The Role and Use of Sketchpad as a Modeling Tool in Secondary Schools

BY

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CONTENTS

ABSTRACT

Over the last decade or two, there has been a discernible move to include modeling in the mathematics curricula in schools. This has come as the result of the demand that society is making on educational institutions to provide workers that are capable of relating theoretical knowledge to that of the real world. Successful industries are those that are able to effectively overcome the complexities of real world problems they encounter on a daily basis.

This research study focused, to some extent, on the different definitions of modeling and some of the processes involved. Various examples are given to illustrate some of the methods employed in the process of modeling. More importantly, this work attempted to build on existing research and tested some of these ideas in a teaching environment. This was done in order to investigate the feasibility of introducing mathematical concepts within the context of dynamic geometry. Learners, who had not been introduced to specific concepts, such as concurrency, equidistant, and so on, were interviewed using *Sketchpad* and their responses were analyzed. The research focused on a few aspects. It attempted to determine whether learners were able to use modeling to solve a given real world problem. It also attempted to establish whether learners developed a better understanding when using *Sketchpad*.

Several useful implications have evolved from this work that may influence both the teaching and learning of geometry in school. Initially these learners showed that, to a large extent, they could not relate mathematics to the real world and vice versa. But a pertinent

finding of this research showed that, with guidance, these learners could apply themselves creatively. Furthermore it reaffirmed the idea that learners can be taught from the general to the more specific, enabling them to develop a better understanding of concepts being taught.

Perhaps the findings and suggestions may be useful to pre-service and in-service educators, as well as curriculum developers.

CHAPTER ONE

Introduction

Mathematics can be regarded as the "*bread and butter of all sciences*" (Cundy and Rollett, 1961 : 1). If the "bread and butter" is missing from the diet of education then this is likely to lead to deficiencies in the scientific education of the youth. This research focuses on the "*bread and butter*" aspect, namely mathematical modeling, using *Sketchpad* as a mediating tool to assist in the understanding of mathematics. There is very little doubt that mathematics has much to contribute to society as a whole. Real-world problems will remain problems without the tremendous problem solving ability and potential of mathematics. Blomhøj (1991 : 186) stressed the need for modeling in society when he stated that "*society demands an educated manpower, the need for a democratic competence to the application of mathematical models in society*". Not only does society demand an educated manpower, it stresses the need for a useful core of workers who will develop creative solutions for real world problems that already exist and those that are encountered continuously.

Despite the research and efforts of many great mathematicians and mathematics educators, there are questions that can still be asked: "*Why is there so much dissatisfaction with the mathematics graduates we produce?*" (Arora and Rogerson, 1991 : 111) and "*... despite the tremendous and far-reaching changes in mathematics education, why is there a growing dissatisfaction all around the world with the mathematical competency of the high school graduates that we produce? Why are the literates from school often so mathematically illiterate?* (Arora and Rogerson, 1991 : 112). This has nothing to do with mathematical abilities but rather with the way mathematics curricula are designed and the way mathematics is taught. The failure of mathematics curriculum developers to include the solution of real-life problems in mathematics classrooms is a major disadvantage to all mathematics graduates. However, this is changing. According to Arora and Rogerson (1991 : 114) "*the relevance of teaching mathematical modelling is being emphasised by various mathematical educationists and voiced in national and international conferences and forums*".

It is often difficult to project the cost of experiments in the real world, when confronted by problems. In some cases performing experiments may be too costly or even impossible to do. Mathematical modeling reduces costs and instead of working with large buildings or fields or whatever is necessary for the experiment, only mathematical equations and computer calculation need to be used. In fact, modeling in mathematics has three main objectives for society: to increase understanding of real-world problems, to predict future trends and then to decide on the best methods of resolution of the problem.

Presently, very little real-life mathematics is done at schools in South Africa. There are some aspects in the present curriculum that has some resemblance to real-life and these are word problems for the younger learners and Linear Programming for the grades 11 and 12 learners. Trigonometry and Calculus also has some potential for the inclusion of applications related to real world problems. In 1991, a new core syllabus were introduced in South Africa. One of the stated aims of that syllabus was that mathematics was being done at schools "*to enable pupils ... to solve ... by recognizing a real-world situation as amenable to mathematical representation, formulating an appropriate mathematical model, selecting the mathematical solution and interpreting the result back in the real-*

world situation" (Noble, 1990 : i). However, despite the statements in the curriculum, very little modeling was translated into practice (for example, in textbooks and class lessons and so on). In fact, very little or no modeling activities took place in mathematics classrooms in South Africa.

Recently, with the release of the new Curriculum statements for mathematics (May 2002), it is evident that mathematical modeling is now much more strongly emphasized in the curriculum. Moreover, specific examples are given as possible contexts for modeling. In the Intermediate Phase (grades 4 to 6) and the Senior Phase (grades 7 to 9), under Learning Outcomes 2 there is a stated focus which "*expects the learner to be able to use mathematical models to represent relationships within an ecosystem*" (Mathematical Curriculum Statements Document, May 2002 : 35 and 63).

Mathematical modeling can be "*applied to many areas but these can be categorized into four main areas, namely, those of science, technology, development and personal / corporate organization*" (Ormell, 1991 : 64). Sadovski and Sadovski (1993) state that "*every scientific approach involves construction of models of the phenomena under study*". For this reason, several authors in mathematics and mathematics education have frequently recommended that mathematical modeling be taught so that it may assist with problem solving in real-life situations. But most educators and textbooks in South Africa still do not address this need adequately because of its absence from the traditional curriculum. As a result, very little or no mathematical modeling is presently being done in mathematics classes in South Africa.

Mathematical modeling has as long a history as mathematics itself. It has been with us throughout the ages (Huntley and James, 1990: v). For example, Arora and Rogerson (1991 : 112), quote from Ulysee's Odysee as follows: "*mathematical models have been around since mathematics itself. We remind the reader of the blind shepherd who sat at the mouth of the cave in the morning and put a pebble in his pouch for each sheep that he let out of the cave and, in the evening, took a pebble for each sheep that he let in the cave*".

Despite the weak presence of mathematical modeling in the school curriculum, research into the teaching of mathematics through modeling is still continuing, but not much research has been done using dynamic computer software. This research is, therefore, not implying that mathematical modeling is a new discovery or a new activity, but it will argue that mathematical modeling can and should be incorporated not only within the mathematics curriculum, but also within the context of dynamic geometry (compare Usiskin 1991 : 30). It is necessary as curriculum developers to consider how to weave applications, modeling and computer technology into the curricula for the average learners. Although, an entire section is dedicated later to the definition of a model, it suffices, for now, to state that "*a model is a description of a system intended to predict what happens if certain actions are taken*" (Bratley, Fox and Schrage, 1987 : 1) or if certain actions are required. A model must adequately describe the system under investigation.

An important aspect of mathematical modeling that all learners ought to learn, is the realization that in some cases in the real world a mathematical solution does not apply exactly: it has to be interpreted, often modified to suit particular situations. Besides, in order to use mathematical modeling it is essential that conditions described in the problem must be realistic and adhere to the general laws of the universe. Much of this will be discussed in greater detail in the chapters that follow.

Due to the scarcity of empirical data available on modeling in dynamic geometry it was decided to investigate mathematical modeling in the context of dynamic geometry, using in particular *Sketchpad*. Through the empirical research conducted, the researcher discovered that mathematical modeling can be effectively used as a teaching strategy and useful for promoting conceptual learning. The modeling process allowed the learners to gain insight into the problem that was presented to them, and assisted the learners in gaining a better understanding of concepts such as perpendicular bisectors and concurrency.

In the theoretical part of this study, the role and process of modeling within mathematics, mathematics education and the computer environment was analysed, in order to identify different ways and contexts in which modeling could be presented meaningfully to learners.

In Chapter Two, definitions of Mathematical Modeling are discussed. Incorporated into this chapter is a discussion of the need for mathematical modeling in general and in the school. This chapter also looks at examples of modeling activities and reports of a few teaching experiments relating to mathematical modeling.

Chapter Three looks at the relationship between mathematical modeling and problem solving. An in-depth discussion on the processes involved in mathematical modeling follows. This chapter is essential in understanding what modeling is and how it could be taught and used. A fairly comprehensive discussion of Realistic Mathematics Education is incorporated into this chapter.

Chapter Four, is a discussion of the research process, which includes the collection of data and a discussion of the research methods.

Chapter Five discusses the research findings and Chapter Six concludes the entire thesis.

CHAPTER TWO

Modeling in mathematics

2.1 Definitions of a mathematical model

The Oxford Dictionary (1983) defines a model as a "*simplified description of a system to assist calculations and predictions*". This definition conveys two distinct thoughts. The first is the idea of a model becoming a simplified version of some aspects of reality and secondly, a model being used for its predictive value. The word "*model*" was derived from Latin (*modellus* from *modulus* from *modus*) and means mold or pattern. Although this may appear simplistic, modeling can be extended to include "*relationships between quantities (distances, money, weight, and so on) that can be observed in a mathematical system*" (Ljung and Glad, 1994: 14).

It is necessary at this point to differentiate between different conceptions of a model. The Oxford Dictionary definition alludes to the fact that the nature of a mathematical model can refer to a physical construct, a shape molded out of some material (for example, wire or clay) in order to enhance understanding. It may also refer to a scale drawing that is used to determine precise measurements - this type of model is used often in architectural or engineering design. The other types of models would refer to models that use algebraic equations and those that are extended to use computer programmes.

The NCTM's Discussion Draft of the Principles and Standards for School Mathematics contains a comprehensive discussion on the different connotations of the word "*model*". It is useful to state their complete discussion at this point (NCTM, 1998 : 98-99).

> *The term "model" has many different meanings. So, it is not surprising that the word is used in many different ways in discussions about mathematics education. For example, "model" is used to refer to physical materials with which students work in school, as in manipulative models. The term is also used in ways that suggest exemplification or simulation, as in when an educator "models" the problem-solving process for her students. Yet another usage treats the term as if it were roughly synonymous with representation. Amidst this array of uses and meanings is another that is particular to mathematics, and it is the major focus of this part of the discussion about the role of representation in school mathematics. The term "mathematical model" in this context means a mathematical representation of the elements and relationships within an idealized version of a complex phenomenon. Mathematical models can be used to clarify understandings of the phenomenon and to solve problems. What it means "to model" in this sense includes not only representation, but also acting upon the representation and interpreting the meanings of one's action within the mathematical model and with respect to the phenomenon being modeled.*

This discussion covers the range of meanings that is encompassed by the word model but it is also important to list other meanings as defined by different authors. It should also be pointed out that there are numerous definitions, as well as the fact that generally people from North America spell modeling with a single 'l' and those from the United Kingdom with a 'll'. For the purposes of this research, modeling will be spelt with a single 'l' but if quotations contain a double 'l' then that will be written exactly as it appears.

Most definitions articulate the idea that mathematical models attempt to develop **prototypes** of the real problem - in other words, simplified versions of the real problem.

- "*When we construct a model of something, we create a substitute of that thing - a representation*" (Mortenson, 1985: 1). We will see later that it may be impossible to fully replicate any real world problems. Whilst we may be able to take many variables into consideration, it may not be possible to consider *every* variable. Lancaster (1976: 2) states this clearly: "*real-life phenomena are generally so complicated in relation to the mathematical methods at our disposal that we cannot hope to represent and account for every characteristic. Consequently, some simplifying hypotheses must be made. The moment that we do this, we are leaving the real world and beginning to make a (mathematical) model*". Bender (1978 : 1) also refers to a model as "*something which mimics relevant features of the situation being studied by using the language of mathematics*".
- *Any complete and consistent set of mathematical equations which is thought to correspond to some other entity, its prototype. Being derived from 'modus' (a measure) the word 'model' implies a change of scale in its representation.* (Aris, 1994: 1)
- *A mathematical model is a mathematical construct designed to study a particular real-world system or phenomenon.* (Giodano, Weir, and Fox, 1997: 34)
- *Trying to understand the world around him, man organises his observations and ideas into conceptual frames. These we shall call (mathematical) models.* (Gårding, 1977: 1)

- *A (mathematical) model always represents a simplification of the situation being modeled.* (Greer and Mulhern 1989: 144).

Whilst the definitions above mainly convey the idea of simplified versions of the real problem, other definitions (see listed below) also include the **predictive** aspect of a model. Since the model emulates certain behaviour of the real problem, the intent of constructing such a model is for its predictive value.

- A mathematical model "*mimics certain aspects of observed behaviour, thus enabling, useful predictions to be made.*" (Fowkes and Mahony, 1994: 1)
- "*It* (mathematical model) *is the representation or transformation of a real situation into mathematical terms, in order to understand more precisely, analyse and possibly predict therefrom*". (Arora and Rodgerson, 1991 : 111)
- "*When a mathematical system is constructed in an attempt to study some phenomenon or situation in the real world, it is usually called a mathematical model*". (Olinick, 1978: 3).
- "*A model of a system is a tool we use to answer questions about the system without having to do an experiment*". (Ljung and Glad, 1994: 13)
- "*a model should, at the very least, be able to describe certain phenomena fairly accurately, that is, it should have predictive value.*" Davis and Hersh (1981: 46)

Edwards and Hamson (1989: 3) detail a definition that appropriately encapsulates the role and function of a mathematical model: "*A mathematical model is a model created using mathematical concepts such as functions and equations. When we create mathematical models, we move from the real world into the abstract world of mathematical concepts,* *which is where the model is built. We then manipulate the model using mathematical techniques or computer-aided numerical computation. Finally we re-enter the real world, taking with us the solution to the mathematical problem, which is then translated into a useful solution to the real problem. Note that the start and end are in the real world*". It is noticeable, however, that Edwards and Hamson are, by reference to functions and equations, seemingly restricting their definition to algebraic models. This definition does not include other types of modeling, especially that of dynamic geometry modeling. In a large number of modeling situations it might be possible that dynamic geometry modeling could serve as the basis of the actual modeling.

Therefore in summary, a mathematical model is the construction of a mathematical representation of some real life problem (phenomenon) with the explicit purpose of not only eliciting a better understanding of that problem (phenomenon) but also to produce a satisfactory solution to the problem and hopefully some predictive value. This usually entails the transformation of a natural phenomenon into a mathematical one.

2.2 The need for mathematical modeling and its processes

As mathematics unfolded with the evolution of human beings, modeling became a necessity to solve real world problems. There are various reasons for teaching modeling and perhaps the most important would be the pragmatic reasons. In order to deal with real world problems, people must be able to understand and display sufficient competency in resolution of such problems. Here modeling is useful in that it allows people to become more practical and interactive with problem situations. An obvious advantage of such proficiency in modeling could be that society becomes more adept at handling different kinds of problems, including economic and social problems.

Cross and Moscardini (1985) argue that in the decade prior to the mid-1970's, computer development changed the way industry and commerce operated. The demand for graduates with modeling skills increased. This renaissance in industry and commerce effectively has created a need for mathematical modeling at all levels of mathematics teaching, namely, schools and universities. The message that is being conveyed is simple: society makes demands for workers in industry to be highly functional, in modeling and schools and universities ought to respond by teaching modeling.

> *"Prior to the mid-1970's the emphasis of applicable mathematics in undergraduate courses was, principally, on the analysis of well-established mathematical representations of physical systems. Since the mid-1970's, however, there has been a growing interest in the problems associated with formulating the set of mathematical equations to represent a process or system. In other words, the emphasis is moving towards the development of mathematical models, rather than merely analysing the resulting set of equations, difficult though this may be".* (Cross and Moscardini, 1985: 11).

Blomhøj (1991 : 188-189) in motivating for modeling in the Danish Gymnasium states that *"the foundation for a wide range of important decisions concerning the individual citizen's employment and participation in democratic society is increasingly provided by the use of mathematical models (for example, production planning, economics, and pollution* *control)*". Since society demands graduates with specific skills in modeling, Von Essen (1991 : 196) argues that "*that all mathematics instruction ought to 'give the student a knowledge of some authentic real-life applications of mathematics of essential social significance*'". If this is not the case then one must ask why teach mathematics? The utilitarian nature of mathematics should be its most significant contribution to society. Mathematics is necessary for various reasons, but for millions of people, who will not become mathematicians, the value of mathematics lies in the way they can engage in mathematical activity simply for the benefit of society or themselves.

Finding satisfactory solutions to the problems of society is therefore a very important role of mathematics. Ljung and Glad (1994: 7) for example, argue that, "*more and more engineering work relies on mathematical models of the studied object*". Leigh (1983: (i)) similarly stresses that "*Mathematical models are finding increasing application in a variety of fields*". Cross and Moscardini (1985 : 24) also summarize the need for mathematical modeling and issues a challenge to all mathematics curriculum developers to give consideration to including modeling in the curriculum as follows: "*Thus since mathematical modeling is still developing throughout society, it is vitally important that mathematicians, scientists and engineers be exposed to it. Not only should they be exposed to its utility, but they should also gain some basic experience and skill in using mathematics to provide quantified solutions to an increasing number of society's technical problems*".

The need for teaching and learning modeling cannot be over emphasized because of its proven and future potential utilitarian value to society. Ormell (1991 : 65) makes a plea in this regard as follows: *"Instead of letting the successful mathematical modeling of the past* *be thrown into the waste paper baskets of history, we should mount a campaign to preserve, record, catalogue and celebrate it .... We need an overall view of mathematics as a modeling instrument – that is, in effect, chiefly as an* ***exploratory*** *and* ***previewing*** *discipline".*

Why do mathematical modelling?

What is it for?

A) An introduction to new areas of mathematics.

B) Passing examinations or assignments.

C) Personal interest.

D) Solving "real life" problems.

Who is it for?

i) The educator

ii) The Student

iii) Industry and commerce

Figure 1

In investigations conducted by Abrantes (1991 : 128) he found that "*attention paid to extra-mathematical aspects and to the process of using mathematics in real situations seems to contribute to increased motivation of mathematics and the understanding of its role*". Most often educators face a barrage of questions in classrooms, where learners demand to know why are they doing a particular aspect of mathematics. If learners can be taught to engage in real life situations using the mathematics they learn, then understanding of the real life situation is improved. More importantly it contributes to a better understanding, and perhaps, a better appreciation and acceptance of the role of mathematics in society.

In attempting to justify the use of mathematical modeling, Eyre (1991 : 290) uses a schematic diagram, as shown in Figure 1, to analyse who actually uses modeling. Again we see the fact that society (represented here by Industry and Commerce) demands a workforce fully conversant with mathematical modeling strategies. Purely according to the demands of society, the most productive people are the ones capable of solving real-life problems affecting the quality of life, health and so on. Effective problem solving techniques are more valued in industry than highly qualified personnel with only abstract knowledge of concepts and processes. Michael Sears, a highly respected applied mathematician in South Africa, working in industry, also argued that the material learners learn in traditional university courses does not prepare them to solve the type of problems encountered in many instances in real life (Sears, 2000). In essence, he argues that our current university courses are too theoretical and often lack the tendency to relate problems to the real world.

In 1970, McLone conducted a survey to study the perceptions of employers of mathematics graduates and the graduates themselves, concerning the relevance of degree courses in the United Kingdom. The findings of this survey was reported by McLone in 1973 (Clements, 1989 : 8) and one of his comments relates to industry: "*The request most often made (by employers) is for mathematics graduates with an appreciation of the applicability of their subject in other fields and an ability to express problems, initially stated in non-mathematical terms, in a form amenable to mathematical treatment with subsequent re-expression in a readily understandable form to non-mathematical colleagues*". He went on to say : "*An important aspect of applied mathematics, which is emphasised by all groups, is mathematical modeling, that is the modeling of real situations in mathematical terms*".

Clements (1989 : 6) therefore argues that the teaching of modeling is vital at all levels: "*There is however, within the discipline of mathematical education today, an identifiable group of educators and lecturers who, to a greater or lesser extent, believe that the teaching of mathematical modeling is vital to the development of an ability to use mathematics and that such teaching is necessarily a distinct activity from the teaching of other topics in mathematics*". Cross and Moscardini (1985 : 11) expressed similar views by stating that: "*As mathematics, science and engineering students are now more likely to become involved in the development and application of mathematical models, it is useful if they are exposed to the ideas involved in modeling of processes and systems during their under-graduate career*".

Although we are aware that, since the origin of mathematics, real life problems have been modeled using mathematical methods, the scale and demand for its use has increased dramatically in modern times. Mathematics has become the basic language of modern science. According to Fowkes and Mahony (1994 : 1) "*the use of mathematics to test ideas and make predictions about the real world has a long and distinguished record in the physical sciences; so much so that mathematics has become the basic language of physics and its applications in engineering*".

Besides the physical sciences and engineering, there are many other fields, which have employed the use of mathematical modeling in drawing postulations and finding solutions, for example, ecology and industry. Voltera, for example in 1926 used mathematical modeling to **qualitatively** explain changes in the shark population in the Mediterranean under circumstances in which there was no possibility of obtaining reliable **quantitative data** (see Fowkes and Mahony). Voltera's work was invaluable because the resulting insights provided, had changed the attitudes towards modeling, as well as leading on to some very successful ecological models in current use.

There is little doubt that the application of mathematics to real world problems is invaluable. Banu (1991 : 118) stresses this as follows : "*Mathematical models are now widely used in water resources planning in many developing countries. In Bangladesh a few modeling projects have also been initiated for flood forecasting, flood control, drainage and irrigation schemes. A mathematical study using the model of the Danish Hydraulic Institute was made to assess the impact of flood control polders upon flood level of the Atrai River in Bangladesh.*"

Closely related to solving real world problems is the promotion of general problem solving in mathematics, an area of concern for some time, especially for secondary school mathematics curriculum developers. General problem solving will be discussed in more detail in Chapter 3. However, one of the major problems has been that traditional curricula have been dominated by the belief that learners first require good manipulative skills in mathematics (e.g. factorising, simplifying, solving equations, differentiating, and so on) before they are able to tackle and solve problems (mathematical or real world).

Despite the increased recognition of the importance of relating classroom teaching to the real world, Magan (1989: 107) points out that often examples used in school are contrived and often not relevant to real life. That is not the only difficulty facing educators. The scarcity of resource materials that present suitable real world problems creates further obstacles. The other problem, despite the efforts of mathematics curriculum developers to include real world problems in the teaching of mathematics, is that it is not always easy to teach children problem solving and modeling skills.

Unfortunately many mathematics educators themselves do not clearly differentiate between ordinary mathematics (factorising, differentiating, etc), problem solving and modeling. Teaching basic strategies, formulae and other theory in mathematics is essential, but exercises directly related to the theory do not constitute real world problems. It is important to clearly differentiate between different activities in a mathematics classroom, such as mathematising, modeling and problem solving (Blum, 1991: 10).

Traditional mathematical texts attempt to relate problem-solving exercises to real-life situations in a particular way. Learners are expected to find solutions by making use of the data that is given, together with previously learnt formulae and techniques they have developed. What learners are doing does constitute aspects of mathematical modeling, but only at a very basic level. It is a fallacy if one believes that children have not already been exposed to modeling activities. In many ways, children are not completely unaccustomed to it. Cross and Moscardini (1985 : 16), for example, point out that children playing as parents, nurses and doctors are in some sense already modeling reality (although not necessarily mathematically).

Of course, it is known that teaching mathematical modeling to children is not very easy. Although there may be an upsurge in the interest for mathematical modeling in schools, there are problems that will have to be overcome. Jones (1993 : 770) points out that one of the problems faced by educators when teaching modeling is that the activities often become more complex and lengthy to establish than traditional teaching activities. Despite its implementation educators often offer insufficient opportunities for reinforcing crucial but tangible parts of the modeling process, such as "*interpreting the problem mathematically*" and "*employing the theories and tools of mathematics to obtain a solution to the problem*" (Swetz, 1991 : 358).

However, learners ought to be taught that real-life situations are not necessarily easy to solve and their models may require some mathematical manipulation (compare Lesh and Lamon (1992) in Amit and Hillman (1999 : 18)). Furthermore, the ability of learners to understand and to solve real-world problems does not result automatically from learning pure mathematics but can only be achieved by embedding real situations into mathematics instruction (compare Blum (1991 : 17)). This contrasts with the traditional approach where the real world applications are done only AFTER the theory is learnt. Blum is advocating the reverse: Start with the real world problem or context and develop the mathematical theory from that. This view is aligned to the notion that people learn by doing.

Modeling could aid in the process of allowing learners to relate or develop mathematical theory from real world situations. Indeed, Blum (1991: 26) claims that through modeling "*the world around us may be comprehended better, mathematical concepts may be understood more deeply and more extensively, formative abilities may be advanced, and attitudes towards mathematics may be improved by giving more meaning to it*".

Problems and solutions pertaining to population dynamics, for example, could offer learners the opportunity to develop or utilize mathematical theory to solve real world problems. For example, consider the following question: A game warden needs to determine the population of crabs in a lake. He traps 50 crabs and marks their shells with paint. These marked individuals are released into the lake to mix with unmarked ones. A week later he traps a sample consisting of 18 crabs. Of these 9 are marked. Two weeks later 40 crabs are trapped. Fifteen of these are marked. Three weeks later a sample of 40 crabs is trapped of these 7 are marked. Estimate the crab population of this lake.

Blum (1991: 26) also emphasizes the idea that although modeling in mathematics is not easy it does have positive spin-offs: "*There is a general consensus of opinion that applications and modeling should be an essential part of mathematics instruction at all levels. On the one hand, they* (here he refers to results produced by Kaiser-Messmer, 1986) *show that mathematics instruction does not become easier for students and educators by the inclusion of applications and modeling but that, on the other hand, the world around us can be comprehended better, mathematical concepts may be understood more deeply and more extensively, formative concepts abilities may be advanced, attitudes towards mathematics may be improved by giving more meaning to it*".

This is also emphasized by Miriam Amit, Israel and Susan Hillman (Amit et al, 1999 : 18) when they state, "*the mathematical modeling of real-world problem situations is generally thought of as having the potential to contribute to the development of higher order thinking and provide information about students' usable mathematical knowledge*".

Many more reasons can be advanced for teaching modeling. Niss (1989) as quoted in Usiskin (1991: 31) gives five aims for teaching applications and modeling. These are:

1. To foster creative and problem-solving attitudes, activities and competences;

2. *To generate a critical potential towards the use and misuse of mathematics in applied contexts;*
3. *To provide the opportunity for students to practice applying mathematics that they would need as individuals, citizens, or professionals;*
4. *To contribute to a balanced picture of mathematics;*
5. *To assist in acquiring and understanding mathematical concepts.*

These five aims reiterate the underlying themes of relating mathematics to the real world and enhancing the ideas of learning and understanding new mathematics. In addition, Usiskin (1991: 33) argues that these experiences should begin early in a child's education and continue throughout.

It is noteworthy that Usiskin (1991: 38-39) equates the importance of modeling to that of proof as follows: "*I consider the process of modeling as being analogous - at least in a pedagogical sense - to proof. Each is at the highest level of cognitive activity. Do we want students to learn proof, because that is what mathematicians do? If so, then we should want students to learn modeling, because that is what applied mathematicians do. If we do not want students to learn proof, then perhaps we should not want them to learn to model. My own view is that we do want students to learn some aspects of both proof and modeling, to have experiences with both*".

Davis and Hersh (1981: 78) list five purposes for construction of mathematical models, in their motivation for the need for modeling in the real world. These are:

(1) to obtain answers about what will happen in the physical world,

(2) to influence further experimentation or observation,

(3) to foster conceptual progress and understanding,

(4) to assist the axiomatization of the physical situation and

(5) *to foster mathematics and the art of making mathematical models.*

Recent writings in mathematics education have shown that learners, within certain contexts, themselves display a desire for proof (as a means of explanation). For example, in a study by Mudaly (1999) to determine whether learners felt any need for proof, it was found that learners expressed a desire to find an explanation (deeper understanding) which was independent of their need for conviction. In an analogous way, it is plausible to assume that children might similarly have a need for better understanding of the real world, and desire to solve important problems therein.

According to Davis (1991 : 2), mathematical models *"are descriptive, predictive and prescriptive: that is to say, tell me what it is, tell me what will be, tell me what to do about it".* In other words, the purpose of a mathematical model is to clearly define and develop an understanding of the problem, to solve it using mathematical means, and then to extrapolate the solution back into the real world.

If society demands workers that are proficient in modeling, our reaction must not only be to teach children modeling, but every effort must be made to develop educators so that they would be proficient at teaching modeling as well. Perhaps this motivated the Mathematical Association of America's Committee on the Undergraduate Program in Mathematics (CUPM) to recommend that "*students should have term projects in an operations research course, as independent study, or as an internship in industry*" (Giordano, Weir and Fox, 1997: x). They (CUPM) are further quoted as saying, "*a* ***modeling experience*** *should be included within the common core of all mathematical sciences majors*" and "*the modeling* *experience should begin as early as possible in the student's career and reinforce modeling over the entire period of study*".

2.3 The process / methodology of modeling

"*One of the features that distinguish applied mathematics is its interest in framing important questions about the observed world in a mathematical way. This process of translation into a mathematical form can give a better handle for certain problems than would be otherwise possible. We call this the modeling process*" (Beltrami, 1987 : xiiv). There is little doubt that for effective modeling one needs to focus on the process - essentially the success of model building lies in the process of modeling.

Cross and Moscardini (1985 : 66) emphasise that the process of modeling "*involves not only the development of a set of skills but also experience and intuition*". This is an important consideration when dealing with the process of mathematical modeling. The assumption that modeling mainly deals with solving of mathematical equations is not a correct one. The process of modeling starts firstly from the need to find solutions to real world problems. Therefore, the process of modeling aims to solve real world problems and not just create models to mimic the real world problem just for the sake of it. In fact, Swetz (1991: 358) described this process of modeling as follows: "*...identifying the problem, including the conditions and constraints under which it exists; mathematically interpreting the problem; employing the theories and tools of mathematics to obtain a solution to the problem; interpreting and testing the solution in the context of the problem; refining the solution techniques to obtain a better answer to the problem under consideration if necessary...*".

However, this process of modeling is not an easy one. It often involves a process of creating a miniature or simpler problem, which is analogous to the larger problem, but enabling the modeler to draw exact conclusions, which can be extrapolated to the original real life problem. Although the model attempts to simulate the original problem it cannot truly replicate all the constraints that might be imposed by the problem itself. Like the children that simulate adulthood by playing mothers and fathers, play with dolls, which are non-living entities, and cannot move the way human babies do.

Modeling usually begins with a real life situation, which may be relatively controlled (for example, determining the profit of a manufacturing company), or sometimes in environments in which the modeler cannot control all the conditions (for example determining the population increase of fish in a river). In all cases the modeler is hoping to predict future behaviour of the system under prevailing conditions. In a factory, it would be easier to predict because information is easily available as compared to the conditions surrounding an investigation at the river. Droughts, floods, disease, the number of fishermen and so on cannot be predicted. This initial recognition of the existence of the problem and its analysis is really the first step of the modeling process.

A model is thereafter constructed (a new model or an old model is used) using the relevant information. The modeler identifies the aspects of the problem that are relevant for the investigation. The next step would be to define clearly the aspects that are basic to the study. All information that is thought to be irrelevant is cast aside. For example, in an investigation commissioned to investigate the population dynamics within a pond consisting of fish and other insects, it really does not matter whether the fish are blue or black, but it may be important to consider rainfall patterns.

It is this step that requires some creativity. The modeler now begins to change the focus by converting all relevant aspects to symbolic mathematical terms. The model is then subjected to mathematical analysis and mathematical results are obtained. These results are then interpreted in the real world; a comparison is made between the real world problem and the mathematical model. It may be that there are several models available but the modeler will use that model which closely approximates the real world situation.

So it is clear that the process of mathematical modeling involves the following broad stages. The first is the stage of ***understanding***. The modeler looks at the problem, agrees that such a problem exists in the real world, that a mathematical model can be constructed, and that solutions are possible. This is the stage where the modeler relies largely on his intuition to formulate the final problem and then look at possible models. The second is the ***model construction*** stage. The modeler now looks at the problem and the constraints that exist and starts to develop a possible model. This model may be a new one or it may be old model that will satisfy the needs of the present problem.

The ***analysis of the model*** stage follows. This is the stage in which the model is analyzed for its effectiveness in solving the problem. It may mean that, as a result of this stage, the modeler may have to return to the ***model construction*** stage if the model does not address the formulated problem. Simple changes to the model or a completely new model may result.

The final stage is the ***interpretative stage***. The modeler finally begins to find solutions to the formulated problem. Conclusions drawn at this stage are now ready for consumption.

A few models of the modeling process are presented below in order to demonstrate the four broad stages. Figure 2, represents this process as depicted by Davis and Hersh (1986: 59).

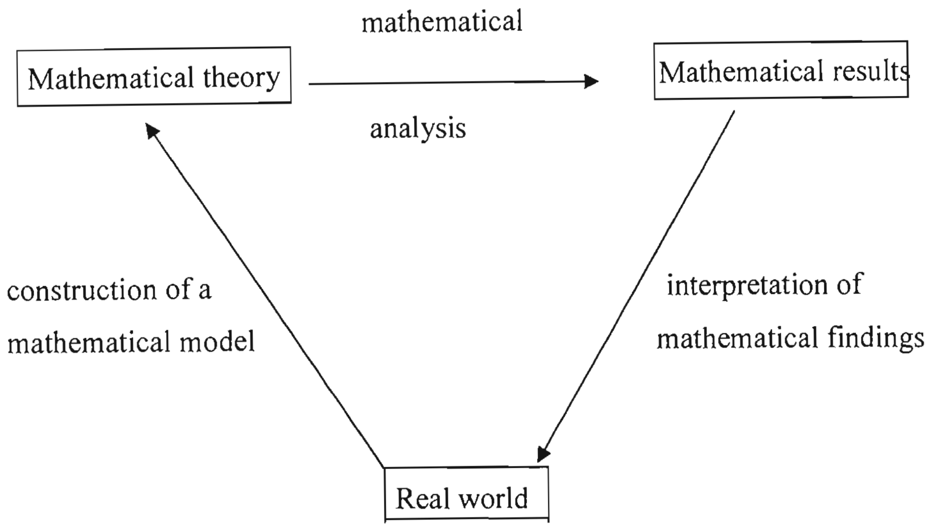


Figure 2

The above model by Davis and Hersh is simple but it conveys the idea that some existing mathematical theory can be used to construct a model of a real world problem and the results obtained from subjecting this model to stringent mathematical tests can be interpreted in the real world.

Similar models were described by Giordano et al (1997: 33) in Figure 3 and Olinick (1978: 4) in Figure 4.

Giordano et al (1997 : 33) explain the modeling process as follows:

1. *Through observation the primary factors involved in the real-world behaviour must be identified, possibly making simplifications.*
2. *Tentative relationships among the factors are conjectured.*

3. *The resultant model is analysed mathematically.*
4. *Mathematical conclusions are then interpreted in terms of the real-world problem.*

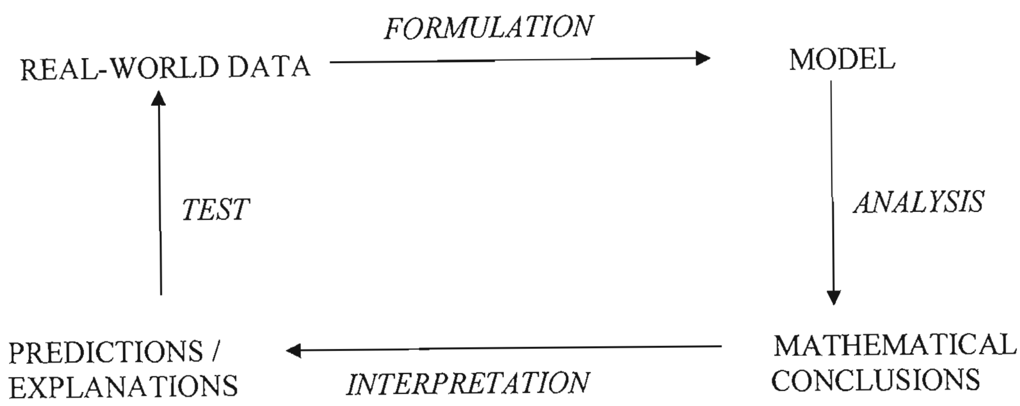


Figure 3

This process does not significantly differ from that outlined by Bender (1978 : 7-8). His four step process is as follows:

1. *Formulate the Problem.* This step is similar to that of Giordano, because in both cases the essential aspects of the problem must be scrutinized and the various factors taken into consideration before the problem is refined and then formulated.
2. *Outline the Model.* After the problem has been formulated, the relationships between the various factors must be taken into consideration and then a conjecture arrived at.
3. *Is it useful?* At this point Giordano had suggested that the problem be analyzed using mathematics whilst Bender required the modeler to step back and look at what was obtained after the model was outlined. This, he suggested, should be done before testing the model, so that it could be determined whether the model could provide the necessary data and whether the modeler could make useful predictions. This is an unnecessary step in Bender's four step process because it might not be possible to judge

the efficiency and usefulness of the model without testing it. And this was exactly what he lists as his fourth step.

4. *Test the Model.* This is the fourth step of the process and it allows for changes if the predictions or conclusions are bad.

Michael Olnick (1994:4) presents a very similar model (Figure 4):

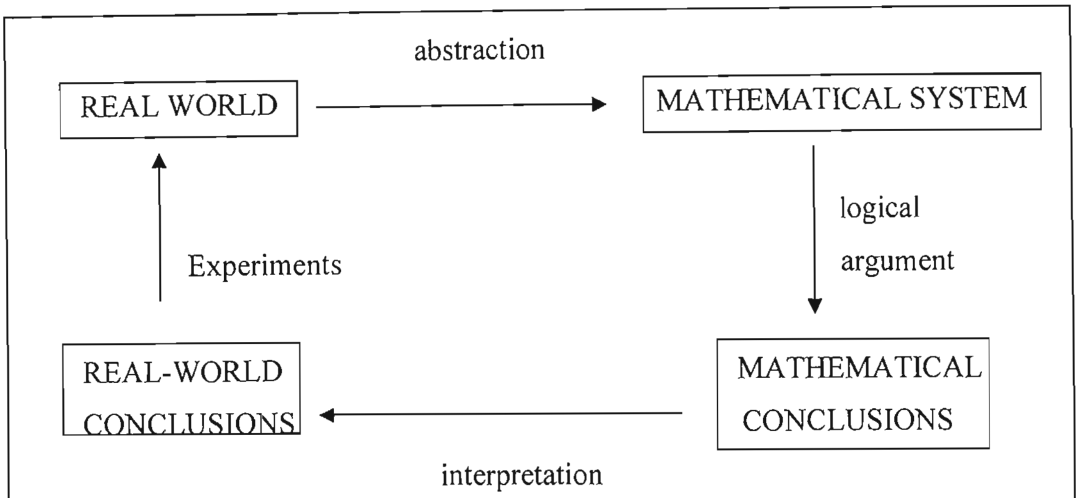


Figure 4

It is clear that the process of modeling involves the construction of a model using the constraints presented by the real world problem. The solution of the mathematical problem leads to the potential solution of the real world problem.

Although the model of Giordano et al (1997: 36-39) was already given above, it is useful to consider the following summary of the process that they make:

1. *Identify the problem. What is it that you want to do or find out? This may involve sorting out large amounts of data.*
2. *Make assumptions. It is difficult to capture every factor that may influence the problem. By making certain assumptions the complexity of the problem is decreased and the task is simplified.*
3. *Solve or interpret the model*

4. *Verify the model. Before use, the model must be tested. Does the model answer the problem? Is the model usable in the practical sense, that is, can we gather the data necessary to operate the model?*
5. *Implement the model*
6. *Maintain the model. Does the problem change with the model, or have some previously insignificant problem become important? Does the model have to be adjusted?*

A simplified model (Figure 5) was presented by Kapur (1988: 5):

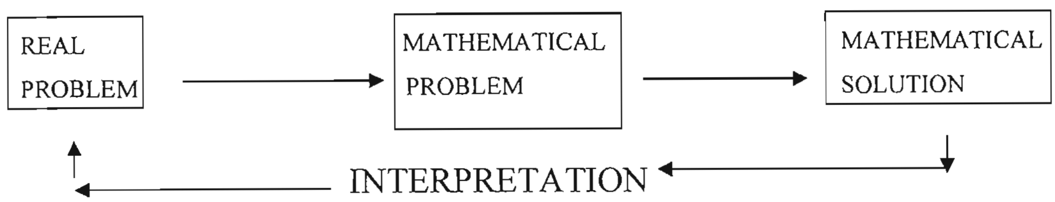


Figure 5

He defined this process (mathematical modeling) by stating that it "*essentially consists of translating real world problems into mathematical problems, solving the mathematical problems and interpreting these solutions in the language of the real figure*". His model above essentially describes his definition of mathematical modeling. This process of modeling was similarly described by Corbitt and Edwards (1979 : 218). Refer to Figure 6.

Corbitt and Edwards (1979 : 218) described their model by stating that a mathematical model consists of a list of variables that describe the real world situation together with the relationships that exist between these variables. A mathematical model is formulated using these variables and the subsequent analysis of the model involves solving the equations that are established. The last step in this model is the interpretation of the results obtained in the context of the real-world situation.

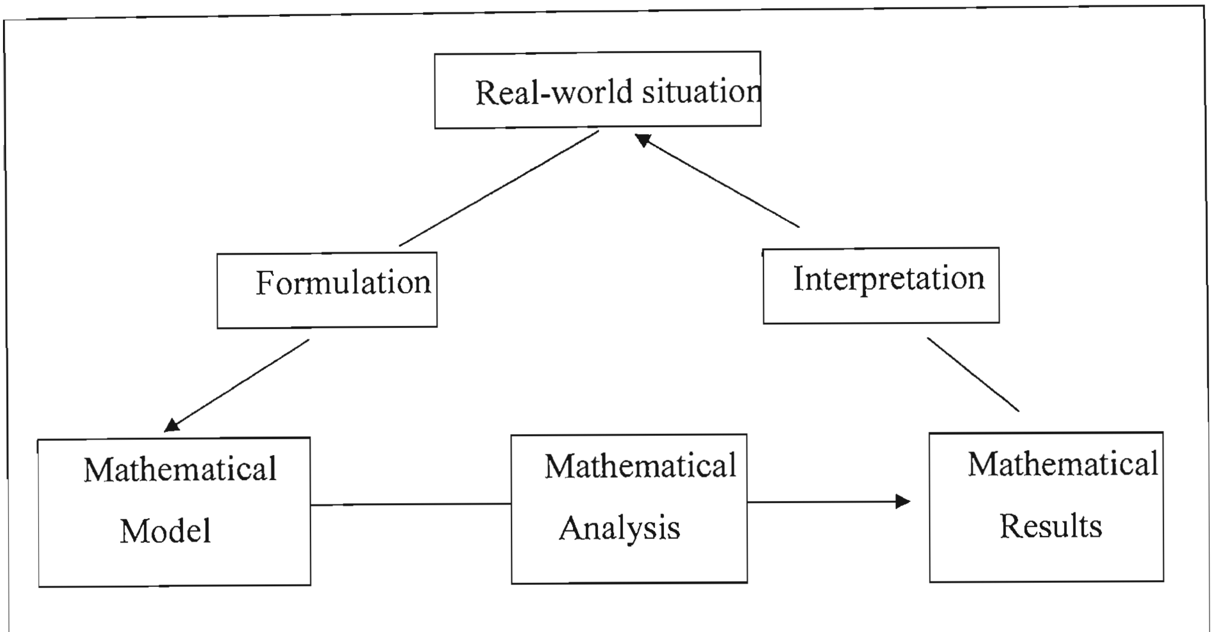


Figure 6

Dym and Ivey (1980 : 5-6) described this process of modeling in a similar way as those described above, but they divided the process into the Real World and the Conceptual World as shown in Figure 7:

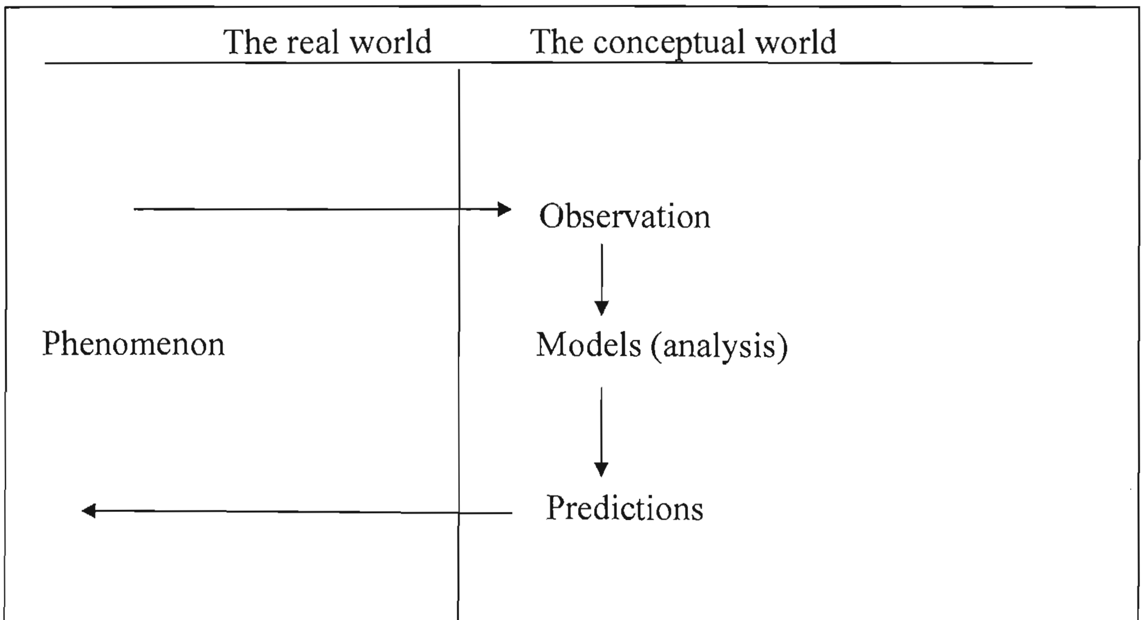


Figure 7

It is evident from the various models presented that there is no one particular methodology when modeling. However, there is broad consensus as to how modeling should be approached. In general terms, the following summary (Huntley and James, 1990, vi) will serve as good starting when determining modeling methodology:

1. Identify the problem. This may also mean understand the problem as well.
2. Formulate the model.
3. Investigate the model.
4. Validate the model.
5. Update the model.

The last point is important because modeling is a dynamic process and any changes to the model must be updated immediately.

Modeling is not easy and therefore some of the important points that need to be considered are listed below:

1. The model is not the real thing. "*The symbol is NOT the thing symbolised; the word is NOT the thing; the map is NOT the territory it stands for.*" (Hayakawa, S. I. As quoted by Dym and Ivy, 1980: 1)
2. Mathematical modeling is a skill that is applied when solving real world problems. There are no clearly defined rules in mathematical modeling. The following essential steps are required when modeling (Edwards and Hamson, 1989 : x):

- Identify the problem variables
- Construct appropriate relations between these variables
- Take measurements and judge the size of quantities

- Collect data and decide how to use them
- Estimate the values of parameters within the model that cannot be measured or calculated from the data.

3. Not every aspect of the system can be taken into account. There are aspects of the system that will have to be ignored. In other words "*the model may be partial, because it may never represent the entire system*" (Cross and Moscardini, 1985: 73). Corbitt and Edwards (1979 : 218) also mentioned the contradiction that might be experienced by the modeler: *If the model is so detailed that it fully represents the physical situation, the mathematical analysis may be too difficult to carry out. If the model is too simple, the results may not be realistic enough for meaningful or reliable use*. It is therefore essential that although not every aspect of the system may be taken into consideration, the modeler must ensure that vital aspects are not left out.
4. Each model is built for a specific purpose or problem. Caution must therefore be exercised when using a previously established model. It is often true that the circumstances surrounding the model may vary from one situation to another.
5. Off course, in general, real world problems cannot simply be converted into a mathematical problem. It would therefore necessitate some manipulation on the part of the modeler, although the important features of the problem remain intact. The model created may only have a "limited range of validity and should not be used outside this range" (Edwards and Hamson, 1989 :3).
6. Modeling is an intellectual activity that requires much time for the modeler to become proficient at it. If the result obtained does not correlate with that expected, it may be necessary to review the methodology used.

Modeling is sometimes much easier if computer models are used. Figure 8 summarizes the process of modeling when using a computer.

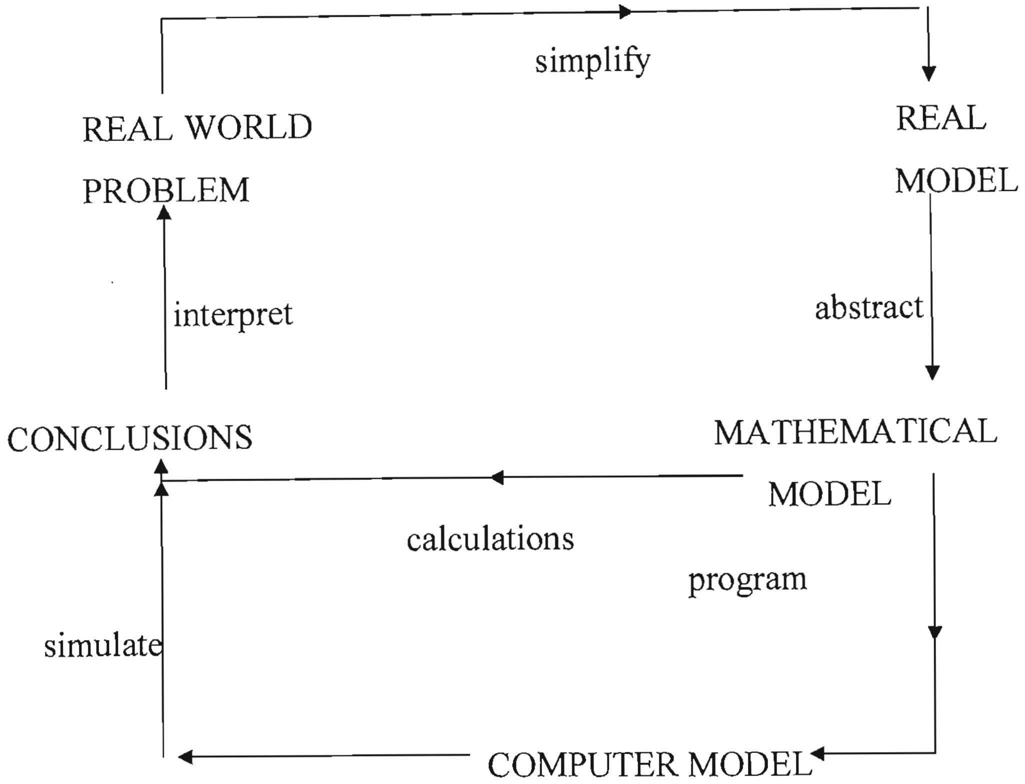


Figure 8

Figure 8 shows how a real-world problem is simplified and using mathematical language and equations it is reduced to a mathematical model. Without the use of a computer, calculations may reveal the conclusions that would make the real world problem much easier to understand. But, with the use of special computer programs, the model can be subjected to various tests within a few minutes. Although the program may not offer a detailed understanding of the problem, it will increase the conviction that the conclusion is valid or not.

Bender (1978 : 3) stated an important observation about the difficulty of modeling process: " *If the wrong things are neglected, the model will be no good. If too much is taken into consideration, the resulting model will be hopelessly complex and probably require* *incredible amounts of data. Sometimes, in desperation a modeler neglects things not because he thinks they are unimportant, but because he cannot handle them and hopes that neglecting them will not invalidate the conclusions*".

2.4 Principles and Standards for School Mathematics: Discussion Draft (Standards 2000)

The National Council of Teachers of Mathematics' draft discussion on the Principles and Standards for school mathematics contains essential literature related to mathematical modeling. The NCTM standards state clearly that modeling real-world phenomena is an integral part of mathematics as follows: "*a comprehensive curriculum is how well that curriculum affords students opportunities to learn about the nature and practice of mathematics. Students need to see that mathematics is a human enterprise that, although often abstract, has tremendous power in explaining and predicting real-world phenomena*" (NCTM, 1998 : 29). They go on further to state that "*one of the most powerful uses of mathematics is the mathematical modeling of phenomena*" and as a result of this "*it is important for students to model a wide variety of phenomena mathematically, beginning in the early grades*" (NCTM, 1998 : 60).

They justifiably argue that "*many of the symbolization-and-analysis activities in which students engage are a simplified form of modeling in which situations are characterized symbolically and then information about the situation is derived analytically*" (NCTM, 1998 : 99). They give two examples to illustrate this point (NCTM, 1998 : 99). The first is when learners are asked to determine the price of a magazine that will yield the highest profit, keeping in mind that every time the price of the magazine is increased the number of people who buy it will decrease. In order to solve such a problem the learner might have to express the profit in the form of an equation and then by using graphical and / or analytical methods determine the price that will yield optimal profit.

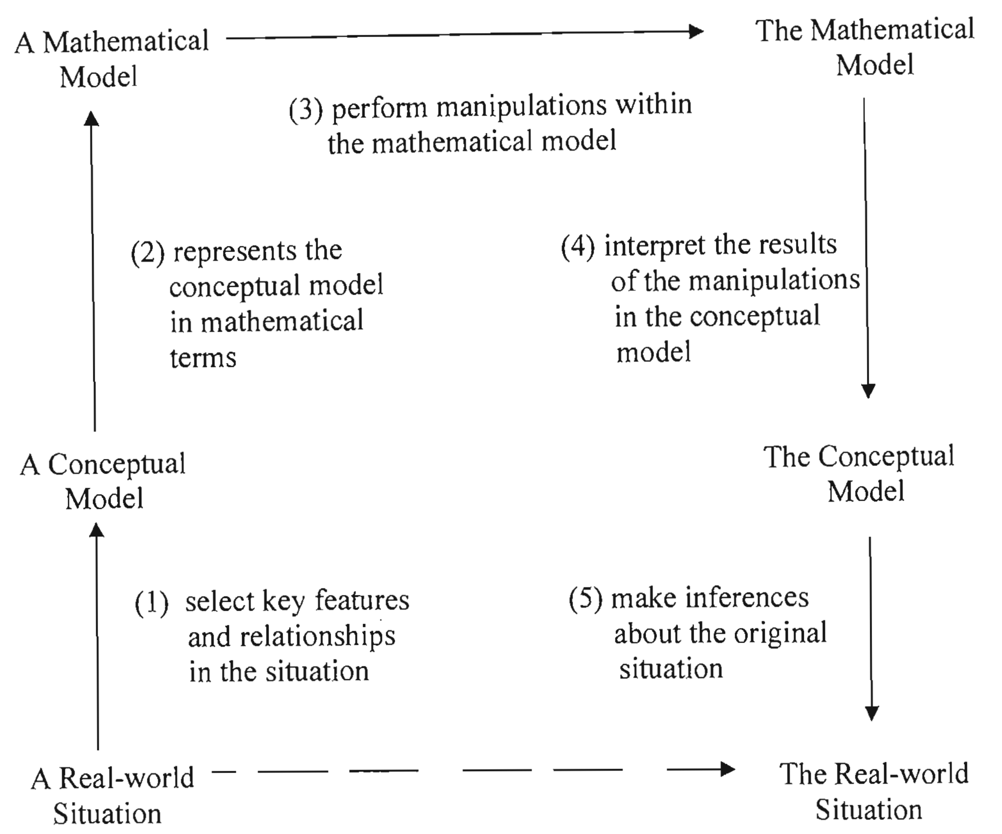


Figure 9

The second example shows how models allow a view of a real-world phenomenon through an analytic structure imposed on it. The question on traffic flow reads as follows: "How long should a traffic light stay green to let a reasonable number of cars to flow through the intersection?" Learners can gather data about how long (on average) it takes the first car to go through, then the second car, then the third and so on. They can represent these data statistically, or they can construct analytic functions to work on the problem in the abstract, considering the wait time before a car starts moving, how long it takes a car to get up to regular traffic speed, and so on. This kind of analysis allows for a simulation of traffic flow, which can be used to make judgments about how long a light should be left green.

The NCTM Standards represent this modeling process as shown in Figure 9 (1998 : 100).

2.5 Mathematical modeling in schools

2.5.1 The importance of modeling in schools

Most children do not use the mathematics they learn at school in solving real life problems. There may be many reasons for this, but a primary reason would be the fact that in many countries, mathematical modeling does not form part of the formal mathematics curriculum. In addition, "*applications*" if present, have usually been relegated to a secondary position. De Villiers (1994 : 34) describes the traditional status quo in South Africa as follows: "*The secondary school curriculum has traditionally focused almost exclusively on developing pupils' manipulative skills (e.g. simplifying, factorizing, solving equations, differentiation, etc.). This focus was in part due to the pervasive belief among teachers (and curriculum developers?) that such technical skills were essential* ***prerequisites*** *for problem solving and mathematical modelling, and therefore had to be mastered*".

However, it is evident that by the nature of its importance, mathematical modeling should really be taught at the school level, so that, children are exposed to many years of modeling experience. This is what Vatter (1994 : 396) is hoping for when he states that "*our hopes for education coincide with those for our children: we want to help them grow into adults* *who will make the world a better place for future generations. This goal might be a lofty one for a high school mathematics teacher, but it is important to keep our ideals clearly in mind as we develop ways to equip our students to live full lives and to make positive contributions to society*". In fact Vatter motivates for a curriculum which incorporates a '*real-life general mathematics course*' which he refers to as "*Civic Mathematics*".

A similar recommendation is being advocated by the National Advisory Committee on Mathematical Education (Pinker, 1979 : 31) in the United States of America: "*the opportunity be provided for students to apply mathematics in as wide a realm as possible - in the social and natural sciences, in consumer and career related areas, as well as in any real life problems that can be subjected to mathematical analysis.*"

Fishman (1993: 628) quotes from the *Curriculum and Evaluation Standards* (NCTM 1989), in order to motivate for the inclusion of modeling in the mathematics curriculum: "*The Curriculum and Evaluation Standards recommends that students learn to recognise and formulate problems, develop problem-solving strategies, and apply the process of mathematical modeling to real-world problems.*"

One of the reasons for the absence of modeling from the curriculum, according to Kerr and Maki (1979 : 1) is the identification of interesting contexts, yet of an appropriate level "*....because mathematical models provide the setting in which mathematics is applied. It is also appropriate because the difficulty in presenting applications in the classroom lies, not vith the mathematics to be applied but with finding interesting settings for those pplications which are at the right level for the students*". However, as pointed out in the evious section, a more important reason is probably the difficulty in teaching modeling as a process. It is imperative, nonetheless, that mathematical modeling "*should be taught as a process*" and not be considered as "*merely a collection of interesting problems and solution schemes*" (Swetz, 1991 : 358).

Whitman (1979 : 59) states the following: "*A goal of mathematics education is to enable students to relate the abstract ideas of mathematics to real or physical situations. This involves expressing a real situation in mathematical terms, manipulating the mathematics in order to gain insight into, and possibly some conclusions about, the real life situation, and then translating the mathematical results back into the real situation.*" Wallace (1985 : 81) similarly notes that "*students must see more than text book problems, because problems in reality do not look like textbook problems no matter how applicable or how creatively written they are*".

These ideas and recommendations effectively argue for the need for the inclusion of mathematical modeling in the secondary school mathematics curriculum. The idea here is to enable the learner to apply all that was learned at school, effectively and appropriately, in the communities in which they reside. Lindsay (1979 : 81) expresses the idea that in order to make learners more interested in mathematics the educator needs to " *bring the outside world into the classroom in simulated situations* ". Of course, she also motivates for mathematical modeling of real life situations to be included in the school mathematics curriculum.

Johnson (1979 : 148) from his experiences with classroom teaching makes three important observations :

1. *"mathematical modeling is an important and useful technique enabling one to manipulate the model to ascertain characteristics of the phenomena being studied"*,
2. *"certain modeling problems are appropriate to the lower levels of mathematics"*, and
3. *"realistic applications can provide additional practice on skills and techniques"*.

Johnson's third observation is important in that it points out that rather than practising skills and techniques in isolation, these can be practised within a modeling context.

The distinction must be made between ordinary word or story problems, which have very little meaning for the learner and problems from real life situations. Thiessen and Wild (1979 : 149) are clear about this when they state that " *the story problems found in elementary curriculum are often impractical and uninteresting because they have no relation to the students' world, either real or imaginary. Practical story problems based on the students' interests, activities, and environment might include situations involving plants, animals, toys, television programs, bicycles, pop bottles and cans, or fuel shortages*".

A further aid to mathematical modeling is computer simulation as argued by Wallace (1985 : 82) : "*computer simulations are an ideal substitute for actual on-site experimentation, which is usually impractical for the average class*". She goes on to state that: "*simulations provide the advantages of studying realistic situations while eliminating most of the problems associated with actual experimentation. Using models that imitate the essential characteristics of a situation, students are able to investigate complex phenomena at their own level of mathematical expertise*".

The dynamic geometry software *Sketchpad* shows a lot of promise as an effective simulation tool which can enable the learner to manipulate variables, whilst simultaneously observing the results. The empirical part of this research will investigate whether learners are able to work effectively and interpret meaningfully a given model in *Sketchpad.*

There is little doubt that mathematical modeling in schools has become as essential as teaching mathematics itself. Cockcroft (1982) in his report states that "*we would wish the word 'numerate' to imply the possession of two attributes. The first is an 'at-homeness' with numbers and an ability to make use of mathematical skills which enable an individual to cope with the practical mathematical demands of his everyday life*" (paragraph 39) and "*mathematics teaching at all levels should include opportunities for ... practical work ... problem solving and investigational work*" (paragraph 243).

In the Preface of their book "Teaching of Mathematical Modelling and Applications" (1991), Niss, Blum and Huntley reinforce the same idea: "*During the last couple of decades,* ***mathematical applications, models and modelling*** *have* ***attracted growing*** *attention in the theory and practice of mathematics teaching at all educational levels. It is fair to say that this theme has now become an established concern in mathematics education and also in the mathematics community at large. It is no longer necessary to fight for recognition of these aspects of mathematical activity and pursuit amongst mathematics teachers, educators and mathematicians. These aspects are now widely considered to be of crucial importance, whether viewed from an educational, a scientific, a societal or a cultural perspective. One piece of evidence of this may be found in the fact that applications, models, modelling, and applied problem solving have become* *increasingly central themes during the latest four international congresses on mathematical education, the ICME's".*

It is perhaps for this reason that the Danish educational authorities decided in August 1988 to change their mathematics curriculum in order to include mathematical modeling (as a compulsory component) in the Danish Gymnasium (high school). Hirsberg and Hermann (1991 : 178 – 180) state that "*this decision, together with others on the content of the mathematics curriculum and syllabus was the result of a long process comprising discussions among teachers and in-service educators, and the testing over a five-year period of an experimental curriculum published by the Ministry of Education*".

Educationists in other countries (for example Italy and Austria) are also starting to see the importance of modeling. Although modeling is not part of the Italian school's curriculum Bottino, Fircheri and Molfino (1991 : 203) state that the importance of modeling has been recognized. They state further that: "*as is well known, mathematical models constitute an important class of tools which allow real situations to be described and interpreted. By investigating these situations, students can be guided to examine them quantitatively, to recognise the common aspects of various phenomena, make assumptions from examination of the data available, and make predictions and vary the consistency of an assumption by analysing critically the result obtained*".

The Italian Educational authorities have begun testing teaching sequences with the aim of including mathematical modeling in the curriculum. In Austria reforms in the mathematics curriculum was introduced at the upper secondary school level called the *Real Gymnasium*

(Ossimitz, 1991 : 213). These reforms affected the grades 9 to 12 learners and the new section introduced is called *System Dynamics*, which is mathematical modeling.

Hirsberg and Hermann (1991 : 181) also give a short description of the aims of modeling that is being done in Denmark: "*The aspect of models and modeling aims at making the students familiar with the building of mathematical models as representations of reality. In addition, the instruction given should enable them to carry out a not too complex modeling process*". The purpose of doing modeling in Danish schools is quite clear. Firstly, modeling is done because it allows the learner to continue with further education, and secondly for the purpose of preparing the child for society. The second reason might be more relevant because of the number of learners who enter work places directly after completing their schooling. These learners would receive no further mathematics instruction so the knowledge acquired at school must be sufficient for the remainder of their lives.

In order to include mathematical modeling in the Danish curriculum, the Danes gave a long list of reasons to demonstrate the importance of teaching learners modeling. Hirsberg and Hermann (1991 : 182) state, with respect to the Danish curriculum and the inclusion of modeling, that : "*the curriculum is intended to help the students understand better the role mathematics plays in society (in the past as well as today) and how mathematics can contribute to the understanding and solution of problems met out of school. Mathematical models, in particular, provide the basis for many decisions in society and play an increasing role in scientific disciplines*".

Jones also argues that : "*What follows is a pedagogical approach designed to increase opportunities for students to engage in mathematical modeling but reduce the time needed by the teacher to reestablish the background for the problem. This approach uses the same real problem but varies the conditions or constraints on the problem in a number of ways. From a teaching-learning perspective it not only follows the initial exploration of the problem but is an extension of it*".

2.5.2 Teaching experiments involving modeling at schools

Research conducted in England by Treilibs, Burkhart and Low (reported in Salzano, 1983: 3), found that when children are taught modeling, the initial stage of the formulation presented the greatest problem. They further found that this might be a problem for the educators too. They identify certain modeling sub-skills that are important to enable learners to model real-life situations. These sub-skills are (in their own words) :

1. *Generating variables : the ability to generate variables or factors which might be pertinent to the problem situation.*
2. *Selection of variables : the ability to distinguish the relative importance of variables in the building of a model of the problem situation.*
3. *Identifying the questions : the ability to identify the specific questions posed by (typically ill defined) realistic problems.*
4. *Generate relationships : the ability to generate the relationships between the variables inherent in the problem situation.*
5. *Selection of relationships : the ability to distinguish the applicability of different relationships to the problem.*

Von Essen (1991 : 196 - 202) describes two curriculum projects that were undertaken by learners in a Danish Secondary school. The first involved a class of learners in the second/third form in 1983-1985. The learners in this project had to learn about modeling especially in industry. In the second form the learners learnt the basic mathematics (like differential calculus) required for the modeling process and aspects of the actual modeling. In the third form the learners worked with the modeling of actual problems in the fishing industry. The second project also involved a class of second/third form learners in 1987-1989. Here again the learners were introduced to the concept of modeling, but instead of working with only modeling over a prolonged period, the time was divided in a way where learners had to work with two different models (two thirds of the time) and then they worked in groups for the remainder of the time. For the two models the learners worked with the models of the structure of the structure of the stars and the other dealt with fishery models (one project had to have some social significance). In the last third of the time the learners worked in groups and they could choose from any theme from the national economy, business economy, population dynamics, fisheries and chaos.

At the end of both the projects the learners had to answer a questionnaire. Almost every learner expressed high degree of satisfaction with the modeling process and stated that they had benefited from the instruction. The learners were happy to have been able to work on their own. In the first project the learners felt that there was too much independence. This resulted in the second project being changed so that learners could work together as well. The learners also commented that they gained sufficient skills and insight during this process and they found the study of models relevant and interesting. An added comment on their questionnaires was the fact that they found the application of mathematics on location important and relevant (they visited the Department of Fisheries and the Astronomical Institute of the University of Aarhus).

Von Essen (1991 : 201) also makes the following cautionary comments regarding the teaching of modeling and the use of projects:

1. *"The learners learn too little compared with the time invested".* This problem could be overcome by proper planning and by incorporating modeling into the normal syllabus. This would imply that as the learners work in the different sections in mathematics, real-life questions must be posed. This will force learners to use the mathematical knowledge they already learned.
2. *"It takes too much time from the teaching of (pure) mathematics".* There is little doubt that teaching modeling will impact on the length of the traditional mathematics syllabus. But when one measures the relevance of modeling in the real world some aspects of the traditional syllabus has to be sacrificed. Perhaps a more relevant suggestion would be to increase the amount of time allocated to mathematics instruction.
3. *"It cannot be tested in a reliable way".* In real life, modeling is conducted in many aspects of society, for instance, building of bridges and so on. How does one test the success of the modeling? The answer lies in the actual product - the bridge that is built. However, there may be many different solutions and assessing learners' work therefore becomes difficult.
4. *"The teachers' knowledge of different applications is inadequate".* Educators ought therefore to be trained in the various aspects of modeling and applications on a continuous basis.

In another experiment conducted to understand children's ways of modeling non-standard (or real world) problems, Nesher and Hershkowitz (1997 : 281 – 287), worked with 480 children of grades 4, 5 and 6 from 15 different schools in Israel. Different children had to work with a single problem and there were six problems altogether. The second problem given to 84 of the children was: *For dinner in a summer camp 17 pizzas were ordered. Some were large and others were small. Each large pizza was divided among four children, and each small pizza was divided between two children. How many children were in the camp, and how many pizzas of each kind were ordered?* The problem had to be solved by each child individually and they were instructed to find different possibilities for the solution. The complete instructions were to draw the story as described in the text. Also to explain in detail the line of thinking, either verbally, by drawing or by mathematical sentences and to write how they would explain the solution to another friend.

The results showed that 79% of the children coped with the problem offering some solutions to the problem. In determining the degree to which they coped, Nesher and Hershkowitz discovered that 47% of the children were able to give one correct solution, 17% were able to give several correct solutions and 2% were able to give solutions that showed a systematic method of inquiry. It was therefore clear from this experiment that even young children in primary grades are able to cope with the non-standard problem given to them. More importantly, the children were able to bring real life considerations into their solutions. For example, they made assumptions that some children would get more than others or they talked about different types of pizzas (that is, different toppings) and for the problem where the number of pizza portions was unknown, the children immediately assumed that the large pizza yielded eight slices, and the small one yielded four, as is common knowledge in Israel. A possible explanation forwarded by Nesher and

Hershkowitz for the large percentage of children who gave only one solution is that in real life one makes only one order at a pizza shop – no one places more than one order for the same number of pizzas.

In an Australian report based on the evaluation and monitoring of a modeling course, Galbraith and Clatworthy (1990 : 143 - 162) made a few significant discoveries. The problem that the learners had to investigate is summarized as follows. The State Government in Brisbane planned to build a road under one of the hills in a place called Bardon. Due to financial reasons, only a single lane (in both directions) could be built. This meant that during peak hour traffic (in the mornings and afternoons) there would be severe traffic flow problems. In order to reduce traffic congestion it was decided that road signs should be erected, advising the road users about the maximum speed that a motorist may travel at and the safe distance to be observed between cars. The learners were then asked to make recommendations.

The written responses of the learners are significant in motivating for teaching modeling at schools. Despite finding the idea of changing the real life problem to a mathematical one difficult, the learners found the act of finding a solution to a real life problem most interesting. They felt that it was a great sense of achievement at the end. There is little doubt that learners feel motivated when they feel that their efforts have attained some success. The learners also felt that the most useful aspect of the entire exercise was the modeling process itself – "*modeling itself because I can now look at a problem and express it mathematically*" or "*the use of modeling or scaling down a problem from real life to a workable situation are both useful in the world outside math*". Many of the learners had changed their opinion of mathematics after going through this course because "*maths is useful and I've learned the human nature of maths*".

These responses indicate that given modeling opportunities together with some guidance, learners enjoy working through real life problems. One of the unexpected findings of this investigation was the fact that the learners were able to use the modeling experience gained in mathematics in other subjects and in their hobbies.

2.5.3 Real world problems at schools

As has been mentioned already, real world problems sometimes do enter South African school classrooms. But there are issues that need to be looked at when learners are asked to work with them. If the problem is to be realistic then it must make sense to the learners. Very often these problems are not very well planned and result in some assertions that may be quite unlikely to be true.

An example of such a question appeared in the November 2002 Matric Examination paper for the standard grade learners in South Africa. The question (an exact quote from the examination paper) reads as follows: *In the accompanying diagram, PT is a 2 m high chalkboard which is 1 m above eye level, QR, of a learner at R. QR = x metres. The angle of elevation of the top of the chalkboard, from R is* θ *and* $\angle PRT = \alpha$. *Prove that* $x = \dfrac{2\cos\theta.\cos(\theta-\alpha)}{\sin\alpha}$ *and if* $\theta = 50^\circ$ *and* $\alpha = 30^\circ$, *calculate, rounded off to* *one decimal digit, how far the learner is sitting away from the wall.*

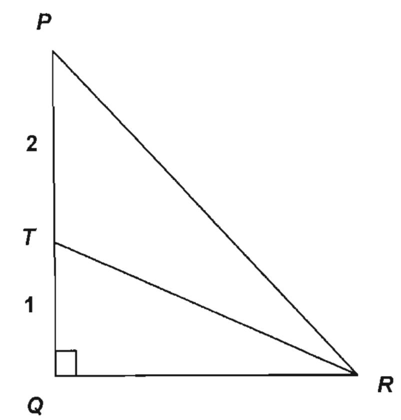


Figure 10

A few points need to be discussed here. Firstly, if the chalkboard is 2 m above eye level, then it implies that the highest part of the board will be 3 m above **eye level**. This would therefore mean that the educator would have to stand on a table to write on the board! Secondly, the learners were asked to calculate the distance the learner is sitting away from the wall. The assumption that is being made is that the chalkboard is in fact in line with the wall. Learners in most South African schools know that this is not really true. Chalkboards are placed against the wall, and in most cases, it protrudes at least 30 mm away from the wall. This should not be a problem if the learners were told in the question to ignore the fact that the board will protrude from the wall. Better still, if they were asked what assumption(s) were being made in the problem (for example, the wall is perpendicular to the floor, the board is in line with the wall, and so on).

But if learners were taught modeling there might be some changes needed when dealing with a mathematical solution, because real world conditions often require consideration of other factors. An example that demonstrates this argument relates to a question like this: *A boy stands 12 km away from a mountain and an angle of elevation of 30^0 is formed as he looks at the top of the mountain. He walks 6 km towards the mountain and looks up at the top and an angle of 45^0 is formed. What is the height of the mountain?* The assumptions that are made are never discussed with the learners, for example, the mountain range is vertical or the road that the boy walked along was totally flat (otherwise 6 km further on he could have been standing on a hill or a valley).

When working with real world problems one needs to understand in most cases assumptions are made in order to simplify the problem at hand. In most cases the real world solutions are only minimally affected by these simplifying assumptions, but learners need to be made aware of them. The problem is that in some cases the simplifying assumptions can lead to completely erroneous solutions. A classic example is the late nineteenth century "proof" that a heavier-than-air flying machine was mathematically impossible (refer to the following websites for information http://www.flyingmachines.org, http://www.ctie.monash.edu.au/hargrave/ader.html and http://www.world-mysteries.com/sar_7.htm. Although the mathematics was correct it was based on the false assumption that air flowing around a moving object behaved like a fluid (for which good models already existed).

2.6 Examples of modeling

Before any examples are discussed it is necessary to state that when solving problems through modeling, various strategies are available to the modeler. The strategies may be one of using a table of data, a formula, graphs, scale drawings, dynamic scaled sketch (by using dynamic geometry programs), a system of linear equations and so on. Although there are many strategies available, the modeler may only be able to use one or two of the strategies when solving a particular problem. Here are three simple examples that use different strategies.

2.6.1 Example 1:

A taxi driver A in Durban charges a basic fare of R1 and 40c for every kilometre or part thereof. What does a journey of 163,5 km cost? A second driver B charges a basic fare of R5,50 and 35c per kilometre or part thereof. Which taxi driver charges a lower tariff? This method of charging taxi fare is not the way it is actually done, but it would be used for the purpose of illustrating a specific method of solving problems.

DISCUSSION: These are a few of the assumptions that are made when answering this question.

- The speed at which the taxis travel is irrelevant.
- The type of taxi does not matter whereas in many instances people are prepared to pay more to travel in a taxi that is mechanically safer as compared to traveling in a taxi where the brakes does not function properly.
- The route each taxi travels may not necessarily be the same provided the distance is the same.

This problem is fairly simple. The type of model is not already specified. The child would have to create or use an old model to arrive at the solution.

One possible way for young learners to solve this problem would be to suggest using a table:

For the moment ignore the basic fare of R1 for Taxi A and R 5.50 for Taxi B.

Km	3	0.5	60	100	163.5
Cost (R) Taxi A	1.20	0.20	24	40	65.40
Cost (R) Taxi B	1.05	0.175 ≈ 0.18	21	35	57.23

Now the child will add the amount of R1 to the total thus giving the solution of R66.40 for Taxi A and R 62.73 for Taxi B. Thus showing that Taxi B is cheaper!

Another method that may be employed by older learners is to develop an equation that incorporates all the given information. This equation might be written as:

Taxi A : Cost = 1 + no. of kilometres x 0.40 where cost is in rands.

Taxi B : Cost = 5.50 + no. of kilometres x 0.35 where cost is in rands.

All the child needs to do now is substitute into the formula and obtain the answer of R66.40 for Taxi A and R62.73 (62.725 is rounded off) for Taxi B. But the important aspect of modeling that there may be many models that may be developed or used in order to arrive at the final solution. If not prescribed, modeling allows the learners to be as creative as they possibly can. In this example the question was relatively simple.

2.6.2 Example 2

A manufacturer makes two types of soft drinks for a catering company in 100 litre units. The first type Pocola has to circulate in mixer A for 5 hours and in mixer B for 1 hour. The second type, Impassionate, has to circulate in mixer A and in mixer B for 2 hours each. Mixer A is available for 16 hours each day and mixer B is available for 8 hours a day. If

Pocola is sold at a profit of R8 per litre and Impassionate is sold at a profit of R10 per litre, calculate the quantity of each kind that has to be made daily to give a maximum profit.

DISCUSSION: The question does not refer to other expenses such as labour, water and electricity bills, rent and so on.

The common model used to find the optimal solution is based on the application of a system of linear equations. In the beginning the modeler might draw a table in order to simplify the interpretation of the given values. This may be done as follows:

Product	No. of 100 l units	Time in Mixer A	Time in Mixer B	Profit per 100 litres
Pocola	x	5	1	R800
Impassionate	y	2	2	R1000
		16	8	P

The table can thus be interpreted as : $x \geq 0$ and $y \geq 0$ because we are talking about quantities of soft drinks.

Also, $5x + 2y \leq 16$ and $x + 2y \leq 8$.

The Profit equation can be written as $P = 800x + 1000y$.

Now using this system of linear equations, graphs can be drawn and the optimum results obtained. By letting P = 1000 a profit line (search line) can also be drawn, and then varying the values of P one looks for its intersection with the feasible domain which gives the maximum profit.

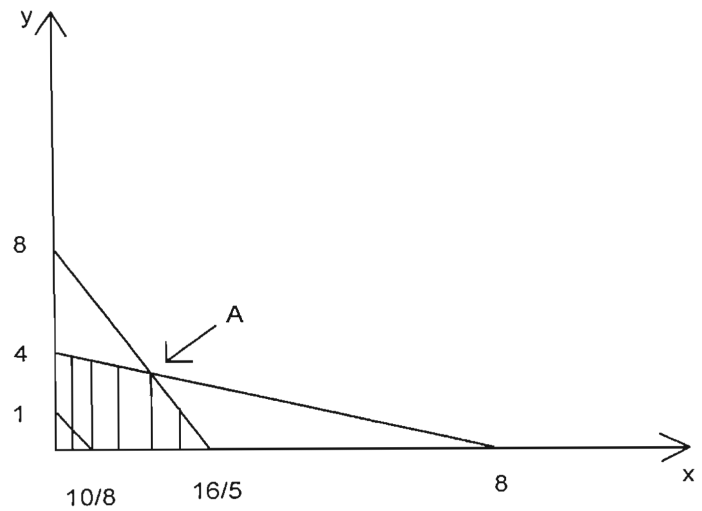


Figure 11

By using the search line it is clear that the solution lies at coordinate A. This coordinate can be obtained from an accurate sketch of the graphs or by solving simultaneously. The coordinate of A is (2 ; 3).

This implies that for maximum profit, under the constraints given, 200 litres of Pocola and 300 litres of Impassionate must be manufactured in order make a maximum profit of R4 600.

Of course, one does not need to use an algebraic method to solve this problem and simple guessing and testing with a calculator would suffice. A spreadsheet could also easily be used to avoid the repetitive, tedious calculations of guessing and testing.

2.6.3 Example 3

A stone is thrown up into the air at a velocity of 100 m/s. Calculate the maximum height reached, and describe the motion of the stone using graphical methods. You may assume that the gravitational constant g is 10 m / s^2.

DISCUSSION : There are a few assumptions that are made in this question. These are as follows: the height of the person throwing the stone is negligible as compared to the height attained by the stone, the stone is thrown vertically upwards, and the flight of the stone is not hampered by other factors, such as resistance created by air, wind or rain and so on.

This question can quite easily be done using a simple spreadsheet such as Excel. Spreadsheet modeling is the process of entering the inputs and decision variables into a spreadsheet and then relating them appropriately using formulae to obtain the outputs. This process allows for easy graphing of important relationships.

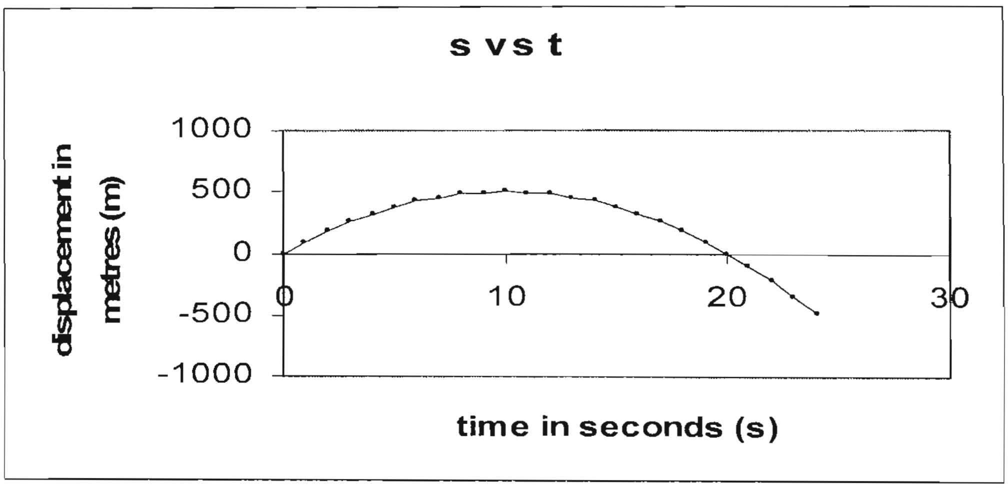
In attempting to model this problem using a spreadsheet it is necessary to firstly create a table of values as shown below. Often, the values are determined by using formulae, which is easily interpreted by the spreadsheet.

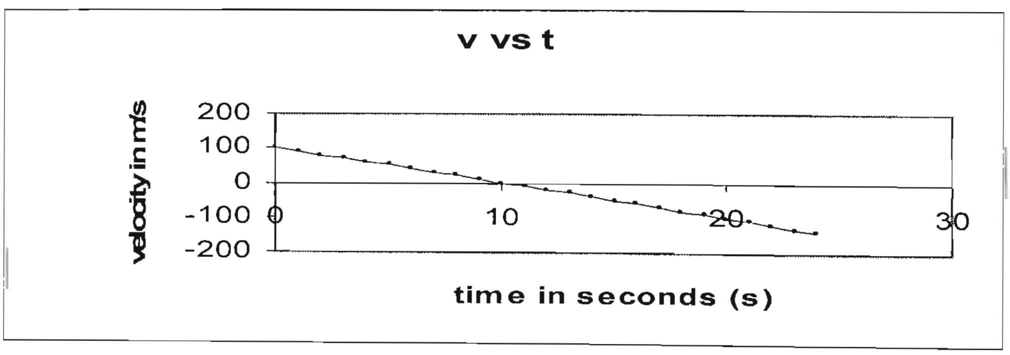
Time	displacement	time	velocity	time	acceleration
0	0	0	100	0	10
1	95	1	90	1	10
2	180	2	80	2	10
3	255	3	70	3	10
4	320	4	60	4	10
5	375	5	50	5	10
6	420	6	40	6	10
7	455	7	30	7	10
8	480	8	20	8	10
9	495	9	10	9	10
10	500	10	0	10	10
11	495	11	-10	11	10

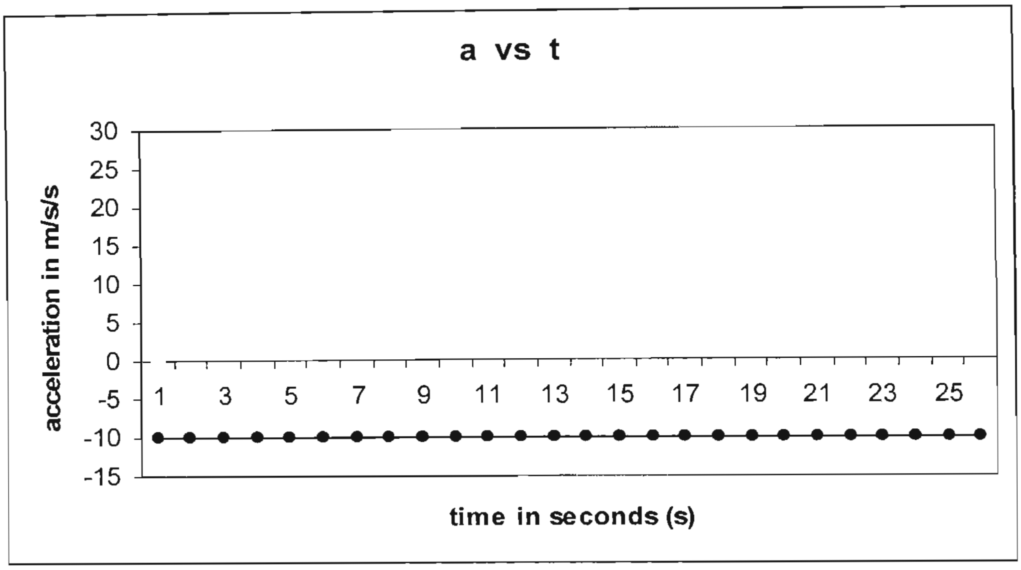
12	480	12	-20	12	10
13	455	13	-30	13	10
14	420	14	-40	14	10
15	375	15	-50	15	10
16	320	16	-60	16	10
17	255	17	-70	17	10
18	180	18	-80	18	10
19	95	19	-90	19	10
20	0	20	-100	20	10
21	-105	21	-110	21	10
22	-220	22	-120	22	10
23	-345	23	-130	23	10
24	-480	24	-140	24	10

Table A

After the table has been constructed graphs can be drawn by selecting the variables required to draw a particular graph. For instance, for the displacement versus time graph, the first and second columns would be highlighted and then using the Chart function a graph can be drawn.







From the first graph above, it can be seen that the maximum displacement of the stone is 500 metres. This can also be calculated using the available (or derived) formulae $s = ut - \frac{1}{2}gt^2$ and $v = u - gt$, where s is displacement, u is the initial velocity, v is the final velocity, t is the time taken and g is the gravitational constant (10 m /s^2). It is also very clear from the graphs that the stone is constantly decreasing in velocity as it reaches the maximum height. In the tenth second the velocity of the stone is 0 m / s. This is the point at which the stone reaches its maximum height. Thereafter the velocity of the stone increases (negatively / downwards) until it reaches the ground.

2.7 Limitations of mathematical modeling

In her article The Dangers of Mathematical Modeling, Köhler (2002 : 140) argues that extrapolating mathematical results from insufficient data "*is a dangerous practice and can* *lead to erroneous predictions*". She conducted an experiment by asking her learners to investigate three different models to describe the growth of the population in the United States of America since 1790. All three types of functions used to determine the models worked equally well with the census data available. According to Köhler, none of the functions had an obvious mathematical flaw. But all three models led to radically different predictions. One predicted that the population would become extinct; the second and third models predicted a population of 288 million and 1.4 billion respectively.

There were many reasons postulated for these discrepancies. The first was the fact that only five data points were available after 1940. This meant that a curve described by so few data points had little meaning for any predictions. There were other factors, which were not considered, but they also affected the population. These are namely the increase in immigrants into the United States, the improved agricultural system might accelerate population growth, and off course no one can state what the effects of wars and natural disasters may mean to the population size.

Clearly Köhler's argument is an important one, but it cannot be used as a means of negating the use of modeling. Obviously, in most modeling situations, adequate data must be available. Insufficient data only tends to give a skewed representation of the reality under investigation. As can be seen from the above paragraph, many factors affected the result. All that this argument does is that it reinforces the idea that a model must replicate a real life problem as closely as possible. Bringing in too many factors may make the model unwieldy to work with. But leaving out essential aspects may present a false solution.

Booss-Bavnbek (1991 : 71) also presents another argument when he warns against the simplistic use of mathematical modeling because of "*the superficial similarity which mathematical models display with respect to their empirical foundation and scientific extra mathematical status, the extent of mastery of the mathematical concepts involved, and the pragmatics of the application situation.*" He goes on further to state that "*these differences are not necessarily decisive for the scientific or practical value or for the credibility of the model, but the adequate choice of assessing their value and credibility requires attention to all these aspects*".

He argues that many of the items used in mathematical modeling are merely man-made "*human constructions*" (1991 : 72), for example the formulae and constants (like Newton's gravitational constant) we use in solving the problems. But this argument becomes invalid, as the value of g can be experimentally verified. Indeed most scientific formulae and constants have been rigorously tested over a period of time and have become accepted as scientific facts. Many theories are based on such formulae and constants and their validity can hardly be in question.

His next argument may be more relevant. He uses the example of modeling the time of maximal weight output in a trout pond. In arriving at a solution he uses the mortality equation ($\frac{dN}{dt} = -MN$) and the metabolic equation ($\frac{dW}{dt} = hW^{2/3} - kW$). But he argues that the solution to this problem is of a tentative character. This is due to the continuous change that occurs in the trout pond. Many factors influence the survival of trout in the pond (like temperature, fishermen, other animals, budgetary constraints). It is difficult to reduce a complex large-scale system to simple equations. This argument is valid, as it is almost impossible when working with any real-world problem to consider every factor that may exert an influence on the resolution of the problem. There is little doubt that care must be taken when modeling, in order to ensure that the major influencing factors are considered. No system will be foolproof, but modeling does allow for a fairly credible outcome if the problem is modeled with care.

Booss-Bavnbek (1991 : 76) also states that modeling cannot be effective if there is an absence of a good theoretical foundation. This is what he calls "*ill-founded and irresponsible modeling*". He provides the following examples to illustrate his view that it is characteristic that the calculations may be acceptably descriptive under normal circumstances but worthless under special circumstances. He claims that the following disasters occurred due to the misguided confidence in the modeling process and technology, and not having a thorough theoretical foundation.

1. *The nuclear disaster at Harrisburg in Chernobyl.*
2. *The Challenger disaster.*
3. *In ship losses due to control failures.*
4. *The crashing of a fully loaded new Airbus during its maiden flight.*

This argument, however, is flawed. In the cases mentioned, Booss-Bavnbek does not take into account human error. There are many nuclear plants in the world but he quotes one disaster. Many space shuttles have successfully returned to earth after trips to space. How many ships and airplanes are lost daily? Very few or none as compared to the number of them out at sea or in the air. In all of the cases he uses in his argument human negligence may have been the cause of the disaster. Failure to recognise a malfunctioning instrument or contractor's performing shoddy work in the manufacturing process does not necessarily mean that the modeling process was incorrect. Without doubt, good-modeling techniques allow the modeler to gain immense confidence in his or her work.

Presently there are no alternate suggestions to replace mathematical modeling as a viable method of stimulating and solving real world problems. Though Boos-Bavnbek's argument is true that as many extraneous factors as possible that could affect the result should be removed, one should be careful not to condemn a method that has been shown to be a useful means of applying mathematics to real world problems.

Boos-Bavnbek argues (1991 : 77) further that "*teaching mathematical modeling seems biased to idealized, completely unrealistic, unrepresentative and misleading 'success' stories. Owing to the pedagogically well-intended selection of examples, those chosen are generally very misleading. They are too simple, too smooth, giving the student the wrong impression of formal concepts matching reality. The students do not see the true problems in modeling, in simplification and generalization. In the formulation, analytical treatment and numerical solution of equations, in the interpretation and control of the results – or they see them only in a systematically reduced form*".

It would appear that the above criticism is aimed more at the poor teaching of modeling than at modeling process as such. It is simply bad teaching if children develop "*misguided confidence*" in modeling and the erroneous belief that models always perfectly match and predict reality. This is also true if teaching indicates that modeling always results in one solution only. Modeling must be taught in a manner that would allow the child to see the various possibilities as solutions, taking into account various factors that can influence the solution. The following simple example below illustrates this point.

Suppose a group of 100 people are to be transported to a party by taxi. How many taxis are required to transport all of them? There are many possible solutions to this problem:

1. If cost is not a factor then each person could be transported in a different taxi. In other words, 100 taxis could be used! Or 50 taxis could transport 2 passengers each or 10 taxis with 10 in each, and various other combinations.

2. Dividing 100 by 12, we get $8\frac{1}{3}$ taxis. Since it is impossible to get $\frac{1}{3}$ of a taxi, 9 taxis are required. Eight of the taxis can carry the normal number but the remaining taxi will take only four passengers. Or 8 taxis can carry 11 passengers but the remaining one carries 12 passengers.

3. Or, if we require $8\frac{1}{3}$ taxis, then maybe 8 taxis should be used. Four of the taxis will carry 12 passengers whilst the other 4 carry 13 passengers. However, this may be against the law, but it saves money.

If children are taught that even when modeling correctly, divergent solutions may result, then they will not develop a "*misguided confidence*" in modeling. If in the above example the extraneous factors affecting the use of the taxis were given, then a satisfactory solution could have been arrived at. If, for example, the question listed cost as an important factor then a particular solution would have resulted. Or if the question specified that all passengers were very fat, then squeezing them into taxis will not be a good thing. Perhaps if the question stipulated that each passenger would be carrying a sizeable amount of luggage, then it could be assumed that more than 12 passengers in a taxi will not be ideal.

In the above it is obvious that there isn't a "best" solution. The ideal solution will be dependent on the precise question to be answered. Similarly there cannot be a "best" model. There may be a model that works well in the way it predicts from the given facts, but there may be other models making different predictions depending on the questions being asked. It must be noted that there is a clear distinction between the model and the real situation because mathematical models are to a large extent artificial. But in many cases it is difficult to determine where the model deviates from reality. In general, researchers ignore the fact that a mathematical model is not the reality they are investigating, because the predictions they derive often compares well with the real situation.

Boos-Bavnbek (1991 : 79) does concede that teaching mathematical modeling is necessary but "*we must find ways to explain the open or hidden but decisive differences in the structure and status of mathematical models and calculations to our students and to the public*". Learners should realize that a model is an imperfect representation of reality because not every extraneous factor can be taken into consideration when modeling. There will be anomalies, but to a large extent models work and because the process is dynamic, new and better models are developing continuously. Models are not perfect, but they do have strong predictive values.

Although this aspect has already been mentioned in paragraph 2.4 it is necessary to raise the point again. A serious concern of educators when teaching of modeling is the fact that there aren't too many questions available, which are suitable for use in the classroom. If modeling is to be closely related to the learners' immediate environment, then questions that could be used must be determined from within those communities. Eyre (1991 : 291) answers this worry with the list of possibilities shown in Figure 12.

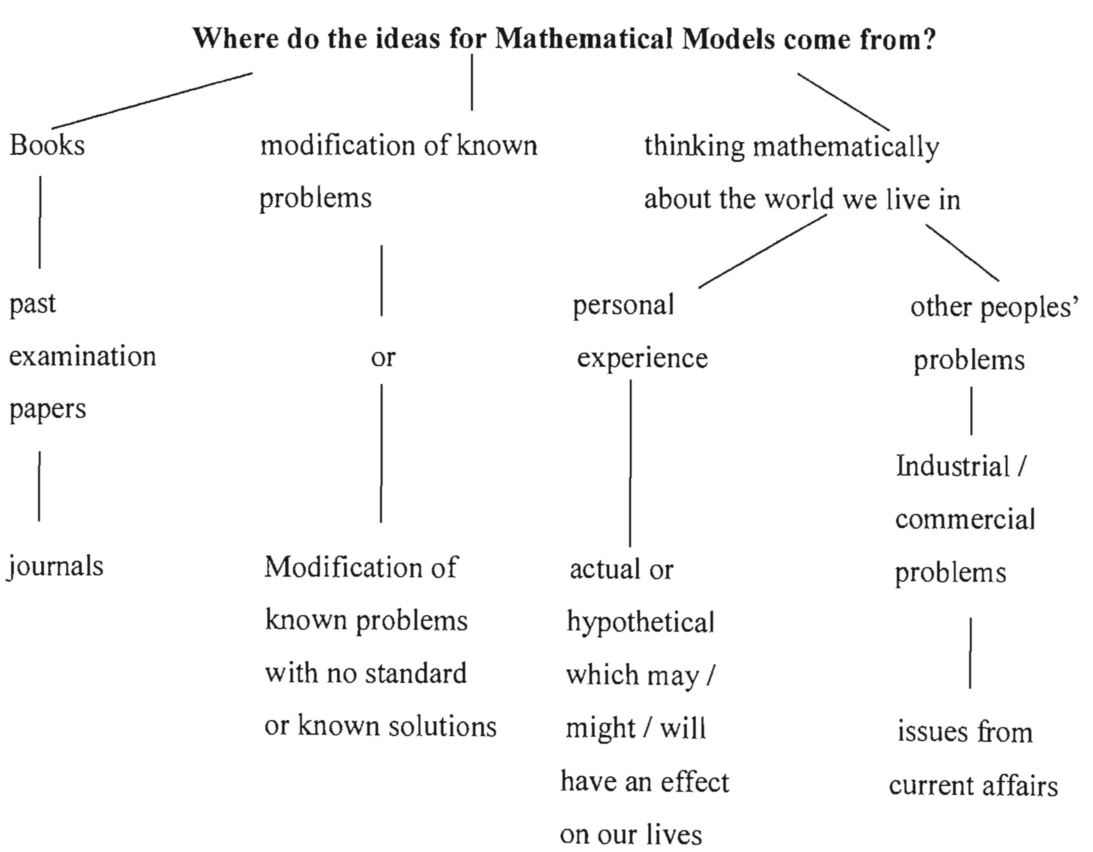


Figure 12

It is noteworthy that the History of Mathematics has been left out, but it seems to be a crucial source of finding "*suitable*" questions.

2.7.1 The role of technology in mathematical modeling

"*By helping people visualize and experiment with mathematical phenomena, modern computing technologies have changed the way all people learn and work. In schools, they can influence how mathematics is learned and taught not only by making calculations and graphing easier but by altering the very nature of what is important to learn. New problems, as well as new ways of investigating all problems, become accessible.*" These statements by Cuoca, Goldenberg and Mark (1995 : 236) indicate clearly the power of technology in the mathematics class. They are clear about technology affecting the "very nature" of the way children learn and dynamic computer programs offer its users a tremendous opportunity of being able to interact with the program so that learning is made easier and the learner is able to immediately visualise the various scenarios and change and adapt the pictures they see in order to reaffirm what they learn.

This view is also reiterated by Arora and Rogerson (1991 : 114) as follows : "*The development of the electronic computer has made possible many new fields of applications of mathematics. The increasing availability of computers in the classroom has had a tremendous impact on the expectations of the computational skills required of all citizens and the type of application problems that can be introduced. In fact, the very meaning of the word solution has undergone a radical change and, with the availability of these new technologies, a more pragmatic, creative and open-ended approach to problem solving can now be adopted in the classroom*". They go on further to state that there is very little doubt that access to computers and computing power (which includes dynamic geometry software) has made possible the teaching of mathematical modeling and applications to real-life problems in the classroom and this will increasingly become the case. A similar view was expressed by Norcliffe (1991 : 121): "*Mathematical modeling has undergone a revolution with the advent of computing. Not only is there the raw computing power of the super computer, to assist with numerical work, and enhanced computer graphics, to see what computations mean, but there are now sophisticated computer algebra packages to help with analytical work*".

As was stated previously, there are many computer programs available which not only makes modeling much simpler, but can greatly assist in the analysis and solution part of the modeling process. If the developed model cannot be solved using ordinary mathematical skills then quite often computer programs are available that can solve it numerically. Otherwise, the computer can be used to simulate the real-world problem and the resulting solution (from the computer) can be re-interpreted back into the real-world situation. De Villiers (1994 : 34) illustrated the modeling process as shown in Figure 13.

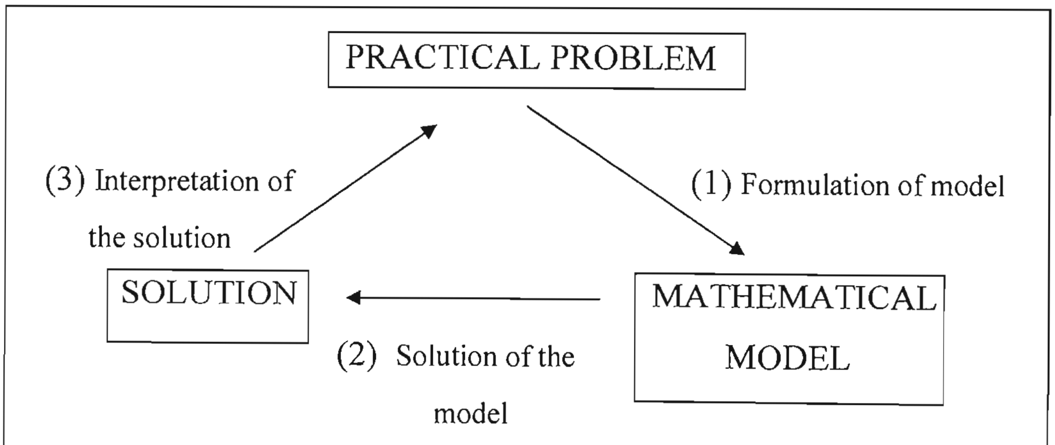


Figure 13

De Villiers (1994 : 34) argues that, in many cases, the computer is of very little assistance in the first and the last steps. During these two steps human ingenuity is required, but it is in the second step (solution stage) that the computer has become a very useful tool : "*The* *availability of computer software (and calculators) that can aid us with the second step therefore strongly challenges the traditional approach which emphasizes technical and manipulative skills at the cost of developing skills in model construction and interpretation. In the global society where computers in the workplace are becoming more and more pervasive, one's "**matheracy**" should no longer be measured so much in terms of one's ability to do routine manipulative skills by hand, as in competency in steps (1) and (3), as well as with proficiency in handling modern technology (for example computer software, calculators, etc.) during step (2).*

In work done by De Villiers (1994 : 41), real-world problems were given to undergraduate and postgraduate learners. In most cases the learners were initially at a complete loss as to how these problems should be solved. They were then encouraged to use scale drawings and other numerical and graphical approaches, and though they managed to solve the problems, they found the tediousness encountered quite frustrating. When the problems were finally investigated using different types of computer software, the learners really appreciated the power of technology.

The usefulness of computer technology in real-world problem solving has been given much attention in the Systemic Initiative For Montana Mathematics and Science (SIMMS) program in the United States of America. "*The SIMMS curriculum emphasizes problem solving, mathematical reasoning, real-world applications, and the appropriate use of technology*" (Cuoco, Goldenberg and Mark, 1991 : 238). The University of Chicago School Mathematics Project (UCSMP) also advocates the use of technology in the mathematics classroom: "*technology is used because of its power to perform mathematical tasks, because it facilitates the learning of concepts ....*" (Cuoco, Goldenberg and Mark, 1991 :

238). The SIMMS program has equipped each classroom with computers and various computer programs, including *The Geometer's Sketchpad.* The UCSMP has equipped each classroom with computers and computer programs such as *GeoExplorer* and *GraphExplorer*.

In the Danish mathematics curriculum, computers play an important role; in fact, a new topic was introduced at Danish schools called computer mathematics. "*Computers are essential for many aspects of modeling*" (Hirsberg and Hermann, 1991 : 185). Clements (1991 : 351) also supports the idea that the computer is an essential component of mathematical modeling when he states that "*modern microcomputers offer a powerful resource to carry out these functions in support of the teaching of mathematical modelling*".

The value of Dynamic Geometric Environments (DGEs) on computer lies not only in the dynamic way in which a real world problem can be modeled, but also in the interactive way in which solutions can be arrived at. Hazzan and Goldenberg (1997 : 49) state that "*DGEs enable one to construct geometrical figures by specifying certain relationships among their components. A distinguishing feature of such environments is their 'dragging mode', which allows one to manipulate geometric constructions by dragging various parts (like points, segments, etc), while preserving the specified relationships. One can then study the unanticipated invariants – properties and relationships that were not explicitly specified, but are consequent to those that defined the construction*". Here they refer to, for example, the concurrency of perpendicular bisectors of triangles. They are clearly referring to the construction of geometric figures and then refining of the figures to observe relationships in the diagram, which would otherwise be impossible in other environments.

Clearly, computer software packages can be valuable mediating artifacts that support mathematics learning. Jones (1997 : 121), for example, points out that "*such a package* (like *Sketchpad*) *allows the user to experience the* ***direct*** *manipulation of geometrical objects (or, at least, the* ***appearance*** *of such direct manipulation)*". Despite the creativity afforded by such packages, it does not completely assist the individual learner to develop chains of thought in understanding and reasoning. The individual is expected to interact with the software in order to develop new ideas and concepts. Although an explanation for the truth of certain characteristics of a geometric figure may be inherent in the figure itself, the learner needs to seek out such explanations because the software is merely a mediating artifact. This therefore does not remove the responsibility of learning away from the learner.

The technology involved in modeling is not limited to computers and computer software. The ordinary calculator and the graphics calculator are also a useful tools. Anderson et al (1999 : 496) state that "*one advantage of student access to graphics calculators is that it allows assessment tasks to be designed around real-world problems that would previously have been regarded as too lengthy or tedious for students equipped with only pen and paper or a standard statistical calculator*".

At the most elementary level computing technology allows one to tackle problems that are more realistic. For example, traditionally problems in the primary school were "simplified" in the sense that real data was usually always rounded off to integers and numbers chosen so that the calculations work out nicely. For example, traditionally primary school learners would not need to divide a 10-digit number by a 6-digit number. Now with the availability of calculators, one need not shy away from problems which involve large numbers and / or cumbersome calculations.

Similarly, the availability of computer algebra software such as *Mathematica*, *Derive* and *Maple* that can factorize, differentiate or integrate polynomial functions, offers one the opportunity to tackle far more realistic and sophisticated problems one would normally not even attempt by pencil and paper means.

However, most importantly, technology makes possible alternative curricula possible with radically new learning sequences. For example, one could approach a conventional topic such as linear functions from a modeling perspective as follows:

Stage 1: Where do linear functions arise in the real world?

- Constant speed versus time
- Weight versus extension of a spring
- Linear revenue, profit, tax and so on.

Stage 2: How does one solve an equation like $mx + c = 0$ or $m_1 x + c_1 = m_2 x + c_2$?

- Guess-and-test successive approximations
    - By hand
    - By graph
    - By calculator or computer
- Formal solution by computer
    - Mathematica
    - Derive
    - Maple

The availability of powerful technologies therefore seriously challenges the traditional "theory first – applications later" approach, as one can start with the real world context and develop the theory later.

2.7.2 The effectiveness of *Sketchpad* as problem solving tool

Sketchpad can be used, in various ways, to effectively solve mathematical and other questions and problems. It should be pointed out at the outset that in traditional classrooms, learners generally work with *static* diagrams. These are figures that are rigid and cannot be moved about in order to change the size. These types of diagrams are the type that learners draw in their books or see in their textbooks. No manipulation of the diagram can take place and in order to view a larger or smaller version of the diagram a new diagram has to be drawn. On the other hand we have *dynamic* computer sketches, which allow for these diagrams to be distorted and changed, subject to the conditions imposed on it. These dynamic sketches can be moved, enlarged, made smaller or specific changes can be made as required. Dynamic sketches allow for many different diagrams to be viewed within a few seconds, having the same conditions imposed on it. It allows for learners to quickly observe patterns and trends. *Sketchpad* is one such software that allows for the drawing of dynamic sketches.

Below are listed some of the different types of problems and uses of *Sketchpad.*

2.7.2.1 "Discovery" of new ideas for learners.

In an experiment quoted by Hirschhorn and Thompson (1996 : 140 – 141), learners were asked to explore relationships from drawing an isosceles trapezoid ABCD and its diagonals AC and BD, which intersect at E. Left to their own devices they generated many conjectures, some of which were:

a) The diagonals are equal.
b) Angle AEB = angle CED.
c) The area of triangle AED = the area of triangle BEC.
d) The perpendicular bisector of AB is the perpendicular bisector of CD.

All these "discoveries" were correct. In fact a few learners used the calculator in *Sketchpad* to show that Area EBC x Area AED = area AEB x Area DEC (refer to Figure 14). This conjecture was thrown open to the entire class, and before a proof was found, a different group of learners found that by distorting the isosceles trapezoid, the result is true for any quadrilateral.

Area EBC = 6.3 cm^2
Area AED = 6.3 cm^2
Area AEB = 4.2 cm^2
Area DEC = 9.4 cm^2
(Area EBC)·(Area AED) = 39.4 cm^4
(Area AEB)·(Area DEC) = 39.4 cm^4

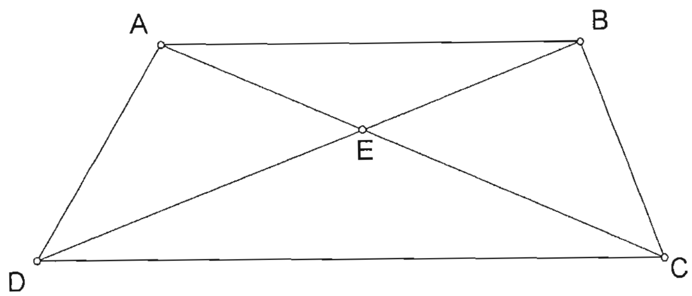


Figure 14

2.7.2.2 Checking conjectures with *Sketchpad* and producing counter-examples

Due to the dynamic nature of *Sketchpad*, learners can easily reason and explore mathematical ideas. The manipulating tools available allow for the distortion of figures according to the reasoning of the learners. Zbiek (1996 : 86) relates an interesting problem which demonstrates that *Sketchpad* can be used to check conjectures. The "*pentagon problem*" appears on the screen of the computer as follows. Three pentagons are constructed in the following way. An arbitrary pentagon is drawn. The second pentagon is constructed by joining the midpoints of the sides of the original pentagon. The third pentagon is constructed by similarly joining the midpoints of the second pentagon. Appearing on the screen are the measurements for the area and perimeter of each of the pentagons. The ratio of the areas and perimeters of the consecutive pentagons are also listed.

From the evidence available it seemed that the ratio of the areas of the consecutive pentagons are equal and similarly for the ratios of the perimeters (refer to Figure 15.1). Therefore a conjecture was developed, which stated that as long as each of the two smaller pentagons are formed by joining of the midpoints of next larger pentagon, the two ratios of the areas will always be equal, and the two ratios of the perimeters will always be equal, no matter what shape the largest pentagon has. The question then asked was : Do you think that this conjecture is valid? Why or why not?

This problem could be approached using several methods. The first method would be to simply drag a vertex around in order to get another pentagon. By doing so many such pentagons can be obtained and the resultant ratios could be observed. It would be clear from observation that this conjecture does not hold true for all pentagons. The second method would be to construct a fourth pentagon using the same method mentioned above. Determining the ratio of the fourth pentagon with that of the third may yield different results. This process may have to be repeated a few times until a discrepancy arises (see Figure 15.3). The third method would be to change the "preferences" function from the nearest tenth to the nearest thousandth. This answer is slightly more precise than the other. The differences in the different ratios will become apparent (see Figure 15.2). Therefore the conjecture is not true.

One could also demonstrate logically that the conjecture is false by using analytical geometry.

Perimeter p1 = 22.6 cm
Perimeter p2 = 18.2 cm
Perimeter p3 = 14.7 cm

$$\frac{(\text{Perimeter p1})}{(\text{Perimeter p2})} = 1.2$$

$$\frac{(\text{Perimeter p2})}{(\text{Perimeter p3})} = 1.2$$

Area p1 = 34.0 cm^2
Area p2 = 22.2 cm^2
Area p3 = 14.5 cm^2

$$\frac{(\text{Area p1})}{(\text{Area p2})} = 1.5$$

$$\frac{(\text{Area p2})}{(\text{Area p3})} = 1.5$$

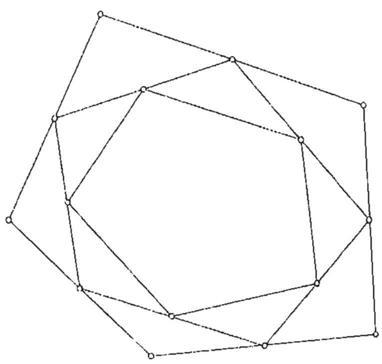


Figure 15.1

Perimeter p1 = 22.643 cm
Perimeter p2 = 18.199 cm
Perimeter p3 = 14.713 cm

$$\frac{(\text{Perimeter p1 })}{(\text{Perimeter p2 })} = 1.244$$

$$\frac{(\text{Perimeter p2 })}{(\text{Perimeter p3 })} = 1.237$$

Area p1 = 33.987 cm ^{2}
Area p2 = 22.168 cm ^{2}
Area p3 = 14.502 cm ^{2}

$$\frac{(\text{Area p1 })}{(\text{Area p2 })} = 1.533$$

$$\frac{(\text{Area p2 })}{(\text{Area p3 })} = 1.529$$

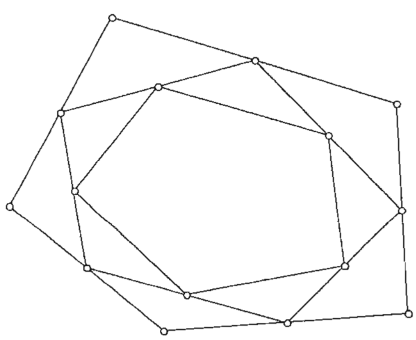


Figure 15.2

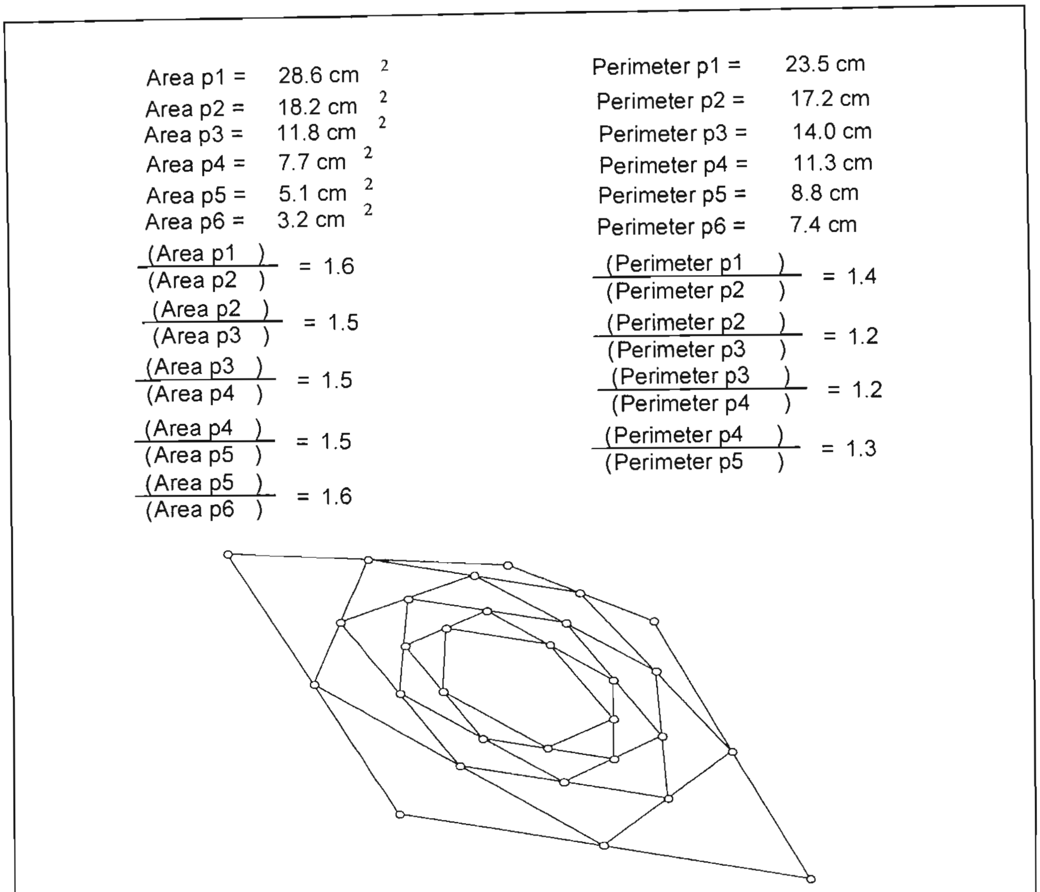


Figure 15.3

2.7.2.3 Using *Sketchpad* for conjecturing

The Geometer's Sketchpad is a useful tool when engaging learners in the exploration of highly complex theorems and relations in geometry. Unlike the traditional method of teaching, where a sample sketch is placed on the board or the learners themselves draw a sketch in their books, *Sketchpad* allows the learners to construct many similar diagrams and observe patterns that are not always obvious using traditional methods. Furthermore, learners can record their constructions as scripts. Scripting constructions has many benefits. The first is that the learner can use the scripts to recall the process that was used in the construction and secondly, the "*students can test whether their constructions work in general or whether they have discovered a special case*" (Giamati, 1995 : 456).

In classroom experimentation, Giamati (1995 : 456) showed that using the "*Geometer's Sketchpad in this type of exploration was invaluable. The students gained a deeper* *understanding of the problem by using their scripts to explore it and make conjectures than they would have if the results had merely been explained to them*". Learners' manipulations allow them to arrive at conjectures quite easily. Although this does not prove a result, the high levels of conviction they develop makes the determining of a reasonable explanation much easier. This is due to the fact that "*explorations that give students the opportunity to make reasonable conjectures and deepen the students' understanding*" of the problem being investigated.

2.7.2.4 Using *Sketchpad* to model in Science

Sketchpad can easily be used to model different aspects in Science and nature, for example the movement of the planets, sun and moon, showing clearly the elliptical trajectories and the different pathways followed by the planets and other heavenly bodies in the universe. This is can be viewed from the Sample Presentations that are provided with the purchase of *The Geometer's Sketchpad*, called the *Cosmos*. This is made easy by the fact that *Sketchpad* has an animation function, which allows for the user to create various kinds of animation.

It is this, together with the Trace function that allows for the simulation of various kinds of motion, for example, the motion and speed of a boat as it is pulled towards the dock or projectile motion. Since Sketchpad handles vectors quite easily, any vector problem in the real world is easily modeled. Making use of the animation function the educator of physical science can demonstrate the Doppler Effect quite easily. These and other such examples may be downloaded from the following website:
http://www.nas.com/~kunkel/gallery/gallery.htm.

The formation of images by a convex lens is an easy example that can quite easily be modeled dynamically using *Sketchpad*. The convex lens can be drawn and the focal length and twice the focal length indicated on both sides of the lens. An object represented by the upward pointing arrow is constructed. Line segments representing rays emanating from the top of the arrow is drawn as depicted in the diagrams below. By dragging the object on the left one can observe the image on the right. The sequence of diagrams below represent the object at different lengths away from the lens. Diagram 1 shows the object greater than two focal lengths away creating an inverted image between one and two focal lengths away. Diagrams 2, 3, 4 and 5 show the object moving closer to the lens and the type of images formed.

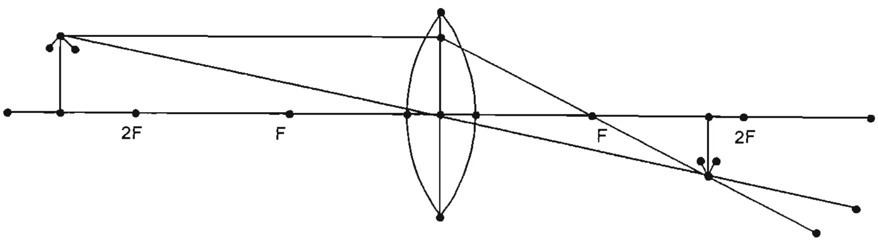


Diagram 1

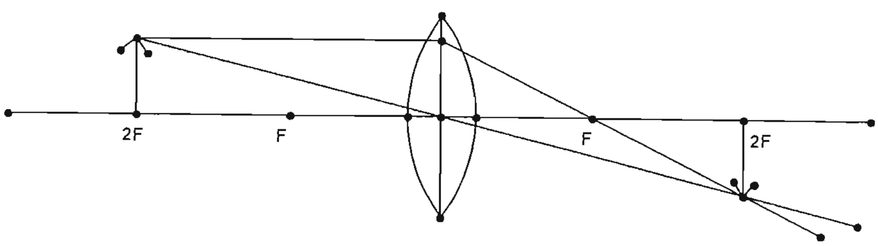


Diagram 2

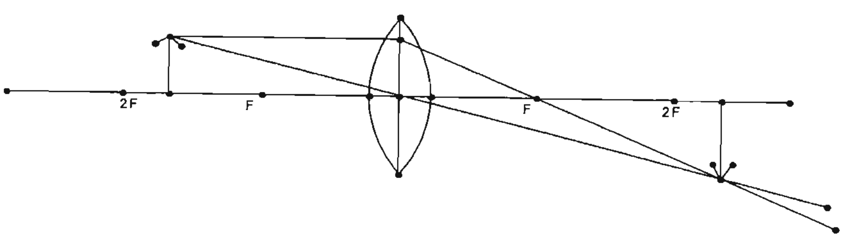


Diagram 3

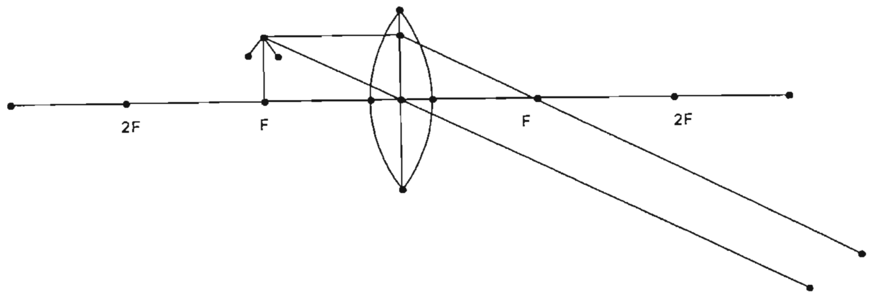


Diagram 4

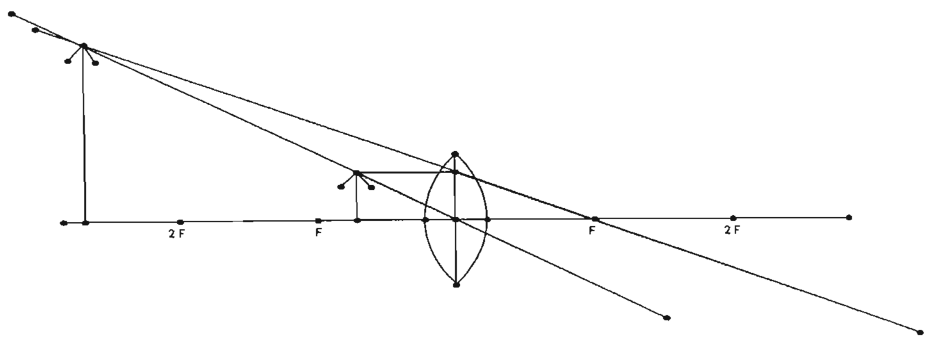


Diagram 5

2.7.2.5 Using *Sketchpad* for determining loci

Sketchpad can be easily used to determine various loci in mathematics. Its greatest use comes via the fact that the locus is no longer an abstract formula but a visual entity that can be seen. This type of work with loci can be extended to real world problems as well. An example of such an application of *Sketchpad* is given in by De Villiers (1999), where a real world problem in rugby is investigated using *Sketchpad.* The problem is as follows: Suppose a try is scored in rugby at point T, then the kicker can kick to the posts from anywhere along line TF perpendicular to the try line PQ. Where is the best place to kick from? (refer to Figure 16.1).

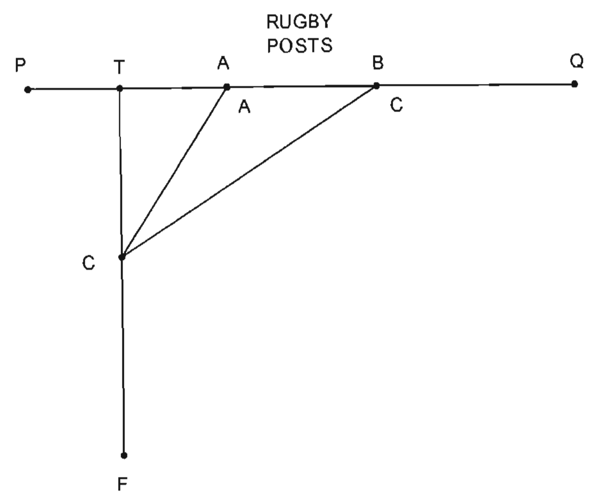


Figure 16.1

It is obvious to rugby place kickers that the ideal position will be when the angle ACB is the largest. This problem is easily solved by constructing the diagram 16.1 using *Sketchpad*, measuring the angle ACB and then dragging C along the line TF until we have the largest angle ACB. This would be the ideal point to kick from.

Another way of solving this problem using *Sketchpad* is to construct a series of circles with AB as chord (refer to Figure 16.2). Remembering the theorem that states that the angles subtended by an arc in the same segment are equal and the noticeable fact that as the circles get larger the angles on the segments increases, it is easy to see that the solution will lie at the point of tangency. A complete discussion of this problem and its solution can be obtained from http://mzone.mweb.co.za/residents/profmd/rugby.pdf.

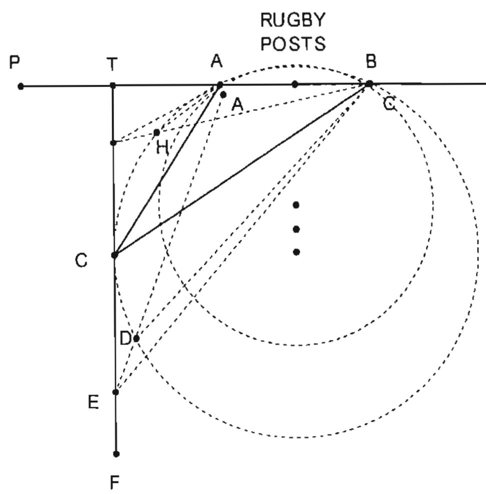


Figure 16.2

Furthermore, De Villiers points out that *Sketchpad*'s trace or locus construction facility can easily be used to investigate the locus of "ideal or best positions", which turn out to be a hyperbola, as shown in Figure 16.3.

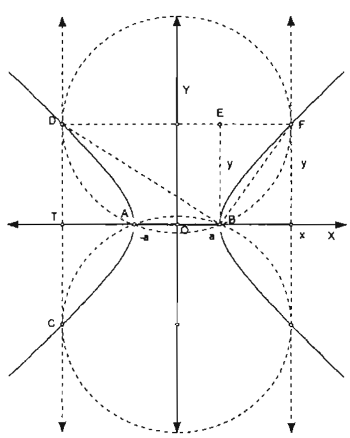


Figure 16.3

2.8 Modeling of real-world problems as a starting point for proof

Modeling should be seen as the very first stage, the beginning of the proving process. Yet proof is often only seen as a means of simply verifying the truth of mathematical statements. According to Hodgson and Riley (2001 : 724) a common perception is that *"proof and real-world problem solving are typically considered to be separate and distinct endeavors"*. It has always been difficult to gauge the relationship between real-world problem solving and proof, yet the clear value of real-world problem solving in the process of proving cannot be underestimated.

Hodgson and Riley (2001 : 724), for example, contend that *"our experience has been that real-world problems supply an important ingredient that seems to be missing from typical classroom instruction on proof. As such, real-world problems may actually be one of the most effective contexts for introducing and eliciting proof. Real-world problems are commonly used as vehicles to introduce or deepen students' understanding of mathematical concepts and relationships"*.

Hodgson's and Riley's argument that real-world problems could be the basis for mathematical proof stems from one step in the modeling process, namely the testing of the solution. They believe that it is essential for the learners to ask *"why is the statement true?"* after they have arrived at the solution. The problem given to their learners was (Hodgson and Riley, 2001 : 724) :

A pool player notices that the centre of the cue ball is positioned exactly two feet from one side of the table and exactly four feet from the centre of the target ball. She wishes to bank the cue ball off the side of the table so that, if it were possible, the centre of the cue ball would pass through the centre of the target ball. Note that the centre of the target ball is also two feet from the side of the table and that both the cue ball and the target ball have a radius of $1\frac{1}{8}$ *inches. At what point on the wall should the centre of the cue ball be aimed to successfully complete the bank shot?*

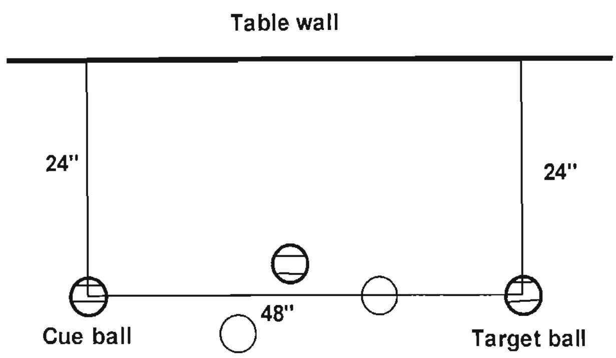


Figure 17

The learners used various methods to arrive at a solution and all learners arrived at the same solution: the cue ball should be aimed at a point on the wall that is halfway between the target and the cue balls. To test their solutions the learners used a hands-on approach. They went to the billiards room and using as much accuracy as possible, they discovered that their solutions were incorrect. This therefore initiated the question "Why?". This became the starting point of a proof. They realized that there were various other factors that had to be considered, for example, the spin of the cue ball.

The question "why?" solicited enough curiosity for the learners to want to work through a proof. There are various functions of proof as pointed out by De Villiers (1990). Some of these are verification, systematization, discovery, communication and self-realization, but the over-riding desire that these learners were demonstrating, was the need for proof as a function of explanation. They wanted to know "why?" their solutions were incorrect. They undoubtedly wanted to know *why* the result was true or false and not *whether* the result was true or false. This is similar to the findings in Mudaly (1999), where research showed that learners have a need for an explanation (deeper understanding) which is independent of their need for conviction. Later in the discussion of the empirical data, it will again be shown that learners do express a desire for wanting an explanation despite the fact that they may be quite convinced of the truth of a conjecture. It is important that learners be encouraged to look at the proof of why a conjecture may or may not be true. This is easily demonstrated by the example in subsection 2.7.2.2, that there may be examples, which seem to be correct but by further investigation the untruth of the conjecture can be shown. Further examples are also listed in Mudaly (1999).

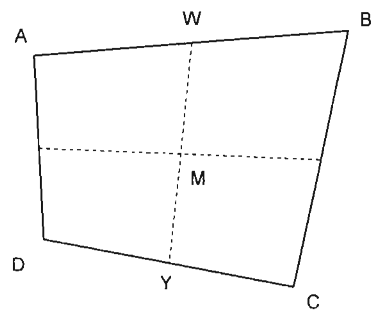
Similarly, Klaoudatos and Papastavridis , discuss a teaching experiment based on Context Oriented Teaching (COT). COT according to Klaoudatos and Papastavridis (forthcoming : 1) is "*a model based on a problem solving framework and on the selection of the appropriate task context*". They observed that COT provided the learner, who had little understanding of the mathematics involved in solving a particular problem, a starting point and a sense of direction (Klaoudatos and Papastavridis, forthcoming : 4). Essentially they conclude that starting with a Context Oriented Question (which is a real world question), the learners use Context Oriented Heuristics to develop Context Oriented Concepts. Context Oriented Conjectures are formulated, which lead to Context Oriented Proofs. Despite framing their arguments within the idea of contexts, they still show that as a starting point to this proof is the modeling activity.

A further significant argument which shows this direct link between modeling and proof is made by Blum when he stated that applications in mathematics (solving of real world problems) provide contexts for what Blum refers to as *reality-related proofs* (page 1). He clearly points out that "***formal** proofs are mostly the final stage in a genetic development – historically as well as epistemologically as well as psychologically*". The researcher therefore contends that formal proof is the final stage of a series of events which ought to begin with the process of modeling.

Another argument related to proof and modeling is given by Hanna and Jahnke (2002 : 1). They explore proofs in a classroom using principles established in physics, such as the centre of gravity. It is their argument that using physics "*adds*" something to the understanding of the theorem, but it does not necessarily give the "*real*" or "*best*" explanation for it. In fact they found (2002 : 8) that "*many students felt that the experiments were **part** of the proof*". This indicates that modeling and experimentation does not necessarily enhance the learners' understanding of the mathematical problem. However, a significant finding by Hanna and Jahnke (2002 : 8) was that "*from the students' responses, most of them found the proof from physics to be convincing, as well as clearer and more readily remembered that the geometric proof*".

In order to illustrate the usefulness of a physics proof, consider the Varignon Theorem, which states that, the midpoints of any arbitrary quadrilateral forms a parallelogram. A purely geometric proof entails constructing one diagonal, joining the midpoints and then proving that one pair of opposite sides of the newly formed quadrilateral is parallel and equal to the constructed diagonal. This then leads to the conclusion that joining the midpoints creates a parallelogram.

Hanna and Jahnke (2002 : 1) use physics to explain the result of the Varignon Theorem as follows. "*Consider the points A, B, C and D (the four vertices of the arbitrary quadrilateral) as weights, each of unit mass, connected by rigid but weightless rods. Such a system, with a total mass of 4, has a centre of gravity. The aim would be to find this centre* *of gravity.. The two subsystems AB and CD each have weight 2, and their respective centres of gravity are their midpoints W and Y. From static considerations we may replace AB and CD by W and Y, each having mass two. But AB and CD make up the whole system of ABCD. Its centre of gravity is therefore the midpoint M of WY. In the same way we can consider ABCD as being made up of BC and DA. Therefore the centre of gravity must also be the midpoint of XZ (where X and Z are midpoints of BC and DA. Since the centre of gravity is unique, this midpoint must be M. This means that M cuts both WY and XZ into equal parts. Thus WXYZ, whose diagonals are WY and XZ, is a parallelogram".*



If learners work with modeling activities then the evidence is pointing to the fact that they seem to increase their understanding of the solution and develop a need to extract an explanation for what they observe.

CHAPTER THREE

Problem Solving, constructivism and realistic mathematics education

3.1 Introduction

This research was conducted in a problem-solving context, underpinned within a strong constructivist framework. The problems that the learners were exposed to were situated in the real world and a problem-centered approach was followed. A main focus of the research was to investigate whether learners would be able to construct their own knowledge through a modeling process using *Sketchpad.*

This chapter therefore discusses the theoretical framework within which this research was embedded. The process of problem solving in the real world and modeling are further analysed, as well as the constructivist underpinning of current efforts to implement modeling in schools. A brief overview of the van Hiele theory is given and some links are made to the process of modeling.

3.2 Problem solving in a real world context

The NCTM (1998 : 243) statement that "*problem solving is the process by which students experience the power and usefulness of mathematics in the world around them*" adequately encapsulates the purpose of doing problem solving in schools. One of the main reasons for doing mathematics would be to solve real world problems. This is what Lester (1980 : 29) implied when he stated that "*there is support for the notion that the ultimate aim of learning mathematics is to be able to solve problems*". Although he does not specifically talk about solving problems in the real world, it is clear that the entire world is pervaded by all kinds of mathematics (financial and economic sectors, industry, computers and so on) and therefore one of the aims of doing mathematics is to solve problems in the real world. When businessmen use graphs, spreadsheets, computer programs and so on to attempt to maximize profit and minimize cost, they use mathematical modeling tools to solve their everyday problems. So problem solving is part of daily life.

The Sixth Standard as determined by the Standards 2000 Writing Group (NCTM, 1998 : 76) states that "*mathematics instructional programs should focus on solving problems as part of understanding mathematics so that all students –*

- *Build new mathematical knowledge through their work with problems;*
- *Develop a disposition to formulate, represent, abstract, generalize in situations within and outside mathematics;*
- *Apply a wide variety of strategies to solve problems and adapt the strategies to new situations;*
- *Monitor and reflect on their mathematical thinking in solving problems.*"

The second bullet represents the idea of also working in real world situations. It conveys the idea that "*people who have developed a mathematical point of view of the world tend to act in ways that are mathematically productive*" (NCTM, 1998 : 77) and this may be extended to the notion that mathematically productive people may be highly productive people in the work environment.

There is a similar strain of thought in Nunokawa's (1995 : 722) belief that problem solving and mathematical modeling are connected when he stated that "*mathematical problem solving is the activity to connect the real world and the mathematical world*". Branca (1980 : 3) also cited "*problem solving as a goal (if not* **the** *goal) of mathematics learning*" and he went on to say that problem solving involves "*applying mathematics to the real world*". Schoenfeld (1980 : 15) similarly stressed the importance of problem solving when he stated that "*the problem solving process is one of the most important aspects of mathematics with which teachers should be concerned*".

This idea of solving problems isn't specific to mathematics. According to Polya (1980 : 1), problem solving is "*human nature itself*". The Standards 2000 Writing Group (NCTM, 1998 : 134) also stated that "*problem solving is natural to small children*". People have their days filled with obstacles that are not immediately attainable, and will therefore require skills to overcome these obstacles. In this process, human beings are problem solvers. As a problem solver one has to overcome obstacles by using information and techniques that one has already experienced or learned. If that is not possible, one then creates a new method. This is in fact similar to the process of modeling. The processes of mathematical modeling and problem solving have much in common. The main difference, according to Treilibs (1979) lies in "*the objectives of each process: the first seeks to solve problems in order to make sense of a situation, the second sees problem solving as an end in itself*".

As a result of the widespread concern for learners' inability to solve problems, school mathematics curricula have been adapted to include problem-solving strategies. There has been a perception that problem solving is difficult at school level and therefore it has been traditionally avoided. Amit, Kelly and Lesh (1994 : 161) reported on research conducted on problem-solving. They stated that "*based on research with hundreds of students in the preceding kinds of "real life" problem solving situations, the following results have emerged consistently: (1) Even students whose prior experiences in school suggested that they are far "below average" in mathematical ability, their performance on such activities routinely shows that they are able to invent (or significantly extend or refine) mathematical models which provide the foundations for the small number of "big ideas" that lie at the heart of mathematics courses in which they enrolled, and (2) the mathematical models that they construct are often more complex and sophisticated than those that previous teaching and testing experiences suggested they were unable to be taught ...*"

Bell (1979 : 49), identified four processes when solving problems as illustrated in Figure 18. The first is the *assimilation phase*. In this phase the actual goal is understood and the data available is collated. The second is the phase where ideas are looked at, taking into consideration the data available and the goal to be attained. This is called the *exploration phase*. In the third phase, the point of *illumination phase*, a connection is made with the data and the goal. In other words the chain that connects the data with the goal is understood. In the final phase, the actual link between the data and the goal is verified by testing the final solution. This is the *verification phase*. If at this point the final solution is shown to be incorrect for certain aspects of the data then phase two will begin all over again.

It is in the assimilation stage that the problem solver understands the problem, taking into account the various aspects given. At this stage importance must be placed on pertinent information whilst irrelevant data will be discarded. It is also at this stage that the actual plan for the resolution of the problem takes place. Previous knowledge is recalled and a cursory relationship is established between what is known and what is expected.

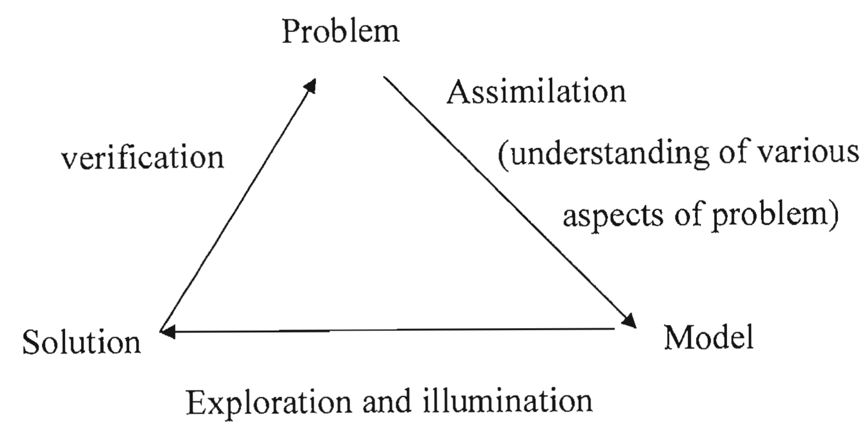


Figure 18

As a result of the assimilation stage, the problem can be transformed into a mathematical form. Solving related problems, which are simpler, may do this. The idea here would be to ascertain patterns that would aid in solving the larger more complex problem. This then is mathematical modeling.

The model established (which may not necessarily be a completely new one) is evaluated. If the verification process reveals that the solution was not suitable then the model is re-evaluated and reassessed (a different model may be used).

3.3 Some categories of modeling

It is useful to distinguish between some of the different categories or types of modeling as distinguished by De Villiers (1993 : 3 - 4). These are:

1. Direct application
2. Analogical application
3. Creative application

The first category is *direct application* (refer to Figure 19) , which describes the immediate recognition of a model for the solution of a problem because the modeler is already familiar with it. This may, for example, involve the simple use of an already established formula such as in the case of falling objects for which the velocity may be calculated by using the formula $v = \frac{1}{2} g\, t^2$, where g represents the gravitational constant and t represents the time taken. In this case there is immediate recognition of the type of problem and a direct application of an already existing model (De Villiers, 1993 : 3). Other examples may be similar but not as straightforward. For instance, if a learner was asked to determine the volume of water in a non- calibrated cylinder. There are a few methods the learner may choose. Measuring the radius of the base and then the height, the volume can easily be calculated by substituting into the formula $V = \pi r^2 h$, where r is the radius and h is the height. Or the learner may simply empty the contents into an already calibrated measuring cylinder and the volume read.

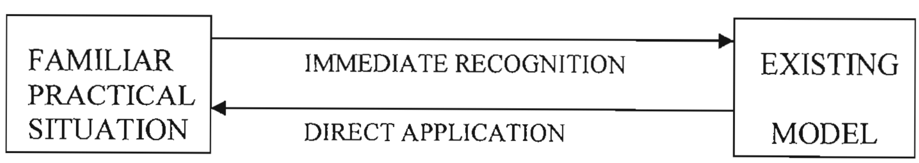


Figure 19

In the second category (refer to Figure 20), recognition of the problem type is not immediate and as a result the modeler must develop a new model which then resembles or is similar to an already existing model. The techniques and methods of the old model is applied to the new model. This category is called *analogical application* (De Villiers, 1993 : 3). Most examples done in Linear Programming at school will typically fall into this category (refer to 2.5.2).

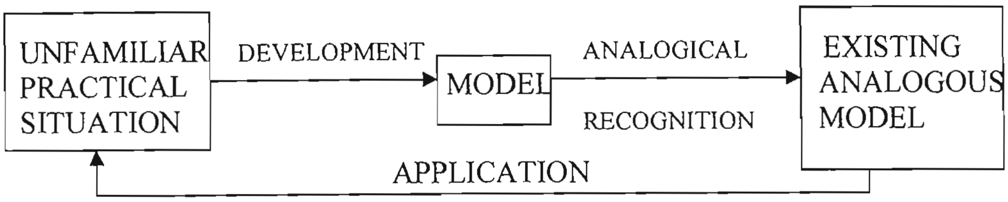


Figure 20

The third category (see Figure 21), *creative application*, is more difficult but it represents "*a teaching strategy which is especially important if mathematics is at all to be represented to pupils as a useful tool by which humans strive to understand their world*" (De Villiers, 1993 : 3). In this type of application a completely new model is constructed consisting of new techniques and concepts. This type of application is especially relevant in this study because the researcher attempted to see whether it could be used to develop and introduce learners to important geometric concepts of equidistance, concurrency, and so on.

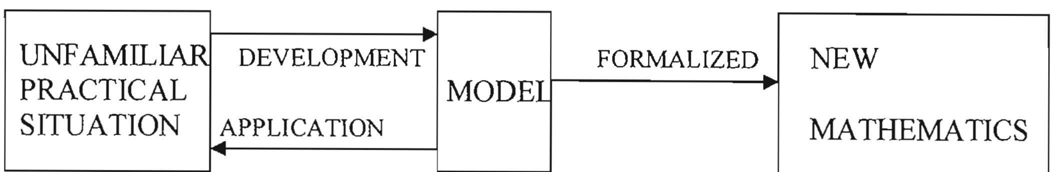


Figure 21

3.4 Constructivism in mathematical modeling in schools

There is very little doubt that children enter mathematics classes with views and ideas of certain mathematical occurrences that they experience in real life. They, for example, may be aware that water reservoirs are generally constructed at the top of the highest hill in a village because the pressure required to feed the water to large distances must be great. They may be aware that in order to establish the height of a mountain, it is basically impossible to start from the top of the mountain with a tape measure and work downwards towards the foot of the mountain. They may not know how the height is calculated, but they could have a sense that mathematics would play a role. Often, mathematics educators make the flawed assumption that learners are empty vessels into which knowledge must be poured. The constructivist view is completely opposite and is based on the theory that learning is an active process and learners construct their own meaning. This therefore implies that learners themselves are responsible for their own learning.

In the constructivist theory it is accepted that "*the learners have their own ideas, that these persist despite teaching and that they develop in a way characteristic of the person and the way they experience things, leads inevitably to the idea that, in learning, people construct their own meaning*" (Brookes, 1994 : 12)

In developing new knowledge, learners often use their pre-existing knowledge. The following, according to Scott (1987 : 7), are key points when considering the construction of meaning by learners.

1. That which is already in the learner's mind, matters. This point simply reiterates the fact that the learner's pre-existing knowledge is important. When modeling, a learner would have to develop strategies based on previous knowledge. If the learner has never worked with equations previously, it would be difficult to model a real life phenomenon based on algebra (consider the equations used in example 2 in 2.4.2).
2. Individuals construct their own meaning. Each learner may be at a distinctly different learning stage. His or her learning experience may be different from the other learners

due to environmental, societal or cultural differences or mental abilities. Talking to a learner from a deep rural area about a reservoir may be difficult if they have no fresh water source in that village. In the modeling experiment in this research, learners talked of shopping complexes in rural areas where no roads existed. Their knowledge of rural environments was minimal. Thus, modeling may be affected by the different meanings that are constructed by different individuals.

3. The construction of meaning is a continuous and active one. Often learners generate ideas, test or evaluate them and then review and reaffirm these ideas or hypotheses. That is exactly what the process of modeling expects. But all the ideas and hypotheses can only be derived from the previous knowledge the learner has. For example, if learners are asked to find a point that is equidistant from two other points, their responses will generally be that the midpoint between the two given points will be equidistant. This comes from their experience of working with midpoints. As this researcher determined, learners were able to find other points that were equidistant as well, not because they worked with it before but because they constructed their own meaning. In this research, they found many points that were equidistant from two points, but it was a revelation to them that these points would lie on the perpendicular bisector.

From a constructivist perspective, educators must therefore carefully consider the learners' prior knowledge when developing modeling activities. For learners to restructure their ideas and knowledge they need to have some already determined starting point. Expecting learners, for example, to differentiate an equation in order to find its gradient at a point would not be advisable for a Grade 10 learner, but may be quite appropriate for a Grade 12 learner. With the constructivist perspective that the learner is responsible for his or her own learning, the role of the educator now becomes more modified. Modeling activities allows the learner to interact with the problem and his or her prior knowledge to construct new meaning for him/herself. Modeling activities takes cognizance of the fact that the educator does not just transfer knowledge, but acts as a facilitator for the learner to construct knowledge.

So according to Constructivism learners change and develop the meaning of that which they experience. Modeling activities, especially using dynamic geometry software, such as *Sketchpad*, offer learners the experience of working with many examples within a few minutes. Learners can "see" the results as they interact with the software. This enables the learner to go from their level of understanding to, either a related level of understanding or a completely new level of understanding. These software programs provide the learners with immediate feedback as they test their ideas. According to Ranson and Martin (1996 : 9), "*reasoning and testing ideas in this way reveals the indispensable mutuality or sociability of learning*". Learners can easily determine a correspondence between what they know and the new knowledge they 'see' unfolding as they work through the exercise.

Often there may be a conflict with old knowledge and the new knowledge they are discovering. Cognitive restructuring of knowledge takes place, where the new knowledge is assimilated using existing schemas that were already established. This is illustrated in the discussion of the empirical findings in Chapter 4.

According to constructivism we quite frequently learn through the process of trial and error (Zecha, 1995 : 82). When faced with a particular problem, the learner engages in different methods to solve the problem. True and useful statements are retained whilst false or untrue statements are discarded. Problems do arise when the learner cannot sufficiently recognize which is useful and which is not. New methods are attempted with the accumulation of the useful knowledge and conjectures are made. Eventually, suitable solutions are attained, which can be justified. This process is remarkably analogous to that of mathematical modeling.

Closely linked to the socio-constructivist theory is the **Problem-centered learning** (PCL) approach, developed in South Africa in the mid 1980's by researchers at the University of Stellenbosch. The PCL approach is based on a socio-constructivist theory of the nature of knowledge and learning and hinges on the following (Olivier, Murray and Human, 1992 : 33):

- The learner is active in the process of acquiring knowledge.
- In acquiring this knowledge, the learner makes use of past experiences and existing knowledge.
- Learning is a social process and the learner acquires new knowledge through interaction with other learners and educators.

PCL does not simply focus on the acquisition of knowledge but also attempts to increase the learners' ability to understand and become good problem solvers.

Olivier, Murray and Human (1992 : 33) stated that "*young children enter school with a wide repertoire of informal mathematical problem-solving strategies that reflect and are based on their understanding of the problem situation and on their existing concepts*". They further stated that "*instead of ignoring or even actively suppressing children's informal knowledge, and imposing formal arithmetic on children, instruction should recognize, encourage and build on the base of children's informal knowledge*".

Research conducted by the Research Unit for Mathematics Education at the University of Stellenbosch (Olivier, Murray and Human, 1992 : 33) showed that the "*majority of children invent powerful non-standard algorithms alongside school-taught standard algorithms; that they prefer to use their own algorithms when allowed to do so and that their success rate when using their own algorithms is significantly higher than the success rate of children who use the standard algorithms or when they themselves use standard algorithms*". This research clearly contributes to the general constructivist theory of children creating their own knowledge from their own experiences and not from the experiences of the educator or textbook author.

3.5 Realistic Mathematics Education

The Hans Freudenthal Institute has since 1971, developed the theory of Realistic Mathematics Education, which was strongly influenced by Hans Freudenthal's concept of mathematics being a human activity. It takes into consideration what mathematics really is, how it should be taught and how learners should actually learn. Realistic Mathematics Education moved away from the idea that the learner is a passive learner and challenged educators to design learning situations and environments that would guide learners to reinvent mathematics, to some extent, in the way it was discovered.

The Realistic theory is closely related to mathematical modeling in that it propounds the notion that

(i) mathematics must be directly related to reality, and

(ii) mathematics must be seen to be a human activity.

In the first instance, when mathematics is to be related to reality, not only is the reference being made to real-world problems but also to the fact that the mathematics must make sense to children, it must remain as close to the concepts that children already have and know. The work they do must appeal to them within the frames of reference that they understand. Selden and Selden (1999 : 9) stated that "*from the perspective of Realistic Mathematics Education, students learn mathematics by mathematizing the subject matter through examining 'realistic' situations, i.e., experientially real contexts for students that draw on their current mathematical understandings*". Clearly, the problem presented precedes the abstract mathematics that is to be learned.

According to Van den Heuvel-Panhuizen (1998 : 1) "*the reason why the Dutch reform of mathematics education was called 'realistic' is not just the connection with real-world, but it is related to the emphasis that Realistic Mathematics Education puts on offering the students problem situations which they can imagine*". They (the Realistic Mathematics educators) place immense emphasis on the idea of making a mathematical idea real in the mind. This therefore implies that the fantasy world of children may also be used suitably and constructively to engage them with mathematical ideas and concepts. Despite the fantasy world of fairy tales for children, children can be taught relevant mathematics using problems within their contexts and understanding of reality. Problems are chosen in a way that the child is able to find and develop their own techniques to solve them.

Selden and Selden (1999 : 9) further stated that there are basically three heuristics that should be used in the designing of curricula surrounding Realistic Mathematics Education. These are:

1. The *reinvention principle*. Here the material must be able to guide the learner to discover at least some of the mathematics for him or herself.
2. *Didactical Phenomenology*. In this case researchers of curriculum should use practical problems as possible starting points for the reinvention process.
3. The construction of *mediating, or emergent models* of learner's informal knowledge and strategies in order to assist learners in generalizing and formalizing their informal mathematics. These would involve various strategies for the transposing of informal knowledge, established from the use of practical problems, to formal mathematics.

Secondly, mathematics must be a human activity where mathematics is presented in a way that resembles the way it was discovered. In other words, mathematics teaching must be organized in such a way that the process of *guided invention* takes the child through the various stages and steps of the discovery of mathematical ideas and concepts.

One of the characteristics of Realistic Mathematics is that it uses modeling as a bridge between the abstract and the real-world situation. Through modeling, learners engage the abstract to interpret real-world situations. A simple example (Zulkardi, 2003 : 7-8) is that of long division (refer to Figure 22). Firstly, one would consider the sharing of sweets amongst children, physically dividing sweets between themselves. Obviously the children would use their situational knowledge to develop strategies for the division. The second stage would be to give them the same problem, but as a written task. Here pencil and paper modeling will take place. The third stage would involve the children in the same division using mathematical methods. Here they work with the problem without thinking of the situation. Finally, the fourth stage might contain the standard written algorithm for long division.

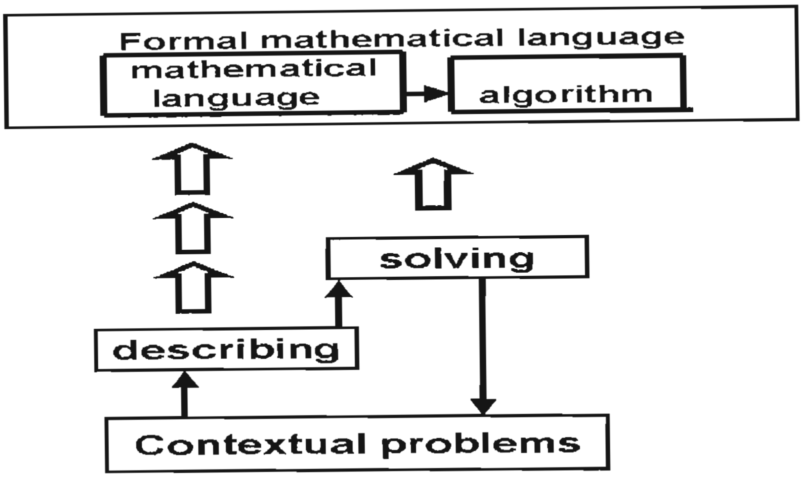


Figure 22: The process of guided reinvention as depicted by Zulkardi (2003 : 4)

From the above discussion it is clear that Realistic Mathematics Education is very closely linked to the constructivist theory. Both (RME and constructivism) is based on the philosophy that learners must be given the freedom to construct their own meanings through the interaction with educators, other learners and their environments. Despite this interaction they use their own strategies to solve problems and by working amongst other learners they are able to share ideas and experiences. Realistic Mathematics Education is based on constructivist theory because it shares the belief that learners learn when there is a strong link between the learner's already established knowledge and the theory which the learner is expected to learn (Wæge, no date : 1).

3.6 The Van Hiele levels of geometric thought

Mathematical modeling falls distinctly within the theory posited by the Van Hieles. In order to understand the relationship between modeling and the Van Hiele theory, it is necessary to discuss the Van Hiele stages.

<u>Level 1</u> : (Basic level) <u>Recognition or Visualization</u>

At this basic level the learner is aware of space and has knowledge of certain basic vocabulary. The learner can recognize, for example, a square, but will not be able to list any properties of the square. So, in other words, the learner recognizes specified shapes holistically, but not by its properties. In this stage of the modeling process, the learner engages in *explorative* activities, attempting to find common relationships that may lead to a hypothesis. It is important to note that the learner at this stage understands the problem posed and attempts to find a visual solution. Using *Sketchpad* in this stage is beneficial to the learner because it allows for the learner to explore various figures and shapes within a few minutes, thus affording the learner visual "proof" of the solution to the problem.

<u>Level 2</u> : <u>Analysis</u>

At this level learners begin to understand that the shapes that they are working with, through observation and playing around with it, have certain properties. The learner can now see that the square is made up of all equal sides, or the rectangle has opposite sides equal. But the learner still cannot find the relevant links between the different figures, for example, the relationship between a square and a rectangle. At this stage in modeling activities the learner begins to *formulate hypotheses* because s/he can see (with guidance) the solution unfolding on the computer screen. There is at this stage no formal deductive proof but by using *Sketchpad* the learner may begin to make deductions that may lead to the formulation of a broad hypothesis.

Level 3 : Ordering or Informal deduction

At this level, learners discern the relationships between and within geometric figures. For example, learners can conclude that if the opposite sides of a quadrilateral are parallel then the quadrilateral is a parallelogram or that a square is a rectangle. So, at this level they can determine the characteristics of an entire class of figures, for example, quadrilaterals. Learners cannot at this level employ deductive strategies to solve geometric problems. They may be able to follow a proof but may not themselves be able to prove. In modeling activities, the learner begins to assimilate the various informal facts that become available through exploration. More especially, if the learner works with *Sketchpad*, these informal facts begin to take on a new meaning, because they can be tested and instantaneous feedback can be obtained.

Level 4 : Deduction

At this level, the pupil understands the significance of deduction as a means of solving geometric problems. They also understand the role of axioms, postulates, definitions and theorems. They can use their knowledge to construct a proof of a statement and understand the relationship between a statement and its converse. In this stage of the modeling process there is illumination of the model as the learner discovers through exploration the solution to the problem and the value of the model. This leads to deductive verification of the problem. This is also an important stage because verification may become easier if the learner shows high levels of conviction.

According to the van Hiele theory in order for the learner to move on to the next stage, s/he must have mastered the previous stage. In other words, Visualization is a prerequisite for Analysis, which is a prerequisite for Ordering, which is a prerequisite for Deduction. In a similar manner, it seems essential for learners engaged in the modeling process to go through the different stages of Exploration, Formulation and Assimilation. These are essential steps in the process and it becomes difficult for the learner if any step is omitted. Each step in the process appears to be a prerequisite for understanding abstract mathematical models.

CHAPTER FOUR

Research methodology and overview

The purpose of this study was to determine whether *Sketchpad* could be useful as a mathematical tool when teaching children to model. Furthermore, this study will test curriculum material that was developed (see De Villiers, 1999) and refined as a result of previous empirical and theoretical research. The material allows the child to work through the problem by guiding the child through stages that are easy and practical. As the child progresses through the worksheet, the child is allowed to record his/her conclusions and conjectures and is led to an explanation (proof).

The theoretical and empirical part of this research has focused on the following major research questions:

1. What is the role and function of mathematical modeling in mathematical sciences, and its potential role in mathematics education?
2. Can learners acquire knowledge about geometric concepts and shapes such as equidistance, perpendicular bisectors and concurrency via creative modeling?
3. Are secondary school learners able to create and use mathematical models to solve geometric problems in the real world without the use of *Sketchpad*? If so, what strategies do they use?
4. Are learners able to use the provided *Sketchpad* sketches effectively to arrive at reasonable solutions?
5. Do learners display greater understanding of the real world problems under question when using *Sketchpad*?

As a result of these questions it was decided to use the method of qualitative analysis by means of one-to-one task based interviews. This method makes it easier to document the high level of information that individual children display when working through a specific problem. Furthermore, this method would allow the researcher greater control to observe and take note of, how each learner answered questions based on the computer manipulation they experienced.

As Novak and Gowin (1984 : 12) stated: "*For this reason most psychologists prefer to do research in the laboratory, where variation in events can be rigidly prescribed or controlled. This approach clearly increases the chances for observing regularities in events and hence for creating new concepts*". The researcher acknowledges that the classroom situation is dynamic, due to the interaction of learners with each other, the educator, the subject content and the environment. By reducing the number of external variables, one narrows the focus, giving generalizations based on findings during task-based interviews greater credibility. Such findings, however, might be able to dictate future classroom practice.

The tasks to be used in the interviews have been conceptualized within an action research paradigm. These tasks were conceptualized mainly as a means of teaching children the different functions of proof by Prof. M. de Villiers, but included aspects of modeling. This research will determine how well they cope with the tasks provided, whether they construct meanings as conceptualized, and whether they would be able to mathematically model a solution. This would also mean that based on initial trials the tasks may have to be redesigned in order to achieve the predetermined goals of the learning activity. In other words, action research acted as a guiding methodology. As Cohen and Manion (1986 : 208) stated : "*Action research is a small-scale intervention in the functioning of the real world and a close examination of the effects of such intervention*". This in effect summarizes the purpose of this experiment. The strategic plan that would be implemented involves the learner interacting with the developed curriculum material, careful observation and thereafter reflection.

The researcher chose to work with learners from Glenover Secondary School due to the convenience of having easy access to the learners themselves and arrangements could easily be made to interview the learners. Two learners were initially chosen for the trial run. Thereafter eight learners, between the ages of 15 and 16 years, were interviewed from Grade 10. Their computer studies class educator selected these learners randomly. They were selected from a group of 60 learners in Grade 10 in March 2000. At this stage, the learners had not written any examinations for the year and therefore their individual academic performances could not be commented on. But based on their previous years mathematics results, the group comprised of 2 learners who had attained between 60 and 70 percent, and the rest of them had marks ranging between 32 and 53 percent. These learners were ideal for this study because the questions are suited to their level, and they had not yet done any theorems based on quadrilaterals and concurrency.

There were a larger number of Indian learners taking computer studies as a subject at this school in the year 2000, as compared to learners from other race groups. All of the learners selected for these interviews were Indian. The school itself is situated in Westcliff, in Chatsworth, which is a predominantly Indian suburb south of Durban. The residents of Westcliff are generally those of the middle to lower income group. The mathematics results at this school has been below average over the previous years, and this was apparent from the matric results and the fact that below 5% of the learners doing mathematics at matric level offer it at the higher grade level. On this basis the average mathematics achievement of learners at this school could be considered to be below average.

Learners were not exposed to this particular problem at the school before and therefore did not know the solution or what to expect beforehand. Learners did not need to have any specific knowledge of mathematical concepts but some knowledge of perpendicular bisectors could have been useful.

A brief synopsis of the interviewing process is necessary in order to place the obtained responses obtained in perspective. Although the learners did not know exactly what to expect, some of them initially displayed an unwillingness to participate in this experiment. They feared failure and felt that they were incapable of working with mathematics in a computer environment because they never experienced any such thing before. The interviews were conducted in the classroom in the school over a period of two months. The interviews were subject to the availability of the learners, because the school only allowed the use of its classroom during school hours and the time for these interviews had to be negotiated with the class educators. The classroom was adequately equipped for the purposes of the interviews, because all that was necessary was a computer. All learners involved were brought together for a short period in order to familiarize them with the general use and application of the computer software – *Sketchpad*. This period of training was inadequate and did not equip them with the skills required to manipulate the program. A longer period of training was not possible for the following reasons:

- the learners came from different classes, and permission could not be obtained to gather the learners at any one given time, and
- only three computers could be used for the purposes of an orientation, which meant that 10 learners would not have been able to satisfactorily enjoy and learn the basics about the software.

In any event, as was stated already, this did not affect the experiment because minimal knowledge was expected from the learners about the software. Each learner was made to feel at ease before the interview commenced, in order to ensure that they would respond in a way that would reflect their understanding of the task provided.

The task that the learners had to work through was based on a relevant real-world topic. All learners were exposed to the different media in South Africa and from prior questioning it was established that all learners were aware of the seriousness of the water born illness called Cholera. Learners were also aware that Cholera was mainly concentrated in rural areas; in particular, the areas were no fresh water was available. So, the identification with the problem was not new and difficult.

All the *Sketchpad* sketches were presented as ready-made models to the learners due mainly to their lack of technical expertise, and to save time. However, the task of constructing such models is also an essential modeling skill and would be an interesting task to ask of learners. For example, it would require them choosing and implementing a reasonable scale and utilizing *Sketchpad's* tools to accurately construct a dynamic scale drawing. The decision to present the sketches directly to them was based on the following reasons:

- The construction of accurate, dynamic *Sketchpad* sketches require a fairly high level of technical expertise.
- Even if learners had a sufficiently high level of expertise, it would have been very time consuming to construct the sketches.
- The research was aimed at ascertaining whether learners could use these given *Sketchpad* sketches effectively to solve real world problems, and not on their ability to construct such sketches themselves.
- Moreover, the research was aimed at investigating whether learners could, through using these given sketches, acquire important concepts such as bisector, circumcentre, circum-circle, cyclic quadrilateral, and some properties of these.

Instead of asking the learners to use the menu to make the various constructions, the learning activity was further simplified with the use of action buttons. This decision was based on the fact that the learners had no (or very little) knowledge of *Sketchpad*. The learners were simply required to click on buttons that were constructed to either hide or show details as requested. This was to simplify the use of *Sketchpad* as a tool for modeling. All constructions that they were required to make were relatively simple and simply relied upon their ability to select points and segments and use the drop down menu.

All measurements were clearly visible on the screen of the computer, so that learners could easily view any changes that might take place. On being seated learners were given the task sheet and were asked to read through it. At the commencement of the interview (when the tape recording began), learners were asked whether they understood the question posed to them. This was done for the following reasons:

- to break the ice and make them feel at ease during the course of the interview, and
- to ensure they knew exactly what they had to determine.

Below is a copy of the tasks that the learners had to work through and the questions that the researcher asked as the interview unfolded.

<u>Task 1</u> : Initial Problem

In a developing country like South Africa, there are many remote villages where people do not have access to safe, clean water and are dependent on nearby streams or rivers for their water supply. With the recent outbreak of cholera in these areas, untreated water from these streams and rivers has become dangerous for human consumption. Suppose you were asked to determine the site for a water reservoir and purification plant so that it would be the same distance away from four remote villages. Where would you recommend the building of this plant?

The diagram they saw on the screen with the easy to use buttons is shown in Figure 23.

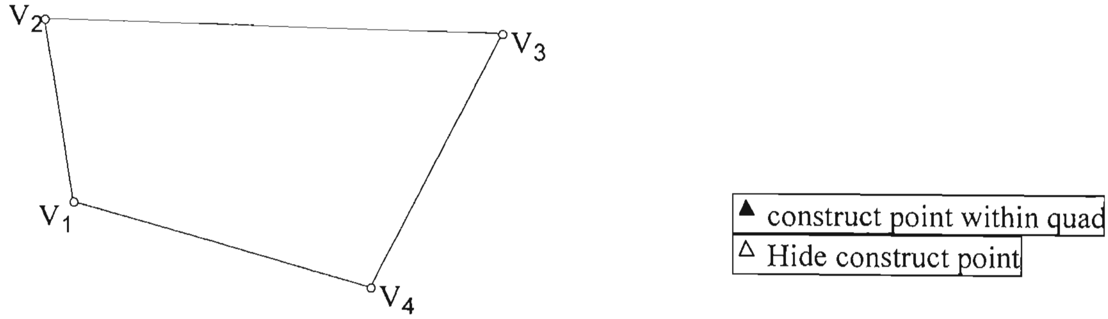


Figure 23

After giving the learner sufficient time the researcher determined whether s/he understood the problem and attempted to extract a conjecture from the learner. The questions asked are listed below.

1.1 By looking at the diagram provided where would you think the reservoir and purification plant should be built?

1.2 What would you do to find the most suitable position of this reservoir?

1.3. If one looks at the real world, the position you choose may not be ideal. What are some of the assumptions you think have been made in this problem that is not necessarily true in the real world or may impact on the "ideal" position? (If necessary the following was added: Real world situations are extremely complex and they usually must be simplified before mathematics can be applied to it. What are some of the assumptions that you think have been made to simplify the problem that may not necessarily be true in real life?)

These are a few assumptions that could have been made (these assumptions were not given to the learners):

a. The villages are all the same size. If one village is ten times the size of all the others, then it may make more sense to put the reservoir closest to the largest village.

b. The sizes of the villages are so small in comparison with the distance to the reservoir, that points can represent their positions.

c. The four villages lie on a flat land, that is, there are no hills, valleys or mountains between the towns.

d. The ideal equidistant point is practical.

e. A river passes close enough for the water to be diverted into the reservoir or water can be diverted from a dam into the reservoir.

f. The cost of building this one large reservoir will not be larger than building four smaller reservoirs in each village.

g. The position for constructing the reservoir is easily reached by road and the pipelines being constructed to carry the water will not encounter any obstacles. In other words, the direction of the pipeline will not be diverted due to obstacles such as rivers, mountains and so on.

1.4 If the learners indicated that the distance from a point to the four vertices be measured and then move point around to find the most suitable position then they were asked to click on "construct point" (button) and drag point O around to find the most suitable position. The learners were then given task one.

TASK 2 : The problem simplified.

The researcher, after establishing that the learner could not find a suitable solution using correct reasoning and arguments, made the problem simpler. The questions and statements that follow were part of the second phase of the interview process.

2. Let us simplify the problem by considering what would happen if there where only two villages (refer to Figure 24). How can we find the point/s which is/are equidistant from two different points?

Figure 24

2.1 Let us suppose that we had only TWO villages. Where do you think we should construct the reservoir so that it would be equidistant to both the villages? The interviewer will also consider saying this: After you have determined the best position for two villages, you may click on the button "Show one possible point".

2.2 What would happen if in reality the position you chose is unsuitable would we still built there? What should we do if there is an obstruction?

2.3 It may be possible that the second position that you determined may also encounter some form of an obstruction. Can you try to find two other positions on the diagram below, which are equidistant from I and J?

2.4 How many positions can you find that will be equidistant from I and J?

2.5 You now have many positions which are equidistant from I and J. Can you find a pattern? You may join the points and measure angles if you want to.

2.6 Concluding question: How would you find ALL THE points which are equidistant from two points?

<u>TASK 3</u> : Return to the original problem.

At this point the researcher returned to the original problem in order to determine whether the learner would use the recently acquired knowledge. It was important at this point to see whether the learners made use of the "new" information they acquired. The idea that the knowledge they acquired in the first task will determine their response for the second task is an important part of designing conceptual learning tasks. These new bits of knowledge become the preconceptions for the next stage. This will be elaborated on in Chapter Six.

These are the questions and statements that were used in the third phase of the interview.

3. Let us now return to the original problem.

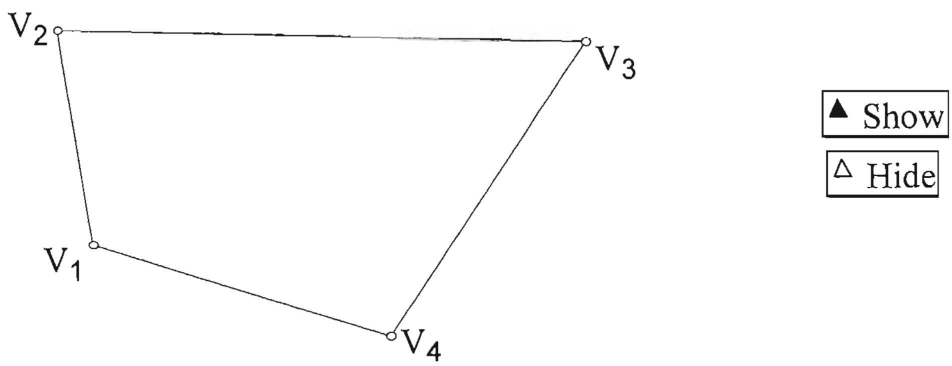


Figure 25

3.1 If we look at four points (refer to Figure 25), how would you attempt to find the ideal position? One way would be to construct a point within the quadrilateral, measure the distances from the point to the vertices V1, V2, V3 and V4 and then drag the point around such that the distances are the same. Can you find an alternative method, which is similar to that which you discovered in number 2 above? Describe your construction method.

3.2 Do you think that you can always find a point equidistant from all four vertices of a quadrilateral, no matter what the shape or size of the quadrilateral? Explain.

3.3 Now drag any one of the vertices of the quadrilateral to determine whether there is ALWAYS such a point. Do you still agree with your answer in 3.2?

The tasks in 3.2 and 3.3 were important. It was conceived with the explicit idea of showing learners that one must be careful of making broad generalizations. Generalizations may hold for certain figures and shapes, but may not necessarily work for other shapes and figures. It is also true that children are often naïve and therefore draw conclusions very easily. It is important to ensure that we constantly draw attention to this when we teach.

Perhaps it is also necessary to place into perspective the reason for choosing the tasks in the order of the quadrilateral first, followed by the triangle. The order of the tasks was conceived with Ausubel's Subsumptive learning in mind, where general ideas are presented and developed first before moving towards incorporating or developing more specific ideas. Subsumption is the process whereby all new information is assimilated into a larger, already existing (more general) body of knowledge (Thompson, 1999 : 2). It refers to the placing of new items of knowledge into a larger, more comprehensive category. Ausubel propounds the theory that all new information is linked to that which already exists, and argues that often meaningful learning is enhanced when the general, more inclusive ideas and concepts are developed first.

This contrasts strongly with the sort of rule of thumb wisdom of traditional curriculum design which often assumes learning should always progress from the more specific to the more general. In the tasks provided in this experiment, the ideas of equidistance, an equidistant point and that such a point may not necessarily always exist, are first developed more generally in the context of a quadrilateral before moving to the triangle as a unique special case. Most traditional curriculum design learning activities usually start with the triangle before moving to the quadrilateral or polygons in general.

One of the dangers of this specific-to-general approach is that learners can often easily develop fixed and rigid (fossilized) misconceptions. For example, from just studying the triangle, learners may easily develop the misconception that any two polygons are similar if their corresponding angles are equal. However, this is ONLY true for triangles and NONE of the other polygons. By starting from the unique, special case children are denied the opportunity to develop a deeper understanding of the GENERAL conditions for similarity.

The questions and statements that follow were part of the fourth phase of the interview.

<u>TASK 4</u> : The problem specialized.

4. Let us now find the ideal position if we only had three different villages (refer to Figure 26).

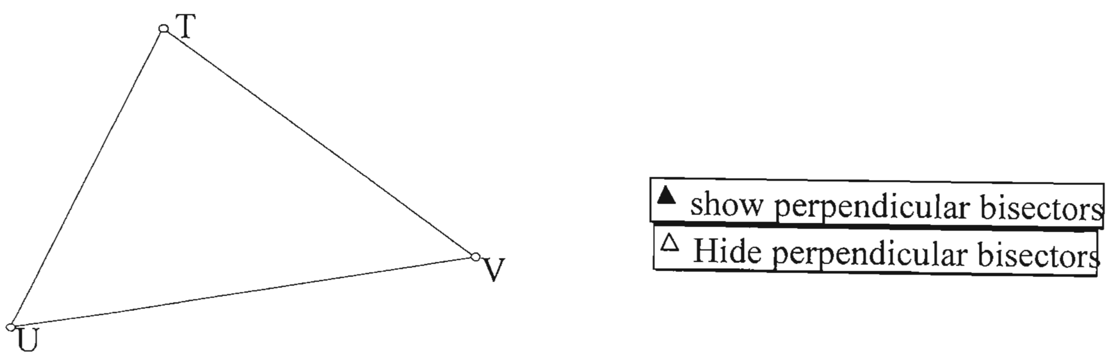


Figure 26

4.1 Before you work with Sketchpad, would you like to guess where the ideal position would be? Explain.

4.2 Construct the perpendicular bisectors and determine the correct BEST location of the reservoir. Does this position correspond with yours in 4.1?

Click on the button to see the perpendicular bisectors.

4.3 Do you think that you can always determine an ideal, EQUIDISTANT position within a triangle? Explain why you think that this may or may not be possible.

4.4 Drag one vertex of the triangle and determine whether there is ALWAYS such a BEST (EQUIDISTANT) position.

4.5 a. How sure are you?

b. Did you find it surprising?

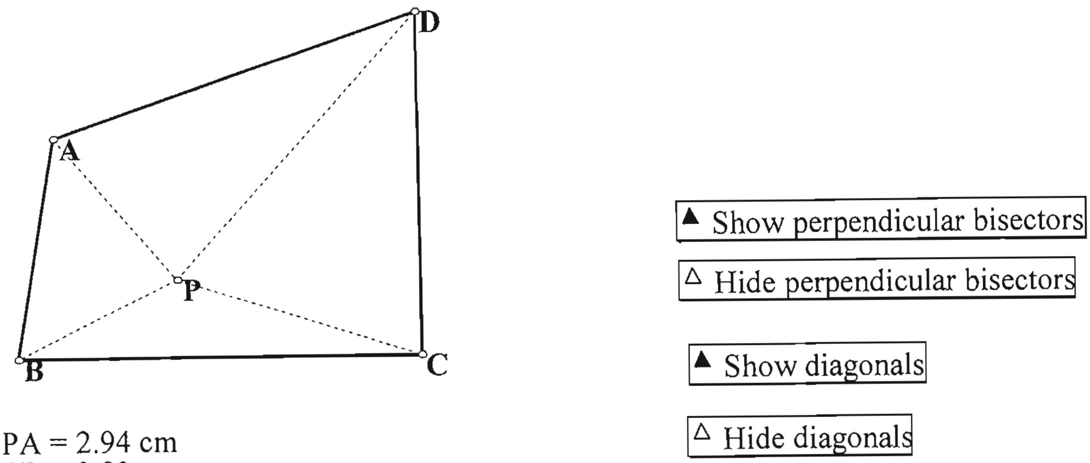
c. Why or why not?

TASK 5 : A variation of the problem.

The fifth and last task that learners had to work through looked similar to the original task. The expectation was that learners would immediately conclude that the same method should be used. This idea of including a different type of question served to test their inclination to only use those methods they are familiar with. Learners are often not trained to try new strategies when solving problems and as a result may not be able to be successful problem solvers.

The final task is quoted below.

> *The government decides instead to build pipelines from the water reservoir to the four villages A, B, C and D. Where should the water reservoir be placed so that the total length of the pipeline is minimized, e.g. so that the SUM of the distances to the four villages is a minimum (as small as possible)* (refer to Figure 27)*?*



PA = 2.94 cm
PB = 2.82 cm
PC = 4.02 cm
PD = 5.62 cm

PA + PB + PC + PD = 15.40 cm

Figure 27

1. The learners would be asked to guess the position of the reservoir.
2. They will then be asked to explain how they would find the position.
3. Click on "Show perpendicular bisectors". What do you notice?
4. Click on "Hide perpendicular bisectors".
5. Double click on "Show diagonals". What do you notice?
6. Click on "Hide diagonals".
7. Drag either A, B, C or D. Now repeat steps 1- 5. Can you explain your observations?
7. If the quadrilateral ABCD is a concave quadrilateral, where do you think P should be placed?

The schedule of questions that followed was designed and redesigned after two trial runs. After the first two interviews had been completed it was decided to include the following question: ***Would you like to know why the perpendicular bisectors of a triangle are always concurrent?*** This question was asked after the learners had discovered that the perpendicular bisectors were always concurrent in a triangle. The explanation was therefore not done immediately, but was concluded at the end of the interview with some of the learners.

Each interview was approximately sixty minutes long and each was audio taped. Although these questions were structured around the critical questions, it also allowed for variation in expected responses from the learners, and further probing was done in particular cases.

After the interviews with these learners were completed, the researcher decided that the actual explanation for the concurrency of perpendicular bisectors in a triangle was a necessary aspect of this task. A further two learners were interviewed and their responses were also recorded.

Finally, the data analysis amounted to systematically grouping and summarizing the responses, and providing a coherent organizing framework that encapsulated and explained the way each learner produced meaning whilst working through the tasks provided.

CHAPTER FIVE

Research analysis

5.1 Introduction

The questions, for the purpose of continuity will be repeated.

In a developing country like South Africa, there are many remote villages where people do not have access to safe, clean water and are dependent on nearby streams or rivers for their water supply. With the recent outbreak of cholera in these areas, untreated water from these streams and rivers has become dangerous for human consumption. Suppose you were asked to determine the site for a water reservoir and purification plant so that it would be the same distance away from four remote villages. Where would you recommend the building of this plant?

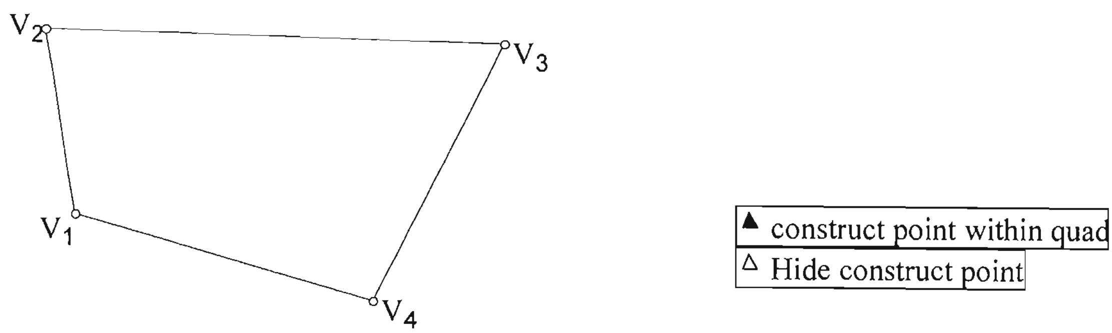


Figure 28

The learners were handed the worksheet and were allowed some time to read and understand the task. Several questions were asked at the beginning just so that the researcher could be sure that the learners understood the question. Most of this was done prior to the conversation being recorded. The interview only continued after the researcher was convinced that the learner was fully conversant with the question. In some instances it did occur that the learner needed clarity in order to improve understanding. This part of the interview was not recorded because these questions had no bearing on the research being conducted.

5.2 Were the learners able to create and use mathematical models to solve the specific real world problem without the use of *Sketchpad*?

Rather than immediately starting with *Sketchpad,* the researcher first asked the learners to attempt to find a solution on their own, using any previous knowledge. All learners "guessed" a solution somewhere in the "middle" of the quadrilateral, but none could find a precise solution. More importantly, few of them even thought of or tried to test their solution. This extract from the interview with Pravanie (see below) was typical of several interviews. The learners seemed to feel that this type of question was not within their ability to solve it and in some cases the learners explicitly said that this was because this type of question had never been taught or asked of them. This is a very significant point. Learners seemed to feel that they could not solve problems not seen before. Moreover it seemed that a real-world context such as this was rather novel to them.

RESEARCHER	Where do you think that we should build the reservoir?
PRAVANIE	I don't know ... all we are only given is this diagram ...
RESEARCHER	Do you think that you will be able to find the most suitable point? You can use any method you know, to do so.
PRAVANIE	I don't know sir ... this is too difficult ... please don't interview me (pleading).
RESEARCHER	Are you saying that you cannot find any way of solving this problem?
PRAVANIE	I can't ... I'm not so good in maths ... maybe at the centre here (pointing to the middle of the quadrilateral).
RESEARCHER	Will you be able to justify your answer? Will you be able to tell me why?
PRAVANIE	(silence)...... not really ...
RESEARCHER	Don't you even want to try?

PRAVANIE	I don't know what to do …

From the above it is clear that Pravanie was very uncomfortable being faced with an unknown problem not seen before, and that she seemed to have been so intimidated that she did not even want to continue with the interview. She seems a typical product of the traditional approach where learners acquire a "learned helplessness" that is, an unwillingness to attempt problems on their own.

Christina's approach though was different (see below). Although she could not determine a precise way of finding a satisfactory solution, she was initially prepared to use pencil and ruler. After she discovered that using the pencil and ruler only gave an approximate solution she simply gave up.

RESEARCHER	Where do you think that we should build the reservoir?
CHRISTINA	Can I use my ruler and pencil?
RESEARCHER	You can use any method you know, to do so.
CHRISTINA	(after a while) I thought it will be here … (pointing to the middle) …(measuring the distances) … but when I measure the distances, it's not the same.
RESEARCHER	Are you saying that you cannot find any way of solving this problem?
CHRISTINA	I think it is in the middle … maybe at the centre here (pointing to the middle of the quadrilateral but not continuing further).

When asked to explain why she thought the solution was somewhere in the middle, she emphatically refused to provide an explanation nor to continue looking for a better solution.

RESEARCHER	Will you be able to justify your answer? Will you be able to tell me why?
CHRISTINA	No! (emphatic)
RESEARCHER	Don't you want to try?
CHRISTINA	No!

Faeeza displayed good self-confidence initially (see below). Her comments such as : " *It's easy to understand*" and "*Ya ... can I measure with my ruler*?" showed that she felt that she was capable of solving the problem. Although she was sure that she could prove that her guess (somewhere in the middle of the quadrilateral) was correct by measuring, she suddenly lost some of her confidence when she discovered that her guess was not correct. Yet again, just like the others, she could not think of another method. It is interesting to note that the learners became irritated when prompted by the researcher to try and find another method. Faeeza's teacher dependence acquired from traditional teaching is also clearly highlighted by her question "... *what method should I use*?"

RESEARCHER	Faeeza, you've read the question, do you understand it?
FAEEZA	It's easy to understand.
RESEARCHER	Where do you think that we should build the reservoir?
FAEEZA	About here (pointing to the centre of the quadrilateral).
RESEARCHER	Can you prove to me that this is the correct point?
FAEEZA	Ya ... can I measure with my ruler?
RESEARCHER	Yes you may.
FAEEZA	(After a while) I don't think I'm measuring correctly ...am I? ...
RESEARCHER	Do you want to perhaps try another method?
FAEEZA	I don't know ... what method should I use?
RESEARCHER	Do you know of any method?
FAEEZA	No!
RESEARCHER	Are you sure?
FAEEZA	I really don't know (irritably).

Similarly, Nigel, Roxanne and Schofield had no idea about what they should be doing.

RESEARCHER	Where do you think that we should build the reservoir?
NIGEL	There (pointing to the middle of the quadrilateral).
RESEARCHER	Can you prove that your answer is correct?
NIGEL	No, I can't.
RESEARCHER	Are you saying that you cannot find **any** way of proving this?
NIGEL	I can't.
RESEARCHER	Don't you want to just try?
NIGEL	I wouldn't know were to start ... I can't.

In all cases the learners could not think of any method (not even successive trial and error) that they could use to verify their guesses. The fossilized teacher-dependence of the learners is aptly summarized by Schofield's comment that: "*We didn't do this in class before ... I can't do it!*" Clearly all these learners seemed to have been only accustomed to teaching-learning situations in which the teacher would always first present a new method or technique and all that is required of them is to practice it. It did not appear at all as if these learners had ever been exposed to a problem-centred (or modeling) approach before, namely, where they are expected to regularly tackle problems not seen before.

Roxanne's interview.

RESEARCHER	Are you saying that there is no way to do it or is it that you don't know how to do it?
ROXANNE	I can't. There might be a way to do it but I can't.
RESEARCHER	Wouldn't you want to try?
ROXANNE	I don't know what to do!

Schofield's interview.

SCHOFIELD	I ... I don't know ...
RESEARCHER	Do you think that there is a way of showing why your answer is correct?
SCHOFIELD	I don't think so.
RESEARCHER	Don't you want to try?
SCHOFIELD	We didn't do this in class before ... I can't do it!

5.3 Learners' conjectures and their justifications

It is perhaps not surprising to note that all learners conjectured that the most suitable point was somewhere in the 'centre', presumably relying on their visual intuition to locate an approximate point equidistant to the vertices.

FAEEZA	Ya. It should be situated towards the middle.
CHRISTINA	In the centre
NIGEL	Towards the middle.
ROXANNE	In the middle.

The inaccuracy of their visual perception created surprise when the learners later discovered that their conjectures were not correct. This surprise in turn created some level of curiosity.

Despite their difficulty or reluctance to use successive guessing and testing with ruler or compass, they nonetheless were able to realize in the *Sketchpad* environment that the distances should be measured from each vertex to the constructed point, and that by dragging the point around one could change the distances until they were the same.

RESEARCHER	In the centre would mean about there in the middle? Is that correct? (Christina nods her head). Now how will we be able to determine whether that is the correct point?
CHRISTINA	We measure from that point to that (pointing to the vertices of the quadrilateral).
RESEARCHER	(after Christina measures the distances) What do you observe?
CHRISTINA	The distances are different…
RESEARCHER	How then can we find a suitable point?
CHRISTINA	Move the point around.
RESEARCHER	Move the point around. (After awhile) Is it easy to find this point?
CHRISTINA	No.
RESEARCHER	So what happens if we move the point around?
CHRISTINA	The distances will change.

Faeeza, who showed much confidence during the interview, was not intimidated by the fact that she was in a new environment (doing mathematics using the computer was new to them). Her reaction to questions and the tone of her replies conveyed the impression that she was quite comfortable with using *Sketchpad*. Other learners displayed similar behaviour, although they hesitated at some stages with their responses.

RESEARCHER	Now, Faeeza, draw a point somewhere in this quadrilateral ... this is where you suggested the building should be. How will we know what the distances are from that point to the various villages?
FAEEZA	Obviously you need to measure it.
RESEARCHER	Will you measure it then? (after a while) Ok...there are all the distances from the point that you chose to the different villages. So what do we do now?
FAEEZA	Drag the point around until we get those distances equal.

Schofield also showed that he was comfortable with using *Sketchpad.*

SCHOFIELD	It is obvious that it must be in the middle. We can show this by measuring the distances from the village to the point that is chosen.
RESEARCHER	Do just that then....(after a while) ... what do you observe?
SCHOFIELD	(silence)... the distances are different...
RESEARCHER	What should we do then to get the distances to be equal?
SCHOFIELD	Move this point around?
RESEARCHER	Go ahead and do that.
SCHOFIELD	(after a long while)... this must be the point.

Roxanne was the only one who initially stated that the ideal point should be at the centre and thereafter asked for the point to be shifted to the side. Although she had not found the accurate point, her moving of the point towards the right was getting closer to the correct position.

RESEARCHER	So do you think that it should be in the middle here? (pointing)
ROXANNE	Ehhh...maybe more towards the side here...(indicating a shift to the right)
RESEARCHER	All right, then how do you think we should go about checking whether it is correct?
ROXANNE	By measuring the distances.
RESEARCHER	Quickly do that ... (after a while) okay. These are all the measurements. But the distances are not equal. How can we get then to be equal?
ROXANNE	You can take the point around and try to find the spot for which the distances will be equal.

5.4 The recognition of real world conditions when modeling

At this point the child was told that real world situations are extremely complex and they usually must be simplified before mathematics can be applied to it. The learners were then asked to give some of the assumptions that they thought may have been made in order to simplify the problem that may not be true in real life. The responses received here showed that given an opportunity learners would be able to reflect on real life conditions as compared to traditional classroom situations where these assumptions are usually not discussed.

The learners being interviewed were able to recognize factors that could have affected the position they chose. At the expense of belaboring this point it may be essential to list the responses of all the learners just so that a clear impression can be obtained with regard to the way the learners construct their reality.

PRAVANIE	Apart from the fact that there might be a valley over there, there could be a mountain, there could be a building already constructed. There could be endangered species.
RESEARCHER	... endangered species of what?
PRAVANIE	Any plants or animals and stuff like that ... a nature reserve.

Rivers and mountains were common responses but the endangered species of plants and animals was an interesting response. It must be remembered that these learners had not seen this question before so their responses were spontaneous.

Christina's responses were similar to earlier responses but Faeeza's and Roxanne's responses about the chief's house or the chief's kraal being situated there was highly realistic.

CHRISTINA	There might be... a hard rock.
RESEARCHER	Yes....
CHRISTINA	Other buildings
RESEARCHER	What kind of buildings do you think?
CHRISTINA	Police station or something....
RESEARCHER	Any other reasons?
CHRISTINA	If there's like a stream or river you won't be able to build

FAEEZA	Maybe there is building ....
RESEARCHER	What kind of building do you think?
FAEEZA	Could be a school, I don't know.... Could be a shopping complex.
RESEARCHER	Remember that these are remote villages ....
FAEEZA	Might be the chief's house......
RESEARCHER	Yes...
FAEEZA	There might be a mountain there ... anything, like a big rock. The cost factor must be too great to get rid of the rocks. It won't be cheap to build there.

ROXANNE	Maybe there is a mine or a building.
RESEARCHER	Remember that these villages are remote villages. What kind of building might be there?
ROXANNE	There might be kraals there or the chief's house...
RESEARCHER	Do you think that there might be other reasons?
ROXANNE	The place might be a mountain, with hard rocks...

Nigel's responses below about a mine being at the exact spot or hazardous nuclear waste being stored there, were responses that were not expected and it showed that given the opportunity, learners could be very creative. This suggests the viability of a more concerted effort be made to encourage learners to think creatively about the real-world and its relationships with mathematics in the mathematics class.

NIGEL	There might be a mine ....
RESEARCHER	Yes, any other reasons?
NIGEL	If the area is very rocky or has mountains.......maybe they are storing hazardous nuclear waste nearby.......

Perhaps a response more suited to a country like South Africa was that of Schofield. He spoke of financial constraints and the social problems by referring to the fact that the people themselves might be unhappy with the location. This showed that learners themselves could be politically mature enough to consider a wide range of issues.

SCHOFIELD	There might be buildings like a school
RESEARCHER	Remember that these are remote villages.
SCHOFIELD	... there may be a river close to the point... if there are no roads then it is going to cost more money to first build roads...the people in the villages may say that they don't want the reservoir at that point...what about very hard rocks...

A matter of concern regarding the learners' responses regarding the unsuitability of the chosen position was the fact that none of the learners focused on the issues more specifically related to mathematics. With the exception of Pravanie none of the other learners stated that it was assumed that the land was flat and though a chosen position may be mathematically ideal, it may not be correct if one considered the possibility of hills and valleys. None of the learners realized that the relative sizes of villages might have some influence on the chosen position. For example, if one village was substantially larger than the others it may make practical sense to put the reservoir closer to it. It is also implicitly assumed that the positions of the reservoir and the villages can be represented by points, that is, their sizes are insignificant compared to the distances in question. If not, this raises several questions. Were the distances being calculated related to the centres of the villages or to the outer boundaries of the villages? Would this therefore not affect the position chosen?

These are critical questions that learners need to focus on when working with real world problems. It seems that their lack of experience in working with real world problems played a role.

The learners' responses to the real-life problems that could be experienced indicated a reasonable level of understanding. In fact, three of the learners directly indicated that mathematics alone cannot always be used for solving real life problems, that is, problems can be experienced.

NIGEL	That sometimes in real-life we may not be able to use exact mathematics to solve problems.
ROXANNE	We are saying that we can work out a place on the computer or by calculating it but it does not mean that it will be the right place.
SCHOFIELD	Maths might be one thing but reality is another ... sometimes we can't use maths on its own.

The other learners could see that a mathematically determined point may be unsuitable in the real world. This does not in any way indicate that mathematics is not effective but it does show that they had some understanding that with real life problems other factors must be considered.

PRAVANIE	Very often we think a certain place will work but when we go there we notice that there is a problem.
CHRISTINA	Sometimes finding the point might not work because ... because the place might not be suitable.
FAEEZA	Because we are trying to show that we can choose a point but that point is not always good... maybe we should only choose the point after we see the place.

5.5 The simplification of the real world problem

It was evident that the learners had no idea as to how they could proceed further with finding an accurate solution to the problem. It was fine that they knew that the point could be dragged around to find an approximate solution, but determining points in this way is time consuming as well, and if one looks at find solutions correct to a few decimal places then finding accurate solutions is essential.

RESEARCHER Now, besides just measuring and dragging the point around like we were doing earlier. There might be other methods in doing this. Do you know or can you think of any other way in which you could find the point?

FAEEZA No.

RESEARCHER Are you sure?

FAEEZA Yes ... I am positive that I don't know of any other way.

Pravanie expressed a similar response but she acknowledged that there might be a method for finding an accurate solution.

PRAVANIE I don't know.....that is the only method I can think of.

RESEARCHER Do you think that there might be a mathematical method of finding the most suitable position?

PRAVANIE Maybe, but I don't know it.

This was the general response from all the learners. In order to simplify the problem, the researcher now asked the learners to consider the same situation with only two villages.

RESEARCHER Let's simplify this problem by taking two villages – let's suppose we had two villages represented in this diagram by I and J, okay? How would I find a point that is exactly the same distance away from I as it is from J?

All the learners responded that the most suitable point would be the midpoint. Christina's responded as follows:

CHRISTINA You find the centre.

RESEARCHER Centre of what?

CHRISTINA Centre of I and J.

RESEARCHER What would that centre be called?

CHRISTINA The midpoint.

RESEARCHER So you would find the midpoint of I and J? Let's find the midpoint of I and J...... there it is , the midpoint is Q. What do you observe?

CHRISTINA Distance QI is 2, 2 the distance of QJ is 2,2 as well, so that means Q is equidistant from I and J.

She showed a lot of confidence and did not hesitate with her responses to the researcher's questions. The responses of the others were similar.

NIGEL	At the centre of the line.
RESEARCHER	What do we call that point?
NIGEL	Midpoint.

When Roxanne was asked to find the most suitable point between two villages her immediate response was:

ROXANNE	Yes, that will be easy.
RESEARCHER	Where will that be?
ROXANNE	In the middle, the midpoint of I & J.

Clearly the simplified version of the problem was easy to solve for the learners. More importantly for the learners, they were getting immediate feedback by using *Sketchpad.* By constructing the midpoint of the segment IJ and measuring the distances to I and J, they could see that it was equidistant.

Again using the idea that the midpoint may not be suitable as a real-life solution, the learners were asked if they could determine another point which was also equidistant from I and J. The purpose was to investigate whether learners could find other points and realize that there were infinitely many and that all lay in a line, that is, the perpendicular bisector. It should be pointed out to the reader that traditionally in South Africa, the perpendicular bisector is usually introduced in Grade 8 or 9 (or sometimes earlier), but only as a construction. In other words simply as a line that passes perpendicularly through the midpoint of a line segment, but its equidistance properties are traditionally never investigated (and perhaps only alluded to in passing).

Most learners gave the same response as Faeeza did.

RESEARCHER As we said earlier, in real life there might be a problem, in the sense that the best position that we chose may not be suitable, or cannot be used for construction, for various reasons. Do you think that I can find another point?

FAEEZA You can locate it around there (pointing), somewhere where it would be equal distance apart.

RESEARCHER Do that construction. ...(after a point and the segments were constructed) Okay how can we check whether it is equidistant apart?

FAEEZA You measure the distance from the village to the point.

RESEARCHER But they are not, what must you do now?

FAEEZA We move it... I'll drag this point around.

RESEARCHER How long should we move it around?

FAEEZA Till we get the .. till the distances are equal.

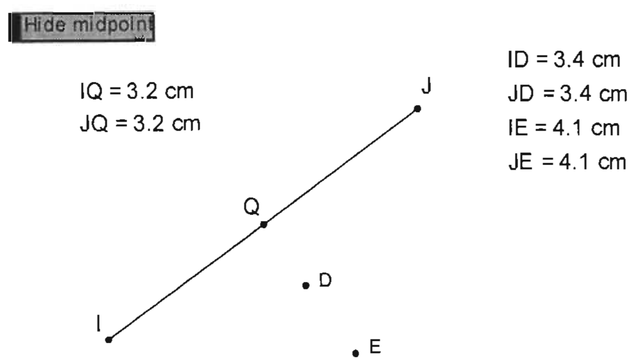


Figure 29

But Christina's response was different. She could see that the solution to the question lay on a line, but she could not determine exactly which line it should be, though she stated that it should be a line through Q.

RESEARCHER Earlier on we said that it might be possible that that point will not work for various reasons. Do you think there might be another point, which might be equidistant from J and I?

CHRISTINA If you move further away.

RESEARCHER If you move further away, and do what?

CHRISTINA Make a straight line on Q

RESEARCHER Would it be just any line?

CHRISTINA I think that it is a line but I'm not sure what line ....

Initially the researcher was unsure what Nigel meant, but it became clear that he was basically saying the same thing as the others. But it did indicate that different learners conceive these ideas differently and they develop their understanding of a situation in their own unique ways.

RESEARCHER	Click on the button for that construction. (after constructing) There you have the midpoint Q it is 2,2 away from I and 2,2 away from J. So it confirms that it is at the centre. Now as we said earlier there might be a problem at that point. What will you do?
NIGEL	I'll draw a triangle.
RESEARCHER	Draw a triangle?
NIGEL	Yes, so that you can measure the distance from I to the apex and from J to the apex.
RESEARCHER	Oh, you want to construct a point here and draw a triangle so that you could measure the two sides of the triangle?
NIGEL	Yes to measure distances from I and J…
RESEARCHER	All you want to do is measure the distances?
NIGEL	Yes.

The learners were quite clear about the fact that if one point was not suitable then another could be used - another point that satisfied the condition of being equidistant from two points. The researcher was initially unsure whether the learners would be able to make the deduction that all points that are equidistant from two points will lie on the perpendicular bisector. But the learners made this connection to the perpendicular bisector quite easily.

RESEARCHER	How many points do you think we can find?
NIGEL	Many….hundreds.
RESEARCHER	What do you notice about all of these points we have drawn?
NIGEL	All the points seem to be on a straight line
RESEARCHER	Construct a line and see if it is on a straight line. (after a while) Do they lie on the same line?
NIGEL	Yes
RESEARCHER	What else do you notice about this line?
NIGEL	It divides….. it cuts the other line in two ……. it is equal
RESEARCHER	What else do you notice?
NIGEL	It looks like a right angle here…….
RESEARCHER	Do you think it is a right angle? Do you want to check?
NIGEL	Yes
RESEARCHER	Measure it ……. (after a while) it is 90 degrees
NIGEL	Yes….I thought so.
RESEARCHER	So what does this mean? If I asked you for more points where would we find them?
NIGEL	On that line.
RESEARCHER	You are telling me that you can find many points on that straight line?

NIGEL	Yes.
RESEARCHER	Describe this line.
NIGEL	It is a straight line. It bisects there, it forms a right angle
RESEARCHER	So what is a line which forms a right angle with another line called?
NIGEL	A perpendicular line.
RESEARCHER	Is this only perpendicular?
NIGEL	It bisects too .... a perpendicular bisector.

Faeeza's response was similar.

RESEARCHER	Faeeza how many points do you think you can get like this?
FAEEZA	Quite a few, it depends on which... millions ...... if we are working with two villages....
RESEARCHER	So if we work with two villages we will have millions of points?
FAEEZA	Ya.
RESEARCHER	What do you notice about the points? Look at the points and what can you see there?
FAEEZA	Going in the same direction.
RESEARCHER	What does that mean?
FAEEZA	(hesitant) They look like they are on a line ....
RESEARCHER	How will I know that for sure?
FAEEZA	Draw a line.....?
RESEARCHER	Then draw a line. Okay, there you are – you were right - they lie on a straight line. What else can you observe about this straight line? Is it special in any way?
FAEEZA	Hmmm.
RESEARCHER	Okay, it might not necessarily be obvious but look at it carefully....
FAEEZA	They look perpendicular.
RESEARCHER	How can you be sure that it is perpendicular? It looks perpendicular, how can we know for sure that it is perpendicular?
FAEEZA	Hmmm................ Measure the angle?
RESEARCHER	Ok, try that. (after the angle is measured) There's the angle there, what is the angle?
FAEEZA	90 degrees.
RESEARCHER	What does that imply?
FAEEZA	It is perpendicular!
RESEARCHER	So if you had 2 villages where should you build a water purification plant that will be equidistant from both the villages?
FAEEZA	Hmmm ... We can go on a straight direction and move further away, it will still be equal distant that you can find...........move on the straight line...
RESEARCHER	How can you describe this line?
FAEEZA	It is perpendicular.
RESEARCHER	Is it only perpendicular?
FAEEZA	It cuts in the middle.
RESEARCHER	What do we call a line that is perpendicular and cuts a line into equal parts?
FAEEZA	Perpendicular bisector.

Roxanne took a little longer to arrive at her conclusion but she knew what transpired and found it only slightly more difficult to articulate her response.

RESEARCHER	How many points do you think?
ROXANNE	I don't know, just a lot.
RESEARCHER	Looking at these points does it show any pattern?
ROXANNE	Yes....they are forming a straight line.
RESEARCHER	How can we know that for certain?
ROXANNE	I can draw a line through these points.
RESEARCHER	(after a while) What do you observe?
ROXANNE	It passes through all the points.
RESEARCHER	What else can you observe about this line?
ROXANNE	I don't know.
RESEARCHER	Look at the line that we constructed. Is there something special about it?
ROXANNE	Well .. it looks like it is forming a 90 degrees angle here.
RESEARCHER	Do you think it might be?
ROXANNE	Yes...it looks like that...
RESEARCHER	How can we be certain?
ROXANNE	Measure the angle. (after a while) It is 90 degrees.
RESEARCHER	So what do we call this line?
ROXANNE	A straight line.
RESEARCHER	Yes it is straight line...but it is a special line.
ROXANNE	A bisector.
RESEARCHER	Yes it is a bisector...but it also has another property.
ROXANNE	Mmm...
RESEARCHER	What is this other property?
ROXANNE	It is perpendicular.
RESEARCHER	Yes...so what can we say about the line...
ROXANNE	It is a perpendicular bisector.

The researcher tried to determine whether the learners really understood the concept of the equidistant property of the perpendicular bisector. The following is Roxanne's response.

RESEARCHER	So if I asked you to find points that are equidistant from two points how will you do it?
ROXANNE	Draw the perpendicular bisector and then any point on the line will be correct.
RESEARCHER	So if we had two villages, where should we build the reservoir?
ROXANNE	Just determine the perpendicular bisector and any point on it will work.
RESEARCHER	Let us say that I drew a perpendicular line here (pointing to a point that is not the midpoint) ... will it work?
ROXANNE	No, it is not the same distance away.

At the end of this section of the interview, learners had discovered that in order to find any point equidistant from two points, the perpendicular bisector could be drawn. It was also evident that this conclusion was relatively easy for them to arrive at.

5.6 Return to the original problem

At this point the researcher was interested to see whether the learners would now immediately use the newly acquired concepts of equidistance and perpendicular bisector they arrived at for the problem with the four villages. The question that the researcher asked was:

RESEARCHER	Let's go on to the original problem, where we had four villages and we want to find a point which is equidistant to the four villages. How do you now think we should do this?

All the responses from the learners will be listed because this is an important aspect of the research. Some of the learners responded with confidence showing that they had very little doubt about the most suitable position.

Pravanie's response was a typical response where the learners simply use the recently acquired knowledge and insight related to the equidistant property of perpendicular bisectors.

PRAVANIE	...You create perpendicular bisectors of V1, V2.
RESEARCHER	V1, V2 or between V1 and V2.
PRAVANIE	Between V1 and V2.
RESEARCHER	Yes.
PRAVANIE	And between V2 & V3, V3 & V4 and V4 & V1.
RESEARCHER	Do you think that will work?
PRAVANIE	Most probably.

RESEARCHER	Click on that button provided.(after a while) So we have the perpendicular bisectors of the four sides. What do you observe?
PRAVANIE	They are all meeting at a point.
RESEARCHER	Does this surprise you?
PRAVANIE	Why should it...they are supposed to meet.
RESEARCHER	So where will you place the reservoir?
PRAVANIE	The point where they meet, will be the ideal place to put it.
RESEARCHER	So how do you know that this point will be equidistant from village 1, village 2, village 3 and village 4?
PRAVANIE	Measure the distances.

Nigel's response showed even more confidence although he was not using the correct technical terminology.

NIGEL	Divide these two lines into equal parts and then draw perpendicular lines and then do it again for the other two lines....
RESEARCHER	Are you saying that I should construct the perpendicular bisectors?
NIGEL	Yes.
RESEARCHER	Click on that button for the construction of the perpendicular bisectors. (after a while) And what do you notice?
NIGEL	It all measures up to one point.
RESEARCHER	Measures?
NIGEL	Meets.

Faeeza's was also confident. She said, "*just draw...*" clearly indicating that she knew exactly how to find the most suitable point.

FAEEZA	Just draw the perpendicular bisectors.
RESEARCHER	Perpendicular bisectors of which lines?
FAEEZA	All.
RESEARCHER	What do you think would happen if we drew the perpendicular bisectors of all?
FAEEZA	You will find a point where it might be equal.
RESEARCHER	What will be equal?
FAEEZA	The distances to V1, V2, V3 and V4.
RESEARCHER	Click on that button to see the construction of these 4 perpendicular bisectors. (after the click)) These are the perpendicular bisectors. What do you observe?
FAEEZA	It all comes to a point.......... They all meet at a common point.

Schofield's response was similar to the responses above.

SCHOFIELD	Construct perpendicular lines.
RESEARCHER	Do we just construct a perpendicular line?

SCHOFIELD	No, we construct all perpendicular bisectors.
RESEARCHER	Click on that button to construct all the perpendicular bisectors and see what happens. (after a while) What do you notice now?
SCHOFIELD	They meet at a point.

Note that he initially said, "*construct perpendicular lines*" but immediately thereafter corrected himself to "*construct all the perpendicular bisectors*". Roxanne, on the other hand, showed some hesitation initially but her response subsequently indicated that her hesitation was unfounded, because she was able to move on, making the correct responses thereafter.

ROXANNE	It will be hard, because you have four points…
RESEARCHER	Is it really that difficult?
ROXANNE	It looks difficult...but I think we should draw their perpendicular lines…no perpendicular bisectors.
RESEARCHER	Why do you want to draw the perpendicular bisectors?
ROXANNE	It worked nicely for the two villages…I don't know…maybe it will work for these villages too.
RESEARCHER	Would you want to see what would happen if we constructed the perpendicular bisectors?
ROXANNE	Yes sir.
RESEARCHER	Then click on that button there (after a while) There we have it...what do you observe?
ROXANNE	All the lines go to one point…

Christina found the same question slightly more difficult, but with some guidance from the researcher she was able to proceed further. This was expected because different learners make meaning of new ideas and concepts differently. But eventually (and this did not take very long) she showed that she understood. As is evident from the interview transcript, this learner required some prompting to arrive at a suitable solution.

RESEARCHER	Let us now go to the original problem. Here we have the quadrilateral again, village 1, village 2, village 3, and village 4. I want to know from you, besides what we did in the beginning… Remember we constructed a point and then we measured the distances for villages 1,2,3 and 4 and we moved it around, besides that method, do you think there is another way in which we can find a suitable position?
CHRISTINA	No

RESEARCHER	Wouldn't you like to try?
CHRISTINA	I don't know what to do.
RESEARCHER	There must be something that you have already learnt that you could use.
CHRISTINA	Is there?... I don't know sir.
RESEARCHER	Let us go back to the one with the 2 villages – How did we find the point that was equidistant between the 2 villages?
CHRISTINA	We used.... (exclaiming) Perpendicular bisector!
RESEARCHER	Do you think then that we might be able to use perpendicular bisectors here?
CHRISTINA	Maybe ...
RESEARCHER	Do you think we should try ?
CHRISTINA	Yes.
RESEARCHER	Click on the button provided and you will see the perpendicular bisectors.
CHRISTINA	It's too complicated (referring to the diagram)...but all the lines meet at one point.

The learners showed no particular emotion when they discovered that the four perpendicular bisectors met at a common point. It was as if they had expected it to happen and it was taken for granted that these lines would meet. None of them stated that this result really surprised them; in fact, most of them conveyed the idea that this was expected. This is perhaps also indicative of the social context within which learners construct meaning for problems given to them. Learners intuitively sense that the problem would not have been asked if there was no precise solution already "pre-existing", and this kind of (unrealistic) anticipation is likely to become a determinant force in their behaviour, unless they are given sufficient experiences to counter it.

Even Christina, who was initially unsure when working with four villages, indicated that the perpendicular bisectors should meet. It might have been interesting at this point if the researcher had tried to probe further in order to elicit from the learners the precise reasons for them believing that the lines must meet.

These are the responses of the learners.

RESEARCHER	Does this surprise you?
PRAVANIE	Why should it? They **are** supposed to meet.
SCHOFIELD	They meet at a point.
RESEARCHER	Does this surprise you?
SCHOFIELD	Not really... they had to meet somewhere...this was the best point.
RESEARCHER	Does it surprise you that these lines are meeting at one point?
ROXANNE	mmm...I don't know... it must meet there...where else can it meet?
RESEARCHER	Does this surprise you?
NIGEL	Just a little bit...but I knew that the lines will meet.
RESEARCHER	Does this surprise you?
FAEEZA	Not really.... It was kind of obvious.
RESEARCHER	Does this surprise you?
CHRISTINA	They have to meet I think.

With the exception of Nigel, all the learners felt that the perpendicular bisectors ought to be concurrent. Only Nigel stated that he was a little bit surprised, but he also stated thereafter that he knew they would meet. This indicates a slight contradiction in his thoughts, but nevertheless he seemed to feel that it was obvious that the lines would meet.

The researcher next introduced the learners to the term "concurrency" as follows. The following transcript with Faeeza was typical.

FAEEZA	It all comes to a point.......... They all meet at a common point.
RESEARCHER	Does this surprise you?
FAEEZA	Not really.... It was kind of obvious.
RESEARCHER	Do you know what this point is called?
FAEEZA	I don't know...midpoint?
RESEARCHER	It is the point of concurrency.

Despite the fact that the learners felt that the perpendicular bisectors will meet, some of them found the fact that they did not meet at the middle somewhat strange. It is clear that Faeeza did not find the fact that they were concurrent strange, but it was the fact that they met at a point away from her initially perceived "middle" of the quadrilateral. It did manage to increase her interest in the problem.

RESEARCHER	Remember what you said initially, you said that the point should be at the centre? Is it different now?
FAEEZA	Yes...it is strange ....
RESEARCHER	Is that interesting?
FAEEZA	Very (emphatically).

Another point that arose from the interview, which might be of interest, is the fact that the computer package *Sketchpad* allowed the learners to become extremely confident about their work. Roxanne, for example, began to communicate with the researcher in a way, which indicated her happiness at knowing something to be true. She was elated when she found that the distances to the four vertices were the same - "*I knew it ... I was right*" and "*Please measure it, **I know** it must be the point*" (her emphasis in bold). Although the transcript does not adequately carry the emotion that she displayed, the researcher was quite taken aback by her enthusiasm. This kind of behaviour is often very difficult to achieve from learners using traditional non-investigative teaching approaches..

RESEARCHER	How will we know for certain that this is the correct point?
ROXANNE	We can measure it, I know it must be the point.
RESEARCHER	All right measure each distance...that is 3.8...again we have 3.8. What do you think the next two distances will be?
ROXANNE	The same...3.8 and 3.8.
RESEARCHER	Go on and check it out ...this is 3.8 and...this is also 3.8.
ROXANNE	I knew it...I was right.

At this point it was already becoming evident to the researcher that the learners were beginning to enjoy working with *Sketchpad* and their initial anxiety was slowly diminishing. This might have been the case for two possible reasons:

1. Sketchpad was allowing them to "see" evidence of their own arguments as they were working and / or

2. The manner in which the solution was being gradually modeled was easy and comfortable for the learners to work with.

Next the researcher informally introduced the learners to the circumcircle, but no formal definitions of circumcentre and circumcircle were given. A circle was constructed from the point of concurrency (the circumcentre) as center with the circumference passing through one of the vertices of the cyclic quadrilateral. The learners showed surprise in their facial expressions and the way in which they responded when they saw that the circle now passed through all the vertices of the quadrilateral.

The following are typical responses.

RESEARCHER	Now can you construct a circle here, using this point of concurrency as the centre, and let us see what happens. (after the circle was constructed) Now what can you say about the circle?
FAEEZA	The four sided figure lies on the boundary of the circle……….
RESEARCHER	Boundary? What is the boundary of a circle called?
FAEEZA	Sorry circumference.
RESEARCHER	Try to construct a circle from here passing through this point (*pointing to a vertex*) … what do you observe?
CHRISTINA	It … it passes through all four villages.
RESEARCHER	Can you construct a circle with the point of concurrency as the centre and move it slowly upwards to this vertex here.(after a while) What do you notice?
NIGEL	The four edges of the quadrilateral meet the circumference of the circle.
RESEARCHER	Edge?…….which are you referring to as the edge? (after learner points) …oh we call that a vertex……..all of these are vertices.

The diagram below is an example of what the learners saw on the computer screen as they worked through with the circumcircle.

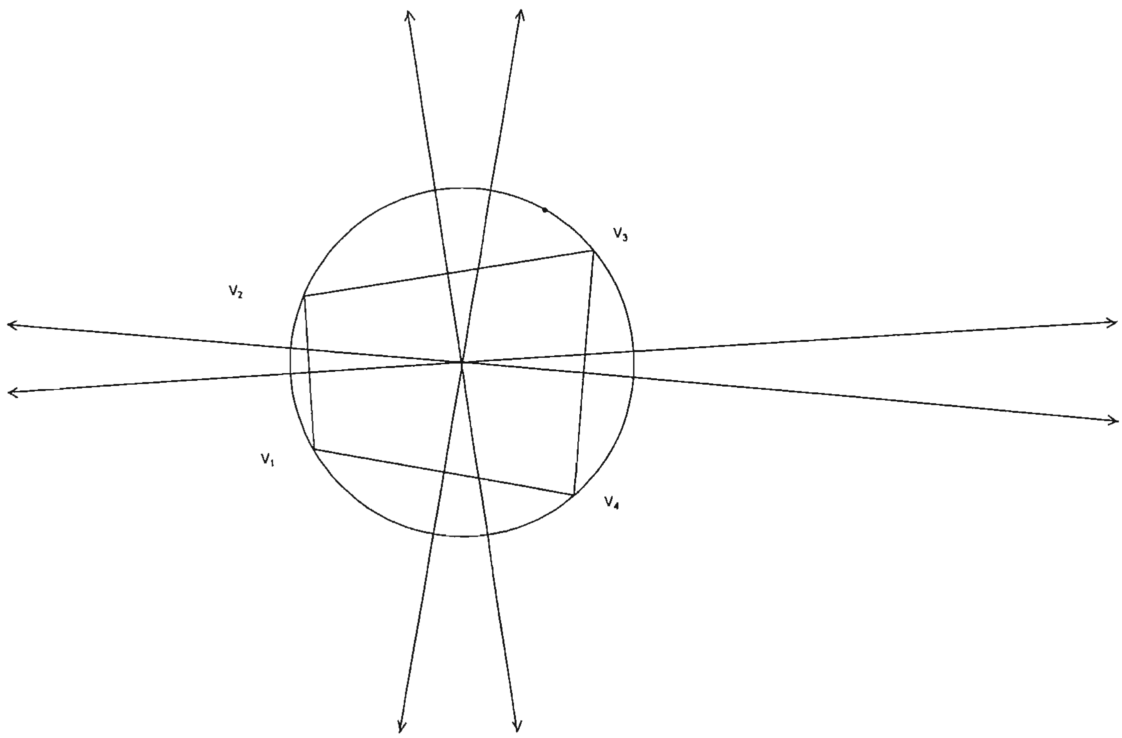


Figure 30

The learners could see that the circle passed through the vertices of the quadrilateral. Perhaps the researcher at this point should have probed further by asking the learners why the circumference of the circle passed through the vertices of the quadrilateral. The fact that the radius represents equal distances from the center to any other point on the circumference might have made it easy for them to see the connection with the point being equidistant from the vertices.

The learners were asked throughout the interview to relate what they were discovering to the real world. It was apparent that they could draw a parallel between the modeling exercise and the real world problem. A few examples, which were typical responses, are listed below.

RESEARCHER	So what does this imply when we go back to the original question?
FAEEZA	The reservoir must be built at the point where the perpendicular bisectors meet.

RESEARCHER	Which perpendicular bisectors?
FAEEZA	The bisectors of the lines joining the villages on the outside.

RESEARCHER	In terms of building the reservoir what must be done?
SCHOFIELD	You just draw lines from one village to another, construct their perpendicular bisectors and where they meet will be the place for building the reservoir.

RESEARCHER	So if we look at the four villages that I gave you initially, where must the reservoir be built?
NIGEL	Find the perpendicular bisectors of the sides and the point of concurrency will be the correct point.

The intention of the next aspect was mainly to see if learners would realize that the perpendicular bisectors of a quadrilateral were not necessarily always concurrent, that is, that for a quadrilateral it was not always possible to find an equidistant point. The dynamic facility of *Sketchpad* not only facilitated this finding quite easily, but assisted some learners to realize that an equidistant point would only exist when the circle passed through the four vertices (that is, the quadrilateral is cyclic).

With *Sketchpad* it was easy to change the size and shape of the given quadrilateral, so that the learners could observe whether the perpendicular bisectors were concurrent and whether the circumference of the circle still passed through the vertices of the quadrilateral.

Roxanne was taken aback by what she observed but understood exactly the implications of what she observed. In fact when she was asked whether she wanted to see more examples she was emphatic about the fact that that was not necessary.

RESEARCHER	Let us see what would happen if we drew a circle from this point of concurrency. Use the circle function there and draw a circle up to this point here. (after a while) Describe what you observe.
ROXANNE	All the points go around…
RESEARCHER	What do you mean by that Roxanne?

ROXANNE	The circle goes around the points of the quadrilateral.
RESEARCHER	Circle goes around the points...?
ROXANNE	See these vertices of the quadrilateral...the circle passes through them.
RESEARCHER	All right...I see what you mean. Do you think that this will always be the same for any quadrilateral?
ROXANNE	Yes.
RESEARCHER	So no matter how big it is, no matter how small it is, what shape it is, it will...
ROXANNE	Always be the same as before.
RESEARCHER	So if I make this a larger or a smaller quadrilateral, will it always work in the same way?
ROXANNE	Yes...it will always be the same.
RESEARCHER	When we say that it will always be the same what are we referring to?
ROXANNE	The lines will meet at the same point ... and the circle will go through these points here (pointing)
RESEARCHER	Can you check? Grab this point here and change the quadrilateral. Now what do you observe? Is this still a quadrilateral?
ROXANNE	Yes it is still a quadrilateral...but something is not working.
RESEARCHER	What is not working?
ROXANNE	They are not meeting.
RESEARCHER	What is not meeting?
ROXANNE	These perpendicular lines and the circle...
RESEARCHER	Yes go on.
ROXANNE	The circle is not touching here and here and here...
RESEARCHER	Maybe there is a problem with this quadrilateral...let us try another one.
ROXANNE	Yeah...(uncertain)
RESEARCHER	Drag this vertex like this. (after a while) Are the points concurrent now?
ROXANNE	Yes.
RESEARCHER	Let us try another one... drag this again. What is happening now?
ROXANNE	It's just like the other one.
RESEARCHER	What do you mean?
ROXANNE	They are not meeting again...and the circle...
RESEARCHER	What about the circle?
ROXANNE	These lines are not concurrent and the circle does not pass through these vertices
RESEARCHER	When do you think that the circle will pass through the vertices?
ROXANNE	When the perpendicular bisectors are concurrent.
RESEARCHER	Are you sure about that?
ROXANNE	Very.
RESEARCHER	Would you want me to try more examples of quadrilaterals?
ROXANNE	No...it is not necessary... I know what I'm saying.

She was initially confident that the perpendicular bisectors would be concurrent for all quadrilaterals and when the quadrilateral began to change some doubt was introduced. Her words "*yes it is still a quadrilateral ... but something is not working*" was a turning point in her interview. Introducing doubt served to create a surprise factor that helped her to grasp an idea that might have otherwise taken much longer. She had clearly also made a connection between the circle passing through all four vertices and the concurrency of the perpendicular bisectors.

Nigel made the same connection immediately and although it is not clearly captured in the transcript of the interview, he was astounded at what he saw. This is exactly what the other learners experienced.

RESEARCHER	So the vertices of this quadrilateral lie on the circumference of the circle. Do you think that if I had any kind of quadrilateral the result would be the same?
NIGEL	Yes (very confidently)
RESEARCHER	So no matter what kind of quadrilateral, the perpendicular bisectors will always be concurrent?
NIGEL	Yes (emphatically)
RESEARCHER	Perhaps we should check. Change the quadrilateral and see what happens to the circle?
NIGEL	It went away from the vert…vertices.
RESEARCHER	Oh and what else do you notice about the perpendicular bisectors?
NIGEL	It does not meet at all.
RESEARCHER	Change it again, and let see if we can make them meet again. Yes they are meeting again but what do you observe?
NIGEL	The perpendicular bisectors are concurrent….
RESEARCHER	What can you say about the vertices of the quad.
NIGEL	It touches…. It lies on the circle.

Schofield's interview was very similar. He showed a lot of confidence in what he was saying until he realized that his hypothesis was not working very well. Notice the change when he said, "*it is still a quadrilateral but...but...*" This was an important teaching point because what he observed was a contradiction to his initial belief. The surprise (more than just his hesitation) he experienced was sufficient to ensure that he informally learnt the new concept of a cyclic quadrilateral and perhaps would remember it for longer. Of course, this could not be tested, as there were no follow up interviews.

RESEARCHER	Yes. Whilst we are at this point, I'd like you to construct a circle using this point of concurrency as the centre. (after a while)...no up to this point. What do you observe?
SCHOFIELD	The four villages are around the circle
RESEARCHER	What do you mean by around the circle?
SCHOFIELD	V1, V2, V3 and V4 are on this part of the circle.
RESEARCHER	You mean that the 4 villages lie on the circumference of the circle?
SCHOFIELD	Yes.
RESEARCHER	Do you think that this is going to be true for any quadrilateral?
SCHOFIELD	Yes
RESEARCHER	Do you think that if we made this quadrilateral a larger one or a smaller one the perpendicular bisectors will still be concurrent?
SCHOFIELD	Yes.
RESEARCHER	Change this quadrilateral by dragging it. ... yes like that... Is it still a quadrilateral and are the perpendicular bisectors meeting?
SCHOFIELD	It is still a quadrilateral but... but ...
RESEARCHER	What is going on?
SCHOFIELD	I don't know ... the perpendicular bisectors are not meeting.
RESEARCHER	Does that surprise you?
SCHOFIELD	Yes, because I thought that it will always meet.
RESEARCHER	What did you notice about the circle?
SCHOFIELD	The circle is not on the quadrilateral.
RESEARCHER	What do you mean by that?
SCHOFIELD	These corner points are not touching the circumference of the circle.
RESEARCHER	When are the vertices on the circumference of the circle?
SCHOFIELD	Only when these lines meet.
RESEARCHER	Which lines are you talking about?
SCHOFIELD	The perpendicular bisectors.
RESEARCHER	How can I say this is a nice way?
SCHOFIELD	The corner points of the quadrilateral...the vertices will be on the circle...if ... if the perpendicular bisectors meet... if they are concurrent.

This surprise factor was also evident in the interview with Pravanie. Although she did not say that she was surprised, she claimed that "*this is strange*". By this she implied that what she saw differed from what she expected and this created some conflict in her mind. Obviously she accepted what she saw because she trusted the authenticity of *Sketchpad.* Furthermore, when asked the next question she remained silent. Her silence does not in any way imply that she did not know what was going on, but simply that she was amazed by what she saw. This becomes clear if one reads her response thereafter. She made the association quite easily and she did not require much prodding to do so.

RESEARCHER	Grab this point and move it. Does the quad still lie on a circle?
PRAVANIE	No ... this is strange...
RESEARCHER	When are they meeting each other?
PRAVANIE	(*silence*)...
RESEARCHER	When do these perpendicular bisectors meet?
PRAVANIE	When the vertices of the quadrilateral lie on the circumference of the circle.
RESEARCHER	So what does this mean?
PRAVANIE	The perpendicular bisectors meet only when the vertices lie on the circumference of a circle.

So the learners who could only initially solve the problem by dragging a point around, were now quite comfortable with working with it after they worked through a simpler problem. In fact, the confidence, which they displayed, exceeded the researcher's expectations. Although this has not been shown, the researcher is confident that this level of conviction and confidence can only be achieved when working with dynamic software such as *Sketchpad*. Faeeza exemplified this confidence in the following extract:

FAEEZA	We only get the lines to be concurrent when the vertices of the quadrilateral are on the circle.
RESEARCHER	Do you think that this is always true?
FAEEZA	Yes (confidently)
RESEARCHER	Do you want to construct more diagrams for you to confirm this?
FAEEZA	No please (emphatic and irritated).
RESEARCHER	Are you convinced?
FAEEZA	Yes...very.

When asked whether she wanted to view more examples, she quickly replied, "*no please*", indicating that what she had seen was sufficient and that there was no need to prolong the point. Furthermore, when asked whether she was convinced, she did not just say "yes" but added that she was "*very*" convinced. Even Nigel showed that he was quite convinced when he exclaimed "*definitely*".

RESEARCHER	Do you think that this is always true?
NIGEL	Definitely!
RESEARCHER	Should I construct more examples for you to test what you are saying?
NIGEL	No sir, I'm quite convinced.
RESEARCHER	What do you think of this?
NIGEL	It is very interesting.

Roxanne, who had hesitated initially, also displayed tremendous confidence. She indicated that she was sure about her hypothesis and when asked whether she needed more examples her reply was "*no... it is not necessary...I know what I'm saying*". Her tone indicated that once she was convinced, there was no point in continuing with the same aspect – she just wanted to go on.

RESEARCHER	When do you think that the circle will pass through the vertices?
ROXANNE	When the perpendicular bisectors are concurrent.
RESEARCHER	Are you sure about that?
ROXANNE	Very.
RESEARCHER	Would you want me to try more examples of quadrilaterals?
ROXANNE	No...it is not necessary... I know what I'm saying.

Schofield's responses were similar. The level of conviction was quite high.

RESEARCHER	Are you convinced that this is true?
SCHOFIELD	Yes, I am.
RESEARCHER	Should I try further examples?
SCHOFIELD	You can, but the answer will always be the same.

All learners displayed high levels of conviction despite the fact that the amount of time they spent in working with the problem was not long as compared to what they would have had to do if they worked with pencil and paper.

An important omission on the part of the researcher was the absence of any further discussion of the implication of their finding that not all quadrilaterals are cyclic, and to investigate what they thought one might do to position the water reservoir in such a case.

This might have provided another valuable lesson on the relationship between mathematics and the real world, and the modeling process.

5.7 Problem two

The learners were next presented the same problem, but now within the context of a triangle. The questions that they were asked were not different from the previous ones and the responses they made were, as expected, in accordance with what they saw in the previous case. The question that they were asked was:

RESEARCHER	Let us now look at this case where we might have 3 villages, suppose we had three villages represented on this diagram by T, U and B. How do you think I would be able to locate the position within that triangle that will be equidistant from T, U and B?

The learners confidently responded that they would again use perpendicular bisectors to locate the required point. In fact, the level of their conviction was very high and this could be determined from the way they responded and the language they used. Consider the language used by Christina in the transcript below.

CHRISTINA	Use perpendicular bisectors.
RESEARCHER	Click on the button there and tell me what you observe.
CHRISTINA	They meet at a point - they are concurrent.
RESEARCHER	So are you saying that this is the most suitable point?
CHRISTINA	Yes, it must be.
RESEARCHER	How can we be sure?
CHRISTINA	Measure the distances.
RESEARCHER	Do you want to measure it?
CHRISTINA	It's up to you sir.....because I'm sure.

When asked whether the point of concurrency was the most suitable point, she replied, "*it **must** be*". Furthermore, when asked whether she wanted the distances measured, she replied, "*it's up to you sir...because I'm sure*". This is the language of a child who is beyond any doubt. She was convinced from the previous examples that this will always be true. Roxanne displayed similar levels of conviction. She stated that she was not surprised that the perpendicular bisectors were concurrent and she did not think that it was necessary to measure the distances because she knew that they would be equal.

ROXANNE	Construct the perpendicular bisectors of all the sides.
RESEARCHER	Do you think that this will work?
ROXANNE	I don't know but it worked for the quadrilateral.
RESEARCHER	Go ahead and construct the perpendicular bisectors ...(after a while)... what do you observe?
ROXANNE	They are concurrent!
RESEARCHER	Does that surprise you?
ROXANNE	No, because I thought that the lines will meet at one point.
RESEARCHER	How do you know that at this point (pointing to the point of concurrency) will be equidistant to all three villages?
ROXANNE	We can measure it.
RESEARCHER	Do you think that we should?
ROXANNE	I don't think that it is necessary.
RESEARCHER	But what if you are wrong?
ROXANNE	I don't think so...but maybe I should just measure it very quickly.
RESEARCHER	All right. (after a while) There you have the measurements. What do you observe?
ROXANNE	As I said, the measurements are the same.

Pravanie knew that using perpendicular bisectors would work but she did not actually mind doing the construction and checking. However, she did state that she knew that the point of concurrency would be the most suitable.

PRAVANIE	We would use perpendicular bisectors again.
RESEARCHER	Well do you want to try it?
PRAVANIE	Yes
RESEARCHER	(*the perpendicular bisectors were constructed*) So you are saying that this point at which the perpendicular bisectors are meeting will be the best position?
PRAVANIE	Yes
RESEARCHER	How can we check?
PRAVANIE	Measure it (*referring to the distances from the point of concurrency to the vertices*).
RESEARCHER	(*after it was measured*) Are they equal?
PRAVANIE	Yes, I knew they would be.

The act of making a hypothesis and receiving feedback immediately removes the negative impact of learners making incorrect assumptions. It was expected that the learners would assume from what they had learned earlier for quadrilaterals that the perpendicular bisectors are also not always concurrent in a triangle.

Schofield's response demonstrates how he thought his observation for quadrilaterals would also apply to triangles; that is, it would work only for certain triangles. The relevant responses are highlighted in bold in the transcript below.

RESEARCHER	Should we construct the perpendicular bisectors?
SCHOFIELD	Yes.
RESEARCHER	Do that then ... (after a while) What do you observe?
SCHOFIELD	The lines are concurrent.
RESEARCHER	But how do we know that the point of concurrency is equidistant from the three villages?
SCHOFIELD	**They will be**...but I can measure it.
RESEARCHER	(after a while) can you see these measurements?
SCHOFIELD	Yes...they are all equal.
RESEARCHER	Does this surprise you?
SCHOFIELD	**No...I knew they will meet.**
RESEARCHER	Do you think that they will meet for all triangles?
SCHOFIELD	**No.**
RESEARCHER	Why?
SCHOFIELD	**It must be the same for the four-sided figures.**
RESEARCHER	What do you mean?
SCHOFIELD	**For the quadrilateral it only worked for some...when the circle touched the four corners...it might be the same for the triangle.**

Faeeza had made similar responses. But Faeeza's response was quite interesting. She initially stated that by constructing the perpendicular bisectors we could find the most suitable position. But she immediately changed her mind to include something that she had just learned for quadrilaterals – a qualification she felt was necessary. She felt that the perpendicular bisectors would only meet if a circle passed through the vertices of the triangle.

FAEEZA	**Use perpendicular bisectors.**

RESEARCHER	Do you think they will meet?
FAEEZA	**They must meet** .... no....... they will only meet if the vertices of the triangle lies on a circle.
RESEARCHER	So the perpendicular bisectors will only be concurrent in certain triangles?
FAEEZA	**Yes ...... only those for which the vertices of the triangle lie on a circle.**
RESEARCHER	Oh ..... construct the perpendicular bisectors........ (after a while) What do you observe?
FAEEZA	The perpendicular bisectors are concurrent.

The aim of the next section was to see if the learners would be surprised to dynamically observe that the perpendicular bisectors were always concurrent irrespective of the shape of the triangle. The level of conviction after seeing a number of different triangles was very high. In fact, most of the learners stated that they were a hundred percent convinced that the perpendicular bisectors were always concurrent. It is the researcher's opinion that this high level of conviction arose as a result of the surprise that the learners experienced when they observed that in contrast to the quadrilaterals the perpendicular bisectors of the sides of any triangle were concurrent, and therefore also that every triangle was concyclic.

Roxanne's responses were of particular interest. She showed a lot of confidence initially in her extrapolation from quadrilaterals to triangles, and she was therefore quite taken aback by what she later observed. It was clear that her observation differed from her expectation and this created a high level of surprise at finding out that the perpendicular bisectors of all triangles are concurrent. Special note must be taken when reading the transcript by her sudden silence and the doubt that followed. It was at this point that she realized that something different had happened. Her exact words were "... **but this is different...it sort of is strange...**". What made this strange? Perhaps it was the immediate recognition that what she observed challenged her belief. This entire process took no more than a few minutes. Although she stated that she was only 99 % convinced, it was clear that she was being extremely cautious and did not want to commit herself to saying that she was a 100

% convinced. This could have been the result of the fact that her firm conviction was just proved incorrect and she was not prepared to make another "mistake". But 99 % conviction was still very high. She felt that this was "strange" because she fashioned her initial hypothesis on what she observed with quadrilaterals. Refer to the transcript below in order to see Roxanne's responses.

RESEARCHER	So if we made this triangle bigger or smaller, or just changed the shape, what do you think would happen?
ROXANNE	A circle wouldn't pass through these points...vertices and the perpendicular lines will not be concurrent.
RESEARCHER	Are you sure about that?
ROXANNE	I already told you sir...it was like that for the quadrilateral.
RESEARCHER	Should we check by changing the size or shape of the triangle?
ROXANNE	Yeah...
RESEARCHER	Drag one vertex and observe what happens ...what do you observe?
ROXANNE	(**silence**)...it still is the same...
RESEARCHER	What do you mean by it still is the same?
ROXANNE	The lines are still concurrent... but let me move it this way and let me see.
RESEARCHER	All right....
ROXANNE	... but...but... it still is the same...even the circle is passing through the vetexes.
RESEARCHER	The word is vertices.
ROXANNE	Vertices...**but this is different...it sort of is strange**...
RESEARCHER	Strange? Why?
ROXANNE	**It is not what I expected...it's like you think it's going to be like the quadrilateral but suddenly it's different.**
RESEARCHER	You look surprised.
ROXANNE	I am...I was so sure that the lines will not be concurrent...
RESEARCHER	Which lines are you talking about?
ROXANNE	The bisectors.
RESEARCHER	Are they just bisectors?
ROXANNE	No, they are perpendicular as well.
RESEARCHER	Are you convinced that this will be true for all triangles?
ROXANNE	Yes...I saw it when I dragged that point around.
RESEARCHER	How many percent convinced are you?
ROXANNE	99%....
RESEARCHER	99%? You are still unsure because you did not say 100%. Why?
ROXANNE	No...sir I am quite convinced but what if there is one triangle somewhere for which it won't work?
RESEARCHER	Do you want to move the triangles around some more?
ROXANNE	No. That is not necessary.
RESEARCHER	So how can you become 100% convinced that it will always work?
ROXANNE	I don't know...I'm quite sure but just in case...I thought that it will be the same as for the quadrilaterals...but I was wrong.

RESEARCHER	So are you quite convinced but you are afraid that you might make a mistake?
ROXANNE	Yes.

Schofield was, initially, also very sure. His conviction arose from his experience with the quadrilateral. He was surprised to see that the lines were always concurrent for a triangle, but was still eager to try it for more triangles – " **No...I think move it a bit more**". He was still unconvinced that the result was true for all triangles. After working through more examples, his response to the question "**are you satisfied that the perpendicular bisectors are always concurrent?**", he replied "**Yes...but I never would have guessed that!**". This process also took only a few minutes but he was very convinced because he saw the changes on the computer screen ("**No doubt...I saw it myself!**"). The implication of this may be that he would not have accepted the truth of the result had he been just told, but the fact that he could dynamically see the result convinced him that the result was true.

RESEARCHER	Do you think that they will meet for all triangles?
SCHOFIELD	No.
RESEARCHER	Why?
SCHOFIELD	It must be the same for the four-sided figures.
RESEARCHER	What do you mean?
SCHOFIELD	For the quadrilateral it only worked for some...when the circle touched the four corners...it might be the same for the triangle.
RESEARCHER	Would you like to check what would happen if you changed the triangle?
SCHOFIELD	Yes.
RESEARCHER	(after a while) What do you observe when you move this point around?
SCHOFIELD	The lines are always concurrent!
RESEARCHER	Do you think that this will always be the case?
SCHOFIELD	No...I think I should move it a bit more.
RESEARCHER	Is there any particular way that you want to move it?
SCHOFIELD	Yes...move this corner about (pointing to the lower left hand corner)
RESEARCHER	Okay ...(after a while) so, what is happening now?
SCHOFIELD	It still is the same...the lines are always concurrency.
RESEARCHER	Concurrent.
SCHOFIELD	Yes.
RESEARCHER	Are you satisfied that the perpendicular bisectors are always concurrent?

SCHOFIELD	Yes...but I never would have guessed that!
RESEARCHER	Why?
SCHOFIELD	I thought that all of the figures will be the same.
RESEARCHER	But they are.
SCHOFIELD	No...no...I mean the different figures...the quadrilateral and the triangle.
RESEARCHER	Are you convinced that the perpendicular bisectors are always concurrent?
SCHOFIELD	Yes.
RESEARCHER	How many percent convinced are you?
SCHOFIELD	100%
RESEARCHER	You have no doubt in your mind?
SCHOFIELD	No doubt...I saw it myself!

When Roxanne realized that her initial hypothesis was untrue she kept silent for a while. The researcher interpreted this as surprise at her sudden realization that her initial belief was wrong. Perhaps the researcher should have asked the following question, "*how many percent convinced are you*?", before allowing the learners to observe the result for other triangles. Although this is mere postulation, the researcher is convinced that most learners, who were interviewed, would have shown a high level of conviction that their initial beliefs were correct. Christina responded in a similar way as Roxanne when she went silent at the same point in the interview. She initially stated that she was 90 % convinced and thereafter changed it to a 100 % after seeing more examples. It was also very difficult to ascertain an exact percentage of their conviction, but from the way they responded, the researcher believes that they were fairly convinced.

RESEARCHER	Do you think that if we change the size of the triangle and the shape of the triangle, do you think it will always remain the same?
CHRISTINA	No.
RESEARCHER	You do not think so?
CHRISTINA	No.
RESEARCHER	Grab this point here and move it around and change the triangle. What do you observe?
CHRISTINA	(Silence)
RESEARCHER	What do you observe?
CHRISTINA	They are still concurrent.
RESEARCHER	What if the lines are meeting on the outside? Drag this vertex across, what do you observe?

CHRISTINA	They are still concurrent.
RESEARCHER	So, what can you say now? Do you think that for every triangle the perpendicular bisectors are concurrent?
CHRISTINA	Yes.
RESEARCHER	You think so? How sure are you? If I ask you how many percent sure are you what would you say?
CHRISTINA	90 %
RESEARCHER	10% unsure? What's making you unsure that it will always be concurrent?
CHRISTINA	The size.
RESEARCHER	What about the size is making you unsure?
CHRISTINA	I'm not sure about very big triangles.
RESEARCHER	Go ahead and make this triangle very big. (After awhile). Does that make any difference?
CHRISTINA	Yes.
RESEARCHER	How many percent sure are you now?
CHRISTINA	Now, 100%.
RESEARCHER	Are you surprised at the result?
CHRISTINA	Yes.
RESEARCHER	Why?
CHRISTINA	No matter how or which way you turn the triangle it still met at that point.

Nigel was also quite surprised at the result and jokingly commented that " **This is very surprising....... I think you tricked me...(laughing)**"

RESEARCHER	Do you think that in every triangle the perpendicular bisectors would be concurrent?
NIGEL	They won't.
RESEARCHER	Why are you saying that they won't?
NIGEL	There are different types of triangles.
RESEARCHER	Let us check. Construct a different triangle by dragging this vertex. Can you see the triangle changing? Is the size changing?
NIGEL	Yes.
RESEARCHER	Is the shape changing?
NIGEL	Yes.
RESEARCHER	And what is happening to the perpendicular bisectors?
NIGEL	It is the same. There is always a midpoint.
RESEARCHER	Midpoint?
NIGEL	A point of concurrency.
RESEARCHER	What do you think of this?
NIGEL	**This is very surprising....... I think you tricked me...(laughing)**
RESEARCHER	Why?
NIGEL	I thought that there will not always be a point of concurrency because of the different sizes and shapes. In the quadrilateral it was different.

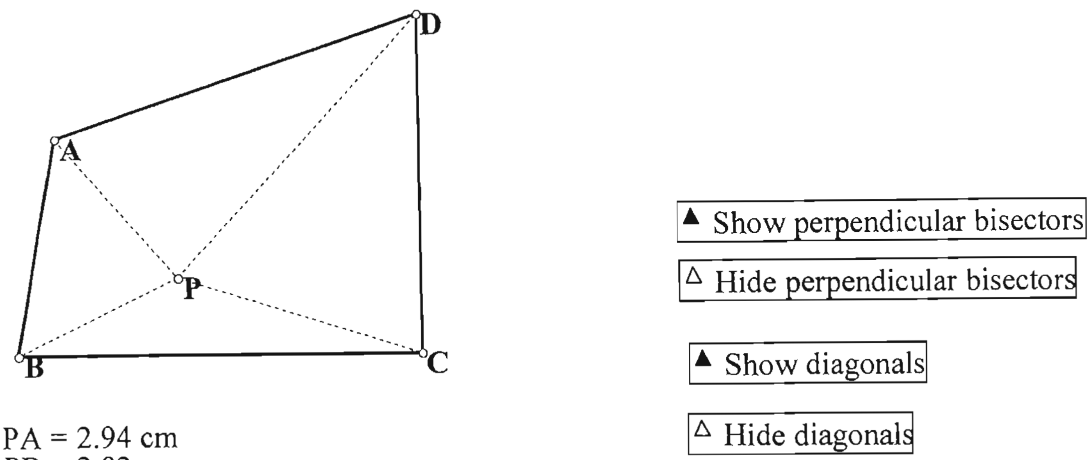
The majority of the learners (83%) were a 100% convinced that the perpendicular bisectors of **all** triangles are concurrent, with the rest being 90 % convinced.

It is also important at this point to mention learners' high levels of conviction are often due to the fact that children tend to over-generalize and gain confidence very quickly (Chazan, 1993 : 361). This is, according to Balacheff (1988 : 218), the result of naïve empiricism, which *"consists of asserting the truth of a result after verifying several cases*". However in order to become mature in mathematics, they need to have experiences that would help them become healthy skeptics. Thus in an environment were one works with mediating artifacts such as *Sketchpad*, which develops high levels of conviction quickly, one needs to develop and encourage learners' skepticism about what they observe. Thus the initial activities that they were engaged in convinced them that what they observed was always true, but the follow up activity with the general quadrilateral just as quickly convinced them that their conjectures do not always hold true. However it is clear from the triangle activity that their skepticism has not yet fully developed to question the general validity of the conjecture for all possible triangles, from only empirical evidence. It would appear that formal proof as a means of verification might not yet be appreciated by these learners. In the follow up activity reported in section 5.9, however, use was made of their need for explanation to introduce them to deductive proof.

5.8 Problem three

At this point learners were given a new *Sketchpad* sketch. They did not know that it was not a cyclic quadrilateral, but for the purposes of this investigation it did not matter whether the diagram was a cyclic quadrilateral or not. The question posed to them is copied below.

The government decides to build pipelines from the water reservoir to the four villages A, B, C and D. Where should the water reservoir be placed so that the total length of the pipeline is minimized, e.g. so that the SUM of the distances to the four villages is a minimum (as small as possible)?



PA = 2.94 cm
PB = 2.82 cm
PC = 4.02 cm
PD = 5.62 cm
PA + PB + PC + PD = 15.40 cm

Figure 31

Apart from obviously greater convenience, an important practical reason for building pipelines from the water reservoir to the villages, instead of having the villagers walk to the reservoir for water collection, can also be found in environmental concerns. For example, having hundreds of villagers on a daily basis walking to the water reservoir and back is bound to have some negative consequences on the natural bush. Numerous paths may be trundled out; increasing the possibility of soil erosion, and plant and animal life may be disturbed, not to speak of the likelihood of litter accumulating along the paths and so on. Note must also be made of the great distances that rural communities have to walk because homes are built in a rural setting, that is, homes are built such that householders would be able to establish vegetable gardens or maintain livestock. This leads to people even in the same village living very far apart, which therefore means that the distances that they would have to walk for water collection would be untenable. Unfortunately the researcher did not question the learners about possible reasons for why the pipelines should be built.

However, the researcher attempted to determine from the learners the reason for obtaining the minimum sum before they were asked to guess the best position.. This question was important in order to establish whether learners understood the economic (financial) context of the question. It was surprising to notice that learners were quick to see the link between minimization requirement and the financial aspects of the real world. Their responses are listed below.

PRAVANIE To save them money – they will use less pipes and they will save on labour.

CHRISTINA To save on the cost.

FAEEZA It will save costs… the less pipe we use the cheaper it will be.

NIGEL To save money.

ROXANNE Because of the costs involved.

SCHOFIELD Save costs…

It was thus quite evident that the learners could identify with some aspects of the real world.

The learners were then asked to find the best position. As anticipated, all the learners responded that perpendicular bisectors should be used.

CHRISTINA	Use the same method - perpendicular bisectors.
FAEEZA	Construct the perpendicular bisectors.
NIGEL	You can use the same rule
RESEARCHER	What rule is that Nigel?
NIGEL	The perpendicular bisectors rule.
ROXANNE	Construct the perpendicular bisectors.
RESEARCHER	Do you think that this is the best way to do it?
ROXANNE	That is the best way!
SCHOFIELD	I can't tell just like that, but we can construct the perpendicular bisectors.

All of the learners listed above confidently stated that the obvious choice of method should be the use of the perpendicular bisectors. Schofield was perhaps a bit unsure of how to locate the best point, but nevertheless suggested that the use of perpendicular bisectors was worth a try.

However, Pravanie's response was interesting. She very quickly surmised that this quadrilateral might not be cyclic, and she immediately requested to test whether this was a cyclic quadrilateral. Although this was a different response, it still showed that she would have used the perpendicular bisectors if this was a cyclic quadrilateral.

PRAVANIE	Since all quadrilaterals do not always have perpendicular bisectors meeting I think we should check whether the vertices of this quadrilateral lie on the circumference of the circle first.
RESEARCHER	Will it make a difference to your answer?
PRAVANIE	Yes. Let's just construct the perpendicular bisectors.
RESEARCHER	(after a while)... What do you observe?
PRAVANIE	The perpendicular bisectors do not meet. This means that the vertices of the quadrilateral do not lie on the circumference of the circle.
RESEARCHER	Okay. What should we do next?
PRAVANIE	I can move this point.
RESEARCHER	Okay. What should we do next?
PRAVANIE	Try moving the point.

Their choice of perpendicular bisectors is hardly unexpected and is simply a confirmation of Ausubelian and other theories, namely, that new experiences and problems of learners are always related to and interpreted according to their past experiences and knowledge. Although they felt strongly about the perpendicular bisectors, the immediate feedback that they received from using *Sketchpad* ensured that they review their method of finding the ideal position. When their hypotheses were tested and it was found that it did not work, they changed their method to simply selecting the point and dragging it around until the sum of the distances became a minimum.

This took a while but eventually they found a point that closely resembled the position that resulted in the sum of the distances being a minimum. At this point they were quick to realize that the ideal position was at the intersection of the diagonals of the quadrilateral. This was an important discovery for them because it was different from using perpendicular bisectors. Since this finding contradicted their initial expectation, it resulted in conceptual restructuring through the learning process of accommodation. Although not articulated, the learners have learnt (at least for the moment) that the minimization of the sum of the distances does not necessarily occur at the point of equidistance (which in this case does not even exist).

PRAVANIE	Yes.(after a while) It will be a minimum here (pointing) ...
RESEARCHER	Are you sure?
PRAVANIE	.... (*after a while*) Yes
RESEARCHER	What does that remind you off?
PRAVANIE	Looks like the diagonals.
RESEARCHER	Why don't you click on that button for the diagonals? What do you observe?
PRAVANIE	Our lines and the diagonals are the same.
RESEARCHER	So what does this mean?
PRAVANIE	This means that the reservoir should be placed at the point where the diagonals meet.

From Pravanie's responses, it is clear that determining the general rule was easy when working with *Sketchpad.* It took a mere 60 seconds to establish that the intersection of the diagonals should be used to determine the ideal position. The responses were similar for all of the learners. Christina also took a few seconds to arrive at the general conclusion.

RESEARCHER	Okay. What do you observe?
CHRISTINA	These... these are the diagonals ......
RESEARCHER	Does it really look like the diagonals?
CHRISTINA	Yes.
RESEARCHER	Click on the button to construct the diagonals. What do you observe?
CHRISTINA	The.... The point where the diagonals meets ...that's the point where you must build the reservoir
RESEARCHER	Are you sure?
CHRISTINA	Yes.
RESEARCHER	What would happen if we changed the size of this quadrilateral? Where do you think the position would be?
CHRISTINA	Where the diagonals intersect.

Two points should be mentioned here. Firstly, the learners found the results quite surprising. Again, this is an essential teaching point. The fact that the result was contrary to what they had initially believed means that they again had to correct their initial perception through the process of accommodation. Christina is a good example of the reaction the learners had when working with *Sketchpad* where they could 'see' the solution dynamically unfolding before them. It is unlikely to have worked so well with pencil and paper, which would have been tedious and hardly as convincing.

RESEARCHER	Do you find this result surprising?
CHRISTINA	Yes... It seems so strange - if I did not **see** it I wouldn't believe it.

The second point that needs to be made is that once they were convinced, they really did not need to work with other examples. This showed a high level of conviction, on the one hand, but also still an undeveloped skepticism on the other hand. Schofield was adamant that he did not need more examples to convince him of the truth. No matter how much the researcher tried to get him to look at another example, he refused. He felt that it was totally unnecessary. This is obviously a matter of concern, because it was impossible that he could say for sure that the result was always true for any conceivable quadrilateral. Again this may be attributed to the fact that *Sketchpad* was a useful tool in convincing them that the result obtained was correct, in a very short period of time and / or simply that they are still naïve, uncritical and tend to over-generalize easily.

RESEARCHER	At the point where the quadrilateral is concave... Do you think it might work for other concave quadrilaterals?
SCHOFIELD	Yes.
RESEARCHER	Must I construct a few more quadrilaterals?
SCHOFIELD	No.
RESEARCHER	Are you sure?
SCHOFIELD	Yes.
RESEARCHER	Maybe just one more?
SCHOFIELD	No...that is not necessary!
RESEARCHER	Are you convinced?
SCHOFIELD	Yes, I am.

Perhaps it would have been appropriate at this point to attempt to elicit some explanation from the learners by asking them to explain why they thought the best solution lay at the intersection of the diagonals. An explanation is simple and utilizes the triangle inequality together with the method of contradiction as shown below:

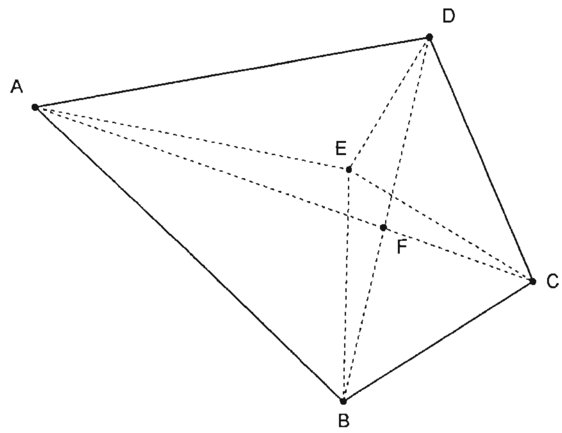


Figure 32

Let us assume the "best" position of the reservoir is at E. Then the pipes must be laid from A to E and from E to C. From the diagram it is clear that AC is smaller than the sum of AE and EC because of the triangle inequality. Therefore the reservoir should be built somewhere on line AC. Similarly, BD is smaller than the sum of BE and ED. Therefore, the reservoir should be built somewhere on line BD. But to satisfy both conditions, it is obvious that the reservoir should be built at F, the intersection of the diagonals.

5.9 Learners' need for an explanation

At the end of problem 2, the researcher asked the learners whether they would like to know why the perpendicular bisectors of the triangle were always concurrent. All learners indicated a desire for an explanation. Their reasons for wanting an explanation were similar and it seems that this desire arose out of the fact that they were surprised at what they had experienced. Due to the limited time available in the computer room at the school and the fact that the explanation did not require the computer, the researcher indicated to each learner that he would return to an explanation later. The learners accepted this and at the end of problem 3 the learners were asked again whether they would want to work through an explanation for the result obtained in problem 2. The learners reaffirmed their desire for an explanation.

In fact, gauging from the tone of their voices, it could be said that most of them were quite enthusiastic about working through an explanation. Only Schofield initially seemed uncertain, but eventually stated that it would be interesting to know why because he was surprised at the results. Faeeza also felt that it would be interesting to know why the result was always true. Parvanie's response was almost similar when she said that it would be useful to know why the result is true. But Christina's need for an explanation was significant. Her response was:

RESEARCHER	Would you want to know why this result is always true?
CHRISTINA	Yes...I can see it is true but maybe if there is a proof for it I'll understand it better.

She showed a very high level of conviction when she stated, "*I can see it is true*" and yet she felt that her understanding would be increased if she worked through a proof. This clearly shows that the level of conviction obtained from working with dynamic geometry software may stimulate further curiosity, which can be used as a starting point for proof.

Roxanne also felt that an explanation would show her why the result obtained for all triangles was different from that obtained for quadrilaterals.

RESEARCHER	Would you want to know why this is always true?
ROXANNE	Yes... maybe it will explain why it was different.

The fact that she saw that the results were distinctly different in Problems 2 and 3 kindled in her the desire to want to know why this was the case. This also clearly indicates that different individuals show different needs when working with proof in geometry. Whilst some felt that it would be useful just to know why, others felt that an explanation will give them greater understanding.

RESEARCHER	Would you like to know why the perpendicular bisectors are always concurrent?
PRAVANIE	I guess that it would be useful to know.

RESEARCHER	Would you like to know why this is always true?
FAEEZA	It might be interesting to know…I can't believe it (showing surprise).

RESEARCHER	Would you like to know why this is always true?
NIGEL	Definitely…maybe I could trick my friends too.

RESEARCHER	Would you like to know why this is always the case?
SCHOFIELD	What do you mean sir?
RESEARCHER	Do you want to know why the perpendicular bisectors are always concurrent?
SCHOFIELD	I don't know…mmm….yes…maybe it will be interesting.
RESEARCHER	You think that this would be interesting?
SCHOFIELD	Yes…I was surprised at the results.

As was already stated, the researcher had returned to this question of explanation after Problem 3. The question of whether they required an explanation was repeated. This was perhaps a good strategy because it served to test whether the learners really wanted an explanation or not. All the learners concurred with their earlier desire for an explanation.

Nigel was enthusiastic about an explanation. He felt that he was doing work that was beyond his grade.

RESEARCHER	Nigel, I did say that we would return to an explanation for why perpendicular bisectors are concurrent for all triangles. Would you still want to know why?
NIGEL	Oh yes.

It should, perhaps be noted again that Ausubel's learning theory, propagates the idea that meaningful learning occurs as the result of the stimulation of the learners' curiosity during the discovery process. It seems as if the difference in their findings for the quadrilateral and triangle stimulated their curiosity, and created a desire for some form of explanation.

The explanation for the concurrency of perpendicular bisectors of all triangles was based on materials developed by De Villiers (1999 : 32). Below are extracts of interviews with the learners.

RESEARCHER	Construct the perpendicular bisector of any side.
DESIGAN	Can I do it for AB?
RESEARCHER	Yes. (*after the construction*) Desigan, what can you tell me about all the points on this perpendicular bisector?
DESIGAN	It is equidistant from A and B.
RESEARCHER	What is equidistant?
DESIGAN	All the points on this line (*pointing to the perpendicular bisector*).
RESEARCHER	What does that really mean to you?
DESIGAN	If you measure the distance from any point on this line to this A and B, the distance will be the same.

In this segment the researcher was simply attempting to get the learners to recall the concepts of perpendicular bisector and equidistance. In a way, it was also a means of determining whether the learners actually understood and remembered what they had done earlier in the interview. Even Vischalan displayed a similar understanding of the concept of equidistance.

RESEARCHER	Look at this triangle on the screen. Construct the perpendicular bisector of side AC. (after the construction) what can you tell me about all the points on this perpendicular bisector?
VISCHALAN	They are the same distance away from A and B.
RESEARCHER	What is the term used to describe same distance away?
VISCHALAN	Equidistance.
RESEARCHER	So what are you saying about all points on this line?
VISCHALAN	All the points on this line (pointing to the perpendicular bisector) are equidistance from A and C..
RESEARCHER	Equidistant not equidistance from A and C. What does that really mean to you?
VISCHALAN	If you calculate the distance from any point to A and then to C the distance will be exactly the same.

It was clear that the learners had a good grasp of this concept (equidistance) and therefore it meant that the researcher could continue with the rest of the explanation. The next part of the explanation was similar in that it required the learners to construct another perpendicular bisector to relate the point of intersections of the two perpendicular bisectors to the three vertices. This relationship between the intersection and the three vertices did not take long to achieve, although in Desigan's case it was obvious that he made a mistake at one point in the interview but he did correct himself.

RESEARCHER	Now construct any other perpendicular bisector.
DESIGAN	(constructing)
RESEARCHER	What can you tell about the points on this line now?
DESIGAN	All the points are the same distance away from B and C.
RESEARCHER	Now look at this point of intersection. What can you say about this point in particular?
DESIGAN	Eh ... eh...
RESEARCHER	Think carefully about the point.
DESIGAN	That point there is the same distance away from A and B and, B and C.
RESEARCHER	A and B and, B and C?
DESIGAN	Yes, it is the same distance away from A, B and C.
RESEARCHER	Are you sure?
DESIGAN	It lies on this line so it must be equidistant from A and B and it lies on that line so it must be equidistant from A and C.
RESEARCHER	If it lies on that line would it be equidistant from A and C?
DESIGAN	No, B and C.

RESEARCHER	Now construct perpendicular bisector of AB.
VISCHALAN	(*constructing*)
RESEARCHER	What can you tell about the points on this line now?
VISCHALAN	All the points are equidistant from B and A.
RESEARCHER	Now look at this point of intersection. What can you say about this point in particular?
VISCHALAN	That is the point of concurrency of these two perpendicular bisectors.
RESEARCHER	Yes, that is true, but think carefully about the point. What is special about it?
VISCHALAN	It is equidistant from A, B and C.
RESEARCHER	Really? Why?
VISCHALAN	It is equidistant from A and C and then it is equidistant from A and B then it must be equidistant from A, B and C.

It was also quite interesting to note the level of reasoning that these learners were able to achieve and their ability to employ the basic transitive property. For example, if $a \otimes b$

and $b \otimes c$, then $a \otimes c$ (where $\otimes$ represents a general binary relationship). It implies that these learners were at the Van Hiele level 3 stage. They could see the deductive logic in the explanation as they were being guided through it. Being able to ascertain that if the point was equidistant from A and B and then from B and C, therefore the point must be equidistant from A, B and C is characteristic of Van Hiele Level 3.

Furthermore, they seemed convinced that their reasoning was correct. The researcher attempted to get them to measure the distance just to check, but the learners felt that this was not necessary.

RESEARCHER	So are you sure that this point of intersection is the same distance away from A, B and C?
VISCHALAN	Yes.
RESEARCHER	Don't you want to measure and check?
VISCHALAN	No...it's not necessary.

The next aspect was particularly important because it would be the real test as to whether the learners understood this concept of equidistance.

RESEARCHER	This you have to think very carefully about. What can you say about the perpendicular bisector of AC?
DESIGAN	All the points will be equidistant from A and C.
RESEARCHER	Yes, that is correct. But look at the other perpendicular bisectors.
DESIGAN	Oh yes, it must pass through the point where these two lines meet (*pointing to the perpendicular bisectors*).

RESEARCHER	What can you say about the perpendicular bisector of BC?
VISCHALAN	(Silence)
RESEARCHER	Think about it... What can you say about the perpendicular bisector of BC?
VISCHALAN	I think ... it will pass through this point of intersection here.

The researcher was aware that the learners may have just guessed the response because they already knew that the perpendicular bisectors of the triangle were concurrent.

Therefore the response that followed was essential in determining whether they were making a response with understanding or not.

RESEARCHER	Really? Do you really think so?
VISCHALAN	Yes, I'm quite sure.
RESEARCHER	Why?
VISCHALAN	Well if I construct the perpendicular bisectors, all the points on that line must be equidistant from B and C.
RESEARCHER	Yes. Go on.
VISCHALAN	What do you mean?
RESEARCHER	You just said that all the points on that line must be equidistant from B and C. So what does that mean?
VISCHALAN	That point of intersection has to pass through the point of intersection ... it has to because that point is also equidistant from B and C.

RESEARCHER	Really? Why?
DESIGAN	Yes, because if all the points on this perpendicular bisector of AC are the same distances away...then ... then this point of intersection is also the same distance away .. then...
RESEARCHER	Yes?
DESIGAN	Then the line must pass through the point of intersection.

It was clear that these learners had actually grasped the concept of equidistant points. It was not surprising though when the researcher asked the learners whether they wanted to see whether their conjecture was true and they indicated that they wanted to. This only showed that they where still skeptics. They knew that they were correct, but they wanted to see it nonetheless. Furthermore, it was interesting to note that Desigan felt that he had himself got the explanation correct when he said "*I didn't take so long to get it right!*". It was important that these learners were actually accepting ownership of the explanation.

RESEARCHER	Do you want to see whether that is true?
DESIGAN	Yes.
RESEARCHER	Construct the perpendicular bisector of AC then.
DESIGAN	(*after constructing*) This is so easy.
RESEARCHER	Was it really that easy?
DESIGAN	I didn't take so long to get it right!

RESEARCHER	Do you want to see whether that is true?
VISCHALAN	Yes.
RESEARCHER	Construct the perpendicular bisector of BC then.
VISCHALAN	(*after constructing*) I was right again.

It was encouraging for the researcher to note that the actual explanation became much easier because of the way the different problems were modeled. The learners made use of their existing knowledge to deductively reason an explanation, even though they were guided through it. The high levels of understanding (confirmed in the way they responded) resulted in the explanation being arrived at with ease. More than just arriving at the explanation, they also felt that this was *their* explanation. It should be noted that this is not the normal textbook proof as done in South African schools (which is based on congruency). However, it is a completely valid proof that seemed to have increased learners' understanding.

CHAPTER SIX

CONCLUSION

6.1 Introduction

This research reaffirmed many of the theories postulated previously. Some of these aspects will be dealt with in this introduction whilst the rest of the chapter will focus on specific research findings which where significant. The aspects that are of importance to the researcher are those related specifically to the question whether learners can model real world problems and the use of *Sketchpad* to conduct this type of modeling.

Clearly, the learners showed that they were able to *assimilate* the information given to them in the problem, taking into account some of the relevant data. All learners did indicate that they understood exactly what was expected of them. Through the careful use of *Sketchpad* the learners were able to explore the different possible solutions that the question could have provided. In this *exploration phase* the learners showed adeptness in the use of *Sketchpad* because they grasped the concepts easily and quickly.

RESEARCHER	What if the point that you chose between I and J is not suitable, as we said earlier? What should we do?
PRAVANIE	Then you can find another point.
RESEARCHER	How would you find the other point?
PRAVANIE	Construct a point.
RESEARCHER	Then construct another point.
PRAVANIE	(*constructing any point*). Now we measure the distant from I to that point.
RESEARCHER	Ok.
PRAVANIE	And from J to that point.
RESEARCHER	Measure the distant from J to that point ....
PRAVANIE	As I did before... I can move the point along.
RESEARCHER	All right are they now equal. (*Pravanie nods her head in agreement*) Let's suppose this point is not suitable.
PRAVANIE	You can choose another point.
RESEARCHER	Okay, do that.

PRAVANIE	And I can measure the distance again, from I to the new point and from J to the new point, and you can move it along until it is equal distant.
RESEARCHER	How many points do you think we can find like this?
PRAVANIE	A lot.
RESEARCHER	A lot? Is there anyway we can find a generalization? Were can we find the other points.
PRAVANIE	On the perpendicular to the line I J

Certainly, being able to work so easily with *Sketchpad*, made the exploration much more exciting and productive, because they were discovering by "doing" and "seeing". Inevitably this lead to patterns being established and hypotheses being made, which, may have been much more difficult to do if ordinary pencil and paper methods were used. This connection between the data provided and the eventual goal of the question was evidently an easy task for them once these patterns became apparent.

CHRISTINA	The position of the reservoir will be at the point where the perpendicular bisectors are concurrent.
RESEARCHER	Okay, but when will they be concurrent, when will the perpendicular bisectors be concurrent?
CHRISTINA	When the vertices lie on the circumference of the circle.
RESEARCHER	So will the perpendicular bisectors always be concurrent?
CHRISTINA	Yes if their vertices lie on the circumference of the circle….otherwise it won't be concurrent.
RESEARCHER	So if we have these four villages as it is drawn here, where should we build the reservoir?
CHRISTINA	At the point where the perpendicular bisectors meet.

Furthermore this research also found that learners can creatively apply themselves in the modeling process. Faced with an unfamiliar real world problem, the learners were able to develop and apply a model (albeit via a guided process), which leads them to attaining new knowledge. This knowledge became formalized through the application and verification of the results obtained. New concepts were effectively developed and introduced to the learners and this was the most important finding of this research.

There are other findings that need to be discussed here. It is important to mention that because these learners were not exposed to this type of modeling activity before, it is the researcher's opinion that their ability to relate real world problems to mathematics of the real world was severely hampered. For example, when asked to provide reasons for the unsuitability of the chosen position as a construction site, the learners gave reasons that were ordinary and not mathematically inclined. No learner indicated that building a reservoir at the chosen site might cost more than building four smaller reservoirs closer to each village. Neither did anyone indicate that all villages may have been of different sizes therefore finding the 'ideal' position, using the concurrency of perpendicular bisectors, would have not been an accurate method. This clearly indicated their inexperience at mathematical modeling.

This research also showed that presenting modeling activities that move from general ideas to more specific ideas can increase understanding. This is not a new discovery, but it does reaffirm Ausubel's Subsumption theory (Chapter 4). The fact that learners first worked with the quadrilateral and then the triangle assisted them in better understanding the significance of the different results obtained. This has already been discussed in Chapter 5.

In many instances this research also showed that learners either enter the learning environment with preconceived ideas or they develop new ideas as they encounter new information, which, in the researcher's opinion, is not always exploited by educators in the classroom. These learners used their previous knowledge of a midpoint to eventually arrive at the conclusion that all points on a perpendicular bisector are equidistant from two given points. Furthermore, they quickly concluded that in order to find the ideal position in the given diagram of a quadrilateral, perpendicular bisectors had to be used. They were adamant that this was the only way to have done it, thus indicating that it was an obvious conclusion. The new knowledge they discovered became so empowering that they felt strongly about their hypothesis that in any quadrilateral the perpendicular bisectors would be concurrent. Of course, it is this very same idea that lead to the surprise that the learners experienced when they found that their hypothesis was not always true. Perhaps, this type of teaching should be explored in more classrooms in order to establish a definite theory regarding teaching by surprise and contradiction.

6.2 The role and function of *Sketchpad* as a mathematical modeling tool and its potential role in mathematics education

A successful mathematical model depends on how well you can see the behaviour of the model. Visualization is useful in modeling and *Sketchpad* allows one to see exactly the changes that take place as the model is created and tested. Dynamic geometry software seems to afford the learner an opportunity to grasp mathematical ideas and concepts far more easily than traditional pencil and paper media. In most instances, instant feedback to learner queries and questions enhanced their ability to grasp new concepts because their doubts were dispelled with ease and their beliefs were reinforced. *Sketchpad* allowed the learners in this experiment to, quickly and easily, work through many different diagrams, which would have been almost impossible if other media traditional pencil and paper media were used. Such computer software, in general, "*should be used to foster those understandings and intuitions*" (NCTM, STANDARDS 2000, 1998 : 40) displayed by learners.

The general usefulness of *Sketchpad* can be summarized as follows:

1. The ease with which the diagrams were constructed and manipulated. This allowed the researcher and the learners the freedom to drag, change diagrams and manipulate the figures as and when required. This may have been impossible to achieve if other mathematical materials were used. This is in agreement with the NCTM STANDARDS 2000 document (1998 : 42) : "*Dynamic geometry programs enhance students' experience of two-dimensional and three-dimensional geometry. Such programs make it easy to generate a large set of instances, and to make conjectures*".
2. The use of buttons saved a lot of time and allowed the learners to see changes at the simple click of a button. Tedious constructions were avoided by the creation of these buttons.
3. The diagrams constructed using *Sketchpad* were clear and made misinterpretation less likely. Pencil and paper sketches might have resulted in many errors such as those created by using instruments incorrectly or by using a thick pencil lead.
4. Measurements were easily obtained when they were required. In many instances, this was fundamental to their understanding. It would have been quite time consuming if learners had to print the diagrams on paper and measured using a ruler. The possibility of them making incorrect measurements was also removed.
5. The manipulation of the diagrams on the screen allowed the learners to grasp properties of figures easily and it was evident that they understood the basic properties after a few minutes.

6. The fact that *Sketchpad* allowed them to develop high levels of conviction within a few minutes meant that more work could be done in a shorter period of time. But, to some extent, the learners expressed a desire for some explanation because they could instantly see that some of their conjectures were incorrect. This immediate feedback surprised them and created in them a need to know why the result was different.
7. The fact that certain relationships could be visualized helped the learners to accept that the concepts were true (*"if I did not see it I would not have believed it"*).

These aspects of *Sketchpad*'s usefulness show that the software has tremendous potential in mathematics education. Geometry teaching could become much easier because of the simple construction and manipulation of sketches. Simple and complex ideas can be understood within a few minutes affording educators greater freedom to work with more challenging problems and allowing the educator more time to work with individual learners.

Carefully planned worksheets enable learners to work comfortably on their own and if learners are *au fait* with the different tools available in *Sketchpad*, constructing and measurement becomes a simple activity. There can be very little doubt that Sketchpad is a tremendous mediating artifact in a mathematics classroom and its potential has not been adequately investigated. Off course, the constraints to using *Sketchpad* in a classroom are real (for example the affordability of computers and the program itself) but this must not allow us to detract from the fact that *Sketchpad* is a useful mathematics software.

It must also be stated that *Sketchpad* allowed the learners to gain sudden insight. An example of such an instance was when the learners realized that the perpendicular bisectors of all triangles were concurrent as compared to their initial conjecture. This is exactly the reason why the learners were surprised with some of the results they experienced. The experience of being surprised was in fact more than just that. It was the point at which the learners discovered relationships that were unfolding on the computer screen. It is this capacity to suddenly gain new insights that makes learning through the use of dynamic computer software all the more worthwhile.

In connecting ideas and concepts the learners were personally constructing their own meanings. Battista (2002 : 333) states that "*true understanding of mathematics arises as students progress through phases of action (physical and mental manipulations), abstraction (process by which actions become mentally solidified so that students can reflect and act on them), and reflection (conscious analysis of one's thinking)*". With the use of *Sketchpad* the learners were able to manipulate (physically) the diagrams and from what they observed, construct meaningful mental responses that aided in the understanding of the concepts that they were discovering and developing.

According to Battista (2002 : 339) "*working in this environment* (*Sketchpad* or other dynamic software) *helps students build increasingly sophisticated mental models for thinking about shapes, models that form the foundation on which genuine understanding of geometry must be constructed*". He goes on further to state that "*such work supports and encourages students' development and understanding of the property-based conceptual system used in geometry to analyze shapes. It encourages students to move to higher levels of thinking instead of having to memorize a laundry list of shape properties. The* *environment involves students as conceptualizing participants, not spectators, in the process of doing geometry*". This is a significant departure from the mundane pencil and paper geometry. The level of conviction attained here is significant and this together with the time frames within which conviction is achieved makes *Sketchpad* a powerful tool in the dynamic geometry context.

Over and above all the useful aspects of *Sketchpad* as a modeling tool, *Sketchpad* contributed most significantly to the third type of model application, namely *creative application* (refer to section 3.2.1). *Sketchpad* allowed for the effective use of modeling introducing the concept of "equidistance".

FAEEZA	You can locate it around there (*pointing*), somewhere where it would be equal distance apart.
RESEARCHER	Do that construction. ...(*after a point and the segments were constructed*) Okay how can we check whether it is equidistant apart?
FAEEZA	You measure the distance from the village to the point.
RESEARCHER	But they are not, what must you do now?
FAEEZA	We move it... I'll drag this point around.
RESEARCHER	How long should we move it around?
FAEEZA	Till we get the .. till the distances are equal.

Thereafter, because of the nature of the software, the learners were able to easily see the link between the idea of equidistance and the concept of perpendicular bisector.

PRAVANIE	And I can measure the distance again, from I to the new point and from J to the new point, and you can move it along until it is equal distant.
RESEARCHER	How many points do you think we can find like this?
PRAVANIE	A lot.
RESEARCHER	A lot? Is there anyway we can find a generalization? Were can we find the other points.
PRAVANIE	On the perpendicular to the line I J
RESEARCHER	So you are saying that it is going to be perpendicular to the line I & J. Maybe we should draw that.... Construct that perpendicular line. (*after a while*) There we go you are right, it passes through all the points. So what have we determined...if we had just two villages where can I find the most suitable point?

PRAVANIE	Just construct the perpendicular line.
RESEARCHER	Is it any perpendicular line? What does this perpendicular line do? I mean I could draw a perpendicular line there isn't that so?
PRAVANIE	It divides the line.
RESEARCHER	What do we call that line?
PRAVANIE	Perpendicular bisector.

The fact that the learners could 'see' the points arranged in a linear fashion enabled them to draw an instant conjecture that these points lay on the straight line. Furthermore, they joined these points to immediately confirm their conjecture. Although by looking at the line they constructed they could tell that it was perpendicular to the line joining the two villages, it was a simple task for them to measure the angle formed in order to reinforce the fact that the line was perpendicular. The link between the equidistant points and the perpendicular bisectors was thus achieved.

The ease with which the learners formed the relationship between the concurrent perpendicular bisectors and the quadrilateral being cyclic contributed to their understanding of the problem.

RESEARCHER	Can you construct a circle with the point of concurrency as the centre and move it slowly upwards to this vertex here.(after a while) What do you notice?
NIGEL	(*after a while*) The four edges of the quadrilateral meet the circumference of the circle.
RESEARCHER	Edge?.......which are you referring to as the edge? (*after learner points*) ...oh we call that a vertex........all of these are vertices.
NIGEL	Ok.
RESEARCHER	So the vertices of this quadrilateral lie on the circumference of the circle. Do you think that if I had any kind of quadrilateral the result would be the same?
NIGEL	Yes (*very confidently*)
RESEARCHER	So no matter what kind of quadrilateral, the perpendicular bisectors will always be concurrent?
NIGEL	Yes (*emphatically*)
RESEARCHER	Perhaps we should check. Change the quadrilateral and see what happens to the circle?
NIGEL	It went away from the vert...vertices.
RESEARCHER	Oh and what else do you notice about the perpendicular bisectors?

NIGEL	It does not meet at all.
RESEARCHER	Change it again, and let see if we can make them meet again. Yes they are meeting again but what do you observe?
NIGEL	The perpendicular bisectors are concurrent....
RESEARCHER	What can you say about the vertices of the quad.
NIGEL	It touches.... It lies on the circle
RESEARCHER	Then can you draw a conclusion from that?
NIGEL	The perpendicular bisectors are concurrent when it lies on the circumference of the circle.
RESEARCHER	Do you think that this is always true?
NIGEL	Definitely!

6.3.1 Are secondary school learners able to create and use mathematical models to solve geometric problems in the real world? If so, what strategies do they use?

Given the results of this experiment it was obvious that learners had not previously been exposed to real world problems which they had to solve using modeling strategies. In fact some learners clearly felt threatened and inadequately prepared to solve such a problem. One learner went as far as asking the researcher not to interview her because the work seemed too difficult. One learner indicated that this type of problem was too difficult whilst another felt that he did not do this type of problem in class. Evidently learners seem to be good imitators of their educators in class. If the educator does a particular type of example then the learners are able to copy those strategies. Unfortunately it currently seems that few educators are engaging learners in mathematical modeling of problems. As a result learners may successfully answer ordinary mathematics questions, but may encounter difficulties when facing questions of the real world.

None of the learners interviewed in this research knew or could devise a precise method of finding a solution to the problem. All some of them could do was to attempt a trial and error approach and measure the distances using a ruler. Although this leads to an approximate solution, it is rather time consuming (and often not accurate). It however generally shows that learners can understand and cope with real world, at least at the trial and error level. This finding therefore shows that teaching via modeling is possible as any real world problem can at least be approached in a trial and error fashion. Indeed, not all real world problems can be solved in a precise way. In many cases, the best solution can only be obtained by trial and error methods.

There is little doubt that much work still has to be done in encouraging the development of mathematical modeling skills in learners. Furthermore, this development must start at an early age. It is often difficult to start encouraging learners to use different modeling strategies at the end of their schooling careers if they were not exposed to such methods already. The Grade 10 learners interviewed in this experiment showed considerable unease initially because of the lack of knowledge of strategies to work with this problem. It is this unease that transmitted itself to the researcher as mild irritation. Their emphatic "*I don't know what to do*" or "*I really don't know*" was said in a tone that could not be captured on the transcript. They were visibly frustrated that the researcher was attempting to coax a solution out of them. This is possibly the result of inadequate preparation for not only tackling real world problems, but also on having to rely on their own ingenuity to invent an appropriate strategy.

The researcher is convinced that exposure to various modeling strategies and dynamic computer software would create a conducive environment for the solution of real world problems and would instill in learners greater confidence when working with different problems. It certainly would prevent learners from becoming overwhelmed by the seemingly insurmountable nature of the problem ("*I don't know sir.... This is too difficult... please don't interview me* (pleading)"). The fact that the learners were able to eventually, through a guided interview, arrive at correct solutions to problems 1, 2 and 3, indicates that with the correct guidance, available strategies and confidence, they could become more successful at solving such problems.

In conclusion of this section it must also be stated that learners showed immense awareness of real world conditions. They were able to recognize why mathematics alone cannot be used in real life, but has to be interpreted and adapted taking local conditions into consideration when considering the suitability of the solutions. It is the opinion of the researcher that this aspect of learning and teaching is often neglected and it might be a useful way of involving learners in a problem whilst at the same time it offers them a look at real world conditions – aspects of which they will encounter as engineers, land surveyors, business people and so on.

Learners obtain a vast amount of experience as they interact with the world and environment around them. When providing reasons for why a particular position was not suitable learners gave reasons that were interesting and realistic such as the position may be the actual homestead of the Chief of the tribe living there. For political and tribal reasons one cannot simply move a Chief out to build a reservoir. Educators ought to start utilizing these experiences to direct the learners' thinking when solving real world problems to finding real world solutions. For example, a problem involving bags of cement may result in a solution of 3,5 bags (50 kilograms each) of cement. Learners are already aware that cement cannot be purchased in smaller quantities – they are only sold in 50-kilogram units. Thus the solution must be 4 bags of cement.

So we know now that children do realize that there are real world conditions. The question that arises is, if and when they do recognize real world conditions, how do they cope with them mathematically? This is an interesting research topic which the researcher has just touched upon with these few problems.

6.3.2 Are learners able to use the provided *Sketchpad* sketches effectively to arrive at reasonable solutions?

It was quite evident that the learners were comfortable with the sketches provided. They at no stage, during the interview, showed signs of not understanding the provided *Sketchpad* sketches or the diagrams that resulted through the dragging or construction of new sketches. The fact that they were able to correctly identify the essential nature of the diagram constructed was a sufficient base to launch new ideas. They were receptive to the new information that they were experiencing and this indicates clearly that the *Sketchpad* sketches had effectively conveyed the ideas that the worksheets had been designed for.

After the trial and error period, the four village problem was simplified by just considering two villages. The learners were quick to recognize that the midpoint was equidistant from the two villages. Thereafter, when the researcher probed for further points the learners indicated a string of points that were also suitable. The fact that they could recognize that these points all lay on a straight line perpendicular to the line joining the two villages gives credence to the effectiveness of the *Sketchpad* sketches. More than just seeing the sketches is the fact that they could verify their conjectures and assumptions by conducting further constructions and measurements. This shows the effectiveness of *Sketchpad* as a tool for experimentation (and thus for modeling as well).

Other examples include the construction of the perpendicular bisectors of the quadrilaterals and triangles, the circles through these quadrilaterals and triangles and the diagonals of the quadrilaterals. In fact the construction of the diagonals in the last problem was very effective in that the learner could see from their manipulation of the diagram that their 'best position' coincided exactly with that of the intersection of the diagonals.

The *Sketchpad* sketches were made larger or smaller depending on what the learner required and at no stage did it alter the basic features of the diagram. By changing the size or shape of the diagrams, the learners were able to produce hundreds of diagrams in just a few seconds. Compared to pencil and paper methods, *Sketchpad* saved much time, energy and materials and at the same time allowed the diagrams to become effective tools to propagate ideas. When the learners could see the diagrams and the changes taking place on the screen it afforded them the opportunity to quickly make judgments that either convinced them that their ideas were valid or it afforded them an opportunity of changing or refining their ideas. An example of such an instance is given below.

RESEARCHER	Do you think that we can always find a point that is equal distances from V1, V2, V3 & V4 using the same method?
PRAVANIE	Yes.
RESEARCHER	So that means, no matter how big the quadrilateral is or how small the quadrilateral is it does not matter? No matter what shape the quadrilateral is, but if we draw the perpendicular bisectors they are going to meet there and that will be the best position?
PRAVANIE	Yes (*emphatically*).
RESEARCHER	Let's test it, let's take anyone of these corners and move it around. You must tell me where the perpendicular bisectors are concurrent?
PRAVANIE	No they are not meeting.
RESEARCHER	They are not?... What can you say about the bisectors...

PRAVANIE	They are not meeting.
RESEARCHER	So does it work in every one?
PRAVANIE	No...

6.3.3 Do learners display greater understanding of the real world problems when using *Sketchpad*?

It is difficult for the researcher to draw any substantial general conclusions from the interviews conducted because no control group was used to compare the learners' ability to work with these real world problems. But it can be said that the learners were able to identify, from the diagrams, the various aspects of the real world problem posed to them. They recognized that the vertices of the triangle and quadrilateral represented the villages. In doing that they were able to understand that the space between them represented open fields because they stated that the kraal of the chief could be there or there might be areas where endangered species of animals roamed about. They did not just see this space as the inside of a triangle or quadrilateral. This is fundamental to the understanding and working with real world problems. They knew that villages and mountains could easily be found in that space.

There are a few ways in which Sketchpad could have contributed to increasing their understanding of the real world problem:

1. *Sketchpad*, without any doubt, aided in the visualization of the problems. Seeing is often believing and this clearly contributed immensely to their basic understanding of the problem and the solution. One of the learners commented that if he did not see it he would not have believed it.

2. The fact that learners could construct a point, measure the distance of the point to the villages and move the point around so that they could see the distances gave them a sufficiently good idea that the ideal position could be anywhere, but they had to work within the constraints of the given problem and find a solution that satisfied the given question.

6.4 Can learners acquire knowledge about geometric concepts and shapes without just being told?

Whilst working through the problem, using *Sketchpad*, the learners informally acquired some information relating to equidistance, concurrency and so on. The learners learnt that when the vertices of any quadrilateral lies on the circumference of a circle then the centre of such a circle is the point of concurrency of the perpendicular bisectors of the sides of the quadrilateral. They were convinced that if the perpendicular bisectors were not concurrent then no circle could pass through the vertices of the quadrilateral and hence that quadrilateral was not cyclic.

The level of their conviction was quite high and they quickly insisted on not working through more examples because they could see that their final conjecture was always true. Even in the case of the triangle the level of conviction that the perpendicular bisectors are always concurrent was quite high. An example of such an instance is given below.

FAEEZA	We only get the lines to be concurrent when the vertices of the quadrilateral are on the circle.
RESEARCHER	Do you think that this is always true?
FAEEZA	Yes (confidently)
RESEARCHER	Do you want to construct more diagrams for you to confirm this?
FAEEZA	No please (emphatic and irritated).

RESEARCHER	Are you convinced?
FAEEZA	Yes...very.

The researcher is convinced that learners can learn useful facts by working with *Sketchpad* and not being told what these facts are. The learners in this experiment showed that through some guidance they were able to discover many useful bits of information that is generally learned only in Grades 11 and 12. It must be remembered that these were only Grade 10 learners and therefore expecting them to imbibe too much of information in a one hour interview might have placed too much of strain on them.

In conclusion, the interviews clearly indicated that learners are able to discover certain geometrical facts and concepts by working with dynamic geometry software without being told of these facts. Moreover, they were able to accept or refute the validity of their conjectures by simply dragging aspects of their diagrams on the computer screen. The surprise that learners encountered whilst seeing the changes taking place on the screen reinforced these ideas and created high levels of conviction. Below is an example of the high level of conviction that was achieved when the learner found that her conjecture was different from what she observed whilst working with *Sketchpad.*

FAEEZA	Well...to get the correct position of the water plant you must use the perpendicular bisectors of all the sides. Where they meet is the position.
RESEARCHER	Remember what you said initially, you said that the point should be at the middle? Is it different now?
FAEEZA	Yes...it is strange ....
RESEARCHER	Is that interesting?
FAEEZA	Very (emphatically)

An important point that should be mentioned here is that despite the learners having a good inductive understanding of the problem, the deductive reasoning contributed to their deep understanding as well. The various specific tasks they did, eventually led them to the required solution, but the actual deductive explanation at the end afforded them the opportunity of gaining further insight. It was clear that the inductive process was made easier by the use of *Sketchpad*, and indeed the deductive process was catalyzed by what the learners could see whilst working with *Sketchpad* as well. It is this combination which facilitated a level of understanding not easily achieved by ordinary pencil and paper methods. Izen (1998 : 720) stated that "*geometry learned by only inductive development is surface learning, whereas deductive proof by itself can be complicated and inaccessible. Together, inductive and deductive reasoning provide the most interesting and intelligible development of geometry*". He further elaborated by stating that "*inductive geometry with computer software certainly makes theorems come alive to students, but proofs lead to the insight and understanding that allow students to extend the concepts that they are studying beyond the uses that they see in examples.*"

6.5 Mathematical preconceptions as a basis of children's conjectures

This experiment clearly showed that the learners brought their experiences into the learning environment. Their preconceptions played an important role in the responses they made. For example, when the learners were first asked to postulate an ideal position for the reservoir, they immediately stated that the position should be towards the middle of the quadrilateral. This stems, perhaps, from the fact that when one wants equal distances then one looks towards the centre of a figure. But the concept of 'centre' could imply the same distance away from all the sides or from the vertices, and it was not clear if the learners had an initial preference to the one or the other.

The researcher therefore postulates that the evidence in this experiment would indicate that learners interact with the real world problem using their preconceptions, for example, they knew that to find a point equidistant from two points they needed to work with the midpoint. Furthermore, they also realized that if the midpoint was not a viable choice, then they could choose other points which were still equidistant from the two points. If these preconceptions conflict (this does not always occur) with the mathematical conceptions then learners are forced to re-evaluate their own knowledge. This was clearly illustrated when the learners discovered that the perpendicular bisectors were always concurrent in triangles. This was contradictory to what they originally anticipated. It is this conflict that created the necessity for a re-evaluation of their existing knowledge. Either with guidance or on their own, learners can establish mathematical models. These models lead to mathematical solutions, which lead to real world solutions. This new knowledge becomes the basis of new preconceptions.

A further example to illustrate this point, occurred when the learners were given the initial real world problem, they guessed that the ideal position would be towards the middle of the figure. Using the measure and drag feature of *Sketchpad*, the likely position (by construction) showed the inaccuracy of their initial guess. Through guidance, they modeled the same problem but using only two villages. The solution to this problem was attained very easily when they discovered that the ideal position can be found at any point on the perpendicular bisector. This (any point on a perpendicular bisector is equidistant to two given points), therefore, became their new knowledge. Thus extrapolated to the real world situation they realized that the solution lay anywhere on the perpendicular line that bisected the line segment between the two villages.

When they encountered the second problem, which was the original question, they knew immediately that they could use perpendicular bisectors. This was their preconception for the new problem. Thus the cycle continued. This can be illustrated by the Figure 33.

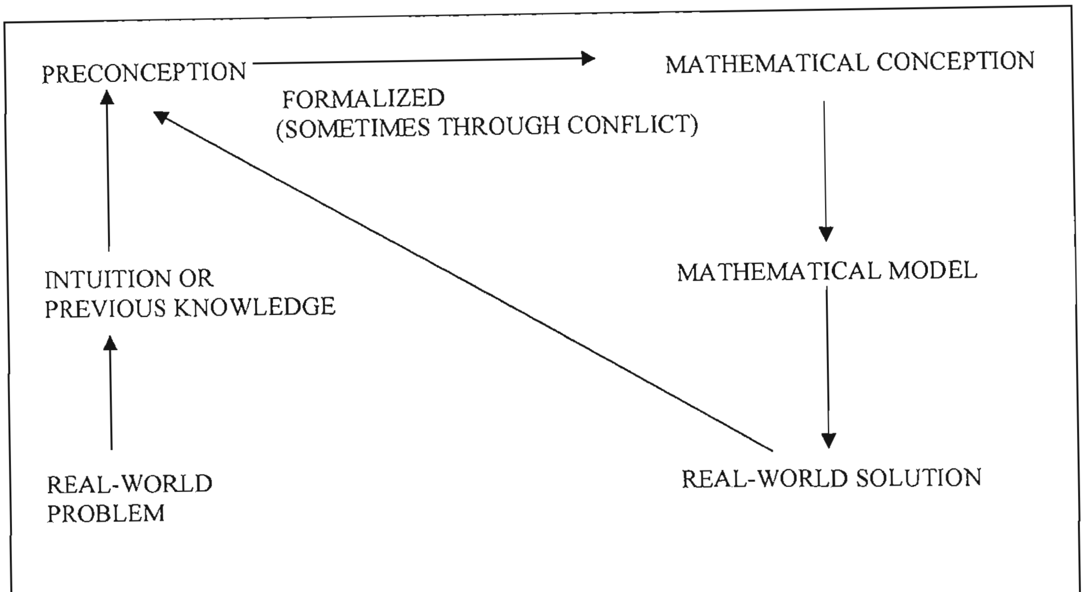
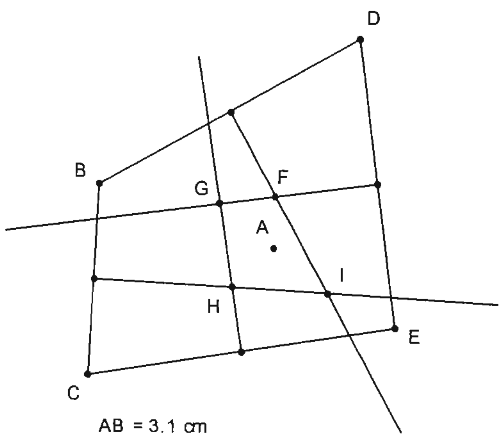


Figure 33

6.6 Shortcomings of this research and recommendations for future research

In the course of the discussion of the findings of this investigation, many shortcomings and omissions were mentioned. The most important shortcoming of this investigation was the failure of the researcher to repeatedly relate some of the findings to the real world. Although the researcher did refer to the real world, there were instances where the direct reference to the real world might have better showed the link between mathematics and the real world. Of course the constant reference to the fact that the learner had to find an ideal position for the reservoir did maintain some relationship between the real world problem and the mathematics involved in its solution.

A second shortcoming of this research was that some ideas could have been pursued further. For example, the researcher could have asked the learners to find the "best" position for a quadrilateral not having an equidistant point (that is, the quadrilateral was not cyclic). This might have resulted in an interesting discussion and investigation around a mathematical solution to a real world problem that is not necessarily precisely obtained. Below is one possible diagram that the learners could have worked with.



AB = 3.1 cm
AC = 3.7 cm
AE = 2.4 cm
AD = 3.7 cm

AB-AC + AB-AE + AB-AD + AC-AE + AE-AD = 4.5 cm

Figure 34

The investigation of a solution would have revolved around the reasonable conjecture that the ideal solution must be found within the feasible region FGHI. F, G, H and I are the circumcenters of triangles BED, CED, BEC and BCD (De Villiers, 2003 : 151). Furthermore, in an attempt to minimize the differences between all the distances, it is necessary to minimize the sum of the absolute values of the differences to obtain the "best" position. This can be done by minimizing the equation

AB-AC + AB-AE + AB-AD + AC-AE + AE-AD = 4.5 cm

This investigation and discussion may have contributed to a better understanding of real world problems.

A third shortcoming of this research was the fact that the research did not attempt to elicit an explanation (or proof) from the learners regarding the last problem that they had to work through. The question that should have been asked is: "Why is the ideal position at the point of intersection of the diagonals of the quadrilateral?" A proof is relatively simple and utilizes the triangle inequality together with the method of contradiction (refer to section 5.8).

A fourth shortcoming of this investigation was the fact that the researcher did not place a lot of emphasis on proof (as justification), but this was not the focus of the investigation. This meant that learners often made conjectures just on the basis of the empirical evidence presented to them by *Sketchpad*. More encouragement of the learners to explain or justify their conclusions might have played a more significant role in the modeling process, as was seen in the subsequent interviews with two learners.

The following are some important recommendations for future research:

1. Research should be conducted in a more classroom-like context, with more that one learner engaging with *Sketchpad* in the modeling process at the same time. The interaction between learners themselves may make the process easier and the researcher postulates that the act of learning from each other would improve their understanding of the different relationships under investigation.
2. A larger longitudinal study needs to be conducted where learners engage with *Sketchpad* in a modeling process together with proof as an essential component of

the research. It would be interesting to investigate whether modeling using *Sketchpad* would improve younger learners' (perhaps Grade 8) understanding of difficult geometric relationships or concepts and whether these learners would be able to work through a guided proof.

3. A more ambitious research project could focus on the learners actually going through the modeling process entirely by themselves using *Sketchpad* and creating their own sketches. Instead of being given ready made diagrams, a new investigation could explore the levels of understanding when learners themselves try to use *Sketchpad* unassisted.
4. At a curriculum development level, it would be important for curriculum developers to investigate the possibility of include dynamic geometry modeling at all schools in South Africa.
5. A further recommendation to educators is that learners should be encouraged to work with problems that naturally pique their curiosity, even if the educator has to select specific problems. This is also what Hiebert, et al (1996 : 12) stated when they concluded that learners "*should be allowed and encouraged to problematize what they study, to define problems that elicit their curiosities and sense making skills*". It may be possible that more learning occurs when learners find their interest stimulated by some aspect of mathematics that may be different from what they already know.

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INTERVIEW WITH PRAVANIE AND RESEARCHER

RESEARCHER	Pravanie, you've read the question, do you understand it?
PRAVANIE	Yes sir, I do.
RESEARCHER	Where do you think that we should build the reservoir?
PRAVANIE	I don't know ... all we are only given is this diagram ...
RESEARCHER	Do you think that you will be able to find the most suitable point? You can use any method you know, to do so.
PRAVANIE	I don't know sir ... this is too difficult ... please don't interview me (pleading).
RESEARCHER	Are you saying that you cannot find any way of solving this problem?
PRAVANIE	I can't ... I'm not so good in maths ... maybe at the centre here (pointing to the middle of the quadrilateral).
RESEARCHER	Will you be able to justify your answer? Will you be able to tell me why?
PRAVANIE	(silence)...... not really ...
RESEARCHER	Don't you even want to try?
PRAVANIE	I don't know what to do ...
RESEARCHER	This is the same diagram you have on the page. Can you point out again the most suitable position of the reservoir by just looking at the diagram.
PRAVANIE	Around here
RESEARCHER	You are pointing towards the middle.
PRAVANIE	Yes around there.
RESEARCHER	So... just there. Are you just guessing? Is there any way to find that suitable position?
PRAVANIE	... Hmmm...
RESEARCHER	Do you think there might be some method of doing it?
PRAVANIE	Actually I am guessing.
RESEARCHER	Is there any mathematical way, you think, we can actually get that most suitable point?
PRAVANIE	No
RESEARCHER	No? You do not think so? ....(after a while Pravanie nods her head) Okay before we actually get to discuss the means of getting a point, the real world is sometimes different from what we do in mathematics. Like, for example, as you pointed to the middle, you said that that might be the ideal point. Construct a point to illustrate where you think the reservoir should be. Now in the real world it might not always be true or viable for us to build a reservoir at that point. What do you think might be some of the problems that might be experienced in the real world if we have to construct a reservoir there?
PRAVANIE	Apart from the fact that there might be a valley over there, there could be a mountain, there could be a building already constructed. There could be endangered species.
RESEARCHER	... endangered species of what?
PRAVANIE	Any plants or animals and stuff like that ... a nature reserve.
RESEARCHER	So you are saying it might be a nature reserve?
PRAVANIE	Yes
RESEARCHER	So are there many more possible problems?
PRAVANIE	Yes I think so.

RESEARCHER	What do think is the purpose of asking all these questions?
PRAVANIE	Very often we think a certain place will work but when we go there we notice that there is a problem.
RESEARCHER	All right ... so we constructed a point in there okay (*pointing to within the quadrilateral*). How can we determine whether that is equidistant from all points?
PRAVANIE	Measure these distances (pointing to the diagram).
RESEARCHER	Let us measure them then.
PRAVANIE	Okay (after a while).
RESEARCHER	So what do you notice?
PRAVANIE	They are all different.
RESEARCHER	How can we make them equal?
PRAVANIE	By moving the point around.
RESEARCHER	Move the point to the position you indicated earlier. Now what do you observe?
PRAVANIE	That point is not the answer because the distances are not the same.
RESEARCHER	So what must we do now?
PRAVANIE	Carry on moving the point until the distances are the same.
RESEARCHER	(after a while)You notice that it is really not that easy, but is takes a bit of time. You have to plot a point measure the distances and drag it along. Do you think that this is the best method of finding the most suitable point? Off course this method is not incorrect but in order to find the accurate position...
PRAVANIE	I don't know.....that is the only method I can think of.
RESEARCHER	Do you think that there might be a mathematical method of finding the most suitable position?
PRAVANIE	Maybe but I don't know it.
RESEARCHER	Let us simplify this problem.
PRAVANIE	Okay
RESEARCHER	We will work with only 2 villages--- How can you find the best position which is an equal distance from J & I (*referring to the diagram already provided*)?
PRAVANIE	Measure the distance between both of them and find the midpoint.
RESEARCHER	You are saying that we should find the midpoint between J & I. Click on the button provided.
PRAVANIE	There it is (*after clicking on the construct a point button*).
RESEARCHER	What if the point that you chose between I and J is not suitable, as we said earlier? What should we do?
PRAVANIE	Then you can find another point.
RESEARCHER	How would you find the other point?
PRAVANIE	Construct a point.
RESEARCHER	Then construct another point.
PRAVANIE	(*constructing any point*). Now we measure the distant from I to that point
RESEARCHER	Ok
PRAVANIE	And from J to that point.
RESEARCHER	Measure the distant from J to that point ....
PRAVANIE	As I did before... I can move the point along.
RESEARCHER	All right are they now equal. (*Pravanie nods her head in agreement*) Let's suppose this point is not suitable.
PRAVANIE	You can choose another point.
RESEARCHER	Okay, do that.

PRAVANIE And I can measure the distance again, from I to the new point and from J to the new point, and you can move it along until it is equal distant.

RESEARCHER How many points do you think we can find like this?

PRAVANIE A lot.

RESEARCHER A lot? Is there anyway we can find a generalization? Were can we find the other points.

PRAVANIE On the perpendicular to the line I J

RESEARCHER So you are saying that it is going to be perpendicular to the line I & J. Maybe we should draw that.... Construct that perpendicular line. (*after a while*) There we go you are right, it passes through all the points. So what have we determined...if we had just two villages where can I find the most suitable point?

PRAVANIE Just construct the perpendicular line.

RESEARCHER Is it any perpendicular line? What does this perpendicular line do? I mean I could draw a perpendicular line there isn't that so?

PRAVANIE It divides the line.

RESEARCHER What do we call that line?

PRAVANIE Perpendicular bisector.

RESEARCHER That's good Pravanie, you are doing well...and now let us go back to the original problem. Here is the original problem and the question still remains. Where would we find a point that is most suitable because it has to be equi-distant from village 1, village 2, village 3, and village 4. Where do you think we should build it now? You said initially that somewhere in the centre here, but now we are trying to establish a good mathematical method of actually finding the point.

PRAVANIE ...You create perpendicular bisectors of V1, V2.

RESEARCHER V1, V2 or between V1 and V2.

PRAVANIE Between V1 and V2

RESEARCHER Yes

PRAVANIE And between V2 & V3, V3 & V4 and V4 & V1.

RESEARCHER Do you think that will work?

PRAVANIE Most probably.

RESEARCHER Click on that button provided.(*after a while*) So we have the perpendicular bisectors of the 4 sides. What do you observe?

PRAVANIE They are all meeting at a point.

RESEARCHER Does this surprise you?

PRAVANIE Why should it...they are supposed to meet.

RESEARCHER So where will you place the reservoir?

PRAVANIE The point where they meet, will be the ideal place to put it.

RESEARCHER So how do you know that this point will be equidistant from village 1, village 2, village 3 and village 4?

PRAVANIE Measure the distances.

RESEARCHER Measure it. (*as the measurements were being done*) 3 point 8. That's 3 point 8... That's 3 point 8 and the other is 3 point 8. It does work. Do you think that we can always find a point that is equal distance from V1, V2, V3 & V4 using the same method?

PRAVANIE Yes.

RESEARCHER So that means, no matter how big the quadrilateral is or how small the quadrilateral is does not matter? No matter what shape the quadrilateral is, but if

	we draw the perpendicular bisectors they are going to meet there and that will be the best position?
PRAVANIE	Yes (*emphatically*).
RESEARCHER	Let's test it, let's take anyone of these corners and move it around. You must tell me where the perpendicular bisectors are concurrent?
PRAVANIE	No they are not meeting.
RESEARCHER	They are not?... What can you say about the bisectors...
PRAVANIE	They are not meeting.
RESEARCHER	So does it work in every one?
PRAVANIE	No...
RESEARCHER	Why do you think it did not work in every quadrilateral?
PRAVANIE	I'm not sure....
RESEARCHER	What do you think is significant about this point here?
PRAVANIE	I don't know.
RESEARCHER	Draw a circle from this point where they meet through that point there. What do you notice?
PRAVANIE	...mmm .... The quadrilateral intersects the circle.
RESEARCHER	In other words the vertices of the quadrilateral...
PRAVANIE	Lie on the circumference of the circle.
RESEARCHER	Grab this point and move it. Does the quad still lie on a circle?
PRAVANIE	No ... this is strange...
RESEARCHER	When are they meeting each other?
PRAVANIE	(*silence*)...
RESEARCHER	When do these perpendicular bisectors meet?
PRAVANIE	When the vertices of the quadrilateral lie on the circumference of the circle.
RESEARCHER	So what does this mean?
PRAVANIE	The perpendicular bisectors meet only when the vertices lie on the circumference of a circle.
RESEARCHER	That was not so difficult was it? What we established thus far?
PRAVANIE	The perpendicular bisectors will give you equal distances and the circle will only pass through the vertexes of the quadrilateral if the perpendicular bisectors meet.
RESEARCHER	What does this mean in terms of the reservoir and the four villages?
PRAVANIE	The reservoir should be built at the point where the four perpendicular bisectors meet. But if that point does not work then we must find another very close.
RESEARCHER	Let us change this now. Consider what would happen if we had only 3 villages. How do you think then we would be able to find the best position? Do you want to attempt an answer first?
PRAVANIE	We would use perpendicular bisectors again.
RESEARCHER	Well do you want to try it?
PRAVANIE	Yes
RESEARCHER	(*the perpendicular bisectors were constructed*) So you are saying that this point at which the perpendicular bisectors are meeting will be the best position?
PRAVANIE	Yes
RESEARCHER	How can we check?
PRAVANIE	Measure it (*referring to the distances from the point of concurrency to the vertices*).
RESEARCHER	(*after it was measured*) Are they equal?
PRAVANIE	Yes, I knew they would be.

RESEARCHER Let's just check whether the circle will move through the vertices if we use that as a centre (*pointing to the point of concurrency*).

PRAVANIE Okay (constructing).

RESEARCHER What do you observe?

PRAVANIE It does.

RESEARCHER The circle will move through the vertices, but will you be able to find a best position for **any** triangle? Do you think this will be possible for any triangle?

PRAVANIE No---- unless its vertices lie on the circumference of the circle.

RESEARCHER Okay so the best way to test that is to move one point of the triangle, which ever way, and see whether the perpendicular bisectors meet.

PRAVANIE (after a while) They do! (*emphatically and excitedly*)

RESEARCHER Does this surprise you?

PRAVANIE It does. It is different from the quadrilaterals. It means that ...this applies to any triangle.

RESEARCHER So it means the perpendicular bisectors of any triangle will meet... and, for the quadrilateral what did we establish?

PRAVANIE That perpendicular bisectors won't meet unless the vertices of the quadrilateral lie on the circumference of the circle.

RESEARCHER What does it mean in terms of the villages?

PRAVANIE You can find the perpendicular bisectors of any village and use that point for the building of the reservoir.

RESEARCHER How sure are you of this result? Are you convinced about this?

PRAVANIE I am quite convinced.

RESEARCHER How many percent convinced are you?

PRAVANIE I'm convinced sir...100%.

RESEARCHER How surprising is this result to you?

PRAVANIE Quite surprising because I thought the same that applies for the quadrilateral would apply for the triangle.

RESEARCHER Would you like to know why the perpendicular bisectors are always concurrent?

PRAVANIE I guess that it would be useful to know.

RESEARCHER We will do that later. I want to now change the question a bit... Lets suppose the government decides to build a pipeline from the water reservoir to the 4 villages A, B, C and D. Where should the water reservoir be placed so that the total length of the pipeline is minimised? In other words let's suppose the water reservoir is placed at point P and if you measure the distances from P to A, P to B, P to C & P to D and then you find the sum of it, where will the point be for this sum to be the smallest. Why do you think that they want to minimise it?

PRAVANIE To save them money – they will use less pipes and they will save on labour.

RESEARCHER Yes ... where do you think we should locate that point?

PRAVANIE Since all quadrilaterals do not always have perpendicular bisectors meeting I think we should check whether the vertices of this quadrilateral lie on the circumference of the circle first.

RESEARCHER Will it make a difference to you answer?

PRAVANIE Yes. Let's just construct the perpendicular bisectors.

RESEARCHER (after a while)... What do you observe?

PRAVANIE The perpendicular bisectors do not meet. This means that the vertices of the quadrilateral do not lie on the circumference of the circle.

RESEARCHER Okay. What should we do next?

PRAVANIE	Try moving the point.
RESEARCHER	As we did before?
PRAVANIE	Yes.(after a while) It will be a minimum here (pointing) …
RESEARCHER	Are you sure?
PRAVANIE	…….. (*after a while*) Yes
RESEARCHER	What does that remind you off?
PRAVANIE	Looks like the diagonals.
RESEARCHER	Why don't you click on that button for the diagonals? What do you observe?
PRAVANIE	Our lines and the diagonals are the same.
RESEARCHER	So what does this mean?
PRAVANIE	This means that the reservoir should be placed at the point where the diagonals meet.
RESEARCHER	If I had to change this…. Let's get rid of the diagonals. Let's say we changed the size of this quadrilateral, where do you think it should be placed now (the reservoir)? Do you think it would be where the perpendicular bisectors are?
PRAVANIE	No
RESEARCHER	Do you think we should drag this around? Do you need to go through more examples? Where should the reservoir be placed?
PRAVANIE	No….. Where the diagonals meet.
RESEARCHER	What would happen if you drag this point to the point where the diagonals meet?
PRAVANIE	That would be the exact point.
RESEARCHER	Was that very difficult?
PRAVANIE	No.
RESEARCHER	The last question is what if this was a concave quadrilateral, where are the diagonals meeting now?
PRAVANIE	Outside the…. On that (*pointing*)
RESEARCHER	And we need to find out which is the minimum, drag the point along and notice where it is decreasing, and notice where the minimum point is.
PRAVANIE	Minimum is at A
RESEARCHER	Which point is that?
PRAVANIE	(*silent*)
RESEARCHER	Can you just describe the point?
PRAVANIE	Yes. It is lying on the diagonal from the apex to the point where it is concave.
RESEARCHER	Notice that this is the minimum point. Do you find all of these results surprising?
PRAVANIE	Yes.
RESEARCHER	Why?
PRAVANIE	I never knew that.
RESEARCHER	Pravanie, I did say that we would return to an explanation for why perpendicular bisectors are concurrent for all triangles. Would you still want to know why?
PRAVANIE	Yes, I think that it would be useful to know that.
RESEARCHER	Read through this page and we will work together when you are ready.
PRAVANIE	(after a while) I am ready sir.
RESEARCHER	So we want to prove that the perpendicular bisectors of any triangle are concurrent. What are we given?
PRAVANIE	The triangle ABC and the perpendicular bisectors.
RESEARCHER	In the diagram we will construct perpendicular bisectors OF and OE. We will

join AO, BO and CO. We will also construct OD perpendicular to BC. What is the difference between OF, OE and OD?

PRAVANIE They are all perpendicular but OD does not bisect BC.

RESEARCHER Good. For us to prove that O is the point of concurrency of all the perpendicular bisectors what should we prove?

PRAVANIE Er…er….

RESEARCHER Are the lines meeting at O already according to our diagram?

PRAVANIE Yes.

RESEARCHER Are OF, OE and OD perpendicular bisectors?

PRAVANIE OF and OE are but not OD…OD is perpendicular but not a bisector.

RESEARCHER So what do you think we should prove?

PRAVANIE Er…maybe we should prove that BD is equal to DC.

RESEARCHER That is correct.

PRAVANIE Now follow the instructions and complete the worksheet.

INTERVIEW WITH RESEARCHER AND CHRISTINA

RESEARCHER	Christina, you've read the question, do you understand it?
CHRISTINA	I do.
RESEARCHER	Where do you think that we should build the reservoir?
CHRISTINA	Can I use my ruler and pencil?
RESEARCHER	You can use any method you know, to do so.
CHRISTINA	(after a while) I thought it will be here ... (pointing to the middle) ... but when I measure the distances, it's not the same.
RESEARCHER	Are you saying that you cannot find any way of solving this problem?
CHRISTINA	I think it is in the middle ... maybe at the centre here (pointing to the middle of the quadrilateral).
RESEARCHER	Will you be able to justify your answer? Will you be able to tell me why?
CHRISTINA	No! (emphatic)
RESEARCHER	Don't you want to try?
CHRISTINA	No!
RESEARCHER	Let us see if working with the computer will help. This is the same diagram you have on the page. Can you point out again the most suitable position of the reservoir by just looking at the diagram. Where do you think Christina, within this quadrilateral should we actually build the reservoir?
CHRISTINA	In the centre
RESEARCHER	In the centre would mean about there in the middle? Is that correct? (Christina nods her head). Now how will we be able to determine whether that is the correct point?
CHRISTINA	We measure from that point to that (pointing to the vertices of the quadrilateral).
RESEARCHER	(after Christina measures the distances) What do you observe?
CHRISTINA	The distances are different...
RESEARCHER	How then can we find a suitable point?
CHRISTINA	Move the point around.
RESEARCHER	Move the point around. (After awhile) Is it easy to find this point?
CHRISTINA	No.
RESEARCHER	So what happens if we move the point around?
CHRISTINA	The distances will change.
RESEARCHER	Now, tell me Christina, in real life it may not be possible to build at a point that you think to be correct. Do you think that there might be reasons for which the point you choose might not work?
CHRISTINA	There might be... a hard rock.
RESEARCHER	Yes....?
CHRISTINA	Other buildings
RESEARCHER	What kind of buildings do you think?
CHRISTINA	Police station or something....
RESEARCHER	Any other reasons?
CHRISTINA	If there's like a stream or river you won't be able to build
RESEARCHER	What do you think the point of all this is?
CHRISTINA	Sometimes finding the point might not work because ... because the place might not be suitable.
RESEARCHER	If you look at the same diagram, do you think there is a way in which we can find the most suitable point that is going to be equidistant from all vertices?

CHRISTINA	No
RESEARCHER	Let's simplify this problem by taking two villages-let's suppose we had two villages represented in this diagram by I and J, okay? How would you find a point that is exactly the same distance away from I as it is from J?
CHRISTINA	Find the centre.
RESEARCHER	Centre of what?
CHRISTINA	Centre of I and J
RESEARCHER	What would that centre be called?
CHRISTINA	The midpoint.
RESEARCHER	So you would find the midpoint of J and I? Let's find the midpoint of J, and I by clicking on that button. Do that and tell me what you observe?
CHRISTINA	Distance QI is 2, 2 the distance of QJ is 2,2 as well, so that means Q is equidistant from I and J.
RESEARCHER	Earlier on we said that it might be possible that that point will not work for various reasons. Do you think there might be another point, which might be equidistant from J and I?
CHRISTINA	If you move further away.
RESEARCHER	If you move further away, and do what?
CHRISTINA	Make a straight line on Q
RESEARCHER	Would it be just any line?
CHRISTINA	I think that it is a line but I'm not sure what line ....
RESEARCHER	Let's suppose we want just one other point, which is equidistant from J and I. Where can we find such a point.
CHRISTINA	Construct a point and measure the distances.
RESEARCHER	Why don't you construct a point and measure the distances. (after a while) What do you observe?
CHRISTINA	The distances are the same.
RESEARCHER	It was just coincidental that the distances were immediately equal ......construct another point and measure the distances. (after a while) Now you can see that the distances are different. How can we make the distances the same?
CHRISTINA	Move the point around.
RESEARCHER	Yes that would be correct. What if I said this point is also not good enough, can you get another point?
CHRISTINA	Yes
RESEARCHER	How can we do that?
CHRISTINA	We move further away and find another point.
RESEARCHER	Construct another point, and what shall we do to check?
CHRISTINA	Measure (measuring the distances). That's 3,7 – and that ....3,6.
RESEARCHER	How can I get these two distances to be equal?
CHRISTINA	By moving the point around.
RESEARCHER	Okay do that. ... Now do you think you can get more points?
CHRISTINA	Yes
RESEARCHER	And now – how do you think we can get more points?
CHRISTINA	Mm... all the points are same – on the same line
RESEARCHER	So you are speaking about a line, again? What do you observe about all these points that we have selected?
CHRISTINA	They are on a straight line.
RESEARCHER	How can we determine that for sure?
CHRISTINA	Draw a line through the points.

RESEARCHER	Let's check – construct a line and see if it passes through all the points you constructed.
CHRISTINA	Yes it does.
RESEARCHER	What else do you observe about this line?
CHRISTINA	Perpendicular
RESEARCHER	It's perpendicular? How can we confirm that?
CHRISTINA	Measure and see if it is a right angle.
RESEARCHER	Do that and tell me what you observe.
CHRISTINA	(after measuring) It is a right angle!
RESEARCHER	Very quickly, if I wanted to build a reservoir where should I build it?
CHRISTINA	At a right angle.
RESEARCHER	You can build something at right angles here as well (pointing to any point on the line IJ and indicating a perpendicular line), will it be anywhere on this line?
CHRISTINA	No.
RESEARCHER	Why?
CHRISTINA	It won't be equidistant.
RESEARCHER	Okay ....so what is so special about this line (pointing to the perpendicular bisector).
CHRISTINA	It is a bisector.
RESEARCHER	Bisector, that's correct. So very quickly again tell me where can I build the reservoir?
CHRISTINA	At any point on the perpendicular bisector.
RESEARCHER	Good. Let us now go to the original problem. Here we have the quadrilateral again, village 1, village 2, village 3, and village 4. I want to know from you, besides what we did in the beginning... Remember we constructed a point and then we measured the distances for villages 1,2,3 and 4 and we moved it around, besides that method, do you think there is another way in which we can find a suitable position?
CHRISTINA	No
RESEARCHER	Wouldn't you like to try?
CHRISTINA	I don't know what to do.
RESEARCHER	There must be something that you have already learnt that you could use.
CHRISTINA	Is there?... I don't know sir.
RESEARCHER	Let us go back to the one with the 2 villages – How did we find the point that was equidistant between the 2 villages?
CHRISTINA	We used.... (exclaiming) Perpendicular bisector!
RESEARCHER	Do you think then that we might be able to use perpendicular bisectors here?
CHRISTINA	Maybe
RESEARCHER	Do you think we should try ?
CHRISTINA	Yes
RESEARCHER	Click on the button provided and you will see the perpendicular bisectors.
CHRISTINA	Its too complicated (referring to the diagram)...but all the lines meet at one point.
RESEARCHER	Does this surprise you?
CHRISTINA	They have to meet I think.
RESEARCHER	We say that these lines are concurrent because they meet at one point. Do you think, now that the lines are concurrent, that if we measure the distances from here (pointing to the point of concurrency) to the villages it would be the same?
CHRISTINA	Yes (emphatically)

RESEARCHER Let us check ... measure the distances from that point to village 1, 2, 3 and 4. What do you observe?

CHRISTINA This is 3.8.

RESEARCHER What do you think the next distance will be?

CHRISTINA The same

RESEARCHER There ....you can see that the third distance is the same and the last one what do you think this will be ?

CHRISTINA Must be the same.

RESEARCHER As you can see, the distances are all the same.

CHRISTINA Yes

RESEARCHER Try to construct a circle from here passing through this point (*pointing to a vertex*) ... what do you observe?

CHRISTINA It ... it passes through all four villages.

RESEARCHER In other words do the vertices of this quadrilateral lie on the circumference of the circle?

CHRISTINA Yes

RESEARCHER Now do you think this will be the case for any quadrilateral? Do you think for any large quadrilateral or for any small quadrilateral or if I changed the size of the quadrilateral or shape of the quadrilateral, do you think it will always be concurrent like this?

CHRISTINA No

RESEARCHER No? Why not? Why do you think that?

CHRISTINA The.....mmm.....

RESEARCHER (after a while) can you think of a reason?

CHRISTINA No.

Do you want to see what happens if we change the shape and size of this quadrilateral?

Yes.

RESEARCHER Grab and vertex and drag the quad. (whilst she drags one vertex of the quadrilateral around) Are they are all meeting at one point? Are they all concurrent?

CHRISTINA No

RESEARCHER When does it seem that they are concurrent? (as Christina drags further)

CHRISTINA When the vertices lie on the circumference of the circle.

RESEARCHER If I had to ask you quickly summarise what you've learnt up to this point, what would say?

CHRISTINA The position of the reservoir will be at the point where the perpendicular bisectors are concurrent.

RESEARCHER Okay, but when will they be concurrent, when will the perpendicular bisectors be concurrent?

CHRISTINA When the vertices lie on the circumference of the circle.

RESEARCHER So will the perpendicular bisectors always be concurrent?

CHRISTINA Yes if their vertices lie on the circumference of the circle....otherwise it won't be concurrent.

RESEARCHER So if we have these four villages as it is drawn here, where should we build the reservoir?

CHRISTINA At the point where the perpendicular bisectors meet.

RESEARCHER Let us now look at the case where we might have 3 villages, suppose we had 3 villages represented on this diagram by T, U and B. How do you think I would

	be able to locate the position within that triangle that will be equidistant from T, U and B?
CHRISTINA	Use perpendicular bisectors.
RESEARCHER	Click on the button there and tell me what you observe.
CHRISTINA	They meet at a point - they are concurrent.
RESEARCHER	So are you saying that this is the most suitable point?
CHRISTINA	Yes, it must be.
RESEARCHER	How can we be sure?
CHRISTINA	Measure the distances.
RESEARCHER	Do you want to measure it?
CHRISTINA	It's up to you sir.....because I'm sure.
RESEARCHER	Let us measure it. (after a while) What do u observe?
CHRISTINA	The distances are the same...you see you wasted my time!
RESEARCHER	Do you think that if we change the size of the triangle and the shape of the triangle, do you think it will always remain the same?
CHRISTINA	No.
RESEARCHER	You do not think so?
	No.
	Grab this point here and move it around and change the triangle. What do you observe?
CHRISTINA	(Silence)
RESEARCHER	What do you observe?
CHRISTINA	They are still concurrent.
RESEARCHER	What if the lines are meeting on the outside? Drag this vertex across, what do you observe?
CHRISTINA	They are still concurrent.
RESEARCHER	So, what can you say now? Do you think that for every triangle the perpendicular bisectors are concurrent?
CHRISTINA	Yes
RESEARCHER	You think so? How sure are you? If I ask you how many percent sure are you what would you say?
CHRISTINA	90 %
RESEARCHER	10% unsure? What's making you unsure that it will always be concurrent?
CHRISTINA	The size.
RESEARCHER	What about the size is making you unsure?
CHRISTINA	I'm not sure about very big triangles.
RESEARCHER	Go ahead and make this triangle very big. (After awhile). Does that make any difference?
CHRISTINA	Yes.
RESEARCHER	How many percent sure are you now?
CHRISTINA	Now, 100%.
RESEARCHER	Are you surprised at the result?
CHRISTINA	Yes
RESEARCHER	Why?
CHRISTINA	No matter how or which way you turn the triangle it still met at that point.
RESEARCHER	Would you want to know why this result is always true?
CHRISTINA	Yes...I can see it is true but maybe if there is a proof for it I'll understand it better.
RESEARCHER	Would you mind if I returned to that later?

CHRISTINA	No.
RESEARCHER	So returning to the original question, where should we build the reservoir if we have three villages.
CHRISTINA	It will be where the perpendicular bisectors meet and it will always meet for every triangle.
RESEARCHER	I'm going to quickly change the diagram. Let us say that there are 4 villages A, B, C and D. The Government wants to build pipelines from the water reservoir to the four villages. Where should the water reservoir be placed so that the total length of the pipeline will be a minimum? Why do you think we want to minimise it?
CHRISTINA	To save on the cost.
RESEARCHER	Now how do you think we will be able to locate the position where we would get PA plus PB plus PC plus PD to be a minimum?
CHRISTINA	Use the same method - perpendicular bisectors.
RESEARCHER	Click on that button for the perpendicular bisectors ...... now what do you observe?
CHRISTINA	No, the perpendicular bisectors do not meet.
RESEARCHER	So what should we do now?
CHRISTINA	Draw a point and move it around.
RESEARCHER	Ok. Drag this point around and look for the minimum sum okay?
CHRISTINA	It is there...(pointing) ... the minimum sum is 15...
RESEARCHER	Let's move it further and find out...
CHRISTINA	15 seems to be the smallest.
RESEARCHER	Okay. What do you observe?
CHRISTINA	These... these are the diagonals ......
RESEARCHER	Does it really look like the diagonals?
CHRISTINA	Yes.
RESEARCHER	Click on the button to construct the diagonals. What do you observe?
CHRISTINA	The.... The point where the diagonals meets ...that's the point where you must build the reservoir
RESEARCHER	Are you sure?
CHRISTINA	Yes.
RESEARCHER	What would happen if we changed the size of this quadrilateral? Where do you think the position would be?
CHRISTINA	Where the diagonals intersect.
	Try it...grab one vertex and move the quad around. What do you observe?
	(after a while) the minimum is still at the point where the these diagonals meet.
	What if we move this vertex to this point? (creating a convex quad.)
	It meets at the this point her on the outside where the diagonals still meet.
RESEARCHER	Do you find this result surprising?
CHRISTINA	Yes... It seems so strange - if I did not see it I wouldn't believe it.
RESEARCHER	Christina, I did say that we would return to an explanation for why perpendicular bisectors are concurrent for all triangles. Would you still want to know why?
CHRISTINA	Yes.
RESEARCHER	Read through this page and we will work together when you are ready.
CHRISTINA	(after a while Christina looks up)
RESEARCHER	We want to prove that the perpendicular bisectors of any triangle are concurrent. What are we given?

CHRISTINA	The perpendicular bisectors of triangle ABC.
RESEARCHER	Are all the bisectors given.
CHRISTINA	No... OD is not ...
RESEARCHER	In the diagram we will construct perpendicular bisectors OF and OE. We will join AO, BO and CO. We will also construct OD perpendicular to BC. What is the difference between OF, OE and OD?
CHRISTINA	OD is not perpendicular bisector.
RESEARCHER	Yes. For us to prove that O is the point of concurrency of all the perpendicular bisectors what should we prove?
CHRISTINA	That all the lines meet at one point.
RESEARCHER	Yes, that is concurrency, but given our diagram, we know that these lines are meeting at O. There is something that is missing...what is it?
CHRISTINA	I don't know.
RESEARCHER	Are the lines meeting at O already according to our diagram?
CHRISTINA	Yes.
RESEARCHER	Are OF, OE and OD perpendicular bisectors?
CHRISTINA	OD is not.
RESEARCHER	So what do you think we should prove?
CHRISTINA	We should prove that OD bisects BC.
RESEARCHER	Now follow the instructions and complete the worksheet.

INTERVIEW WITH FAEEZA AND RESEARCHER

RESEARCHER	Faeeza, you've read the question, do you understand it?
FAEEZA	It's easy to understand.
RESEARCHER	Where do you think that we should build the reservoir?
FAEEZA	About here (pointing to the middle of the quadrilateral).
RESEARCHER	Can you prove to me that this is the correct point?
FAEEZA	Ya ... can I measure with my ruler?
RESEARCHER	Yes you may.
FAEEZA	(After a while) I don't think I'm measuring correctly ...am I? ...
RESEARCHER	Do you want to perhaps try another method?
FAEEZA	I don't know ... what method should I use?
RESEARCHER	Do you know of any method?
FAEEZA	No!
RESEARCHER	Are you sure?
FAEEZA	I really don't know (irritably).
RESEARCHER	Ok. Let us see if working with the computer will help. This is the same diagram you have on the page. Can you point out again the most suitable position of the reservoir by just looking at the diagram?
FAEEZA	Ya. It should be situated towards the middle.
RESEARCHER	Faeeza just suppose, the point you chose is not suitable. In real life it is not as easy as it is in mathematics. Do you know of any reasons why we may not be able to build water purification works at the point you indicated?
FAEEZA	Maybe there is building ....
RESEARCHER	What kind of building do you think?
FAEEZA	Could be a school, I don't know.... could be a shopping complex.
RESEARCHER	Remember that these are remote villages ....
FAEEZA	Might be the chief's house......
RESEARCHER	Yes...
FAEEZA	There might be a mountain there ... anything, like a big rock. The cost factor must be too great to get rid of the rocks. It won't be cheap to build there.
RESEARCHER	What do you think the purpose of asking these questions is?
FAEEZA	Because we are trying to show that we can choose a point but that point is not always good... maybe we should only choose the point after we see the place.
RESEARCHER	So it is true that in a real life there might be difficulties that we may experience. Now Faeeza draw a point somewhere in this quadrilateral ... this is where you suggested the building should be. How will we know what the distances are from that point to the various villages?
FAEEZA	Obviously you need to measure it.
RESEARCHER	Will you measure it then? (after a while) Ok...there are all the distances from the point that you chose to the different villages. So what do we do now?
FAEEZA	Drag the point around until we get those distances equal.
RESEARCHER	Do that now.
FAEEZA	(after a while) It's taking too, long and I can't get it easily.
RESEARCHER	But you have found a method of finding the point. Now, besides just measuring and dragging the point around like you just did, there might be other methods. Do you know of any other way in which you could find the point?
FAEEZA	No.
RESEARCHER	Are you sure?

FAEEZA	Yes...I'm positive that I don't know of any other way.
RESEARCHER	Let us now make it simpler. Let us say that there are just 2 villages and are represented by I and J, as shown on this diagram. Now I want to find the point, which is equidistant from I and J. Where do you think the best position would be?
FAEEZA	You could build in-between them ... the midpoint.
RESEARCHER	How would you do that?
FAEEZA	Just select the line and construct the midpoint.
RESEARCHER	Click on that button and you will see the midpoint. There is the midpoint; you can see by the measurement, that these 2 portions are equal (pointing to the line segments from the midpoint to I and J). As we said earlier, in real life there might be a problem, in the sense that the best position that we chose may not be suitable, or cannot be used for construction, for various reasons. Do you think that I can find another point?
FAEEZA	You can locate it around there (pointing), somewhere where it would be equal distance apart.
RESEARCHER	Do that construction. ...(after a point and the segments were constructed) Okay how can we check whether it is equidistant apart?
FAEEZA	You measure the distance from the village to the point.
RESEARCHER	But they are not, what must you do now?
FAEEZA	We move it... I'll drag this point around.
RESEARCHER	How long should we move it around?
FAEEZA	Till we get the .. till the distances are equal.
RESEARCHER	Now lets suppose that that point that we chose is as well not suitable, not workable
FAEEZA	Move more further away ...... get another point... like this ...(after constructing another point)
RESEARCHER	Faeeza how many points do you think you can get like this?
FAEEZA	Quite a few, it depends on which... millions ...... if we are working with two villages....
RESEARCHER	So if we work with two villages we will have millions of points?
FAEEZA	Ya.
RESEARCHER	What do you notice about the points? Look at the points and what can you see there?
FAEEZA	Going in the same direction.
RESEARCHER	What does that mean?
FAEEZA	(hesitant) They look like they are on a line ....
RESEARCHER	How will I know that for sure?
FAEEZA	Draw a line.....?
RESEARCHER	Then draw a line. Okay, there you are – you were right - they lie on a straight line. What else can you observe about this straight line? Is it special in any way?
FAEEZA	Hmmm.
RESEARCHER	Okay, it might not necessarily be obvious but look at it carefully....
FAEEZA	They look perpendicular.
RESEARCHER	How can you be sure that it is perpendicular? It looks perpendicular, how can we know for sure that it is perpendicular?
FAEEZA	Hmmm................ Measure the angle?
RESEARCHER	Ok, try that. (after the angle is measured) There's the angle there, what is the

angle?

FAEEZA 90 degrees.

RESEARCHER What does that imply?

FAEEZA It is perpendicular!

RESEARCHER So if you had 2 villages where should you build a water purification plant that will be equidistant from both the villages?

FAEEZA Hmmm ... We can go on a straight direction and move further away, it will still be equal distant that you can find...........move on the straight line...

RESEARCHER How can you describe this line?

FAEEZA It is perpendicular.

RESEARCHER Is it only perpendicular?

FAEEZA It cuts in the middle.

RESEARCHER What do we call a line that is perpendicular and cuts a line into equal parts?

FAEEZA Perpendicular bisector

RESEARCHER Yes that is a perpendicular bisector. So now just to summarise what you said was that if we wanted to find a point that is equidistant from I and J.............

FAEEZA ...... we must build it anywhere along the perpendicular bisector.

RESEARCHER Let's go on to the original problem, where we had 4 villages and we want to find a point which is equidistant to the four villages. Remember you told me that the point is in the middle?

FAEEZA Yes.

RESEARCHER Now besides constructing a point, and measuring the distances and moving it about, how else do you think I can find that point?

FAEEZA Just draw the perpendicular bisectors.

RESEARCHER Perpendicular bisectors of which lines?

FAEEZA All.

RESEARCHER What do you think would happen if we drew the perpendicular bisectors of all?

FAEEZA You will find a point where it might be equal.

RESEARCHER What will be equal?

FAEEZA The distances to V1, V2, V3 and V4.

RESEARCHER Click on that button to see the construction of these four perpendicular bisectors. (after the click) These are the perpendicular bisectors. What do you observe?

FAEEZA It all comes to a point.......... They all meet at a common point.

RESEARCHER Does this surprise you?

FAEEZA Not really.... It was kind of obvious.

RESEARCHER Do you know what this point is called?

FAEEZA I don't know...midpoint?

RESEARCHER It is the point of concurrency.

FAEEZA Ok.

RESEARCHER So they all concurrent - meaning that they meet at one particular point. But how do we know that the point at which they meet is equidistant from V1, V2, V3 and V4?

FAEEZA You measure.

RESEARCHER Let us measure it and check okay?

FAEEZA (whilst measuring) It must be because the distances are all 3.8.

RESEARCHER What does this mean to us?

FAEEZA Well...to get the correct position of the water plant you must use the perpendicular bisectors of all the sides. Where they meet is the position.

RESEARCHER	Remember what you said initially, you said that the point should be at the middle? Is it different now?
FAEEZA	Yes...it is strange ....
RESEARCHER	Is that interesting?
FAEEZA	Very (emphatically)
RESEARCHER	Now can you construct a circle here, using this point of concurrency as the centre, and let it pass through this point here. Let us se see what happens. (after the circle was constructed) Now what can you say about the circle?
FAEEZA	The four sided figure lies on the boundary of the circle..........
RESEARCHER	Boundary? What is the boundary of a circle called?
FAEEZA	Sorry circumference.
RESEARCHER	Now do you think that in every quadrilateral the perpendicular bisectors are going to be concurrent? That means if I drew any other quadrilateral and constructed the perpendicular bisectors will they be concurrent?
FAEEZA	Ya.
RESEARCHER	Let's test it. Grab one of them ... one vertex and move it ... can you see what is going on? Now there's another quadrilateral, can you see the quadrilateral? Are the perpendicular bisectors meeting?
FAEEZA	No.
RESEARCHER	What do you observe?
FAEEZA	The circle is not passing through the vertices of the quadrilateral.
RESEARCHER	Let us move it again. (whilst Faeeza is moving the triangle) What do you observe now?
FAEEZA	The perpendicular bisectors are concurrent and the quadrilateral passes through the circle.
RESEARCHER	Is it not true that the circle is passing through vertices of the quadrilateral?
FAEEZA	Yes.
RESEARCHER	So what does this mean?
FAEEZA	We only get the lines to be concurrent when the vertices of the quadrilateral are on the circle.
RESEARCHER	Do you think that this is always true?
FAEEZA	Yes (confidently)
RESEARCHER	Do you want to construct more diagrams for you to confirm this?
FAEEZA	No please (emphatic and irritated).
RESEARCHER	Are you convinced?
FAEEZA	Yes...very.
RESEARCHER	So what does this imply when we go back to the original question?
FAEEZA	The reservoir must be built at the point where the perpendicular bisectors meet.
RESEARCHER	Which perpendicular bisectors?
FAEEZA	The bisectors of the lines joining the villages on the outside.
RESEARCHER	Let's change this a little bit now, we've dealt with 2 villages, we've dealt with 4, let's see what will happen if we have 3 villages. Let's suppose I have 3 villages and they form a triangle like this. How do you think we could find the point, which is equidistant to all 3 villages?
FAEEZA	Use perpendicular bisectors.
RESEARCHER	Do you think they will meet?
FAEEZA	They must meet .... no....... they will only meet if the vertices of the triangle lies on a circle.
RESEARCHER	So the perpendicular bisectors will only be concurrent in certain triangles?

FAEEZA	Yes ...... only those for which the vertices of the triangle lie on a circle.
RESEARCHER	Ok ..... let us construct the perpendicular bisectors........ (after a while) What do you observe?
FAEEZA	The perpendicular bisectors are concurrent.
RESEARCHER	Let us draw a circle then. (after a while)
FAEEZA	Yes I was right...you can draw a circle.
RESEARCHER	But you did say that these bisectors would not always be concurrent. Only for those triangles, which have a circle passing through its vertices?
FAEEZA	Yes.
RESEARCHER	Ok ...... let us change the size of this triangle. What do you observe?
FAEEZA	The lines are still meeting.
RESEARCHER	So what can we say?
FAEEZA	I don't know sir...move it some more ...(after a while) ....I don't know about this but the lines are always current.
RESEARCHER	Concurrent.
FAEEZA	Yes concurrent.
RESEARCHER	What about this circle?
FAEEZA	It always passes through the vertices.
RESEARCHER	Do you think that this is a trick?
FAEEZA	No.
RESEARCHER	Do you think that the perpendicular bisectors are always concurrent for triangles?
FAEEZA	Yes.
RESEARCHER	What if we turned the triangle around?
FAEEZA	But they are still meeting on the outside.
RESEARCHER	Are you sure?
FAEEZA	Yes.
RESEARCHER	How many percent convinced are you?
FAEEZA	100% because when you move it in all directions, it meets at a common point.
RESEARCHER	Would you like to know why this is always true?
FAEEZA	It might be interesting to know...I can't believe it (showing surprise).
RESEARCHER	Would you mind if I explained that later?
FAEEZA	No.
RESEARCHER	Okay, let me change this problem, suppose I have 4 villages now, A, B, C and D. The government decides to build pipelines from the water reservoir to the 4 villages, A, B, C and D. Where should the water reservoir be placed so that the total length of the pipeline is minimised? Firstly, why should we minimise on the sum?
FAEEZA	It will save costs... the fewer pipes we use the cheaper it will be.
RESEARCHER	Yes it will be cheaper. So how can we find that point?
FAEEZA	Construct the perpendicular bisectors.
RESEARCHER	Do you think that that method will work?
FAEEZA	I don't know.... I never saw this problem before.
RESEARCHER	Do that then... (after a while). What do you observe?
FAEEZA	The perpendicular bisectors are not concurrent.
RESEARCHER	So what should we do now?
FAEEZA	Use the long method. Construct a point and drag it around.
RESEARCHER	Ok ... do that.
FAEEZA	(after a while) yes the sum is changing...it is

increasing......decreasing...decreasing..........increasing again.....decreasing.......I think this point here is right.

RESEARCHER Do you think that this is very accurate?

FAEEZA I think it is close enough.

RESEARCHER But is it accurate?

FAEEZA I don't know whether it is the accurate point.

RESEARCHER Look at the lines you constructed and tell me whether it reminds of anything?

FAEEZA These are the lines joining the vertices.

RESEARCHER What are these lines called?

FAEEZA Diagonals ...?

RESEARCHER Yes they are called diagonals. Do you think that that is the way to get the solution?

FAEEZA Yes.

RESEARCHER Do you think that if we construct the diagonals we'll get the exact point?

FAEEZA Yes I think so.

RESEARCHER Click on that button to construct the diagonals then...(after a while)...what do you observe?

FAEEZA The point we dragged is very close to the point where the diagonals meet.

RESEARCHER So how would you find the most suitable point?

FAEEZA Construct diagonals.

RESEARCHER Do you think that if I change the quadrilateral, the same result will apply?

FAEEZA Yes...it must.

RESEARCHER Why are you so certain?

FAEEZA I don't know....I just think that it will work.

RESEARCHER Wouldn't you want to try it for other quadrilaterals?

FAEEZA No...it will only waste time.

RESEARCHER But how can you be sure that it always works?

FAEEZA Ok..ok...let's try other diagrams.

RESEARCHER If you think that it is not necessary then I won't...

FAEEZA No let us try just one more.

RESEARCHER Would it work for this quadrilateral?

FAEEZA But sir, what is that?

RESEARCHER It is called a concave quadrilateral.....does it have four sides?

FAEEZA Yes...

RESEARCHER Then it is a quadrilateral.

FAEEZA But it looks different.

RESEARCHER Yes it does...do you think it works for this one?

FAEEZA Let us just drag that point first....

RESEARCHER What do you observe?

FAEEZA It's the smallest at A.

RESEARCHER So what is A?

FAEEZA That is where diagram is concave.

RESEARCHER Do you think that this will always be the case?

FAEEZA I think so...

RESEARCHER Would you like to try another one?

FAEEZA Yes.

RESEARCHER Just drag this point...is it still a concave quadrilateral?

FAEEZA Yes.

RESEARCHER How should we check whether the rule applies here?

FAEEZA	Move the point around and then to the point where it is concave.
RESEARCHER	Good ... now what do you observe?
FAEEZA	That is the same point! The minimum point will be at the point of intersection of the diagonals. In a concave one the position is where it meets, at the concave part.
RESEARCHER	Are you sure?
FAEEZA	Yes...definitely!
RESEARCHER	Faeeza, I did say that we would return to an explanation for why perpendicular bisectors are concurrent for all triangles. Would you still want to know why?
FAEEZA	Yes ... I thought you forgot.
RESEARCHER	Read through this page and we will work together when you are ready.
FAEEZA	(after a while) ok sir
RESEARCHER	So we want to prove that the perpendicular bisectors of any triangle are concurrent. What are we given?
FAEEZA	We are given this triangle ABC.
RESEARCHER	In the diagram we will construct perpendicular bisectors OF and OE. We will join AO, BO and CO. We will also construct OD perpendicular to BC. What is the difference between OF, OE and OD?
FAEEZA	OD is a broken line.
RESEARCHER	Yes according to the diagram it is, but I want you to read 1.3 and 1.4 again. What does 1.3 tell us?
FAEEZA	That OF and OE are perpendicular bisectors.
RESEARCHER	What does that mean?
FAEEZA	OF is perpendicular to AB and OE is perpendicular to AC and .....AF = BF and ....AE = EC.
RESEARCHER	What does 1.4 tell us?
FAEEZA	OD is perpendicular to BC ....and ...oh it's not bisecting...
RESEARCHER	Yes. For us to prove that O is the point of concurrency of all the perpendicular bisectors what should we prove?
FAEEZA	I think that we should prove that BD is equal to DC.
RESEARCHER	That is correct.

INTERVIEW WITH NIGEL

RESEARCHER	Nigel, you've read the question, do you understand it?
NIGEL	Yes I do.
RESEARCHER	Where do you think that we should build the reservoir?
NIGEL	There (pointing to the middle of the quadrilateral).
RESEARCHER	Can you prove that your answer is correct?
NIGEL	No I can't.
RESEARCHER	Are you saying that you cannot find **any** way of proving this?
NIGEL	I can't.
RESEARCHER	Don't you want to just try?
NIGEL	I wouldn't know were to start ... I can't.
RESEARCHER	Let us see if working with the computer will help. This is the same diagram you have on the page. Can you point out again the most suitable position of the reservoir by just looking at the diagram?
NIGEL	Should be built an equal distance away from all 4 villages... say about somewhere around here.
RESEARCHER	How would you describe the position?
NIGEL	Towards the middle.
RESEARCHER	Just construct any point towards the middle, as you are pointing it out to me. How will we know if this is equidistant from the four points of the quadrilateral? How can we check?
NIGEL	I can measure it.
RESEARCHER	(after a while) You have completed the construction but are the distances the same?
NIGEL	No.
RESEARCHER	Now how can we actually determine the point that is equidistant?
NIGEL	You can move the point around until they (referring to the distances) are all equal.
RESEARCHER	Why don't you drag this point around then ... What is happening to the distances?
NIGEL	They are changing.
RESEARCHER	So when will you stop the dragging?
NIGEL	When the distances are equal...but this is taking to long...
RESEARCHER	This is certainly a long process and we may find it difficult to find the exact point. Is there a mathematical way of doing it?
NIGEL	Not that I know of.
RESEARCHER	Let's suppose this point that you chose is the correct one. In real life it often occurs that there might be problems with the point that you have chosen. What reasons do you think might influence the construction of the reservoir?
NIGEL	There might be a mine ....
RESEARCHER	Yes ... any other reasons?
NIGEL	If the area is very rocky or has mountains.......maybe they are storing hazardous nuclear waste nearby.......
RESEARCHER	So what do you think is the point of this question?
NIGEL	That sometimes in real-life we may not be able to use exact mathematics to solve problems.
RESEARCHER	So do you think that there is an easier way of finding the most suitable point?
NIGEL	No.
RESEARCHER	Let us simplify the problem then. Let us assume that we are now working with 2 villages as depicted in this picture. Villages I and J. Where would you locate a

	reservoir if it had to be the same distance from J and I.
NIGEL	At the centre of the line.
RESEARCHER	What do we call that point?
NIGEL	Midpoint.
RESEARCHER	Click on the button for that construction. (after constructing) There you have the midpoint Q it is 2,2 away from I and 2,2 away from J. So it confirms that it is at the centre. Now as we said earlier there might be a problem at that point. What will you do?
NIGEL	I'll draw a triangle
RESEARCHER	Draw a triangle?
NIGEL	Yes, so that you can measure the distance from I to the apex and from J to the apex.
RESEARCHER	Oh, you want to construct a point here and draw a triangle so that you could measure the two sides of the triangle?
NIGEL	Yes to measure distances from I and J...
RESEARCHER	All you want to do is measure the distances?
NIGEL	Yes.
RESEARCHER	Do you think that it is necessary to construct the entire triangle?
	No... just a point.
	Do that then... how can we find the exact position for it to be equidistant?
NIGEL	Move it (the point) around.......like here.....they are equal now.
RESEARCHER	Yes they are but what if that point was a problem?
NIGEL	You would do the same on the other side.
RESEARCHER	So you would draw another point, but on the other side? Do that and measure the distance ...(after a while) they are exactly the same and if that point is problem? Do you think that we might be able to find another point?
NIGEL	We might find a point in between these 2.
RESEARCHER	(after a while) ok .....we are doing the same thing...if all of these points were a problem would we be able to find others?
NIGEL	Yes
RESEARCHER	How many points do you think we can find?
NIGEL	Many....hundreds.
RESEARCHER	What do you notice about all of these points we have drawn?
NIGEL	All the points seem to be on a straight line
RESEARCHER	Construct a line and see if it is on a straight line. (after a while) Do they lie on the same line?
NIGEL	Yes
RESEARCHER	What else do you notice about this line?
NIGEL	It divides..... it cuts the other line in two .... it is equal
RESEARCHER	What else do you notice?
NIGEL	It looks like a right angle here.......
RESEARCHER	Do you think it is a right angle? Do you want to check
NIGEL	Yes
RESEARCHER	Measure it .... (after a while) it is 90degrees
NIGEL	Yes....I thought so.
RESEARCHER	So what does this mean? If I asked you for more points where would we find them?
NIGEL	On that line.
RESEARCHER	You are telling me that you can find many points on that straight line?

NIGEL	Yes.
RESEARCHER	Describe this line.
NIGEL	It is a straight line. It bisects there , it forms a right angle
RESEARCHER	So what is a line which forms a right angle with another line called?
NIGEL	A perpendicular line.
RESEARCHER	Is this only perpendicular?
NIGEL	It bisects too …….. a perpendicular bisector.
RESEARCHER	Very quickly now if we had 2 villages like that and we want to find positions that are equal distances from each of the villages what would we do?
NIGEL	Find the perpendicular bisector….and then any point on it will work.
RESEARCHER	Let us now go back to the original problem. The original problem said that we had four villages, like this. We want to know, besides constructing a point and measuring the distances and then moving the point around like we did earlier, how do you think we would be able to find the most suitable position which is equidistant from all 4 villages?
NIGEL	Divide these 2 lines into equal parts and then draw perpendicular lines and then do it again for the other 2 lines….
RESEARCHER	Are you saying that I should construct the perpendicular bisectors?
NIGEL	Yes.
RESEARCHER	Click on that button for the construction of the perpendicular bisectors. (after a while) And what do you notice?
NIGEL	It all measures up to one point.
RESEARCHER	Measures?
NIGEL	Meets.
RESEARCHER	Does this surprise you?
NIGEL	Just a little bit…but I knew that the lines will meet.
RESEARCHER	When many lines meet at a point what do we call that point?
NIGEL	I don't know.
RESEARCHER	We say that they are concurrent. In this case they meet at a common point. Do you think that, that point now is equidistant from V1, V2, V3, and V4?
NIGEL	Yes
RESEARCHER	Are you sure?
NIGEL	Let's measure it.
RESEARCHER	Go ahead and measure it … you have 3.8, another 3.8…do we need to measure any more?
NIGEL	Yes let us measure these 2.
RESEARCHER	(*after a while*) what do we notice?
NIGEL	They are all equal.
RESEARCHER	Can you construct a circle with the point of concurrency as the centre and move it slowly upwards to this vertex here.(after a while) What do you notice?
NIGEL	(*after a while*) The four edges of the quadrilateral meet the circumference of the circle.
RESEARCHER	Edge?…….which are you referring to as the edge? (*after learner points*) …oh we call that a vertex……..all of these are vertices.
NIGEL	Ok.
RESEARCHER	So the vertices of this quadrilateral lie on the circumference of the circle. Do you think that if I had any kind of quadrilateral the result would be the same?
NIGEL	Yes (*very confidently*)
RESEARCHER	So no matter what kind of quadrilateral, the perpendicular bisectors will always

	be concurrent?
NIGEL	Yes (*emphatically*)
RESEARCHER	Perhaps we should check. Change the quadrilateral and see what happens to the circle?
NIGEL	It went away from the vert…vertices.
RESEARCHER	Oh and what else do you notice about the perpendicular bisectors?
NIGEL	It does not meet at all.
RESEARCHER	Change it again, and let see if we can make them meet again. Yes they are meeting again but what do you observe?
NIGEL	The perpendicular bisectors are concurrent….
RESEARCHER	What can you say about the vertices of the quad.
NIGEL	It touches…. It lies on the circle
RESEARCHER	Then can you draw a conclusion from that?
NIGEL	The perpendicular bisectors are concurrent when it lies on the circumference of the circle.
RESEARCHER	Do you think that this is always true?
NIGEL	Definitely!
RESEARCHER	Should I construct more examples for you to test what you are saying?
NIGEL	No sir, I'm quite convinced.
RESEARCHER	What do you think of this?
NIGEL	It is very interesting.
RESEARCHER	So if we look at the four villages that I gave you initially, where must the reservoir be built?
NIGEL	Find the perpendicular bisectors of the sides and the point of concurrency will be the correct point.
RESEARCHER	Good, let us change this problem further. We have worked with 2 villages and we have worked with 4 villages let us see what happens if we had 3 villages. There is a triangle here representing the 3 villages T, U and B. If I ask you the same question, how can I find a point that is equidistant from T, U and B, how will find the point?
NIGEL	Use the other method.
RESEARCHER	What is the other method?
NIGEL	The perpendicular bisectors.
RESEARCHER	Are saying that we should use the perpendicular bisectors?
NIGEL	Yes.
RESEARCHER	Click on the button for that construction. (After a while) Are the perpendicular bisectors concurrent?
NIGEL	Yes.
RESEARCHER	Do you think that in every triangle the perpendicular bisectors would be concurrent?
NIGEL	They won't.
RESEARCHER	Why are you saying that they won't?
NIGEL	There are different types of triangles.
RESEARCHER	Let us check. Construct a different triangle by dragging this vertex. Can you see the triangle changing? Is the size changing?
NIGEL	Yes
RESEARCHER	Is the shape changing
NIGEL	Yes
RESEARCHER	And what is happening to the perpendicular bisectors?

NIGEL	It is the same. There is always a midpoint.
RESEARCHER	Midpoint?
NIGEL	A point of concurrency.
RESEARCHER	What do you think of this?
NIGEL	This is very surprising……. I think you tricked me…(laughing)
RESEARCHER	Why?
NIGEL	I thought that there will not always be a point of concurrency because of the different sizes and shapes. In the quadrilateral it was different.
RESEARCHER	What if we turned this triangle around…. Like this…what happens then?
NIGEL	But they are still meeting…but on the outside.
RESEARCHER	If I asked you how many percent convinced are you what would you say?
NIGEL	I have no doubt….100%.
RESEARCHER	Would you like to know why this is always true?
NIGEL	Definitely…maybe I could trick my friends too.
RESEARCHER	Can we do the explanation a little later on?
NIGEL	Yes.
	So if we had three villages where should we build the reservoir?
	At the point of concurrency of all the perpendicular bisectors. We are lucky here because we will always have a point of concurrency.
RESEARCHER	I am going to now change the diagram. Let's talk about 4 villages A, B, C, and D. Let us say that the government decides to build pipelines now from the water reservoir to the 4 villages A, B, C and D. Where should the water reservoir be placed so that the total length of the pipeline is minimised? In other words, the sum of the distances is at minimum. Firstly, why do you think they would want it to be at a minimum?
NIGEL	To save money.
RESEARCHER	That is an important thing. Now the question is where you do think we should build it and how can I get the sum of the distances to be a minimum?
NIGEL	You can use the same rule
RESEARCHER	What rule is that Nigel?
NIGEL	The perpendicular bisectors rule.
RESEARCHER	Are you saying that we should construct the perpendicular bisectors of the sides?
NIGEL	Yes.
RESEARCHER	Click on the button for perpendicular bisectors. (after a while) There we go ……..are the perpendicular bisectors concurrent?
NIGEL	No
RESEARCHER	So does that work? What should we do now?
NIGEL	No it does not work but yaybe just construct a point and move it about.
RESEARCHER	Ok… do that…(after a while) What is the smallest sum?
NIGEL	The smallest sum is 15
RESEARCHER	If you look at these lines here does it remind you of anything?
NIGEL	It reminds me of the lines going across to opposite sides.
RESEARCHER	What do we call these lines?
NIGEL	Diagonals.
RESEARCHER	Do you think that they may be diagonals?
NIGEL	Yes.
RESEARCHER	Let us construct the diagonals and compare them to what we already have. Click on that button for the diagonals (after a while) What do you observe?

NIGEL	It is the diagonals! (Triumphant!)
RESEARCHER	You say that minimum sum is obtained where the diagonals…?
NIGEL	Meet.
RESEARCHER	Do you think that if you change the size of this quadrilateral we will get the same result?
NIGEL	Yes.
RESEARCHER	Do you think that we should check?
NIGEL	Yes.
RESEARCHER	(after a while) you have now dragged this vertex and changed the quad…what do you observe?
NIGEL	The result is still the same.
RESEARCHER	What if I had a concave quadrilateral like that? Where do you think it would be now?
NIGEL	I've never seen a concave quadrilateral but I think that it would be the same
RESEARCHER	You think it is going to be the same.
NIGEL	Yes.
RESEARCHER	There is your 1 diagonal, from B to D and the other from B to C. I have to extend that line and so that they will meet. Drag this point and see where it would be a minimum. Are you looking at the sum?
NIGEL	Yes. 12,4
RESEARCHER	Yes…
NIGEL	12.3, 12.1, 11.7
RESEARCHER	Where is that point?
NIGEL	A
RESEARCHER	You say on A, and where is that actually?
NIGEL	If you extend your diagonals you can see that it would meet at that point there.
RESEARCHER	It is the point where the quadrilateral is concave isn't it? Nigel, I did say that we would return to an explanation for why perpendicular bisectors are concurrent for all triangles. Would you still want to know why?
NIGEL	Oh yes…is this standard 8 work?
RESEARCHER	No, you will do this in grade 11 but you are doing quite well with it. Read through this page and we will work together when you are ready.
NIGEL	(after a while) ok I'm finished.
RESEARCHER	So we want to prove that the perpendicular bisectors of any triangle are concurrent. What are we given?
NIGEL	OF is perpendicular to AB and OE is perpendicular to AC. Also OE is perpendicular to OD.
RESEARCHER	In the diagram we constructed perpendicular bisectors OF and OE. We will join AO, BO and CO. We will also construct OD perpendicular to BC. What is the difference between OF, OE and OD?
NIGEL	OD is a broken line.
RESEARCHER	Yes in the diagram it is but read 1.3 and 1.4 again and tell me what the difference is?
NIGEL	OD is not a perpendicular bisector….it is only perpendicular.
RESEARCHER	For us to prove that O is the point of concurrency of all the perpendicular bisectors what should we prove?
NIGEL	Hmm….hmm…
RESEARCHER	Are the lines meeting at O already according to our diagram?
NIGEL	Yes.

RESEARCHER	Are OF, OE and OD perpendicular bisectors?
NIGEL	OD is not ...OD is perpendicular but not a bisector.
RESEARCHER	So what do you think we should prove?
NIGEL	OD is perpendicular a bisector.
RESEARCHER	That is correct.
NIGEL	Now follow the instructions and complete the worksheet.

INTERVIEW WITH ROXANNE AND RESEARCHER

RESEARCHER	Roxanne, you've read the question, do you understand it?
ROXANNE	Yes.
RESEARCHER	Where do you think that we should build the reservoir?
ROXANNE	In the centre.
RESEARCHER	Is there any way of proving that the reservoir should be in the middle?
ROXANNE	I don't think so.
RESEARCHER	Are you saying that there is no way to do it or is it that you don't know how to do it?
ROXANNE	I can't. There might be a way to do it but I can't.
RESEARCHER	Wouldn't you want to try?
ROXANNE	I don't know what to do!
RESEARCHER	Let us see if working with the computer will help. This is the same diagram you have on the page. Can you point out again the most suitable position of the reservoir by just looking at the diagram?
ROXANNE	In the middle.
RESEARCHER	So do you think that it should be in the middle here? (pointing)
ROXANNE	Ehhh...maybe more towards the side here...(indicating a shift to the right)
RESEARCHER	All right, then how do you think we should go about checking whether it is correct?
ROXANNE	By measuring the distances.
RESEARCHER	Quickly do that ... (after a while) okay. These are all the measurements. But the distances are not equal. How can we get then to be equal?
ROXANNE	You can take the point around and try to find the spot for which the distances will be equal.
RESEARCHER	So you mean we should drag this point around?
ROXANNE	Yes.
RESEARCHER	Do that now.
ROXANNE	This takes long to get the right position.
RESEARCHER	It does take long. It might not be the best way to do it. Let us suppose that we do find the point. Very often in real life the point that we find mathematically might not be suitable for various reasons. What do you think might be some of the reasons for this point not being suitable?
ROXANNE	Maybe there is a mine or a building.
RESEARCHER	Remember that these villages are remote villages. What kind of building might be there?
ROXANNE	There might be kraals there or the chief's house...
RESEARCHER	Do you think that there might be other reasons?
ROXANNE	The place might be a mountain, with hard rocks...
RESEARCHER	What do you think the purpose of asking this question?
ROXANNE	We are saying that we can work out a place on the computer or by calculating it but it does not mean that it will we the right place.
RESEARCHER	Now, besides just measuring and dragging the point around like we were doing earlier. There might be other methods in doing this. Do you know of any other way in which you could find the point?
ROXANNE	No.
RESEARCHER	All right then let us try to make that problem simpler. Let us suppose that we only had two villages and if you look at this diagram here it is represented by I and J. If I said to you that there are only two villages, represented by I and J

	and I want you to find a point that is equidistant from I and J. Would you be able to find such a point?
ROXANNE	Yes, that will be easy.
RESEARCHER	Where will that be?
ROXANNE	In the middle, the midpoint of I & J.
RESEARCHER	The midpoint of I & J?
ROXANNE	Yes.
RESEARCHER	Click on that button for the midpoint ... okay, and you can see by the distances represented here that that the points are equidistant. Now as we said earlier, that that the point we chose...
ROXANNE	... might be a problem?
RESEARCHER	Yes. What should we do then?
ROXANNE	We can bring it to the centre over here maybe about so far, in the centre though.
RESEARCHER	So are you saying that we should find another point which is equidistant from I and J?
ROXANNE	Yes.
RESEARCHER	Construct that and show it to me. (after a while) How will I know whether that distance is correct?
ROXANNE	Measure it and move it around.
RESEARCHER	So go on and drag the point
ROXANNE	Yes... yes here.
RESEARCHER	All right...but what if this point is not suitable?
ROXANNE	Do the same thing again on this side.
RESEARCHER	On the opposite side?
ROXANNE	Yes.
RESEARCHER	(after a while) ok...you seem to have it. What if this point was also not suitable?
ROXANNE	Just construct another one....you can construct as many as you like.
RESEARCHER	How many points do you think?
ROXANNE	I don't know, just a lot.
RESEARCHER	Looking at these points does it show any pattern?
ROXANNE	Yes....they are forming a straight line.
RESEARCHER	How can we know that for certain?
ROXANNE	I can draw a line through these points.
RESEARCHER	(after a while) What do you observe?
ROXANNE	It passes through all the points.
RESEARCHER	What else can you observe about this line?
ROXANNE	I don't know.
RESEARCHER	Look at the line that we constructed. Is there something special about it?
ROXANNE	Well .. it looks like it is forming a 90 degrees angle here.
RESEARCHER	Do you think it might be?
ROXANNE	Yes...it looks like that...
RESEARCHER	How can we be certain?
ROXANNE	Measure the angle. (after a while) It is 90 degrees.
RESEARCHER	So what do we call this line?
ROXANNE	A straight line.
RESEARCHER	Yes it is straight line...but it is a special line.
ROXANNE	A bisector.
RESEARCHER	Yes it is a bisector...but it also has another property.

ROXANNE	Mmm…
RESEARCHER	What is this other property?
ROXANNE	It is perpendicular.
RESEARCHER	Yes…so what can we say about the line…
ROXANNE	It is a perpendicular bisector.
RESEARCHER	So if I asked you to find points that are equidistant from two points how will you do it?
ROXANNE	Draw the perpendicular bisector and then any point on the line will be correct.
RESEARCHER	So if we had two villages, where should we build the reservoir?
ROXANNE	Just determine the perpendicular bisector and any point on it will work.
RESEARCHER	Let us say that I drew a perpendicular line here, will it work?
ROXANNE	No, it is not the same distance away.
RESEARCHER	Let us look at the original problem. How would you find the most suitable point, besides of course drawing a point, measuring and dragging it around like we did earlier? Do you think there might be another way of getting this point?
ROXANNE	It will be hard, because you have four points…
RESEARCHER	Is it really that difficult?
ROXANNE	It looks difficult...but I think we should draw their perpendicular lines…no perpendicular bisectors.
RESEARCHER	Why do you want to draw the perpendicular bisectors?
ROXANNE	It worked nicely for the two villages…I don't know…maybe it will work for these villages too.
RESEARCHER	Would you want to see what would happen if we constructed the perpendicular bisectors?
ROXANNE	Yes sir.
RESEARCHER	Then click on that button there (after a while) There we have it...what do you observe?
ROXANNE	All the lines go to one point…
RESEARCHER	When you say go what do you mean?
ROXANNE	They come to…they meet at one point.
RESEARCHER	Do you think that this is the most suitable point?
ROXANNE	Yes sir.
RESEARCHER	Does it surprise you that these lines are meeting at one point?
ROXANNE	mmm…I don't know… it must meet there…where else can it meet?
RESEARCHER	How will we know for certain that this is the correct point?
ROXANNE	We can measure it, I know it must be the point.
RESEARCHER	All right measure each distance…that is 3.8…again we have 3.8. What do you think the next two distances will be?
ROXANNE	The same…3.8 and 3.8.
RESEARCHER	Go on and check it out …this is 3.8 and...this is also 3.8.
ROXANNE	I knew it…I was right.
RESEARCHER	Yes you were. Roxanne, do you know what we call this point were these lines meet?
ROXANNE	Central point?
RESEARCHER	No…is it the centre of this quadrilateral?
ROXANNE	No.
RESEARCHER	We call this point the point of concurrency…we say that these perpendicular bisectors are concurrent at this point.
ROXANNE	Concurrent?

RESEARCHER	Yes. Let us see what would happen if we drew a circle from this point of concurrency. Use the circle function there and draw a circle up to this point here. (after a while) Describe what you observe.
ROXANNE	All the points go around...
RESEARCHER	What do you mean by that Roxanne?
ROXANNE	The circle goes around the points of the quadrilateral.
RESEARCHER	Circle goes around the points...?
ROXANNE	See these vertices of the quadrilateral...the circle passes through them.
RESEARCHER	All right...I see what you mean. Do you think that this will always be the same for any quadrilateral?
ROXANNE	Yes.
RESEARCHER	So no matter how big it is, no matter how small it is, what shape it is, it will...
ROXANNE	Always be the same as before.
RESEARCHER	So if I make this a larger or a smaller quadrilateral, will it always work in the same way?
ROXANNE	Yes...it will always be the same.
RESEARCHER	When we say that it will always be the same what are we referring to?
ROXANNE	The lines will meet at the same point ... and the circle will go through these points here (pointing)
RESEARCHER	Can you check? Grab this point here and change the quadrilateral. Now what do you observe? Is this still a quadrilateral?
ROXANNE	Yes it is still a quadrilateral...but something is not working.
RESEARCHER	What is not working?
ROXANNE	They are not meeting.
RESEARCHER	What is not meeting?
ROXANNE	These perpendicular lines and the circle...
RESEARCHER	Yes go on.
ROXANNE	The circle is not touching here and here and here...
RESEARCHER	Maybe there is a problem with this quadrilateral...let us try another one.
ROXANNE	Yeah...(uncertain)
RESEARCHER	Drag this vertex like this. (after a while) Are the points concurrent now?
ROXANNE	Yes.
RESEARCHER	Let us try another one... drag this again. What is happening now?
ROXANNE	It's just like the other one.
RESEARCHER	What do you mean?
ROXANNE	They are not meeting again...and the circle...
RESEARCHER	What about the circle?
ROXANNE	These lines are not concurrent and the circle does not pass through these vertices
RESEARCHER	When do you think that the circle will pass through the vertices?
ROXANNE	When the perpendicular bisectors are concurrent.
RESEARCHER	Are you sure about that?
ROXANNE	Very.
RESEARCHER	Would you want me to try more examples of quadrilaterals?
ROXANNE	No...it is not necessary... I know what I'm saying.
RESEARCHER	So what can we say about the building of the reservoir if we had these four villages?
ROXANNE	You will have to build it just at the point where the perpendicular bisectors meet.

RESEARCHER	Let us now try to find the most suitable point if we had three villages. Where do you think that that point will be?
ROXANNE	In the middle because the triangle's all sides are equal.
RESEARCHER	But it does not necessarily mean that all sides are equal, you can have a scalene triangle.
ROXANNE	Yes that's why you have to measure it first.
RESEARCHER	Why don't you do that? ...(after a while)...can you see those measurements?
ROXANNE	Yes...it is a scalene triangle.
RESEARCHER	So where do you think the most suitable point will be?
ROXANNE	Construct the perpendicular bisectors of all the sides.
RESEARCHER	Do you think that this will work?
ROXANNE	I don't know but it worked for the quadrilateral.
RESEARCHER	Go ahead and construct the perpendicular bisectors ...(after a while)... what do you observe?
ROXANNE	They are concurrent!
RESEARCHER	Does that surprise you?
ROXANNE	No, because I thought that the lines will meet at one point.
RESEARCHER	How do you know that at this point (pointing to the point of concurrency) will be equidistant to all three villages?
ROXANNE	We can measure it.
RESEARCHER	Do you think that we should?
ROXANNE	I don't think that it is necessary.
RESEARCHER	But what if you are wrong?
ROXANNE	I don't think so...but maybe I should just measure it very quickly.
RESEARCHER	All right. (after a while) There you have the measurements. What do you observe?
ROXANNE	As I said, the measurements are the same.
RESEARCHER	What do you think will happen if we changed the triangle? Do you think that these lines will still be concurrent?
ROXANNE	No...in the quadrilateral it changed...only if a circle can pass through its vertices.
RESEARCHER	Would you like to try to construct a circle around these vertices?
ROXANNE	Yes...a circle should go through here ...here and here.
RESEARCHER	(after a while) now what do you observe?
ROXANNE	The circle passes through these vertices.
RESEARCHER	So if we made this triangle bigger or smaller, or just changed the shape, what do you think would happen?
ROXANNE	A circle wouldn't pass through these points...vertices and the perpendicular lines will not be concurrent.
RESEARCHER	Are you sure about that?
ROXANNE	I already told you sir...it was like that for the quadrilateral.
RESEARCHER	Should we check by changing the size or shape of the triangle?
ROXANNE	Yeah...
RESEARCHER	Drag one vertex and observe what happens ...what do you observe?
ROXANNE	(silence)...it still is the same...
RESEARCHER	What do you mean by it still is the same?
ROXANNE	The lines are still concurrent... but let me move it this way and let me see.
RESEARCHER	All right....
ROXANNE	... but...but... it still is the same...even the circle is passing through the

vetexes.

RESEARCHER The word is vertices.

ROXANNE Vertices...but this is different...it sort of is strange...

RESEARCHER Strange? Why?

ROXANNE It is not what I expected...it's like you think it's going to be like the quadrilateral but suddenly it's different.

RESEARCHER You look surprised.

ROXANNE I am...I was so sure that the lines will not be concurrent...

RESEARCHER Which lines are you talking about?

ROXANNE The bisectors.

RESEARCHER Are they just bisectors?

ROXANNE No, they are perpendicular as well.

RESEARCHER Are you convinced that this will be true for all triangles?

ROXANNE Yes...I saw it when I dragged that point around.

RESEARCHER How many percent convinced are you?

ROXANNE 99%....

RESEARCHER 99%? You are still unsure because you did not say 100%. Why?

ROXANNE No...sir I quite convinced but what if there is one triangle somewhere for which it won't work?

RESEARCHER Do you want to move the triangles around some more?

ROXANNE No. That is not necessary.

RESEARCHER So how can you become 100% convinced that it will always work?

ROXANNE I don't know...I'm quite sure but just in case...I thought that it will be the same as for the quadrilaterals...but I was wrong.

RESEARCHER So are you quite convinced but you are afraid that you might make a mistake?

ROXANNE Yes.

RESEARCHER Would you want to know why this is always true?

ROXANNE Yes... Ok, maybe it will explain why it was different.

RESEARCHER Can I do that at the end? I'd like to do something else first.

ROXANNE Yes.

So how would you explain the building of the reservoir if you had three villages?

You can still use the perpendicular bisectors. It will work for all the triangles.

RESEARCHER Okay, let us change our question. Suppose we have four villages, they are represented by A, B, C and D. Let us suppose that there is a reservoir at P. We want to find the best position for P so that the sum of all the pipes that we have is a minimum. Why do you think we want it to be a minimum?

ROXANNE Because of the costs involved.

RESEARCHER Good. Where do you think we should locate point P?

ROXANNE Construct the perpendicular bisectors.

RESEARCHER Do you think that this is the best way to do it?

ROXANNE That is the best way!

RESEARCHER Ok...click on that button for the perpendicular bisectors

ROXANNE ...(after a while)... But they are not concurrent.

RESEARCHER So what does that mean?

ROXANNE I don't know where the point is going to be...maybe...maybe we should construct a point...then measure the distances and add the distances up and then find the point...

RESEARCHER What do you mean and then find the point?

ROXANNE	You can drag the point and look for the smallest distances.
RESEARCHER	Do you mean look for the smallest sum?
ROXANNE	Yes.
RESEARCHER	Construct the point quickly and measure the distances... (after a while) now drag the point around and find the best position..
ROXANNE	I think it is about there...the sum is 15...that is the smallest.
RESEARCHER	Look at these broken lines. How would you describe them?
ROXANNE	They look like the diagonals.
RESEARCHER	Do they?
ROXANNE	Yes.
RESEARCHER	Would you like to construct the diagonals?
ROXANNE	Yes.
RESEARCHER	All right...click on that button for the diagonals. (after a while) What do you observe?
ROXANNE	The lines are the same.
RESEARCHER	What do you mean by the lines are the same?
ROXANNE	The lines we had are the same as the diagonals.
RESEARCHER	So which would be the most accurate point?
ROXANNE	Where the diagonals bisect each other.
RESEARCHER	Bisect? Will the diagonals bisect each other?
ROXANNE	No where they meet.
RESEARCHER	In other words where the diagonals intersect.
ROXANNE	Yes.
RESEARCHER	Do you think that this will always be true for all quadrilaterals?
ROXANNE	I don't know...maybe I should move it around first.
RESEARCHER	Don't you want to try to guess first?
ROXANNE	No. I want to see what happens first.
RESEARCHER	All right do that...(after a while)
ROXANNE	Yes it works for all.
RESEARCHER	Do you think that you have seen enough already?
ROXANNE	Yes.
RESEARCHER	Ok...let us change this quadrilateral ... (after a while)
ROXANNE	That is a funny quadrilateral... (laughs)
RESEARCHER	This is called a concave quadrilateral. So where do you think the point will be now?
ROXANNE	...(after a while)...there...right there.
RESEARCHER	Where is that...describe the point?
ROXANNE	Where it goes inwards...
RESEARCHER	At the point where the quadrilateral is concave... Do you think it might work for other concave quadrilaterals?
ROXANNE	Yes.
RESEARCHER	Do you want to construct a few more quadrilaterals?
ROXANNE	No.
RESEARCHER	Are you sure?
ROXANNE	Yes.
RESEARCHER	Should we try just one more?
ROXANNE	It is not necessary, but let us try it.
RESEARCHER	(after a while) what do you observe?
ROXANNE	...yes it works.

RESEARCHER	What works?
ROXANNE	It is still at the point where the quadrilateral is concave.
RESEARCHER	Are you convinced?
ROXANNE	Yes sir.
RESEARCHER	Should we try some more?
ROXANNE	No.
RESEARCHER	I did say that we would return to an explanation for why perpendicular bisectors are concurrent for all triangles. Would you still want to know why?
ROXANNE	Yes.
RESEARCHER	Read through this page and we will work together when you are ready.
ROXANNE	(after a while) Ii looks complicated.
RESEARCHER	We'll work through it together. We want to prove that the perpendicular bisectors of any triangle are concurrent. What are we given?
ROXANNE	OF, OE and OD are perpendicular.
RESEARCHER	Is that all that we are given?
ROXANNE	Yes…no…no…also AF = AB, AE = CE and BD = DC.
RESEARCHER	Are we actually given that BD = CD? Read 1.3 and 1.4.
RESEARCHER	OD is only perpendicular but if there is concurrency….?
ROXANNE	That is exactly what we want to prove. For us to prove that O is the point of concurrency of all the perpendicular bisectors what should we prove?
RESEARCHER	Er…er….
ROXANNE	Are the lines meeting at O already concurrent according to our diagram?
RESEARCHER	Yes.
ROXANNE	Are OF, OE and OD perpendicular bisectors?
RESEARCHER	Two of them are but OD is only perpendicular.
ROXANNE	So what do you think we should prove?
RESEARCHER	That BD is equal to DC.
RESEARCHER	Yes that is correct. Now follow the instructions and complete the worksheet.

INTERVIEW WITH RESEARCHER AND SCHOFIELD

RESEARCHER	Schofield do you understand the question?
SCHOFIELD	Yes sir.
RESEARCHER	By looking at the question and the diagram on that page, where do you think that the water reservoir should be built?
SCHOFIELD	Somewhere in the centre.
RESEARCHER	Would you be able to justify that answer? Can you tell me why you think that that is the point?
SCHOFIELD	I ... I don't know ...
RESEARCHER	Do you think that there is a way of showing why your answer is correct?
SCHOFIELD	I don't think so.
RESEARCHER	Don't you want to try?
SCHOFIELD	We didn't do this in class before ... I can't do it!
RESEARCHER	Let us see if working with the computer will help. This is the same diagram you have on the page. Can you point out again the most suitable position of the reservoir by just looking at the diagram?
SCHOFIELD	Somewhere in the centre.
RESEARCHER	Why do you think that? Or is there anyway that you think that we will be able to check whether that is a correct point or not.
SCHOFIELD	It is obvious that it must be in the middle. We can show this by measuring the distances from the village to the point that is chosen.
RESEARCHER	Do just that then....(after a while) ... what do you observe?
SCHOFIELD	(silence)... the distances are different...
RESEARCHER	What should we do then to get the distances to be equal?
SCHOFIELD	Move this point around?
RESEARCHER	Go ahead and do that.
SCHOFIELD	(after a long while)... this must be the point.
RESEARCHER	Is that point in the middle?
SCHOFIELD	No sir...
RESEARCHER	Do you think that this is a very good mathematical way of getting the most suitable point?
SCHOFIELD	No...
RESEARCHER	Do you know of any mathematical way of getting an accurate answer?
SCHOFIELD	No...we never did this before.
RESEARCHER	Before we go on let us suppose the point that you chose is correct. In real life you might not be able to build the purification and the water works there. What are some of the possible problems you may encounter?
SCHOFIELD	There might be buildings like a school
RESEARCHER	Remember that these are remote villages.
SCHOFIELD	... there may be a river close to the point... if there are no roads then it is going to cost more money to first build roads...the people in the villages may say that they don't want the reservoir at that point...what about very hard rocks...
RESEARCHER	What do you think the purpose of asking these questions is?
SCHOFIELD	Maths might be one thing but reality is another ... sometimes we can't use maths on its own.
RESEARCHER	Let us try to simplify the problem. Let us suppose that there are only two villages I and J. How would I get the point that is equidistant from I and J?

SCHOFIELD	Find the point in the centre here (pointing)
RESEARCHER	What is that point called?
SCHOFIELD	Midpoint.
RESEARCHER	Click on that button to construct the midpoint then. How can you be sure that this point is the midpoint?
SCHOFIELD	Just measure the distances.
RESEARCHER	Measure that quickly...(after a while)...what do you observe?
SCHOFIELD	The distances are the same.
RESEARCHER	As we said earlier that there may be a problem with the site that you chose? What do you think we should do then?
SCHOFIELD	Maybe we should get another point next to that, alongside it.
RESEARCHER	That means on this line here (pointing to segment IJ)?
SCHOFIELD	No...out of the line...in front here.
RESEARCHER	Construct that point... how do we know that the point is equidistant from I and J?
SCHOFIELD	Measure it.
RESEARCHER	(after measuring) ... Is this point equidistant to J and I?
SCHOFIELD	No but I can move it about until they become equal.
RESEARCHER	So we drag this point until the distances are equal. Let us suppose that this point that we chose now is not suitable what do we do?
SCHOFIELD	Construct another point.
RESEARCHER	Where should that point be?
SCHOFIELD	Anywhere here, just like we did before.
RESEARCHER	Do that then... (after a while) ... are these distances the same?
SCHOFIELD	No but we move it about like this.
RESEARCHER	How many points do you think we can get like this?
SCHOFIELD	A lot.
RESEARCHER	Do you really think that you can get a lot?
SCHOFIELD	Yes.
RESEARCHER	What do you observe about these points that we already have here?
SCHOFIELD	The points are ...lying on a straight line....
RESEARCHER	Do you think that they are? How could we check?
SCHOFIELD	Yes they look like they are. We can draw a line through here.
RESEARCHER	Draw a straight line there and check ... we do have a straight line passing through those points. What else can you say about this line?
SCHOFIELD	The line looks like it is perpendicular to I and J.
RESEARCHER	Measure the angle and check. What do you notice?
SCHOFIELD	That angle is equal to 90 degrees.
RESEARCHER	What does this mean to us?
SCHOFIELD	We can get many more points from this line... that means if we can get a line perpendicular to I and J and then we would be able to construct a reservoir anywhere on that line.
RESEARCHER	Must the line just be perpendicular?
SCHOFIELD	No it must pass through the midpoint here as well.
RESEARCHER	What would this line be called?
SCHOFIELD	A perpendicular line.
RESEARCHER	Just a perpendicular line?
SCHOFIELD	A bisector as well.

RESEARCHER So what would you call that line?
SCHOFIELD A perpendicular bisector.
RESEARCHER So what can you say about the building of the reservoir if you had two villages?
SCHOFIELD You can build the reservoir anywhere on this perpendicular bisector…but I already said that.
RESEARCHER Let's get back to the original problem. Look at this diagram…these represents the four villages. How can we get the most suitable position?
SCHOFIELD We can use the same method.
RESEARCHER And what is that?
SCHOFIELD Construct perpendicular lines.
RESEARCHER Do we just construct a perpendicular line?
SCHOFIELD No, we construct all perpendicular bisectors.
RESEARCHER Click on that button to construct all the perpendicular bisectors and see what happens. (after a while) What do you notice now?
SCHOFIELD They meet at a point.
RESEARCHER Does this surprise you?
SCHOFIELD Not really… they had to meet somewhere…this was the best point.
RESEARCHER How do you know that, that point is equidistant from V1, V2, V3, and V4.
SCHOFIELD We can measure it
RESEARCHER Should we measure it?
SCHOFIELD Yes.
RESEARCHER (after a while) what do you observe?
SCHOFIELD The distances are the same – it's 3.8.
RESEARCHER What do we call this point at which they meet?
SCHOFIELD Point of intersection.
RESEARCHER Yes they do intersect at a common point. But this point has a special name. Do you know it?
SCHOFIELD No.
RESEARCHER We call this point the point of concurrency.
SCHOFIELD Concurrency?
RESEARCHER Yes. Whilst we are at this point, I'd like you to construct a circle using this point of concurrency as the centre. (after a while)…no up to this point. What do you observe?
SCHOFIELD The 4 villages are around the circle
RESEARCHER What do you mean by around the circle?
SCHOFIELD V1, V2, V3 and V4 are on this part of the circle.
RESEARCHER You mean that the 4 villages lie on the circumference of the circle?
SCHOFIELD Yes.
RESEARCHER Do you think that this is going to be true for any quadrilateral?
SCHOFIELD Yes
RESEARCHER Do you think that if we made this quadrilateral a larger one or a smaller one the perpendicular bisectors will still be concurrent?
SCHOFIELD Yes.
RESEARCHER Change this quadrilateral by dragging it. … yes like that… Is it still a quadrilateral and are the perpendicular bisectors meeting?
SCHOFIELD It is still a quadrilateral but… but …
RESEARCHER What is going on?
SCHOFIELD I don't know … the perpendicular bisectors are not meeting.
RESEARCHER Does that surprise you?

SCHOFIELD	Yes, because I thought that it will always meet.
RESEARCHER	What did you notice about the circle?
SCHOFIELD	The circle is not on the quadrilateral.
RESEARCHER	What do you mean by that?
SCHOFIELD	These corner points are not touching the circumference of the circle.
RESEARCHER	When are the vertices on the circumference of the circle?
SCHOFIELD	Only when these lines meet.
RESEARCHER	Which lines are you talking about?
SCHOFIELD	The perpendicular bisectors.
RESEARCHER	How can I say this is a nice way?
SCHOFIELD	The corner points of the quadrilateral...the vertices will be on the circle...if ... if the perpendicular bisectors meet... if they are concurrent.
RESEARCHER	Are you convinced that this is true?
SCHOFIELD	Yes I am.
RESEARCHER	Should we try further examples?
SCHOFIELD	You can but the answer will always be the same.
RESEARCHER	In terms of building the reservoir what must be done?
SCHOFIELD	You just draw lines from one village to another, construct their perpendicular bisectors and where they meet will be the place for building the reservoir.
RESEARCHER	Let us change this question. Suppose we only had three villages. How would you find the position that will be equidistant from the three villages?
SCHOFIELD	Construct the perpendicular bisectors of the sides.
RESEARCHER	Do you think that they will be concurrent?
SCHOFIELD	I don't know...they might be concurrent.
RESEARCHER	Should we construct the perpendicular bisectors?
SCHOFIELD	Yes.
RESEARCHER	Do that then ... (after a while) What do you observe?
SCHOFIELD	The lines are concurrent.
RESEARCHER	But how do we know that the point of concurrency is equidistant from the three villages?
SCHOFIELD	They will be...but I can measure it.
RESEARCHER	(after a while) can you see these measurements?
SCHOFIELD	Yes...they are all equal.
RESEARCHER	Does this surprise you?
SCHOFIELD	No...I knew they will meet.
RESEARCHER	Do you think that they will meet for all triangles?
SCHOFIELD	No.
RESEARCHER	Why?
SCHOFIELD	It must be the same for the four-sided figures.
RESEARCHER	What do you mean?
SCHOFIELD	For the quadrilateral it only worked for some...when the circle touched the four corners...it might be the same for the triangle.
RESEARCHER	Would you like to check what would happen if you changed the triangle?
SCHOFIELD	Yes.
RESEARCHER	(after a while) What do you observe when you move this point around?
SCHOFIELD	The lines are always concurrent!
RESEARCHER	Do you think that this will always be the case?
SCHOFIELD	No...I think I should move it a bit more.
RESEARCHER	Is there any particular way that you want to move it?

SCHOFIELD	Yes...move this corner about (pointing to the lower left hand corner)
RESEARCHER	Okay ...(after a while) so, what is happening now?
SCHOFIELD	It still is the same...the lines are always concurrency.
RESEARCHER	Concurrent.
SCHOFIELD	Yes.
RESEARCHER	Are you satisfied that the perpendicular bisectors are always concurrent?
SCHOFIELD	Yes...but I never would have guessed that!
RESEARCHER	Why?
SCHOFIELD	I thought that all of the figures will be the same.
RESEARCHER	But they are.
SCHOFIELD	No...no...I mean the different figures...the quadrilateral and the triangle.
RESEARCHER	Are you convinced that the perpendicular bisectors are always concurrent?
SCHOFIELD	Yes.
RESEARCHER	How many percent convinced are you?
SCHOFIELD	100%
RESEARCHER	You have no doubt in your mind?
SCHOFIELD	No doubt...I saw it myself!
RESEARCHER	Would you like to know why this is always the case?
SCHOFIELD	What do you mean sir?
RESEARCHER	Do you want to know why the perpendicular bisectors are always concurrent?
SCHOFIELD	I don't know...mmm....yes...maybe it will be interesting.
RESEARCHER	You think that this would be interesting?
SCHOFIELD	Yes...I was surprised at the results.
RESEARCHER	Will you mind if we did that at the end of this session?
SCHOFIELD	Not a problem.
RESEARCHER	So if you had three villages where would you build the reservoir?
SCHOFIELD	Still at the point of concurrency of the perpendicular bisectors but for three villages we will always have a point of concurrency. It's different for the quadrilateral.
RESEARCHER	Let us change our question. Suppose we have four villages, A, B, C, and D represents them. Let us suppose that there is a reservoir at P. We want to find the best position for P so that the sum of all the pipes that we have is a minimum. Why do you think we want it to be a minimum?
SCHOFIELD	Save costs...
RESEARCHER	That is correct. Where do you think that the best position for P would be?
SCHOFIELD	I can't tell just like that, but we can construct the perpendicular bisectors.
RESEARCHER	Why do you think it is necessary to construct the perpendicular bisectors?
SCHOFIELD	Because where they meet will give us the best position.
RESEARCHER	So are you sure that we should construct the perpendicular bisectors?
SCHOFIELD	Yes.
RESEARCHER	Do that then (after a while)...What do you observe?
SCHOFIELD	They...they don't meet.
RESEARCHER	Why do you think that they don't meet?
SCHOFIELD	Because a circle will not pass through these points here.
RESEARCHER	You are pointing to the vertices of the quadrilateral. Do you think that we can find the point?
SCHOFIELD	Yes ... if we construct a point and then measure the distances and then drag that point. Then you can add the distances and look for the minimum total.
RESEARCHER	You mean the minimum sum?

SCHOFIELD	Yes.
RESEARCHER	Ok. Construct that point. (after a while)... Can you see this sum here?
SCHOFIELD	Yes sir.
RESEARCHER	So where should we drag the point?
SCHOFIELD	In all directions.
RESEARCHER	Go on and stop when you are ready.
SCHOFIELD	(after a while) ...the minimum is 15.
RESEARCHER	Are you satisfied that this should be the point?
SCHOFIELD	Yes.
RESEARCHER	Do these lines remind you of anything?
SCHOFIELD	Yes, they look like they go to the opposite sides of this figure.
RESEARCHER	Do you know what these lines are called?
SCHOFIELD	Hmm...hmmm...I forgot!
RESEARCHER	They are called diagonals. Would you like for us to construct the diagonals ... just to check?
SCHOFIELD	Yes.
RESEARCHER	Click here...(after a while)... now what do you observe?
SCHOFIELD	Well I was very close.
RESEARCHER	Should we measure the distances from each village to the point of intersection of the diagonals?
SCHOFIELD	No we can see that it will be correct.
RESEARCHER	But don't you want to be sure?
SCHOFIELD	I am sure....ok, never mind measure it.
RESEARCHER	(after a while) So these measurements confirm what you have said.
SCHOFIELD	I told you it was not necessary.
RESEARCHER	Do you think that you can use this method for all quadrilaterals?
SCHOFIELD	Yes.
RESEARCHER	So if I drew a large quadrilateral, how would you find a point such that the sum of those distances will be a minimum?
SCHOFIELD	Construct the diagonals. Where they meet will be the point.
RESEARCHER	Should we do that then?
SCHOFIELD	The answer is obvious.
RESEARCHER	Is there no need to try then?
SCHOFIELD	No!
RESEARCHER	Do you think that it would be true for any quadrilateral then?
SCHOFIELD	Yes. That's what I said.
RESEARCHER	Ok...let me change this quadrilateral ... (after a while)
SCHOFIELD	Is that a quadrilateral?
RESEARCHER	This is called a concave quadrilateral. So where do you think that the point will be now?
SCHOFIELD	I think we need to move point P around ... about there ... no there (pointing).
RESEARCHER	Where is that...describe the point?
SCHOFIELD	Where it is folded...
RESEARCHER	At the point where the quadrilateral is concave... Do you think it might work for other concave quadrilaterals?
SCHOFIELD	Yes.
RESEARCHER	Must I construct a few more quadrilaterals?
SCHOFIELD	No.
RESEARCHER	Are you sure?

SCHOFIELD	Yes.
RESEARCHER	Maybe just one more?
SCHOFIELD	No...that is not necessary!
RESEARCHER	Are you convinced?
SCHOFIELD	Yes I am
RESEARCHER	I did say that we would return to an explanation for why perpendicular bisectors are concurrent for all triangles. Would you still want to know why?
SCHOFIELD	Yes I think so.
RESEARCHER	Read through this page and we will work together when you are ready.
SCHOFIELD	(after a while) Must I write on this page?
RESEARCHER	No it is ok...I'll tell you when to write. So we want to prove that the perpendicular bisectors of any triangle are concurrent. What are we given?
SCHOFIELD	All the perpendicular bisectors.
RESEARCHER	Are we given all the perpendicular bisectors?
SCHOFIELD	Yes.
RESEARCHER	Is OD a bisector
SCHOFIELD	No.
RESEARCHER	In the diagram we will construct perpendicular bisectors OF and OE. We will join AO, BO and CO. We will also construct OD perpendicular to BC. What is the difference between OF, OE and OD?
SCHOFIELD	I don't know..
RESEARCHER	Read 1.3 and 1.4 again. What does 1.3 state?
SCHOFIELD	FO and EO are perpendicular bisectors.
RESEARCHER	What does 1.4 state?
SCHOFIELD	OD is perpendicular to BC.
RESEARCHER	So in order to prove that O is the point of concurrency of all the perpendicular bisectors what should we prove?
SCHOFIELD	Er...er....
RESEARCHER	Are the lines meeting at O already according to our diagram?
SCHOFIELD	Yes.
RESEARCHER	Are OF, OE and OD perpendicular bisectors?
SCHOFIELD	OF and OE are but not OD...OD is perpendicular but not a bisector.
RESEARCHER	So what do you think we should prove?
SCHOFIELD	OD bisects BC.
RESEARCHER	That is correct. Now follow the instructions and complete the worksheet

INTERVIEW WITH DESIGAN AND RESEARCHER

RESEARCHER	Let us change this now. Consider what would happen if we had only 3 villages. How do you think then we would be able to find the best position? Do you want to attempt an answer first?
DESIGAN	Yes, let me construct the perpendicular bisectors.
RESEARCHER	Well do you think that would work
DESIGAN	It must work ... it worked for the quadrilateral.... No... no...it will work for some triangles.
RESEARCHER	Really? Which triangles would it work for?
DESIGAN	The ones for which a circle will pass through its points.
RESEARCHER	Do you want to construct the perpendicular bisectors?
DESIGAN	Yes... (after clicking on the button) ... they are concurrent!
RESEARCHER	What is concurrent?
DESIGAN	All the perpendicular bisectors.
RESEARCHER	What does that mean?
DESIGAN	A circle must pass through this point here and this point here and here. (pointing to the vertices).
RESEARCHER	Do you think that you were just lucky?
DESIGAN	Yes I think so.
RESEARCHER	Do you think that the perpendicular bisectors would be concurrent for all triangles?
DESIGAN	No!
RESEARCHER	So are you saying that this point at which the perpendicular bisectors are meeting will be the best position?
DESIGAN	Yes
RESEARCHER	How can we check?
DESIGAN	Measure the distances.
RESEARCHER	Go ahead and measure it (*after it was measured*) What do you observe.
DESIGAN	I expected them to be equal because these (pointing to the perpendicular bisectors) lines are concurrent.
RESEARCHER	Let's just check whether the circle will pass through the vertices if we use that as a centre (*pointing to the point of concurrency*).
DESIGAN	Okay (constructing).
RESEARCHER	What do you observe?
DESIGAN	This is very obvious...the circle must pass through.
RESEARCHER	The circle will pass through the vertices, but will you be able to find a best position for **any** triangle? Do you think this will be possible for any triangle?
DESIGAN	Well ... only if a circle passes through the vertices.
RESEARCHER	Okay so the best way to test that is to move one point of the triangle, which ever way, and see whether the perpendicular bisectors meet.
DESIGAN	(Silence for a while) This is different! (uncertain)
RESEARCHER	What is different?
DESIGAN	The triangles are different from the quadrilaterals.
RESEARCHER	Why are the different?
DESIGAN	It ... looks like the perpendicular bisectors are concurrent for all triangles.
RESEARCHER	Does this surprise you?
DESIGAN	Yes ... It is different from the quadrilaterals. It means that ...you can't say that

	the triangles and quadrilaterals have a similar property.
RESEARCHER	What property are you talking about?
DESIGAN	Only in quadrilaterals were the vertices fall on a circle ... will the perpendicular bisectors meet....but for all triangles they meet (referring to the perpendicular bisectors).
RESEARCHER	So it means the perpendicular bisectors of any triangle will meet?
DESIGAN	Yes...
RESEARCHER	What does it mean in terms of the villages?
DESIGAN	You can join the three villages and then find the perpendicular bisectors. Where they meet is the important point for us to use.
RESEARCHER	How sure are you of this result? Are you convinced about this?
DESIGAN	I am quite convinced.
RESEARCHER	How many percent convinced are you?
DESIGAN	I'm convinced sir...100%.
RESEARCHER	How surprising is this result to you?
DESIGAN	I'm still surprised.
RESEARCHER	Would you like to know why the perpendicular bisectors are always concurrent?
DESIGAN	If you tell me why.
RESEARCHER	Let us work together. Look at this triangle on the screen again. Construct the perpendicular bisector of any side.
DESIGAN	Can I do it for AB?
RESEARCHER	Yes. (after the construction) Desigan, what can you tell me about all the points on this perpendicular bisector?
DESIGAN	It is equidistant from A and B.
RESEARCHER	What is equidistant?
DESIGAN	All the points on this line (pointing to the perpendicular bisector).
RESEARCHER	What does that really mean to you?
DESIGAN	If you measure the distance from any point on this line to this A and B, the distance will be the same.
RESEARCHER	Now construct any other perpendicular bisector.
DESIGAN	(constructing)
RESEARCHER	What can you tell about the points on this line now?
DESIGAN	All the points are the same distance away from B and C.
RESEARCHER	Now look at this point of intersection. What can you say about this point in particular?
DESIGAN	Eh ... eh...
RESEARCHER	Think carefully about the point.
DESIGAN	That point there is the same distance away from A and B and, B and C.
RESEARCHER	A and B and, B and C?
DESIGAN	Yes, it is the same distance away from A, B and C.
RESEARCHER	Are you sure?
DESIGAN	It lies on this line so it must be equidistant from A and B and it lies on that line so it must be equidistant from A and C.
RESEARCHER	If it lies on that line would it be equidistant from A and C?
DESIGAN	No B and C.
RESEARCHER	So are you sure that this point of intersection is the same distance away from A, B and C?
DESIGAN	Yes.
RESEARCHER	This you have to think very carefully about. What can you say about the

	perpendicular bisector of AC?
DESIGAN	All the points will be equidistant from A and C.
RESEARCHER	Yes, that is correct. But look at the other perpendicular bisectors.
DESIGAN	Oh yes, it must pass through the point where these two lines meet (pointing to the perpendicular bisectors).
RESEARCHER	Really?
DESIGAN	Yes because if all the points on this perpendicular bisector of AC are the same distances away…then … then this point of intersection is also the same distance away .. then…
RESEARCHER	Yes?
DESIGAN	Then the line must pass through the point of intersection.
RESEARCHER	Do you want to see whether that is true?
DESIGAN	Yes.
RESEARCHER	Construct the perpendicular bisector of AC then.
DESIGAN	(after constructing) This is so easy.
RESEARCHER	Was it really that easy?
DESIGAN	I didn't take so long to get it right!

INTERVIEW WITH VISCHALAN AND RESEARCHER

RESEARCHER	Let us change this now. Consider what would happen if we had only three villages. How do you think then we would be able to find the best position?
VISCHALAN	By constructing the perpendicular bisectors.
RESEARCHER	Do you think that would work?
VISCHALAN	It should work but I'm worried about if a circle does not pass through these points (pointing to vertices).
RESEARCHER	Do you think that we may have a problem with that?
VISCHALAN	Yes...just like the quadrilaterals.
RESEARCHER	Why don't you construct the perpendicular bisectors by clicking on the button there?
VISCHALAN	(after clicking on the button) They meet.
RESEARCHER	What do you mean?
VISCHALAN	All the perpendicular bisectors are concurrency.
RESEARCHER	Concurrency?
VISCHALAN	Yes, the perpendicular bisectors are meeting at a same point.
RESEARCHER	You mean concurrent?
VISCHALAN	Yes.
RESEARCHER	Do you think that a circle will pass through the vertices?
VISCHALAN	A circle must pass through them.
RESEARCHER	Why don't you construct the circle?
VISCHALAN	(after construction) I told you it will!
RESEARCHER	Do you think that the perpendicular bisectors would be concurrent for all triangles?
VISCHALAN	I don't think so.
RESEARCHER	You don't think so or are you sure?
VISCHALAN	No it won't ... there will be a similar problem...like the quadrilateral.
RESEARCHER	The best way to test that is to move one point of the triangle and see whether the perpendicular bisectors meet.
VISCHALAN	(Silence) This is strange! (uncertain)
RESEARCHER	Why is it strange?
VISCHALAN	I don't know ... the bisectors are always meeting.
RESEARCHER	Do you mean the perpendicular bisectors? (Vischalan nods)
VISCHALAN	I didn't expect it.
RESEARCHER	Does this surprise you?
VISCHALAN	Yes ... it does. Can I move this point here? (pointing to a vertex)
RESEARCHER	Why do you want to do that?
VISCHALAN	This is really surprising... I want to be really sure.
RESEARCHER	Please do that. (after a while) Are you satisfied?
VISCHALAN	(silently nodding)
RESEARCHER	Why do you find this so surprising?
VISCHALAN	Because it is different for the quadrilateral.
RESEARCHER	So does it mean that the perpendicular bisectors of any triangle will meet?
VISCHALAN	Yes.
RESEARCHER	What does it mean in terms of the villages?
VISCHALAN	If the villages are situated like this triangle then all you have to do is join the villages, find the midpoints between then and construct the perpendicular

	bisectors.
RESEARCHER	Are you convinced about this result?
VISCHALAN	Yes... very.
RESEARCHER	How many percent convinced are you?
VISCHALAN	100%.
RESEARCHER	You mean you have no doubt at all?
VISCHALAN	No I'm convinced.
RESEARCHER	Would you like to know why the perpendicular bisectors are always concurrent?
VISCHALAN	Yes I would.
RESEARCHER	Why?
VISCHALAN	I don't know ... I guess it will explain why it is so different from the quadrilateral.
RESEARCHER	Look at this triangle on the screen. Construct the perpendicular bisector of side AC. (after the construction) what can you tell me about all the points on this perpendicular bisector?
VISCHALAN	They are the same distance away from A and B.
RESEARCHER	What is the term used to describe same distance away?
VISCHALAN	Equidistance.
RESEARCHER	So what are you saying about all points on this line?
VISCHALAN	All the points on this line (pointing to the perpendicular bisector) are equidistance from A and C..
RESEARCHER	Equidistant not equidistance from A and C. What does that really mean to you?
VISCHALAN	If you calculate the distance from any point to A and then to C the distance will be exactly the same.
RESEARCHER	Now construct perpendicular bisector of AB.
VISCHALAN	(constructing)
RESEARCHER	What can you tell about the points on this line now?
VISCHALAN	All the points are equidistant from B and A.
RESEARCHER	Now look at this point of intersection. What can you say about this point in particular?
VISCHALAN	That is the point of concurrency of these two perpendicular bisectors.
	Yes, that is true, but think carefully about the point. What is special about it?
VISCHALAN	It is equidistant from A, B and C.
RESEARCHER	Really? Why?
VISCHALAN	It is equidistant from A and C and then it is equidistant from A and B then it must be equidistant from A, B and C.
RESEARCHER	So are you sure that this point of intersection is the same distance away from A, B and C?
VISCHALAN	Yes.
RESEARCHER	What can you say about the perpendicular bisector of BC?
VISCHALAN	(Silence)
	Think about it... What can you say about the perpendicular bisector of BC?
VISCHALAN	I think ... it will pass through this point of intersection here.
RESEARCHER	Really? Do you really think so?
VISCHALAN	Yes, I'm quite sure.
RESEARCHER	Why?
VISCHALAN	Well if I construct the perpendicular bisectors, all the points on that line must be equidistant from B and C.
RESEARCHER	Yes. Go on.

VISCHALAN	What do you mean? You just said that all the points on that line must be equidistant from B and C. So what does that mean?
VISCHALAN	That point of intersection has to pass through the point of intersection ... it has to because that point is also equidistant from B and C.
RESEARCHER	Do you want to see whether that is true?
VISCHALAN	Yes.
RESEARCHER	Construct the perpendicular bisector of BC then.
VISCHALAN	(after constructing) I was right again.
RESEARCHER	Yes you were.