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Grothendieck Universes[1]

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Summary. The foundation of the Mizar Mathematical Library [2], is first-order Tarski-Grothendieck set theory. However, the foundation explicitly refers only to Tarski's Axiom A, which states that for every set X there is a Tarski universe U such that $X \in U$. In this article, we prove, using the Mizar [3] formalism, that the Grothendieck name is justified. We show the relationship between Tarski and Grothendieck universe.

First we prove in Theorem (17) that every Grothendieck universe satisfies Tarski's Axiom A. Then in Theorem (18) we prove that every Grothendieck universe that contains a given set X, even the least (with respect to inclusion) denoted by `GrothendieckUniverse` X, has as a subset the least (with respect to inclusion) Tarski universe that contains X, denoted by the `Tarski-Class` X. Since Tarski universes, as opposed to Grothendieck universes [5], might not be transitive (called `epsilon-transitive` in the Mizar Mathematical Library [1]) we focused our attention to demonstrate that `Tarski-Class` $X \subsetneq$ `GrothendieckUniverse` X for some X.

Then we show in Theorem (19) that `Tarski-Class` X where X is the singleton of any infinite set is a proper subset of `GrothendieckUniverse` X. Finally we show that `Tarski-Class` $X =$ `GrothendieckUniverse` X holds under the assumption that X is a transitive set.

The formalisation is an extension of the formalisation used in [4].

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1. Grothendieck Universes Axioms

From now on X, Y, Z denote sets, x, y, z denote objects, and A, B, C denote ordinal numbers.

Let us consider X. We say that X is power-closed if and only if

(Def. 1) if $Y \in X$, then $2^Y \in X$.

We say that X is union-closed if and only if

(Def. 2) if $Y \in X$, then $\bigcup Y \in X$.

We say that X is Family-Union-closed if and only if

(Def. 3) for every Y and for every function f such that $\operatorname{dom} f = Y$ and $\operatorname{rng} f \subseteq X$ and $Y \in X$ holds $\bigcup \operatorname{rng} f \in X$.

Note that every set which is Tarski is also power-closed and subset-closed and every set which is transitive and Tarski is also union-closed and Family-Union-closed and every set which is transitive and Family-Union-closed is also union-closed and every set which is transitive and power-closed is also subset-closed.

A Grothendieck is a transitive, power-closed, Family-Union-closed set.

2. Grothendieck Universe Operator

Let X be a set. A Grothendieck of X is a Grothendieck defined by

(Def. 4) $X \in it$.

Let G_1, G_2 be Grothendiecks. One can verify that $G_1 \cap G_2$ is transitive, power-closed, and Family-Union-closed.

Now we state the proposition:

(1) Let us consider Grothendiecks G_1, G_2 of X. Then $G_1 \cap G_2$ is a Grothendieck of X.

Let X be a set. The functor GrothendieckUniverse(X) yielding a Grothendieck of X is defined by

(Def. 5) for every Grothendieck G of X, $it \subseteq G$.

The scheme *ClosedUnderReplacement* deals with a set $\mathcal{X}$ and a Grothendieck $\mathcal{U}$ of $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding a set and states that

(Sch. 1) $\{\mathcal{F}(x)$, where x is an element of $\mathcal{X} : x \in \mathcal{X}\} \in \mathcal{U}$

provided

- if $Y \in \mathcal{X}$, then $\mathcal{F}(Y) \in \mathcal{U}$.

In the sequel U denotes a Grothendieck. Now we state the proposition:

(2) Let us consider a function f. If $\operatorname{dom} f \in U$ and $\operatorname{rng} f \subseteq U$, then $\operatorname{rng} f \in U$.
Proof: Set $A = \operatorname{dom} f$. Define $\mathcal{S}(\text{set}) = \{f(\$_1)\}$. Consider s being a function such that $\operatorname{dom} s = A$ and for every X such that $X \in A$ holds $s(X) = \mathcal{S}(X)$. $\operatorname{rng} s \subseteq U$. $\bigcup s \subseteq \operatorname{rng} f$. $\operatorname{rng} f \subseteq \bigcup s$. □

3. Set of all Sets up to Given Rank

Let x be an object. The functor $\mathbf{R}\text{rank}(x)$ yielding a transitive set is defined by the term

(Def. 6) $\mathbf{R}_{\text{rk}(x)}$.

Now we state the propositions:

(3) $X \in \mathbf{R}_A$ if and only if there exists B such that $B \in A$ and $X \in 2^{\mathbf{R}_B}$.
Proof: If $X \in \mathbf{R}_A$, then there exists B such that $B \in A$ and $X \in 2^{\mathbf{R}_B}$. □

(4) $Y \in \mathbf{R}\text{rank}(X)$ if and only if there exists Z such that $Z \in X$ and $Y \in 2^{\mathbf{R}\text{rank}(Z)}$.
Proof: If $Y \in \mathbf{R}\text{rank}(X)$, then there exists Z such that $Z \in X$ and $Y \in 2^{\mathbf{R}\text{rank}(Z)}$. □

(5) If $x \in X$ and $y \in \mathbf{R}\text{rank}(x)$, then $y \in \mathbf{R}\text{rank}(X)$.

(6) If $Y \in \mathbf{R}\text{rank}(X)$, then there exists x such that $x \in X$ and $Y \subseteq \mathbf{R}\text{rank}(x)$. The theorem is a consequence of (4).

(7) $X \subseteq \mathbf{R}\text{rank}(X)$.

(8) If $X \subseteq \mathbf{R}\text{rank}(Y)$, then $\mathbf{R}\text{rank}(X) \subseteq \mathbf{R}\text{rank}(Y)$.

(9) If $X \in \mathbf{R}\text{rank}(Y)$, then $\mathbf{R}\text{rank}(X) \in \mathbf{R}\text{rank}(Y)$.

(10) (i) $X \in \mathbf{R}\text{rank}(Y)$, or

(ii) $\mathbf{R}\text{rank}(Y) \subseteq \mathbf{R}\text{rank}(X)$.

(11) (i) $\mathbf{R}\text{rank}(X) \in \mathbf{R}\text{rank}(Y)$, or

(ii) $\mathbf{R}\text{rank}(Y) \subseteq \mathbf{R}\text{rank}(X)$.

(12) If $X \in U$ and $X \approx A$, then $A \in U$.
Proof: Define $\mathcal{P}[\text{ordinal number}] \equiv$ for every X such that $X \approx \$_1$ and $X \in U$ holds $\$_1 \in U$. For every ordinal number A such that for every ordinal number C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$. For every ordinal number O, $\mathcal{P}[O]$. □

(13) If $X \in Y \in U$, then $X \in U$.

(14) If $X \in U$, then $\mathbf{R}\text{rank}(X) \in U$.

Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ for every set A such that $\mathrm{rk}(A) \in \$_1$ and $A \in U$ holds $\mathbf{R}\mathrm{rank}(A) \in U$. For every A such that for every C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$. For every ordinal number O, $\mathcal{P}[O]$. □

(15) If $A \in U$, then $\mathbf{R}_A \in U$.
Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ if $\$_1 \in U$, then $\mathbf{R}_{\$_1} \in U$. For every A such that for every C such that $C \in A$ holds $\mathcal{P}[C]$ holds $\mathcal{P}[A]$. For every ordinal number O, $\mathcal{P}[O]$. □

4. Tarski vs. Grothendieck Universe

Now we state the propositions:

(16) If $X \subseteq U$ and $X \notin U$, then there exists a function f such that f is one-to-one and $\mathrm{dom}\, f = \mathrm{On}\, U$ and $\mathrm{rng}\, f = X$.
Proof: For every set x such that $x \in \mathrm{On}\, U$ holds x is an ordinal number and $x \subseteq \mathrm{On}\, U$. Reconsider $\Lambda = \mathrm{On}\, U$ as an ordinal number. There exists a function *THE* such that for every set x such that $\emptyset \neq x \subseteq X$ holds $\mathit{THE}(x) \in x$. Consider *THE* being a function such that for every set x such that $\emptyset \neq x \subseteq X$ holds $\mathit{THE}(x) \in x$. Define $\mathcal{R}(\text{set}) = \{\mathrm{rk}(x)$, where x is an element of $\$_1 : x \in \$_1\}$. For every set A and for every object x, $x \in \mathcal{R}(A)$ iff there exists a set a such that $a \in A$ and $x = \mathrm{rk}(a)$.

Define $\mathcal{Q}[\text{set}, \text{object}] \equiv \$_2 \in X \setminus \$_1$ and for every ordinal number B such that $B \in \mathcal{R}(X \setminus \$_1)$ holds $\mathrm{rk}(\$_2) \subseteq B$. Define $\mathcal{F}(\text{transfinite sequence}) = \mathit{THE}(\{x$, where x is an element of $X : \mathcal{Q}[\mathrm{rng}\, \$_1, x]\})$. Consider f being a transfinite sequence such that $\mathrm{dom}\, f = \Lambda$ and for every ordinal number A and for every transfinite sequence L such that $A \in \Lambda$ and $L = f{\restriction}A$ holds $f(A) = \mathcal{F}(L)$. For every ordinal number A such that $A \in \Lambda$ holds $\mathcal{Q}[\mathrm{rng}(f{\restriction}A), f(A)]$. f is one-to-one. $\mathrm{rng}\, f \subseteq X$. $X \subseteq \mathrm{rng}\, f$. □

(17) Every Grothendieck is Tarski.
Proof: If $X \notin U$, then $X \approx U$. □

Let us note that every set which is transitive, power-closed, and Family-Union-closed is also universal and every set which is universal is also transitive, power-closed, and Family-Union-closed.

Now we state the propositions:

(18) Let us consider a Grothendieck G of X. Then $\mathbf{T}(X) \subseteq G$.

(19) Let us consider an infinite set X. Then $X \notin \mathbf{T}(\{X\})$.
Proof: Define $\mathcal{B}(\text{set}, \text{set}) = \$_2 \cup 2^{\$_2}$. Consider f being a function such that $\mathrm{dom}\, f = \mathbb{N}$ and $f(0) = \{\{A\}, \emptyset\}$ and for every natural number n, $f(n+1) = \mathcal{B}(n, f(n))$. Set $U = \bigcup f$. Define $\mathcal{M}[\text{object}, \text{object}] \equiv \$_1 \in f(\$_2)$ and $\$_2 \in \mathrm{dom}\, f$ and for every natural numbers i, j such that $i < j = \$_2$

holds $\$_1 \notin f(i)$. For every object x such that $x \in U$ there exists an object y such that $\mathcal{M}[x, y]$.

Consider M being a function such that $\operatorname{dom} M = U$ and for every object x such that $x \in U$ holds $\mathcal{M}[x, M(x)]$. U is subset-closed. For every X such that $X \in U$ holds $2^X \in U$. Define $\mathcal{D}[\text{natural number}] \equiv f(\$_1)$ is finite. For every natural number n such that $\mathcal{D}[n]$ holds $\mathcal{D}[n+1]$. For every natural number n, $\mathcal{D}[n]$. For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is countable. For every X such that $X \subseteq U$ holds $X \approx U$ or $X \in U$. $A \notin U$. □

(20) Let us consider an infinite set X. Then $\mathbf{T}(\{X\}) \subset$ GrothendieckUniverse $(\{X\})$. The theorem is a consequence of (18) and (19).

(21) (i) GrothendieckUniverse(X) is a universal class, and

(ii) for every universal class U such that $X \in U$ holds GrothendieckUniverse$(X) \subseteq U$.

(22) Let us consider a transitive set X. Then $\mathbf{T}(X) =$ GrothendieckUniverse(X). The theorem is a consequence of (18).

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