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Formalization of Quasilattices

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Summary. The main aim of this article is to introduce formally one of the generalizations of lattices, namely quasilattices, which can be obtained from the axiomatization of the former class by certain weakening of ordinary absorption laws. We show propositions QLT-1 to QLT-7 from [15], presenting also some short variants of corresponding axiom systems. Some of the results were proven in the Mizar [1], [2] system with the help of Prover9 [14] proof assistant.

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0. Introduction

For years, lattice theory was quite dynamically developed area of mathematics represented formally in the Mizar Mathematical Library. The first Mizar article in this topic was [18], and the monographs of two authors were stimulating source for formalization efforts: Birkhoff [3] (especially at the very beginning), and then Grätzer [12], [13]. The chosen approach was just the algebraic one, with two operation of binary supremum and infimum, and the induced ordering relation as a generated Mizar predicate.

Initially, the formalization efforts within lattice theory were not very systematic, but during the project of translating "Compendium of Continuous Lattices" [5] into Mizar formalism with a number of people involved, a lot of work was done to provide the alternative approach for lattices, with relational structures as the starting point (as it was claimed in [4]).

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The series of Mizar articles with MML identifiers beginning with `YELLOW` (with numerals), e.g. [7] was written to explore this specific field in a more detailed way, but the structures behind both approaches are different (although from the informal viewpoint the difference is meaningless [10]). Still however, the correspondence between relational structures and lattices in the form of the Mizar structure `LattRelStr` with binary operations and the underlying ordering relation available as parallel selectors in the merged structure was studied [8]. An overview of the mechanization of lattice theory in the repository of Mizar texts can be found in [6]. Most of described efforts were done more or less manually.

Our work can be seen as a step towards a Mizar support for [15] or [16], where original proof objects by Otter/Prover9 were used. Some preliminary works in this direction were already done in [9] by present authors. We use the interface `ott2miz` [17] which allows for the automated translation of proofs; these automatically generated proofs are usually quite lenghty, even after native enhancements done by internal Mizar software for library revisions.

In the present development, we deal with the parts of Chap. 6 "Lattice-like algebras" of [15], pp. 111–135, devoted to quasilattices.

The class of quasilattices (QLT) can be characterized from the standard set of axioms for lattices (with idempotence for the join and meet operations included), where absorption laws are replaced by the pair of link laws (called `QLT1` and `QLT2` in the Mizar source – compare Def. 1 and Def. 2). Def. 8 and Def. 9 provide standard examples of structures which are quasilatices, but not necessarily lattices (absorption laws do not hold). In the latter one, the lattice operations are given by

$\sqcup$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$\sqcap$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

Then we prove, using Mizar formalism, the new form of distributivity for QLT, that the standard distributivity implies its dual, and self-dual, a bit longer, form of distributivity (QLT-1, QLT-2, QLT-3). Later we characterize Bowden's inequality (which forces quasilattices, and hence lattices, to be distributive – QLT-4) and some modularity conditions (QLT-5 and QLT-6) – both in the form of the equations (taking into account automatic treatment of the equality predicate in Mizar [11] and the design of Prover9 this is more feasible), and in the more common (at least from informal point of view) form of implication with inequality. The final section shows that the meet operation need not be unique in QLT (although in the class of lattices, starting with the same join operation, the other operation is uniquely defined).

1. PRELIMINARIES

From now on L denotes a non empty lattice structure and v_3, v_{101}, v_{100}, v_{102}, v_{103}, v_2, v_1, v_0 denote elements of L.

Let L be a non empty lattice structure. We say that L satisfies QLT1 if and only if

(Def. 1) for every elements v_0, v_2, v_1 of L, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$.

We say that L satisfies QLT2 if and only if

(Def. 2) for every elements v_0, v_2, v_1 of L, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$.

We say that L is QLT-distributive if and only if

(Def. 3) for every elements v_1, v_2, v_0 of L, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcap v_2)) = v_0 \sqcap (v_1 \sqcup v_2)$.

Observe that every non empty lattice structure which is trivial is also QLT-distributive and satisfies also QLT1 and QLT2 and every non empty lattice structure which is trivial is also join-idempotent and meet-idempotent and there exists a non empty lattice structure which is join-commutative, join-associative, join-idempotent, meet-commutative, meet-associative, and meet-idempotent and satisfies QLT1 and QLT2.

Let L be a join-commutative, non empty lattice structure. One can verify that L satisfies QLT1 if and only if the condition (Def. 4) is satisfied.

(Def. 4) for every elements v_0, v_1, v_2 of L, $v_0 \sqcap v_1 \sqsubseteq v_0 \sqcap (v_1 \sqcup v_2)$.

Note that $\{0, 1, 2\}$ is real-membered and every element of $\{0, 1, 2\}$ is real.

Let x, y be elements of $\{0, 1, 2\}$. The functor OpEx2(x, y) yielding an element of $\{0, 1, 2\}$ is defined by the term

(Def. 5) $\begin{cases} 1, & \textbf{if } x = 1 \text{ or } y = 1, \\ \min(x, y), & \textbf{if } x \neq 1 \text{ and } y \neq 1. \end{cases}$

The functors: QLTEx1 and QLTEx2 yielding binary operations on $\{0, 1, 2\}$ are defined by conditions

(Def. 6) for every elements x, y of $\{0, 1, 2\}$, if $x = y$, then QLTEx1$(x, y) = x$ and if $x \neq y$, then QLTEx1$(x, y) = 0$,

(Def. 7) for every elements x, y of $\{0, 1, 2\}$, if $x = 1$ or $y = 1$, then QLTEx2$(x, y) = 1$ and if $x \neq 1$ and $y \neq 1$, then QLTEx2$(x, y) = \min(x, y)$,

respectively. Now we state the proposition:

(1) QLTEx1 $\neq$ QLTEx2.

The functors: QLTLattice1 and QLTLattice2 yielding strict, non empty lattice structures are defined by terms

(Def. 8) $\langle \{0, 1, 2\}, \text{QLTEx1}, \text{QLTEx1} \rangle$,

(Def. 9) $\langle \{0, 1, 2\}, \text{QLTEx1}, \text{QLTEx2} \rangle$, respectively. Let us note that QLTEx1 is commutative, associative, and idempotent and QLTEx2 is commutative, associative, and idempotent and QLTLattice1 is join-commutative, join-associative, and join-idempotent and QLTLattice1 is meet-commutative, meet-associative, and meet-idempotent.

Let us consider elements v_0, v_1 of QLTLattice1. Now we state the propositions:

(2) If $v_1 = 0$, then $v_0 \sqcap v_1 = v_1$.

(3) If $v_1 = 0$, then $v_0 \sqcup v_1 = v_1$.

Observe that QLTLattice1 satisfies QLT1 and QLTLattice1 satisfies QLT2 and every element of QLTLattice2 is real and QLTLattice2 is join-commutative, join-associative, and join-idempotent and QLTLattice2 is meet-commutative, meet-associative, and meet-idempotent.

Observe also that QLTLattice2 satisfies QLT1 and QLTLattice2 satisfies QLT2 and QLTLattice2 is non join-absorbing and QLTLattice2 is non meet-absorbing and QLTLattice1 is non join-absorbing and QLTLattice1 is non meet-absorbing.

A quasilattice is a join-commutative, join-associative, meet-commutative, meet-associative, join-idempotent, meet-idempotent, non empty lattice structure satisfying QLT1 and QLT2.

2. Properties of Quasilattices: QLT-1

Now we state the propositions:

(4) Suppose for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_1, v_2, and v_0, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcap v_2)) = v_0 \sqcap (v_1 \sqcup v_2)$. $(v_1 \sqcap v_2) \sqcup (v_1 \sqcap v_3) = v_1 \sqcap (v_2 \sqcup v_3)$.

(5) If L is meet-commutative, join-idempotent, join-associative, join-commutative, and QLT-distributive and satisfies QLT1 and QLT2, then L is distributive. The theorem is a consequence of (4).

Observe that every non empty lattice structure which is meet-commutative, join-idempotent, join-associative, join-commutative, and QLT-distributive and satisfies QLT1 and QLT2 is also distributive.

3. QLT-2

Now we state the propositions:

(6) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_0 \sqcap v_1) \sqcup (v_0 \sqcap v_2)$. $v_1 \sqcup (v_2 \sqcap v_3) = (v_1 \sqcup v_2) \sqcap (v_1 \sqcup v_3)$.

(7) If L is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, and distributive and satisfies QLT2, then L is distributive'. The theorem is a consequence of (6).

Let us observe that every non empty lattice structure which is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, and distributive and satisfies QLT2 is also distributive'.

4. QLT-3

Let us consider L. We say that L is QLT-selfdistributive if and only if

(Def. 10) for every v_2, v_1, and v_0, $(((v_0 \sqcap v_1) \sqcup v_2) \sqcap v_1) \sqcup (v_2 \sqcap v_0) = (((v_0 \sqcup v_1) \sqcap v_2) \sqcup v_1) \sqcap (v_2 \sqcup v_0)$.

Now we state the proposition:

(8) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_2, v_1, and v_0, $(((v_0 \sqcap v_1) \sqcup v_2) \sqcap v_1) \sqcup (v_2 \sqcap v_0) = (((v_0 \sqcup v_1) \sqcap v_2) \sqcup v_1) \sqcap (v_2 \sqcup v_0)$. $v_1 \sqcup (v_2 \sqcap v_3) = (v_1 \sqcup v_2) \sqcap (v_1 \sqcup v_3)$.

Let us note that every non empty lattice structure which is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, join-commutative, and QLT-selfdistributive and satisfies QLT1 and QLT2 is also distributive'.

5. QLT-4: Bowden Inequality

Let us consider L. We say that L satisfies Bowden inequality if and only if

(Def. 11) for every elements x, y, z of L, $(x \sqcup y) \sqcap z \sqsubseteq x \sqcup (y \sqcap z)$.

Let L be a join-commutative, non empty lattice structure. Observe that L satisfies Bowden inequality if and only if the condition (Def. 12) is satisfied.

(Def. 12) for every elements v_0, v_2, v_1 of L, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcup ((v_0 \sqcup v_1) \sqcap v_2) = v_0 \sqcup (v_1 \sqcap v_2)$.

Now we state the proposition:

(9) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcup ((v_0 \sqcup v_1) \sqcap v_2) = v_0 \sqcup (v_1 \sqcap v_2)$. $v_1 \sqcup (v_2 \sqcap v_3) = (v_1 \sqcup v_2) \sqcap (v_1 \sqcup v_3)$.

Note that every non empty lattice structure which is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, and join-commutative and satisfies QLT1, QLT2, and Bowden inequality is also distributive'.

6. Preliminaries to QLT-5: Modularity for Quasilattices

Let us consider L. We say that L is QLT-selfmodular if and only if

(Def. 13) for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcup (v_2 \sqcap (v_0 \sqcup v_1)) = (v_0 \sqcup v_1) \sqcap (v_2 \sqcup (v_0 \sqcap v_1))$.

Let L be a join-idempotent, non empty lattice structure and a, b be elements of L. Let us note that the predicate $a \sqsubseteq b$ is reflexive.

Let us consider v_1, v_2, and v_3. Now we state the propositions:

(10) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_0, v_1, and v_2 such that $v_0 \sqcup v_1 = v_1$ holds $v_0 \sqcup (v_2 \sqcap v_1) = (v_0 \sqcup v_2) \sqcap v_1$. Then $(v_1 \sqcap v_2) \sqcup (v_1 \sqcap v_3) = v_1 \sqcap (v_2 \sqcup (v_1 \sqcap v_3))$.

(11) Suppose for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$

and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcup (v_0 \sqcap v_2) = v_0 \sqcap (v_1 \sqcup (v_0 \sqcap v_2))$. Then if $v_1 \sqcup v_2 = v_2$, then $v_1 \sqcup (v_3 \sqcap v_2) = (v_1 \sqcup v_3) \sqcap v_2$.

Let L be a meet-idempotent, join-idempotent, meet-commutative, join-commutative, meet-associative, join-associative, non empty lattice structure satisfying QLT1 and QLT2. Observe that L is modular if and only if the condition (Def. 14) is satisfied.

(Def. 14) for every elements v_1, v_2, v_3 of L, $(v_1 \sqcap v_2) \sqcup (v_1 \sqcap v_3) = v_1 \sqcap (v_2 \sqcup (v_1 \sqcap v_3))$.

7. QLT-5

Now we state the proposition:

(12) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcup (v_2 \sqcap (v_0 \sqcup v_1)) = (v_0 \sqcup v_1) \sqcap (v_2 \sqcup (v_0 \sqcap v_1))$. $(v_1 \sqcap v_2) \sqcup (v_1 \sqcap v_3) = v_1 \sqcap (v_2 \sqcup (v_1 \sqcap v_3))$.

Let us note that every non empty lattice structure which is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, join-commutative, and QLT-selfmodular and satisfies QLT1 and QLT2 is also modular.

8. QLT-6

Now we state the proposition:

(13) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcap (v_1 \sqcup v_2)) \sqcup (v_0 \sqcap v_1) = v_0 \sqcap (v_1 \sqcup v_2)$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_0, v_2, and v_1, $(v_0 \sqcup (v_1 \sqcap v_2)) \sqcap (v_0 \sqcup v_1) = v_0 \sqcup (v_1 \sqcap v_2)$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap v_2) \sqcup v_1 = ((v_2 \sqcup v_1) \sqcap v_0) \sqcup v_1$. $(v_1 \sqcap v_2) \sqcup (v_1 \sqcap v_3) = v_1 \sqcap (v_2 \sqcup (v_1 \sqcap v_3))$.

Let us consider L. We say that L is QLT-selfmodular' if and only if

(Def. 15) for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap v_2) \sqcup v_1 = ((v_2 \sqcup v_1) \sqcap v_0) \sqcup v_1$.

Observe that every non empty lattice structure which is meet-idempotent, meet-associative, meet-commutative, join-idempotent, join-associative, join-commutative, and QLT-selfmodular' and satisfies QLT1 and QLT2 is also modular.

9. The Counterexample Needed to Prove QLT-7

Now we state the proposition:

(14) There exist quasilattices L_1, L_2 such that

(i) the carrier of L_1 = the carrier of L_2, and

(ii) the join operation of L_1 = the join operation of L_2, and

(iii) the meet operation of $L_1 \neq$ the meet operation of L_2.

The theorem is a consequence of (1).

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