.3 hold and let $\mathbf{P}$ be locally connected. . For every $x_j \in \mathbb{X}_{\mathfrak{r}}$ let*

$$\mathsf{S}_j := \mathsf{S}(x_j) := d_j^{-1} \sup_{p_i \in \mathbb{Y} \cap \mathfrak{A}_{2,j}} \sup_{Y \in \mathbb{AX}_{\text{flat}}(p_i, x_j)} \text{Length}(Y) .$$

Defining $\mathsf{R}_0(x_j) := \mathsf{R}_0(x_j, \mathfrak{A}_{2,j})$ and

$$\mathbb{AX}(\mathbf{Q}) := \bigcup_{x_j \in \mathbf{Q}^{\sim\sim}} \mathbb{B}_{\mathsf{R}_0(x_j)}(x_j) \tag{7.4}$$

it holds

$$\frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}^{\sim\sim}} |f(u)|^r \leq C_1 (C_2 + C_3) \left(\int_{\mathbb{AX}(\mathbf{Q})} |\nabla u|^p \right)^{\frac{r}{p}} ,$$

where for some $s \in (r,p)$

$$C_1 = \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{AX}(\mathbf{Q})}\left(\sum_{x_j\in\mathbb{X}_{\mathfrak{r}}(\mathbf{Q})}\chi_{\mathbb{B}_{\mathsf{R}_0(x_j,\mathfrak{A}_{2,j})}(x_j)}d_j^{d+\frac{s-1}{s}}\mathsf{S}_j^{\frac{s-1}{s}}\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}}$$

$$\leq \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbb{AX}(\mathbf{Q})}\left(\sum_{x_j\in\mathbb{X}_{\mathfrak{r}}(\mathbf{Q})}\chi_{\mathbb{B}_{\mathsf{S}_j d_j}(x_j)}d_j^{d+\frac{s-1}{s}}\mathsf{S}_j^{\frac{s-1}{s}}\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}}$$

$$C_2 = \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}_{\mathfrak{r}}\cap\mathbb{R}^d_3}|\partial_l\phi_0|^{\frac{sr}{s-r}}\right)^{\frac{s-r}{s}},$$

$$C_3 = \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{P}\cap\mathbf{Q}}\left(\sum_{j:\,\partial_l\Phi_j<0}d_j^{\frac{r(d-1)+drs}{s-r}}\chi_{\nabla\Phi_j\neq 0}\right)^{\frac{s}{s-r}}\right)^{\frac{s-r}{s}}.$$

Definition 7.8. We call S_j the statistical stretch factor.

Corollary 7.9. *It holds* $\mathsf{R}_0(x_j)\leq d_j\mathsf{S}_j$.

Corollary 7.10. *If* $u\in W^{1,p}(\mathbf{P})$ *satisfies* $u\equiv 0$ *on* $\mathbb{R}^d\backslash\mathbf{Q}$ *then* $\mathcal{U}$ *has support on* $\mathbb{AX}(\mathbf{Q})$.

Proof. This follows since

$$\mathbb{AX}(\mathbf{Q})\supset\bigcup_{j\sim\sim\mathbf{Q}}\mathfrak{A}_{1,j}\,.$$

□

Proof of Theorem 7.7. With regard to Lemma 7.4, we observe that $f_j=\Phi_j$ with $\mathbb{X}(\mathbf{Q})=\mathbb{X}_{\mathfrak{r}}(\mathbf{Q})$ and $\chi_{f_j}(p_i)=1$ only if $p_i\in\mathfrak{A}_{1,j}$. Furthermore, $W(Y)\leq 1$ and we define

$$L_j:=\sup_{p_i\in\mathbb{Y}\cap\mathfrak{A}_{2,j}}\ \sup_{Y\in\mathbb{AX}_{\text{flat}}(p_i,x_j)}\text{Length}(Y_{\text{flat}})\,.$$

Hence we find for given x_j using Corollary 7.9:

$$\sum_i\chi_{f_j}(p_i)\tilde\rho_i^d\sum_{Y\in\mathbb{AX}_{\text{flat}}(p_i,x_j)}\chi_{Y_{\text{flat}}}W(Y)\,\text{Length}(Y_{\text{flat}})^{\frac{s-1}{s}}\leq\chi_{\mathbb{B}_{\mathsf{R}_0(x_j,\mathfrak{A}_{1,j})}(x_j)}\left|\mathfrak{A}_{1,j}\right|L_j^{\frac{s-1}{s}}$$

$$\leq\chi_{\mathbb{B}_{\mathsf{S}_j d_j}(x_j)}\left|\mathfrak{A}_{1,j}\right|L_j^{\frac{s-1}{s}}\,.$$

Also with regard to Lemma 7.4 we find for given x_j

$$\sum_{\substack{x_k\sim\sim x_j\\ x_k\in\mathbb{X}_{\mathfrak{r}}(\mathbf{Q})}}\sum_{Y\in\mathbb{AX}_{\text{flat}}(x_k,x_j)}W(Y)\text{Length}(Y_{\text{flat}})^{\frac{s-1}{s}}\leq\chi_{\mathbb{B}_{\mathsf{R}_0(x_j,\mathfrak{A}_{2,j})}(x_j)}\left|\mathfrak{A}_{2,j}\right|L_j^{\frac{s-1}{s}}$$

$$\leq\chi_{\mathbb{B}_{\mathsf{S}_j d_j}(x_j)}\left|\mathfrak{A}_{2,j}\right|L_j^{\frac{s-1}{s}}\,.$$

The statement now follows from the definition of S_j, Lemmas 6.4 and 6.5. □

Finally, the following result allows us to estimate the difference of $\mathbf{Q}$ and $\mathbb{AX}(\mathbf{Q})$.

Theorem 7.11. *Let the Assumptions of Theorem 6.3 hold, let $\mathbf{Q}$ have a C^1-boundary and let $\mathbb{AX}(\mathbf{Q})$ be given by (7.4). Furthermore, let R_0 be ergodic such that for every $\varepsilon > 0$*

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}(1+\varepsilon)^k\,\mathbb{E}\big(\mathsf{R}_0(x_j)\geq(1+\varepsilon)^k n\big)=0\,. \tag{7.5}$$

Then

$$\lim_{n\to\infty}\frac{|n\mathbf{Q}|}{|\mathbb{AX}\,(n\mathbf{Q})|}\to 1\,.$$

Remark 7.12. Condition (7.5) is satisfied if e.g. $\mathbb{E}(\mathsf{R}_0(x_j)\geq r_0)\leq r_0^{-a}$ for some $a>1$ as then

$$\sum_{k=1}^{\infty}(1+\varepsilon)^k\,\mathbb{E}\big(\mathsf{R}_0(x_j)\geq(1+\varepsilon)^k n\big)\leq\frac{1}{n^{\alpha}}\sum_{k=1}^{\infty}\left(\frac{1}{(1+\varepsilon)^{a-1}}\right)^k\,.$$

Proof. Since $n\mathbf{Q}\subset\mathbb{AX}(n\mathbf{Q})$ we have to estimate the excess mass of $\mathbb{AX}(n\mathbf{Q})$ over $|n\mathbf{Q}|$. If we define

$$\begin{aligned}\mathbb{X}_{n\mathbf{Q}}&:=\left\{x_j\in\mathbb{X}_{\mathfrak{r}}\cap\mathbf{Q}:\,\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\backslash\,(n\mathbf{Q})\neq\emptyset\right\}\,,\\ \mathbb{X}_{n\mathbf{Q}^{\complement}}&:=\left\{x_j\in\mathbb{X}_{\mathfrak{r}}\backslash\mathbf{Q}:\,\mathbb{B}_{\mathfrak{r}}(\mathfrak{A}_{1,j})\cap(n\mathbf{Q})\neq\emptyset\right\}\,,\end{aligned}$$

we find

$$|\mathbb{AX}(n\mathbf{Q})\backslash\,(n\mathbf{Q})|\leq\sum_{x_j\in\mathbb{X}_{n\mathbf{Q}}\cup\mathbb{X}_{n\mathbf{Q}^{\complement}}}\left|\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\right|\,,$$

and we thus derive an estimate on the contribution from $\mathbb{X}_{n\mathbf{Q}}$ and $\mathbb{X}_{n\mathbf{Q}^{\complement}}$ respectively.

Let $\varepsilon>0$. Then for $\mathbf{Q}_{n,k}^{\varepsilon}:=\big((1+\varepsilon)^k n\mathbf{Q}\big)\setminus\big((1+\varepsilon)^{k-1}n\mathbf{Q}\big)$

$$\begin{aligned}\sum_{x_j\in\mathbb{X}_{n\mathbf{Q}^{\complement}}}\left|\mathbb{B}_{\mathsf{R}_0(x_j,\mathfrak{A}_{2,j})}(x_j)\right|&\leq\sum_{x_j\in\mathbb{X}_{\mathfrak{r}}\cap\mathbf{Q}_{n,1}^{\varepsilon}}\left|\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\right|+\sum_{k=2}^{\infty}\sum_{\substack{x_j\in\mathbb{X}_{\mathfrak{r}}\cap\mathbf{Q}_{n,k}^{\varepsilon}\\ d(x_j)\geq(1+\varepsilon)^{k-1}n}}\left|\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\right|\\ &\leq\sum_{x_j\in\mathbb{X}_{\mathfrak{r}}\cap\mathbf{Q}_{n,1}^{\varepsilon}}\left|\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\right|+\sum_{k=2}^{\infty}\sum_{\substack{x_j\in\mathbb{X}_{\mathfrak{r}}\cap\mathbf{Q}_{n,k}^{\varepsilon}\\ \mathsf{R}_0(x_j)\geq(1+\varepsilon)^{k-1}n}}\left|\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\right|\\ &\leq\sum_{x_j\in\mathbb{X}_{\mathfrak{r}}\cap\mathbf{Q}_{n,1}^{\varepsilon}}\left|\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\right|+\sum_{k=2}^{\infty}\sum_{\substack{x_j\in\mathbb{X}_{\mathfrak{r}}\cap\mathbf{Q}_{n,k}^{\varepsilon}\\ \mathsf{R}_0(x_j)\geq(1+\varepsilon)n}}\left|\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\right|\end{aligned}$$

Due to the ergodic theorem, we obtain for every $n_0\in\mathbb{N}$

$$\begin{aligned}\frac{1}{|n\mathbf{Q}|}\sum_{\substack{x_j\in\mathbb{X}_{\mathfrak{r}}\cap\mathbf{Q}_{n,k}^{\varepsilon}\\ \mathsf{R}_0(x_j)\geq(1+\varepsilon)^{k-1}n}}\left|\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\right|&\leq\frac{1}{|n\mathbf{Q}|}\sum_{\substack{x_j\in\mathbb{X}_{\mathfrak{r}}\cap\mathbf{Q}_{n,k}^{\varepsilon}\\ \mathsf{R}_0(x_j)\geq(1+\varepsilon)^{k-1}n_0}}\left|\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\right|\\ &\to\Big((1+\varepsilon)^k-(1+\varepsilon)^{k-1}\Big)\,\mathbb{E}\big(\mathsf{R}_0(x_j)\geq(1+\varepsilon)^{k-1}n_0\big)\\ &\leq\varepsilon\,(1+\varepsilon)^{k-1}\,\mathbb{E}\big(\mathsf{R}_0(x_j)\geq(1+\varepsilon)^{k-1}n_0\big)\end{aligned}$$

and similarly

$$\lim_{n\to\infty}\frac{1}{|n\mathbf{Q}|}\sum_{x_j\in\mathbb{X}_{\mathfrak{r}}\cap\mathbf{Q}_{n,1}^{\varepsilon}}\left|\mathbb{B}_{\mathsf{R}_0(x_j)}(x_j)\right|=\varepsilon\mathbb{E}(\mathsf{R}_0)\,.$$

Since the above estimates hold for every ε and every n_0, we find

$$\frac{1}{|n\mathbf{Q}|}\sum_{x_j\in\mathbb{X}_{n\mathbf{Q}^\complement}}\left|\mathbb{B}_{\mathsf{R}_0(x_j,\mathfrak{A}_{2,j})}(x_j)\right|\to 0\,.$$

In a similar way, we prove

$$\frac{1}{|n\mathbf{Q}|}\sum_{x_j\in\mathbb{X}_{n\mathbf{Q}}}\left|\mathbb{B}_{\mathsf{R}_0(x_j,\mathfrak{A}_{2,j})}(x_j)\right|\to 0\,.$$

□

8 Sample Geometries

8.1 Boolean Model for the Poisson Ball Process

Recalling Example 2.36 we consider a Poisson point process $\mathbb{X}_{\mathrm{pois}}(\omega)=(x_i(\omega))_{i\in\mathbb{N}}$ with intensity λ (recall Example 2.36). To each point x_i a random ball $B_i=\mathbb{B}_1(x_i)$ is assigned and the family $\mathbb{B}:=(B_i)_{i\in\mathbb{N}}$ is called the Poisson ball process. We then denote $\mathbf{P}\,(\omega):=\mathbb{R}^d\backslash\overline{\bigcup_i B_i}$ and seek for a corresponding uniform extension operator. The following argumentation will be strongly based on the so called void probability. This is the probability $\mathbb{P}_0(A)$ to not find any point of the point process in a given open set A and is given by (2.34) i.e. $\mathbb{P}_0(A):=e^{-\lambda|A|}$.

The void probability for the ball process is given accordingly by

$$\mathbb{P}_0(A):=e^{-\lambda\left|\overline{\mathbb{B}_1(A)}\right|},\qquad \overline{\mathbb{B}_1(A)}:=\left\{x\in\mathbb{R}^d\,:\,\operatorname{dist}(x,A)\leq 1\right\}\,,$$

which is the probability that no ball intersects with $A\subset\mathbb{R}^d$.

Theorem 8.1. *Let $\mathbf{P}\,(\omega):=\bigcup_i B_i(\omega)$ and define*

$$\begin{aligned}\tilde{\delta}(x)&:=\min\left\{\delta(\tilde{x})\,:\,\tilde{x}\in\partial\mathbf{P}\text{ s.t. }x\in\mathbb{B}_{\frac18\delta(\tilde{x})}(\tilde{x})\right\}\,,\\ \tilde{\hat{\rho}}(x)&:=\min\left\{\hat{\rho}(\tilde{x})\,:\,\tilde{x}\in\partial\mathbf{P}\text{ s.t. }x\in\mathbb{B}_{\frac18\hat{\rho}(\tilde{x})}(\tilde{x})\right\}\,,\end{aligned}$$

where $\min\emptyset:=0$ for convenience. Then $\partial\mathbf{P}$ is almost surely locally (δ,M) regular and for every $\gamma<1$, $\beta<d+2$ and $1\leq r<2$ and $2\frac{sr}{2(s-1)-sr}\leq d+2$ it holds

$$\mathbb{E}\big(\delta^{-\gamma}\big)+\mathbb{E}\Big(\tilde{\delta}^{-\gamma-1}\Big)+\mathbb{E}\Big(\tilde{M}^{\beta}\Big)+\mathbb{E}\Big(\tilde{\hat{\rho}}^{-\frac{rs}{s-1}}\Big)<\infty\,.$$

Furthermore, it holds $\hat{d}\leq d-1$ and $\alpha=0$ in inequalities (5.8) and (5.17). The same holds if $\mathbf{P}\,(\omega):=\mathbb{R}^d\backslash\overline{\bigcup_i B_i}(\omega)$ with α replaced by d.

Remark 8.2. We observe that the union of balls has better properties than the complement.

Proof. We study only $\mathbf{P}\,(\omega):=\bigcup_i B_i(\omega)$ since $\mathbb{R}^d\backslash\overline{\bigcup_i B_i}(\omega)$ is the complement sharing the same boundary. Hence, in case $\mathbf{P}(\omega)=\mathbb{R}^d\backslash\overline{\bigcup_i B_i}(\omega)$, all calculations remain basically the same. However, in the first case, we assume that $\mathfrak{r}(y_k)=\frac14\tilde{\rho}(y_k)$, which we cannot assume in the other case, where $\mathfrak{r}(y_k)$ is proportional to $\tilde{\rho}_k\tilde{M}_k^{-1}$. This is the reason for the different α in the two cases.

In what follows, we use that the distribution of balls is mutually independent. That means, given a ball around $x_i \in \mathbb{X}_{\rm pois}$, the set $\mathbb{X}_{\rm pois}\setminus\{x_i\}$ is also a Poisson process. W.l.o.g. , we assume $x_i = x_0 = 0$ with $B_0 := \mathbb{B}_1(0)$. First we note that $p \in \partial B_0 \cap \partial \mathbf{P}$ if and only if $p \in \partial B_0 \backslash \mathbf{P}$, which holds with probability $\mathbb{P}_0(\mathbb{B}_1(p)) = \mathbb{P}_0(B_0)$. This is a fixed quantity, independent from p.

Now assuming $p \in \partial B_0 \backslash \mathbf{P}$, the distance to the closest ball besides B_0 is denoted

$$r(p) = \mathrm{dist}(p, \partial\mathbf{P}\backslash\partial B_0)$$

with a probability distribution

$$\mathbb{P}_{\rm dist}(r) := \mathbb{P}_0(\mathbb{B}_{1+r}(p))/\mathbb{P}_0(\mathbb{B}_1(p))\,.$$

It is important to observe that ∂B_0 is r-regular in the sense of Lemma 2.12. Another important feature in view of Lemma 4.4 is $r(p) < \Delta(p)$. In particular, $\delta(p) > \frac{1}{2}r(p)$ and ∂B_0 is $(\delta, 1)$-regular in case $\delta < \sqrt{\frac{1}{2}}$. Hence, in what follows, we will derive estimates on $r^{-\gamma}$, which immediately imply estimates on $\delta^{-\gamma}$.

Estimate on γ: A lower estimate for the distribution of $r(p)$ is given by

$$\mathbb{P}_{\rm dist}(r) := \mathbb{P}_0(\mathbb{B}_{1+r}(p))/\mathbb{P}_0(\mathbb{B}_1(p)) \approx 1 - \lambda\left|\mathbb{S}^{d-1}\right| r\,. \tag{8.1}$$

This implies that almost surely for $\gamma < 1$

$$\limsup_{n\to\infty} \frac{1}{(2n)^d}\int_{(-n,n)^d\cap\partial\mathbf{P}} r(p)^{-\gamma}\,\mathrm{d}\mathcal{H}^{d-1}(p) < \infty\,,$$

i.e. $\mathbb{E}(\delta^{-\gamma}) < \infty$.

Intersecting balls: Now assume there exists x_i, $i \neq 0$ such that $p \in \partial B_i \cap \partial B_0$. W.l.o.g. assume $x_i = x_1 := (2x, 0, \dots, 0)$ and $p = \left(\sqrt{1-x^2}, 0, \dots, 0\right)$. Then

$$\delta(p) \leq \delta_0(p) := 2\sqrt{1-x^2}$$

and p is at least $M(p) = \frac{x}{\sqrt{1-x^2}}$-regular. Again, a lower estimate for the probability of r is given by (8.1) on the interval $(0, \delta_0)$. Above this value, the probability is approximately given by $\lambda\left|\mathbb{S}^{d-1}\right|\delta_0$ (for small δ_0i.e. $x \approx 1$). We introduce as a new variable $\xi = 1 - x$ and obtain from $1 - x^2 = \xi(1+x)$ that

$$\delta_0 \leq C\xi^{\frac{1}{2}} \quad \text{and} \quad M(p) \leq C\xi^{-\frac{1}{2}}\,. \tag{8.2}$$

No touching: At this point, we observe that M is almost surely locally finite. Otherwise, we would have $x = 1$ and for every $\varepsilon > 0$ we had $x_1 \in \mathbb{B}_{2+\varepsilon}(x_0)\backslash\mathbb{B}_{2-\varepsilon}(x_0)$. But

$$\mathbb{P}_0(\mathbb{B}_{2+\varepsilon}(x_0)\backslash\mathbb{B}_{2-\varepsilon}(x_0)) \approx 1 - \lambda 2\left|\mathbb{S}^{d-1}\right|\varepsilon \;\to\; 1 \qquad \text{as } \varepsilon \to 0\,.$$

Therefore, the probability that two balls "touch" (i.e. that $x = 1$) is zero. The almost sure local boundedness of M now follows from the countable number of balls.

Extension to $\tilde{\delta}$: We again study each ball separately. Let $p \in \partial B_0 \backslash \overline{\mathbf{P}}$ with tangent space T_p and normal space N_p. Let $x \in N_p$ and $\tilde{p} \in \partial B_0$ such that $x \in \mathbb{B}_{\frac{1}{8}\delta(\tilde{p})}(\tilde{p})$, then also $p \in \mathbb{B}_{\frac{1}{8}\delta(\tilde{p})}(\tilde{p})$ and $\delta(p) \in (\frac{7}{8}, \frac{7}{6})\delta(\tilde{p})$ and $\delta(\tilde{p}) \in (\frac{7}{8}, \frac{7}{6})\delta(p)$ by Lemma 2.12. Defining

$$\tilde{\delta}_i(x) := \min\left\{\delta(\tilde{x}) \;:\; \tilde{x} \in \partial B_i\backslash\mathbf{P} \text{ s.t. } x \in \mathbb{B}_{\frac{1}{8}\delta(\tilde{x})}(\tilde{x})\right\}\,,$$

we find

$$\tilde{\delta}^{-\gamma} \le \sum_i \chi_{\tilde{\delta}_i>0} \tilde{\delta}_i^{-\gamma} \,.$$

Studying δ_0 on ∂B_0 we can assume $M \le M_0$ in (4.10) and we find

$$\int_{\mathbf{P}} \chi_{\tilde{\delta}_0>0} \tilde{\delta}_0^{-\gamma-1} \le C \int_{\partial B_0 \setminus \mathbf{P}} \delta^{-\gamma} \,.$$

Hence we find

$$\int_{\mathbf{P}} \tilde{\delta}^{-\gamma-1} \le \sum_i \int_{\mathbf{P}} \chi_{\tilde{\delta}_i>0} \tilde{\delta}_i^{-\gamma-1} \le \sum_i C \int_{\partial B_i \setminus \mathbf{P}} \delta^{-\gamma} \,.$$

Estimate on β**:** For two points $x_i, x_j \in \mathbb{X}_{\text{pois}}$ let $\mathrm{Circ}_{ij} := \partial B_i \cap \partial B_j$ and $\mathbb{B}_{\frac{1}{8}\tilde{\delta}}(\mathrm{Circ}_{ij}) := \bigcup_{p \in \mathrm{Circ}_{ij}} \mathbb{B}_{\frac{1}{8}\tilde{\delta}(p)}(p)$. For the fixed ball $B_i = B_0$ we write Circ_{0j} and obtain $|\mathrm{Circ}_{0j}| \le C\delta_0^d$ with δ_0 from (8.2). Therefore, we find

$$\int_{\mathrm{Circ}_{0j}} (1+M(p))^\beta \le \delta_0^d (1+M(p))^\beta \le C\xi^{-\frac{1}{2}(\beta-d)} \,.$$

We now derive an estimate for $\mathbb{E}\Big(\int_{\mathbb{B}_{1+\mathfrak{r}}(0)} \tilde{M}^\beta\Big)$.

To this aim, let $q \in (0,1)$. Then $x \in \mathbb{B}_{2-q^{k+1}}(0) \setminus \mathbb{B}_{2-q^k}(0)$ implies $\xi \ge q^{k+1}$ and

$$\begin{aligned}
\int_{\mathbb{B}_{1+\mathfrak{r}}(0)} \tilde{M}^\beta &\le C + \sum_{k=1}^\infty \sum_{x_j \in \mathbb{B}_{2-q^{k+1}}(0) \setminus \mathbb{B}_{2-q^k}(0)} \int_{\mathrm{Circ}_{0j}} (1+M(p))^\beta \\
&\le C + \sum_{k=1}^\infty \sum_{x_j \in \mathbb{B}_{2-q^{k+1}}(0) \setminus \mathbb{B}_{2-q^k}(0)} C\left(q^{k+1}\right)^{-\frac{1}{2}(\beta-d)}
\end{aligned}$$

The only random quantity in the latter expression is $\#\left\{x_j \in \mathbb{B}_{2-q^{k+1}}(0) \setminus \mathbb{B}_{2-q^k}(0)\right\}$. Therefore, we obtain with $\mathbb{E}(\mathbb{X}(A)) = \lambda\,|A|$ that

$$\begin{aligned}
\mathbb{E}\Big(\int_{\mathbb{B}_{1+\mathfrak{r}}(0)} \tilde{M}^\beta\Big) &\le C\left(1 + \sum_{k=1}^\infty \left(q^k - q^{k+1}\right)\left(q^{k+1}\right)^{-\frac{1}{2}(\beta-d)}\right) \\
&\le C\left(1 + \sum_{k=1}^\infty \left(q^k\right)^{-\frac{1}{2}(\beta-d-2)}\right) \,.
\end{aligned}$$

Since the point process has finite intensity, this property carries over to the whole ball process and we obtain the condition $\beta < d+2$ in order for the right hand side to remain bounded.

Estimate on $\tilde{\gamma}$**:** We realize that $\tilde{\hat{\rho}} \ge \frac{\tilde{\delta}}{\tilde{M}} \ge \frac{\tilde{r}}{\tilde{M}}$. Hence we obtain from Hölder's inequality

$$\mathbb{E}\Big(\tilde{\hat{\rho}}^{-\frac{rs}{s-1}}\Big) \le \mathbb{E}\Big(\tilde{\delta}^{-\tilde{s}}\Big)^{\frac{1}{q}} \mathbb{E}\Big(\tilde{M}^{\frac{sr}{(s-1)}p}\Big)^{\frac{1}{p}} \,,$$

where $\tilde{s} = \frac{rs}{s-1}q$ and $\frac{1}{p} + \frac{1}{q} = 1$. From the right hand side of the last inequality, we infer boundedness of the first expectation value for $\tilde{s} < 2$ implying $q < \frac{2(s-1)}{sr}$. Since we have to require $q > 1$, this implies $r < 2$ and $s > \frac{2}{2-r}$. On the other hand, we know that the second expectation is finite if

$\frac{sr}{(s-1)}p < d+2$. For $q = \frac{2(s-1)}{sr}$, we obtain the lower bound for $p = \frac{q}{q-1}$ and hence we conclude the sufficient condition

$$2\frac{1}{\frac{2(s-1)}{sr}-1} \leq d+2\,,$$

which implies our claim.

Estimate on $\hat{d}$: We have to estimate the local maximum number of $A_{3,k}$ overlapping in a single point in terms of $\tilde{M}$. We first recall that $\hat{\rho}(p) \approx 8\tilde{M}(p)\tilde{\rho}(p)$. Thus large discrepancy between $\hat{\rho}$ and $\tilde{\rho}$ occurs in points where $\tilde{M}$ is large. This is at the intersection of at least two balls. Despite these "cusps", the set $\partial\mathbf{P}$ consists locally on the order of $\hat{\rho}$ of almost flat parts. Arguing like in Lemma 5.2 resp. Remark 5.3 this yields $\hat{d} \leq d-1$.

Estimate on α: Given two points y_1, y_2 with radii $\mathfrak{r}(y_1)$, $\mathfrak{r}(y_2)$, $\mathbb{B}_{y_i} := \mathbb{B}_{\mathfrak{r}(y_i)}(y_i)$ and $\mathcal{M}_{y_i}u := |\mathbb{B}_{y_1}|^{-1}\int_{\mathbb{B}_{y_1}} u$ we find

$$|\mathcal{M}_{y_1}u - \mathcal{M}_{y_2}u| \leq \frac{|y_1-y_2| + \left|\left(\frac{\mathfrak{r}(y_2)}{\mathfrak{r}(y_1)}-1\right)\mathfrak{r}(y_1)\right|}{|\mathbb{B}_{y_1}|}\int_{\mathrm{conv}\left(\mathbb{B}_{y_1}\cup\mathbb{B}_{y_2}\right)} |\nabla u|\ .$$

By our initial assumptions on $\mathfrak{r}(y_i)$ we prove our claim on α. □

It remains to verify bounded average connectivity of the Boolean set $\mathbf{P}\,(\omega) := \bigcup_i B_i(\omega)$ or its complement. In what follows we restrict to the Boolean set and use the following result.

Theorem 8.3. *[37]Let $\mathbf{P}$ have a connected component and let $\mathbb{G}(\mathbb{X}_{\mathrm{pois}})$ be the graph on $\mathbb{X}_{\mathrm{pois}}$ constructed from $x \sim y$ iff $\mathbb{B}_1(x)\cap\mathbb{B}_1(y) \neq \emptyset$. Let $\tilde{\mathbf{P}}$ be the connected component of $\mathbf{P}$ and $\tilde{\mathbb{X}}_{\mathrm{pois}} := \mathbb{X}_{\mathrm{pois}}\cap\tilde{\mathbf{P}}$. For $x, y \in \tilde{\mathbb{X}}_{\mathrm{pois}}$ let $d(x,y)$ be the graph distance. Then for every $\varepsilon > 0$ there exists $\mu > 1$, $\nu > 0$ such that*

$$\mathbb{P}\left(\frac{d(x,y)}{\mu\,|x-y|} \notin (1-\varepsilon, 1+\varepsilon)\right) \leq e^{-\nu|x-y|}\,.$$

The latter result enables us to prove the following.

Lemma 8.4. *Using the notation of Theorem 8.3, let $x, y \in \tilde{\mathbb{X}}_{\mathrm{pois}}$ and $a > 2$. Then*

$$\mathbb{P}(d(x,y) \geq 4\mu a\,|x-y|\,(1+\varepsilon)) \leq 2e^{-\frac{\nu}{2}a|x-y|}\,.$$

In other words, the probability that the distance between x and y on the grid is stretched by more than $5\mu a$ is decreasing exponentially in a.

Proof. Let $x, y \in \tilde{\mathbb{X}}_{\mathrm{pois}}$. Let $a > 2$ and let $n \in \mathbb{N}$ such that $a \in [2^n, 2^{n+1})$. With probability $1-\exp\left(-\lambda\left|\mathbb{S}^{d-1}\right|\left(2^{dn+d}-2^d\right)|x-y|^d\right) > \frac{1}{2}$ there exists $z \in \mathbb{B}_{2^{n+1}|x-y|}(x)\backslash\mathbb{B}_{2^n|x-y|}(x)$. For such z it holds

$$\begin{aligned}2^n\,|x-y| \leq |z-x| < 2^{n+1}\,|x-y|\\ 2^n\,|x-y| \leq |z-y| < \left(2^{n+1}+1\right)|x-y|\end{aligned}$$

In particular, we obtain for $a_{n+1} := 2^{n+1}+1$

$$\begin{aligned}\frac{d(x,y)}{4\mu a\,|x-y|} \leq \frac{d(x,y)}{2\mu a_{n+1}\,|x-y|} &\leq \frac{d(x,z)}{2\mu a_{n+1}\,|x-y|} + \frac{d(z,y)}{2\mu a_{n+1}\,|x-y|}\\ &\leq \frac{d(x,z)}{2\mu\,|x-z|} + \frac{d(z,y)}{2\mu\,|z-y|}\end{aligned}$$

Hence, assuming $1+\varepsilon \leq \frac{d(x,y)}{4\mu a|x-y|}$ we find that at least one of the conditions $\frac{d(x,z)}{\mu|x-z|} \geq 1+\varepsilon$ or $\frac{d(z,y)}{\mu|z-y|} \geq 1+\varepsilon$ has to hold, which implies

$$\mathbb{P}\left(1+\varepsilon \leq \frac{d(x,y)}{4\mu a\,|x-y|}\right) \leq \mathbb{P}\left(\frac{d(x,z)}{\mu\,|x-z|} \geq 1+\varepsilon \text{ or } \frac{d(z,y)}{\mu\,|z-y|} \geq 1+\varepsilon\right).$$

Now it holds under the condition that z exists

$$\mathbb{P}\left(\frac{d(x,z)}{\mu\,|x-z|} \geq 1+\varepsilon \text{ or } \frac{d(z,y)}{\mu\,|z-y|} \geq 1+\varepsilon\right) < e^{-\nu|x-z|}+e^{-\nu|y-z|} < 2e^{-\nu 2^n|x-y|} < 2e^{-\frac{\nu}{2}a|x-y|},$$

which implies the statement. □

We construct a suitable graph $(\mathbb{Y},\mathbb{G}(\mathbf{P}))$. For this we choose $\mathbb{X}_{\mathfrak{r}} := \mathbb{X}_{\mathfrak{r}}(\mathbf{P})$ according to Lemma 2.50 and define

$$\mathbb{Y}_{\mathrm{pois}} = \mathbb{Y}_{\partial\mathbb{X}} \cup \partial\mathbb{X} \cup \mathbb{X}_{\mathfrak{r}} \cup \mathbb{X}_{\mathrm{pois}}\,.$$

For $\mathbb{Y}_{\partial\mathbb{X}}$ and $\partial\mathbb{X}$ we choose the standard neighborhood relation. Furthermore, we say for $y \in \mathbb{Y}_{\partial\mathbb{X}}$ and $x \in \mathbb{X}_{\mathfrak{r}}$ that $y \sim x$ iff there exists $\tilde{x} \in \mathbb{X}_{\mathrm{pois}}$ with $x, y \in \mathbb{B}_1(\tilde{x})$ and for $x \in \mathbb{X}_{\mathfrak{r}}$, $\tilde{x} \in \mathbb{X}_{\mathrm{pois}}$ we say $x \sim \tilde{x}$ iff $x \in \mathbb{B}_1(\tilde{x})$. This graph is called $\mathbb{G}_{\mathrm{pois}}$.

Theorem 8.5. *Let $\mathbf{P}$ be the connected component of $\bigcup_i B_i(\omega)$. Then $\mathbf{P}$ is locally connected and for $(\mathbb{Y}_{\mathrm{pois}},\mathbb{G}_{\mathrm{pois}})$ we find for every $\gamma>0$ that $\mathbb{E}\big(\mathsf{S}_j^\gamma\big) \leq \infty$.*

Proof. We write $a = \mathfrak{r}^{-1}$. Let $x_1 \in \mathbb{X}_{\mathfrak{r}}$ with diameter d_1 of the Voronoi cell and let $\mathbb{X}_{\mathfrak{r},1} := \mathbb{X}_{\mathfrak{r}} \cap \mathbb{B}_{3d_1}(x)$. We can chose $\mathbb{X}_{\mathrm{pois},1} \subset \mathbb{X}_{\mathrm{pois}} \cap \mathbb{B}_{3d_1+a\mathfrak{r}}(x)$ with $\#\mathbb{X}_{\mathrm{pois},1} = \#\mathbb{X}_{\mathfrak{r},1}$ such that $\mathbb{X}_{\mathfrak{r},1} \subset \mathbb{B}_1(\mathbb{X}_{\mathrm{pois},1})$. Note in particular, that $\#\mathbb{X}_{\mathrm{pois},1} \leq Cd_1^d$. Now let $y \in \mathbb{Y}_{\partial\mathbb{X}} \cap \mathbb{B}_{3d}(x)$ and let $Y = (y_1,\dots,y_k) \in \mathbb{AX}(x,y)$. If $y_1 = y$, then $y_2 \in \mathbb{X}_{\mathfrak{r},1}$ and, w.l.o.g., $y_3, y_{k-1} \in \mathbb{X}_{\mathrm{pois},1}$. For the graph distances it holds

$$\begin{aligned} d(x,y) &\leq d(x,y_{k-1}) + d(y_{k-1},y_3) + d(y_2,y_3) + d(y_1,y_2) \\ &\leq a\mathfrak{r} + d(y_{k-1},y_3) + a\mathfrak{r} + 4\sqrt{d}\,\mathfrak{r}\,. \end{aligned}$$

In case $d(y_{k-1},y_3) \leq 1$ we conclude with $d(x,y) \leq \left(3a+4\sqrt{d}\right)\mathfrak{r} \leq \left(3a+4\sqrt{d}\right)d_1$. If $d(y_{k-1},y_3) \geq 4\sqrt{d}$ we obtain $d(x,y) \leq 4d(y_{k-1},y_3)$.

Hence, because $\#\mathbb{X}_{\mathrm{pois},1} \leq Cd_1^d$, it only remains to observe that Lemma 8.4 yields an exponential decrease for the probability of large stretch factors for $d(y_{k-1},y_3)$. □

8.2 Delaunay Pipes for a Matern Process

For two points $x, y \in \mathbb{R}^d$, we denote

$$P_r(x,y) := \left\{ y+z \in \mathbb{R}^d :\, 0 \leq z\cdot(x-y) \leq |x-y|^2\,,\, \left| z - z\cdot(x-y)\frac{x-y}{|x-y|} \right| < r \right\},$$

the cylinder (or pipe) around the straight line segment connecting x and y with radius $r>0$.

Recalling Example 2.36 we consider a Poisson point process $\mathbb{X}_{\mathrm{pois}}(\omega) = (x_i(\omega))_{i\in\mathbb{N}}$ with intensity λ (recall Example 2.36) and construct a hard core Matern process $\mathbb{X}_{\mathrm{mat}}$ by deleting all points with a mutual distance smaller than $d\mathfrak{r}$ for some $\mathfrak{r}>0$ (refer to Example 2.37). From the remaining point process $\mathbb{X}_{\mathrm{mat}}$ we construct the Delaunay triangulation $\mathbb{D}(\omega) := \mathbb{D}(X_{\mathrm{mat}}(\omega))$ and assign to each $(x,y) \in \mathbb{D}$ a random number $\delta(x,y)$ in $(0,\mathfrak{r})$ in an i.i.d. manner from some probability distribution $\delta(\omega)$. We finally define

$$\mathbf{P}(\omega) := \bigcup_{(x,y)\in\mathbb{D}(\omega)} P_{\delta(x,y)}(x,y) \bigcup_{x\in\mathbb{X}_{\mathrm{mat}}} \mathbb{B}_{\mathfrak{r}}(x)$$

the family of all pipes generated by the Delaunay grid "smoothed" by balls with the fix radius $\mathfrak{r}$ around each point of the generating Matern process.

Since the Matern process is mixing and δ is mixing, Lemma 2.22 yields that the whole process is still ergodic.

Remark 8.6. The family of balls $\mathbb{B}_{\mathfrak{r}}(x)$ can also be dropped from the model. However, this would imply we had to remove some of the points from $\mathbb{X}_{\mathrm{mat}}$ for the generation of the Voronoi cells. This would cause technical difficulties which would not change much in the result, as the probability for the size of Voronoi cells would still decrease subexponentially.

Lemma 8.7. *$\mathbb{X}_{\mathrm{mat}}$ is a point process for $\mathbf{P}(\omega)$ that satisfies Assumption 6.1 and $\mathbf{P}$ is isotropic cone mixing for $\mathbb{X}_{\mathrm{mat}}$ with exponentially decreasing $f(R) \leq Ce^{-R^d}$. Furthermore, assume there exists $C_\delta, a_\delta > 0$ such that $\mathbb{P}(\delta(x,y) < \delta_0) \leq C_\delta e^{-a_\delta \frac{1}{\delta_0}}$, then $\mathbb{P}(\tilde{M} > M_0) \leq Ce^{-aM_0}$ for some $C, a > 0$.*

Proof. Isotropic cone mixing: For $x, y \in 2d\mathfrak{r}\mathbb{Z}^d$ the events $\left(x + [0,1]^d\right) \cap \mathbb{X}_{\mathrm{mat}}$ and $\left(y + [0,1]^d\right) \cap \mathbb{X}_{\mathrm{mat}}$ are mutually independent. Hence

$$\mathbb{P}\big(\big(k2dr\,[-1,1]^d\big) \cap \mathbb{X}_{\mathrm{mat}} = \emptyset\big) \leq \mathbb{P}\big([-1,1]^d \cap \mathbb{X}_{\mathrm{mat}} = \emptyset\big)^{k^d} .$$

Hence the open set $\mathbf{P}$ is isotropic cone mixing for $\mathbb{X} = \mathbb{X}_{\mathrm{mat}}$ with exponentially decaying $f(R) \leq Ce^{-R^d}$.

Estimate on δ: There exists $C > 0$ such that $\mathbf{P}$ is $(\delta(x,y), C\delta(x,y)^{-1})$-regular in every $x \in \partial P_{\delta(x,y)}(x,y)$. Since the distribution of $\delta(x,y)$ is independent from x and y, this implies that $\mathbb{P}(\delta < \delta_0) \leq C_\delta e^{-a_\delta \frac{1}{\delta_0}}$.

Estimate on the distribution of M: By definition of the Delaunay triangulation, two pipes intersect only if they share one common point $x \in \mathbb{X}_{\mathrm{mat}}$.

Given three points $x, y, z \in \mathbb{X}_{\mathrm{mat}}$ with $x \sim y$ and $x \sim z$, the highest local Lipschitz constant on $\partial\left(P_{\delta(x,y)}(x,y) \cup P_{\delta(x,z)}(x,z)\right)$ is attained in

$$\tilde{x} = \arg\max\left\{|x - \tilde{x}| : \tilde{x} \in \partial P_{\delta(x,y)}(x,y) \cap \partial P_{\delta(x,z)}(x,z)\right\} .$$

It is bounded by

$$\max\left\{\arctan\left(\frac{1}{2}\sphericalangle\left((x,y),(x,z)\right)\right), \frac{1}{\delta(x,y)}, \frac{1}{\delta(x,z)}\right\} ,$$

where $\alpha := \sphericalangle\left((x,y),(x,z)\right)$ in the following denotes the angle between (x,y) and (x,z), see Figure 6. If d_x is the diameter of the Voronoi cell of x, we show that a necessary (but not sufficient) condition that the angle α can be smaller than some α_0 is given by

$$d_x \geq C\frac{1}{\sin\alpha_0} , \tag{8.3}$$

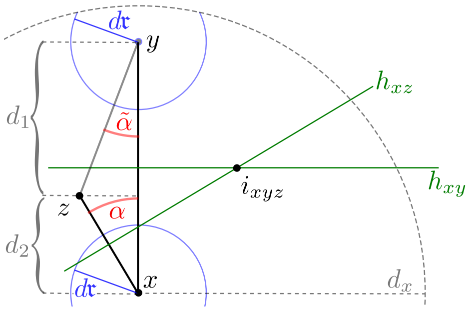

Figure 6: Sketch of the proof of Lemma 8.7 and estimate (8.3).

where $C > 0$ is a constant depending only on the dimension d. Since for small α we find $M \approx \frac{1}{\sin\alpha}$, and since the distribution for d_x decays subexponentially, also the distribution for M decays subexponentially.

Proof of (8.3): Given an angle $\alpha > 0$ and $x \in \mathbb{X}_{\mathrm{mat}}$ we derive a lower bound for the diameter of $G(x)$ such that for two neighbors y, z of x it can hold $\sphericalangle((x,y),(x,z)) \leq \alpha$. With regard to Figure 6, we assume $|x-y| \geq |x-z|$.

Writing $d_x := d(x)$ the diameter of $G(x)$ and $\tilde\alpha = \sphericalangle((x,z),(z,y))$, w.l.o.g let $y = (d_1 + d_2, 0, \dots, 0)$, where $d_1 + d_2 < d_x$ and $d_1 = |y-z|\cos\tilde\alpha$. Hence we can assume that z takes the form $z = (d_2, -|y-z|\sin\tilde\alpha, 0\dots 0)$ and in what follows, we focus on the first two coordinates only. The boundaries between the cells x and z and x and y lie on the planes

$$h_{xz}(t) = \frac{1}{2}z + t\begin{pmatrix} |y-z|\sin\tilde\alpha \\ d_2 \end{pmatrix}, \quad h_{xy}(s) = \frac{1}{2}y + s\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

respectively. The intersection of these planes has the first two coordinates

$$i_{xyz} := \left(\frac{d_1+d_2}{2}, -\frac{1}{2}|y-z|\sin\tilde\alpha + \frac{1}{2}\frac{d_1 d_2}{|y-z|\sin\tilde\alpha}\right)$$

. Using the explicit form of d_2, the latter point has the distance

$$|i_{xyz}|^2 = \frac{1}{4}|y-z|^2 + \frac{1}{4}d_2^2 + \frac{1}{4}\frac{d_2^2\cos^2\tilde\alpha}{\sin^2\tilde\alpha}$$

to the origin $x = 0$. Using $|y-z|\sin\tilde\alpha = |z|\sin\alpha$ and $d_2 = |y| - |z|\cos\alpha$ we obtain

$$|i_{xyz}|^2 = \frac{1}{4}\left(|y-z|^2\left(1 + \frac{(|y|-|z|\cos\alpha)^2\cos^2\tilde\alpha}{|z|^2\sin^2\alpha}\right) + (|y|-|z|\cos\alpha)^2\right).$$

Given y, the latter expression becomes small for $|y-z|$ small, with the smallest value being $|y-z| = d\mathfrak{r}$. But then

$$\cos^2\tilde\alpha = 1 - \sin^2\tilde\alpha = 1 - \frac{(|z|\sin\alpha)^2}{|y-z|^2}$$

and hence the distance becomes

$$|i_{xyz}|^2 = \frac{1}{4}\left((d\mathfrak{r})^2\left(1 + \frac{(|y|-|z|\cos\alpha)^2\left((d\mathfrak{r})^2 + |z|^2\sin^2\alpha\right)}{(d\mathfrak{r})^2|z|^2\sin^2\alpha}\right) + (|y|-|z|\cos\alpha)^2\right).$$

We finally use $|y| = |z|\cos\alpha - \sqrt{(d\mathfrak{r})^2 - |z|^2\sin^2\alpha}$ and obtain

$$|i_{xyz}|^2 = \frac{1}{4}\left((d\mathfrak{r})^2\left(1 + \frac{\left((d\mathfrak{r})^4 - |z|^4\sin^4\alpha\right)}{(d\mathfrak{r})^2|z|^2\sin^2\alpha}\right) + \left((d\mathfrak{r})^2 - |z|^2\sin^2\alpha\right)\right).$$

The latter expression now needs to be smaller than d_x. We observe that the expression on the right hand side decreases for fixed α if $|z|$ increases.

On the other hand, we can resolve $|z|\,(y) = |y|\cos\alpha - \sqrt{|y|^2\sin^2\alpha + (d\mathfrak{r})^2}$. From the conditions $|y| \le d_x$ and $|i_{xyz}| \le d_x$, we then infer (8.3). □

Lemma 8.8. *Let $\mathbb{Y}$ be constructed from Lemma 4.25 for $\mathbb{X}_\mathfrak{r} = \mathbb{X}_{\mathrm{mat}}$ with the corresponding standard graph $\mathbb{G}_{\mathrm{simple}}(\mathbf{P})$ (see Definition 4.29). Let the admissible paths $\mathbb{AX}(y,x)$, $x \in \mathbb{X}_\mathfrak{r}$, $y \in \mathbb{Y}$, be the set of shortest paths on the graph between x and y. Then there exists $C > 0$ such that for every $x_j \in \mathbb{X}_\mathfrak{r}$ it holds $\mathsf{R}_0(x_j, \mathfrak{A}_{2,j})/d_j + \mathsf{S}_j \le C$. In particular, for every $1 < s < p$ it holds*

$$\lim_{n\to\infty} \frac{1}{|n\mathbf{Q}|} \int_{\mathbb{AX}(n\mathbf{Q})} \left(\sum_{x_j \in \mathbb{X}_\mathfrak{r}(\mathbf{Q})} \chi_{\mathbb{B}_{\mathsf{R}_0(x_j,\mathfrak{A}_{2,j})}(x_j)} d_j^{d+\frac{s-1}{s}} \mathsf{S}_j^{\frac{s-1}{s}} \right)^{\frac{p}{p-s}} < \infty \,.$$

Proof. Since the admissible paths are the shortest paths, there exists $C > 0$ such that for every $Y \in \mathbb{AX}(y, x_j)$, $x_j \in \mathbb{X}_\mathfrak{r}$, $y \in \mathbb{Y} \cap \mathbb{B}_{\frac{\mathfrak{r}}{2}}(\mathfrak{A}_{1,j})$ it holds $\mathrm{Length} Y \le C\,|x_j - y|$. Furthermore, for $x_k \in \mathbb{X}_\mathfrak{r}$ with $x_k \sim\sim x_j$ we find $|x_k - x_j| \le 2d_j$ and since x_k and x_j are connected through a path lying inside $\mathbb{B}_{2d_j}(x_j)$ possibly crossing other $x_i \in \mathfrak{A}_{2,j} \cap \mathbb{B}_{2d_j}(x_j)$ we can assume for the same C that for every $Y \in \mathbb{AX}(x_k, x_j)$, $x_k \sim\sim x_j$ it holds $\mathrm{Length} Y \le C\,|x_k - x_j|$. This provides a uniform bound on $\mathsf{R}_0(x_j, \mathfrak{A}_{2,j}) + \mathsf{S}_j d_j \le C d_j$. The lemma now follows from Lemma 4.21 and Theorem 7.11. □

9 Sobolev Spaces on the Probability Space $(\Omega, \mathbb{P})$

Based on Assumption 2.14, we want to achieve a better understanding of the mapping $f \mapsto f_\omega$. For this we make the following basic assumption throughout this section.

Assumption 9.1. *Let $(\Omega, \sigma, \mathbb{P})$ be a probability space satisfying Assumption 2.14 and let τ be a dynamical system on Ω in the sense of Definition 2.15.*

For the introduction of traces of $W^{1,p}(\Omega)$-functions below we will need the following (uncommon) stronger assumption. It is motivated by Theorem 2.33, which states that we can assume Ω to be a separable metric space.

Assumption 9.2. *Let $(\Omega, \sigma, \mathbb{P})$ be a probability space satisfying Assumption 2.14 and let τ be a dynamical system on Ω in the sense of Definition 2.15. Furthermore, let Ω be a separable metric space such that σ is the completion of the Borel algebra $\mathcal{B}(\Omega)$ under the construction of the Lebesgue space $L^1(\Omega; \mathbb{P})$.*

Assumption 9.2 will pay of due to the second part of the following lemma, which is a fundamental property of separable σ-algebras.

Lemma 9.3. *Let (A, Σ, μ) be a measure space with a countably generated σ-algebra Σ. Then for every $1 \le p < \infty$ the space $L^p(A; \mu)$ is separable. If A is a separable metric space and Σ the completion of the Borel algebra with respect to μ then $C_b(A) \hookrightarrow L^p(\Omega; \mu)$ densely and continuously, where $C_b(\Omega)$ are the bounded continuous functions on Ω.*

The following lemma is a fundamental observation which will be frequently used throughout the rest of this work. It relies on the following notation. For $f:\Omega\to X$, X a metric space, and $\omega\in\Omega$ we define the *realization* f_ω of f as

$$f_\omega:\mathbb{R}^d\to X\,,\qquad x\mapsto f(\tau_x\omega)\,.$$

Then we find the following behavior.

Lemma 9.4. *Let Assumption 9.1 hold and let $f\in L^p(\Omega)$ for $1\le p\le\infty$. Then for almost every $\omega\in\Omega$ and for every bounded domain $\mathbf{Q}$ it holds $f_\omega\in L^p(\mathbf{Q})$.*

Proof. For $1\le p<\infty$ observe that

$$\begin{aligned}\mathcal{L}(\mathbf{Q})\int_\Omega|f(\omega)|^p\,\mathrm{d}\mathbb{P}(\omega)&=\int_{\mathbf{Q}}\int_\Omega|f(\omega)|^p\,\mathrm{d}\mathbb{P}(\omega)\mathrm{d}x=\int_{\mathbf{Q}}\int_\Omega|f(\tau_x\omega)|^p\,\mathrm{d}\mathbb{P}(\omega)\mathrm{d}x\\&=\int_\Omega\int_{\mathbf{Q}}|f(\tau_x\omega)|^p\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\,.\end{aligned}$$

From Fubini's theorem it follows that $\int_{\mathbf{Q}}|f(\tau_x\omega)|^p\,\mathrm{d}x$ exists for a.e. $\omega\in\Omega$. For $p=\infty$ the statement follows since $\int_{\mathbf{Q}}|f(\tau_x\omega)|^p\,\mathrm{d}x$ exists for every $p<\infty$. □

9.1 The Semigroup T on $L^p(\Omega)$ and its Generators

For every $x\in\mathbb{R}^d$ we define the mapping

$$\mathrm{T}(x):\ f\mapsto\mathrm{T}(x)f\,,$$

through $\mathrm{T}(x)f(\omega):=f(\tau_x\omega)$. This mapping is well defined for every measurable function $f:\ \Omega\to\mathbb{R}$. Moreover, we have the following properties.

Lemma 9.5. *Let Assumption 9.1 hold. For every $1\le p<\infty$, the family $(\mathrm{T}(x))_{x\in\mathbb{R}^d}$ is a strongly continuous unitary group on $L^p(\Omega)$.*

Proof. Every $\mathrm{T}(x)$ is linear on $L^p(\Omega)$ and the group property follows from $(\mathrm{T}(x)\mathrm{T}(y)f)(\omega)=f(\tau_x\tau_y\omega)=\mathrm{T}(x+y)f(\omega)$. Since τ_x is measure preserving, we find $\|f\|_{L^p(\Omega)}=\|\mathrm{T}(x)f\|_{L^p(\Omega)}$ and hence $\mathrm{T}(x)$ is unitary.

In order to prove the strong continuity, observe

$$\begin{aligned}\|\mathrm{T}(x)f-f\|^p_{L^p(\Omega)}&=\int_\Omega|f(\tau_x\omega)-f(\omega)|^p\,\mathrm{d}\mathbb{P}(\omega)\\&=\int_\Omega\int_{\mathbb{Y}}|f(\tau_{x+y}\omega)-f(\tau_y\omega)|^p\,\mathrm{d}y\,\mathrm{d}\mathbb{P}(\omega)\\&=\int_\Omega\int_{\mathbb{Y}}|f_\omega(x+y)-f_\omega(y)|^p\,\mathrm{d}y\,\mathrm{d}\mathbb{P}(\omega)\,,\end{aligned}$$

where we used that τ_y preserves measure and Fubini's theorem. By Lemma 9.4 $f_\omega\in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for almost every $\omega\in\Omega$ and for such ω it holds $\lim_{|h|\to0}\|f_\omega-f_\omega(\cdot+h)\|_{L^p(4\mathbb{Y})}=0$. Furthermore, for $|x|<\frac12$ we have

$$\int_{\mathbb{Y}}|f_\omega(x+y)-f_\omega(y)|^p\,\mathrm{d}y<2p\int_{2\mathbb{Y}}|f_\omega(y)|^p\,\mathrm{d}y\,.$$

Thus, the Lebesgue dominated convergence theorem yields

$$\sup_{|x|<t}\|\mathrm{T}(x)f-f\|_{L^p(\Omega)}^p\to 0\qquad\text{as } t\to 0\,.$$

$\square$

For $i=1,\dots,d$, let $\mathbf{e}_i$ be the i-th canonical basis vector in $\mathbb{R}^d$. Since $\mathrm{T}(x)$ define a strongly continuous group we can draw the conclusion that the operators $\mathrm{T}_i(t)f:=\mathrm{T}(t\mathbf{e}_i)f$, define d independent one-parameter strongly continuous semigroups on $L^p(\Omega)$ that commute with each other and jointly generate $(\mathrm{T}(x))_{x\in\mathbb{R}^d}$ on $L^p(\Omega)$. Each of these one-parameter groups has a generator D_i defined by

$$\mathrm{D}_i f(\omega)=\lim_{t\to0}\frac{\mathrm{T}_i(t)f(\omega)-f(\omega)}{t}=\lim_{t\to0}\frac{f(\tau_{t\mathbf{e}_i}\omega)-f(\omega)}{t}\,.$$

The expression $\mathrm{D}_i f$ is called i-th derivative of f and is skew adjoint:

$$\int_\Omega g\mathrm{D}_i f\mathrm{d}\mathbb{P}=-\int_\Omega f\mathrm{D}_i g\mathrm{d}\mathbb{P}\,.$$

The joint domain of all D_i in $L^p(\Omega)$ is denote

$$W^{1,p}(\Omega):=\{f\in L^p(\Omega)\,|\,\forall i=1,\dots,d\,:\,\,\mathrm{D}_i f\in L^p(\Omega)\}\,,$$

with the natural norm

$$\|f\|_{W^{1,p}(\Omega)}:=\|f\|_{L^p(\Omega)}+\sum_{i=1}^d\|\mathrm{D}_i f\|_{L^p(\Omega)}\,.$$

In case $p=2$, this is a Hilbert space with scalar product

$$\langle f,g\rangle_{W^{1,2}(\Omega)}^2:=\int_\Omega fg\mathrm{d}\mathbb{P}+\sum_{i=1}^d\int_\Omega\mathrm{D}_i f\mathrm{D}_i g\,\mathrm{d}\mathbb{P}\,.$$

We finally denote $\mathrm{D}_\omega f:=(\mathrm{D}_1 f,\dots,\mathrm{D}_d f)^T$ the gradient with respect to ω and by $-\mathrm{div}_\omega$ the adjoint of D_ω. Sometimes we write $\nabla_\omega f:=\mathrm{D}_\omega f$ to underline the gradient aspect. Similar to distributional derivatives in $\mathbb{R}^d$, we may define $\mathrm{D}_\omega^k f$ through iterated application of D_ω and

$$W^{k,p}(\Omega):=\left\{f\in L^p(\Omega)\,|\,\forall j=1,\dots,k\,:\,\,\mathrm{D}_\omega^j f\in L^p(\Omega)^{d^j}\right\}\,.$$

In case Assumption 9.2 holds, we denote

$$C_b^1(\Omega):=\left\{f\in C_b(\Omega):\,\nabla f\in C_b(\Omega;\mathbb{R}^d)\right\}\,.$$

Lemma 9.6. *For every $f\in W^{1,p}(\Omega)$ for almost every $\omega\in\Omega$ it holds $f_\omega\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$. In particular, for every bounded domain $\mathbf{Q}\subset\mathbb{R}^d$ it holds*

$$\forall\psi\in C_c^1(\mathbf{Q})\,:\qquad\int_{\mathbf{Q}}f_\omega\partial_i\psi=-\int_{\mathbf{Q}}\psi\left(\mathrm{D}_i f\right)_\omega\,.\tag{9.1}$$

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ and let $g \in L^q(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$. Using Lebesgue's dominated convergence theorem it follows:

$$\begin{aligned}
\int_\Omega g(\omega)\int_{\mathbb{R}^d} f_\omega \partial_i\psi \mathrm{d}\mathcal{L}\mathrm{d}\mathbb{P}(\omega) &= \int_\Omega g(\omega)\int_{\mathbb{R}^d} f_\omega(x)\lim_{t\to 0}\frac{\psi(x+t\mathbf{e}_i)-\psi(x)}{t}\mathrm{d}x\mathrm{d}\mathbb{P}(\omega)\\
&= \lim_{t\to 0}\int_\Omega g(\omega)\int_{\mathbb{R}^d} f_\omega(x)\frac{\psi(x+t\mathbf{e}_i)-\psi(x)}{t}\mathrm{d}x\mathrm{d}\mathbb{P}(\omega)\\
&= \lim_{t\to 0}\int_\Omega g(\omega)\int_{\mathbb{R}^d} \psi(x+t\mathbf{e}_i)\frac{f_\omega(x)-f_\omega(x+t\mathbf{e}_i)}{t}\mathrm{d}x\mathrm{d}\mathbb{P}(\omega)\,.
\end{aligned}$$

Since τ_x preserves measure, we obtain

$$\begin{aligned}
\int_\Omega\int_{\mathbb{R}^d} f_\omega \partial_i\psi \mathrm{d}\mathcal{L}\mathrm{d}\mathbb{P}(\omega) &= \lim_{t\to 0}\int_{\mathbb{R}^d}\int_\Omega g(\tau_{-x}\omega)\psi(x+t\mathbf{e}_i)\frac{f(\omega)-\mathrm{T}_i f(\omega)}{t}\mathrm{d}x\mathrm{d}\mathbb{P}(\omega)\\
&= \lim_{t\to 0}\int_\Omega \frac{f(\omega)-\mathrm{T}_i f(\omega)}{t}\int_{\mathbb{R}^d} g(\tau_{-x}\omega)\psi(x+t\mathbf{e}_i)\mathrm{d}x\mathrm{d}\mathbb{P}(\omega)\\
&= -\int_{\mathbb{R}^d}\int_\Omega g(\tau_{-x}\omega)\psi(x)\mathrm{D}_i f(\omega)\mathrm{d}x\mathrm{d}\mathbb{P}(\omega)\\
&= -\int_\Omega g(\omega)\int_{\mathbb{R}^d}\left(\mathrm{D}_i f\right)_\omega \psi\mathrm{d}\mathcal{L}\mathrm{d}\mathbb{P}(\omega)\,.
\end{aligned}$$

Using a countable dense subset $(\psi_i)_{i\in\mathbb{N}} \subset L^q(\mathbb{R}^d)$, $\psi_i \in C_c^\infty(\mathbb{R}^d)$ and a suitable family of testfunctions $(g_j)_{j\in\mathbb{N}} \subset L^q(\Omega)$, we obtain that for almost every $\omega\in\Omega$ equation (9.1) holds for every ψ_i. Hence, by density, it holds for all $\psi \in C_c^1(\mathbf{Q})$. □

Lemma 9.7. *Let $1\le p<\infty$ and let $\eta \in C_c^\infty(\mathbb{R}^d)$. For every $f\in L^p(\Omega)$ let*

$$\left(\eta * f\right)(\omega) := \int_{\mathbb{R}^d}\eta(x)f(\tau_x\omega)\mathrm{d}x\,.$$

Then for every $k\in\mathbb{N}$ it holds $\eta * f \in W^{k,p}(\Omega)$ with $\mathrm{D}_i(\eta * f) = (\partial_i\eta) * f$ and almost every realization of $\mathscr{I}_\delta f$ is an element of $C^\infty(\mathbb{R}^d)$. Furthermore, the estimates

$$\|\eta * f\|_{L^p(\Omega)}^p \le \|\eta\|_{L^1(\mathbb{R}^d)}\|f\|_{L^p(\Omega)}\,,\qquad \|\mathrm{D}_i(\eta * f)\|_{L^p(\Omega)}^p \le \|\partial_i\eta\|_{L^1(\mathbb{R}^d)}\|f\|_{L^p(\Omega)} \tag{9.2}$$

hold and we have $\mathrm{D}_i\left(\eta * f\right) = \eta * \mathrm{D}_i f$.

Proof. Let $k\in\mathbb{N}$ and observe

$$\begin{aligned}
\|\eta * f\|_{L^p(\Omega)}^p &= \int_\Omega (2k)^{-d}\int_{(-k,k)^d}\left|\left(\eta * f\right)(\tau_y\omega)\right|^p\mathrm{d}y\,\mathrm{d}\mathbb{P}(\omega)\\
&\le \int_\Omega (2k)^{-d}\int_{(-k,k)^d}\int_{\mathbb{R}^d}\left|\eta(x)f(\tau_{y+x}\omega)\right|^p\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\mathbb{P}(\omega)\,.
\end{aligned}$$

Due to the convolution inequality we have

$$\begin{aligned}
\|\eta * f\|_{L^p(\Omega)}^p &\le \|\eta\|_{L^1(\mathbb{R}^d)}(2k)^{-d}\int_\Omega \|f_\omega\|_{L^p((-k-1,k+1)^d)}^p\mathrm{d}\mathbb{P}(\omega)\\
&\le \|\eta\|_{L^1(\mathbb{R}^d)}\left(\frac{k+1}{k}\right)^{-d}\int_\Omega |f(\omega)|^p\,\mathrm{d}\mathbb{P}(\omega)
\end{aligned}$$

and since k is arbitrary, the we obtain $\|\eta * f\|_{L^p(\Omega)}^p \leq \|\eta\|_{L^1(\mathbb{R}^d)} \|f\|_{L^p(\Omega)}$, the first part of (9.2).

In order to show $\mathscr{I}_\delta f \in W^{k,p}(\Omega)$ observe

$$\frac{1}{t}\left(\eta * f(\tau_{t\mathbf{e}_i}\omega) - \eta * f(\omega)\right) = \int_{\mathbb{R}^d} \frac{1}{t}\left(\eta(x+t\mathbf{e}_i) - \eta(x)\right) f(\tau_x\omega)\,.$$

Taking the limit $t \to 0$ in $L^p(\Omega)$ on both sides using Lebesgue's dominated convergence theorem implies

$$\mathrm{D}_i\left(\eta * f\right) = \int_{\mathbb{R}^d} \partial_i\eta(x) f(\tau_x\omega)\,, \tag{9.3}$$

and hence $\mathrm{D}_i\left(\mathscr{I}_\delta f\right) \in L^p(\Omega)$ with $\mathrm{D}_i(\eta * f) = (\partial_i\eta) * f$ and the second part of (9.2) follows. Equation (9.3) also shows that

$$\left(\eta * f\right)(\tau_y\omega) = \int_{\mathbb{R}^d} \eta(x) f\left(\tau_{x+y}\omega\right) \mathrm{d}x = \int_{\mathbb{R}^d} \eta(x-y) f\left(\tau_x\omega\right) \mathrm{d}x$$

and hence almost every realization of $\eta * f$ has C^∞-regularity. Furthermore, (9.3) implies

$$\begin{aligned}
\mathrm{D}_i\left(\mathscr{I}_\delta f\right) &= \lim_{t\to 0} \frac{1}{t}\left(\left(\eta * f\right)(\tau_{t\mathbf{e}_i}\omega) - \left(\eta * f\right)(\omega)\right) \\
&= \eta * \lim_{t\to 0} \frac{1}{t}\left(f(\tau_{t\mathbf{e}_i}\omega) - f(\omega)\right) \\
&= \eta * \mathrm{D}_i f\,,
\end{aligned}$$

where we used continuity of $f \mapsto \eta * f$ and strong convergence of $\frac{1}{t}\left(f(\tau_{t\mathbf{e}_i}\omega) - f(\omega)\right) \to \mathrm{D}_i f$. □

Similar to $L^p(\mathbb{R}^d)$- and Sobolev spaces on $\mathbb{R}^d$, we can introduce a family of smoothing operators. Let $(\eta_\delta)_{\delta>0}$ be a standard sequence of mollifiers which are symmetric w.r.t. 0 and define

$$\mathscr{I}_\delta :\, L^p(\Omega) \to L^p(\Omega)\,, \qquad \mathscr{I}_\delta f(\omega) := \int_{\mathbb{R}^d} \eta_\delta(x) f\left(\tau_x\omega\right) \mathrm{d}x\,. \tag{9.4}$$

Lemma 9.8. *For every $\delta > 0$, $1 \leq p < \infty$, the operator $\mathscr{I}_\delta$ is unitary and selfadjoint. For every $f \in L^p(\Omega)$, $k \in \mathbb{N}$ it holds $\mathscr{I}_\delta f \in W^{k,p}(\Omega)$, $\mathscr{I}_\delta f \to f$ strongly in $L^p(\Omega)$, and almost every realization of $\mathscr{I}_\delta f$ is an element of $C^\infty(\mathbb{R}^d)$. Finally, for $f \in W^{1,p}(\Omega)$ it holds*

$$\lim_{\delta\to 0} \|\mathscr{I}_\delta f - f\|_{W^{1,p}(\Omega)} = 0 \tag{9.5}$$

and $\mathrm{D}_i\mathscr{I}_\delta f = \mathscr{I}_\delta \mathrm{D}_i f$.

Proof. The selfadjointness follows from the definition of $\mathscr{I}_\delta$, symmetry of η_δ and invariance of $\mathbb{P}$ w.r.t. τ_x. All other parts except for (9.5) follow from Lemma 9.7.

Finally, observe that the the convolution inequality and the strong continuity of $\mathrm{T}(x)$ yield

$$\begin{aligned}
\int_\Omega |\mathscr{I}_\delta f - f|^p &= \int_\Omega \left|\int_{\mathbb{R}^d} \eta_\delta(x)\left(f(\tau_x\omega) - f(\omega)\right)\mathrm{d}x\right|^p \mathrm{d}\mathbb{P}(\omega) \\
&\leq \int_\Omega \|\eta_\delta\|_{L^1(\mathbb{R}^d)}^p \int_{[-\delta,\delta]^d} |(f(\tau_x\omega) - f(\omega))|^p\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega) \\
&\leq \int_{[-\delta,\delta]^d} \int_\Omega |(f(\tau_x\omega) - f(\omega))|^p\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x \\
&\to 0\,.
\end{aligned}$$

Since $\mathrm{D}_i\mathscr{I}_\delta f = \mathscr{I}_\delta \mathrm{D}_i f$, it also holds $\mathrm{D}_i\mathscr{I}_\delta f \to \mathrm{D}_i f$ strongly in $L^p(\Omega)$. □

9.2 Gradients and Solenoidals

We denote by $L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ the set of measurable functions $f:\mathbb{R}^d\to\mathbb{R}^d$ such that $f|_{\mathbf{U}}\in L^p(\mathbf{U};\mathbb{R}^d)$ for every bounded domain $\mathbf{U}$ and we define

$$\begin{aligned}
L^p_{\mathrm{pot,loc}}(\mathbb{R}^d) &:= \left\{u\in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)\mid \forall\mathbf{U}\text{ bounded domain, }\exists\varphi\in H^1(\mathbf{U})\,:\,u=\nabla\varphi\right\}\,,\\
L^p_{\mathrm{sol,loc}}(\mathbb{R}^d) &:= \left\{u\in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)\mid \int_{\mathbb{R}^d}u\cdot\nabla\varphi=0\ \forall\varphi\in C^1_c(\mathbb{R}^d)\right\}\,.
\end{aligned}$$

Remark 9.9. The space $L^p_{\mathrm{pot,loc}}(\mathbb{R}^d)$ is invariant under convolution. This follows immediately from the fact that if $u=\nabla\varphi$ locally, then $\eta_\delta * u=\nabla\left(\eta_\delta*\varphi\right)$.

Recalling the notation for a realization $u_\omega(x):=u(\tau_x\omega)$ for $u\in L^p(\Omega)$, we can then define corresponding spaces on Ω through

$$\begin{aligned}
L^p_{\mathrm{pot}}(\Omega) &:= \left\{u\in L^p(\Omega;\mathbb{R}^d)\,:\,u_\omega\in L^p_{\mathrm{pot,loc}}(\mathbb{R}^d)\text{ for }\mathbb{P}-\text{a.e. }\omega\in\Omega\right\}\,,\\
L^p_{\mathrm{sol}}(\Omega) &:= \left\{u\in L^p(\Omega;\mathbb{R}^d)\,:\,u_\omega\in L^p_{\mathrm{sol,loc}}(\mathbb{R}^d)\text{ for }\mathbb{P}-\text{a.e. }\omega\in\Omega\right\}\,,\\
\mathcal{V}^p_{\mathrm{pot}}(\Omega) &:= \left\{u\in L^p_{\mathrm{pot}}(\Omega)\,:\,\int_\Omega u\,\mathrm{d}\mathbb{P}=0\right\}\,.
\end{aligned}\tag{9.6}$$

The spaces $L^p_{\mathrm{pot}}(\Omega)$ and $W^{1,p}(\Omega)$ are connected as the following theorem shows.

Theorem 9.10. *For $1<p,q<\infty$ with $\frac1p+\frac1q=1$ the spaces $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ and $L^p_{\mathrm{sol}}(\Omega)$ are closed and it holds*

$$\left(\mathcal{V}^p_{\mathrm{pot}}(\Omega)\right)^\perp=L^q_{\mathrm{sol}}(\Omega)\,,\qquad \left(L^p_{\mathrm{sol}}(\Omega)\right)^\perp=\mathcal{V}^q_{\mathrm{pot}}(\Omega)\tag{9.7}$$

in the sense of duality. Furthermore, $W^{1,p}(\Omega)$ lies densely in $L^p(\Omega)$ and

$$\mathcal{V}^p_{\mathrm{pot}}(\Omega)=\mathrm{closure}_{L^p}\left\{\mathrm{D}u\mid u\in W^{1,p}(\Omega)\right\}\,.\tag{9.8}$$

Proof. The density of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ follows from Lemma 9.8. We furthermore observe that $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ is invariant with respect to $\mathscr{I}_\delta$. In fact, let $u\in\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ and consider $\omega\in\Omega$ such that $u_\omega\in L^p_{\mathrm{pot,loc}}(\mathbb{R}^d)$. Then

$$\left(\mathscr{I}_\delta u\right)_\omega(x)=\int_{\mathbb{R}^d}\eta_\delta(y)u\left(\tau_{x+y}\omega\right)\mathrm{d}y$$

and hence $\left(\mathscr{I}_\delta u\right)_\omega\in L^p_{\mathrm{pot,loc}}(\mathbb{R}^d)$ due to Remark 9.9. Furthermore, the space $L^p_{\mathrm{sol}}(\Omega)$ is closed as can be seen from the continuity of the expression

$$L^p(\Omega;\mathbb{R}^d)\to\mathbb{R}\,,\qquad u\mapsto\int_\Omega g(\omega)\int_{(-1,1)^d}u(\tau_x\omega)\cdot\nabla\varphi(x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\,,$$

where $\varphi\in W^{1,q}(\mathbb{R}^d)$ and $g\in L^q(\Omega)$ are arbitrary.

It remains to show (9.7), (9.8) and closedness of $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$.

Step 1: We first show that $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ and $L^q_{\mathrm{sol}}(\Omega)$ are mutually orthogonal in the sense of duality. Let $\upsilon\in\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ and $p\in L^q_{\mathrm{sol}}(\Omega)$ and chose $\omega\in\Omega$ such that for $\upsilon^\varepsilon(x)=\upsilon(\tau_{\frac{x}{\varepsilon}}\omega)$, $p^\varepsilon(x)=p(\tau_{\frac{x}{\varepsilon}}\omega)$ and $\upsilon^\varepsilon\cdot p^\varepsilon$ the ergodic theorem **??** holds. Thus, we get $\upsilon^\varepsilon\cdot p^\varepsilon\rightharpoonup\mathbb{E}(\upsilon\cdot p|\mathscr{I})$ weakly in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. It remains to show that $\upsilon^\varepsilon\cdot p^\varepsilon\rightharpoonup^*0$. Since $\upsilon\in L^p_{\mathrm{pot}}(\Omega)$, we find for every $\varepsilon>0$ some $u^\varepsilon\in W^{1,p}(\mathbf{Q})$ such that $\nabla u^\varepsilon=\upsilon^\varepsilon$ and $\int_{\mathbf{Q}}u^\varepsilon=0$. By the ergodic theorem $\nabla u^\varepsilon=\upsilon^\varepsilon\rightharpoonup^*\mathbb{E}(\upsilon|\mathscr{I})=0$ and

$u^\varepsilon \rightharpoonup u$ has average 0. Due to the PoincarÃ© inequality and the compact embedding $W^{1,p}(\mathbf{Q}) \hookrightarrow L^p(\mathbf{Q})$, we find $u^\varepsilon \to 0$ strongly in $L^p(\mathbf{Q})$. Therefore, for all $\psi \in C_c^\infty(\mathbf{Q})$, we find

$$\int_{\mathbf{Q}} \psi v^\varepsilon \cdot p^\varepsilon \, dx = \int_{\mathbf{Q}} \psi p^\varepsilon \cdot \nabla u^\varepsilon \, dx = -\int_{\mathbf{Q}} u^\varepsilon p^\varepsilon \cdot \nabla \psi \, dx \to 0 \quad \text{for } \varepsilon \to 0 .$$

Therefore, we obtain

$$L^q_{\mathrm{sol}}(\Omega) \subset \left(\mathcal{V}^p_{\mathrm{pot}}(\Omega)\right)^\perp \quad \text{and} \quad \mathcal{V}^p_{\mathrm{pot}}(\Omega) \subset (L^q_{\mathrm{sol}}(\Omega))^\perp . \tag{9.9}$$

Step 2: We prove (9.7) and closedness of $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ in case $p = 2$. From Step 1 we know that $L^2_{\mathrm{sol}}(\Omega) \subset \left(\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right)^\perp$ and it remains to show that $\left(\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right)^\perp \subseteq L^2_{\mathrm{sol}}(\Omega)$. Let $u \in L^2(\Omega;\mathbb{R}^d)$ and use the decomposition $u = u_{\mathrm{pot}} + \tilde{u}$ where $u_{\mathrm{pot}} \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$ and $\tilde{u} \in \left(\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right)^\perp$. Since $\mathscr{I}_\delta$ is symmetric and $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$ is invariant with respect to $\mathscr{I}_\delta$, we observe that

$$\forall v \in \mathcal{V}^2_{\mathrm{pot}}(\Omega) : \quad \langle \mathscr{I}_\delta \tilde{u},\, v\rangle = \langle \tilde{u},\, \mathscr{I}_\delta v\rangle = 0$$

and hence $\mathscr{I}_\delta \tilde{u} \in \left(\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right)^\perp$. In particular, for every $\varepsilon > 0$ and every $\phi \in L^2(\Omega)$ it holds

$$0 = \langle \mathscr{I}_\delta \tilde{u},\, \mathrm{D}_\omega \mathscr{I}_\varepsilon \phi\rangle = -\left\langle \mathrm{div}_\omega \mathscr{I}_\delta \tilde{u},\, \mathscr{I}_\varepsilon \phi\right\rangle$$

and as $\varepsilon \to 0$ it holds

$$0 = -\left\langle \mathrm{div}_\omega \mathscr{I}_\delta \tilde{u},\, \phi\right\rangle .$$

Since $\phi \in L^2(\Omega)$ was arbitrary, this implies $\sum_i \mathrm{D}_i \mathscr{I}_\delta \tilde{u} = 0$ almost everywhere, i.e. $\mathscr{I}_\delta \tilde{u} \in L^2_{\mathrm{sol}}(\Omega)$. Since $\mathscr{I}_\delta \tilde{u} \to \tilde{u}$ as $\delta \to 0$, the closedness of $L^2_{\mathrm{sol}}(\Omega)$ implies $\tilde{u} \in L^2_{\mathrm{sol}}(\Omega)$. Hence $L^2_{\mathrm{sol}}(\Omega) \supset \mathcal{V}^2_{\mathrm{pot}}(\Omega)^\perp$ and Step 1 implies $L^2_{\mathrm{sol}}(\Omega) = \mathcal{V}^2_{\mathrm{pot}}(\Omega)^\perp$ and closedness of $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$.

Step 3: For $p \in [1,2]$ we deduce from Step 2

$$\left(\mathcal{V}^p_{\mathrm{pot}}(\Omega)\right)^\perp \subseteq L^q(\Omega;\mathbb{R}^d) \cap \left(\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right)^\perp = L^q(\Omega;\mathbb{R}^d) \cap L^2_{\mathrm{sol}}(\Omega) \subseteq L^q_{\mathrm{sol}}(\Omega) . \tag{9.10}$$

Interchanging the role of $\mathcal{V}_{\mathrm{pot}}$ and L_{sol} yields

$$(L^p_{\mathrm{sol}}(\Omega))^\perp \subseteq \mathcal{V}^q_{\mathrm{pot}}(\Omega) . \tag{9.11}$$

Inclusions (9.9), (9.10) and (9.11) imply (9.7).

Step 4: For $1 < p < \infty$ we denote

$$V := \left\{ \mathrm{D}\phi \,|\, \phi \in W^{1,p}(\Omega) \right\} \subset L^p_{\mathrm{pot}}(\Omega) .$$

Let $u \in L^q(\Omega;\mathbb{R}^d)$ satisfy

$$\forall \phi \in W^{1,p}(\Omega) : \qquad \langle u,\, \mathrm{D}_\omega \phi\rangle = 0 .$$

According to Lemma 9.8, D_i and $\mathscr{I}_\delta$ commute for $\phi \in W^{1,p}(\Omega)$. Furthermore, $\mathscr{I}_\delta \phi \in W^{1,p}(\Omega)$ and hence

$$0 = \langle u,\, \mathrm{D}_\omega \mathscr{I}_\delta \phi\rangle = \langle u,\, \mathscr{I}_\delta \mathrm{D}_\omega \phi\rangle = -\left\langle \mathrm{div}_\omega \mathscr{I}_\delta u,\, \phi\right\rangle .$$

Since $\phi \in W^{1,p}(\Omega)$ was arbitrary and $W^{1,p}(\Omega)$ is dense in $L^p(\Omega)$, it follows $\mathrm{div}_\omega \mathscr{I}_\delta u = 0$, which implies $u \in L^q_{\mathrm{sol}}(\Omega)$ by closedness of $L^q_{\mathrm{sol}}(\Omega)$. To conclude, we have shown $L^q_{\mathrm{sol}}(\Omega) = \left(\mathcal{V}^p_{\mathrm{pot}}(\Omega)\right)^\perp \subseteq V^\perp \subseteq L^q_{\mathrm{sol}}(\Omega)$, and hence (9.8). $\square$

9.3 Stampaccia's Lemma

Lemma 9.11 (Stampaccia)**.** *Let $G : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous and let $u \in W^{1,p}(\Omega)$. Then $G \circ u \in W^{1,p}(\Omega)$.*

Proof. Let $u \in W^{1,p}(\Omega)$. It holds

$$\begin{aligned}\limsup_{h\to 0}\left\|\frac{T_{h\mathbf{e}_i}G(u)-G(u)}{h}\right\|_{L^p(\Omega)} &= \limsup_{h\to 0}\left\|\frac{G\left(T_{h\mathbf{e}_i}u\right)-G(u)}{h}\right\|_{L^p(\Omega)}\\ &\leq \limsup_{h\to 0}\left\|\frac{G\left(T_{h\mathbf{e}_i}u\right)-G\left(u\right)}{T_{h\mathbf{e}_i}u-u}\right\|_{\infty}\left\|\frac{T_{h\mathbf{e}_i}u-u}{h}\right\|_{L^p(\Omega)}\\ &\leq \left\|G'\right\|_{\infty}\left\|\mathrm{D}_i u\right\|_{L^p(\Omega)}\,.\end{aligned}$$

Hence, we find that there exists $w_i \in L^p(\Omega)$ such that $\frac{1}{h}\left(T_{h\mathbf{e}_i}G(u)-G(u)\right) \rightharpoonup w_i$ weakly along a further subsequence. Testing this limit with a function $\varphi \in W^{1,q}(\Omega)$, we obtain that $w = (w_i)_{i=1\dots d}$ is the weak derivative of $G(u)$ as

$$\begin{aligned}\int_\Omega w_i\varphi \mathrm{d}\mathbb{P} &= \lim_{h\to 0}\int_\Omega \frac{1}{h}\left(T_{h\mathbf{e}_i}G(u)-G(u)\right)\varphi \mathrm{d}\mathbb{P}\\ &= -\lim_{h\to 0}\int_\Omega \frac{1}{h}\left(T_{h\mathbf{e}_i}\varphi-\varphi\right)G(u)\mathrm{d}\mathbb{P} = -\int_\Omega G(u)\mathrm{D}_i\varphi \mathrm{d}\mathbb{P}\,.\end{aligned}$$

□

Remark 9.12. Lemma 9.11 is well known in Sobolev theory in $\mathbb{R}^d$ and is due to Stampaccia. It can be found for example in the book by Evans [10]. Stampaccia [34] also showed for functions $u \in W^{1,p}(\mathbb{R}^d)$ that $\nabla\left(G\circ u\right) = G'(u)\nabla u$. However, to proof such a result in the case of general Ω goes beyond the scope of this chapter.

Theorem 9.13. *For every $1 \leq p < \infty$ the embedding $W^{1,\infty}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ is dense. In particular,*

$$\mathcal{V}^p_{\mathrm{pot}}(\Omega) = \mathrm{closure}_{L^p}\left\{\mathrm{D}u \mid u \in W^{1,\infty}(\Omega)\right\}\,.$$

Proof. Let $u \in W^{1,p}(\Omega)$ and let $k \in \mathbb{N}$. By Lemma 9.11 we obtain that the function $u_k := \max\left\{-k, \min\left\{k,u\right\}\right\}$ satisfies $u_k \in W^{1,p}(\Omega)$ and $\left\|u_k\right\|_\infty \leq k$. Since $u_k \to u$ as $k \to \infty$, it remains to show that u_k can be approximated by functions in $W^{1,\infty}(\Omega)$. To see this, note that for $u_k^\delta := \mathscr{I}_\delta u_k$ it holds

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\left(\mathscr{I}_\delta u_k(\tau_{t\mathbf{e}_i}\omega)-\mathscr{I}_\delta u_k(\omega)\right) &= \lim_{t\to 0}\int_{\mathbb{R}^d}\frac{1}{t}\left(\eta_\delta(x+t\mathbf{e}_i)-\eta_\delta(x)\right)u_k(\tau_x\omega)\\ &= \int_{\mathbb{R}^d}u_k(\tau_x\omega)\partial_i\eta_\delta(x)\,.\end{aligned}$$

and since $\eta_\delta \in C_c^\infty(\mathbb{R}^d)$ we find $u_k^\delta \in W^{1,\infty}(\Omega)$. Since $u_k^\delta \to u_k$ in $W^{1,p}(\Omega)$ as $\delta \to 0$ by Lemma 9.8, the theorem is proved. □

The last Theorem has an important implication for the existence of suitable countable and dense family of functions.

Theorem 9.14. *Let Assumption 9.1 hold. For every $1 \leq p < \infty$ there exists a countable dense family of functions $(u_k)_{k\in\mathbb{N}} \subset W^{1,p}(\Omega)$ such that $(u_k)_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega)$ and $(u_k)_{k\in\mathbb{N}}$ is stable under addition and scalar multiplication with $q \in \mathbb{Q}$. Furthermore, every u_k has almost surely bounded and continuously differentiable realizations with $\|u_{k,\omega}\|_{W^{1,\infty}(\mathbb{R}^d)} \leq \|u_k\|_{W^{1,\infty}(\Omega)}$. If additionally Assumption 9.2 holds, then $(u_k)_{k\in\mathbb{N}}$ can be chosen such that for every k it holds $u_k \in C_b^1(\Omega)$, $\nabla_\omega u_k \in C_b(\Omega)$.*

Proof. Let $(v_k)_{k\in\mathbb{N}} \subset W^{1,p}(\Omega)$ be dense. Then for every k consider $v_{k,n} := \max\{-n, \min\{n, v_k\}\}$ and for $m \in \mathbb{N}$ define $v_{k,n,m} := \mathscr{I}_{\frac{1}{m}} v_{k,n} = \eta_{\frac{1}{m}} * v_{k,n}$. Then $\|\mathrm{D}_i v_{k,n,m}\|_\infty \leq \left\|\partial_i \eta_{\frac{1}{m}}\right\|_\infty \|v_{k,n}\|_\infty$. Moreover, for every $\varepsilon > 0$ and every $\phi \in W^{1,p}(\Omega)$ there exists k with $\|v_k - \phi\|_{W^{1,p}(\Omega)} \leq \frac{\varepsilon}{3}$, n with $\|v_k - v_{k,n}\|_{W^{1,p}(\Omega)} \leq \frac{\varepsilon}{3}$ and m with $\|v_{k,n} - v_{k,n,m}\|_{W^{1,p}(\Omega)} \leq \frac{\varepsilon}{3}$. Based on the countable family $(v_{k,n,m})_{k,n,m\in\mathbb{N}}$, we find that

$$(u_k)_{k\in\mathbb{N}} := \left\{ \sum_{k,n,m=1}^{N} \lambda_{k,n,m} v_{k,n,m} :\ \lambda_{k,n,m} \in \mathbb{Q},\ N \in \mathbb{N} \right\}$$

satisfies all demanded properties.

If Assumption 9.2 holds we find $(c_l)_{l\in\mathbb{N}} \subset C_b(\Omega) \cap L^p(\Omega)$ dense in $L^p(\Omega)$. For every v_k like above and every $\delta > 0$ we observe by Lemma 9.7 that

$$\begin{aligned} \|\eta_\delta * (v_k - c_l)\|_{L^p(\Omega)} &\leq \|v_k - c_l\|_{L^p(\Omega)} , \\ \|\mathrm{D}_i(\eta_\delta * (v_k - c_l))\|_{L^p(\Omega)} &\leq \|\partial_i \eta_\delta\|_{L^1(\mathbb{R}^d)} \|v_k - c_l\|_{L^p(\Omega)} . \end{aligned}$$

Hence the family $(c_{l,j})_{l,j\in\mathbb{N}} := \left(\eta_{\frac{1}{j}} * c_l\right)_{l,j\in\mathbb{N}}$ is countable and dense $W^{1,p}(\Omega)$. From here we can proceed similarly with the modification that $c_{l,j}$ are already in $W^{1,\infty}(\Omega)$. Based on the countable family $(c_{l,j})_{l,j\in\mathbb{N}}$, we find that

$$(u_k)_{k\in\mathbb{N}} := \left\{ \sum_{l,j=1}^{N} \lambda_{l,j} c_{l,j} :\ \lambda_{l,j} \in \mathbb{Q},\ N \in \mathbb{N} \right\}$$

satisfies all demanded properties.

The bound $\|u_{k,\omega}\|_{W^{1,\infty}(\mathbb{R}^d)} \leq \|u_k\|_{W^{1,\infty}(\Omega)}$ and continuous differentiability of realizations are a direct consequence of the construction of u_k. □

9.4 Traces and Extensions

For the remainder of this section, we make the following assumption.

Assumption 9.15. *Under the Assumption 9.1 let $\mathbf{P}(\omega)$ be a random open set with boundary $\Gamma(\omega) := \partial\mathbf{P}(\omega)$ such that $\Gamma(\omega)$ is a random closed set. The corresponding prototypes $\mathbf{P}, \Gamma \subset \Omega$ in the sense of Theorem 2.33 have Palm measures $\chi_{\mathbf{P}}\mathbb{P}$ and $\mu_{\Gamma,\mathcal{P}}$ respectively.*

We then introduce the following function spaces.

Definition 9.16. Under the Assumption 9.15 we introduce for $1 \leq p \leq \infty$ the space

$$\begin{aligned} W^{1,p}(\mathbf{P}) := \big\{ u \in L^p(\mathbf{P};\mathbb{P}) :\ &\text{for a.e. } \omega \text{ holds } u_\omega \in W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega)) \text{ and} \\ &\text{there exists } \mathrm{D}u \in L^p(\mathbf{P})^d \text{ s.t. for a.e. } \omega :\ \nabla u_\omega = (\mathrm{D}u)_\omega \big\} , \\ \|u\|_{W^{1,p}(\mathbf{P})} := \|u\|_{L^p(\mathbf{P})} + \|\mathrm{D}u\|_{L^p(\mathbf{P})} . \end{aligned}$$

Based on Definition 9.16, we also introduce the following properties of $\mathbf{P}$ and Γ.

Definition 9.17. We say for the corresponding prototypes $\mathbf{P},\Gamma\subset\Omega$ in the sense of Theorem 2.33 that

1 $\mathbf{P}$ has the weak (r,p)-extension property for $1\leq r\leq p$ if Assumption 9.1 holds and there exists a continuous linear operator $\mathcal{U}_\Omega:W^{1,p}(\mathbf{P})\to W^{1,r}(\Omega)$ such that $(\mathcal{U}_\Omega u)|_{\mathbf{P}}=u$. In this context, we define

$$\begin{aligned}W^{1,r,p}(\Omega,\mathbf{P})&:=\left\{u\in W^{1,r}(\Omega):\ u|_{\mathbf{P}}\in L^p(\mathbf{P}),\,\mathrm{D}_\omega u\in L^p(\mathbf{P};\mathbb{R}^d)\right\}\,,\\ \mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})&:=\mathrm{closure}_{L^p}\left\{\mathrm{D}u\mid u\in W^{1,p}(\mathbf{P})\right\}\,,\\ \mathcal{V}^{r,p}_{\mathrm{pot}}(\mathbf{P})&:=\left\{\mathrm{D}u\in\mathcal{V}^r_{\mathrm{pot}}(\Omega)\mid \mathrm{D}u\in\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})\right\}\,.\end{aligned}$$

2 $\mathbf{P}$ has the strong (r,p)-extension property for $1\leq r\leq p$ if Assumption 9.1 holds and there exists a continuous linear operator $\mathcal{U}_\Omega:W^{1,p}(\mathbf{P})\to W^{1,r}(\Omega)$ such that $(\mathcal{U}_\Omega u)|_{\mathbf{P}}=u$ and such that

$$\left\|\mathrm{D}_\omega\mathcal{U}_\Omega u\right\|_{L^r(\Omega)}\leq C\left\|\mathrm{D}_\omega u\right\|_{L^p(\Omega)}\,.$$

3 Γ has the strong (r,p)-trace property for $1\leq r\leq p$ if Assumption 9.2 holds and there exists a continuous linear operator $\mathcal{T}_\Omega:W^{1,1,p}(\Omega)\to L^r(\Gamma;\mu_{\Gamma,\mathcal{P}})$ such that for every $u\in C_b(\Omega)$ it holds $\mathcal{T}_\Omega u=u|_\Gamma$ in the sense of $\mu_{\Gamma,\mathcal{P}}$.

We already mention at this point a very important property which holds for $\mathbf{P}=\Omega$, but which we are not able to reproduce for general $\mathbf{P}$ in this work. Hence we formulate it as a conjecture, and will avoid to use it in the remainder of this work. Fortunately, it turns out to be non-essential up to uniqueness properties of the homogenized problem in Section 10.6.

Conjecture 9.18. *If $\mathbf{P}$ has the strong extension property it holds*

$$\mathbb{R}^d\cap\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})=\emptyset\,.$$

Theorem 9.19. *Let Assumptions 9.2 and 1.3 hold for the random open set $\mathbf{P}(\omega)$ with $1\leq r<p$ and let τ be ergodic. Then Γ has the strong (r,p)-trace property.*

In order to prove Theorem 9.19 we first need the following lemma.

Lemma 9.20. *Let Assumption 9.1 hold and let $1\leq r<p$, then there exists a family $(u_k)_{k\in\mathbb{N}}\subset W^{1,\infty}(\Omega)$ which is dense in $W^{1,r,p}(\Omega,\mathbf{P})$. If Assumption 9.2 holds then we can additionally assume $(u_k)_{k\in\mathbb{N}}\subset W^{1,\infty}(\Omega)\cap C^1_b(\Omega)$. In both cases $(u_k)_{k\in\mathbb{N}}$ is stable under addition and scalar multiplication with $q\in\mathbb{Q}$. Furthermore, every u_k has almost surely bounded and continuously differentiable realizations with $\|u_{k,\omega}\|_{W^{1,\infty}(\mathbb{R}^d)}\leq\|u_k\|_{W^{1,\infty}(\Omega)}$*

Proof. By Theorem 9.14 there exists $(u_k)_{k\in\mathbb{N}}\subset W^{1,\infty}(\Omega)$ which is at the same time dense in $W^{1,r}(\Omega)$ and $W^{1,p}(\Omega)$. The statement now follows from $W^{1,r}(\Omega)\supset W^{1,r,p}(\Omega,\mathbf{P})\supset W^{1,p}(\Omega)$. If Assumption 9.2 holds Theorem 9.14 yields $(u_k)_{k\in\mathbb{N}}\subset W^{1,\infty}(\Omega)\cap C^1_b(\Omega)$. □

Proof of Theorem 9.19. Let $(u_k)_{k\in\mathbb{N}}\subset W^{1,\infty}(\Omega)\cap C^1_b(\Omega)$ be dense in $W^{1,1,p}(\Omega,\mathbf{P})$ according to Theorem 9.14. For each $u\in(u_k)_{k\in\mathbb{N}}$ the function $u|_\Gamma$ is well defined. Writing $\mathbf{Q}_n:=[-n,n]^d$ and using Theorem 5.9 as well as the Ergodic Theorems we find

$$\begin{aligned}\int_\Gamma |u|^r \,\mathrm{d}\mu_{\Gamma,\mathcal{P}} &= \frac{1}{(2n)^d}\int_{\mathbf{Q}_n}\int_\Gamma |u|^r \,\mathrm{d}\mu_{\Gamma,\mathcal{P}} = \frac{1}{|\mathbf{Q}_n|}\mathbb{E}\int_{\mathbf{Q}_n\cap\partial\mathbf{P}(\omega)} |\mathcal{T}u_\omega|^r \\ &\leq \mathbb{E}\left(C_\omega\left(\frac{1}{|\mathbf{Q}_n|}\int_{\mathbf{Q}_{n+1}\cap\mathbf{P}(\omega)}|u_\omega|^p+|\nabla u_\omega|^p\right)^{\frac{r}{p}}\right) \\ &\leq \mathbb{E}\Big(C_\omega^{\frac{p}{p-r}}\Big)^{\frac{p-r}{p}}\,\mathbb{E}\left(\frac{1}{|\mathbf{Q}_n|}\int_{\mathbf{Q}_{n+1}\cap\mathbf{P}(\omega)}|u_\omega|^p+|\nabla u_\omega|^p\right)^{\frac{r}{p}} \\ &\to \mathbb{E}\Big(C_\omega^{\frac{p}{p-r}}\Big)^{\frac{p-r}{p}}\left(\int_\Omega |u|^p+|\nabla_\omega u|^p\,\mathrm{d}\mathbb{P}\right)^{\frac{r}{p}}\end{aligned}$$

as $n\to\infty$. Using the definition of C_ω in Theorem 5.9 we conclude. □

A generalization of Theorem 9.19 to the general case of Assumption 9.1 is difficult, since the trace property does not apply for general L^∞-functions, even in $\mathbb{R}^d$. However, for the sake of homogenization, there exists a workaround.

Definition 9.21. We say for the corresponding prototypes $\mathbf{P},\Gamma\subset\Omega$ in the sense of Theorem 2.33 that Γ has the weak (r,p)-trace property for $1\leq r\leq p$ if Assumption 9.1 holds and for every family of functions $(u_k)_{k\in\mathbb{N}}\subset W^{1,\infty}(\Omega)$ according to Lemma 9.20 which is dense in $W^{1,1,p}(\Omega,\mathbf{P})$ there exists a continuous linear operator $\mathcal{T}_\Omega : W^{1,1,p}(\Omega,\mathbf{P})\to L^r(\Gamma;\mu_{\Gamma,\mathcal{P}})$ such that for almost every $\omega\in\Omega$ and every u_k it holds $(\mathcal{T}_\Omega u_k)_\omega = \mathcal{T}u_{k,\omega}$ on $\Gamma(\omega)$.

Theorem 9.22. *Let Assumption 9.1 hold, let τ be ergodic and let $\Gamma(\omega)$ be almost surely locally (δ,M)-regular satisfying Assumption 1.3. Then Γ has the weak (r,p)-trace property.*

Proof. We define $\mathcal{T}_\Omega u_k$ pointwise in ω through $(\mathcal{T}_\Omega u_k)_\omega = \mathcal{T}u_{k,\omega}$ and observe that $\mathcal{T}_\Omega$ is bounded by the argument in the proof of Theorem 9.19. It thus remains to show that $\mathcal{T}_\Omega u_k$ is measurable, because then, we can simply extend $\mathcal{T}_\Omega$ to $W^{1,1,p}(\Omega,\mathbf{P})$.

We use Lemma 2.29 and obtain that $\Gamma_\delta(\omega):=\mathbb{B}_\delta(\Gamma(\omega))$ is a RACS with prototype Γ_δ due to Theorem 2.33. We observe that $\Gamma=\bigcap_\delta\Gamma_\delta$ as well as (by definition) $\mathcal{T}_\Omega u_k=\inf_\delta\chi_{\Gamma_\delta}u_k$, hence $\mathcal{T}_\Omega u_k$ is measurable. □

We will now turn our focus to the extension properties. We start with an important implication by the strong extension property.

Theorem 9.23. *Let Assumption 9.1 hold, let τ be ergodic and let $\mathbf{P}$ have the strong (r,p)-extension property. Then the operator $\mathcal{U}_\Omega$ can be extended to a continuous operator $\mathcal{U}_\Omega : \mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})\to\mathcal{V}^{r,p}_{\mathrm{pot}}(\Omega,\mathbf{P})$. More precisely we can identify $\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$ with*

$$\begin{aligned}\tilde{\mathcal{V}}^p_{\mathrm{pot}}(\mathbf{P}) &= \mathrm{closure}_{L^{r,p}(\Omega,\mathbb{P})}\left\{\mathcal{U}_\Omega\mathrm{D}_\omega u:\ u\in W^{1,p}(\Omega)\right\}\,, \\ &= \mathrm{closure}_{L^{r,p}(\Omega,\mathbb{P})}\left\{\mathcal{U}_\Omega\mathrm{D}_\omega u:\ u\in W^{1,r,p}(\Omega;\mathbf{P})\right\}\,, \\ \|\xi\|_{L^{r,p}(\Omega,\mathbb{P})} &= \|\xi\|_{L^r(\Omega)}+\|\xi\|_{L^p(\mathbf{P})}\,.\end{aligned}$$

This means that for $\phi\in\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$ and $\tilde\phi\in\tilde{\mathcal{V}}^p_{\mathrm{pot}}(\mathbf{P})$ it holds $\tilde\phi|_{\mathbf{P}}=\phi$ iff $\tilde\phi=\mathcal{U}_\Omega\phi$.

Proof. The first part follows immediately from the definition of the spaces and of the strong extension property.

For the second part, remark that $\mathcal{U}_\Omega W^{1,p}(\mathbf{P}) \subset W^{1,r,p}(\Omega,\mathbf{P})$ and $(\mathcal{U}_\Omega\phi)|_{\mathbf{P}} = \phi$. Furthermore, $W^{1,p}(\Omega)$ is dense in $W^{1,r,p}(\Omega,\mathbf{P})$ by Lemma 9.20. Finally, $\mathcal{U}_\Omega\mathcal{U}_\Omega\phi = \mathcal{U}_\Omega\phi$ and for $\phi \in \mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$ and $\tilde\phi \in \tilde{\mathcal{V}}^p_{\mathrm{pot}}(\mathbf{P})$ it holds $\tilde\phi|_{\mathbf{P}} = \phi$ iff $\tilde\phi = \mathcal{U}_\Omega\phi$. □

Theorem 9.24. *Let Assumption 9.1 hold, let τ be ergodic and let $\Gamma(\omega)$ be almost surely locally (δ, M)-regular satisfying Assumption 1.5 for $1 < r < p_0 < p_1 < p$. Then Γ has the weak (r,p)-extension property.*

Theorem 9.25. *Let Assumption 9.1 hold, let τ be ergodic and let $\Gamma(\omega)$ be almost surely locally (δ, M)-regular satisfying Assumption 1.8 for $1 < r < p_0 < p_1 < p$. Then Γ has the strong (r,p)-extension property.*

We will prove Theorems 9.24 and 9.25 in Section 10.5 using homogenization theory.

9.5 The Outer Normal Field of $\mathbf{P}$

Theorem 9.26. *Let Assumptions 9.2 and 9.15 hold and let Γ have the strong (r,p)-trace property for $1 < r < p$. Let τ be ergodic, let $\Gamma(\omega)$ be almost surely locally (δ, M)-regular and let $\nu_{\Gamma(\omega)}$ be the outer normal of $\mathbf{P}(\omega)$ on $\Gamma(\omega)$. Then there exists a measurable function $\nu_\Gamma : \Gamma \to \mathbb{S}^{d-1}$ such that almost surely $\nu_{\Gamma(\omega)}(x) = \nu_\Gamma(\tau_x\omega)$. Furthermore, for $f \in C^1_b(\Omega;\mathbb{R}^d)$ and $\phi \in C^1_b(\Omega)$ it holds*

$$\int_{\mathbf{P}} \mathrm{div}_\omega(f\phi)\,\mathrm{d}\mathbb{P} = \int_\Gamma \phi f\cdot\nu_\Gamma\,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,. \tag{9.12}$$

If Γ satisfies the weak $(1,p)$-extension property, the equation (9.12) extends to $\phi \in W^{1,1,p}(\Omega,\mathbf{P})$ and $f \in C^1_b(\Omega;\mathbb{R}^d)$ or to $f \in W^{1,1,p}(\Omega,\mathbf{P})^d$ and $\phi \in C^1_b(\Omega)$.

Proof. For $\delta > 0$ define $\chi_\delta(\omega) := (\eta_\delta * \chi_{\mathbf{P}})(\omega)$. We observe that

$$|\mathrm{D}_\omega\chi_\delta|(\tau_x\omega) = |\mathrm{D}_\omega(\eta_\delta*\chi_{\mathbf{P}})|(\tau_x\omega) = |\eta_\delta * (\mathrm{D}_\omega\chi_{\mathbf{P}})(\tau.\omega)|(x) = \left|\eta_\delta * \nabla\chi_{\mathbf{P}(\omega)}\right|(x)\,, \tag{9.13}$$

and hence for almost every ω we have $|\mathrm{D}_\omega\chi_\delta| \to \left|\nabla\chi_{\mathbf{P}(\omega)}\right| = \mathcal{H}^{d-1}(\Gamma(\omega)\cap\,\cdot\,)$ weakly. Then for $\varphi \in C^\infty_c(\mathbb{R}^d)$ and $f \in C_b(\Omega)$ it holds by the Palm formula and (9.13)

$$\begin{aligned}
\int_{\mathbb{R}^d}\varphi\int_\Omega f\,|\mathrm{D}_\omega\chi_\delta| &= \int_\Omega\int_{\mathbb{R}^d} f(\tau_x\omega)\varphi(x)\,|\mathrm{D}_\omega\chi_\delta|(\tau_x\omega)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\\
&= \int_\Omega\int_{\mathbb{R}^d} f(\tau_x\omega)\varphi(x)\left|\eta_\delta*\nabla\chi_{\mathbf{P}(\omega)}\right|(x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\\
&\le \int_\Omega\int_{\mathbb{R}^d} f(\tau_x\omega)\varphi(x)\left(\left|\nabla\chi_{\mathbf{P}(\omega)}\right|(\mathbb{B}_\delta(\mathrm{supp}\varphi))\right)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\,,
\end{aligned}$$

where $\left|\nabla\chi_{\mathbf{P}(\omega)}\right| = \mathcal{H}^{d-1}(\,\cdot\cap\Gamma(\omega)) = \mu_{\Gamma(\omega)}$. From the ergodic theorem, the $\mathbb{P}$-almost sure pointwise weak convergence and the Lebesgue dominated convergence theorem, we conclude

$$\begin{aligned}
\int_{\mathbb{R}^d}\varphi\int_\Omega f\,|\mathrm{D}_\omega\chi_\delta| \to &\int_\Omega\int_{\mathbb{R}^d} f(\tau_x\omega)\varphi(x)\,\mathrm{d}\mu_{\Gamma(\omega)}(x)\,\mathrm{d}\mathbb{P}(\omega)\\
&= \int_{\mathbb{R}^d}\varphi\int_\Omega f\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,,
\end{aligned}$$

which implies $\int_\Omega f\,|\mathrm{D}_\omega\chi_\delta| \to \int_\Omega f\mathrm{d}\mu_{\Gamma,\mathcal{P}}$. In a similar way, we show $\int_\Omega f\mathrm{D}_\omega\chi_\delta \to \int_\Omega f\mathrm{d}\widetilde{\mu_{\Gamma,\mathcal{P}}}$, where $\widetilde{\mu_{\Gamma,\mathcal{P}}}$ is a $\mathbb{R}^d$-valued measure on Γ. Furthermore, for every e_i in the canonical basis of $\mathbb{R}^d$, $\mathrm{e}_i\cdot\widetilde{\mu_{\Gamma,\mathcal{P}}} \ll \mu_{\Gamma,\mathcal{P}}$, which implies by the Radon-Nikodym theorem the existence of a measurable ν_Γ with values in $\mathbb{S}^{d-1}$ such that $\widetilde{\mu_{\Gamma,\mathcal{P}}} = \nu_\Gamma\mu_{\Gamma,\mathcal{P}}$. The property $\nu_{\Gamma(\omega)}(x) = \nu_\Gamma(\tau_x\omega)$ follows from the fact that $\widetilde{\mu_{\Gamma,\mathcal{P}}}$ is the Palm measure of $\nabla\chi_{\mathbf{P}(\omega)}$.

For $f\in C_b^1(\Omega;\mathbb{R}^d)$ and $\phi\in C_b^1(\Omega,\mathbf{P})$ and $\varphi\in C_c^\infty(\mathbb{R}^d)$ it holds

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi\int_{\mathbf{P}}\mathrm{div}_\omega(f\phi) &= \int_\Omega\int_{\mathbb{R}^d}\varphi(x)\,\mathrm{div}(f\phi)_\omega\\ &= \int_\Omega\int_{\Gamma(\omega)}\varphi(x)\,\phi_\omega f_\omega\cdot\nu_{\Gamma(\omega)}\\ &= \int_{\mathbb{R}^d}\int_\Gamma\varphi(x)\,\phi f\cdot\nu_\Gamma\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,,\end{aligned}$$

which implies (9.12) by a density argument. □

Definition 9.27. Let Γ have the strong (r,p)-Trace property for $1<r<p$ and the weak $(1,p)$-extension property. We say that $f\in L^p(\mathbf{P};\mathbb{R}^d)$ has the weak normal trace $f_\nu\in L^r(\Gamma)$ and weak divergence $\mathrm{div}_\omega f\in L^1(\mathbf{P})$ if for all $\phi\in C_b^1(\Omega)$

$$\int_{\mathbf{P}}\left(\phi\mathrm{div}_\omega f + f\cdot\nabla_\omega\phi\right)\,\mathrm{d}\mathbb{P} = \int_\Gamma\phi f_\nu\,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,.$$

Theorem 9.28. *Let Assumptions 9.2 and 9.15 hold and for some $r\in(1,2)$ let Γ have the strong $(r,2)$-Trace property and the weak $(r,2)$-extension property and let $\Gamma^\varepsilon(\omega)$ have the strong uniform trace property (see Definition 10.10 below). Let τ be ergodic, let $\Gamma(\omega)$ be almost surely locally (δ,M)-regular and let $\nu_{\Gamma(\omega)}$ be the outer normal of $\mathbf{P}(\omega)$ on $\Gamma(\omega)$. Then there exists $u_\Omega\in W^{1,r}(\Omega)\cap W^{1,2}(\mathbf{P};\mathbb{R}^d)$, such that $\nabla_\omega u_\Omega$ has a weak normal trace $f_\nu\in L^1(\Gamma)$ and weak divergence u_Ω, i.e.*

$$\forall\phi\in C_b^1(\omega):\quad\int_{\mathbf{P}}\left(\phi u_\Omega + \nabla u_\Omega\cdot\nabla_\omega\phi\right)\,\mathrm{d}\mathbb{P} = \int_\Gamma\phi f_\nu\,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,.$$

The last theorem is less trivial than one might think. In particular, we lack a PoincarÃ©-type inequality on Ω, which is typically used to prove corresponding results in $\mathbb{R}^d$. We shift the proof to Section 10.5.

10 Two-Scale Convergence and Application

As we have already explained in the introduction, there have been several approaches to the introduction of two-scale convergence in stochastic homogenization. In this work, we chose a modification of [15] because it does not rely on compactness of the underlying probability space.

10.1 General Setting

For the rest of this work, we consider a stationary random measure $\omega\to\mu_\omega$ with Palm measure $\mu_{\mathcal{P}}$ and we define

$$\mu_\omega^\varepsilon(A) := \varepsilon^d\mu_\omega\big(\varepsilon^{-1}A\big)\,. \tag{10.1}$$

For the corresponding Lebesgue spaces we write $L^p(\Omega;\mu_{\mathcal{P}})$ or $L^p(\mathbf{Q};\mu^\varepsilon_\omega)$, where $\mathbf{Q}\subset\mathbb{R}^d$ is a convex domain with C^1-boundary. If $\mu_\omega=\mathcal{L}$, i.e. $\mu_{\mathcal{P}}=\mathbb{P}$, or $\mu_\omega=\chi_{\mathbf{P}(\omega)}\mathcal{L}$ we omit the notion of μ^ε_ω and $\mu_{\mathcal{P}}$.

In our applications, $\mathrm{d}\mu_\omega=\chi_{\mathbf{P}(\omega)}\mathrm{d}\mathcal{L}$ for the characteristic function of the prototype $\mathbf{P}\subset\Omega$ of the random set $\mathbf{P}(\omega)$ with Palm measure $\chi_{\mathbf{P}}\mathbb{P}$ or $\mathrm{d}\mu_\omega=\mathrm{d}\mu_{\Gamma(\omega)}:=\chi_{\Gamma(\omega)}\mathrm{d}\mathcal{H}^{d-1}$, with Palm measure located on $\Gamma\subset\Omega$, the prototype of $\Gamma(\omega):=\partial\mathbf{P}(\omega)$. If we explicitly study the latter case, we write $\mu_{\Gamma,\mathcal{P}}$ for the Palm measure.

Moreover, in view of (10.1), we write $\mu^\varepsilon_{\Gamma(\omega)}(A):=\varepsilon^d\mu_{\Gamma(\omega)}(\varepsilon^{-1}A)=\varepsilon\mathcal{H}^{d-1}(A\cap\varepsilon\Gamma(\omega))$. In case of $\mu_\omega=\chi_{\mathbf{P}(\omega)}\mathcal{L}$, we drop the notation μ^ε_ω.

Assumption 10.1. *Let $(\Omega,\sigma,\mathbb{P})$ be a probability space with ergodic dynamical system $(\tau_x)_{\in\mathbb{R}^d}$ in the sense of Definition 2.15. Let $1<q,p<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$ and*

$$\Phi_{\mathcal{P},q}\subset L^q(\Omega;\mu_{\mathcal{P}})$$

be a countable dense subset, which is stable under scalar multiplication and linear combination. Finally, let Ω_Φ be such that (2.31) holds for all $\varphi\in C(\overline{\mathbf{Q}})$, $\omega\in\Omega_\Phi$, $f\in\Phi_{\mathcal{P},q}$.

Remark. In some proofs below we will assume w.l.o.g. that some particular, essentially bounded functions are elements of $\Phi_{\mathcal{P},q}$. These will always be countably many and hence Ω_Φ has to be changed only by a set of measure 0.

Definition 10.2. Let Assumption 10.1 hold. Let $\omega\in\Omega_\Phi$ and let $u^\varepsilon\in L^p(\mathbf{Q};\mu^\varepsilon_\omega)$ for all $\varepsilon>0$. We say that (u^ε) converges (weakly) in two scales to $u\in L^p(\mathbf{Q};L^p(\Omega;\mu_{\mathcal{P}}))$ and write $u^\varepsilon\xrightharpoonup{2s}u$ if $\sup_{\varepsilon>0}\|u^\varepsilon\|_{L^p(\mathbf{Q};\mu^\varepsilon_\omega)}<\infty$ and if for every $\psi\in\Phi_{\mathcal{P},q}$, $\varphi\in C(\overline{\mathbf{Q}})$ there holds with $\phi_{\omega,\varepsilon}(x):=\varphi(x)\psi(\tau_{\frac{x}{\varepsilon}}\omega)$

$$\lim_{\varepsilon\to0}\int_{\mathbf{Q}}u^\varepsilon(x)\phi_{\omega,\varepsilon}(x)\mathrm{d}\mu^\varepsilon_\omega(x)=\int_{\mathbf{Q}}\int_\Omega u(x,\tilde\omega)\varphi(x)\psi(\tilde\omega)\,\mathrm{d}\mu_{\mathcal{P}}(\tilde\omega)\,\mathrm{d}x\,.$$

We note that the definition of two-scale convergence in [15] is formulated more generally, in particular for a more general class of test-functions.

Lemma 10.3 ([15] Lemma 4.4-1.)**.** *Let Assumption 10.1 hold. Let $\omega\in\Omega$ and $u^\varepsilon\in L^p(\mathbf{Q};\mu^\varepsilon_\omega)$ be a sequence of functions such that $\|u^\varepsilon\|_{L^p(\mathbf{Q})}\le C$ for some $C>0$ independent of ε. Then there exists a subsequence of $(u^{\varepsilon'})_{\varepsilon'\to0}$ and $u\in L^p(\mathbf{Q};L^p(\Omega;\mu_{\mathcal{P}}))$ such that $u^{\varepsilon'}\xrightharpoonup{2s}u$ and*

$$\|u\|_{L^p(\mathbf{Q};L^p(\Omega;\mu_{\mathcal{P}}))}\le\liminf_{\varepsilon'\to0}\left\|u^{\varepsilon'}\right\|_{L^p(\mathbf{Q};\mu^\varepsilon_\omega)}\,. \tag{10.2}$$

Sketch of proof. The proof is standard and has been carried out in various publications under various assumptions [2, 14, 15, 18, 39]. The important point is the separability of $C(\overline{\mathbf{Q}})$, which allows to pass to the limit for a countable number of test functions $(\varphi_k)_{k\in\mathbb{N}}\in C(\overline{\mathbf{Q}})$ first, and then apply a density argument. ☐

Furthermore, we will need the following result on the lower estimate in homogenization of convex functionals using two-scale convergence, which was obtained in [17].

Lemma 10.4. *Let Assumption 10.1 hold and let μ_ω be a random measure. Let $f:\ \mathbf{Q}\times\Omega\times\mathbb{R}^N\to\mathbb{R}$ be a convex functional in $\mathbb{R}^d$. For almost all $\omega\in\Omega_{\Phi_p}$ the following holds: Let $u^\varepsilon\in L^q(\mathbf{Q};\mu^\varepsilon_\omega)$ be a sequence such that $\|u^\varepsilon\|_{L^q(\mathbf{Q};\mu^\varepsilon_\omega)}\le C$ for some $0<C<\infty$ and such that $u^\varepsilon\xrightharpoonup{2s}u\in L^q(\mathbf{Q}\times\Omega;\mathcal{L}\otimes\mu_{\mathcal{P}})$. Then, it holds*

$$\int_{\mathbf{Q}}\int_\Omega f(x,\tilde\omega,u(x,\tilde\omega))\,d\mu_{\mathcal{P}}(\tilde\omega)\,dx\le\liminf_{\varepsilon\to0}\int_{\mathbf{Q}}f(x,\tau_{\frac{x}{\varepsilon}}\omega,u^\varepsilon(x))\,d\mu^\varepsilon_\omega(x)\,.$$

10.2 The "Right" Choice of Oscillating Test Functions

In what follows, we will have to deal with two-scale limits of functions on $\mathbb{R}^d$, but also on $\mathbf{P}(\omega)$ or $\Gamma(\omega)$. Hence we deal with two-scale convergence w.r.t. to $\mathbb{P}$, $\chi_{\mathbf{P}}\mathbb{P}$ and $\mu_{\Gamma,\mathcal{P}}$. In order to keep notation of the set(s) of testfunctions short and concise, we make the following choice:

$$\Phi_q = \Phi_{\mathbb{P},q} = (u_k)_{k\in\mathbb{N}}$$

is the set countable set of functions $(u_k)_{k\in\mathbb{N}} \subset W^{1,p}(\Omega)\cap W^{1,\infty}(\Omega)$ from Theorem 9.14. Hence $(u_k)_{k\in\mathbb{N}}$ is dense in $L^p(\Omega)$ and $(\nabla u_k)_{k\in\mathbb{N}}$ is dense in $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ (see Theorem 9.10).

If Γ has the strong or weak (r,p)-trace property, using Theorem 9.19 and 9.22 we define

$$\Phi_{r,\Gamma} = \mathcal{T}_\Omega\Phi_p \cup \tilde{\Phi}_{r,\Gamma}\,,$$

where $\tilde{\Phi}_{r,\Gamma}$ is dense in $L^r(\Gamma,\mu_{\Gamma,\mathcal{P}})$. In case of Assumption 9.2, we note that $\mathcal{T}_\Omega\Phi_p$ is dense in $L^r(\Gamma,\mu_{\Gamma,\mathcal{P}})$ because $C_b(\omega)$ is dense in $L^r(\Gamma,\mu_{\Gamma,\mathcal{P}})$. However, in case of Assumption 9.1 it is not clear that $\mathcal{T}_\Omega\Phi_p$ is dense in $L^r(\Gamma,\mu_{\Gamma,\mathcal{P}})$, which is why $\tilde{\Phi}_{r,\Gamma}$ is needed.

10.3 Homogenization of Gradients

In what follows, we introduce two-scale convergence of gradients. This result has been proven in various work under various assumptions, see e.g. [2] for the periodic case and [39, 29, 15] in the stochastic case. We provide the proof here for self-containedness of this outline.

Theorem 10.5. *Under Assumption 10.1 for almost every $\omega\in\Omega$ the following holds:*

If $u^\varepsilon \in W^{1,p}(\mathbf{Q};\mathbb{R}^d)$ for all ε and if there exists $0<C_u<\infty$ with

$$\sup_{\varepsilon>0}\|u^\varepsilon\|_{L^p(\mathbf{Q})} + \varepsilon^\gamma\|\nabla u^\varepsilon\|_{L^p(\mathbf{Q})} < C_u$$

Then there exists $u\in L^p(\mathbf{Q}L^p(\Omega;\mathbb{P}))$ such that $u^\varepsilon \stackrel{2s}{\rightharpoonup} u$. Depending on the choice of γ, the following holds:

1. *If $\gamma = 0$, then $u \in W^{1,p}(\mathbf{Q})$ with $u^\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\mathbf{Q})$ and there exists $\boldsymbol{\upsilon}_1 \in L^p(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\Omega))$ such that $\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla_x u + \boldsymbol{\upsilon}_1$ weakly in two scales.*
2. *If $\gamma\in(0,1)$ then $\varepsilon^\gamma\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \boldsymbol{\upsilon}_1$ for some $\boldsymbol{\upsilon}_1\in L^p(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\Omega))$.*
3. *If $\gamma = 1$ then $u\in L^p(\mathbf{Q};W^{1,p}(\Omega))$ and $\varepsilon\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \mathrm{D}_\omega u$.*
4. *If $\gamma>1$ then $\varepsilon^\gamma\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} 0$.*

Lemma 10.6. *Under Assumption 10.1 for almost all $\omega\in\Omega$ the following holds: Let $p>1$ and $(u^\varepsilon)_{\varepsilon>0}$ be a sequence of functions satisfying*

$$\sup_{\varepsilon>0}\|u^\varepsilon\|_{L^p(\mathbf{Q})} < +\infty\,,\qquad \lim_{\varepsilon\to0}\varepsilon\|\nabla u^\varepsilon\|_{L^p(\mathbf{Q})} = 0\,. \tag{10.3}$$

If $u^\varepsilon \stackrel{2s}{\rightharpoonup} u$ along a subsequence, then $u\in L^p(\mathbf{Q})$ is independent of Ω.

Proof. We obtain that $u^\varepsilon \stackrel{2s}{\rightharpoonup} u \in L^p(\mathbf{Q};L^p(\Omega))$ along a subsequence. We show that u does not depend on the Ω-coordinate using ergodicity. We recall that $\tau_\bullet$ are all measure preserving for $\mathbb{P}$. Hence, for any $\varphi \in C_c^\infty(\mathbf{Q})$ and $\psi \in \Phi_q$, we find for any $a \in \mathbb{Q}^d$ it holds

$$\begin{aligned}
\int_{\mathbf{Q}}\int_\Omega &(u(x,\tau_a\omega)-u(x,\omega))\varphi(x)\psi(\omega)\,d\mathbb{P}(\omega)\,dx \\
&= \int_{\mathbf{Q}}\int_\Omega u(x,\omega)\varphi(x)\left(\psi(\tau_{-a}\omega)-\psi(\omega)\right)\,d\mathbb{P}(\omega)\,dx \\
&= \lim_{\varepsilon\to 0}\int_{\mathbf{Q}} u^\varepsilon(x)\varphi(x)\left(\psi(\tau_{\frac{-\varepsilon a+x}{\varepsilon}}\omega)-\psi(\tau_{\frac{x}{\varepsilon}}\omega)\right)\,dx \\
&= \lim_{\varepsilon\to 0}\int_{\mathbf{Q}} \left(u^\varepsilon(x+\varepsilon a)\varphi(x+\varepsilon a)-u^\varepsilon(x)\varphi(x)\right)\psi(\tau_{\frac{x}{\varepsilon}}\omega)\,dx \\
&= \lim_{\varepsilon\to 0}\int_{\mathbf{Q}} u^\varepsilon(x+\varepsilon a)\left(\varphi(x+\varepsilon a)-\varphi(x)\right)\psi(\tau_{\frac{x}{\varepsilon}}\omega)\,dx \\
&\quad + \lim_{\varepsilon\to 0}\int_{\mathbf{Q}} \left(u^\varepsilon(x+\varepsilon a)-u^\varepsilon(x)\right)\varphi(x)\psi(\tau_{\frac{x}{\varepsilon}}\omega)\,dx\,.
\end{aligned}$$

The first integral on the right hand side can be easily estimated through

$$\|u^\varepsilon\|_{L^p(\mathbf{Q})}\left\|\psi(\tau_{\frac{\cdot}{\varepsilon}}\omega)\right\|_{L^q(\mathbf{Q})}\varepsilon|a|\,\|\nabla\varphi\|_\infty \to 0 \quad \text{as } \varepsilon\to 0\,.$$

The second integral can be estimated through

$$\|\varphi\|_\infty\int_{\mathbf{Q}}\left|\int_0^\varepsilon \nabla u^\varepsilon(x+ta)\cdot a\,\mathrm{d}t\right|\,\left|\psi(\tau_{\frac{x}{\varepsilon}}\omega)\right|\,dx \le \|\varphi\|_\infty\,\varepsilon\,\|\nabla u^\varepsilon\|_{L^p(\mathbf{Q})}\,|a|\,\left\|\psi(\tau_{\frac{\cdot}{\varepsilon}}\omega)\right\|_{L^q(\mathbf{Q})}\,.$$

Due to (10.3) the right hand side of the above inequality converges to 0. Since φ and ψ were arbitrary, we obtain $u(x,\tau_a\omega)=u(x,\omega)$ for every $a\in\mathbb{R}^d$. Hence u is invariant under all translations τ_a, which implies for almost every $x\in\mathbf{Q}$ that $u(x,\cdot)=const$ by ergodicity of $\tau_\bullet$. □

Based on Lemma 10.6 we can now prove Theorem 10.5.

Proof of Theorem 10.5. We note that $u^\varepsilon \stackrel{2s}{\rightharpoonup} u \in L^p(\mathbf{Q};L^p(\Omega))$ and $\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \boldsymbol{v} \in L^p(\mathbf{Q};L^p(\Omega;\mathbb{R}^d))$ along a subsequence.

Proof of 1: We consider a countable set $\Phi_{\mathrm{sol}} \subset L^q_{\mathrm{sol}}(\Omega)$ which is dense in $L^q_{\mathrm{sol}}(\Omega)$. Then, by definition of $L^p_{\mathrm{sol}}(\Omega)$ we find for all $b\in\Phi_{\mathrm{sol}}$ and all $\varphi\in C_c^\infty(\mathbf{Q})$

$$\int_{\mathbf{Q}}(\varphi\nabla u^\varepsilon + u^\varepsilon\nabla\varphi)\cdot b(\tau_{\frac{\bullet}{\varepsilon}}\omega)\,d\mathcal{L} = \int_{\mathbf{Q}}\nabla\left(u^\varepsilon\varphi\right)\cdot b(\tau_{\frac{\bullet}{\varepsilon}}\omega)\,d\mathcal{L} = 0\,.$$

We take the limit $\varepsilon\to 0$ on the left hand side and obtain

$$\int_{\mathbf{Q}}\left(\varphi(x)v(x,\tilde\omega)+u\nabla\varphi(x)\right)\cdot b(\tilde\omega)\,d\mathbb{P}(\tilde\omega)\,dx = 0\,.$$

After integration by parts, this implies

$$\int_{\mathbf{Q}}\varphi(x)\left(\nabla u(x)-v(x,\tilde\omega)\right)\cdot b(\tilde\omega)\,d\mathbb{P}(\tilde\omega)\,dx = 0\,.$$

As $\varphi \in C_c^\infty(\mathbf{Q})$ and $b \in \Phi_{\mathrm{sol}}$ were arbitrary and since $\Phi_{\mathrm{sol}} \subset L^q_{\mathrm{sol}}(\Omega)$ is dense, the last equation and Lemma 9.10 imply that $\nabla u(x) - \boldsymbol{v}(x,\cdot) \in \mathcal{V}^p_{\mathrm{pot}}(\Omega)$ for almost every $x \in \mathbf{Q}$.

Proof of 2: We apply Part 1 to $\tilde u^\varepsilon := \varepsilon^\gamma u^\varepsilon$. Evidently, $\tilde u^\varepsilon \stackrel{2s}{\rightharpoonup} 0$ and hence there exists $\boldsymbol{v}_1 \in L^p(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\Omega))$ such that $\varepsilon^\gamma \nabla u^\varepsilon = \nabla \tilde u^\varepsilon \stackrel{2s}{\rightharpoonup} \boldsymbol{v}_1$.

Proof of 3: Let $\psi \in L^q_{\mathrm{sol}}(\Omega)$ and $\varphi \in C^1_0(\mathbf{Q})$. Then we have

$$\int_{\mathbf{Q}} \varepsilon \nabla u^\varepsilon \cdot \varphi\psi\left(\tau_{\frac{\bullet}{\varepsilon}}\omega\right) \mathrm{d}\mathcal{L} = -\int_{\mathbf{Q}} u^\varepsilon \psi\left(\tau_{\frac{\bullet}{\varepsilon}}\omega\right) \cdot \varepsilon \nabla_x \varphi \mathrm{d}\mathcal{L}\,.$$

As $\varepsilon \to \infty$ we obtain

$$\int_{\mathbf{Q}}\int_{\Omega} \boldsymbol{v}(x,\tilde\omega)\cdot\varphi(x)\psi(\tilde\omega)\,\mathrm{d}\mathbb{P}(\tilde\omega)\,\mathrm{d}x = 0$$

and since this holds for every $\psi \in L^q_{\mathrm{sol}}(\Omega)$ and $\varphi \in C^1_0(\mathbf{Q})$, we obtain that $\boldsymbol{v}(x,\omega) \in L^p(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\Omega))$. Furthermore, for a countable dense family $\psi \in W^{1,p}(\Omega)$ and $\varphi \in C^1_0(\mathbf{Q})$ we obtain

$$\int_{\mathbf{Q}} \varepsilon\partial_i u^\varepsilon(x)\varphi(x)\psi\left(\tau_{\frac{x}{\varepsilon}}\omega\right)\,\mathrm{d}x = -\int_{\mathbf{Q}} u^\varepsilon(x)\psi\left(\tau_{\frac{x}{\varepsilon}}\omega\right)\cdot\varepsilon\partial_i\varphi(x)\mathrm{d}x - \int_{\mathbf{Q}} u^\varepsilon(x)\mathrm{D}_i\psi\left(\tau_{\frac{x}{\varepsilon}}\omega\right)\varphi(x)\,\mathrm{d}x$$

and in the limit

$$\int_{\mathbf{Q}}\int_{\mathbb{Y}} \boldsymbol{v}_i(x,y)\cdot\varphi(x)\psi(\omega)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x = -\int_{\mathbf{Q}}\int_{\mathbb{Y}} u(x,y)\mathrm{D}_i\psi(\omega)\,\varphi(x)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x\,.$$

This implies $\boldsymbol{v}_i = \mathrm{D}_i u$.

Proof of 4: Part 3 implies that $\tilde u^\varepsilon := \varepsilon^{\gamma-1}$ satisfies $\tilde u^\varepsilon \stackrel{2s}{\rightharpoonup} 0$ and $\varepsilon^\gamma\nabla u^\varepsilon = \varepsilon\nabla\tilde u^\varepsilon \stackrel{2s}{\rightharpoonup} \mathrm{D}_\omega 0 = 0$. □

Important in the context of convergence of gradients is also the following recovery lemma, obtained in [19, Section 2.3] for the L^2-case.

Lemma 10.7. *Let Assumption 10.1 hold. Let $\upsilon \in \mathcal{V}^p_{\mathrm{pot}}(\Omega)$, $1 < p < \infty$ and let $\mathbf{Q}$ be a bounded convex domain. For almost every ω there exists $C > 0$ such that the following holds: For every $\varepsilon > 0$ there exists a unique $V^\omega_\varepsilon \in W^{1,p}(\mathbf{Q})$ with $\nabla V^\omega_\varepsilon(x) = \upsilon(\tau_{\frac{x}{\varepsilon}}\omega)$, $\int_{\mathbf{Q}} V^\omega_\varepsilon = 0$ and $\|V_\varepsilon\|_{W^{1,p}(\mathbf{Q})} \le C\|\upsilon\|_{L^p_{\mathrm{pot}}(\Omega)}$ for all $\varepsilon > 0$. Furthermore,*

$$\lim_{\varepsilon\to 0}\|V^\omega_\varepsilon\|_{L^p(\mathbf{Q})} = 0\,.$$

Sketch of Proof, see [19]. By definition of $L^p_{\mathrm{pot}}(\Omega)$ there exists for almost every $\omega \in \Omega$ a function $V^\omega_\varepsilon \in W^{1,p}(\mathbf{Q})$ with $\nabla V^\omega_\varepsilon(x) = \upsilon(\tau_{\frac{x}{\varepsilon}}\omega)$, $\int_{\mathbf{Q}} V^\omega_\varepsilon = 0$. By a standard contradiction argument, there exists a constant $C > 0$ such that

$$\forall V \in W^{1,p}(\mathbf{Q}):\quad \|V\|_{L^p(\mathbf{Q})} \le C\left(\|\nabla V\|_{L^p(\mathbf{Q})} + \left|\int_{\mathbf{Q}} V\right|\right)\,.$$

The last inequality implies that $V^\omega_\varepsilon \rightharpoonup V$ weakly in $W^{1,p}(\mathbf{Q})$ and $V^\omega_\varepsilon \to V$ strongly in $L^p(\mathbf{Q})$. Furthermore, the Ergodic Theorem 2.26 yields for every $f \in C(\overline{\mathbf{Q}})$

$$\int_{\mathbf{Q}} f\cdot\nabla V^\omega_\varepsilon = \int_{\mathbf{Q}} f\cdot\upsilon(\tau_{\frac{x}{\varepsilon}}\omega) \to \int_{\mathbf{Q}} f\cdot\int \upsilon\,\mathrm{d}\mathbb{P} = \int_{\mathbf{Q}} f\cdot 0 = 0\,.$$

Hence $\nabla V = 0$ and since $\int_{\mathbf{Q}} V = 0$ it follows $V = 0$. □

10.4 Uniform Extension- and Trace-Properties

For the rest of this section, we make the following assumptions. Under the Assumptions 9.1 and 10.1 and using the notations introduced in Section 10.1 we introduce $\mathbf{P}^\varepsilon(\omega) := \varepsilon\mathbf{P}(\omega)$, $\mathbf{Q}_1^\varepsilon(\omega) := \mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega)$ and $\Gamma^\varepsilon(\Omega) := \mathbf{Q}\cap\varepsilon\Gamma(\omega)$.

Following (10.1) we recall the definition

$$\mu^\varepsilon_{\Gamma(\omega)}(A) := \varepsilon^n\mathcal{H}^{d-1}\left(\frac{A}{\varepsilon}\cap\Gamma(\omega)\right) = \varepsilon\mathcal{H}^{d-1}(A\cap\Gamma^\varepsilon(\omega))$$

Definition 10.8 (Uniform Dirichlet extension property)**.** Let $\mathbf{Q}$ be a bounded open convex domain with Lipschitz boundary. We say for $1\leq r\leq p$ that $\mathbf{P}^\varepsilon(\omega)$ has the uniform (r,p)-Dirichlet extension property on $\mathbf{Q}$ if for almost every ω there exists $C_\omega>0$ and a linear extension operator

$$\mathcal{U}:\,W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega))\to W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$$

such that

$$\mathcal{U}_\varepsilon[u](x) := \mathcal{U}[u(\varepsilon\cdot)]\left(\frac{x}{\varepsilon}\right)$$

satisfies the following: For every $u\in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}_1^\varepsilon(\omega))$

$$\|\nabla\mathcal{U}_\varepsilon u\|_{L^r(\mathbf{Q})}\leq C_\omega\|\nabla u\|_{L^p(\mathbf{Q}_1^\varepsilon(\omega))}\,,\quad \|\mathcal{U}_\varepsilon u\|_{L^r(\mathbf{Q})}\leq C_\omega\|u\|_{L^p(\mathbf{Q}_1^\varepsilon(\omega))}$$

and

$$\|\mathcal{U}_\varepsilon u\|_{W^{1,r}(\mathbb{R}^d\backslash\mathbf{Q})}\to 0\,. \tag{10.4}$$

Theorem 7.11 shows that virtually every random geometry to which the theory of Sections 5–7 applies has the (r,p)- extension property on bounded convex C^1-domains $\mathbf{Q}$. In particular, we obtain the following reformulation of Theorem 1.9.

Theorem 10.9. *For $1\leq r<\tilde{s}<s<p\leq\infty$ let $\mathbf{P}(\omega)$ be almost surely (δ,M)-regular (Def. 4.2) and isotropic cone mixing for $\mathfrak{r}>0$ and $f(R)$ (Def. 4.17) as well as locally connected and satisfy $\mathbb{P}(\mathsf{S}>S_0)\leq f_s(S_0)$ such that Assumption 1.8 holds. Then for almost every ω the set $\mathbf{P}^\varepsilon$ has the uniform (r,p)-Dirichlet extension property on $\mathbf{Q}$.*

Proof. This is almost the statement of Theorem 1.9 except for (10.4). However, for $u\in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{P}(\omega)\cap n\mathbf{Q})$ and $m_n := |\mathbb{AX}(n\mathbf{Q})|$ we obtain note that estimates (1.3)–(1.4) can be extended to

$$\frac{1}{m_n}\int_{\mathbb{AX}(n\mathbf{Q})}|\mathcal{U}u|^r\leq C(\omega)\left(\frac{1}{m_n}\int_{\mathbf{P}(\omega)\cap\mathbb{AX}(n\mathbf{Q})}|u|^p\right)^{\frac{r}{p}} = C(\omega)\left(\frac{|n\mathbf{Q}|}{m_n}\frac{1}{|n\mathbf{Q}|}\int_{\mathbf{P}(\omega)\cap(n\mathbf{Q})}|u|^p\right)^{\frac{r}{p}}\,,$$
$$\frac{1}{m_n}\int_{\mathbb{AX}(n\mathbf{Q})}|\mathcal{U}u|^r\leq C(\omega)\left(\frac{1}{m_n}\int_{\mathbf{P}(\omega)\cap\mathbb{AX}(n\mathbf{Q})}|\nabla u|^p\right)^{\frac{r}{p}} = C(\omega)\left(\frac{|n\mathbf{Q}|}{m_n}\frac{1}{|n\mathbf{Q}|}\int_{\mathbf{P}(\omega)\cap(n\mathbf{Q})}|\nabla u|^p\right)^{\frac{r}{p}}\,,$$

and the statement follows from Theorem 7.11 and Corollary 7.10. □

There exists a weaker notion of extension property, which is for some applications sufficient.

Definition 10.10 (Uniform weak extension property)**.** Let $\mathbf{Q}$ be a bounded open convex domain with Lipschitz boundary. We say for $1\leq r\leq p$ that $\mathbf{P}^\varepsilon(\omega)$ has the uniform weak (r,p)-extension property on $\mathbf{Q}$ if for almost every ω there exists $C_\omega>0$ and a linear extension operator

$$\mathcal{U}:\,W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega))\to W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$$

such that

$$\mathcal{U}_\varepsilon[u](x) := \mathcal{U}[u(\varepsilon\cdot)]\left(\frac{x}{\varepsilon}\right)$$

satisfies the following: For every $u \in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon_1(\omega))$

$$\varepsilon \left\|\nabla\mathcal{U}_\varepsilon u\right\|_{L^r(\mathbf{Q})} + \left\|\mathcal{U}_\varepsilon u\right\|_{L^r(\mathbf{Q})} \leq C_\omega\Big(\varepsilon \left\|\nabla u\right\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))} + \left\|u\right\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\Big)\,.$$

Theorem 10.11. *For $1 \leq r < p_0 < p_1 < p < \infty$ let $\mathbf{P}(\omega)$ be almost surely (δ, M)-regular (Def. 4.2) such that Assumption 1.5 holds. Then for almost every ω the set $\mathbf{P}^\varepsilon$ has the weak uniform (r,p)-extension property on $\mathbf{Q}$.*

Proof. After rescaling, this is the statement of Theorem 1.6. □

Similarly to the extension property, we may introduce a uniform trace property.

Definition 10.12 (Uniform trace property)**.** Let $\mathbf{Q}$ be a bounded open convex domain with Lipschitz boundary. We say for $1 \leq r \leq p$ that $\Gamma^\varepsilon(\omega)$ has the uniform (r,p)-trace property on $\mathbf{Q}$ if for almost every ω there exists $C_\omega > 0$ such that the trace operators

$$\mathcal{T}_\varepsilon :\, W^{1,p}(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\omega)) \to L^r(\mathbf{Q} \cap \Gamma^\varepsilon)$$

satisfy the estimate

$$\left\|\mathcal{T}_\varepsilon u\right\|_{L^r(\Gamma^\varepsilon\cap\mathbf{Q})} \leq C_\omega \left(\left\|u\right\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))} + \varepsilon\left\|\nabla u\right\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\right)\,.$$

Theorem 10.13. *Let $\mathbf{P}(\omega)$ be a stationary ergodic random open set which is almost surely (δ, M)-regular (Def. 4.2) such that Assumption 1.3 holds. For $1 \leq r < p_0 < p < \infty$ and $\mathbf{Q} \subset \mathbb{R}^d$ a bounded domain with Lipschitz boundary. Then for almost every ω the set $\mathbf{P}^\varepsilon$ has the uniform (r,p)-trace property on $\mathbf{Q}$.*

Proof. After rescaling, this is the statement of Theorem 1.4. □

10.5 Homogenization on Domains with Holes

In what follows, we will naturally deal with two-scale limits of functions defined solely on $\mathbf{Q}^\varepsilon_1$. Hence we introduce the following definition.

Definition 10.14. Let $1 < p \leq \infty$ and $u^\varepsilon \in L^p(\mathbf{Q}^\varepsilon_1(\omega))$ for all $\varepsilon > 0$. We say that (u^ε) converges (weakly) in two scales to $u \in L^p(\mathbf{Q}; L^p(\mathbf{P}))$ and write $u^\varepsilon \xrightarrow{2s} u$ if $\sup_{\varepsilon>0}\left\|u^\varepsilon\right\|_{L^2(\mathbf{Q}^\varepsilon_1(\omega))} < \infty$ and if for every $\psi \in \Phi_q$ and $\varphi \in C(\overline{\mathbf{Q}})$ there holds with $\phi_{\omega,\varepsilon}(x) := \varphi(x)\psi(\tau_{\frac{x}{\varepsilon}}\omega)$

$$\lim_{\varepsilon\to0}\int_{\mathbf{Q}^\varepsilon_1} u^\varepsilon\phi_{\omega,\varepsilon} = \int_{\mathbf{Q}}\int_\Omega \chi_{\mathbf{P}}u\varphi\psi\,\mathrm{d}\mathbb{P}\,\mathrm{d}\mathcal{L}\,.$$

The latter definition coincides with Definition 10.2 for $\mathrm{d}\mu_\omega = \chi_{\mathbf{P}(\omega)}\mathrm{d}\mathcal{L}$, which can be verified using the ergodic theorem. Hence, we find the following lemma:

Lemma 10.15. *Let $1 < p \leq \infty$ and $u^\varepsilon \in L^p(\mathbf{Q}_1^\varepsilon(\omega))$ be a sequence such that $\sup_{\varepsilon>0} \|u^\varepsilon\|_{L^p(\mathbf{Q}_1^\varepsilon(\omega))} < \infty$. Then there exists $u \in L^p(\mathbf{Q}; L^p(\mathbf{P}))$ and a subsequence $\varepsilon' \to 0$ such that $u^{\varepsilon'} \xrightarrow{2s} u$.*

Furthermore, if $u^\varepsilon \in L^p(\mathbf{Q})$ is a sequence such that $\sup_{\varepsilon>0} \|u^\varepsilon\|_{L^p(\mathbf{Q})} < \infty$ and $u^{\varepsilon'} \xrightarrow{2s} u$ along a subsequence $\varepsilon' \to 0$ for some $u \in L^p(\mathbf{Q}; L^p(\Omega))$, then $u^{\varepsilon'} \chi_{\mathbf{Q}_1^{\varepsilon'}(\omega)} \xrightarrow{2s} \chi_{\mathbf{P}} u$.

Proof. This follows immediately from Lemma 10.3 extending u^ε by 0 to $\mathbf{Q}$ and on noting that $\psi \in \Phi_q$ implies w.l.o.g. $\chi_{\mathbf{P}} \psi \in \Phi_q$. □

Lemma 10.16. *Let $\mathbf{P}(\omega)$ be a random open domain such that $\mathbf{P}^\varepsilon(\omega)$ has the weak uniform (r,p)-extension property on $\mathbf{Q}$ for $1 < r < p < \infty$. Then for almost every $\omega \in \Omega$ the following holds: If $u^\varepsilon \in W^{1,p}(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\omega); \mathbb{R}^d)$ for all ε with*

$$\sup_\varepsilon \left(\|u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\omega))} + \varepsilon \|\nabla u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\omega))} \right) < C$$

for C independent from $\varepsilon > 0$ then there exists a subsequence denoted by $u^{\varepsilon'}$ and a function $u \in L^p(\mathbf{Q}; W^{1,r}(\Omega)) \cap L^p(\mathbf{Q} \times \mathbf{P})$ such that

$$\mathcal{U}_{\varepsilon'} u^{\varepsilon'} \xrightarrow{2s} u \quad \text{and} \quad \varepsilon \nabla \mathcal{U}_{\varepsilon'} u^{\varepsilon'} \xrightarrow{2s} \nabla_\omega u \tag{10.5}$$

as well as

$$u^{\varepsilon'} \xrightarrow{2s} u \quad \text{and} \quad \varepsilon \nabla u^{\varepsilon'} \xrightarrow{2s} \chi_{\mathbf{P}} \nabla_\omega u \tag{10.6}$$

as $\varepsilon \to 0$.

Proof. We find

$$\begin{aligned} \sup_\varepsilon & \left(\|\mathcal{U}_\varepsilon u^\varepsilon\|_{L^r(\mathbf{Q} \cap \mathbf{P}^\varepsilon(\omega))} + \varepsilon \|\nabla \mathcal{U}_\varepsilon u^\varepsilon\|_{L^r(\mathbf{Q} \cap \mathbf{P}^\varepsilon(\omega))} \right) \\ & \leq C \sup_\varepsilon \left(\|u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\omega))} + \varepsilon \|\nabla u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\omega))} \right) \end{aligned} \tag{10.7}$$

Theorem 10.5 and Definition 10.10 imply now for some limit function $u \in L^r(\mathbf{Q}; W^{1,r}(\Omega))$ that (10.5) and (10.6) hold. □

We are now able to provide the:

Proof of Theorem 9.24. Theorem 10.11 shows that $\mathbf{P}^\varepsilon(\omega)$ satisfies the uniform weak extension property. Hence, if $(u_k)_{k\in\mathbb{N}}$ is a countable dense subset of $W^{1,p}(\Omega)$, we find a set of full measure $\tilde{\Omega} \subset \Omega$ such that for every $k \in \mathbb{N}$ and every $\omega \in \tilde{\Omega}$ the realizations $u_{k,\omega}$ are well defined elements of $W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega))$, the extension operator defined in (5.14) is uniformly bounded and hence $\mathcal{U}_\varepsilon$ defined in Definition 10.10 is uniformly bounded, too. We can thus use the two-scale convergence method as a tool.

Given such ω, we define $u^\varepsilon(x) := u_k\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ and by Lemma 10.16 we find $\tilde{u} \in L^p(\mathbf{Q}; W^{1,r}(\Omega)) \cap L^p(\mathbf{Q} \times \mathbf{P})$ such that $\mathcal{U}_\varepsilon u^\varepsilon \to \tilde{u}_k$ and $\varepsilon \nabla \mathcal{U}_\varepsilon u^\varepsilon \to \nabla_\omega \tilde{u}_k$. Furthermore, we find

$$\begin{aligned} \|\tilde{u}_k\|_{L^r(\mathbf{Q}\times\Omega)} + \|\nabla_\omega \tilde{u}_k\|_{L^r(\mathbf{Q}\times\Omega)} & \leq \liminf_{\varepsilon\to 0} \left(\|\mathcal{U}_\varepsilon u^\varepsilon\|_{L^r(\mathbf{Q})} + \varepsilon \, \|\nabla \mathcal{U}_\varepsilon u^\varepsilon\|_{L^r(\mathbf{Q})} \right) \\ & \leq C \liminf_{\varepsilon\to 0} \left(\|u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\omega))} + \varepsilon \, \|\nabla u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\omega))} \right) \\ & = C \left(\|u_k\|_{L^p(\mathbf{Q}\times\Omega)} + \|\nabla_\omega u_k\|_{L^p(\mathbf{Q}\times\Omega)} \right) . \end{aligned}$$

Since the operator $u_k \to \tilde{u}_k$ is linear and bounded, it can be extended to the whole of $W^{1,p}(\mathbf{P})$. □

Proof of Theorem 9.28. For every $\varepsilon > 0$ there exists a unique u^ε that solves

$$\begin{aligned} -\varepsilon^2\nabla u^\varepsilon + u^\varepsilon &= 0 && \text{on } \mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\Omega)\,, \\ -\varepsilon\nabla u^\varepsilon \cdot \nu_{\Gamma^\varepsilon(\omega)} &= 1 && \text{on } \Gamma^\varepsilon(\omega) \cap \mathbf{Q}\,, \\ u^\varepsilon &= 0 && \text{on } \partial\mathbf{Q}\,. \end{aligned}$$

Deriving apriori estimates in the usual way, for some $C > 0$ independent from ε it holds

$$\varepsilon \left\|\nabla u^\varepsilon\right\|_{L^2(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\Omega))} + \left\|u^\varepsilon\right\|_{L^2(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\Omega))} \leq C$$

and thus according to Lemma 10.16 we find $u \in L^p(\mathbf{Q}; W^{1,r}(\Omega)) \cap L^p(\mathbf{Q}\times\mathbf{P})$ such that

$$\mathcal{U}_{\varepsilon'}u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} u \quad \text{and} \quad \varepsilon\nabla\mathcal{U}_{\varepsilon'}u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} \nabla_\omega u$$

along a subsequence $u^{\varepsilon'}$ which we again denote u^ε in the following, for simplicity. But then for $\phi \in C_b^1(\Omega)$ and $\psi \in C_c^1(\mathbf{Q})$ it follows

$$\begin{aligned} \varepsilon\int_{\mathbf{Q}\cap\Gamma^\varepsilon(\omega)} \phi(\tau_{\frac{x}{\varepsilon}}\omega)\psi(x)\,\mathrm{d}\mathcal{H}^{d-1}(x) &= -\varepsilon^2\int_{\mathbf{Q}\cap\Gamma^\varepsilon(\omega)} \phi(\tau_{\frac{x}{\varepsilon}}\omega)\psi(x)\nabla u^\varepsilon(x)\cdot\nu_{\Gamma(\omega)}(\tau_{\frac{x}{\varepsilon}}\omega)\,\mathrm{d}\mathcal{H}^{d-1}(x) \\ &= \int_{\mathbf{Q}_1^\varepsilon(\omega)} \varepsilon\nabla u^\varepsilon\cdot\left(\nabla_\omega\phi(\tau_{\frac{x}{\varepsilon}}\omega)\psi(x) + \varepsilon\phi(\tau_{\frac{x}{\varepsilon}}\omega)\nabla\psi(x)\right)\,\mathrm{d}x \\ &\quad + \int_{\mathbf{Q}_1^\varepsilon(\omega)} u^\varepsilon\phi(\tau_{\frac{x}{\varepsilon}}\omega)\psi(x)\,\mathrm{d}x \\ &\to \int_{\mathbf{Q}}\int_{\mathbf{P}} \left(\nabla_\omega u\cdot\nabla_\omega\phi\psi + u\phi\psi\right)\,. \end{aligned}$$

Since the left hand side of the above calculation converges to $\int_{\mathbf{Q}}\int_\Gamma \phi\psi\,\mathrm{d}\mu_{\Gamma,\mathcal{P}}$ and ψ was arbitrary, we conclude. □

Proof of Theorem 9.25. Let $\mathbf{Q} = \mathbb{B}_2(0)$ and let $\phi \in C_c^\infty(\mathbf{Q})$ with $\phi|_{\mathbb{B}_1(0)} = 1$, $\phi \geq 0$. According to Theorem 10.9, $\mathbf{P}^\varepsilon$ has the uniform (r,p)-Dirichlet extension property. The theorem now follows from part 2 of the following Lemma. □

Lemma 10.17. *Let $\mathbf{P}(\omega)$ be a random open domain such that $\mathbf{P}^\varepsilon(\omega)$ has the uniform (r,p)-Dirichlet extension property on $\mathbf{Q}$ for $1 < r < p < \infty$. Then for almost every $\omega \in \Omega$ the following holds:*

1 *If $u^\varepsilon \in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega);\mathbb{R}^d)$ for all ε with $\sup_\varepsilon \left\|u^\varepsilon\right\|_{L^p(\mathbf{Q}_1^\varepsilon(\omega))} + \left\|\nabla u^\varepsilon\right\|_{L^p(\mathbf{Q}_1^\varepsilon(\omega))} < C$ for C independent from $\varepsilon > 0$ then there exists a subsequence denoted by $u^{\varepsilon'}$ and functions $u \in W_0^{1,r}(\mathbf{Q};\mathbb{R}^d) \cap L^p(\mathbf{Q})$ and $\upsilon \in L^r(\mathbf{Q};\mathcal{V}^r_{\mathrm{pot}}(\Omega))$ such that*

$$u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} \chi_{\mathbf{P}}u \quad \text{and} \quad \nabla u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} \chi_{\mathbf{P}}\nabla u + \chi_{\mathbf{P}}\upsilon \qquad \text{as } \varepsilon \to 0\,, \tag{10.8}$$

$$\mathcal{U}_{\varepsilon'}u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} u \quad \text{and} \quad \nabla\mathcal{U}_{\varepsilon'}u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} \nabla u + \upsilon \qquad \text{as } \varepsilon \to 0\,. \tag{10.9}$$

Furthermore, $\mathcal{U}_{\varepsilon'}u^{\varepsilon'} \rightharpoonup u$ weakly in $W^{1,r}(\mathbf{Q})) \cap L^p(\mathbf{Q})$.

2 *$\mathbf{P}$ has the strong (r,p)-extension property with $\mathcal{U}_\Omega\phi = \mathrm{ts}{-}\lim_{\varepsilon\to0}\mathcal{U}_\varepsilon\phi(\tau_{\frac{x}{\varepsilon}}\omega)$ for $\phi \in W^{1,p}(\mathbf{P})$.*

3 If $p \geq 2$ and the Assumptions of Theorem 9.28 are satisfied and $\Gamma^\varepsilon(\omega)$ additionally has the uniform (s,p)-trace property for some $s>1$ then

$$\mathcal{T}_{\varepsilon'}u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} u \quad \textit{in } L^s(\Gamma^\varepsilon\cap\mathbf{Q};\mu^\varepsilon_{\Gamma(\omega)})\,.$$

If, even further, $\Gamma^\varepsilon(\omega)$ has the uniform (s,r)-trace property with r from Part 1, then

$$\lim_{\varepsilon\to0}\left\|\mathcal{T}_{\varepsilon'}u^{\varepsilon'}-\mathcal{T}_{\varepsilon'}u\right\|_{L^s(\Gamma^{\varepsilon'}\cap\mathbf{Q};\mu^{\varepsilon'}_{\Gamma(\omega)})}\to0\,. \tag{10.10}$$

Proof. In what follows, convergences always hold along subsequently chosen subsequences of u^ε, which we always relabel by u^ε.

Proof of 1: Let $\frac1r+\frac1q=1$. Then Theorem 10.5 and the assumption that (w.l.o.g.) $\chi_{\mathbf{P}}\Phi_q\subset\Phi_q$ yields that for some $u\in W^{1,r}(\mathbf{Q};\mathbb{R}^d)$ and $v\in L^r(\mathbf{Q};L^r_{\mathrm{pot}}(\Omega))$

$$\mathcal{U}_\varepsilon u^\varepsilon \stackrel{2s}{\rightharpoonup} u \quad\text{and}\quad \nabla\mathcal{U}_\varepsilon u^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla u+v \qquad \text{as } \varepsilon\to0\,.$$

Due to (10.4) we find $u\in W^{1,r}_0(\mathbf{Q};\mathbb{R}^d)$. This yields (10.8).

Proof of 2: For $u\in W^{1,p}(\mathbf{P})$ with $u^\varepsilon(x):=u\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ we find for almost every ω that $\mathcal{U}_\varepsilon$ from Definition 10.8 satisfies

$$\begin{aligned}\varepsilon\left\|\nabla\mathcal{U}_\varepsilon(\phi u^\varepsilon)\right\|_{L^r(\mathbf{Q})}&\leq C\left(\varepsilon\left\|u^\varepsilon\nabla\phi\right\|_{L^p(\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega))}+\varepsilon\left\|\phi\nabla u^\varepsilon\right\|_{L^p(\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega))}\right)\\ \left\|\mathcal{U}_\varepsilon(\phi u^\varepsilon)\right\|_{L^r(\mathbf{Q})}&\leq C\left\|u^\varepsilon\phi\right\|_{L^p(\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega))}\end{aligned} \tag{10.11}$$

As $\varepsilon\to0$, Lemma 10.16 yields $u^\varepsilon\phi \stackrel{2s}{\rightharpoonup} \tilde u$, $\nabla\mathcal{U}_\varepsilon(\phi u^\varepsilon)\stackrel{2s}{\rightharpoonup}\mathrm{D}_\omega\tilde u$, where $\tilde u\in L^p(\mathbf{Q};W^{1,r,p}(\Omega,\mathbf{P}))$. Moreover, inequality (10.11) implies in the limit that

$$\left\|\mathrm{D}_\omega\tilde u\right\|_{L^{r,p}_{\mathrm{pot}}(\Omega,\mathbf{P})}\leq C\left\|\mathrm{D}_\omega u\right\|_{L^p_{\mathrm{pot}}(\mathbf{P})}\,.$$

Hence we can set $\mathcal{U}_\Omega\mathrm{D}_\omega u:=\int_{\mathbf{Q}}\mathrm{D}_\omega\tilde u$. By density, this operator extends to $\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$.

Proof of 3: Now let $p\geq2$ and let the Assumptions of Theorem 9.28 be satisfied and let $\Gamma^\varepsilon(\omega)$ additionally have the uniform (s,p)-trace property for some $s>1$. If u_Ω is the function from Theorem 9.28 we observe for $u^\varepsilon_\Omega(x):=u_\Omega\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ for every $\psi\in C^\infty_c(\mathbf{Q})$ and $\phi\in C^1_b(\Omega)$ with $\phi^\varepsilon(x):=\phi\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ that

$$\begin{aligned}\int_{\mathbf{Q}\cap\Gamma^\varepsilon(\omega)}u^\varepsilon\psi\phi^\varepsilon\,\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)}&=\varepsilon\int_{\mathbf{Q}\cap\Gamma^\varepsilon(\omega)}u^\varepsilon\psi\phi^\varepsilon\varepsilon\nabla_\omega u^\varepsilon_\Omega\cdot\nu_{\Gamma^\varepsilon(\omega)}\,\mathrm{d}\mathcal{H}^{d-1}\\&=\int_{\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega)}\left(u^\varepsilon\psi\phi^\varepsilon u^\varepsilon_\Omega+\varepsilon\nabla u^\varepsilon_\Omega\cdot\left(u^\varepsilon\phi^\varepsilon\varepsilon\nabla\psi+\psi\phi^\varepsilon\varepsilon\nabla u^\varepsilon+\psi u^\varepsilon\varepsilon\nabla\phi^\varepsilon\right)\right)\\&\to\int_{\mathbf{Q}}\int_{\mathbf{P}}\left(u\psi\phi u_\Omega+\psi u\nabla_\omega u_\Omega\cdot\nabla_\omega\phi\right)\\&=\int_{\mathbf{Q}}\int_\Gamma u\psi\,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,.\end{aligned}$$

Since ψ and ϕ were arbitrary and $\nabla_\omega(u\psi)=0$ we conclude

$$\lim_{\varepsilon\to0}\int_{\mathbf{Q}\cap\Gamma^\varepsilon(\omega)}u^\varepsilon\psi\phi^\varepsilon\,\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)}=\int_{\mathbf{Q}}\int_\Gamma u\phi\psi\,.$$

In order to show (10.10) note that

$$\|\mathcal{T}_\varepsilon u^\varepsilon - \mathcal{T}_\varepsilon u_\delta\|_{L^s(\Gamma^\varepsilon\cap\mathbf{Q};\mu^\varepsilon_{\Gamma(\omega)})} \leq \|u^\varepsilon - u\|_{L^r(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))} + \varepsilon\,\|\nabla\,(u^\varepsilon - u)\|_{L^r(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\;.$$

Since the first term on the right hand side converges to zero and $\|\nabla\,(u^\varepsilon - u)\|_{L^r(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}$ is bounded, the claim follows. □

10.6 Homogenization of p-Laplace Equations

Assumption 10.18. *For the rest of this work, let Assumptions 1.3, 1.8 and 9.2 hold for some $1 < r < p$ and $p \geq 2$. This implies that $\mathbf{P}$ and Γ satisfy the strong (r,p)-extension and the strong (r,p)-trace property, as well as the weak uniform (r,p)-extension and the uniform (r,p)-Dirichlet extension property with the uniform (r,p)-trace property. In particular, we can apply all of the above developed theory.*

In what follows, we will consider the homogenization of the following functionals:

$$\mathcal{E}_{\varepsilon,\omega}(u) = \int_{\mathbf{Q}^\varepsilon(\omega)}\left(\frac{1}{p}\,|\nabla u|^p + \frac{1}{p}\,|u|^p - g\,u\right) + \int_{\Gamma^\varepsilon(\omega)} F(u(x))\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)}(x)\,,$$

where F is a convex function with $\partial F = f$, $F(\cdot) \geq F_0 > -\infty$ for some constant $F_0 \in \mathbb{R}$ and we assume that $|\partial F(A)|$ is bounded on bounded subsets $A \subset \mathbb{R}$. Note that compared to (1.1) we add the term $|u|^p$ in order to reduce technical difficulties. However, we will discuss how to treat the case of missing $|u|^p$ in Remark 10.22. Minimizers of $\mathcal{E}_{\varepsilon,\omega}$ satisfy the partial differential equation system

$$\begin{aligned} -\mathrm{div}\left(a\,|\nabla u^\varepsilon|^{p-2}\,\nabla u^\varepsilon\right) + |u|^{p-1} &= g && \text{on } \mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)\,,\\ u &= 0 && \text{on } \partial\mathbf{Q}\,,\\ |\nabla u^\varepsilon|^{p-2}\,\nabla u^\varepsilon\cdot\nu_{\Gamma^\varepsilon(\omega)} &= f(u^\varepsilon) && \text{on } \Gamma^\varepsilon(\omega)\,. \end{aligned} \tag{10.12}$$

and we will see that homogenization of the latter system is equivalent with a two-scale Γ-convergence of $\mathcal{E}_{\varepsilon,\omega}$. In particular, we find the following

Theorem 10.19. *Let Assumption 10.18 hold. Then, for almost every $\omega \in \Omega$ and*

$$\mathcal{E}(u,\upsilon) := \int_{\mathbf{Q}}\int_{\mathbf{P}}\frac{1}{p}\left(|\nabla u + \upsilon|^p + |u|^p\right) - \int_{\mathbf{Q}}\int_{\mathbf{P}} g\,u + \int_{\mathbf{Q}}\int_{\Gamma} F(u)\mathrm{d}\mu_{\Gamma,\mathcal{P}}$$

we find $\mathcal{E}_{\varepsilon,\omega} \xrightarrow{2s\Gamma} \mathcal{E}$ in the following sense

1 *For $u^\varepsilon \rightharpoonup u$ weakly in $L^p(\mathbf{Q})$, $u^\varepsilon \in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))$ with $\sup_\varepsilon \mathcal{E}_{\varepsilon,\omega}(u^\varepsilon) < \infty$, there holds $u \in W^{1,r}_0(\mathbf{Q})$ and there exists $\upsilon \in L^r(\mathbf{Q};\mathcal{V}^r_{\mathrm{pot}}(\Omega,\mathbf{P}))$ such that $\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_{\mathbf{P}}\cdot(\nabla u + \upsilon)$ and*

$$\mathcal{E}(u,\upsilon) \leq \liminf_{\varepsilon\to 0}\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)\,.$$

2 *For each pair $(u,\upsilon) \in W^{1,r}_0(\mathbf{Q}) \times L^r(\mathbf{Q};\mathcal{V}^r_{\mathrm{pot}}(\Omega))$ with $\mathcal{E}(u,\upsilon) < +\infty$ there exists a sequence $u^\varepsilon \in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))$ such that $u^\varepsilon \rightharpoonup |\mathbf{P}|\,u$ weakly in $L^p(\mathbf{Q})$, $\mathcal{U}_\varepsilon u^\varepsilon \rightharpoonup u$ weakly in $W^{1,r}(\mathbf{Q})$ and $\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_{\mathbf{P}}\cdot(\nabla u + \upsilon)$ weakly in two scales and*

$$\mathcal{E}(u,\upsilon) = \lim_{\varepsilon\to 0}\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)\,.$$

Proof. 1. Evidently,

$$\int_{\mathbf{Q}^\varepsilon(\omega)} \left(\frac{1}{p}\left|\nabla u^\varepsilon\right|^p + \frac{1}{p}\left|u^\varepsilon\right|^p\right) \leq C\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)$$

for C independent from ε. Hence the statement follows from Lemmas 10.17 and 10.4 on particularly noting that $u^\varepsilon \xrightarrow{2s} u$ in $L^s(\Gamma^\varepsilon(\omega);\mu^\varepsilon_{\Gamma(\omega)})$.

2. Step a: Let $(u_k)_{k\in\mathbb{N}} \subset C_b^1(\Omega)$ be the countable dense family in $W^{1,p}(\Omega)$ according to Theorem 9.14. Furthermore, let $(\phi_j)_{j\in\mathbb{N}} \subset C_c^\infty(\mathbf{Q})$ be dense in $W_0^{1,p}(\mathbf{Q})$. Then the span of the functions $\phi_j \nabla_\omega u_k$ is dense in $L^r(\mathbf{Q};\mathcal{V}^r_{\mathrm{pot}}(\Omega))$. Writing $S = \mathrm{span}\phi_j\nabla_\omega u_k$ we show statement 2. for $(u,\upsilon) \in (\phi_j)_{j\in\mathbb{N}} \times S$. However, for such (u,υ) we find $V \in \mathrm{span}\phi_j u_k$ such that $\upsilon = \nabla_\omega V$ and $V^\varepsilon(x) := V(x,\tau_{\frac{x}{\varepsilon}}\omega)$ is well defined and measurable for every ω. For simplicity of notation, we assume $V = \phi_j u_k$

In particular, we have for $u^\varepsilon = u + \varepsilon V^\varepsilon$ that $u^\varepsilon \xrightarrow{2s} u$ and $\nabla u^\varepsilon = \nabla u + \varepsilon\nabla\phi_j\, u_k\big(\tau_{\frac{x}{\varepsilon}}\omega\big) + \phi_j\nabla_\omega u_k\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ and hence $u^\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\mathbf{Q})$ and $\nabla u^\varepsilon \xrightarrow{2s} \nabla u + \phi_j\nabla_\omega u_k$. Using essential boundedness of $\nabla\phi_j\, u_k\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$, the ergodic theorem now yields

$$\begin{aligned}\lim_{\varepsilon\to0}\int_{\mathbf{Q}^\varepsilon(\omega)}\left|\nabla u^\varepsilon\right|^p &= \lim_{\varepsilon\to0}\int_{\mathbf{Q}}\chi_{\mathbf{P}}\big(\tau_{\frac{x}{\varepsilon}}\omega\big)\left|\nabla u + \phi_j\nabla_\omega u_k\big(\tau_{\frac{x}{\varepsilon}}\omega\big)\right|^p \\ &= \int_{\mathbf{Q}}\int_{\mathbf{P}}\left|\nabla u + \upsilon\right|^p .\end{aligned}$$

Similarly, we show $\int_{\mathbf{Q}^\varepsilon(\omega)}|u^\varepsilon|^p \to \int_{\mathbf{Q}}\int_{\mathbf{P}}|u|^p$ and $\int_{\mathbf{Q}^\varepsilon(\omega)} gu^\varepsilon \to \int_{\mathbf{Q}}\int_{\mathbf{P}} gu$.

Step b: By Lemma 10.17 we find $\mathcal{T}_\varepsilon u^\varepsilon \xrightarrow{2s} u$. Unfortunately, this is not enough to pass to the limit in the integral $\int_{\Gamma^\varepsilon(\omega)} F(u(x))\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)}(x)$. However, we can make use of

$$F(u) + \partial F(u)\varepsilon V^\varepsilon \leq F(u+\varepsilon V^\varepsilon) \leq F(u) + \partial F(u+\varepsilon V^\varepsilon)\varepsilon V^\varepsilon .$$

Since $\sup_\varepsilon \left\|V^\varepsilon\right\|_\infty + \left\|u\right\|_\infty < \infty$ we find

$$\left\|\partial F(u)\right\|_\infty + \sup_\varepsilon\left\|\partial F(u+\varepsilon V^\varepsilon)\right\|_\infty \leq C < \infty$$

and hence

$$F(u) - \varepsilon C \leq F(u+\varepsilon V^\varepsilon) \leq F(u) + \varepsilon C .$$

This implies by the ergodic theorem

$$\int_{\Gamma^\varepsilon(\omega)} F(u+\varepsilon V^\varepsilon)\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)}(x) \to \int_{\mathbf{Q}}\int_\Gamma F(u)\mathrm{d}\mu_{\Gamma,\mathcal{P}} ,$$

and hence 2. for $(u,\upsilon) \in (\phi_j)_{j\in\mathbb{N}} \times S$.

Step c: We pick up an idea of [9], Proposition 6.2. For general $(u,\upsilon) \in W_0^{1,r}(\mathbf{Q}) \times L^r(\mathbf{Q};\mathcal{V}^r_{\mathrm{pot}}(\Omega))$ with $\mathcal{E}(u,\upsilon) < +\infty$ let $(u_n,\upsilon_n) \in (\phi_j)_{j\in\mathbb{N}} \times S$ with

$$\left\|(u,\upsilon) - (u_n,\upsilon_n)\right\|_{W_0^{1,r}(\mathbf{Q})\times L^r(\mathbf{Q};\mathcal{V}^r_{\mathrm{pot}}(\Omega))} \leq \frac{1}{n} \tag{10.13}$$

and

$$\left|\mathcal{E}(u,\upsilon) - \mathcal{E}(u_n,\upsilon_n)\right| \leq \frac{1}{n} . \tag{10.14}$$

We achieve this in the following way: First we introduce $M_F := \sup F^{-1}(-\infty, M)$ and cut $u_M := \min\{u, M_F\}$. Furthermore, we set $\upsilon_M(x,\omega) = \chi_{(-\infty,M_F)}(u(x))\,\upsilon(x,\omega)$, i.e. $u_M = M_F$ implies $\upsilon = 0$. Then u_M and υ_M are still in the same respective spaces. Furthermore, as $M \to \infty$ we find $\mathcal{E}(u_M, \upsilon_M) \to \mathcal{E}(u, \upsilon)$ by the Lebesgue dominated convergence theorem. Now, by the properties of F, we can approach (u_M, υ_M) by elements $(u_{M,\delta}, \upsilon_{M,\delta}) \in (\phi_j)_{j\in\mathbb{N}} \times S$. By the Lebesgue dominated convergence theorem we get convergence in the $|\cdot|^p$-terms and using the convexity of F and local boundedness of ∂F like in Step b we show that $\mathcal{E}(u_{M,\delta}, \upsilon_{M,\delta}) \to \mathcal{E}(u_M, \upsilon_M)$. Successively choosing M and δ, we find $(u_n, \upsilon_n) \in (\phi_j)_{j\in\mathbb{N}} \times S$ satisfying 10.13–10.14..

We set $\varepsilon_0(\omega) = 1$ and for each $(u_n, \upsilon_n) \in (\phi_j)_{j\in\mathbb{N}} \times S$ we find by Steps a and b for almost every ω some $\varepsilon_n(\omega) \le \frac{1}{2}\varepsilon_{n-1}(\omega)$ such that for $\varepsilon < \varepsilon_n(\omega)$ and $u^\varepsilon_{n,\omega} = u_n(x) + \varepsilon V_n(x, \tau_{\frac{x}{\varepsilon}}\omega)$ it holds

$$\left|\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon_{n,\omega}) - \mathcal{E}(u_n, \upsilon_n)\right| \le \frac{1}{n}\,.$$

The set $\tilde{\Omega} \subset \Omega$ such that all $\varepsilon_n(\omega)$ are well defined has measure 1. For such ω we choose $u^\varepsilon = u^\varepsilon_{n,\omega}$ if $\varepsilon \in (\varepsilon_{n+1}, \varepsilon_n)$. Then

$$\left|\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon) - \mathcal{E}(u, \upsilon)\right| \le \frac{2}{n} \qquad \text{for } \varepsilon < \varepsilon_n\,.$$

which implies the claim. □

Theorem 10.20. *Let Assumption 10.18 hold. Then for almost every ω the following holds: For every $\varepsilon > 0$ let $u^\varepsilon_{\min} \in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))$ be the unique minimizer of $\mathcal{E}_{\varepsilon,\omega}$. Then*

$$\sup_{\varepsilon>0} \|u^\varepsilon_{\min}\|_{W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))} + \mathcal{E}_{\varepsilon,\omega}(u^\varepsilon_{\min}) \le \infty$$

and for every subsequence such that $\mathcal{U}_\varepsilon u^\varepsilon_{\min} \rightharpoonup u$ weakly in $L^p(\mathbf{Q})$ and weakly in $W^{1,r}(\mathbf{Q})$ with $\upsilon \in L^r(\mathbf{Q}; \mathcal{V}^r_{\mathrm{pot}}(\Omega, \mathbf{P}))$ such that $\nabla u^\varepsilon_{\min} \stackrel{2s}{\rightharpoonup} \nabla u + \upsilon$. It further holds $u \in W^{1,r}_0(\mathbf{Q})$ and (u, υ) is a global minimizer of $\mathcal{E}$ in $W^{1,r}_0(\mathbf{Q}) \times \mathcal{V}^r_{\mathrm{pot}}(\Omega)$.

Remark 10.21. Unfortunately, we are not able to prove uniqueness of homogenized solution due to a lack of coercivity in the respective case. However, note that in case Conjecture 9.18 holds, one can immediately prove that both $\nabla u \in L^p(\mathbf{Q})$ and $\upsilon \in L^p(\mathbf{Q}; \mathcal{V}^{r,p}_{\mathrm{pot}}(\Omega, \mathbf{P}))$, which allows to show the uniqueness of the minimizer by a standard coercivity argument.

Proof. In what follows, we denote

$$W_r := W^{1,r}_0(\mathbf{Q}), \quad \mathcal{V}_r := \mathcal{V}^r_{\mathrm{pot}}(\Omega)\,,$$

and note that every of the following countable steps works for almost every ω.

Step 1: Let $(u, \upsilon) \in W_\infty \times \mathcal{V}_p \subset W_r \times \mathcal{V}_r$. Then $\mathcal{E}(u, \upsilon) < +\infty$ and hence by standard arguments $\mathcal{E}$ has a at least one local minimizer (u_R, υ_R) on every closed ball of sufficiently large radius R in $W_r \times \mathcal{V}_r$

$$\overline{\mathbb{B}}^{W_r\times\mathcal{V}_r}_R(0) := \left\{(u, \upsilon) \in W_r \times \mathcal{V}_r :\ \|u\|_{W_r} + \|\upsilon\|_{\mathcal{V}_r} \le R\right\}\,.$$

By Theorem 10.19.2 there exists a recovery sequence $u^\varepsilon \in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))$ such that $u^\varepsilon \rightharpoonup |\mathbf{P}|\, u_R$ weakly in $L^p(\mathbf{Q})$, $\mathcal{U}_\varepsilon u^\varepsilon \rightharpoonup u_R$ weakly in $W^{1,r}(\mathbf{Q})$ and $\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_{\mathbf{P}} \cdot (\nabla u_R + \upsilon_R)$ weakly in two scales and

$$\mathcal{E}(u_R, \upsilon_R) = \lim_{\varepsilon\to 0} \mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)\,.$$

Step 2: We conclude for the minimizers

$$\liminf_{\varepsilon\to0}\|u^\varepsilon_{\min}\|_{W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))}\leq\liminf_{\varepsilon\to0}\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon_{\min})\leq\liminf_{\varepsilon\to0}\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)\leq\mathcal{E}(u_R,\upsilon_R)\,,$$

which at the same time implies by Theorem 10.19.1 that $\mathcal{U}_\varepsilon u^\varepsilon\rightharpoonup u$ weakly in $L^p(\mathbf{Q})$ and $W^{1,r}(\mathbf{Q})$ and there exists $\upsilon\in L^r(\mathbf{Q};\mathcal{V}^r_{\mathrm{pot}}(\Omega,\mathbf{P}))$ such that $\nabla u^\varepsilon\stackrel{2s}{\rightharpoonup}\chi_{\mathbf{P}}\cdot(\nabla u+\upsilon)$ and

$$\begin{aligned}\|u\|_{W_r}+\|\upsilon\|_{\mathcal{V}_r}&\leq C\,\mathcal{E}(u_R,\upsilon_R)\,,\\ \mathcal{E}(u,\upsilon)&\leq\mathcal{E}(u_R,\upsilon_R)\,,\end{aligned}$$

with C independent from (u_R,υ_R). Since also $\|u^\varepsilon\|_{W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))}\leq\mathcal{E}(u_R,\upsilon_R)$, we conclude

$$\|u_R\|_{W_r}+\|\upsilon_R\|_{\mathcal{V}_r}\leq C\,\mathcal{E}(u_R,\upsilon_R)\,,$$

Step 3: Similarly, if (u_{R^*},υ_{R^*}) is a further minimizer on any ball $\overline{\mathbb{B}}^{W_r\times\mathcal{V}_r}_{R^*}(0)$ with $\mathcal{E}(u_{R^*},\upsilon_{R^*})\leq\mathcal{E}(u_R,\upsilon_R)$ we can conclude

$$\|u_{R^*}\|_{W_r}+\|\upsilon_{R^*}\|_{\mathcal{V}_r}\leq C\,\mathcal{E}(u_R,\upsilon_R)$$

from the argument of Step 2 and a suitable recovery sequence.

Step 4: Hence, repeating Step 1 among the local minimizers, there exists a global minimizer $(\bar u,\bar\upsilon)\in\overline{\mathbb{B}}^{W_r\times\mathcal{V}_r}_{C\,\mathcal{E}(u_R,\upsilon_R)}(0)$.

Step 5: Repeating the argument of Step 2 we hence find that every sequence of minimizers of $\mathcal{E}_{\varepsilon,\omega}$ satisfies the claim. □

Remark 10.22. If the term $|u|^p$ in the above arguments is dropped, we first need to embed $\mathcal{U}_\varepsilon u^\varepsilon$ uniformly into $W^{1,s}(\mathbf{Q})$. From here, we need s large enough such that $\mathbf{P}^\varepsilon$ still has the uniform (r,s)-trace property. This will not affect the basic structure of the proofs, however it makes the presentation more complicated and less readable.

Nomenclature

We use the following notations:

$x\sim y$, x and y are neighbors, see Definition 2.43

$A_{1,k},A_{2,k},A_{3,k}$, see Equation (5.1)

$\mathcal{A}\left(0,\mathbf{P},\rho\right):=\left\{(\tilde x,-x_d+2\phi(\tilde x))\;:\;(\tilde x,x_d)\in\mathbb{B}_\rho(0)\backslash\mathbf{P}\right\}$ (Lemma 2.2)

$\mathbb{AX}(y,x)$, the Admissible paths from $y\in\mathbb{Y}\setminus\{x\}$ to $x\in\mathbb{X}_r$, see Definition 4.24

$\mathbb{B}_r(x)$ the Ball around x with radius r (Section 2)

$\mathbb{C}_{\nu,\alpha,R}(x)$ the Cone with apix x, direction ν, opening angle α and hight R (Section 2)

$\mathrm{conv}A$ the convex hull of A (Section 2)

Convex averaging sequence, see Definition 2.17

(δ,M)-regularity, see Definition 4.2

$\tilde\delta$, see Equation (5.2)

$\mathcal{U}$ for local and global extension operators (Lemma 2.2)

$\mathbb{X}$, $\mathbb{Y}$ Families of points (Section 2)

$\mathbb{X}_r(\omega) = \mathbb{X}_r(\mathbf{P}(\omega)) = 2r\mathbb{Z}^d \cap \mathbf{P}_{-r}(\omega)$, (2.36)

$\partial\mathbb{X}$, $\hat{\mathbb{X}}$, see Notation 4.26

Y_{flat}, see Notation 4.33

$\mathbb{Y}_{\partial\mathbb{X}}$, see Notation 4.26

$\mathring{\mathbb{Y}}$, $\partial\mathbb{Y}$, $\mathbb{Y}$, see Notation 4.26

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