View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Texas A&M Repository

SURGERY ON FRAMES

A Dissertation

by

NGA QUYNH NGUYEN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2008

Major Subject: Mathematics

SURGERY ON FRAMES

A Dissertation

by

NGA QUYNH NGUYEN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	David Larson
Committee Members,	Roger Smith
	Joe Ward
	Wilbert Wilhelm
Head of Department,	Al Boggess

August 2008

Major Subject: Mathematics

ABSTRACT

Surgery on Frames. (August 2008)

Nga Quynh Nguyen, B.S., Hanoi State University

Chair of Advisory Committee: Dr. David Larson

In this dissertation, we investigate methods of modifying a tight frame sequence on a finite subset of the frame so that the result is a tight frame with better properties. We call this a surgery on the frame. There are basically three types of surgeries: transplants, expansions, and contractions. In this dissertation, it will be necessary to consider surgeries on not-necessarily-tight frames because the subsets of frames that are excised and replaced are usually not themselves tight frames on their spans, even if the initial frame and the final frame are tight. This makes the theory necessarily complicated, and richer than one might expect.

Chapter I is devoted to an introduction to frame theory. In Chapter II, we investigate conditions under which expansion, contraction, and transplant problems have a solution. In particular, we consider the equiangular replacement problem. We show that we can always replace a set of three unit vectors with a set of three complex unit equiangular vectors which has the same Bessel operator as the Bessel operator of the original set. We show that this can not always be done if we require the replacement vectors to be real, even if the original vectors are real. We also prove that the minimum angle between pairs of vectors in the replacement set becomes largest when the replacement set is equiangular. Iterating this procedure can yield a frame with smaller maximal frame correlation than the original. Frames with optimal maximal frame correlation are called Grassmannian frames and no general method is known at the present time for constructing them. Addressing this, in Chapter III we introduce a spreading algorithm for finite unit tight frames by replacing vectors

three-at-a-time to produce a unit tight frame with better maximal frame correlation than the original frame. This algorithm also provides a “good” orientation for the replacement sets. The orientation part ensures stability in the sense that if a selected set of three unit vectors happens to already be equiangular, then the algorithm gives back the same three vectors in the original order. In chapter IV and chapter V, we investigate two special classes of frames called push-out frames and group frames. Chapter VI is devoted to some mathematical problems related to the ”cocktail party problem ”.

To my family for their love and support

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor, Professor David R. Larson, for his invaluable encouragement and guidance during my graduate study and in the preparation of this dissertation.

I also would like to thank all the members of my advisory committee, Professor Roger Smith, Professor Joe Ward and Professor Wilbert Wilhelm for their service on my advisory committee.

I also would like to thank Dr. Troy Henderson for helping with programming.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION TO FRAME THEORY

Frames for a Hilbert space were formally defined by Duffin and Schaeffer [9] in 1952 to study some deep problems in nonharmonic Fourier series. Their ideas did not generate much general interest outside of nonharmonic Fourier series until the landmark paper of Daubechies, Grossmann and Meyer [8] in 1986. Since then the theory of frames began to be more widely studied. Recent references for frames and the closely related topics of wavelets and wavelet frames that we have used include [2],[4],[5],[7],[11]. We have also used several textbooks and research monographs for basis theory and notation in the subjects of operator theory [18],[21],[22], matrix analysis [17],[27], and group representation [23].

Frames have traditionally been used in signal processing. Today, frames have many useful applications in mathematics and engineering such as sampling theory, image processing, data transmission with erasures, as well as operator theory. What makes frames a useful tool in these areas is their overcompleteness, which allows representations of vectors which are resiliant to additive noise, give stable reconstruction after erasures, and give freedom to capture significant signal characteristics.

A *frame* for a Hilbert space H is a sequence $\{x_j\}_{j\in\mathbb{J}}$ in H, for a countable index set $\mathbb{J}$ with the property that there exist positive constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{j\in\mathbb{J}} |\langle\, x\,,\, x_j\,\rangle|^2 \leq B\|x\|^2 \tag{I.1}$$

holds for every $x \in H$. We call the largest A and the smallest B for which (VI.1)

This dissertation follows the style of SIAM Journal on Control and Optimization.

holds the lower and upper frame bounds for the frame, repectively. A frame is called *tight* when $A = B$, and a frame is Parseval when $A = B = 1$. If we only require the right-hand side of the inequality (VI.1), then $\{x_j\}_{j\in\mathbb{J}}$ is called a *Bessel sequence*. In the case that (VI.1) holds only for all the $x \in \overline{\text{span}}\{x_j\}_{j\in\mathbb{J}}$, then we say that $\{x_j\}_{j\in\mathbb{J}}$ is a *frame sequence*. In a finite dimensional space, a finite frame is just a finite spanning set, and every finite set is a frame sequence. If all the frame vectors have the same norm then we call it an *equal-norm frame*, and if the frame vectors are all norm one we call it a *unit norm frame*. A set $\{x_j\}_{j\in\mathbb{J}}$ of unit norm vectors is called *equiangular* if there is a constant $c \in [0, 1]$ such that $|\langle x_k , x_l \rangle| = c$ when $k \neq l$ and *strictly equiangular* if $\langle x_k , x_l \rangle = c$ when $k \neq l$. The *analysis operator* $\Theta_X : H \to \ell^2(\mathbb{J})$ for a Bessel sequence $X = \{x_j\}_{j\in\mathbb{J}}$ is defined by

$$\Theta_X(x) = \sum_{j\in\mathbb{J}} \langle x , x_j \rangle e_j, \quad x \in H,$$

where $\{e_j\}$ is the standard orthonormal basis for the $\ell^2(\mathbb{J})$-sequence space. The adjoint operator $\Theta_X^* : \ell^2(\mathbb{J}) \to H$ of the analysis operator Θ_X is called the *synthesis operator*. It is easy to check that $\Theta_X^*(\sum_{j\in\mathbb{J}} c_j e_j) = \sum_{j\in\mathbb{J}} c_j x_j$. We can verify that $\Theta_X^*\Theta_X = \sum_{j\in\mathbb{J}} x_j \otimes x_j$, where $x \otimes y$ is the elementary tensor rank-one operator defined by $(x \otimes y)(h) = \langle h , y \rangle x$ for $h \in H$. The operator $x \otimes x$ is a projection if and only if $||x|| = 1$. If $x = (x_1, x_2, ..., x_k)^T$ and $y = (y_1, y_2, ..., y_k)^T$ then

$$x \otimes y = \begin{pmatrix} x_1\bar{y}_1 & x_1\bar{y}_2 & \cdots & x_1\bar{y}_k \\ x_2\bar{y}_1 & x_2\bar{y}_2 & \cdots & x_1\bar{y}_k \\ \cdots & \cdots & \cdots & \cdots \\ x_k\bar{y}_1 & x_k\bar{y}_2 & \cdots & x_k\bar{y}_k \end{pmatrix}.$$

The operator $S_X = \Theta_X^*\Theta_X : H \to H$ is called the *frame operator*. For a Bessel sequence $X = \{x_j\}_{j\in\mathbb{J}}$, we call the operator $B_X = \sum_{j\in\mathbb{J}} x_j \otimes x_j$ the *Bessel operator* for the sequence X. The operator $G_X = \Theta_X\Theta_X^* : \ell^2(\mathbb{J}) \to \ell^2(\mathbb{J})$ is called the *Grammian operator*. It is useful to note that [15], in a finite dimensional space, the Grammian matrix for a frame $X = \{x_j\}_{j=1}^k$ is

$$G = \begin{pmatrix} \langle x_1 , x_1 \rangle & \langle x_2 , x_1 \rangle & \cdots & \langle x_k , x_1 \rangle \\ \langle x_1 , x_1 \rangle & \langle x_2 , x_1 \rangle & \cdots & \langle x_k , x_1 \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle x_1 , x_k \rangle & \langle x_2 , x_k \rangle & \cdots & \langle x_k , x_k \rangle \end{pmatrix}.$$

If $n \in \mathbb{N}$, we denote H_n the n dimensional Hilbert (real or complex) space.

We say that frames $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ on Hilbert spaces H, K, respectively, are *unitarily equivalent* if there is a unitary operator $U : H \to K$ such that $Ux_j = y_j$ for all $j \in \mathbb{J}$. We say that they are *similar* if there is a bounded linear invertible operator $T : H \to K$ such that $Tx_j = y_j$ for all $j \in \mathbb{J}$. The following result tells us that every frame is similar to a Parseval frame.

Lemma I.1. ([14]) Let $X = \{x_j\}_{j\in\mathbb{J}}$ be a frame for a Hilbert space H with frame operator S_X. Then $\{S_X^{-1/2}x_j\}_{j\in\mathbb{J}}$ is a Parseval frame for H.

We can characterize a frame through its analysis operator, synthesis operator, frame operator as follows.

Proposition I.2. ([3],[14]) Suppose $\{x_j\}_{j\in\mathbb{J}}$ is a sequence of vectors in a Hilbert space H. The following are equivalent:

1)$\{x_j\}_{j\in\mathbb{J}}$ is a frame for H.

2)The analysis operator $\Theta : H \to \ell^2(\mathbb{J})$ is linear, bounded, bounded from below.

3)The synthesis operator $\Theta^* : \ell^2(\mathbb{J}) \to H$ is linear, bounded and surjective.

4)The frame operator $S : H \to H$ is positive, self-adjoint, invertible.

We also can characterize a Parseval frame through its analysis operator, synthesis operator, frame operator and Grammian operator as follows.

Proposition I.3. ([3],[6]) Suppose $\{x_j\}_{j\in\mathbb{J}}$ is a sequence of vectors in a Hilbert space H. The following are equivalent:

1)$\{x_j\}_{j\in\mathbb{J}}$ is a Parseval frame for H

2)The analysis operator Θ is an isometry from H into $\ell^2(\mathbb{J})$.

3)The synthesis operator $\Theta^* : \ell^2(\mathbb{J}) \to H$ is a partial isometry.

4)The frame operator $S : H \to H$ is the identity.

5)The Grammian operator $G : \ell^2(\mathbb{J}) \to \ell^2(\mathbb{J})$ is an orthogonal projection with range $\Theta(H)$.

One of the most important properties of a frame is the ability to recover every element in the Hilbert space as a combination of a frame vectors. In [13], it is proved that if $X = \{x_j\}_{j\in\mathbb{J}}$ is a frame for H then

$$x = \sum_{j\in\mathbb{J}} \langle\, x \,,\, x_j \,\rangle S^{-1}x_j = \sum_{j\in\mathbb{J}} \langle\, x \,,\, S^{-1}x_j \,\rangle x_j$$

for all $x \in H$. The collection of vectors $X^* = \{S^{-1}x_j\}_{j\in\mathbb{J}}$ is called the canonical dual frame of X.

Lemma I.4. Suppose $X = \{x_j\}_{j=1}^k$ and $Y = \{y_j\}_{j=1}^k$ are frames in H_n and G_X, G_Y are their Grammian operators, respectively. Then $G_X = G_Y$ if and only if there is a unitary operator U such that $y_j = Ux_j$ for $j = 1, 2, ..., k$.

Proof. For the "only if" part, define $U = \Theta_{Y^*}^*\Theta_X : H_n \to H_n$ where Θ_X is the analysis operator for X and $\Theta_{Y^*}^*$ is the synthesis operator for the canonical dual frame Y^* of Y. We will prove that U is an unitary operator and $y_j = Ux_j$ for all j.

For all $x \in H_n$, we have $Ux = \sum_{j=1}^k \langle\, x \,,\, x_j \,\rangle y_j^*$ and so for any l,

$$Ux_l = \sum_{j=1}^k \langle\, x_l \,,\, x_j \,\rangle y_j^*$$

Since $G_X = G_Y$, $\langle\, x_l \,,\, x_j \,\rangle = \langle\, y_l \,,\, y_j \,\rangle$ for $l \neq j$. Thus, $Ux_l = \sum_{j=1}^k \langle\, y_l \,,\, y_j \,\rangle y_j^* = y_l$ for all l.

We have $\Theta_Y^*\Theta_{Y^*}(x) = \sum_{j=1}^k \langle x , y_j^* \rangle y_j = x$ for all $x \in H_n$ which implies that $\Theta_Y^*\Theta_{Y^*} = I$. Similarly, $\Theta_{Y^*}^*\Theta_Y(x) = \sum_{j=1}^k \langle x , y_j \rangle y_j^* = x$ for all $x \in H_n$ which implies that $\Theta_{Y^*}^*\Theta_Y = I$.

We have $UU^* = \Theta_{Y^*}^*\Theta_X\Theta_X^*\Theta_{Y^*} = \Theta_{Y^*}^*G_X\Theta_{Y^*} = \Theta_{Y^*}^*G_Y\Theta_{Y^*} = \Theta_{Y^*}^*\Theta_Y\Theta_Y^*\Theta_{Y^*} = I.I = I$. Therefore, for all $x \in H_n$, $||U^*x||^2 = ||\langle UU^*x , x \rangle|| = ||x||^2$. So $U^* : H_n \to H_n$ is an isometry and injective operator which imply that U^* is an unitary operator. So U is an unitary operator as well.

For the "if " part, since $y_j = Ux_j$ for all j and each row vectors of Θ_X and Θ_Y are $\bar{x_j}^T$ and $\bar{y_j}^T$, respectively. So $\Theta_X U^* = \Theta_Y$ and $G_Y = \Theta_Y\Theta_Y^* = \Theta_X U^* U \Theta_X^* = \Theta_X\Theta_X^* = G_X$. □

The following lemma is well known.

Lemma I.5. If $\{x_j\}$ is a unit norm tight frame of k vectors in a n dimensional space H_n, then the frame bound is $\frac{k}{n}$ and we have

$$\sum_{j=1}^{k} x_j \otimes x_j = \frac{k}{n}\ I$$

where I is the identity on H_n. So for a uniform norm orthogonal basis with norm b, the frame bound is b^2.

We will need to use the following proposition which was shown in [10],[20].

Proposition I.6. Let $A \in \mathcal{B}(H)$ be a finite rank positive operator with integer trace k. If $k \geq \text{rank}(A)$, then A is the sum of k projections of rank one.

Proof. We will construct unit vectors $x_1, x_2, ..., x_k$ such that $A = \sum_{j=1}^k x_j \otimes x_j$. The proof uses induction on k. Let $n = \text{rank}(A)$ and write $H_n = \text{ran}(A)$. If $k = 1$, then A is a rank-1 projection. Assume that $k \geq 2$. Select an orthonormal basis $\{e_j\}_{j=1}^n$

for H_n such that A can be written on H_n as a diagonal matrix with positive entries $a_1 \geq a_2 ... \geq a_n > 0$.

Case 1 : $k > n$. In this case, we have $a_1 > 1$ so we can take $x_k = e_1$. Then $A - (x_k \otimes x_k) = \text{diag}(a_1 - 1, a_2, ..., a_n)$ has positive diagonal entries, rank n, and trace $k - 1 \geq n$. By the inductive hypothesis, the result holds.

Case 2 : $k = n$. We have $a_1 \geq 1$ and $a_n \leq 1$. Given any finite rank, self adjoint $R \in \mathcal{B}(H)$, let $\mu_n(R)$ denote the n-th largest eigenvalue of R counting multiplicity. Note that $\mu_n(A - (e_1 \otimes e_1)) \geq 0, \mu_n(A - (e_n \otimes e_n)) \leq 0$ and $\mu_n(A - (x \otimes x))$ is a continuous function of $x \in H_n$. Hence, there is $y \in H_n$ such that $\mu_n(A - (y \otimes y)) = 0$. Choose $x_k = y$. Note that $A - (x_k \otimes x_k) \geq 0$ and

$$\text{trace}(A - (x_k \otimes x_k)) = n - 1,$$

$$\text{rank}(A - (x_k \otimes x_k)) = n - 1 = k - 1.$$

Again, by the inductive hypothesis, the result holds.

□

In [13], the following proposition is proved.

Proposition I.7. i) Let $\mathbb{J}$ be a countable (or finite) index set. If $\{e_j\}_{j \in \mathbb{J}}$ is an orthonormal basis for a Hilbert space K and P is the orthogonal projection from K onto a closed subspace H, then $\{Pe_j\}_{j \in \mathbb{J}}$ is a Parseval frame for H.

ii) Suppose that $\{x_j\}_{j \in \mathbb{J}}$ is a Parseval frame for a Hilbert space H. Then there exists a Hilbert space $K \supseteq H$ and and an orthonormal basis $\{e_j\}_{j \in \mathbb{J}}$ for K such that $x_j = Pe_j$, where P is the orthogonal projection from K onto H.

Let T be a positive operator in $\mathcal{B}(H)$ and $\{e_j\}_{j \in \mathbb{J}}$ be an orthonormal basis for H. Let $v_j = T^{1/2} e_j$. Then $T = T^{1/2}(\sum_{j \in \mathbb{J}} e_j \otimes e_j) T^{1/2} = \sum_{j \in \mathbb{J}} (T^{1/2} e_j) \otimes (T^{1/2} e_j)$. Thus every positive operator can be written in the form $T = \sum_{j \in \mathbb{J}} v_j \otimes v_j$ where the sum is convergent in the strong operator topology. In this connection, the following lemma proves useful.

Lemma I.8. ([15]) Let T be a positive operator on H. Suppose $T = \sum_{j\in\mathbb{J}} v_j \otimes v_j$, where the series has either finitely or countably many terms and converges in the strong operator topology. Then $\overline{\text{ran}(T)} = \overline{\text{span}\{v_j\}}$.

Proof. Let P be the orthogonal projection of H onto $\overline{\text{ran}(T)}$, and let $P^\perp = I - P$. Then we have:

$$0 = P^\perp T P^\perp = \sum_{j\in\mathbb{J}} P^\perp v_j \otimes P^\perp v_j.$$

That $P^\perp v_j \otimes P^\perp v_j$ is a positive operator implies $P^\perp v_j = 0$ for all j. Thus $v_j \in P(H) = \overline{\text{ran}(T)}$, so $\overline{\text{span}\{v_j\}} \subseteq \overline{\text{ran}(T)}$.

Now suppose that $\overline{\text{span}\{v_j\}}$ is a proper subset of $\overline{\text{ran}(T)}$. Then we could find a unit vector $z \in \overline{\text{ran}(T)}$ that is perpendicular to each v_j. Let $z_j = Tw_j \in \text{ran}(T)$ be such that $z_j \to z$, and let $Q = z \otimes z$. Then $Qz_j \to Qz = z$. But for each j we also have

$$Qz_j = QTw_j = \sum_{l\in\mathbb{J}} \langle w_j\,,\, v_l \rangle Q v_l = 0$$

So this implies that $z = 0$, which is a contradiction.

□

For a unit norm frame $\{x_j\}_{j=1}^k$ in H_n, we define the *maximal frame correlation* $\mathcal{M}(\{x_j\}_{j=1}^k)$ by $\mathcal{M}(\{x_j\}_{j=1}^k) = \max\{|\langle x_m\,,\, x_l \rangle| : m \neq l\}$. A sequence of vectors $\{x_j\}_{j=1}^k$ in H_n is called a *Grassmannian frame* if it is a solution to $\min\{\mathcal{M}(\{x_j\}_{j=1}^k)\}$ where the minimum is taken over all unit norm frames $\{x_j\}_{j=1}^k$ in H_n. In other words, Grassmannian frame is the unit norm frame which makes the smallest angle between vectors as large as possible. A compactness argument shows that Grassmannian frames exist. However, constructing Grassmannian frames can be difficult.

The concept of Grassmannian frames is related to various areas in mathematics and engineering[24].

Theorem I.9. ([24]) Let $\{x_j\}_{j=1}^k$ be a unit frame in H_n. Then

$$\mathcal{M}(\{x_j\}_{j=1}^k) \geq \sqrt{\frac{k-n}{n(k-1)}}$$

Equality holds if and only if $\{x_j\}_{j=1}^k$ is an equiangular tight frame.

Furthermore,

If $H = \mathbb{R}$, equality can only hold if $k \leq \frac{n(n+1)}{2}$.

If $H = \mathbb{C}$, equality can only hold if $k \leq n^2$.

We call unit norm frames that meet the bound with equality *optimal Grassmannian frames.*

This dissertation will provide a spreading method which allows one to replace three vectors of a given unit norm frames at a time to achieve a better distribution which might lead to a construction of Grassmannian frames. To this end, I need to consider the conditions under which I could make a replacement for a set of vectors to get better properties. I call the replacement process a surgery on the frame. There are basically three types of surgeries: transplants, expansions, and contractions. It will be necessary to consider surgeries on not-necessarily-tight frames because the subsets of frames that are excised and replaced are usually not themselves tight frames on their spans, even if the initial frame and the final frame are tight. This makes the theory necessarily complicated, and richer than one might expect.

CHAPTER II

THE (P,Q)-REPLACEMENT PROBLEM

Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame. If we remove p vectors from the frame and replace this set with a set of q vectors, we call the operation a (p,q)-replacement surgery on the frame. We call the p vectors removed the "exised" set and the q replacement vectors the "replacement" set. There are three possibilities: $p > q$, $p = q$, $p < q$. It is clear that if the excised and replaced sets have the same Bessel operator, then the frame operator for the new frame is unchanged from the old frame operator. In this case the frame bounds are unchanged and, in particular, if the original frame is tight then the new frame is tight. In this chapter we consider only surgeries on tight frames.

Not all (p,q)-replacement surgeries we want to consider preserve the frame operator. If $\{x_j\}_{j\in\mathbb{J}}$ is an equal-norm frame and if we want the new frame to also be equal-norm, then unless we replaced the entire set, the new vectors must have the same norm as the original. If the original frame is tight and we require the new frame to be tight as well, then if $p \neq q$ the new frame bound must be different from the old frame bound. This follows immediately from Lemma I.5. So the equal-norm tight frame (p,q)-replacement problem will require change in frame bound unless $p = q$. This will be true for the case $p > q$ (contraction) and $p < q$ (expansion). By scaling an equal-norm frame we can assume that all frame vectors have norm one.

First we consider the possibilities to have a tight frame from an arbitrary sequence of vectors which does not form a tight frame by inserting another set of vectors with arbitrary norms into the sequence. It turns out that this can always be done if we insert $n-1$ vectors where n is the dimension of the space.

Lemma II.1. Suppose $\{x_l\}_{l=1}^{k}$ is a sequence of vectors in $\mathbb{C}^n$ with $k \geq 1$ which does not form a tight frame. Then we can always find $n-1$ vectors $\{y_j\}_{j=1}^{n-1}$ such that the sequence $\{x_l\}_{l=1}^{k} \bigcup \{y_j\}_{j=1}^{n-1}$ is a tight frame.

Proof. Let $B = \sum_{l=1}^{k} x_l \otimes x_l$ and $m = ||B||$. Then $mI - B$ is a positive, singular operator with rank less than n. So we can find $n-1$ vectors $\{y_j\}_{j=1}^{n-1}$ such that $mI - B = \sum_{j=1}^{n-1} y_j \otimes y_j$. Therefore, $\{x_l\}_{l=1}^{k} \bigcup \{y_j\}_{j=1}^{n-1}$ has the frame operator mI and forms a tight frame. □

However, if the vectors in the original set all have norm 1 and we want to find a set of unit vectors such that by taking the union with the original set we have a unit norm tight frame, then we may require more vectors than in the non-unit case.

Lemma II.2. Let $\{x_l\}_{l=1}^{k}$ be a sequence of unit vectors in $\mathbb{C}^n$ with $k \geq 1$ which does not form a tight frame, and let B be its Bessel operator. If $\{y_j\}_{j=1}^{q}$ is a sequence of unit vectors such that $\{x_l\}_{l=1}^{k} \bigcup \{y_j\}_{j=1}^{q}$ is a tight frame then $q \geq n||B|| - k$. Conversely, if $q \geq \max\{n||B|| - k, n\}$ then we can find q unit vectors to insert in the original set to make a tight frame.

Proof. Suppose that $q < n||B|| - k$. Since $\frac{k+q}{n} < ||B||$, we have $\sum_{j=1}^{q} y_j \otimes y_j = \frac{k+q}{n} I - \sum_{l=1}^{k} x_l \otimes x_l$ which is not a positive operator, a contradiction. Now suppose that $q \geq \max\{n||B|| - k, n\}$. Then $A = \frac{k+q}{n} I - \sum_{l=1}^{k} x_l \otimes x_l$ is a positive operator with $\text{rank}(A) \leq n \leq q = tr(A)$. Therefore, by proposition I.6, there are unit vectors $\{y_j\}_{j=1}^{q}$ such that $\sum_{j=1}^{q} y_j \otimes y_j = \frac{k+q}{n} I - \sum_{l=1}^{k} x_l \otimes x_l$. So $\{x_l\}_{l=1}^{k} \bigcup \{y_j\}_{j=1}^{q}$ is a tight frame. □

1. The tight unit norm contraction problem (The case $p > q$)

The following proposition will give the basic principle for the (p, q)-contraction problem.

Lemma II.3. Suppose that $\{x_l\}_{l=1}^k$ is a unit norm tight frame in H_n. Necessary and sufficient conditions in order to replace a subset of p vectors $\{x_j\}_{j\in M}$, where M has cardinality p, with q unit vectors $\{y_m\}_{m=1}^q$, for $p > q$, such that the new sequence remains a tight frame are

$$\sum_{j\in M} x_j \otimes x_j \geq \frac{p-q}{n} I$$

and

$$\text{rank}(\sum_{j\in M} x_j \otimes x_j - \frac{p-q}{n} I) \leq q$$

Proof. For the necessary condition, by Lemma I.5, we have:

$$\begin{aligned} \sum_{l=1}^{k} x_l \otimes x_l &= \frac{k}{n}\ I \\ \sum_{m=1}^{q} y_m \otimes y_m + \sum_{l\in\{1,\ldots,k\}\setminus M} x_l \otimes x_l &= \frac{k-p+q}{n}\ I \end{aligned}$$

By subtracting both sides of the above equations and changing sides, we have:

$$\sum_{j\in M} x_j \otimes x_j - \frac{p-q}{n}\ I = \sum_{m=1}^{q} y_m \otimes y_m$$

So $\sum_{j\in M} x_j \otimes x_j \geq \frac{p-q}{n} I$ and $\text{rank}(\sum_{j\in M} x_j \otimes x_j - \frac{p-q}{n} I) \leq q$.

For the sufficient condition, by Proposition I.6, we can find q unit vectors $\{y_m\}_{m=1}^q$ such that

$$\sum_{j\in M} x_j \otimes x_j - \frac{p-q}{n}\ I = \sum_{m=1}^{q} y_m \otimes y_m$$

Therefore, $\sum_{m=1}^q y_m \otimes y_m + \sum_{l\in\{1,\ldots,k\}\setminus M} x_l \otimes x_l = \frac{k-p+q}{n}\ I$ and $\{y_m\}_{m=1}^q \bigcup \{x_l\}_{l\in\{1,\ldots,k\}\setminus M}$ is a tight frame. □

The above Lemma gives a practical way to test whether a solution of a (p,q)-contraction problem exists.

Corollary II.4. A necessary condition for the existence of a solution of a (p,q)-

contraction problem is that the exised set must span the entire space H_n. In particular, $p \geq n$.

For $p \geq n$, a solution to a tight unit norm contraction problem may or may not exist depending on the properties of a given frame .

Example .1. Let

$$x_1 = (\frac{\sqrt{2}}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}})^T, x_2 = (-\frac{\sqrt{2}}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}})^T, x_3 = (0, \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T, x_4 = (0, -\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T$$

Then $\{x_1, x_2, x_3, x_4\}$ is a set of unit vectors. Note that by dilating this to

$$(\frac{\sqrt{2}}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T, (-\frac{\sqrt{2}}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T, (0, \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})^T, (0, -\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})^T$$

we obtain an orthogonal basis for $\mathbb{R}^4$ with uniform norm $\frac{2}{\sqrt{3}}$ (and hence frame bound 4/3). Thus $\{x_1, x_2, x_3, x_4\}$ is a unit tight frame with frame bound 4/3. It is easy to check that we can not remove any 2 vectors and replace with 1 vector but we can always remove 3 vectors and replace with 2 other vectors.

Remark 1. Completely analysing the case $p \geq n$ is an interesting problem for further work.

2. The tight unit norm expansion problem (The case $p < q$)

Lemma II.5. Necessary and sufficient conditions for the existence of a solution to the tight unit norm expansion problem are $q \geq n$.

Proof. Suppose that $q \geq n$. Without loss of generality, we can assume that we remove $\{x_j\}_{j=1}^p$. Denote $B = \sum_{j=1}^p x_j \otimes x_j + \frac{q-p}{n} I$. Since B is positive operator and $\text{rank}(B) = n \leq q = \text{trace}(B)$, by Proposition I.6, there are unit vectors $\{y_l\}_{l=1}^q$ such that $\sum_{l=1}^{q} y_l \otimes y_l = B$. Therefore, we have:

$$\begin{aligned}
\sum_{l=1}^{q} y_l \otimes y_l + \sum_{j=p+1}^{k} x_j \otimes x_j &= \sum_{j=1}^{p} x_j \otimes x_j + \frac{q-p}{n}\, I + \sum_{j=p+1}^{k} x_j \otimes x_j \\
&= \sum_{j=1}^{k} x_j \otimes x_j + \frac{q-p}{n}\, I \\
&= \frac{k}{n}\, I + \frac{q-p}{n}\, I \\
&= \frac{k+q-p}{n}\, I.
\end{aligned}$$

This shows that $\{y_1, ..., y_q, x_{p+1}, ..., x_k\}$ is a unit norm tight frame.

Suppose $q < n$ and there exist unit vectors $\{y_j\}_{j=1}^{q}$ such that $\sum_{j=1}^{q} y_j \otimes y_j = B$. Since $\operatorname{rank}(B) = n$ and $\operatorname{rank}(\sum_{j=1}^{q} y_j \otimes y_j) < n$, we have a contradiction. $\square$

3. The tight unit norm transpant problem (The case $p = q$)

Definition 1. 1) Let F be a unit tight frame in a real or complex Hilbert space. A subset $A \subset F$ is called *rigid* if whenever we replace A with another set A' of the same cardinality such that the new sequence is also unit tight frame then the vectors in A' are the same as those in A up to a permutation and a possible multiplication by scalars of modulus 1.

2) Suppose that $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ are Bessel sequences of vectors. We say $\{y_j\}_{j\in\mathbb{J}}$ are *geometrically equivalent* to $\{x_j\}_{j\in\mathbb{J}}$ if there are scalars $\{d_i\}_{j\in\mathbb{J}}$ of modulus 1, a permutation Π of $\mathbb{J}$ and a unitary U which commutes with $\sum_{j=1}^{3} x_j \otimes x_j$ such that $y_j = d_j U x_{\Pi(j)}$ for all $j \in \mathbb{J}$. We can easily check that geometrical equivalence is an equivalence relationship.

3) Let F be a unit tight frame in a real or complex Hilbert space. A subset $A \subset F$ is called *stable* if whenever we replace A with another set A' of the same cardinality such that the new sequence is also unit tight frame then A' must be geometrically

equivalent to A.

Singleton sets are rigid and therefore stable since in order to replace one element, say, $\{x_1\}$ with another element $\{y_1\}$, we will need $x_1 \otimes x_1 = y_1 \otimes y_1$ which implies that $x_1 = \lambda y_1$. Since $||x_1|| = ||y_1|| = 1$, we get $|\lambda| = 1$. Orthonomal sets are never rigid because we always can replace them with another orthonomal set which spans the same space. However, orthonomal sets are stable. A set that contains a non-rigid set is non-rigid but a set that contains a non-stable set can be non-stable. For example, the frame $\{Pe_j\}_{j=1}^4$ is stable while $\{Pe_j\}_{j=1}^3$ is not where $\{e_j\}_{j=1}^4$ is the standard orthonomal basis for $\mathbb{R}^4$ and P is the orthogonal projection of $\mathbb{R}^4$ onto $\mathbb{R}^3$ spanned by $\sum_{j=1}^4 e_j$. A tight frame is not necessary stable. For example, it is easy to see that the set of 5 vectors $x_1 = (1,0)^T, x_2 = \left(\cos(\frac{2\pi}{5}), \sin(\frac{2\pi}{5})\right)^T, x_3 = \left(\cos(\frac{4\pi}{5}), \sin(\frac{4\pi}{5})\right)^T, x_4 = \left(\cos(\frac{6\pi}{5}), \sin(\frac{6\pi}{5})\right)^T, x_5 = \left(\cos(\frac{8\pi}{5}), \sin(\frac{8\pi}{5})\right)^T$ can be replaced by the set of 5 vectors $y_1 = (1,0)^T, y_2 = (0,1)^T, y_3 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^T, y_4 = \left(\cos(\frac{11\pi}{12}), \sin(\frac{11\pi}{12})\right)^T, y_5 = \left(\cos(\frac{19\pi}{12}), \sin(\frac{19\pi}{12})\right)^T$ and they are not geometrically equivalent. We will show later that a set of three linearly independent vectors must be non-rigid. We will also show that in the real case every set of 2 non-orthogonal linearly independent vectors must be rigid but in the complex case, it is non-rigid.

Lemma II.6. Let $x_1, x_2 \in \mathbb{C}^2$ be arbitrary unit vectors. If $B = x_1 \otimes x_1 + x_2 \otimes x_2$ then eigenvalues of B are $1 \pm |\langle x_1 , x_2 \rangle|$

Proof. Without loss of generality, we can assume that $x_1 = (1,0)^T, x_2 = (\alpha, \beta)^T$ where $|\alpha|^2 + |\beta|^2 = 1$. Then

$$B = \begin{pmatrix} 1 + |\alpha|^2 & \alpha\bar{\beta} \\ \bar{\alpha}\beta & |\beta|^2 \end{pmatrix}$$

whose characteristic polynomial is $x^2 - 2x + |\beta|^2 = 0$ and eigenvalues are $1 \pm |\alpha| =$

$1 \pm |\langle x_1 , x_2 \rangle|$. □

It follows immediately the following result.

Corollary II.7. If $x_1, x_2, u_1, u_2 \in \mathbb{C}^2$ are arbitrary unit vectors and $x_1 \otimes x_1 + x_2 \otimes x_2 = u_1 \otimes u_1 + u_2 \otimes u_2$ then $|\langle x_1 , x_2 \rangle| = |\langle u_1 , u_2 \rangle|$.

Remark 2. The converse direction is not true. For example, we can check that $x_1 = (1,0)^T, x_2 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^T, u_1 = (\frac{\sqrt{3}}{2}, \frac{1}{2})^T, u_2 = (\frac{\sqrt{6}-\sqrt{2}}{4}, \frac{\sqrt{6}+\sqrt{2}}{4})^T$ have $|\langle x_1 , x_2 \rangle| = |\langle u_1 , u_2 \rangle|$ but $x_1 \otimes x_1 + x_2 \otimes x_2 \neq u_1 \otimes u_1 + u_2 \otimes u_2$

Lemma II.8. If x, y, z, w are unit vectors in H_n then $\{z, w\}$ is geometrically equivalent to $\{x, y\}$ if and only if $x \otimes x + y \otimes y = z \otimes z + w \otimes w$.

Proof. Suppose that $\{z, w\}$ is geometrically equivalent to $\{x, y\}$, that is there are scalars d_1, d_2 of modulus 1, and a unitary U such that $z = d_1 Ux, w = d_2 Uy$ and $U(x \otimes x + y \otimes y) = (x \otimes x + y \otimes y)U$. Then $z \otimes z + w \otimes w = d_1 Ux \otimes d_1 Ux + d_2 Uy \otimes d_2 Uy = U(x \otimes x + y \otimes y)U^* = (x \otimes x + y \otimes y)UU^* = x \otimes x + y \otimes y$.

Suppose that $x \otimes x + y \otimes y = z \otimes z + w \otimes w$ then by Lemma II.7, $|\langle z , w \rangle| = |\langle x , y \rangle|$ which implies that $\langle x , y \rangle = d\langle z , w \rangle$ where $|d| = 1$. Let $t = dz$ then $\langle x , y \rangle = \langle t , w \rangle$. Two Grammian matrices

$$G_{\{x,y\}} = \begin{pmatrix} 1 & \langle x , y \rangle \\ \langle y , x \rangle & 1 \end{pmatrix}$$

and

$$G_{\{t,w\}} = \begin{pmatrix} 1 & \langle t , w \rangle \\ \langle w , t \rangle & 1 \end{pmatrix}$$

are equal which implies that $\{x, y\}$ and $\{t, w\}$ are unitarily equivalent.

□

Corollary II.9. Any set of 2 unit vectors in H_n is stable.

The following lemma shows more details about the collection of sets of 2 unit vectors which have the same Bessel operator. We will use it in section II.4.

Lemma II.10. The collection of all unit vectors $u_1, u_2 \in \mathbb{C}^2$ such that $u_1 \otimes u_1 + u_2 \otimes u_2 = \tilde{B} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$ where β_1, β_2 are positive numbers and $\beta_1 + \beta_2 = 2$ is

$$\{(e^{i\alpha}\sqrt{\frac{\beta_1}{2}}, e^{i\mu}\sqrt{1-\frac{\beta_1}{2}})^T, (e^{i\omega}\sqrt{\frac{\beta_1}{2}}, -e^{i(\omega-\alpha+\mu)}\sqrt{1-\frac{\beta_1}{2}})^T : \alpha, \omega, \mu \text{ arbitrary}\}$$

Proof. Without loss of generality, we can assume that

$$u_1 = (x_1, e^{-i\theta}\sqrt{1-x_1^2})^T, u_2 = (x_2, e^{-i\gamma}\sqrt{1-x_2^2})^T$$

where $x_1, x_2 \geq 0$. Therefore, $x_1^2 + x_2^2 = \beta_1, e^{i\theta}x_1\sqrt{1-x_1^2} + e^{i\gamma}x_2\sqrt{1-x_2^2} = 0$. So $|e^{i\theta}x_1\sqrt{1-x_1^2}| = |e^{i\gamma}x_2\sqrt{1-x_2^2}|$ and hence, $x_1\sqrt{1-x_1^2} = x_2\sqrt{1-x_2^2}$. Since $x_2 = \sqrt{\beta_1 - x_1^2}$, we have $x_1\sqrt{1-x_1^2} = \sqrt{\beta_1 - x_1^2}\sqrt{1-\beta_1+x_1^2}$. By squaring both sides, we have $x_1^2(1-x_1^2) = (\beta_1 - x_1^2)(1-\beta_1+x_1^2)$ which implies that $x_1 = \sqrt{\frac{\beta_1}{2}}$. Therefore, $x_2 = \sqrt{\frac{\beta_1}{2}}$ and $e^{i\theta}\sqrt{\frac{\beta_1}{2}}\sqrt{1-\frac{\beta_1}{2}} + e^{i\gamma}\sqrt{\frac{\beta_1}{2}}\sqrt{1-\frac{\beta_1}{2}} = 0$. Since $0 < \beta_1 < 2$, we have $e^{i\theta} + e^{i\gamma} = 0$. It follows that $\theta = \gamma + \Pi$. So $u_1 = (\sqrt{\frac{\beta_1}{2}}, e^{-i\theta}\sqrt{1-\frac{\beta_1}{2}})^T, u_2 = (\sqrt{\frac{\beta_1}{2}}, -e^{-i\theta}\sqrt{1-\frac{\beta_1}{2}})^T$ where θ is any angle. By Lemma II.8, if $v_1, v_2 \in \mathbb{C}^2$ are unit vectors such that $v_1 \otimes v_1 + v_2 \otimes v_2 = \tilde{B}$ then there exist scalars $e^{i\nu_1}, e^{i\nu_2}$, a permutation Π of $\{1, 2\}$ and a unitary 2×2 matrix U which commutes with $\tilde{B}$ such that $v_j = e^{i\nu_j}Uu_{\Pi(j)}$. Since unitary matrix U commutes with the diagonal matrix $\tilde{B}$, U must be diagonal as well. Therefore, the result follows immediately. □

Now we will prove some general lemmas.

Lemma II.11. Suppose B is a positive operator with rank n and u is unit vector in H_n such that $B - u \otimes u$ has rank $n-1$. Then there is a unit vector $x \in H_n$ such that $\|Bx\| = \|B^{1/2}x\|$.

Proof. Since $B-u\otimes u$ is singular, there is a unit vector x such that $(B-u\otimes u)x=0$ which implies $Bx = \langle\, x\,,\, u\,\rangle u$. Since B has full rank, $Bx \neq 0$. So $\langle\, x\,,\, u\,\rangle \neq 0$ and $u = \frac{1}{\langle\, x\,,\, u\,\rangle}Bx$. Let $\mu = \langle\, x\,,\, u\,\rangle$. Hence, $u = \frac{1}{\mu}Bx$ and $Bx = \langle\, x\,,\, u\,\rangle u = \langle\, x\,,\, \frac{1}{\mu}Bx\,\rangle \frac{1}{\mu}Bx = \frac{1}{|\mu|^2}\langle\, x\,,\, Bx\,\rangle Bx$. It implies that

$$|\mu| = \sqrt{\langle\, Bx\,,\, x\,\rangle} = \sqrt{\langle\, B^{1/2}B^{1/2}x\,,\, x\,\rangle} = \sqrt{\langle\, B^{1/2}x\,,\, B^{1/2}x\,\rangle} = \|B^{1/2}x\|$$

Since $\|u\| = 1$, we have $||Bx|| = |\mu|$and hence, $\|Bx\| = \|B^{1/2}x\|$. □

Lemma II.12. i) Suppose B is a positive operator with rank n and u is a unit vector in $\mathrm{ran}(B)$. Then $B-u\otimes u$ is singular if and only if $\|B^{-1/2}u\| = 1$ where the inverse is taken on $\mathrm{ran}(B)$.

ii) Suppose $\{u_j\}_{j=1}^n$ are linearly independent unit vectors and $B = \sum_{j=1}^n u_j \otimes u_j$. Then $u_j \in \mathcal{S}_1 \cap B^{1/2}(\mathcal{S}_1)$ where $\mathcal{S}_1$ is the unit sphere in H_n.

Proof. i) For the forward direction, by the proof of Lemma II.8, we have $Bx = \langle\, x\,,\, u\,\rangle u$ and therefore, $x = \langle\, x\,,\, u\,\rangle B^{-1}u$ which implies that

$$\langle\, x\,,\, u\,\rangle = \langle\, x\,,\, u\,\rangle\langle\, B^{-1}u\,,\, u\,\rangle = \langle\, x\,,\, u\,\rangle\|B^{-1/2}u\|^2$$

Since $\langle\, x\,,\, u\,\rangle \neq 0$, $\|B^{-1/2}u\| = 1$. For the backward direction, suppose that $\|u\| = 1$ and $\|B^{-1/2}u\| = 1$. Then

$$(B-u\otimes u)B^{-1}u = u-(u\otimes u)B^{-1}u = (1-\langle\, B^{-1}u\,,\, u\,\rangle)u = (1-\langle\, B^{-1/2}u\,,\, B^{-1/2}u\,\rangle)u = 0$$

Hence, $(B-u\otimes u)B^{-1}u = 0$ and $B-u\otimes u$ is singular.

ii) This follows directly from part i). □

Lemma II.13. Suppose F is a unit norm tight frame in $\mathbb{R}^n$ and A is a non-orthogonal subset of F with cardinality 2. Then A is rigid.

Proof. Suppose $F = \{x_j\}_{j=1}^k$ is a unit norm tight frame and $A = \{x_1, x_2\}$ is a non-orthogonal set which can be replaced by $A' = \{y_1, y_2\}$ such that $\{y_1, y_2, x_3, ..., x_k\}$ is also a unit norm tight frame. By comparing the frame operators for the original and the replaced tight frame, we get $x_1 \otimes x_1 + x_2 \otimes x_2 = y_1 \otimes y_1 + y_2 \otimes y_2$. Let $B = x_1 \otimes x_1 + x_2 \otimes x_2$. If x_1, x_2 are linearly dependent then $x_2 = wx_1$ for a scalar w of modulus 1, so $x_2 \otimes x_2 = x_1 \otimes x_1$. It follows that $y_1 \otimes y_1 + y_2 \otimes y_2 = 2x_1 \otimes x_1$ which is rank-1 operator. Since $y_1 \otimes y_1 + y_2 \otimes y_2 \geq 0$, y_1, y_2 are in its range. So $y_1 = w_1 x_1, y_2 = w_2 x_1$ where w_1, w_2 are scalars of modulus 1. Hence A is rigid. If x_1, x_2 are linearly independent then so are y_1, y_2. By Lemma II.12, $x_1, x_2, y_1, y_2 \in \mathcal{S}_1 \cap B^{1/2}(\mathcal{S}_1)$. This is the intersection of a circle of radius 1 and an ellipse centered at 0 in a two dimensional real space. So there are vectors p_1, p_2 such that $\{x_1, x_2, y_1, y_2\} = \{\pm p_1, \pm p_2\}$. It implies that A is rigid. $\square$

Lemma II.14. Let $F = \{x_j\}_{j \in \mathbb{J}}$ be a tight frame for $\mathbb{C}^n$. Then any subset of 2 non-orthogonal linearly independent vectors is non-rigid.

Proof. Let A be any subset of 2 non-orthogonal linearly independent vectors, say, $A = \{x_{j_1}, x_{j_2}\}$ and B be the Bessel operator, $B = x_{j_1} \otimes x_{j_1} + x_{j_2} \otimes x_{j_2}$. Suppose U is a unitary operator in $\mathcal{B}(H)$ that commutes with B. Let $y_{j_1} = Ux_{j_1}, y_{j_2} = Ux_{j_2}$. Then

$$\begin{aligned} y_{j_1} \otimes y_{j_1} + y_{j_2} \otimes y_{j_2} &= Ux_{j_1} \otimes Ux_{j_1} + Ux_{j_2} \otimes Ux_{j_2} \\ &= U(x_{j_1} \otimes x_{j_1})U^* + U(x_{j_2} \otimes x_{j_2})U^* \\ &= UBU^* = B \end{aligned}$$

So $\{y_{j_1}, y_{j_2}\}$ is a replacement set for $\{x_{j_1}, x_{j_2}\}$. Suppose

$$B = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to an orthonomal basis $\{g_1, ..., g_n\}$. So $B = a_1 g_1 \otimes g_1 + a_2 g_2 \otimes g_2 = x_{j_1} \otimes x_{j_1} + x_{j_2} \otimes x_{j_2}$ and therefore, $x_{j_1}, x_{j_2} \in \text{span}\{g_1, g_2\}$. Write $x_{j_1} = (\alpha_1, \alpha_2, 0, ..., 0)^T, x_{j_2} = (\beta_1, \beta_2, 0, ..., 0)^T$. Since $\{x_{j_1}, x_{j_2}\}$ are non-orthogonal, we have $a_1 \neq a_2$. Then the commutant

$$\begin{aligned} \{B\}' &= \{C \in \mathcal{B}(H) : CB = BC\} \\ &= \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & D \end{pmatrix} : d_1, d_2 \in \mathbb{C}, \text{D is an arbitrary matrix} \right\} \end{aligned}$$

If $U \in \{B\}'$ then $Ux_{j_1} = (d_1\alpha_1, d_2\alpha_2, 0, ..., 0)^T, Ux_{j_2} = (d_1\beta_1, d_2\beta_2, 0, ..., 0)^T$.

It is easy to check that for any complex numbers $d_1 \neq \pm d_2$ of modulus 1, the set $\{(d_1\alpha_1, d_2\alpha_2, 0, ..., 0)^T, (d_1\beta_1, d_2\beta_2, 0, ..., 0)^T\}$ is not a permutation with perhaps scalar multiples of modulus 1 of the original set A. Thus, we can replace $\{x_{j_1}, x_{j_2}\}$ with $\{Ux_{j_1}, Ux_{j_2}\}$ such that a new sequence is also tight frame.

□

Proposition II.15. Suppose $F = \{x_j\}_{j=1}^k$ is a unit tight frame in H_n and A is a subset of F consisting of 3 vectors or more. Then A is not rigid except when the dimension of the space spanned by A is 1.

Proof. In order to prove that $A = \{x_1, x_2, x_3\}$ is not rigid, we will find a set $A' = \{y_1, y_2, y_3\}$ different from A such that $x_1 \otimes x_1 + x_2 \otimes x_2 + x_3 \otimes x_3 = y_1 \otimes y_1 + y_2 \otimes y_2 + y_3 \otimes y_3$. First consider the case A is a linear independent set. Then $B = x_1 \otimes x_1 + x_2 \otimes x_2 + x_3 \otimes x_3$

is a positive operator with full rank 3. Let y_1 be any element in

$$(\mathcal{S}_1 \cap B^{1/2}(\mathcal{S}_1)) \setminus \{d_1x_1, d_2x_2, d_3x_3 : d_j \in \mathbb{C}, |d_j| = 1, j = 1, 2, 3\}.$$

By Lemma II.12, $B - y_1 \otimes y_1$ is singular. Moreover, $B - y_1 \otimes y_1$ is positive. Indeed, assume $y_1 = B^{1/2}u$ where $u \in \mathcal{S}_1$. Therefore, for any $x \in \mathbb{R}^n$, we have $\langle (B - y_1 \otimes y_1)x\, ,\, x \rangle = \langle Bx\, ,\, x \rangle - |\langle x\, ,\, y_1 \rangle|^2 = \|B^{1/2}x\|^2 - |\langle x\, ,\, B^{1/2}u \rangle|^2 = \|B^{1/2}x\|^2 - |\langle B^{1/2}x\, ,\, u \rangle|^2$ but $|\langle B^{1/2}x\, ,\, u \rangle| \leq \|B^{1/2}x\|\|u\| = \|B^{1/2}x\|$. So $B - y_1 \otimes y_1$ is positive. Since $B - y_1 \otimes y_1$ is positive with rank 2 and trace 2, there exist unit vectors y_2, y_3 such that $B - y_1 \otimes y_1 = y_2 \otimes y_2 + y_3 \otimes y_3$. Hence in this case A is not rigid. Now assume that A spanning two dimensional space H_2. Select an orthonormal basis e_1, e_2 for H_2 such that B can be written as a diagonal matrix with positive entries $\lambda_1 \geq \lambda_2 > 0$. Since trace$(B) = 3$, we have $\lambda_1 + \lambda_2 = 3$ and $\lambda_1 \geq 3/2$. Let $y_1 = e_1$. So $B - y_1 \otimes y_1$ is positive operator with rank 2 and trace 2. As before, there exist unit vectors y_2, y_3 such that $B - y_1 \otimes y_1 = y_2 \otimes y_2 + y_3 \otimes y_3$. If $d_1e_1 \notin A$ for any scalar d_1 of modulus 1 then $\{y_j\}_{j=1}^3$ is a replacement set which is not a permutation with perhaps scalar multiples of modulus 1 of A and so A is not rigid. If $d_1e_1 \in A$ for some scalar d_1 of modulus 1 then let l be a positive number such that $B - e_1 \otimes e_1 \geq lI$, and let $\beta > 0$ be any nonzero real number such that for the vector $y = \sqrt{1 - \beta^2}e_1 + \beta e_2$, we have $||y \otimes y - e_1 \otimes e_1|| < l$ and no scalar multiple of y is contained in A. Then $B - y \otimes y = (B - e_1 \otimes e_1) + (e_1 \otimes e_1 - y \otimes y) \geq lI + (e_1 \otimes e_1 - y \otimes y) \geq 0$ because $e_1 \otimes e_1 - y \otimes y$ is a self-adjoint operator of norm less than l. Let $y_1 = y$. So as above, there are unit vectors y_2, y_3 such that $B - y_1 \otimes y_1 = y_2 \otimes y_2 + y_3 \otimes y_3$.

□

4. Equiangular replacement

Equiangular tight frames have applications in signal processing, communications and coding theory [24]. Recent literature for equiangular frames includes [16],[19],[25],[26].

We show that we can always replace a set of three unit vectors with a set of three complex unit equiangular vectors which has the same Bessel operator as the Bessel operator of the original set. We show that this can not always be done if we require the replacement vectors to be real, even if the original vectors are real.

Proposition II.16. Let $F = \{x_j\}_{j=1}^k$ be a unit norm tight frame in $\mathbb{R}^n$, $k \geq 3$. Let $A \subset F$ with cardinality 3. Let B be the Bessel operator for A, that is, $B = \sum\{x \otimes x \ : x \in A\}$. If two eigenvalues are equal then we can replace A with an equiangular set of 3 unit vectors. The converse direction is also true: If A can be replaced by an equiangular set of three "real" unit vectors, then B must have two equal eigenvectors.

[Note: in the Proposition II.17 we will show that this two equal eigenvalue restriction can be removed by using complex unit vectors.]

Proof. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of B. Since B is positive trace 3, we have $\lambda_1, \lambda_2, \lambda_3 \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 3$. Since two eigenvalues of B are equal, we can assume that $\lambda_1 = \lambda_2 = 1+\alpha, \lambda_3 = 1-2\alpha$ where $0 \leq \alpha \leq 1/2$ or $-1 < \alpha \leq 0$. Let $x_1 = (1,0,0)^T, x_2 = (\alpha, \sqrt{1-\alpha^2}, 0)^T, x_3 = (\alpha, -\frac{\alpha\sqrt{1+\alpha}}{\sqrt{1-\alpha}}, \frac{\sqrt{(1+\alpha)(1-2\alpha)}}{\sqrt{(1-\alpha)}})^T$. We can check that the operator $\sum_{j=1}^3 (x_j \otimes x_j)$ has eigenvalues the same as eigenvalues of B. Therefore, there is a unitary operator U such that $B = U \sum_{j=1}^3 (x_j \otimes x_j) U^* = \sum_{j=1}^3 (Ux_j \otimes Ux_j)$. Since the set $\{x_1, x_2, x_3\}$ is equiangular, so is $\{Ux_1, Ux_2, Ux_3\}$. Therefore, we can replace A with an equiangular set of 3 unit vectors $\{Ux_1, Ux_2, Ux_3\}$. For the converse direction, suppose $B = \sum_{j=1}^3 (x_j \otimes x_j)$ where x_1, x_2, x_3 are unit vectors and there is a constant c such that $|\langle x_k , x_l \rangle| = c$ for $k \neq l$. Then the Grammian matrix G is of the form

$$G = \begin{pmatrix} 1 & \pm c & \pm c \\ \pm c & 1 & \pm c \\ \pm c & \pm c & 1 \end{pmatrix}$$

which has two eigenvalues the same. Since the eigenvalues of B are equal to the eigenvalues of G, two eigenvalues of B are equal.

□

For the complex case, we will prove that we can always replace any subset of three vectors in a unit norm tight frame with an equiangular set of three unit vectors such that the resulting sequence is also a unit norm tight frame. Moreover, we will give a formula to calculate replacement vectors from the eigenvalues of the Bessel operator for the original subset. First we will prove a general result.

Proposition II.17. Let B be a positive operator of trace 3 with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$. Then there is an equiangular set of three complex unit vectors u_1, u_2, u_3 such that $B = \sum_{j=1}^{3}(u_j \otimes u_j)$

Proof. Suppose that

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

with respect to an orthonomal basis e_1, e_2, e_3 for $\mathbb{C}^3$. So $\sum_{j=1}^{3} \lambda_j = 3$.

$$\begin{aligned} 9 &= (\lambda_1 + \lambda_2 + \lambda_3)^2 \\ &= (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + (2\lambda_1\lambda_2 + 2\lambda_1\lambda_3 + 2\lambda_2\lambda_3) \\ &\geq (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + (2\lambda_1\lambda_2 + 2\lambda_1\lambda_3 + 2\lambda_2\lambda_3) \\ &= 3\lambda_1\lambda_2 + 3\lambda_1\lambda_3 + 3\lambda_2\lambda_3 \end{aligned}$$

So $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \leq 3$. Let

$$c = \sqrt{\frac{3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3}{3}}$$

and

$$M = \frac{\lambda_1\lambda_2\lambda_3 + 2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3}{2}$$

Obviously that $c \in [0, 1)$. We will prove that $|M| \leq c^3$. Indeed,

$$\begin{aligned} 3c^2(1-\lambda_1) - (1-\lambda_1)^3 &= (3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3)(1-\lambda_1) - (1-\lambda_1)^3 \\ &= 2 + \lambda_1\lambda_2\lambda_3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 + \lambda_1^2(\lambda_2+\lambda_3) - 3\lambda_1^2 + \lambda_1^3 \\ &= 2 + \lambda_1\lambda_2\lambda_3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 + \lambda_1^2(3-\lambda_1) - 3\lambda_1^2 + \lambda_1^3 \\ &= 2 + \lambda_1\lambda_2\lambda_3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 \\ &= 2M \end{aligned}$$

Also it is easy to check that

$$\begin{aligned} (1-\lambda_1)^2 + (1-\lambda_1)(1-\lambda_2) + (1-\lambda_2)^2 &= (1-\lambda_1)^2 + (1-\lambda_1)(1-\lambda_3) + (1-\lambda_3)^2 \\ &= (1-\lambda_2)^2 + (1-\lambda_2)(1-\lambda_3) + (1-\lambda_3)^2 \\ &= 3c^2 \end{aligned}$$

Therefore, $(1-\lambda_1)^3 - 3c^2(1-\lambda_1) = (1-\lambda_2)^3 - 3c^2(1-\lambda_2) = (1-\lambda_1)^3 - 3c^2(1-\lambda_1)$ which is called m. So $m = -2M$. There are two posibilities: 1) $0 \leq \lambda_1 \leq \lambda_2 \leq 1, \lambda_3 \geq 1$. We have $3(1-\lambda_2)^2 \leq (1-\lambda_1)^2 + (1-\lambda_1)(1-\lambda_2) + (1-\lambda_2)^2 = 3c^2 \leq 3(1-\lambda_1)^2$. So $1 - \lambda_1 \geq c,\ 0 \leq 1 - \lambda_2 \leq c$. Then $m = (1-\lambda_2)[(1-\lambda_2)^2 - 3c^2] \leq 0$. Let $x = 1 - \lambda_1$. Then $c \leq x \leq 1$. So $m(x) = x^3 - 3c^2x$ attains a minimum $-2c^3$ at $x = c$. Since $m(1) = 1 - 3c^2 \geq -2c^3$ for all $c \in [0, 1]$, we have $-2c^3 \leq m \leq 0$. So $|M| \leq c^3$.

2) $\lambda_1 \leq 1 \leq \lambda_2 \leq \lambda_3$.

We have $3(1-\lambda_2)^2 \leq (1-\lambda_2)^2 + (1-\lambda_2)(1-\lambda_3) + (1-\lambda_3)^2 = 3c^2 \leq 3(1-\lambda_3)^2$. So

$-c \leq 1-\lambda_2 \leq 0, 1-\lambda_3 \leq -c$. Then $m = (1-\lambda_2)[(1-\lambda_2)^2 - 3c^2] \geq 0$. Let $x = 1-\lambda_3$. Then $-1 \leq x \leq -c$. So $m(x) = x^3 - 3c^2x$ attains a maximum $2c^3$ at $x = -c$. Since $m(-1) = -1 + 3c^2 \leq 2c^3$ for all $c \in [0,1]$, we have $0 \leq m \leq 2c^3$. So $|M| \leq c^3$.

Now let ω be an angle such that $\cos\omega = \frac{M}{c^3}$ and let

$$u_1 = (1,0,0)^T, u_2 = (c\ e^{-i\omega}, \sqrt{1-c^2}, 0)^T$$

$$u_3 = (c\ e^{-i\omega}, \frac{ce^{i\omega} - c^2}{\sqrt{1-c^2}}, \frac{\sqrt{(1-c^2)^2 - |ce^{i\omega} - c^2|^2}}{\sqrt{1-c^2}})^T$$

Then $\langle\, u_1\,,\, u_2\,\rangle = \langle\, u_1\,,\, u_3\,\rangle = \overline{\langle\, u_2\,,\, u_3\,\rangle} = c\ e^{i\omega}$. So $\{u_1, u_2, u_3\}$ is an equiangular set of three unit vectors. The Grammian matrix for $\{u_i\}_{i=1}^3$ is

$$G = \begin{pmatrix} 1 & c\ e^{i\omega} & c\ e^{i\omega} \\ c\ e^{-i\omega} & 1 & c\ e^{-i\omega} \\ c\ e^{-i\omega} & c\ e^{i\omega} & 1 \end{pmatrix}$$

Then G has the characteristic polynomial

$$\begin{aligned} (1-\lambda)^3 - 3c^2(1-\lambda) + 2c^3\cos\omega &= (1-\lambda)^3 - 3c^2(1-\lambda) + 2M \\ &= -\lambda^3 + 3\lambda^2 - (3-3c^2)\lambda - 3c^2 + 1 + 2M \\ &= -\lambda^3 + 3\lambda^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda + \lambda_1\lambda_2\lambda_3 \\ &= -(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) \end{aligned}$$

So G has eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$. Since the eigenvalues of $\sum_{j=1}^3 (u_j \otimes u_j)$ are equal to the eigenvalues of B, as the proof in the previous proposition, B is the Bessel operator of an equiangular set of three unit vectors. □

We immediately obtain the following result.

Corollary II.18. Let F be a unit norm tight frame of k vectors ($k \geq 3$) in $\mathbb{C}^n$ and A be any subset of F consisting of three vectors. Then we can replace A with an equiangular set of three unit vectors such that the new sequence is also unit norm tight frame.

Remark 3. Suppose $\{x_1, x_2, x_3\}$ are three unit vectors in $\mathbb{C}^n$ forming an equiangular set. We can find scalars of modulus one d_1, d_2, d_3 such that $\langle\, d_1x_1\,,\, d_2x_2\,\rangle = \langle\, d_1x_1\,,\, d_3x_3\,\rangle = \langle\, d_2x_2\,,\, d_3x_3\,\rangle$. More precisely, if $\langle\, x_1\,,\, x_2\,\rangle = ae^{i\theta_1}, \langle\, x_2\,,\, x_3\,\rangle = ae^{i\theta_2}, \langle\, x_1\,,\, x_3\,\rangle = ae^{i\theta_3}$ then $d_1 = e^{i(\theta_1+\theta_2-2\theta_3)}, d_2 = e^{i(\theta_1-\theta_3)}, d_3 = 1$. This is not necessarily true if the set has more than three vectors in $\mathbb{C}^n$ and if $\{x_1, x_2, x_3\}$ are three unit vectors in $\mathbb{R}^n, n \geq 2$. For example, let $x_1 = (1, 0, 0)^T, x_2 = \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}, 0\right)^T, x_3 = \left(\frac{\sqrt{5}}{5}, \frac{1}{2} - \frac{\sqrt{5}}{10}, \sqrt{\frac{5+\sqrt{5}}{10}}\right)^T, x_4 = \left(\frac{\sqrt{5}}{5}, -\frac{1}{2} - \frac{\sqrt{5}}{10}, \sqrt{\frac{5-\sqrt{5}}{10}}\right)^T$.We can check that $||x_1|| = ||x_2|| = ||x_3|| = ||x_4|| = 1$ and $\langle\, x_1\,,\, x_2\,\rangle = \langle\, x_1\,,\, x_3\,\rangle = \langle\, x_1\,,\, x_4\,\rangle = \langle\, x_2\,,\, x_3\,\rangle = \langle\, x_3\,,\, x_4\,\rangle = -\langle\, x_2\,,\, x_4\,\rangle = \frac{\sqrt{5}}{5}$. So $\{x_1, x_2, x_3, x_4\}$ is equiangular set in $\mathbb{R}^3$ but we can not rescale so that $\{x_1, x_2, x_3, x_4\}$ is strictly equiangular in $\mathbb{R}^3$.

Any set of 3 unit vectors $\{x_1, x_2, x_3\}$ which is not equiangular is not stable. Indeed, by Corollary II.18, there exists $\{x_1', x_2', x_3'\}$ which is equiangular and $\sum_{j=1}^3 x_j' \otimes x_j' = \sum_{j=1}^3 x_j \otimes x_j$. The stability implies that there are scalars $\{d_1, d_2, d_3\}$ of modulus 1, a permutation Π of $\{1, 2, 3\}$ and a unitary U which commutes with $\sum_{j=1}^3 x_j \otimes x_j$ such that $y_j = d_jUx_{\Pi(j)}$ for all $j = 1, 2, 3$. Hence $|\langle\, x_j\,,\, x_l\,\rangle| = |\langle\, x'_{\Pi(j)}\,,\, x'_{\Pi(l)}\,\rangle|$ for $j \neq l$. This leads to a contradiction since $\{x_1', x_2', x_3'\}$ is equiangular but $\{x_1, x_2, x_3\}$ is not.

We will characterize all equiangular sets of three unit vectors which have the same Bessel operator.

Lemma II.19. Suppose $X = \{x_j\}_{j=1}^3$ and $Y = \{y_j\}_{j=1}^3$ are two sets of unit norm vectors in $\mathbb{C}^n$. If X and Y are equiangular sets with the same Bessel opeators $B_X = B_Y$ then X and Y are geometrictly equivalent.

Proof. For the forward direction, by remark 2, we can assume that $\langle\, x_1\,,\, x_2\,\rangle =$

$\langle x_1 , x_3 \rangle = \langle x_2 , x_3 \rangle = a$ and $\langle y_1 , y_2 \rangle = \langle y_1 , y_3 \rangle = \langle y_2 , y_3 \rangle = b$ for some complex numbers a, b. Since the eigenvalues of the Grammian operator and Bessel operator are the same and $B_X = B_Y$, the eigenvalues of the Grammian operators G_X and G_Y are the same.

If three eigenvalues of the Grammian operator G_X are the same then $B_X = B_Y = I$ and so X and Y are orthonormal bases. Hence, X and Y are geometrically equivalent.

Now assume that at least two eigenvalues of G_X are different. The characteristics polynomial of G_X is

$$(1-\lambda)^3 - 3|a|^2(1-\lambda) + 2|a|^2\mathcal{R}e(a) = 0$$

and the characteristics polynomial of G_Y is

$$(1-\lambda)^3 - 3|b|^2(1-\lambda) + 2|b|^2\mathcal{R}e(b) = 0$$

Since these characteristics polynomials are the same, the following equation

$$(1-\lambda)(3|b|^2 - 3|a|^2) + 2|a|^2\mathcal{R}e(a) - 2|b|^2\mathcal{R}e(b) = 0$$

has at least two solutions. It follows that $3|b|^2 - 3|a|^2 = 0$ and $2|a|^2\mathcal{R}e(a) - 2|b|^2\mathcal{R}e(b) = 0$. Hence, $|a| = |b|$ and $\mathcal{R}e(a) = \mathcal{R}e(b)$ which imply that either $a = b$ or $a = \bar{b}$.

If $a = b$ then $G_X = G_Y$ and therefore X and Y are unitarily equivalent by lemma I.4. Then there is some unitary operator U such that $x_j = Uy_j, j = 1, 2, 3$. We have $\sum_{j=1}^3 y_j \otimes y_j = \sum_{j=1}^3 x_j \otimes x_j = U(\sum_{j=1}^3 y_j \otimes y_j)U^*$. So U commutes with $\sum_{j=1}^3 x_j \otimes x_j$.

If $a = \bar{b}$ then $G_{\{x_3,x_2,x_1\}} = G_{\{y_1,y_2,y_3\}}$. So $\{x_3, x_2, x_1\}$ and $\{y_1, y_2, y_3\}$ are unitarily equivalent. Then there is some unitary operator U such that $x_3 = Uy_1, x_2 = Uy_2, x_1 = Uy_3$. We have $\sum_{j=1}^3 y_j \otimes y_j = \sum_{j=1}^3 x_j \otimes x_j = U(\sum_{j=1}^3 y_j \otimes y_j)U^*$. So U commutes with $\sum_{j=1}^{3} x_j \otimes x_j$.

$\square$

We wish to characterize all sets of three linearly independent, unit vectors whose Bessel operator is equal to a given positive invertible operator B.

Lemma II.20. Suppose B is a positive invertible operator in $\mathcal{B}(H)$. Let $\mathcal{F}_B = \{C \in \mathcal{B}(H) : CBC^* = B\}$. Then $\mathcal{F}_B = \{B^{1/2}UB^{-1/2}$: U is an arbitrary unitary operator$\}$

Proof. Let $T \in \mathcal{F}_B$ and let $A = TB^{1/2}$. Then $A^* = B^{1/2}T^*$ and $AA^* = B$. So $|A^*| = (AA^*)^{1/2} = B^{1/2}$. By polar decomposition $A^* = U|A^*| = UB^{1/2}$ where U is an unitary operator. Hence, $U = A^*B^{-1/2} = B^{1/2}T^*B^{-1/2}$ and $T^* = B^{-1/2}UB^{1/2}$ which implies that $T = B^{1/2}U^*B^{-1/2}$. Now for any unitary operator U, we have

$$(B^{1/2}UB^{-1/2})B(B^{-1/2}U^*B^{1/2}) = B$$

Therefore, $B^{1/2}UB^{-1/2} \in \mathcal{F}_B$. $\square$

Corollary II.21. Suppose that $\{x_j\}_{j=1}^{3}$ are unit vectors in $\mathbb{C}^3$ which are linearly independent and $B = \sum_{j=1}^{3} x_j \otimes x_j$. Suppose that $\{y_j\}_{j=1}^{3}$ are unit vectors in $\mathbb{C}^3$. Then $\sum_{j=1}^{3} y_j \otimes y_j = B$ if and only if $\{y_j\}_{j=1}^{3} = \{B^{1/2}UB^{-1/2}x_j\}_{j=1}^{3}$ where U is some unitary operator from $\mathbb{C}^3$ to $\mathbb{C}^3$.

Proof. For the "only if " part, since $\{x_j\}_{j=1}^{3}$ are unit vectors which are linearly independent, $\{y_j\}_{j=1}^{3}$ are linearly independent as well. Then we can define uniquely a map $T : \mathbb{C}^3 \to \mathbb{C}^3$ such that $y_j = T(x_j)$ for $j = 1, 2, 3$. Then $B = \sum_{j=1}^{3} y_j \otimes y_j = T(\sum_{j=1}^{3} x_j \otimes x_j)T^* = TBT^*$. Therefore, $T \in \mathcal{S}_B$ and by lemma II.20, $T = B^{1/2}UB^{-1/2}$ for some unitary operator U. Thus, $y_j = B^{1/2}UB^{-1/2}x_j$ for $j = 1, 2, 3$.

For the "if " part, if $y_j = B^{1/2}UB^{-1/2}x_j$ for $j = 1, 2, 3$ then

$$\begin{aligned}
\sum_{j=1}^{3} y_j \otimes y_j &= \sum_{j=1}^{3} B^{1/2}UB^{-1/2}x_j \otimes B^{1/2}UB^{-1/2}x_j \\
&= (B^{1/2}UB^{-1/2})(\sum_{j=1}^{3} x_j \otimes x_j)(B^{1/2}UB^{-1/2})^* \\
&= (B^{1/2}UB^{-1/2})B(B^{-1/2}U^*B^{1/2}) \\
&= B^{1/2}U(B^{-1/2}BB^{-1/2})U^*B^{1/2} \\
&= B^{1/2}UU^*B^{1/2} = B
\end{aligned}$$

□

Lemma II.22. Let $\tilde{x_j} = (\tilde{x}_{j1}, \tilde{x}_{j2}, \tilde{x}_{j3})^T$ be a set of three non-zero linearly independent vectors. Suppose that some off-diagonal element of $\tilde{A} = \sum_{j=1}^{3} x_j \otimes x_j$ is zero. Suppose also that the matrix

$$M = \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{21} & \tilde{x}_{31} \\ \tilde{x}_{12} & \tilde{x}_{22} & \tilde{x}_{32} \\ \tilde{x}_{13} & \tilde{x}_{23} & \tilde{x}_{33} \end{pmatrix}$$

has the property that every row and column contains a zero element. Then either $\tilde{x_1} \perp \tilde{x_2}, \tilde{x_1} \perp \tilde{x_3}$ or $\tilde{x_2} \perp \tilde{x_3}$.

Proof. Suppose that no pair in $\{\tilde{x_1}, \tilde{x_2}, \tilde{x_3}\}$ is orthogonal. Since $\tilde{x_j} \neq 0$ for all $j = 1, 2, 3$, no column contains all 0. Since $\tilde{x_j}$ are linearly independent, no row contains all 0. Since $\tilde{x_1}, \tilde{x_2}$ are not orthogonal, at least one of the pairs of numbers $\{\tilde{x}_{1j}, \tilde{x}_{2j}\}$ is a nonzero pair (that is, both numbers are nonzero) for some $j \in \{1, 2, 3\}$. Similarly, at least one of the pairs of numbers $\{\tilde{x}_{2k}, \tilde{x}_{3k}\}$ is a nonzero pair for some $k \in \{1, 2, 3\}$ and at least one of the pairs of numbers $\{\tilde{x}_{1l}, \tilde{x}_{3l}\}$ is a nonzero pair for some $l \in \{1, 2, 3\}$. Since each row has a zero element, we have $j \neq k \neq l$. By permutting the orthonormal basis for the representation, without loss of generality, we can assume that $j = 1, k = 2, l = 3$. Thus, $\tilde{x}_{11}, \tilde{x}_{21}, \tilde{x}_{22}, \tilde{x}_{32}, \tilde{x}_{13}, \tilde{x}_{33}$ are nonzero. Again, since each row has a zero element, $\tilde{x}_{31} = \tilde{x}_{12} = \tilde{x}_{23} = 0$. We have

$$\begin{aligned} \tilde{A} &= \sum_{j=1}^{3} \tilde{x}_j \otimes \tilde{x}_j \\ &= \begin{pmatrix} |\tilde{x}_{11}|^2 & \sum_{j=1}^{3} \tilde{x}_{j1}\overline{\tilde{x}_{j2}} & \sum_{j=1}^{3} \tilde{x}_{j1}\overline{\tilde{x}_{j3}} \\ \sum_{j=1}^{3} \tilde{x}_{j2}\overline{\tilde{x}_{j1}} & |\tilde{x}_{22}|^2 & \sum_{j=1}^{3} \tilde{x}_{j2}\overline{\tilde{x}_{j3}} \\ \sum_{j=1}^{3} \tilde{x}_{j3}\overline{\tilde{x}_{j1}} & \sum_{j=1}^{3} \tilde{x}_{j2}\overline{\tilde{x}_{j1}} & |\tilde{x}_{33}|^2 \end{pmatrix} \end{aligned}$$

By hypothesis, some off-diagonal element of $\tilde{A}$ is zero, say, $\tilde{A}_{12} = 0$. Then $\sum_{j=1}^{3} \tilde{x}_{j1}\overline{\tilde{x}_{j2}} = 0$. From $\tilde{x}_{31} = \tilde{x}_{12} = 0$, we have $\tilde{x}_{21}\overline{\tilde{x}_{22}} = 0$. Therefore, either $\tilde{x}_{21} = 0$ or $\tilde{x}_{22} = 0$, a contradiction. Hence, some pair in $\{\tilde{x}_j\}_{j=1}^{3}$ must be orthogonal.

A similar argument shows that if any $\tilde{A}_{jl} = 0$ for $j \neq l$, then some pair in $\{\tilde{x}_j\}_{j=1}^{3}$ must be orthogonal. $\square$

Lemma II.23. The intersection $\mathcal{S}_{\{a,b,c\}}$ of an ellipsoid $\mathcal{E}_{\{a,b,c\}} = \{(x, y, z)^T \in \mathbb{C}^3 : \frac{|x|^2}{a} + \frac{|y|^2}{b} + \frac{|z|^2}{c} = 1\}$ and the unit sphere $\mathcal{S} = \{(x, y, z)^T \in \mathbb{C}^3 : |x|^2 + |y|^2 + |z|^2 = 1\}$ is connected if it is nonempty where $a > b > c > 0$.

Proof. If $a > b > c > 1$ then $\frac{|x|^2}{a} + \frac{|y|^2}{b} + \frac{|z|^2}{c} \leq |x|^2 + |y|^2 + |z|^2$ and the equality holds only if $(x, y, z)^T = 0 \notin \mathcal{S}$. So $\mathcal{S}_{\{a,b,c\}} = \emptyset$. Similarly, if $1 > a > b > c > 0$ then $\mathcal{S}_{\{a,b,c\}} = \emptyset$ as well. So $a \geq 1$ and $c \leq 1$. Note that if $(x, y, z)^T \in \mathcal{S}_{\{a,b,c\}}$ then $(|x|, |y|, |z|)^T \in \mathcal{S}_{\{a,b,c\}}$. Suppose $x = |x|e^{i2\Pi\alpha}, y = |y|e^{i2\Pi\beta}, z = |z|e^{i2\Pi\gamma}$ where $0 \leq \alpha, \beta, \gamma < 1$. Then $\nu(t) = (xe^{-i2\Pi\alpha t}, ye^{-i2\Pi\beta t}, ze^{-i2\Pi\gamma t})$ is a continuous path connecting $(x, y, z)^T$ and $(|x|, |y|, |z|)^T$ where $0 \leq t \leq 1$.

Let $\mathcal{S}^+_{\{a,b,c\}} = \{(x, y, z)^T \in \mathcal{S}_{\{a,b,c\}} : x, y, z \geq 0\} \subset \mathcal{S}_{\{a,b,c\}} \cap \mathbb{R}^3 \subset (\mathcal{E}_{\{a,b,c\}} \cap \mathbb{R}^3) \cap (\mathcal{S} \cap \mathbb{R}^3)$. Since the intersection between the real unit sphere and the real ellipsoid is the union of two curves and only one of them contains positive points, i.e. points with all positive coordinates, $\mathcal{S}^+_{\{a,b,c\}}$ is connected. $\square$

More precisely, we will give a formula for the continuous path connecting two points in $\mathcal{S}^+_{\{a,b,c\}}$ in Lemma II.23

Suppose $(\hat{x},\hat{y},\hat{z})^T \in \mathcal{S}^+_{\{a,b,c\}}$. We want to express $\hat{x},\hat{z}$ in terms of $\hat{y}$. We have

$$\frac{\hat{x}^2}{a}+\frac{\hat{y}^2}{b}+\frac{\hat{z}^2}{c}=1=\hat{x}^2+\hat{y}^2+\hat{z}^2$$

So $\dfrac{\hat{x}^2}{a}+\dfrac{\hat{z}^2}{c}=1-\dfrac{\hat{y}^2}{b}, \hat{x}^2+\hat{z}^2=1-\hat{y}^2$ and therefore, $\hat{y}\leq \min\{\sqrt{b},1\}$.
It follows that, if $\frac{1-\hat{y}^2}{c}-(1-\frac{\hat{y}^2}{b})\geq 0$, i.e $\hat{y}\leq\sqrt{\frac{(1-c)b}{b-c}}$, then

$$\hat{x}=\sqrt{\frac{-1+\frac{1}{c}-\frac{\hat{y}^2}{c}+\frac{\hat{y}^2}{b}}{\frac{1}{c}-\frac{1}{a}}}$$

We have $\hat{z}^2=1-\hat{y}^2-\hat{x}^2$ which implies that if $\hat{x}^2\leq 1-\hat{y}^2$, i.e, $\hat{y}\leq\sqrt{\frac{(a-1)b}{a-b}}$, then

$$\hat{z}=\sqrt{(1-\hat{y}^2)-\frac{\frac{1-\hat{y}^2}{c}-(1-\frac{\hat{y}^2}{b})}{\frac{1}{c}-\frac{1}{a}}}=\sqrt{\frac{1-\frac{1}{a}-\frac{\hat{y}^2}{b}+\frac{\hat{y}^2}{a}}{\frac{1}{c}-\frac{1}{a}}}$$

Thus, for any $\hat{y}\leq K=\min\left\{\sqrt{b},1,\sqrt{\frac{(1-c)b}{b-c}},\sqrt{\frac{(a-1)b}{a-b}}\right\}$, we have

$$\hat{x}=\sqrt{\frac{-1+\frac{1}{c}-\frac{\hat{y}^2}{c}+\frac{\hat{y}^2}{b}}{\frac{1}{c}-\frac{1}{a}}}$$

$$\hat{z}=\sqrt{\frac{1-\frac{1}{a}-\frac{\hat{y}^2}{b}+\frac{\hat{y}^2}{a}}{\frac{1}{c}-\frac{1}{a}}}$$

Now let $(\hat{x}_0,\hat{y}_0,\hat{z}_0)^T,(\hat{x}_1,\hat{y}_1,\hat{z}_1)^T\in\mathcal{S}^+_{\{a,b,c\}}$. By the above argument, $0\leq\hat{y}_0,\hat{y}_1\leq K$. Let $0\leq t\leq 1$ and $\hat{y}(t)=(1-t)\hat{y}_0+t\hat{y}_1$. Then $0\leq\hat{y}(t)\leq K$. Let

$$\hat{x}(t)=\sqrt{\frac{-1+\frac{1}{c}-\frac{\hat{y}(t)^2}{c}+\frac{\hat{y}(t)^2}{b}}{\frac{1}{c}-\frac{1}{a}}}$$

$$\hat{z}(t)=\sqrt{\frac{1-\frac{1}{a}-\frac{\hat{y}(t)^2}{b}+\frac{\hat{y}(t)^2}{a}}{\frac{1}{c}-\frac{1}{a}}}$$

Then $(\hat{x}(t), \hat{y}(t), \hat{z}(t))^T$ is a continuous path in $\mathcal{S}^+_{\{a,b,c\}}$ connecting $(\hat{x}_0, \hat{y}_0, \hat{z}_0)^T$ and $(\hat{x}_1, \hat{y}_1, \hat{z}_1)^T$.

To summarize, let $(x_0, y_0, z_0)^T, (x_1, y_1, z_1)^T \in \mathcal{S}_{\{a,b,c\}}$. Write

$$x_0 = |x_0|e^{i2\Pi\alpha_0}, y_0 = |y_0|e^{i2\Pi\beta_0}, z_0 = |z_0|e^{i2\Pi\gamma_0}$$

$$x_1 = |x_1|e^{i2\Pi\alpha_1}, y_1 = |y_1|e^{i2\Pi\beta_1}, z_1 = |z_1|e^{i2\Pi\gamma_1}$$

where $0 \le \alpha_j, \beta_j, \gamma_j < 1$ for $j = 1, 2$.
Let $\hat{x}_0 = |x_0|, \hat{y}_0 = |y_0|, \hat{z}_0 = |z_0|, \hat{x}_1 = |x_1|, \hat{y}_1 = |y_1|, \hat{z}_1 = |z_1|$.
We define

$$\alpha(t) = (1-t)\alpha_0 + t\alpha_1, \beta(t) = (1-t)\beta_0 + t\beta_1, \gamma(t) = (1-t)\gamma_0 + t\gamma_1$$

Let

$$\hat{y}(t) = (1-t)\hat{y}_0 + t\hat{y}_1$$
$$\hat{x}(t) = \sqrt{\frac{-1 + \frac{1}{c} - \frac{\hat{y}(t)^2}{c} + \frac{\hat{y}(t)^2}{b}}{\frac{1}{c} - \frac{1}{a}}}$$
$$\hat{z}(t) = \sqrt{\frac{1 - \frac{1}{a} - \frac{\hat{y}(t)^2}{b} + \frac{\hat{y}(t)^2}{a}}{\frac{1}{c} - \frac{1}{a}}}$$

and

$$x(t) = \hat{x}(t)e^{i2\Pi\alpha(t)}, y(t) = \hat{y}(t)e^{i2\Pi\beta(t)}, z(t) = \hat{z}(t)e^{i2\Pi\gamma(t)}$$

Then $p(t) = (x(t), y(t), z(t))^T \in \mathcal{S}_{\{a,b,c\}}$ for $0 \le t \le 1$ is a continuous path connecting $(x_0, y_0, z_0)^T$ and $(x_1, y_1, z_1)^T$.

The following proposition shows the path connectivity between sets of three unit vectors which have the same Bessel operator.

Proposition II.24. Given a positive operator B with eigenvalues $\lambda_1 \ge \lambda_2 \ge \lambda_3 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 3$. Suppose that $\{x_j\}_{j=1}^3$ are unit vectors which are linearly independent and $\{x'_j\}_{j=1}^3$ are unit vectors which are linearly independent in $\mathbb{C}^n$ such that $\sum_{j=1}^3 x_j \otimes x_j = \sum_{j=1}^3 x'_j \otimes x'_j = B$. There are continuous paths $\{p_j(t)\}_{j=1}^3$ of unit vectors such that $p_j(0) = x_j, p_j(1) = x'_j$ and $\sum_{j=1}^3 p_j(t) \otimes p_j(t) = B$ for any $0 \leq t \leq 1$

Proof. Note that if

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

then $\mathcal{S}_1 \cap B^{1/2}(\mathcal{S}_1) = \mathcal{S}_{(\lambda_1,\lambda_2,\lambda_3)}$.

From Lemma II.12, $\{x_j\}_{j=1}^3 \subset \mathcal{S}_1 \cap B^{1/2}(\mathcal{S}_1)$ and $\{x'_j\}_{j=1}^3 \subset \mathcal{S}_1 \cap B^{1/2}(\mathcal{S}_1)$ as well. Suppose that $\{f_1, f_2, f_3\}$ are corresponding eigenvectors which form an orthonormal basis under which B can be written as

$$\tilde{B} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Therefore, $(f_1\ f_2\ f_3)^* B (f_1\ f_2\ f_3) = \tilde{B}$.

Let $\tilde{x}_j = (f_1\ f_2\ f_3)^* x_j$ and $\tilde{x}'_j = (f_1\ f_2\ f_3)^* x'_j$. Then $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \in \mathcal{S}_{(\lambda_1,\lambda_2,\lambda_3)}$ and

$$\sum_{j=1}^3 \tilde{x}_j \otimes \tilde{x}_j = (f_1\ f_2\ f_3)^* B (f_1\ f_2\ f_3) = \tilde{B}$$

Similarly, $\tilde{x}'_1, \tilde{x}'_2, \tilde{x}'_3 \in \mathcal{S}_{(\lambda_1,\lambda_2,\lambda_3)}$ and $\sum_{j=1}^3 \tilde{x}'_j \otimes \tilde{x}'_j = \tilde{B}$.

Let $\tilde{p}_1(t)$ be the continuous path in Lemma II.23 which lies in $\mathcal{S}_{(\lambda_1,\lambda_2,\lambda_3)}$ and connects $\tilde{x}_1$ and $\tilde{x}'_1$. Then $p_1(t) = (f_1\ f_2\ f_3)\tilde{p}_1(t) \in \mathcal{S}_1 \cap B^{1/2}(\mathcal{S}_1)$ is a continuous path connecting x_1 and x'_1. Let $A(t) = B - p_1(t) \otimes p_1(t)$. Since $p_1(t) \in \mathcal{S}_1 \cap B^{1/2}(\mathcal{S}_1)$, $A(t)$ is a positive operator which has rank 2, trace 2. For each $0 \leq t \leq 1$, let $g_1(t), g_2(t)$ be the orthonormal basis of eigenvectors of $A(t)$ under which $A(t)$ can be written as

$$\tilde{A}(t) = \begin{pmatrix} \nu_1(t) & 0 & 0 \\ 0 & \nu_2(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let

$$\hat{x}_2 = (g_1(0) \;\; g_2(0) \;\; g_3(0))^* x_2, \hat{x}_3 = (g_1(0) \;\; g_2(0) \;\; g_3(0))^* x_3$$

$$\hat{x}_2' = (g_1(1) \;\; g_2(1) \;\; g_3(1))^* x_2', \hat{x}_3' = (g_1(1) \;\; g_2(1) \;\; g_3(1))^* x_3'$$

Then

$$\begin{aligned}
\tilde{A}(0) &= (g_1(0) \;\; g_2(0) \;\; g_3(0))^* A(0)(g_1(0) \;\; g_2(0) \;\; g_3(0)) \\
&= (g_1(0) \;\; g_2(0) \;\; g_3(0))^* (B - x_1 \otimes x_1)(g_1(0) \;\; g_2(0) \;\; g_3(0)) \\
&= (g_1(0) \;\; g_2(0) \;\; g_3(0))^* (x_2 \otimes x_2 + x_3 \otimes x_3)(g_1(0) \;\; g_2(0) \;\; g_3(0)) \\
&= (g_1(0) \;\; g_2(0) \;\; g_3(0))^* x_2 \otimes x_2 (g_1(0) \;\; g_2(0) \;\; g_3(0)) \\
&+ (g_1(0) \;\; g_2(0) \;\; g_3(0))^* x_3 \otimes x_3 (g_1(0) \;\; g_2(0) \;\; g_3(0)) \\
&= (g_1(0) \;\; g_2(0) \;\; g_3(0))^* x_2 \otimes (g_1(0) \;\; g_2(0) \;\; g_3(0))^* x_2 \\
&+ (g_1(0) \;\; g_2(0) \;\; g_3(0))^* x_3 \otimes (g_1(0) \;\; g_2(0) \;\; g_3(0))^* x_3 \\
&= \hat{x}_2 \otimes \hat{x}_2 + \hat{x}_3 \otimes \hat{x}_3
\end{aligned}$$

Similarly, $\tilde{A}(1) = \hat{x}_2' \otimes \hat{x}_2' + \hat{x}_3' \otimes \hat{x}_3'$.

Suppose that

$$\hat{x}_2 = \begin{pmatrix} e^{i\alpha}\sqrt{\frac{\beta_1(0)}{2}} \\ e^{i\mu}\sqrt{1 - \frac{\beta_1(0)}{2}} \\ 0 \end{pmatrix}, \hat{x}_3 = \begin{pmatrix} e^{i\omega}\sqrt{\frac{\beta_1(0)}{2}} \\ -e^{i(\omega-\alpha+\mu)}\sqrt{1 - \frac{\beta_1(0)}{2}} \\ 0 \end{pmatrix}.$$

$$\hat{x}_2' = \begin{pmatrix} e^{i\alpha'}\sqrt{\frac{\beta_1(1)}{2}} \\ e^{i\mu'}\sqrt{1-\frac{\beta_1(1)}{2}} \\ 0 \end{pmatrix}, \hat{x}_3 = \begin{pmatrix} e^{i\omega'}\sqrt{\frac{\beta_1(1)}{2}} \\ -e^{i(\omega'-\alpha'+\mu')}\sqrt{1-\frac{\beta_1(1)}{2}} \\ 0 \end{pmatrix}.$$

Let

$$\alpha(t) = (1-t)\alpha + t\alpha',\ \mu(t) = (1-t)\mu + t\mu',\ \omega(t) = (1-t)\omega + t\omega'.$$

Let

$$\hat{p}_2(t) = \begin{pmatrix} e^{i\alpha(t)}\sqrt{\frac{\beta_1(t)}{2}} \\ e^{i\mu(t)}\sqrt{1-\frac{\beta_1(t)}{2}} \\ 0 \end{pmatrix}, \hat{p}_3(t) = \begin{pmatrix} e^{i\omega(t)}\sqrt{\frac{\beta_1(t)}{2}} \\ -e^{i(\omega(t)-\alpha(t)+\mu(t))}\sqrt{1-\frac{\beta_1(t)}{2}} \\ 0 \end{pmatrix}.$$

Then $\hat{p}_2(t) \otimes \hat{p}_2(t) + \hat{p}_2(t) \otimes \hat{p}_2(t) = \tilde{A}(t)$. Let

$$p_2(t) = (g_1(t) \;\; g_2(t) \;\; g_3(t))\hat{p}_2(t),\ p_3(t) = (g_1(t) \;\; g_2(t) \;\; g_3(t))\hat{p}_3(t)$$

Therefore, $\sum_{j=2}^{3} p_j(t) \otimes p_j(t) = A(t)$ and so $\sum_{j=1}^{3} p_j(t) \otimes p_j(t) = B$. We can check that $\hat{p}_2(t)$ is a continuous path connecting $\hat{x}_2$ and $\hat{x}_2'$ and hence, $p_2(t)$ is a continuous path connecting x_2 and x_2'. Similarly, $p_3(t)$ is a continuous path connecting x_3 and x_3'.

□

When two sets of three unit vectors are both equiangular, the following proposition shows that after a permutation, we can connect them by an equiangular path.

Proposition II.25. Given a positive operator B with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 3$. Suppose that $\{x_j\}_{j=1}^{3}$ are unit vectors which are linearly independent, equiangular and $\{x_j'\}_{j=1}^{3}$ are unit vectors which are linearly independent, equiangular in $\mathbb{C}^n$ such that $\sum_{j=1}^{3} x_j \otimes x_j = \sum_{j=1}^{3} x_j' \otimes x_j' = B$. There are continuous paths $\{p_j(t)\}_{j=1}^3$ of equiangular unit vectors and a permutation Π of $\{1,2,3\}$ such that $p_j(0) = x_{\Pi(j)}, p_j(1) = x'_j$ and $\sum_{j=1}^3 p_j(t) \otimes p_j(t) = B$ for any $0 \le t \le 1$.

Proof. By Lemma II.8, there are scalars $\{e^{i\mu_j}\}_{j=1}^3$, a permutation Π of $\{1,2,3\}$ and a unitary operator U which commutes with B such that $x'_j = e^{i\mu_j} U x_{\Pi(j)}$ for all $j = 1,2,3$. Let $\{f_1, f_2, f_3\}$ be the orthonormal basis of eigenvectors of U such that U can be written as

$$\tilde{U} = \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & e^{i\gamma} \end{pmatrix}.$$

Let $\tilde{x}_j = (f_1\ f_2\ f_3)^* x_j$ and $\tilde{x}'_j = (f_1\ f_2\ f_3)^* x'_j$. Then

$$\begin{aligned} \tilde{x}'_j &= (f_1\ f_2\ f_3)^* x'_j = (f_1\ f_2\ f_3)^* e^{i\mu_j} (f_1\ f_2\ f_3) \tilde{U} (f_1\ f_2\ f_3)^* x_{\Pi(j)} \\ &= e^{i\mu_j} \tilde{U} (f_1\ f_2\ f_3)^* x_{\Pi(j)} = e^{i\mu_j} \tilde{U} \tilde{x}_{\Pi(j)} \end{aligned}$$

Therefore, $\tilde{x}'_j = e^{i\mu_j} \tilde{U} \tilde{x}_{\Pi(j)}$. We have

$$\begin{aligned} \sum_{j=1}^3 \tilde{x}_j \otimes \tilde{x}_j &= \sum_{j=1}^3 (f_1\ f_2\ f_3)^* x_j \otimes (f_1\ f_2\ f_3)^* x_j \\ &= (f_1\ f_2\ f_3)^* (\sum_{j=1}^3 x_j \otimes x_j)(f_1\ f_2\ f_3) \\ &= (f_1\ f_2\ f_3)^* B (f_1\ f_2\ f_3) \end{aligned}$$

Denote $\hat{B} = (f_1\ f_2\ f_3)^* B (f_1\ f_2\ f_3)$. Since $UB = BU$, we have $(f_1\ f_2\ f_3)\tilde{U}(f_1\ f_2\ f_3)^* B = B(f_1\ f_2\ f_3)\tilde{U}(f_1\ f_2\ f_3)^*$ from which, it follows that

$$\tilde{U}\hat{B} = \tilde{U}(f_1\ f_2\ f_3)^* B (f_1\ f_2\ f_3) = (f_1\ f_2\ f_3)^* B (f_1\ f_2\ f_3) \tilde{U} = \hat{B}\tilde{U}$$

For $0 \leq t \leq 1$, let

$$\tilde{U}(t) = \begin{pmatrix} e^{i\alpha t} & 0 & 0 \\ 0 & e^{i\beta t} & 0 \\ 0 & 0 & e^{i\gamma t} \end{pmatrix}.$$

Let $\tilde{p}_j(t) = e^{i\mu_j t}\tilde{U}(t)\tilde{x}_{\Pi(j)}$. Then $\tilde{p}_j(t), j = 1, 2, 3$ are continuous and $\tilde{p}_j(0) = \tilde{x}_{\Pi(j)}$ and $\tilde{p}_j(1) = e^{i\mu_j}\tilde{U}\tilde{x}_{\Pi(j)} = \tilde{x}'_j$. We have $||\tilde{p}_j(t)|| = 1, |\langle p_1(t), p_2(t) \rangle| = |\langle p_1(t), p_3(t) \rangle| = |\langle p_2(t), p_3(t) \rangle|$ and

$$\begin{aligned} \sum_{j=1}^{3} \tilde{p}_j(t) \otimes \tilde{p}_j(t) &= \sum_{j=1}^{3} \tilde{U}(t)\tilde{x}_{\Pi(j)} \otimes \tilde{U}(t)\tilde{x}_{\Pi(j)} \\ &= \tilde{U}(t)(\sum_{j=1}^{3} \tilde{x}_{\Pi(j)} \otimes \tilde{x}_{\Pi(j)})\tilde{U}(t)^* \\ &= \tilde{U}(t)\hat{B}\tilde{U}(t)^* = \hat{B} \end{aligned}$$

Let $p_j(t) = (f_1\ f_2\ f_3)\tilde{p}_j(t)$. Then $||p_j(t)|| = ||\tilde{p}_j(t)|| = 1$, $\{p_j(t)\}_{j=1}^{3}$ is an equiangular set and

$$\sum_{j=1}^{3} p_j(t) \otimes p_j(t) = \sum_{j=1}^{3} (f_1\ f_2\ f_3)\tilde{p}_j(t) \otimes (f_1\ f_2\ f_3)\tilde{p}_j(t) = (f_1\ f_2\ f_3)\hat{B}(f_1\ f_2\ f_3)^* = B.$$

We can check that $p_j(t), j = 1, 2, 3$ are continuous and

$$p_j(0) = (f_1\ f_2\ f_3)\tilde{p}_j(0) = (f_1\ f_2\ f_3)\tilde{x}_{\Pi(j)} = x_{\Pi(j)}$$

and similarly, $p_j(1) = x'_j$. □

Remark 4. If $X = \{x_j\}_{j=1}^{k}$ and $Y = \{y_j\}_{j=1}^{k}$ are geometrically equivalent then there is a unitary operator U that is the product of a permutation and a diagonal unitary such that

$$UG_XU^* = G_Y$$

Lemma II.26. Given a positive operator B of trace 3 with eigenvalues $0 \leq \lambda_1 \leq$

$\lambda_2 \leq \lambda_3$. Suppose that $\{x_j\}_{j=1}^3$ are unit vectors having B as the Bessel operator. Let $\mathcal{M}(\{x_j\}_{j=1}^3)$ be the maximal frame correlation for $\{x_j\}_{j=1}^3$. Minimize $\mathcal{M}(\{x_j\}_{j=1}^k)$ over all set of three unit vectors with Bessel operator B. Then the maximal frame correlation is smallest when $\{x_j\}_{j=1}^3$ are equiangular.

Proof. Suppose that $\{w_j\}_{j=1}^3$ are three unit equiangular vectors with Bessel operator B. Then $|\langle w_1 , w_2 \rangle| = |\langle w_1 , w_3 \rangle| = |\langle u_2 , u_3 \rangle| = c$ where

$$c = \sqrt{\frac{3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3}{3}}$$

Assume that $\langle x_1 , x_2 \rangle = b, \langle x_1 , x_3 \rangle = d, \langle x_2 , x_3 \rangle = e$ and $|b| \geq |d| \geq |e|$. Then the characteristics polynomial of the Grammian operator $G_{\{x_i\}_{i=1}^3}$ for $\{x_i\}_{i=1}^3$ is

$$(1-\lambda)^3 - (1-\lambda)(|b|^2 + |d|^2 + |e|^2) + be\bar{d} + d\bar{b}\bar{e} = 0$$

and the characteristics polynomial of the Grammian operator $G_{\{w_i\}_{i=1}^3}$ for $\{w_i\}_{i=1}^3$ is

$$(1-\lambda)^3 - 3c^2(1-\lambda) + 2c^3 \cos\omega = 0$$

where ω is the angle defined in Proposition II.17

Since two Grammian operators have the same eigenvalues, the above two characteristics polynomials are the same. It follows that $|b|^2 + |d|^2 + |e|^2 = 3c^2$. Therefore, $3c^2 \leq 3|b|^2$ and $c \leq |b|$. □

Therefore, each set of three unit vectors which do not lie in the same line can be replaced with another set of three unit vectors which has the same Bessel operator as the Bessel operator of the original set. In particular, we can always replace the original set with an equiangular set of three unit vectors. Moreover, the minimum angle between pairs of vectors in the replacement set becomes largest when the replacement set is equiangular. So iterating this procedure might lead to a construction of Grassmannian frames.

CHAPTER III

A SPREADING ALGORITHM FOR FINITE UNIT NORM TIGHT FRAMES

Suppose that $\{y_j\}_{j=1}^k$ is a unit norm tight frame in $\mathbb{C}^n$. We will replace vectors three-at-a-time to produce a unit norm tight frame with better maximal frame correlation than the original frame.

Suppose that y_1, y_2, y_3 are linearly independent unit vectors in $\mathbb{C}^n$. We wish to built an algorithm whose output is a set of three equiangular unit vectors with "good" direction in the sense that when the input is an equiangular set then the output is exactly the input in the same order.

Step 1 : Gram-Schmidt them, obtaining an orthonormal basis $\{h_1, h_2, h_3\}$ for span $\{y_1, y_2, y_3\}$. Let x_1, x_2, x_3 be the coordinate vectors in $\mathbb{C}^3$ for y_1, y_2, y_3, respectively.

Step 2 : Let $A = \sum_{j=1}^3 x_j \otimes x_j$. Compute the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A. Suppose $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Compute an orthonormal basis $\{f_1, f_2, f_3\}$ of eigenvectors for A corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Step 3 : Let

$$c = \sqrt{\frac{3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3}{3}}$$

and

$$M = \frac{\lambda_1\lambda_2\lambda_3 + 2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3}{2}$$

and $\omega = \arccos(\frac{M}{c^3})$.

Let

$$u_1 = (1,0,0)^T, u_2 = (c\ e^{-i\omega}, \sqrt{1-c^2}, 0)^T$$

$$u_3 = (c\ e^{-i\omega}, \frac{ce^{i\omega} - c^2}{\sqrt{1-c^2}}, \frac{\sqrt{(1-c^2)^2 - |ce^{i\omega} - c^2|^2}}{\sqrt{1-c^2}})^T$$

Step 4 : Let $B = \sum_{j=1}^{3} u_j \otimes u_j$. Compute the eigenvalues of B and compute an orthonormal basis $\{g_1, g_2, g_3\}$ of eigenvectors for B.

Step 5 : Let $W = (f_1\ f_2\ f_3).(g_1\ g_2\ g_3)^*$. Then W is a unitary matrix.

Step 6 : Let $x_1' = Wu_1, x_2' = Wu_2, x_3' = Wu_3$ in $\mathbb{C}^3$. Then x_1', x_2', x_3' are unit, equiangular and $\sum_{j=1}^{3} x_j' \otimes x_j' = A$.

We want to orient x_1', x_2', x_3' obtaining $\hat{x}_1, \hat{x}_2, \hat{x}_3$ such that when x_1, x_2, x_3 are equiangular, we have $\hat{x}_j = x_j$ for all $j = 1, 2, 3$.

Step 7 : Let $\mathcal{S}_A = A^{1/2} S_1(\mathbb{C}^3) \cap S_1(\mathbb{C}^3)$ where $S_1(\mathbb{C}^3)$ is the unit sphere in $\mathbb{C}^3$. Let $\{f_j\}_{j=1}^{3}$ be an orthonormal basis of eigenvectors of A such that A can be written as

$$\tilde{A} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

where $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 3$. We define

$$\mathcal{S}_{(\lambda_1,\lambda_2,\lambda_3)} = \left\{ (x, y, z)^T \in \mathbb{C}^3 : \frac{|x|^2}{\lambda_1} + \frac{|y|^2}{\lambda_2} + \frac{|z|^2}{\lambda_3} = 1 \right\} \bigcap S_1(\mathbb{C}^3)$$

Note that $x_1, x_2, x_3 \in \mathcal{S}_A$.

Step 8 : For $j = 1, 2, 3$, let $\tilde{x_j} = (f_1\ f_2\ f_3)^{-1} x_j$ and $\tilde{x_j}' = (f_1\ f_2\ f_3)^{-1} x_j'$. Then $\tilde{x_1}, \tilde{x_2}, \tilde{x_3} \in \mathcal{S}_{(\lambda_1,\lambda_2,\lambda_3)}$ and $\tilde{x_1}', \tilde{x_2}', \tilde{x_3}' \in \mathcal{S}_{(\lambda_1,\lambda_2,\lambda_3)}$. Therefore, $\sum_{j=1}^{3} \tilde{x_j} \otimes \tilde{x_j} = \tilde{A} = \sum_{j=1}^{3} \tilde{x_j}' \otimes \tilde{x_j}'$.

If $v = (x, y, z)^T \in \mathbb{C}^3$ then we define $|v| = (|x|, |y|, |z|)^T \in \mathbb{R}_+^3$.

Step 9 : Let $\tilde{w_1}$ be the $\tilde{x_j}'$ such that

$$|||\tilde{x_1}| - |\tilde{w_1}||| = \min\{|||\tilde{x_1}| - |\tilde{x_j}'||| : j = 1, 2, 3\}$$

If there are more than one $\tilde{x_j}'$ satisfying this minimum condition choose the first one having $||\tilde{x_1} - \tilde{x_j}'||$ the smallest possible. Then let $\tilde{w_2} \in \{\tilde{x_1}', \tilde{x_2}', \tilde{x_3}'\} \setminus \{\tilde{w_1}\}$ for which

$$|||\tilde{x_2}| - |\tilde{w_2}||| = \min\{|||\tilde{x_2}| - |\tilde{x_j}'||| : \tilde{x_j}' \neq \tilde{w_1}\}$$

If there are more than one $\tilde{x_j}'$ satisfying this minimum condition, we choose the one having $||\tilde{x_2} - \tilde{x_j}'||$ the smaller. Then let $\tilde{w_3}$ be the remaining vector in $\{\tilde{x_1}', \tilde{x_2}', \tilde{x_3}'\}$. Thus, $\sum_{j=1}^{3} \tilde{w_j} \otimes \tilde{w_j} = \sum_{j=1}^{3} \tilde{x_j}' \otimes \tilde{x_j}' = \tilde{A}$ and $\{\tilde{w_j}\}_{j=1}^{3}$ are unit, equiangular.
Define the phase $arg(v)$ of a vector $v \in \mathbb{C}$ as follows.

$$arg(v) = \begin{cases} 1, & \text{if } v = 0; \\ \frac{v}{|v|} & \text{if } v \neq 0. \end{cases}$$

Write $\tilde{x_j} = (\tilde{x_{j1}}, \tilde{x_{j2}}, \tilde{x_{j3}})^T, \tilde{w_j} = (\tilde{w_{j1}}, \tilde{w_{j2}}, \tilde{w_{j3}})^T$ for $j = 1, 2, 3$. We consider several cases.

Case 1 Assume that the vectors $|\tilde{x_j}|_{j=1}^{3}$ are distinct.

Case (1.1) Assume all components of $\tilde{x_1}$ are different from 0.

Step 10.1.1 : We will construct a vector $\tilde{z_1}$ such that $|\tilde{z_1}| = |\tilde{w_1}|$ which has the same phase as $\tilde{x_1}$ as follows. Write $\tilde{x_1} = (\gamma_1|\tilde{x_{11}}|, \gamma_2|\tilde{x_{12}}|, \gamma_3|\tilde{x_{13}}|)^T$ and $\tilde{w_1} = (\delta_1|\tilde{w_{11}}|, \delta_2|\tilde{w_{12}}|, \delta_3|\tilde{w_{13}}|)$ where $|\gamma_j| = 1 = |\delta_j|$ for $j = 1, 2, 3$.
Let $\tilde{z_1} = (\gamma_1|\tilde{w_{11}}|, \gamma_2|\tilde{w_{12}}|, \gamma_3|\tilde{w_{13}}|)^T$ and

$$\Gamma = \begin{pmatrix} \frac{\gamma_1}{\delta_1} & 0 & 0 \\ 0 & \frac{\gamma_2}{\delta_2} & 0 \\ 0 & 0 & \frac{\gamma_3}{\delta_3} \end{pmatrix}$$

Then Γ is an unitary matrix and $\tilde{z_1} = \Gamma\tilde{w_1}$. Let $\tilde{z_2} = \Gamma\tilde{w_2}, \tilde{z_3} = \Gamma\tilde{w_3}$. Then $|\tilde{w_j}| = |\tilde{z_j}|$. Since $\sum_{j=1}^{3} \tilde{w_j} \otimes \tilde{w_j} = \tilde{A}$ and Γ commutes with $\tilde{A}$, we have $\sum_{j=1}^{3} \tilde{z_j} \otimes \tilde{z_j} = \sum_{j=1}^{3} \Gamma\tilde{w_j} \otimes \Gamma\tilde{w_j} = \Gamma\tilde{A}\Gamma^* = \tilde{A}$. Note that $\{\tilde{z_j}\}_{j=1}^{3}$ are unit, equiangular.

Step 11.1.1 : Write $\tilde{z_j} = (\tilde{z_{j1}}, \tilde{z_{j2}}, \tilde{z_{j3}})^T$. Let

$$\nu_2 = \frac{arg(\tilde{x}_{21})arg(\tilde{z}_{11})}{arg(\tilde{z}_{21})arg(\tilde{x}_{11})}, \nu_3 = \frac{arg(\tilde{x}_{31})arg(\tilde{z}_{11})}{arg(\tilde{z}_{31})arg(\tilde{x}_{11})}$$

Let $\tilde{v_1} = \tilde{z_1}, \tilde{v_2} = \nu_2\tilde{z_2}, \tilde{v_3} = \nu_3\tilde{z_3}$. Then $\{\tilde{v_j}\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$ and $\sum_{j=1}^3 \tilde{v_j} \otimes \tilde{v_j} = \sum_{j=1}^3 \tilde{z_j} \otimes \tilde{z_j} = \tilde{A}$.

Step 12.1.1 : Let $\hat{x_j} = (f_1\ f_2\ f_3)\tilde{v_j}$ for $j = 1, 2, 3$. Then $\sum_{j=1}^3 \hat{x_j} \otimes \hat{x_j} = A$ and $\{\hat{x_j}\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.1.1 : Suppose that $\hat{x_j} = (\alpha_j, \beta_j, \mu_j)^T$. Then $y_j' = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^3 y_j' \otimes y_j' = \sum_{j=1}^3 y_j \otimes y_j$. Stop, the algorithm for Case (1.1) is complete.

Case(1.2) Assume $\tilde{x_1}$ has a zero component and all components of $\tilde{x_2}$ are different from 0.

Step 10.1.2 : We will construct a vector $\tilde{z_2}$ such that $|\tilde{z_2}| = |\tilde{w_2}|$ which has the same phase as $\tilde{x_2}$ as follows. Write $\tilde{x_2} = (\gamma_1|\tilde{x_{21}}|, \gamma_2|\tilde{x_{22}}|, \gamma_3|\tilde{x_{23}}|)^T$ and $\tilde{w_2} = (\delta_1|\tilde{w_{21}}|, \delta_2|\tilde{w_{22}}|, \delta_3|\tilde{w_{23}}|)$ where $|\gamma_j| = 1 = |\delta_j|$ for $j = 1, 2, 3$.

Let $\tilde{z_2} = (\gamma_1|\tilde{w_{21}}|, \gamma_2|\tilde{w_{22}}|, \gamma_3|\tilde{w_{23}}|)^T$ and

$$\Gamma = \begin{pmatrix} \frac{\gamma_1}{\delta_1} & 0 & 0 \\ 0 & \frac{\gamma_2}{\delta_2} & 0 \\ 0 & 0 & \frac{\gamma_3}{\delta_3} \end{pmatrix}$$

Then Γ is an unitary matrix and $\tilde{z_2} = \Gamma\tilde{w_2}$. Let $\tilde{z_1} = \Gamma\tilde{w_1}, \tilde{z_3} = \Gamma\tilde{w_3}$. Then $|\tilde{w_j}| = |\tilde{z_j}|$. Since $\sum_{j=1}^3 \tilde{w_j} \otimes \tilde{w_j} = \sum_{j=1}^3 \tilde{x_j}' \otimes \tilde{x_j}' = \tilde{A}$ and Γ commutes with $\tilde{A}$, we have $\sum_{j=1}^3 \tilde{z_j} \otimes \tilde{z_j} = \sum_{j=1}^3 \Gamma\tilde{w_j} \otimes \Gamma\tilde{w_j} = \Gamma\tilde{A}\Gamma^* = \tilde{A}$. Note that $\{\tilde{z_j}\}_{j=1}^3$ are unit, equiangular.

Step 11.1.2 : Write $\tilde{z_j} = (\tilde{z_{j1}}, \tilde{z_{j2}}, \tilde{z_{j3}})^T$. Let

$$\nu_1 = \frac{arg(\tilde{z}_{21})arg(\tilde{x}_{11})}{arg(\tilde{x}_{21})arg(\tilde{z}_{11})}, \nu_3 = \frac{arg(\tilde{z}_{21})arg(\tilde{x}_{31})}{arg(\tilde{x}_{21})arg(\tilde{z}_{31})}$$

Let $\tilde{v_1} = \nu_1\tilde{z_1}, \tilde{v_2} = \tilde{z_2}, \tilde{v_3} = \nu_3\tilde{z_3}$. Then $\{\tilde{v_j}\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$ and $\sum_{j=1}^3 \tilde{v_j} \otimes \tilde{v_j} = \sum_{j=1}^3 \tilde{z_j} \otimes \tilde{z_j} = \tilde{A}$.

Step 12.1.2 : Let $\hat{x_j} = (f_1\ f_2\ f_3)\tilde{v_j}$ for $j = 1, 2, 3$. Then $\sum_{j=1}^3 \hat{x_j} \otimes \hat{x_j} = A$ and $\{\hat{x_j}\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.1.2 : Suppose that $\hat{x_j} = (\alpha_j, \beta_j, \mu_j)^T$. Then $y_j' = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^3 y_j' \otimes y_j' = \sum_{j=1}^3 y_j \otimes y_j$. Stop, the algorithm for Case (1.2) is complete.

Case(1.3) Assume $\tilde{x_1}$ and $\tilde{x_2}$ have a zero component and all components of $\tilde{x_3}$ are different from 0.

Step 10.1.3 : We will construct a vector $\tilde{z_3}$ such that $|\tilde{z_3}| = |\tilde{w_3}|$ which has the same phase as $\tilde{x_3}$ as follows. Write $\tilde{x_3} = (\gamma_1|\tilde{x_{31}}|, \gamma_2|\tilde{x_{32}}|, \gamma_3|\tilde{x_{33}}|)^T$ and $\tilde{w_3} = (\delta_1|\tilde{w_{31}}|, \delta_2|\tilde{w_{32}}|, \delta_3|\tilde{w_{33}}|)$ where $|\gamma_j| = 1 = |\delta_j|$ for $j = 1, 2, 3$.

Let $\tilde{z_3} = (\gamma_1|\tilde{w_{31}}|, \gamma_2|\tilde{w_{32}}|, \gamma_3|\tilde{w_{33}}|)^T$ and

$$\Gamma = \begin{pmatrix} \frac{\gamma_1}{\delta_1} & 0 & 0 \\ 0 & \frac{\gamma_2}{\delta_2} & 0 \\ 0 & 0 & \frac{\gamma_3}{\delta_3} \end{pmatrix}$$

Then Γ is an unitary matrix and $\tilde{z_3} = \Gamma\tilde{w_3}$. Let $\tilde{z_1} = \Gamma\tilde{w_1}, \tilde{z_2} = \Gamma\tilde{w_2}$. Then $|\tilde{w_j}| = |\tilde{z_j}|$. Since $\sum_{j=1}^3 \tilde{w_j} \otimes \tilde{w_j} = \sum_{j=1}^3 \tilde{x_j}' \otimes \tilde{x_j}' = \tilde{A}$ and Γ commutes with $\tilde{A}$, we have $\sum_{j=1}^3 \tilde{z_j} \otimes \tilde{z_j} = \sum_{j=1}^3 \Gamma\tilde{w_j} \otimes \Gamma\tilde{w_j} = \Gamma\tilde{A}\Gamma^* = \tilde{A}$. Note that $\{\tilde{z_j}\}_{j=1}^3$ are unit, equiangular.

Step 11.1.3 : Write $\tilde{z_j} = (\tilde{z_{j1}}, \tilde{z_{j2}}, \tilde{z_{j3}})^T$. Let

$$\nu_1 = \frac{arg(\tilde{z}_{31})arg(\tilde{x}_{11})}{arg(\tilde{x}_{31})arg(\tilde{z}_{11})}, \nu_3 = \frac{arg(\tilde{z}_{31})arg(\tilde{x}_{21})}{arg(\tilde{x}_{31})arg(\tilde{z}_{21})}$$

Let $\tilde{v_1} = \nu_1\tilde{z_1}, \tilde{v_2} = \nu_2\tilde{z_2}, \tilde{v_3} = \tilde{z_3}$. Then $\{\tilde{v_j}\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$ and $\sum_{j=1}^3 \tilde{v_j} \otimes \tilde{v_j} = \sum_{j=1}^3 \tilde{z_j} \otimes \tilde{z_j} = \tilde{A}$.

Step 12.1.3 : Let $\hat{x_j} = (f_1\ f_2\ f_3)\tilde{v_j}$ for $j = 1, 2, 3$. Then $\sum_{j=1}^3 \hat{x_j} \otimes \hat{x_j} = A$ and

$\{\hat{x}_j\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.1.3 : Suppose that $\hat{x_j} = (\alpha_j, \beta_j, \mu_j)^T$. Then $y'_j = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^3 y'_j \otimes y'_j = \sum_{j=1}^3 y_j \otimes y_j$. Stop, the algorithm for Case (1.3) is complete.

Case(1.4) Assume that $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ each have at least one zero component and that $\tilde{x}_{11} \neq 0, \tilde{x}_{21} \neq 0, \tilde{x}_{31} \neq 0$.

Step 10.1.4 : Write $\tilde{x}_{11} = \gamma_1|\tilde{x}_{11}|, \tilde{x}_{21} = \gamma_2|\tilde{x}_{21}|, \tilde{x}_{31} = \gamma_3|\tilde{x}_{31}|$, $\tilde{w}_{11} = \delta_1|\tilde{w}_{11}|, \tilde{w}_{21} = \delta_2|\tilde{w}_{21}|, \tilde{w}_{31} = \delta_3|\tilde{w}_{31}|$ where $|\gamma_j| = 1 = |\delta_j|$ for $j = 1, 2, 3$. Let $\tilde{z}_j = \frac{\gamma_j}{\delta_j}\tilde{w}_j$ for $j = 1, 2, 3$. Note that $\{\tilde{z}_j\}_{j=1}^3$ are unit, equiangular and $\sum_{j=1}^3 \tilde{z}_j \otimes \tilde{z}_j = \tilde{A}$.

Step 11.1.4 : Let

$$\nu_2 = \frac{arg(\tilde{x}_{11})arg(\tilde{z}_{12})}{arg(\tilde{z}_{11})arg(\tilde{x}_{12})}, \nu_3 = \frac{arg(\tilde{x}_{11})arg(\tilde{z}_{13})}{arg(\tilde{z}_{11})arg(\tilde{x}_{13})}$$

and

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\nu_2} & 0 \\ 0 & 0 & \frac{1}{\nu_3} \end{pmatrix}$$

Let $\tilde{v_j} = \Gamma\tilde{z}_j$. Then $\{\tilde{v}_j\}_{j=1}^3$ are unit, equiangular and $\sum_{j=1}^3 \tilde{z}_j \otimes \tilde{z}_j = \tilde{A}$.

Step 12.1.4 : Let $\hat{x_j} = (f_1\ f_2\ f_3)\tilde{v_j}$ for $j = 1, 2, 3$. Then $\sum_{j=1}^3 \hat{x_j} \otimes \hat{x_j} = A$ and $\{\hat{x_j}\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.1.4: Suppose that $\hat{x_j} = (\alpha_j, \beta_j, \mu_j)^T$. Then $y'_j = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^3 y'_j \otimes y'_j = \sum_{j=1}^3 y_j \otimes y_j$. Stop, the algorithm for Case (1.4) is complete.

Case(1.5) Assume that the hypotheses in Cases (1.1)-(1.4) all failed and $\tilde{x}_{12} \neq 0, \tilde{x}_{22} \neq 0, \tilde{x}_{32} \neq 0$.

Step 10.1.5 : Write $\tilde{x}_{12} = \gamma_1|\tilde{x}_{12}|, \tilde{x}_{22} = \gamma_2|\tilde{x}_{22}|, \tilde{x}_{32} = \gamma_3|\tilde{x}_{32}|$, $\tilde{w}_{12} = \delta_1|\tilde{w}_{12}|, \tilde{w}_{22} = \delta_2|\tilde{w}_{22}|, \tilde{w}_{32} = \delta_3|\tilde{w}_{32}|$ where $|\gamma_j| = 1 = |\delta_j|$ for $j = 1, 2, 3$. Let $\tilde{z}_j = \frac{\gamma_j}{\delta_j}\tilde{w}_j$ for $j = 1, 2, 3$.

Note that $\{\tilde{z_j}\}_{j=1}^3$ are unit, equiangular and $\sum_{j=1}^3 \tilde{z_j} \otimes \tilde{z_j} = \tilde{A}$.

Step 11.1.5 : Let

$$\nu_1 = \frac{arg(\tilde{z}_{11})arg(\tilde{x}_{12})}{arg(\tilde{x}_{11})arg(\tilde{z}_{12})}, \nu_3 = \frac{arg(\tilde{x}_{12})arg(\tilde{z}_{13})}{arg(\tilde{z}_{12})arg(\tilde{x}_{13})}$$

and

$$\Gamma = \begin{pmatrix} \frac{1}{\nu_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\nu_3} \end{pmatrix}.$$

Let $\tilde{v_j} = \Gamma \tilde{z_j}$. Then $\{\tilde{v_j}\}_{j=1}^3$ are unit, equiangular and $\sum_{j=1}^3 \tilde{z_j} \otimes \tilde{z_j} = \tilde{A}$.

Step 12.1.5 : Let $\hat{x_j} = (f_1 \ f_2 \ f_3)\tilde{v_j}$ for $j = 1, 2, 3$. Then $\sum_{j=1}^3 \hat{x_j} \otimes \hat{x_j} = A$ and $\{\hat{x_j}\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.1.5 : Suppose that $\hat{x_j} = (\alpha_j, \beta_j, \mu_j)^T$. Then $y_j' = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^3 y_j' \otimes y_j' = \sum_{j=1}^3 y_j \otimes y_j$. Stop, the algorithm for Case (1.5) is complete.

Case(1.6) Assume that the hypotheses in Cases (1.1)-(1.5) all failed and $\tilde{x}_{13} \neq 0, \tilde{x}_{23} \neq 0, \tilde{x}_{33} \neq 0$.

Step 10.1.6: Write $\tilde{x}_{13} = \gamma_1|\tilde{x}_{13}|, \tilde{x}_{23} = \gamma_2|\tilde{x}_{23}|, \tilde{x}_{33} = \gamma_3|\tilde{x}_{33}|, \tilde{w}_{13} = \delta_1|\tilde{w}_{13}|, \tilde{w}_{23} = \delta_2|\tilde{w}_{23}|, \tilde{w}_{33} = \delta_3|\tilde{w}_{33}|$ where $|\gamma_j| = 1 = |\delta_j|$ for $j = 1, 2, 3$. Let $\tilde{z_j} = \frac{\gamma_j}{\delta_j}\tilde{w_j}$ for $j = 1, 2, 3$. Note that $\{\tilde{z_j}\}_{j=1}^3$ are unit, equiangular and $\sum_{j=1}^3 \tilde{z_j} \otimes \tilde{z_j} = \tilde{A}$.

Step 11.1.6 : Let

$$\nu_1 = \frac{arg(\tilde{z}_{11})arg(\tilde{x}_{13})}{arg(\tilde{x}_{11})arg(\tilde{z}_{13})}, \nu_2 = \frac{arg(\tilde{z}_{12})arg(\tilde{x}_{13})}{arg(\tilde{x}_{12})arg(\tilde{z}_{13})}$$

and

$$\Gamma = \begin{pmatrix} \frac{1}{\nu_1} & 0 & 0 \\ 0 & \frac{1}{\nu_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $\tilde{v_j} = \Gamma \tilde{z_j}$. Then $\{\tilde{v_j}\}_{j=1}^{3}$ are unit, equiangular and $\sum_{j=1}^{3} \tilde{z_j} \otimes \tilde{z_j} = \tilde{A}$

Step 12.1.6 : Let $\hat{x_j} = (f_1\ f_2\ f_3)\tilde{v_j}$ for $j = 1, 2, 3$. Then $\sum_{j=1}^{3} \hat{x_j} \otimes \hat{x_j} = A$ and $\{\hat{x_j}\}_{j=1}^{3}$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.1.6 : Suppose that $\hat{x_j} = (\alpha_j, \beta_j, \mu_j)^T$. Then $y'_j = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^{3} y'_j \otimes y'_j = \sum_{j=1}^{3} y_j \otimes y_j$. Stop, the algorithm for Case (1.6) is complete.

Case(1.7) Assume that the hypotheses in Cases (1.1)-(1.6) all failed. Then we do the following:

Step 12.1.7 : Let $\hat{x_j} = (f_1\ f_2\ f_3)\tilde{w_j}$ for $j = 1, 2, 3$. Then $\sum_{j=1}^{3} \hat{x_j} \otimes \hat{x_j} = A$ and $\{\hat{x_j}\}_{j=1}^{3}$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.1.7 : Suppose that $\hat{x_j} = (\alpha_j, \beta_j, \mu_j)^T$. Then $y'_j = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^{3} y'_j \otimes y'_j = \sum_{j=1}^{3} y_j \otimes y_j$. Stop, the algorithm for Case (1.7) is complete.

Case 2 : There is one pair of vectors in $\{|\tilde{x_1}|, |\tilde{x_2}|, |\tilde{x_3}|\}$ the same.

Case 2.1 : $|\tilde{x_1}| = |\tilde{x_2}| \neq |\tilde{x_3}|$.

Write

$$\tilde{x_1} = (e^{i\theta_1}x, e^{i\theta_2}y, e^{i\theta_3}z)^T$$

$$\tilde{x_2} = (e^{i\alpha_1}x, e^{i\alpha_2}y, e^{i\alpha_3}z)^T$$

$$\tilde{x_3} = (e^{i\beta_1}x', e^{i\beta_2}y', e^{i\beta_3}z')^T$$

$$\tilde{w}_1 = (e^{i\theta'_1}|\tilde{w}_{11}|, e^{i\theta'_2}|\tilde{w}_{12}|, e^{i\theta'_3}|\tilde{w}_{13}|)^T$$

$$\tilde{w}_2 = (e^{i\alpha'_1}|\tilde{w}_{21}|, e^{i\alpha'_2}|\tilde{w}_{22}|, e^{i\alpha'_3}|\tilde{w}_{23}|)^T$$

$$\tilde{w}_3 = (e^{i\beta'_1}|\tilde{w}_{31}|, e^{i\beta'_2}|\tilde{w}_{32}|, e^{i\beta'_3}|\tilde{w}_{33}|)^T$$

Case 2.1.1 : $x' \neq 0, y' \neq 0, z' \neq 0$

Let

$$\Gamma = \begin{pmatrix} e^{i(\beta_1-\beta_1')} & 0 & 0 \\ 0 & e^{i(\beta_2-\beta_2')} & 0 \\ 0 & 0 & e^{i(\beta_3-\beta_3')} \end{pmatrix}$$

Let $\tilde{z_j} = \Gamma\tilde{w_j}$ for $j = 1, 2, 3$. Note that $\tilde{z_3}$ has the same phase as $\tilde{x_3}$ and $|\tilde{w_j}| = |\tilde{z_j}|$. Similar to previous cases, we have $\sum_{j=1}^{3} \tilde{z_j} \otimes \tilde{z_j} = \tilde{A}$ and $\{\tilde{z_j}\}_{j=1}^{3}$ are unit, equiangular.

Step 11.2.1.1 : Write $\tilde{z_j} = (\tilde{z_{j1}}, \tilde{z_{j2}}, \tilde{z_{j3}})^T$. Write $\tilde{z_1} = (e^{i\mu_1}|\tilde{z_{11}}|, e^{i\mu_2}|\tilde{z_{12}}|, e^{i\mu_3}|\tilde{z_{13}}|)^T$

1) If $\mu_1 - \alpha_1 = \mu_2 - \alpha_2 = \mu_3 - \alpha_3$ then let

$$\nu_1 = \frac{arg(\tilde{z}_{31})arg(\tilde{x}_{21})}{arg(\tilde{x}_{31})arg(\tilde{z}_{11})}, \nu_2 = \frac{arg(\tilde{z}_{31})arg(\tilde{x}_{11})}{arg(\tilde{x}_{31})arg(\tilde{z}_{21})}$$

2) Otherwise let

$$\nu_1 = \frac{arg(\tilde{z}_{31})arg(\tilde{x}_{11})}{arg(\tilde{x}_{31})arg(\tilde{z}_{11})}, \nu_2 = \frac{arg(\tilde{z}_{31})arg(\tilde{x}_{21})}{arg(\tilde{x}_{31})arg(\tilde{z}_{21})}$$

Let $\tilde{v_1}' = \nu_1\tilde{z_1}, \tilde{v_2}' = \nu_2\tilde{z_2}, \tilde{v_3}' = \tilde{z_3}$. Then $\{\tilde{v_j}'\}_{j=1}^{3}$ are unit, equiangular in $\mathbb{C}^3$ and $\sum_{j=1}^{3} \tilde{v_j}' \otimes \tilde{v_j} = \sum_{j=1}^{3} \tilde{z_j} \otimes \tilde{z_j} = \tilde{A}$.

Step 12.2.1.1 : Let $\tilde{v_3} = \tilde{v_3}'$. Let $\tilde{v_1} \in \{\tilde{v_1}', \tilde{v_2}'\}$ be such that

$$||\tilde{x_1} - \tilde{v_1}|| = min\{||\tilde{x_1} - \tilde{v_j}'|| : j = 1, 2\}$$

and $\tilde{v_2}$ be the remaining in $\{\tilde{v_1}', \tilde{v_2}', \tilde{v_3}'\}$.

Step 13.2.1.1 : Let $\hat{x_j} = (f_1\ f_2\ f_3)\tilde{v_j}$ for $j = 1, 2, 3$. Then $\sum_{j=1}^{3} \hat{x_j} \otimes \hat{x_j} = A$ and $\{\hat{x_j}\}_{j=1}^{3}$ are unit, equiangular in $\mathbb{C}^3$.

Step 14.2.1.1 : Suppose that $\hat{x_j} = (\alpha_j, \beta_j, \mu_j)^T$. Then $y_j' = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^{3} y_j' \otimes y_j' = \sum_{j=1}^{3} y_j \otimes y_j$. Stop.

Case 2.1.2 : $x' = z' = 0$

Step 12.2.1.2 : Let $\hat{x_j} = (f_1\ f_2\ f_3)\tilde{w_j}$ for $j = 1, 2, 3$. Then $\sum_{j=1}^{3} \hat{x_j} \otimes \hat{x_j} = A$ and

$\{\hat{x}_j\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.1.7 : Suppose that $\hat{x}_j = (\alpha_j, \beta_j, \mu_j)^T$. Then $y'_j = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^3 y'_j \otimes y'_j = \sum_{j=1}^3 y_j \otimes y_j$. Stop.

Case 2.1.3 : $z' = 0, x' \neq 0, y' \neq 0$

Step 10.2.1.3 : We will construct a vector $\tilde{z}_3$ such that $|\tilde{z}_3| = |\tilde{w}_3|$ which has the same phase as $\tilde{x}_3$ as follows. Let

$$\Gamma = \begin{pmatrix} e^{i(\beta_1 - \beta'_1)} & 0 & 0 \\ 0 & e^{i(\beta_2 - \beta'_2)} & 0 \\ 0 & 0 & e^{i(-\zeta_2 + \zeta'_2 + \beta_1 - \beta'_1)} \end{pmatrix}$$

Let $\tilde{z}_j = \Gamma \tilde{w}_j$ for $j = 1, 2, 3$. Then $|\tilde{w}_j| = |\tilde{z}_j|$ and $\sum_{j=1}^3 \tilde{z}_j \otimes \tilde{z}_j = \tilde{A}$. Note that $\{\tilde{z}_j\}_{j=1}^3$ are unit, equiangular.

Step 11.2.1.3 : Let

$$\nu_1 = e^{i(\theta_1 - \theta'_1 - \beta_1 + \beta'_1)}, \nu_2 = e^{i(\alpha_1 - \alpha'_1 - \beta_1 + \beta'_1)}$$

Let $\tilde{v}_1 = \nu_1 \tilde{z}_1, \tilde{v}_2 = \nu_2 \tilde{z}_2, \tilde{v}_3 = \tilde{z}_3$. Then $\{\tilde{v}_j\}_{j=1}^3$ are unit, equiangular and $\sum_{j=1}^3 \tilde{v}_j \otimes \tilde{v}_j = \sum_{j=1}^3 \tilde{z}_j \otimes \tilde{z}_j = \tilde{A}$.

Step 12.2.1.3 : Let $\hat{x}_j = (f_1\ f_2\ f_3)\tilde{v}_j$ for $j = 1, 2, 3$. Then $\sum_{j=1}^3 \hat{x}_j \otimes \hat{x}_j = A$ and $\{\hat{x}_j\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.2.1.3 : Suppose that $\hat{x}_j = (\alpha_j, \beta_j, \mu_j)^T$. Then $y'_j = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^3 y'_j \otimes y'_j = \sum_{j=1}^3 y_j \otimes y_j$. Stop.

Case 2.1.4 : $x' = 0, y' \neq 0, z' \neq 0$

Step 10.2.1.4 : We will construct a vector $\tilde{z}_3$ such that $|\tilde{z}_3| = |\tilde{w}_3|$ which has the same phase as $\tilde{x}_3$ as follows. Let

$$\Gamma = \begin{pmatrix} e^{i(\zeta_1-\zeta_1'+\beta_2-\beta_2')} & 0 & 0 \\ 0 & e^{i(\beta_2-\beta_2')} & 0 \\ 0 & 0 & e^{i(\beta_3-\beta_3')} \end{pmatrix}$$

Let $\tilde{z}_j = \Gamma\tilde{w}_j$ for $j = 1,2,3$. Then $|\tilde{w}_j| = |\tilde{z}_j|$ and $\sum_{j=1}^{3} \tilde{z}_j \otimes \tilde{z}_j = \tilde{A}$. Note that $\{\tilde{z}_j\}_{j=1}^{3}$ are unit, equiangular.

Step 11.2.1.4 : Let

$$\nu_1 = e^{i(\theta_2-\theta_2'-\beta_2+\beta_2')}, \nu_2 = e^{i(\alpha_2-\alpha_2'-\beta_2+\beta_2')}$$

Let $\tilde{v}_1 = \nu_1\tilde{z}_1, \tilde{v}_2 = \nu_2\tilde{z}_2, \tilde{v}_3 = \tilde{z}_3$. Then $\{\tilde{v}_j\}_{j=1}^{3}$ are unit, equiangular and $\sum_{j=1}^{3} \tilde{v}_j \otimes \tilde{v}_j = \sum_{j=1}^{3} \tilde{z}_j \otimes \tilde{z}_j = \tilde{A}$.

Step 12.2.1.4 : Let $\hat{x}_j = (f_1\ f_2\ f_3)\tilde{v}_j$ for $j = 1,2,3$. Then $\sum_{j=1}^{3} \hat{x}_j \otimes \hat{x}_j = A$ and $\{\hat{x}_j\}_{j=1}^{3}$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.2.1.4 : Suppose that $\hat{x}_j = (\alpha_j, \beta_j, \mu_j)^T$. Then $y_j' = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1,2,3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^{3} y_j' \otimes y_j' = \sum_{j=1}^{3} y_j \otimes y_j$. Stop.

Case 2.1.5 : $y' = 0, x' \neq 0, z' \neq 0$

Step 10.2.1.5 : We will construct a vector $\tilde{z}_3$ such that $|\tilde{z}_3| = |\tilde{w}_3|$ which has the same phase as $\tilde{x}_3$ as follows. Let

$$\Gamma = \begin{pmatrix} e^{i(\beta_1-\beta_1')} & 0 & 0 \\ 0 & e^{i(\zeta_1'-\zeta_1+\beta_1-\beta_1')} & 0 \\ 0 & 0 & e^{i(\beta_3-\beta_3')} \end{pmatrix}$$

Let $\tilde{z}_j = \Gamma\tilde{w}_j$ for $j = 1,2,3$. Then $|\tilde{w}_j| = |\tilde{z}_j|$ and $\sum_{j=1}^{3} \tilde{z}_j \otimes \tilde{z}_j = \tilde{A}$. Note that $\{\tilde{z}_j\}_{j=1}^{3}$ are unit, equiangular.

Step 11.2.1.5 : Let

$$\nu_1 = e^{i(\theta_1 - \theta_1' - \beta_1 + \beta_1')}, \nu_2 = e^{i(\alpha_1 - \alpha_1' - \beta_1 + \beta_1')}$$

Let $\tilde{v}_1 = \nu_1 \tilde{z}_1, \tilde{v}_2 = \nu_2 \tilde{z}_2, \tilde{v}_3 = \tilde{z}_3$. Then $\{\tilde{v}_j\}_{j=1}^3$ are unit, equiangular and $\sum_{j=1}^3 \tilde{v}_j \otimes \tilde{v}_j = \sum_{j=1}^3 \tilde{z}_j \otimes \tilde{z}_j = \tilde{A}$.

Step 12.2.1.5 : Let $\hat{x}_j = (f_1\ f_2\ f_3)\tilde{v}_j$ for $j = 1, 2, 3$. Then $\sum_{j=1}^3 \hat{x}_j \otimes \hat{x}_j = A$ and $\{\hat{x}_j\}_{j=1}^3$ are unit, equiangular in $\mathbb{C}^3$.

Step 13.2.1.5 : Suppose that $\hat{x}_j = (\alpha_j, \beta_j, \mu_j)^T$. Then $y_j' = \alpha_j h_1 + \beta_j h_2 + \mu_j h_3$ for $j = 1, 2, 3$ are three unit vectors in $\mathbb{C}^n$ which are equiangular and $\sum_{j=1}^3 y_j' \otimes y_j' = \sum_{j=1}^3 y_j \otimes y_j$. Stop.

Case 2.2 : $|\tilde{x_1}| = |\tilde{x_3}| \neq |\tilde{x_2}|$.

Similar to *Case 2.1*

Case 2.3 : $|\tilde{x_2}| = |\tilde{x_3}| \neq |\tilde{x_1}|$.

Similar to *Case 2.1*

First, we will prove that when $\{y_j\}_{j=1}^3$ is equiangular in $\mathbb{C}^n$, $y_j' = y_j$ for all $j = 1, 2, 3$. Since $\{y_j\}_{j=1}^3$ is equiangular in $\mathbb{C}^n$, $\{x_j\}_{j=1}^3$ is equiangular in $\mathbb{C}^3$ and $\{\tilde{x_j}\}_{j=1}^3$ in $\mathbb{C}^3$ is equiangular as well. First, we will show that when $\{\tilde{x_j}\}_{j=1}^3$ is equiangular in $\mathbb{C}^n$, $|\tilde{w_j}| = |\tilde{x_j}|$ for $j = 1, 2, 3$. Because $\{\tilde{x_j}\}_{j=1}^3$ and $\{\tilde{x_j}'\}_{j=1}^3$ are equiangular and have the same Bessel operator $\tilde{A}$, they are geometrically equivalent by Lemma (II.19). Therefore, there are scalars $\{d_j\}_{j=1}^3$ of modulus 1, a permutation Π of $\{1, 2, 3\}$ and a unitary operator U which commutes with $\tilde{A}$ such that $\tilde{x_j} = d_j U \tilde{x}'_{\Pi(j)}$ for all $j = 1, 2, 3$. Since $\tilde{A}$ is diagonal matrix with $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and U commutes with $\tilde{A}$, U must be diagonal. Let $U = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}$ where $|\omega_1| = |\omega_2| = |\omega_3| = 1$. Hence, $|\tilde{x_j}| = |\tilde{x}'_{\Pi(j)}|$. From the construction of w_j we have $|\tilde{w_j}| = |\tilde{x_j}|$.

Now we will consider each case to see that $y'_j = y_j$ for all $j = 1, 2, 3$. Since in cases (1.1)-(1.7), the steps 12 and 13 are the same, we will prove in each case $\tilde{v_j} = \tilde{x_j}$. From that and $\hat{x_j} = (f_1\ f_2\ f_3)\tilde{v_j}$ and $x_j = (f_1\ f_2\ f_3)\tilde{x_j}$, we have $\hat{x_j} = x_j$. It follows immediately that $y'_j = y_j$ by step 13.

For *Case(1.1)*, if $\{\tilde{x_j}\}_{j=1}^3$ is equiangular in $\mathbb{C}^3$ then $|\tilde{x_j}| = |\tilde{w_j}| = |\tilde{z_j}| = |\tilde{v_j}|$. Since $\tilde{z_1}$ has the same phase as $\tilde{x_1}$ we have $\tilde{z_1} = \tilde{x_1}$ and so $\tilde{v_1} = \tilde{x_1}$. Since $\sum_{j=1}^3 \tilde{z_j} \otimes \tilde{z_j} = \sum_{j=1}^3 \tilde{x_j} \otimes \tilde{x_j} = \tilde{A}$, there exist scalars $\{d_j\}_{j=1}^3$ of modulus 1, a permutation Π of $\{1, 2, 3\}$ and an unitary matrix ψ commuting with $\tilde{A}$ such that $\tilde{z_j} = d_j \psi \tilde{x}_{\Pi(j)}$. Because $\tilde{A}$ is diagonal with $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and ψ commutes with $\tilde{A}$, ψ must be diagonal. Let $\psi = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$ where α_j are complex numbers of modulus 1. So $|\tilde{x_j}| = |\tilde{z_j}| = |\tilde{x}_{\Pi(j)}|$. Since $|\tilde{x_1}| \neq |\tilde{x_2}| \neq |\tilde{x_3}|$, $\Pi = 1$. Therefore, $\tilde{z_j} = d_i \psi \tilde{x}_j$ for $j = 1, 2, 3$. From $\tilde{z_1} = \tilde{x_1}$ and all components of $\tilde{x_1}$ are different from 0, it follows that $1 = d_1\alpha_1 = d_1\alpha_2 = d_1\alpha_3$. Thus, $\alpha_1 = \alpha_2 = \alpha_3 = \bar{d_1}$ and we denote this number α. So for $j = 2, 3$, we have $\tilde{z_j} = d_j \alpha \tilde{x}_j$. Hence,

$$d_2\alpha = \frac{arg(\tilde{z}_{21})}{arg(\tilde{x}_{21})} = \frac{arg(\tilde{z}_{21})arg(\tilde{x}_{11})}{arg(\tilde{x}_{21})arg(\tilde{z}_{11})} = \frac{1}{\nu_2},\ d_3\alpha = \frac{arg(\tilde{z}_{31})}{arg(\tilde{x}_{31})} = \frac{arg(\tilde{z}_{31})arg(\tilde{x}_{11})}{arg(\tilde{x}_{31})arg(\tilde{z}_{11})} = \frac{1}{\nu_3}$$

Thus, $\tilde{v_2} = \nu_2 \tilde{z_2} = \tilde{x_2}, \tilde{v_3} = \nu_3 \tilde{z_3} = \tilde{x_3}$. Therefore, $\tilde{v_j} = \tilde{x_j}$.

For *Case(1.2)* and *Case(1.3)*, by using the similar argument as *Case(1.1)*, we also have $\tilde{v_j} = \tilde{x_j}$.

For *Case(1.4)*, if $\{\tilde{x_j}\}_{j=1}^3$ is equiangular in $\mathbb{C}^3$ then $|\tilde{x_j}| = |\tilde{w_j}| = |\tilde{z_j}| = |\tilde{v_j}|$ and $\tilde{x}_{j1} = \tilde{z}_{j1} = \tilde{v}_{j1}, j = 1, 2, 3$. Since $\sum_{j=1}^3 \tilde{z_j} \otimes \tilde{z_j} = \sum_{j=1}^3 \tilde{x_j} \otimes \tilde{x_j} = \tilde{A}$, there exist scalars $\{d_j\}_{j=1}^3$ of modulus 1, a permutation Π of $\{1, 2, 3\}$ and an unitary matrix ψ commuting with $\tilde{A}$ such that $\tilde{z_j} = d_j \psi \tilde{x}_{\Pi(j)}$. Because $\tilde{A}$ is diagonal with $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and

ψ commutes with $\tilde{A}$, ψ must be diagonal. Let $\psi = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$ where α_j are complex numbers of modulus 1. So $|\tilde{x_j}| = |\tilde{z_j}| = |\tilde{x}_{\Pi(j)}|$. Since $|\tilde{x_1}| \neq |\tilde{x_2}| \neq |\tilde{x_3}|$, $\Pi = 1$. Therefore, $\tilde{z_j} = d_i \psi \tilde{x}_j$ for $j = 1, 2, 3$. From $\tilde{x}_{11} \neq 0, \tilde{x}_{21} \neq 0, \tilde{x}_{31} \neq 0$, it follows that $1 = d_1\alpha_1 = d_2\alpha_1 = d_3\alpha_1$. Thus, $d_1 = d_2 = d_3 = \bar{\alpha_1}$ and we denote this number d. Hence,

$$d\alpha_2 = \frac{arg(\tilde{z}_{12})}{arg(\tilde{x}_{12})} = \frac{arg(\tilde{x}_{11})arg(\tilde{z}_{12})}{arg(\tilde{z}_{11})arg(\tilde{x}_{12})} = \nu_2,\ d\alpha_3 = \frac{arg(\tilde{z}_{13})}{arg(\tilde{x}_{13})} = \frac{arg(\tilde{x}_{11})arg(\tilde{z}_{13})}{arg(\tilde{z}_{11})arg(\tilde{x}_{13})} = \nu_3$$

Thus, $\tilde{v}_{12} = \frac{1}{\nu_2}\tilde{z}_{12} = \frac{1}{d\alpha_2}\tilde{z}_{12} = \tilde{x}_{12}$. Similarly, we have $\tilde{v}_{22} = \tilde{x}_{22}, \tilde{v}_{32} = \tilde{x}_{32}$ and $\tilde{v}_{j3} = \tilde{x}_{j3}, j = 1, 2, 3$. Therefore, $\tilde{v_j} = \tilde{x_j}$.

For *Case(1.5)* and *Case(1.6)*, by using the similar argument as *Case(1.4)*, we also have $\tilde{v_j} = \tilde{x_j}$.

For *Case(1.7)*, suppose that $\{\tilde{x_j}\}_{j=1}^3$ is an equiangular set. Let M be a matrix whose columns are $\tilde{x_j}$, that is,

$$M = \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{21} & \tilde{x}_{31} \\ \tilde{x}_{12} & \tilde{x}_{22} & \tilde{x}_{32} \\ \tilde{x}_{13} & \tilde{x}_{23} & \tilde{x}_{33} \end{pmatrix}$$

In the case (1.7), each column and each row has a zero element. By Lemma II.22, there exists a pair of vectors in $\{\tilde{x_j}\}_{j=1}^3$ is orthogonal which is impossible because $\{\tilde{x_j}\}_{j=1}^3$ must form an orthonormal basis for $\mathbb{C}^3$ and so $\lambda_1 = \lambda_2 = \lambda_3 = 1$ which contradicts to the hypothesis that $\{\lambda_j\}_{j=1}^3$ are distinct. Hence, $\{\tilde{x_j}\}_{j=1}^3$ can not be an equiangular set.

For Case(2.1.1), if $\{\tilde{x_j}\}_{j=1}^3$ is equiangular in $\mathbb{C}^3$ then $|\tilde{x_j}| = |\tilde{w_j}| = |\tilde{z_j}| = |\tilde{v_j}| = |\tilde{v_j}'|$. Since $\tilde{z_3}$ has the same phase as $\tilde{x_3}$ we have $\tilde{z_3} = \tilde{x_3}$ and so $\tilde{v_3} = \tilde{v_3}' = \tilde{x_3}$.

Since $\sum_{j=1}^{3}\tilde{z_j}\otimes\tilde{z_j} = \sum_{j=1}^{3}\tilde{x_j}\otimes\tilde{x_j} = \tilde{A}$, there exist scalars $\{d_j\}_{j=1}^{3}$ of modulus 1, a permutation Π of $\{1,2,3\}$ and an unitary matrix ψ commuting with $\tilde{A}$ such that $\tilde{z_j} = d_j\psi\tilde{x}_{\Pi(j)}$. Because $\tilde{A}$ is diagonal with $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and ψ commutes with $\tilde{A}$, ψ must be diagonal. Let $\psi = \begin{pmatrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\\ 0 & 0 & \alpha_3\end{pmatrix}$ where α_j are complex numbers of modulus 1. So $|\tilde{x_j}| = |\tilde{z_j}| = |\tilde{x}_{\Pi(j)}|$. Since $|\tilde{x_1}| = |\tilde{x_2}| \neq |\tilde{x_3}|$, $\Pi = 1$ or $\Pi(1) = 2, \Pi(2) = 1, \Pi(3) = 3$. From $\tilde{z_3} = \tilde{x_3}$ and all components of $\tilde{x_3}$ are different from 0, it follows that $1 = d_3\alpha_1 = d_3\alpha_2 = d_3\alpha_3$. Thus, $\alpha_1 = \alpha_2 = \alpha_3 = \bar{d_3}$ and we denote this number α.

If $\Pi = 1$ then $\tilde{z_j} = d_i\psi\tilde{x}_j$ for $j = 1,2,3$. So for $j = 1,2$, we have $\tilde{z_j} = d_j\alpha\tilde{x}_j$. Hence,

$$d_1\alpha = \frac{arg(\tilde{z}_{11})}{arg(\tilde{x}_{11})} = \frac{arg(\tilde{x}_{31})arg(\tilde{z}_{11})}{arg(\tilde{z}_{31})arg(\tilde{x}_{11})} = \frac{1}{\nu_1},\ d_2\alpha = \frac{arg(\tilde{z}_{21})}{arg(\tilde{x}_{21})} = \frac{arg(\tilde{x}_{31})arg(\tilde{z}_{21})}{arg(\tilde{z}_{31})arg(\tilde{x}_{21})} = \frac{1}{\nu_2}$$

Thus, $\tilde{v_1}' = \nu_1\tilde{z_1} = \tilde{x_1}, \tilde{v_2}' = \nu_2\tilde{z_2} = \tilde{x_2}$. Therefore, from the construction of $\tilde{v_j}$, we have $\tilde{v_j} = \tilde{v_j}' = \tilde{x_j}$ for all $j = 1,2,3$.

If $\Pi(1) = 2, \Pi(2) = 1, \Pi(3) = 3$ then $\tilde{z_1} = d_1\alpha\tilde{x_2}, \tilde{z_2} = d_2\alpha\tilde{x_1}, \tilde{z_3} = \tilde{x_3}$. Hence,

$$d_1\alpha = \frac{arg(\tilde{z}_{11})}{arg(\tilde{x}_{21})} = \frac{arg(\tilde{x}_{31})arg(\tilde{z}_{11})}{arg(\tilde{z}_{31})arg(\tilde{x}_{21})} = \frac{1}{\nu_1},\ d_2\alpha = \frac{arg(\tilde{z}_{21})}{arg(\tilde{x}_{11})} = \frac{arg(\tilde{x}_{31})arg(\tilde{z}_{21})}{arg(\tilde{z}_{31})arg(\tilde{x}_{11})} = \frac{1}{\nu_2}$$

Since $\tilde{v_1}' = \nu_1\tilde{z_1} = \tilde{x_2}, \tilde{v_2}' = \nu_2\tilde{z_2} = \tilde{x_1}, \tilde{v_3}' = \tilde{z_3} = \tilde{x_3}$ and from the construction of $\tilde{v_j}$, we have $\tilde{v_j} = \tilde{v_j}' = \tilde{x_j}$ for all $j = 1,2,3$.

For Case (2.1.2), suppose that $\{\tilde{x_j}\}_{j=1}^{3}$ is equiangular in $\mathbb{C}^3$. Since $\sum_{j=1}^{3}\tilde{x_j}\otimes\tilde{x_j} = \tilde{A}$, we have

$$e^{i(\theta_1-\theta_2)}xy + e^{i(\alpha_1-\alpha_2)}xy + e^{i(\beta_1-\beta_2)}x'y' = 0 \qquad \text{(III.1)}$$

$$e^{i(\theta_1-\theta_3)}xz + e^{i(\alpha_1-\alpha_3)}xz + e^{i(\beta_1-\beta_3)}x'z' = 0 \qquad \text{(III.2)}$$

$$e^{i(\theta_2-\theta_3)}yz + e^{i(\alpha_2-\alpha_3)}yz + e^{i(\beta_2-\beta_3)}y'z' = 0 \tag{III.3}$$

$$2x^2 + x'^2 = \lambda_1 \tag{III.4}$$

$$2y^2 + y'^2 = \lambda_2 \tag{III.5}$$

$$2z^2 + z'^2 = \lambda_3 \tag{III.6}$$

From $x' = z' = 0$, it follows that $y' = 1$. From (III.4), (III.6), $x = \sqrt{\frac{\lambda_1}{2}} \neq 0, z = \sqrt{\frac{\lambda_3}{2}} \neq 0$. If $y = 0$ then $\tilde{x_1} \perp \tilde{x_3}$. Since $\{\tilde{x_j}\}_{j=1}^3$ is equiangular in $\mathbb{C}^3$, they form an orthonomal basis in $\mathbb{C}^3$ which in turn implies that $\lambda_1 = \lambda_2 = \lambda_3 = 1$, a contradiction. So $y \neq 0$. From (III.1), (III.2), (III.3), we have

$$e^{i(\theta_1-\theta_2)} + e^{i(\alpha_1-\alpha_2)} = 0 \tag{III.7}$$

$$e^{i(\theta_1-\theta_3)} + e^{i(\alpha_1-\alpha_3)} = 0 \tag{III.8}$$

$$e^{i(\theta_2-\theta_3)} + e^{i(\alpha_2-\alpha_3)} = 0 \tag{III.9}$$

From elementary geometry in the plane and (III.7), we have

$$\theta_1 - \theta_2 = \zeta_1 + 2l_1\Pi, \;\; \alpha_1 - \alpha_2 = \Pi + \zeta_1 + 2m_1\Pi. \tag{III.10}$$

Similarly, from (III.8) we have

$$\theta_1 - \theta_3 = \zeta_2 + 2l_2\Pi, \;\; \alpha_1 - \alpha_3 = \Pi + \zeta_2 + 2m_2\Pi. \tag{III.11}$$

From (III.9), we have

$$\theta_2 - \theta_3 = \zeta_3 + 2l_3\Pi, \;\; \alpha_2 - \alpha_3 = \Pi + \zeta_3 + 2m_3\Pi. \tag{III.12}$$

By subtracting the first equation of (III.10) from the first equation of (III.11), we have

$$\theta_2 - \theta_3 = \zeta_2 - \zeta_1 + 2(l_2 - l_1)\Pi$$

By subtracting the second equation of (III.10) from the second equation of (III.11), we have

$$\alpha_2 - \alpha_3 = \zeta_2 - \zeta_1 + 2(m_2 - m_1)\Pi$$

So we have a contradiction to (III.12)

For Case 2.1.3, from (III.6), $z = \sqrt{\frac{\lambda_3}{2}} \neq 0$. If $x = 0$ then from (III.1), we have $x'y' = 0$, a contradiction. Similarly if $y = 0$ then from (III.1), we have $x'y' = 0$, a contradiction. So x, y, z are nonzero.

By using the same argument as before, we have (III.8),(III.9) and (III.11),(III.12). Thus,

$$\tilde{x_1} = \begin{pmatrix} e^{i(\theta_3+\zeta_2)}x \\ e^{i(\theta_3+\zeta_3)}y \\ e^{i\theta_3}z \end{pmatrix}, \tilde{x_2} = \begin{pmatrix} e^{i(\alpha_3+\Pi+\zeta_2)}x \\ e^{i(\alpha_3+\Pi+\zeta_3)}y \\ e^{i\alpha_3}z \end{pmatrix} = \begin{pmatrix} -e^{i(\alpha_3+\zeta_2)}x \\ -e^{i(\alpha_3+\zeta_3)}y \\ e^{i\alpha_3}z \end{pmatrix}, \tilde{x_3} = \begin{pmatrix} e^{i\beta_1}x' \\ e^{i\beta_2}y' \\ 0 \end{pmatrix}$$

Since $|\tilde{w}_j| = |\tilde{x}_j|$ for all $j = 1, 2, 3$ and $\sum_{j=1}^{3} \tilde{w}_j \otimes \tilde{w}_j = \tilde{A}$, by repeating the same argument as for $\{\tilde{x}_j\}_{j=1}^{3}$, we have

$$e^{i(\theta_1'-\theta_3')} + e^{i(\alpha_1'-\alpha_3')} = 0 \tag{III.13}$$

$$e^{i(\theta_2'-\theta_3')} + e^{i(\alpha_2'-\alpha_3')} = 0 \tag{III.14}$$

$$\theta_1' - \theta_3' = \zeta_2' + 2l_2'\Pi,\ \alpha_1' - \alpha_3' = \Pi + \zeta_2' + 2m_2'\Pi. \tag{III.15}$$

$$\theta_2' - \theta_3' = \zeta_3' + 2l_3'\Pi,\ \alpha_2' - \alpha_3' = \Pi + \zeta_3' + 2m_3'\Pi. \tag{III.16}$$

$$\tilde{w}_1 = \begin{pmatrix} e^{i(\theta_3'+\zeta_2')}x \\ e^{i(\theta_3'+\zeta_3')}y \\ e^{i\theta_3'}z \end{pmatrix}, \tilde{w}_2 = \begin{pmatrix} -e^{i(\alpha_3'+\zeta_2')}x \\ -e^{i(\alpha_3'+\zeta_3')}y \\ e^{i\alpha_3'}z \end{pmatrix}, \tilde{w}_3 = \begin{pmatrix} e^{i\beta_1'}x' \\ e^{i\beta_2'}y' \\ 0 \end{pmatrix}$$

From the construction of $\tilde{z}_j$, we have $\tilde{z}_3 = \tilde{x}_3$ and

$$\tilde{z}_1 = \begin{pmatrix} -e^{i(\theta_3'+\zeta_2'+\beta_1-\beta_1')}x \\ -e^{i(\theta_3'+\zeta_3'+\beta_2-\beta_2')}y \\ e^{i(\theta_3'-\zeta_2+\zeta_2'+\beta_1-\beta_1')}z \end{pmatrix}, \tilde{z}_2 = \begin{pmatrix} -e^{i(\alpha_3'+\zeta_2'+\beta_1-\beta_1')}x \\ -e^{i(\alpha_3'+\zeta_3'+\beta_2-\beta_2')}y \\ e^{i(\alpha_3'-\zeta_2+\zeta_2'+\beta_1-\beta_1')}z \end{pmatrix}$$

Since $\sum_{j=1}^{2} \tilde{z}_j \otimes \tilde{z}_j = \sum_{j=1}^{2} \tilde{x}_j \otimes \tilde{x}_j$, we have

$$2e^{i(\zeta_2-\zeta_3)}xy = 2e^{i(\zeta_2'-\zeta_3'+\beta_1-\beta_1'-\beta_2+\beta_2')}xy$$

Therefore, $\zeta_2' - \zeta_3' + \beta_1 - \beta_1' - \beta_2 + \beta_2' = \zeta_2 - \zeta_3 + 2l\Pi$. We have

$$\nu_1\tilde{z}_1 = \begin{pmatrix} -e^{i(\theta_3'+\zeta_2'+\theta_1-\theta_1')}x \\ -e^{i(\theta_3'+\zeta_3'+\beta_2-\beta_2'+\theta_1-\theta_1'-\beta_1+\beta_1')}y \\ e^{i(\theta_3'-\zeta_2+\zeta_2'+\theta_1-\theta_1')}z \end{pmatrix}$$

$$\nu_2\tilde{z}_2 = \begin{pmatrix} -e^{i(\alpha_3'+\zeta_2'+\alpha_1-\alpha_1')}x \\ -e^{i(\alpha_3'+\zeta_3'+\beta_2-\beta_2'+\alpha_1-\alpha_1'-\beta_1+\beta_1')}y \\ e^{i(\alpha_3'-\zeta_2+\zeta_2'+\alpha_1-\alpha_1')}z \end{pmatrix}$$

Since

$$\theta_3' + \zeta_2' + \theta_1 - \theta_1' = \theta_3' + \theta_1' - \theta_3' + 2l_2'\Pi + \theta_1 - \theta_1' = \theta_1 + 2l_2'\Pi = \zeta_2 + \theta_3 + 2(l_2' + l_2)\Pi$$

and

$$\begin{aligned} \theta_3' + \zeta_3' + \beta_2 - \beta_2' + \theta_1 - \theta_1' - \beta_1 + \beta_1' &= -\zeta_2' - 2l_2\Pi + \zeta_3' + \beta_2 - \beta_2' + \theta_1 - \beta_1 + \beta_1' \\ &= -\zeta_2 + \zeta_3 - 2l_2\Pi + \theta_1 \\ &= \theta_3 - \theta_1 + \zeta_3 + \theta_1 = \theta_3 + \zeta_3 \end{aligned}$$

and

$$\begin{aligned}\theta_3' - \zeta_2 + \zeta_2' + \theta_1 - \theta_1' &= \theta_3' - \zeta_2 + \theta_1' - \theta_3' + 2l_2'\Pi + \theta_1 - \theta_1' \\ &= -\zeta_2 + \theta_1 + 2l_2'\Pi \\ &= \theta_3 + 2(l_2' - l_2)\Pi\end{aligned}$$

we have $\nu_1\tilde{z}_1 = \tilde{x}_1$. Similarly, $\nu_2\tilde{z}_2 = \tilde{x}_2$.

Remark 5. i) We will prove that the case $|\tilde{x_1}| = |\tilde{x_2}| = |\tilde{x_3}|$ can't happen. Suppose that $\tilde{x_1} = (e^{i\theta_1}x, e^{i\theta_2}y, e^{i\theta_3}z)^T, \tilde{x_2} = (e^{i\alpha_1}x, e^{i\alpha_2}y, e^{i\alpha_3}z)^T, \tilde{x_3} = (e^{i\beta_1}x, e^{i\beta_2}y, e^{i\beta_3}z)^T$ where $x, y, z \in \mathbb{R}$. Since $\sum_{j=1}^{3} \tilde{x_j} \otimes \tilde{x_j} = \tilde{A}$, we have

$$e^{i\theta_1}x.e^{-i\theta_1}y + e^{i\alpha_1}x.e^{-i\alpha_2}y + e^{i\beta_1}x.e^{-i\beta_2}y = 0$$

$$e^{i\theta_2}y.e^{-i\theta_3}z + e^{i\alpha_2}y.e^{-i\alpha_3}z + e^{i\beta_2}y.e^{-i\beta_3}z = 0$$

$$e^{i\theta_1}x.e^{-i\theta_3}z + e^{i\alpha_1}x.e^{-i\alpha_3}z + e^{i\beta_1}x.e^{-i\beta_3}z = 0$$

Since $\{\tilde{x_j}\}_{j=1}^{3}$ are linearly independent, x, y, z are nonzero numbers. So it follows that

$$e^{i(\theta_1-\theta_2)} + e^{i(\alpha_1-\alpha_2)} + e^{i(\beta_1-\beta_2)} = 0 \tag{III.17}$$

$$e^{i(\theta_1-\theta_3)} + e^{i(\alpha_1-\alpha_3)} + e^{i(\beta_1-\beta_3)} = 0 \tag{III.18}$$

$$e^{i(\theta_2-\theta_3)} + e^{i(\alpha_2-\alpha_3)} + e^{i(\beta_2-\beta_3)} = 0 \tag{III.19}$$

By multiplying (III.17) by $e^{i(-\theta_1+\theta_2)}$, (III.18) by $e^{i(-\theta_1+\theta_3)}$, (III.19) by $e^{i(-\theta_2+\theta_3)}$, we have

$$1 + e^{i(\theta_2-\theta_1+\alpha_1-\alpha_2)} + e^{i(\theta_2-\theta_1+\beta_1-\beta_2)} = 0 \tag{III.20}$$

$$1 + e^{i(\theta_3-\theta_1+\alpha_1-\alpha_3)} + e^{i(\theta_3-\theta_1+\beta_1-\beta_3)} = 0 \tag{III.21}$$

$$1 + e^{i(\theta_3-\theta_2+\alpha_2-\alpha_3)} + e^{i(\theta_3-\theta_2+\beta_2-\beta_3)} = 0 \tag{III.22}$$

Let $\mu_1 = \theta_2 - \theta_1 + \alpha_1 - \alpha_2, \nu_1 = \theta_2 - \theta_1 + \beta_1 - \beta_2$. So $1 + e^{i\mu_1} + e^{i\nu_1} = 0$. By elementary geometry in a plane, we have either $\mu_1 = \frac{2\Pi}{3} + 2l_1\Pi,\ \nu_1 = \frac{4\Pi}{3} + 2m_1\Pi$ or $\mu_1 = \frac{4\Pi}{3} + 2l_1\Pi,\ \nu_1 = \frac{2\Pi}{3} + 2m_1\Pi$ for l_1, m_1 are integer numbers. So either

$$\theta_2 - \theta_1 + \alpha_1 - \alpha_2 = \frac{2\Pi}{3} + 2l_1\Pi,\ \theta_2 - \theta_1 + \beta_1 - \beta_2 = \frac{4\Pi}{3} + 2m_1\Pi$$

or

$$\theta_2 - \theta_1 + \alpha_1 - \alpha_2 = \frac{4\Pi}{3} + 2l_1\Pi,\ \theta_2 - \theta_1 + \beta_1 - \beta_2 = \frac{2\Pi}{3} + 2m_1\Pi.$$

Similarly, we can prove that either

$$\theta_3 - \theta_1 + \alpha_1 - \alpha_3 = \frac{2\Pi}{3} + 2l_2\Pi, \theta_3 - \theta_1 + \beta_1 - \beta_3 = \frac{4\Pi}{3} + 2m_2\Pi$$

or

$$\theta_3 - \theta_1 + \alpha_1 - \alpha_3 = \frac{4\Pi}{3} + 2l_2\Pi,\ \theta_3 - \theta_1 + \beta_1 - \beta_3 = \frac{2\Pi}{3} + +2m_2\Pi$$

and either

$$\theta_3 - \theta_2 + \alpha_2 - \alpha_3 = \frac{2\Pi}{3} + 2l_3\Pi,\ \theta_3 - \theta_2 + \beta_2 - \beta_3 = \frac{4\Pi}{3} + 2m_3\Pi$$

or

$$\theta_3 - \theta_2 + \alpha_2 - \alpha_3 = \frac{4\Pi}{3} + 2l_3\Pi,\ \theta_3 - \theta_2 + \beta_2 - \beta_3 = \frac{2\Pi}{3} + 2m_3\Pi$$

1) Suppose

$$\theta_2 - \theta_1 + \alpha_1 - \alpha_2 = \frac{2\Pi}{3} + 2l_1\Pi,\ \theta_2 - \theta_1 + \beta_1 - \beta_2 = \frac{4\Pi}{3} + 2m_1\Pi \qquad \text{(III.23)}$$

$$\theta_3 - \theta_1 + \alpha_1 - \alpha_3 = \frac{2\Pi}{3} + 2l_2\Pi,\ \theta_3 - \theta_1 + \beta_1 - \beta_3 = \frac{4\Pi}{3} + 2m_2\Pi \qquad \text{(III.24)}$$

$$\theta_3 - \theta_2 + \alpha_2 - \alpha_3 = \frac{2\Pi}{3} + 2l_3\Pi,\ \theta_3 - \theta_2 + \beta_2 - \beta_3 = \frac{4\Pi}{3} + 2m_3\Pi \qquad \text{(III.25)}$$

By subtracting the first equation of (III.24) from the second equation of (III.24), we have $\beta_1 - \beta_3 + \alpha_3 - \alpha_1 = \frac{2\Pi}{3} + 2(m_2 - l_2)\Pi$. On the other hand, by adding the first equation of (III.23) to the first equation of (III.25), we have $\alpha_1 - \theta_1 + \theta_3 - \alpha_3 = \frac{4\Pi}{3} + 2(l_1 + l_3)\Pi$. By subtracting this equation from the second equation of (III.24), we have $\beta_1 - \beta_3 + \alpha_3 - \alpha_1 = 2(m_2 - l_1 - l_3)\Pi$. So we have a contradiction.

2) Suppose

$$\theta_2 - \theta_1 + \alpha_1 - \alpha_2 = \frac{4\Pi}{3} + 2l_1\Pi,\ \theta_2 - \theta_1 + \beta_1 - \beta_2 = \frac{2\Pi}{3} + 2m_1\Pi \qquad \text{(III.26)}$$

$$\theta_3 - \theta_1 + \alpha_1 - \alpha_3 = \frac{2\Pi}{3} + 2l_2\Pi,\ \theta_3 - \theta_1 + \beta_1 - \beta_3 = \frac{4\Pi}{3} + 2m_2\Pi \qquad \text{(III.27)}$$

$$\theta_3 - \theta_2 + \alpha_2 - \alpha_3 = \frac{2\Pi}{3} + 2l_3\Pi,\ \theta_3 - \theta_2 + \beta_2 - \beta_3 = \frac{4\Pi}{3} + 2m_3\Pi \qquad \text{(III.28)}$$

By subtracting the first equation of (III.27) from the second equation of (III.27), we have $\beta_1 - \beta_3 + \alpha_3 - \alpha_1 = \frac{2\Pi}{3} + 2(m_2 - l_2)\Pi$. On the other hand, by adding the first equation of (III.26) to the first equation of (III.28), we have $\alpha_1 - \theta_1 + \theta_3 - \alpha_3 = 2\Pi + 2(l_1 + l_3)\Pi$. By subtracting this equation from the second equation of (III.27), we have $\beta_1 - \beta_3 + \alpha_3 - \alpha_1 = -\frac{2\Pi}{3} + 2(m_2 - l_1 - l_3)\Pi$. So we have a contradiction.

For other cases we handle similarly. So $|\tilde{x_1}| = |\tilde{x_2}| = |\tilde{x_3}|$ can't happen.

ii) We will prove that in Case 2.1, it is impossible that $y' = z' = 0$. Indeed, since $\sum_{j=1}^{3} \tilde{x_j} \otimes \tilde{x_j} = \tilde{A}$, we have

$$e^{i(\theta_1-\theta_2)}xy + e^{i(\alpha_1-\alpha_2)}xy + e^{i(\beta_1-\beta_2)}x'y' = 0 \qquad \text{(III.29)}$$

$$e^{i(\theta_1-\theta_3)}xz + e^{i(\alpha_1-\alpha_3)}xz + e^{i(\beta_1-\beta_3)}x'z' = 0 \qquad \text{(III.30)}$$

$$e^{i(\theta_2-\theta_3)}yz + e^{i(\alpha_2-\alpha_3)}yz + e^{i(\beta_2-\beta_3)}y'z' = 0 \qquad \text{(III.31)}$$

$$2x^2 + x'^2 = \lambda_1 \qquad \text{(III.32)}$$

$$2y^2 + y'^2 = \lambda_2 \qquad \text{(III.33)}$$

$$2z^2 + z'^2 = \lambda_3 \qquad \text{(III.34)}$$

Due to $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\sum_{j=1}^{3} \lambda_j = 3$ we have $\lambda_1 > 1, \lambda_3 < 1$.

If $y' = z' = 0$ then $x' = 1$. From (III.32), (III.33), (III.34), $x = \sqrt{\frac{\lambda_1 - x'2}{2}} \neq 0, y = \sqrt{\frac{\lambda_2}{2}} \neq 0, z = \sqrt{\frac{\lambda_3}{2}} \neq 0$. From (III.29), (III.30), (III.31), we have

$$e^{i(\theta_1-\theta_2)} + e^{i(\alpha_1-\alpha_2)} = 0 \tag{III.35}$$

$$e^{i(\theta_1-\theta_3)} + e^{i(\alpha_1-\alpha_3)} = 0 \tag{III.36}$$

$$e^{i(\theta_2-\theta_3)} + e^{i(\alpha_2-\alpha_3)} = 0 \tag{III.37}$$

From elementary geometry in the plane and (III.35), we have

$$\theta_1 - \theta_2 = \zeta_1 + 2l_1\Pi,\ \alpha_1 - \alpha_2 = \Pi + \zeta_1 + 2m_1\Pi. \tag{III.38}$$

Similarly, from (0.36) we have

$$\theta_1 - \theta_3 = \zeta_2 + 2l_2\Pi,\ \alpha_1 - \alpha_3 = \Pi + \zeta_2 + 2m_2\Pi. \tag{III.39}$$

From (0.37), we have

$$\theta_2 - \theta_3 = \zeta_3 + 2l_3\Pi,\ \alpha_2 - \alpha_3 = \Pi + \zeta_3 + 2m_3\Pi. \tag{III.40}$$

By subtracting the first equation of (III.36) from the first equation of (III.37), we have

$$\theta_2 - \theta_3 = \zeta_2 - \zeta_1 + 2(l_2 - l_1)\Pi$$

By subtracting the second equation of (III.36) from the second equation of (0.37), we have

$$\alpha_2 - \alpha_3 = \zeta_2 - \zeta_1 + 2(m_2 - m_1)\Pi$$

So we have a contradiction to (III.40).

Similarly, if $x' = y' = 0$ then $z' = 1$. From (III.6), since $\lambda_3 < 1$ and $z' = 1$, we have a contradiction.

CHAPTER IV

PUSH-OUT FRAMES

Let $\mathbb{J} = \{1, 2, ..., k\}$ and $\{x_j\}_{j\in\mathbb{J}}$ be a frame on a Hilbert space H_n. We define a *push-out* of $\{x_j\}_{j\in\mathbb{J}}$ to be a frame $\{z_i\}_{j\in\mathbb{J}}$ on $H_n \oplus \mathbb{R}$ (or $H_n \oplus \mathbb{C}$) of the form $z_j = x_j \oplus b$ for some fixed $b \neq 0$ in $\mathbb{R}$ (or $b \in \mathbb{C}$). Not every frame has a push-out which is a frame. We call the frame $\{x_j\}_{j\in\mathbb{J}}$ which has a push-out frame to be a *root frame*.

Let $F = \{x_j\}_{j\in\mathbb{J}}$ be a sequence of vectors in H_n. A space $\mathcal{D}_F = \text{span}\{x_j - x_l : j \neq l \in \mathbb{J}\}$ is called the *difference space* of F. Let $\delta_F = \dim(\text{span}(F) \ominus \mathcal{D}_F)$. Then for any finite sequence F, we have $\delta_F = 0$ or 1 since $\mathcal{D}_F = \text{span}\{x_j - x_l : j \neq l \in \mathbb{J}\} = \text{span}\{x_j - x_1 : j \neq 1 \in \mathbb{J}\}$. Therefore, $\text{span}(x_1, \mathcal{D}_F) = \text{span}(F)$.

Remark 6. A sequence $\{x_j\}_{j\in\mathbb{J}}$ is a push-out frame on H_n if and only if there is a 1-dimensional subspace E of H_n such that $P_E x_j$ is a constant vector for all $j \in \mathbb{J}$. Indeed, the forward direction is obvious. For the other direction, write $P_E x_j = w, j \in \mathbb{J}$ for some $w \in H_n$. Let $y_j = P_{E^\perp} x_j \in P_{E^\perp} H_n$. Then $\{y_j\}_{j\in\mathbb{J}}$ is a frame in $P_{E^\perp} H_n$ and $x_j = y_j \oplus w$.

Equivalently, a sequence $\{x_j\}_{j\in\mathbb{J}}$ is a push-out frame on H_n if and only if there is a vector $w \neq 0$ in H_n such that $\langle\, x_j \,,\, w \,\rangle$ = constant ,$\forall j \in \mathbb{J}$. To see this, let $E = \text{span}\{w\}$ and write $P_E x_j = \frac{\langle\, x_j \,,\, w \,\rangle}{|w|} \frac{w}{|w|}$.

Lemma IV.1. A sequence F is a push-out frame if and only if $\delta_F = 1$.

Proof. Since F is a push-out frame if anf only if there is a vector $w \neq 0$ such that $\langle\, x_j \,,\, w \,\rangle = \langle\, x_l \,,\, w \,\rangle$ for $j \neq l$ which is equivalent to $\langle\, x_j - x_l \,,\, w \,\rangle = 0$ for $j \neq l$. Therefore, $\mathcal{D}_F$ is a proper subspace of span(F).

□

Example .2. 1) A basis $\mathcal{E} = \{e_j\}_{j=1}^n$ of H_n is a push-out frame on H_n since e_1 can not be written as a linear combination of $\{e_j - e_1 : j = 1, ..., n\}$.

2) The standard orthonormal basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$ is a push-out frame of the Mercedes-Benz frame $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ in $\mathbb{R}^2$ where $\tilde{e}_1 = (\sqrt{\frac{2}{3}}, 0)^T, \tilde{e}_2 = (-\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{2})^T, \tilde{e}_2 = (-\frac{1}{\sqrt{6}}, -\frac{\sqrt{2}}{2})^T$. (A Mercedes-Benz frame is a Parseval frame of three equal-norm vectors whose angle between every pair of vectors is 120 degree.)

Remark 7. Let $\mathcal{F}(H_n)$ be the set of all frames on H_n, $\mathcal{R}(H_n)$ be the set of all root frames on H_n, $\mathcal{P}(H_n)$ be the set of all push-out frames on H_n. It is clear from the above that $\mathcal{F}(H_n) = \mathcal{R}(H_n) \cup \mathcal{P}(H_n)$ and $\mathcal{R}(H_n) \cap \mathcal{P}(H_n) = \emptyset$.

Definition 2. A sequence of vectors is called an *ultra tight root frame* if it is a tight root frame which has a push-out to a tight frame.

We can characterize all ultra tight root frames as follows.

Proposition IV.2. A frame $F = \{x_j\}_{j\in\mathbb{J}}$ is an ultra tight root frame if and only if it is a tight frame and $\sum_{j\in\mathbb{J}} x_j = 0$.

Proof. Suppose that $F = \{x_j\}_{j\in\mathbb{J}}$ is an ultra tight root frame. Then there exists a push-out frame $E = \{z_j\}_{j\in\mathbb{J}}$ on $K = H_n \oplus \mathbb{R}$ (or $H_n \oplus \mathbb{C}$). Thus, there is a vector $w \neq 0$ in $\mathbb{R}$ (or $\mathbb{C}$) such that $P^{\perp} z_j = x_j$ and $P z_j = w$ for all $j \in \mathbb{J}$ where P is the orthogonal projection on span$\{w\}$. Since F is a tight frame on H_n, by Lemma I.5, we have $\sum_{j\in\mathbb{J}} x_j \otimes x_j = \lambda I_{H_n}$ and since F is a tight frame on K, we have $\sum_{j\in\mathbb{J}} z_j \otimes z_j = \alpha I_K$ where I_{H_n}, I_K are identities of H_n, K respctively. We have $\alpha I_K = \sum_{j\in\mathbb{J}} z_j \otimes z_j = \sum_{j\in\mathbb{J}} x_j \otimes x_j + \sum_{j\in\mathbb{J}} x_j \otimes w + w \otimes \sum_{j\in\mathbb{J}} x_j + kw \otimes w = \lambda I_{H_n} + \sum_{j\in\mathbb{J}} x_j \otimes w + w \otimes \sum_{j\in\mathbb{J}} x_j + kw \otimes w$. For any $x \in H_n$, $\alpha I_K(x) = \lambda I_{H_n}(x) + \sum_{j\in\mathbb{J}} x_j \otimes w(x) + w \otimes \sum_{j\in\mathbb{J}} x_j(x) + kw \otimes w(x)$. Since $w \perp H_n$ and $x \in H_n$, we have $\sum_{j\in\mathbb{J}} x_j \otimes w(x) = 0 = kw \otimes w(x)$ and hence, $(\alpha - \lambda)x = \langle x \, , \sum_{j\in\mathbb{J}} x_j \rangle w$. Since $w \perp x$, we get $(\alpha - \lambda)x = 0 = \langle x \, , \sum_{j\in\mathbb{J}} x_j \rangle w$ for any $x \in H_n$. Therefore, $\alpha = \lambda$ and

$\langle x , \sum_{j\in\mathbb{J}} x_j \rangle = 0$. Since x is arbitrary in H_n, we have $\sum_{j\in\mathbb{J}} x_j = 0$.

Now suppose that $\{x_j\}_{j\in\mathbb{J}}$ is tight frame on H_n with frame bound λ and $\sum_{j\in\mathbb{J}} x_j = 0$. Then $\sum_{j\in\mathbb{J}} x_j \otimes x_j = \lambda I_{H_n}$. Let $w = \sqrt{\frac{\lambda}{k}}$. Then $\sum_{j\in\mathbb{J}}(x_j \oplus w) \otimes (x_j \oplus w) = \sum_{j\in\mathbb{J}} x_j \otimes x_j + \sum_{j\in\mathbb{J}} x_j \otimes w + w \otimes \sum_{j\in\mathbb{J}} x_j + kw \otimes w$. Since $\sum_{j\in\mathbb{J}} x_j = 0$, we have $\sum_{j\in\mathbb{J}}(x_j \oplus w) \otimes (x_j \oplus w) = \sum_{j\in\mathbb{J}} x_j \otimes x_j + kw \otimes w = \lambda I_{H_n} \oplus \lambda 1 = \lambda(I_H \oplus 1)$. Let $z_j = x_j \oplus w$. Then $\{z_j\}_{j\in\mathbb{J}}$ is a tight push-out frame of $\{x_j\}_{j\in\mathbb{J}}$. □

Definition 3. A frame $\{x_J\}_{j\in\mathbb{J}}$ is called a *scaled push-out* (*scaled root frame*) of a frame $\{z_j\}_{j\in\mathbb{J}}$ if there are scalars $w_j, j \in \mathbb{J}$ of modulus 1 such that $\{w_j x_j\}_{j\in\mathbb{J}}$ is a push-out frame (a root frame) of $\{z_j\}_{j\in\mathbb{J}}$.

Remark 8. 1)A frame $\{z_j\}_{j\in\mathbb{J}}$ is a scaled push-out on H_n if and only if it is a frame and there is $w \neq 0$ in H_n such that $|\langle x_j , w \rangle| = \text{constant}, \forall j \in \mathbb{J}$.

2)Suppose $\{z_j\}_{j\in\mathbb{J}}$ is an equiangular uniform frame. Then it is a scaled push-out frame. Indeed, let $w = z_1$. Then $\frac{|\langle z_j , z_1 \rangle|}{|z_j||z_1|} = \frac{|\langle z_l , z_1 \rangle|}{|z_l||z_1|}$ and therefore, $|\langle z_j , z_1 \rangle| = |\langle z_l , z_1 \rangle|$ for $j \neq l$.

3)A Parseval uniform frame $\{x_j\}$ of $n+1$ vectors in C^n is a scaled root frame since by Proposition I.7, there exists $\lambda_1, \lambda_2, ..., \lambda_{n+1}$ in $\mathbb{C}^n$ such that $\{x_j \oplus \lambda_j\}$ is an orthonormal basis in $\mathbb{C}^{n+1}$. Thus, $|\lambda_j| = c$ for $c = \sqrt{1 - ||x_j||^2}\ \forall j$. Let $a_j = \frac{\lambda_j}{c}$. So $|a_j| = 1$. It follows that $\{x_j \oplus a_j c\}$ is an orthonormal basis for $\mathbb{C}^{n+1}$, so is $\{\bar{a}_j x_j \oplus c\}$. Hence, $\{\bar{a}_j x_j\}$ is a root frame.

Example .3. A Parseval uniform frame of $n+1$ vectors in $\mathbb{R}^n$ may be a push-out frame. Let $x_1 = \frac{\sqrt{2}}{2}(1, 0, \frac{\sqrt{2}}{2})^T, x_2 = \frac{\sqrt{2}}{2}((-1, 0, \frac{\sqrt{2}}{2})^T, x_3 = \frac{\sqrt{2}}{2}(0, 1, \frac{\sqrt{2}}{2})^T, x_4 = \frac{\sqrt{2}}{2}(0, -1, \frac{\sqrt{2}}{2})^T$. Then $\{x_1, x_2, x_3, x_4\}$ is a Parseval uniform frame of four vectors in $\mathbb{R}^3$ which is a push-out frame.

We also note that every frame that contains a root frame as a subset is a root frame.

Definition 4. A root frame is called *minimal* if no proper subset is root frame itself.

Proposition IV.3. 1)A sequence $F = \{x_i : i = 1, ..., k\}$ is a minimal root frame in $\mathbb{R}^n$, $k \geq n+1$ if and only if span$(F) = \mathbb{R}^n = \mathcal{D}_F$ (the difference space) and no vector in F is convex combination of the rest.

2) A minimal root frame F in $\mathbb{R}^n$ is a root frame that has a push-out to a basis.

3)A minimal root frame in $\mathbb{R}^n$ have cardinality $n+1$.

Proof. 1) F is root frame if and only if span$(F) = \mathbb{R}^n = \mathcal{D}_F$. Therefore, F is minimal root frame if and only if there is $c \neq 0$ such that $\dim(\text{span}\{(x_j \oplus c) : j = 1, 2, ..., k\}) = n+1$ and any subset of $k-1$ vectors of $A = \{(x_j \oplus c) : j = 1, 2, ..., k\}$ spans a n-dimensional subspace. Equivalently, for every j, $(x_j \oplus c)$ is not a linear combination of $A \setminus \{(x_j \oplus c)\}$. That means, there does not exist a set of real scalars $\{a_l : l \in \{1, 2, ..., k\} \setminus \{j\}\}$ such that $(x_j \oplus c) = \sum_{l \neq j} a_l(x_l \oplus c)$. It is equivalent to say that for any j, x_j can not be written as $\sum_{l \neq j} a_l x_l$ with $\sum_{l \neq j} a_l = 1$, which is equivalent to x_j is not a convex combination of $F \setminus \{x_j\}$ for any j.

2) Since F is a minimal root frame, the above push-out frame A spans a $(n+1)$-dimensional space and every proper subset of A spans a space of dimension less than $n+1$. Therefore, no proper subset of A is a basis for $\mathbb{R}^{n+1}$. Because every spanning set for a finite dimensional space has a subset which is basis, A itself is a basis for $\mathbb{R}^{n+1}$.

3) It follows directly from part 2. $\square$

Proposition IV.4. 1) For any frames $F = \{x_j\}_{j=1}^k$ in H_n except bases, there are scalars $\{a_j\}_{j=1}^k$ such that the scaled frame $A = \{\frac{x_j}{a_j}\}_{j=1}^k$ is a root frame.

2)For any frames $F = \{x_j\}_{j=1}^k$ in H_n except frames contain zero vectors, there are scalars $\{a_j\}_{j=1}^k$ such that the scaled frame $A = \{\frac{x_j}{a_j}\}_{j=1}^k$ is a push-out frame.

Proof. 1) If F is a basis then $k = n$ and any push-out sequence of scaled basis consisting of n vectors cannot span $(n+1)$-dimensional space. Thus, a scaled basis is not a root frame. Suppose F is a frame which is not a basis. Then span$(F) = H_n$. Let M be a $k \times n$ matrix whose rows are x_j. Consider n column vectors $y_1, y_2, ..., y_n$ of M in k-dimensional space. Since $k > n$, there is a vector $a = (a_1, a_2, ..., a_k)$ in H_k with $a_j \neq 0, \forall j$ such that $a \notin \text{span}\{y_1, y_2, ..., y_n\}$. Then $\{\frac{x_j}{a_j}\}_{j=1}^k$ is a scaled frame which is root frame.

2) It is obvious that if F contains zero vectors, then any scaled frame also contains zero vectors and hence can't be a push-out frame. Now suppose F is a frame for H_n but not a push-out frame with $x_j \neq 0, \forall j$. Then $k > n$. Let $\mathcal{E}$ be a subspace of H_n with dimension $n-1$ which doesn't contain vectors $\{x_j\}_{j=1}^k$. Let P be the orthogonal projection onto $\mathcal{E}$ and let $a_j = P^{\perp}(x_j), \forall\ j$. Since $\mathcal{E}$ doesn't contain vectors $\{x_j\}_{j=1}^k$, we have $a_j \neq 0, \forall j$. Hence $\{\frac{x_j}{a_j}\}_{j=1}^k$ is a scaled frame which is a push-out of the frame $\{P(\frac{x_j}{a_j})\}_{j=1}^k$. □

Proposition IV.5. Suppose $F = \{x_j\}_{j=1}^k$ is a strictly equiangular root frame in $\mathbb{R}_n$. Then F is a tight frame.

Proof. Suppose $\langle\, x_j\,,\, x_l\,\rangle = a, \forall\ j \neq l$ and $||x_j|| = b, \forall\ j$. Let $S = \sum_{j=1}^k x_j \otimes x_j$. Then

$$\begin{aligned} Sx_j &= ax_1 + ax_2 + ... + b^2 x_j + ... + ax_k \\ Sx_l &= ax_1 + ax_2 + ... + b^2 x_l + ... + ax_k \end{aligned}$$

By substracting two above equations, we get:

$$S(x_j - x_l) = b^2(x_j - x_l) + a(x_l - x_j) = (b^2 - a)(x_j - x_l), \forall\ j \neq l\ .$$

Since F is a root frame for $\mathbb{R}_n$, span$\{x_j - x_l : j \neq l\} = \mathbb{R}_n$, $S = (b^2 - a)I$ and hence F is tight. □

Remark 9. 1) In $\mathbb{R}^2$, by multiplying by -1 and rotating if necessary, every unit norm equiangular frame is a scalar multiple of a Mercedes-Benz frame which is tight and

$|\langle x_j , x_l \rangle| = \frac{1}{2}$ for $j \neq l$ but in $\mathbb{C}^2$, it is no longer true. Consider $x_1 = (1,0)^T, x_2 = \left(\frac{1}{\sqrt{3}}, i\frac{\sqrt{2}}{\sqrt{3}}\right)^T, x_3 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} - \frac{i}{\sqrt{6}}\right)^T$ which is a non-tight unit norm equiangular frame in $\mathbb{C}^2$ with $|\langle x_j , x_l \rangle| = \frac{1}{\sqrt{3}}$ for $j \neq l$.

One natural question is that if every equal-norm equiangular frame of three vectors in $\mathbb{C}^3$ is geometrically equivalent to a push-out of a scalar multiple of a Mercedes-Benz frame in $\mathbb{C}^2$. The answer is negative. One example of a non-tight equiangular frame in $\mathbb{C}^3$ of three vectors is

$$\left\{ x_1 = (-i, 0, 1)^T, x_2 = \left(\frac{1}{\sqrt{3}}, i\frac{\sqrt{2}}{\sqrt{3}}, 1\right)^T, x_3 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{6}}, 1\right)^T \right\}$$

which is a push-out of a non-tight unit norm equiangular frame in $\mathbb{C}^2$.

Example .4. Let $x_1 = (1,0)^T, x_2 = \left(\frac{1}{\sqrt{3}}, i\frac{\sqrt{2}}{\sqrt{3}}\right)^T, x_3 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{6}}\right)^T$, $x_4 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{6}}\right)^T$. Then $\{x_j\}_{j=1}^4$ is an equiangular frame in $\mathbb{C}^2$ which is an optimal Grassmannian frame since $\mathcal{M}(\{x_j\}_{j=1}^4) = \frac{\sqrt{3}}{3}$.

Remark 10. In an infinite dimensional separable Hilbert space, it is still true that $\delta_X = 0$ or 1 for any set X. However, any infinite Bessel sequence $X = \{x_j\}_{j=1}^\infty$ which spans an infinite dimensional space has $\delta_X = 0$ because if there is some vector $u \neq 0$ in $\overline{\text{span}}(X)$ and $u \perp \mathcal{D}_X$, then $\langle u , x_j - x_l \rangle = 0, \forall\, j \neq l$. So $\langle u , x_j \rangle = c, \forall\, j$. It follows that $\sum_{j=1}^\infty |\langle u , x_j \rangle|^2 = \infty$ if $c \neq 0$. Since X is a Bessel sequence, $\sum_{j=1}^\infty |\langle u , x_j \rangle|^2 < \infty$. So $\overline{\text{span}}(X) = \mathcal{D}_X$ and thus, $\delta_X = 0$.

In particular, any infinite frame is a root frame. But a Schauder basis X in an infinite dimensional separable space H can have $\delta_X = 1$. For example, let H be an infinite dimensional space with othonormal basis $\{e_j\}_{j=1}^\infty$ and let $x_j = j^2 e_j, \forall\, j$. Then $\{x_j\}_{j=1}^\infty$ is a Schauder basis but not a frame. Let $x = \sum_{j=1}^\infty \frac{e_j}{j^2}$. Then $\langle x , x_j \rangle = \langle x , x_l \rangle = 1, \forall\, j \neq l$ and therefore $\langle x , x_j - x_l \rangle = 0, \forall j \neq l$ and $\overline{\text{span}}\{x_j - x_l : j \neq l\} \neq H$. It implies that $\delta_X = 1$.

CHAPTER V

GROUP FRAMES

Let G be a group and $\mathcal{U}(\mathcal{B}(H))$ be the group of all unitary operators on a Hilbert space H. A unitary representation Π of G on H is a group homomorphism from G into $\mathcal{U}(\mathcal{B}(H))$. In other words, for every $g, h \in G$, $\Pi(g), \Pi(h)$ are unitary operators on H such that $\Pi(g)\Pi(h) = \Pi(gh)$ and $\Pi(g^{-1}) = \Pi(g)^{-1}$.

Let $\Pi : G \to \mathcal{U}(\mathcal{B}(H))$ be a unitary representation of G on H. If there is $x \in H$ such that $\{\Pi(g)x\}_{g\in G}$ is a frame for H then the frame is called a *group frame* and the vector is called a *frame vector* for G.

Lemma V.1. Suppose $\mathcal{U}$ is a countable group of unitary operators on H which has a frame vector x, and let S be the frame operator of $\{\mathcal{U}x\}$. Then S commutes with $\mathcal{U}$.

Proof. We have:

$$S = \sum_{U\in\mathcal{U}} (Ux) \otimes (Ux) = \sum_{U\in\mathcal{U}} U(x \otimes x)U^*.$$

For every $V \in \mathcal{U}$,

$$VSV^* = \sum_{U\in\mathcal{U}} VU(x \otimes x)U^*V^* = \sum_{U_1\in\mathcal{U}} U_1(x \otimes x)U_1^* = S.$$

where $U_1 = VU$. So $VS = SV$. □

We use the fact that every positive operator Q has a unique positive square root $Q^{1/2}$ which commutes with every operator in $\mathcal{B}(H)$ that commutes with Q.

The left regular representation Π_L is a map from G to $\mathcal{U}(\ell^2(G))$ defined by $\Pi_L(g)(\xi_h) = \xi_{gh}$ where $g, h \in G$ and $\{\xi_g\}_{g\in G}$ is the standard orthonormal basis in $\ell^2(G)$, that is, $\xi_g(h) = 0$ if $h \neq g$ and $\xi_g(g) = 1$.

Lemma V.2. Suppose G is a countable group and $\{x_g\}_{g\in G}$ is a frame indexed by G for H satisfying

$$\langle x_{hg1}, x_{hg2}\rangle = \langle x_{g1}, x_{g2}\rangle \tag{V.1}$$

for all $h, g_1, g_2 \in G$. Let Θ be the analysis operator of the frame $\{x_g\}_{g\in G}$. Then $\mathrm{ran}(\Theta)$ is invariant under $\Pi_L(G)$.

Proof. Using $\Pi_L(h)\{\lambda_g\}_{g\in G} = \{\lambda_{h^{-1}g}\}_{g\in G}$ for every sequence $\{\lambda_g\}_{g\in G} \in \ell^2(G)$ and (V.1), we have for every $h \in G$,

$$\begin{aligned}
\Pi_L(h)\Theta(x_l) &= \Pi_L(h)\{\langle\, x_l\,,\, x_g\,\rangle\}_{g\in G} \\
&= \{\langle\, x_l\,,\, x_{h^{-1}g}\,\rangle\}_{g\in G} \\
&= \{\langle\, x_{h^{-1}g'}\,,\, x_{h^{-1}g}\,\rangle\}_{g\in G} \\
&= \{\langle\, x_{g'}\,,\, x_g\,\rangle\}_{g\in G} \\
&= \Theta(x'_g)
\end{aligned}$$

where $g' = hl$. Therefore, since $H = \overline{\mathrm{span}}(\{x_g\}_{g\in G})$, $\Theta(H)$ is closed and $\Pi_L(h)$ is continuous, we have for every $h \in G$, $\Pi_L(h)(\Theta(H)) \subseteq \Theta(H)$.

□

Proposition V.3. Suppose $\{x_g\}_{g\in G}$ is a Parseval frame for H indexed by G and for every $g_1, g_2, h \in G$, (V.1) holds. Then the frame is a group frame for a unitary representation of G on H.

Proof. Let Θ be the analysis operator of $\{x_g\}_{g\in G}$ and P be the orthogonal projection from $\ell^2(G)$ onto the range of Θ which is a closed subspace of $\ell^2(G)$. Let $\{\xi_g\}$ be the standard orthonormal basis of $\ell^2(G)$. Then range of Θ is invariant under $\Pi_L(G) : \ell^2(G) \to \ell^2(G)$. Let e be the identity element of G. Then for any $g \in G$, since $\Pi_L(g)P = P\Pi_L(g)$ we have $\Pi_L(g)P\xi_e = P\Pi_L(g)\xi_e = P\xi_{ge} = P\xi_g$. Since $\{x_g\}$ is a

Parseval frame, Θ is an isometry and $x_g = \Theta^*|_{\Theta(H)} P\xi_g$ for all $g \in G$ where $\Theta^*|_{\Theta(H)}$ is the adjoint operator of Θ restricted on the range of Θ. Let $\rho : G \to \mathcal{U}(\mathcal{B}(H))$ is defined by $\rho(g) = \Theta^*|_{\Theta(H)}\Pi_L(g)\Theta$. Then $x_g = \Theta^*|_{\Theta(H)}P\xi_g = \Theta^*|_{\Theta(H)}\Pi_L(g)P\xi_e = \Theta^*|_{\Theta(H)}\Pi_L(g)\Theta(x_e)$ because $P\xi_e = \Theta(x_e)$. So $x_g = \rho(g)(x)$ for any $g \in G$ and therefore, $\{x_g\}$ is a group frame for a unitary representation ρ of G on H. □

Lemma V.4. If $\{x_g\}_{g\in G}$ is a group frame for a unitary representation satisfying the condition (V.1) then the corresponding canonical Parseval frame $S^{-1/2}(x_g)$ where S is the frame operator of $\{x_g\}$ also satisfy (V.1).

Proof. Suppose there are a unitary representation ρ and a vector x such that $x_g = \rho(g)x$. Since S commutes with $\rho(g)$, S^{-1} also commutes with $\rho(g)$ for all $g \in G$. So $S^{-1/2}$ commutes with $\rho(g)$ for all $g \in G$ as well. Let $y_g = S^{-1/2}(x_g)$. Then we have $y_g = S^{-1/2}\rho(g)x = \rho(g)S^{-1/2}x$. Thus,

$$\begin{aligned}
\langle\, y_{hg_1}\,,\, y_{hg_2}\,\rangle &= \langle\, \rho(hg_1)S^{-1/2}x\,,\, \rho(hg_2)S^{-1/2}x\,\rangle \\
&= \langle\, \rho(h)\rho(g_1)S^{-1/2}x\,,\, \rho(h)\rho(g_2)S^{-1/2}x\,\rangle \\
&= \langle\, \rho(g_1)S^{-1/2}x\,,\, \rho(g_2)S^{-1/2}x\,\rangle \\
&= \langle\, y_{g_1}\,,\, y_{g_1}\,\rangle
\end{aligned}$$

□

Remark 11. 1) If a frame $\{x_g\}$ indexed by a group G ($\{x_g\}$ is not necessary a group frame) satisfies (V.1), the corresponding canonical Parseval frame $S^{-1/2}(x_g)$ does not necessarily satisfy (V.1). For example, consider a frame F consisting of 4 vectors

$$x_1 = (1,0,0)^T, x_2 = (-\frac{1}{2}, 0, \frac{\sqrt{3}}{2})^T, x_3 = (0, \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}})^T, x_4 = (-\frac{1}{2}, -\sqrt{\frac{2}{3}}, -\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}})$$

We have

$$\langle x_1 , x_2 \rangle = \langle x_2 , x_3 \rangle = \langle x_3 , x_4 \rangle = \langle x_4 , x_1 \rangle = -1/2$$

and

$$\langle x_2 , x_4 \rangle = \langle x_1 , x_3 \rangle = 0 \ .$$

So F satisfy the condition (V.1). But $\langle S^{-1/2}x_3 , S^{-1/2}x_4 \rangle = -1$ while $\langle S^{-1/2}x_1 , S^{-1/2}x_2 \rangle = -1/4$. So the corresponding canonical Parseval frame does not satisfy (V.1).

2)Note that if we drop the condition that $\{x_g\}_{g\in G}$ is a Parseval frame then the Proposition (V.3) fails. The frame F in part 1 is not a Parseval frame, satisfying condition (V.1). This is not a group frame because if there is a unitary representation ρ and a vector x such that $x_j = \rho(j)x$ then $S^{-1/2}(x_j)$ must satisfy (V.1) also by Lemma (V.4) but $S^{-1/2}(x_j)$ does not.

Proposition V.5. If $\{x_g\}_{g\in G}$ is a Parseval frame such that range Θ_X is invariant under the left regular representation $\Pi_L(G)$ then we have $\langle x_{hg1} , x_{hg2} \rangle = \langle x_{g1} , x_{g2} \rangle$ for every $h, g_1, g_2 \in G$ and there is a faithful unitary representation ρ of G on H such that $\{x_g\}_{g\in G}$ is a group frame.

Proof. Since $\text{ran}(\Theta_X)$ is invariant under $\Pi_L(G)$, we have $P\Pi_L(G) = \Pi_L(G)P$ where P is the orthogonal projection from $\ell^2(G)$ onto $\text{ran}(\Theta_X)$. That $\{x_g\}_{g\in G}$ is a Parseval frame implies $x_g = P\xi_g$ and hence

$$\begin{aligned}
\langle x_{hg_1} , x_{hg_2} \rangle &= \langle P\xi_{hg_1} , P\xi_{hg_2} \rangle \\
&= \langle P\Pi_L(h)\xi_{g_1} , P\Pi_L(h)\xi_{g_2} \rangle \\
&= \langle \Pi_L(h)P\xi_{g_1} , \Pi_L(h)P\xi_{g_2} \rangle \\
&= \langle P\xi_{g_1} , P\xi_{g_2} \rangle \\
&= \langle x_{g_1} , x_{g_2} \rangle
\end{aligned}$$

By Proposition (V.3), $\{x_g\}_{g\in G}$ is a group frame and $x_g = \rho(g)x$ where ρ is a unitary representation of G on H and $x \in H$. Let $M = \mathrm{ran}(\Theta_X)$ and let $\tilde{\phi} = \Pi_L(G)|_M$. Since $\Pi_L(G) : M \to M$ is a unitary representation, so is $\tilde{\phi} : G \to \mathcal{B}(M)$. Define $\phi : G \to \mathcal{B}(H)$ by $\phi(g) = \Theta^*\tilde{\phi}(g)\Theta$ for any $g \in G$. Note that $\phi(g) = \Theta^*\Pi_L(G)\Theta$. So ϕ is a unitary representation. It is obvious that ϕ is one to one. So ϕ is a faithful representation. We want to prove that $\phi(g)x_e = x_g$ for all $g \in G$. We have

$$\Theta(x_g) = P\xi_g = P\Pi_L(g)\xi_e = P^2\Pi_L(g)\xi_e = P\Pi_L(g)P\xi_e = \Theta\Theta^*\Pi_L(g)\Theta x_e = \Theta\phi(g)x_e$$

So $x_g = \phi(g)x_e$. □

Remark 12. If we drop the condition that $\{x_g\}$ is a Parseval frame then Proposition (V.3) is no longer true. We consider the following example. Let $G = \mathbf{Z}_3, H = \mathbb{R}^2, x_0 = (1,0)^T, x_1 = (0,-1)^T, x_2 = (-1,1)^T$. Then $\{x_0, x_1, x_2\}$ is a frame with range $\Theta_X = \mathrm{span}\{(1,0,-1)^T, (0,-1,1)^T\}$ which is invariant under the left regular representation $\Pi_L(G)$. Since $\langle x_0, x_1\rangle \neq \langle x_1, x_2\rangle$, this frame is not a group frame for any unitary representation. However, if we consider a mapping $\rho : G \to \mathcal{B}^{-1}(\mathbb{R}^2)$ defined by $\rho(j) = A^j$ where $A : \mathbb{R}^2 \to \mathbb{R}^2$ has a matrix representation

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

then ρ is a group representation (not unitary representation). Let $x = (1,0)^T$. Then x is a frame vector, that is, $x_j = \rho(j)x$ for all $j \in \mathbf{Z}_3$.

CHAPTER VI

COCKTAIL PARTY PROBLEM

The "cocktail party problem" is the problem of how one can separate one sound - perhaps a voice - from a group of other recorded sounds, perhaps a multitude of voices at a cocktail party. Recently, Cassaza, Edidin and Balan [1] gave a solution to this problem by constructing certain Parseval frames for a finite dimensional Hilbert space which permits signal reconstruction from the absolute values of the frame coefficients. In this chapter, we will discuss the mathematics involved in the "cocktail party problem".

Definition 5. 1) A frame $\{x_j\}_{j=1}^k$ in a n-dimensional Hilbert space H_n is said to have $|\Theta|$-*property* if the map $g : H_n \to \mathbb{C}^k$ defined by $g(x) = (|\langle\, x\,,\, x_l\,\rangle|)_{l=1}^k \in \mathbb{C}^k$ is one to one modulo multiples of scalar modulus 1, that is, if $x, y \in H_n$ and $g(x) = g(y)$ then $x = \lambda y$ for some scalar λ with $|\lambda| = 1$.

2) Let $\mathcal{E}$ be a basis for n-dimensional vector space X. A subspace M is said to be *oblique* to $\mathcal{E}$ if the map $f : X \to \mathbb{C}^n$ is one to one modulo multiples of scalar modulus 1 on M, where f is the nonlinear map defined by $f(x) = (|a_j|)_{j=1}^n$ where $(a_j)_{j=1}^n$ is the coefficient vector of x with respect to $\mathcal{E}$.

Lemma VI.1. A frame $\{x_j\}_{j=1}^k$ for H_n has $|\Theta|$-property if and only if the range of analysis operator Θ is oblique with respect to the standard orthonormal basis for $\mathbb{C}^k$

Proof. For the forward direction, assume that $\{x_j\}_{j=1}^k$ has $|\Theta|$-property, M is the range of analysis operator Θ and $y_1, y_2 \in M$ with $f(y_1) = f(y_2)$. So there exist $z_1, z_2 \in H_n$ such that $y_1 = \Theta(z_1), y_2 = \Theta(z_2)$. Hence $|\langle\, z_1\,,\, x_j\,\rangle| = |\langle\, z_2\,,\, x_j\,\rangle|$ for all $j = 1, 2, ..., k$ and so $g(z_1) = g(z_2)$ which implies $z_1 = \lambda z_2$ with $|\lambda| = 1$. Therefore,

$y_1 = \lambda y_2$ and M is oblique with respect to the standard orthonormal basis for $\mathbb{C}^k$.

Now assume that ran(Θ) is oblique with respect to the standard orthonormal basis for $\mathbb{C}^k$ and $y_1, y_2 \in H^n$ such that $g(y_1) = g(y_2)$ which implies $|\langle y_1 , x_j \rangle| = |\langle y_2 , x_j \rangle|$ for all $j = 1, 2, ..., k$. Therefore, $f(\Theta(y_1)) = f(\Theta(y_2))$ and $\Theta(y_1) = \lambda\Theta(y_2)$ with $|\lambda| = 1$. Since Θ is one to one, $y_1 = \lambda y_2$.

□

Remark 13. If N is a subspace of M and M is oblique with respect to $\mathcal{E}$ then so is N.

Lemma VI.2. If some $e_j \in \mathcal{E}$ is in M and if the dimension of M is greater than 2 then M is not oblique with respect to $\mathcal{E}$.

Proof. Without loss of generality we can assume that $e_1 \in \mathcal{E} \cap M$. Let $v = \sum c_j e_j$ be a vector in M linearly independent to e_1. By subtracting a scalar multiple of e_1 if necessary, we can assume $c_1 = 0$. Then $x_1 = v + e_1, x_2 = v - e_1$ are in M and $f(x_1) = f(x_2)$ but x_1, x_2 are linearly independent. □

Definition 6. 1) A subspace $E \subset X$ is *diagonal* with respect to a basis $\mathcal{E}$ if E is a linear span of basis vectors from $\mathcal{E}$. If I is a nonempty subset of $\{1, 2, ..., n\}$, denote $E_I = span\{e_j, j \in I\}$.

2) If E, F are subspaces of X such that $E \cap F = \{0\}$, we will say that E, F are *disjoint*. A pair E, F is called a *nontrivial disjoint pair* if $E \cap F = \{0\}, E \neq \{0\}, F \neq \{0\}$.

Lemma VI.3. If I, J are disjoint nonempty subsets of $\{1, 2, ..., n\}$ such that $M \cap E_I \neq \{0\}$ and $M \cap E_J \neq \{0\}$ then M is not oblique with respect to $\mathcal{E}$.

Proof. Let u, v be nonzero vectors in $M \cap E_I$ and $M \cap E_J$, respectively. Let $x_1 = u + v, x_2 = u - v$. Then $f(x_1) = f(x_2)$ but x_2 is not a scalar multiple of x_1. □

In the real case, Lemma (VI.3) has a converse. Therefore we can characterize the obliqueness.

Proposition VI.4. Let X be a n dimensional real vector space with a basis $\mathcal{E}$ and M be a subspace of X. Then M is oblique with respect to $\mathcal{E}$ if and only if for all diagonal subspaces E_I either $M \cap E_I \neq \{0\}$ or $M \cap E_{I^c} \neq \{0\}$ where $I^c = \{1, 2, ..., n\} \setminus I$

Proof. The forward direction comes from Lemma (VI.3). Now suppose M is not oblique with respect to $\mathcal{E}$. Then there are $x, y \neq 0 \in M, x \notin \{\pm y\}$ such that $f(x) = f(y)$. We write $x = \sum a_j e_j,\ y = \sum b_j e_j,\ a_j,\ b_j \in \mathbb{R}$. Then $|a_j| = |b_j|$ for all j and $b_j = a_j$ or $b_j = -a_j$ for all j. Let $L = \{j : b_j = a_j \neq 0\}, J = \{j : b_j = -a_j \neq 0\}, K = \{j : b_j = a_j = 0\}$. Then $x + y \in E_L \neq \{0\}, x - y \in E_J \neq \{0\}$. So $M \cap E_L \neq \{0\}, M \cap E_J \neq \{0\}$ which implies $M \cap E_{L^c} \neq \{0\}$, a contradiction. □

Lemma VI.5. Suppose that A_1, A_2 are positive operators and let $A = A_1 + A_2$. If $x \in H$ and $Ax = 0$ then $A_1 x = 0 = A_2 x$.

Proof. Since $0 = \langle Ax\,,\, x \rangle = \langle A_1 x\,,\, x \rangle + \langle A_2 x\,,\, x \rangle$ and $\langle A_1 x\,,\, x \rangle \geq 0, \langle A_2 x\,,\, x \rangle \geq 0$, we have $\langle A_1 x\,,\, x \rangle = 0 = \langle A_2 x\,,\, x \rangle$ which implies

$$\langle A_1^{1/2} x\,,\, A_1^{1/2} x \rangle = 0 = \langle A_2^{1/2} x\,,\, A_2^{1/2} x \rangle$$

. So $||A_1^{1/2} x|| = 0 = ||A_2^{1/2} x||$ and $A_1^{1/2} x = 0 = A_2^{1/2} x$. Then $A_1 x = A_1^{1/2}(A_1^{1/2} x) = 0 = A_2^{1/2}(A_2^{1/2} x) = A_2 x$ □

Proposition VI.6. Suppose that P, Q are orthogonal projections with complementary rank in $\mathcal{B}(H)$, i.e. $\text{rank}(Q) = \dim(H) - \text{rank}(P)$ for a finite dimensional Hilbert space H. Then $P(H) \cap Q(H) = \{0\}$ if and only if $P + Q$ is an invertible operator in $\mathcal{B}(H)$.

Proof. Assume that $P+Q$ is an invertible operator in $\mathcal{B}(H)$. Then $(P+Q)(H) = P(H)+Q(H) = H$. Since $\dim H = \dim(P(H)+Q(H)) = \dim P(H) + \dim Q(H) - \dim(P(H)\cap Q(H)) = \dim H - \dim(P(H)\cap Q(H))$, we have $\dim(P(H)\cap Q(H)) = 0$ which implies that $P(H)\cap Q(H) = \{0\}$.

Now suppose that $P(H)\cap Q(H) = \{0\}$. Then $\dim H = \dim(P(H)+Q(H))$ which implies that $P(H)+Q(H) = H$. So $P+Q$ is surjective and $H = P(H)\oplus Q(H)$. If $(P+Q)(x) = 0$ then by Lemma (VI.5), $P(x) = Q(x) = 0$ and therefore $x \in Q(H)\cap P(H) = \{0\}$. So $x = 0$ and $P+Q$ is injective. Hence, $P+Q$ is invertible. □

Corollary VI.7. If P, Q are orthogonal projections in a finite dimensional Hilbert space H with complementary rank then $P(H)\cap Q(H) = \{0\}$ if and only if $\det(P+Q) > 0$.

Proposition VI.8. If M is a n-dimensional subspace of a k-dimensional space H then the set of all subspaces of H of dimension $(k-n)$ that are disjoint from M is open in the set of all subspaces of dimension $(k-n)$ with the topology on subspaces induced by metric $d(M,L) = ||P_M - P_L||$ where P_M, P_L are the orthogonal projections onto M, L, respectively.

Proof. Suppose that N is a $(k-n)$ dimensional subspace of H that is disjoint from M. By Corollary (VI.7), $\det(P_M + P_N) > 0$. Since det is a continuous function, if $||P_W - P_N||$ is small enough then $\det(P_M + P_W) > 0$. From [12], $\dim(P_W(H)) = \dim(P_N(H))$. Therefore, W is a subspace of dimension $(k-n)$ that is disjoint from M □

Lemma VI.9. Let X, Y be two closed subspaces of H. Then

$$d(X,Y) = \max\{\sup\{d(x,Y) : x \in X, ||x|| \leq 1\}, \sup\{d(X,y) : y \in Y, ||y|| \leq 1\}\}$$

Proof. We have $d(x,Y) = ||x - P_Y x|| = ||(I - P_Y)x||$ and

$$\sup\{d(x,Y) : x \in X, ||x|| \leq 1\} = \sup\{\frac{||(I-P_Y)x||}{||x||} : x \neq 0 \in X\}$$

$$\begin{aligned} d(X,Y) = ||P_Y - P_X|| &= \sup\left\{\frac{||(P_Y - P_X)h||}{||h||} : h \neq 0 \in H\right\} \\ &= \sup\left\{\frac{\sqrt{||P_Y(I-P_X)h||^2 + ||(I-P_Y)P_Xh||^2}}{||h||} : h \neq 0 \in H\right\} \\ &\geq \sup\left\{\frac{\sqrt{||P_Y(I-P_X)h||^2 + ||(I-P_Y)P_Xh||^2}}{||h||} : h \neq 0 \in X\right\} \\ &= \sup\left\{\frac{||(I-P_Y)h||}{||h||} : h \neq 0 \in X\right\} \end{aligned}$$

Let $\rho_Y = \sup\left\{\frac{||(I-P_Y)h||}{||h||} : h \neq 0 \in X\right\}$ and hence $d(X,Y) \geq \rho_Y$.

Similarly, let $\rho_X = \sup\left\{\frac{||(I-P_X)h||}{||h||} : h \neq 0 \in Y\right\}$ and hence $d(X,Y) \geq \rho_X$. So $d(X,Y) \geq \max\{\rho_X, \rho_Y\}$.

Now we show that $d(X,Y) \leq \max\{\rho_X, \rho_Y\}$. From the definition of ρ_Y, we have $||(I-P_Y)P_Xh|| \leq \rho_Y ||P_Xh||$ for any $h \in H$. So

$$||(I-P_Y)P_Xh||^2 \leq \rho_Y^2 ||P_Xh||^2 \tag{VI.1}$$

for any $h \in H$

On the other hand, we have

$$\begin{aligned} ||P_Y(I-P_X)h||^2 &= \langle\, P_Y(I-P_X)h\,,\, P_Y(I-P_X)h\,\rangle \\ &= \langle\, P_Y(I-P_X)h\,,\, (I-P_X)h\,\rangle = \langle\, (I-P_X)P_Y(I-P_X)h\,,\, (I-P_X)h\,\rangle \\ &\leq ||(I-P_X)P_Y(I-P_X)h||.||(I-P_X)h|| \end{aligned}$$

By the definition of ρ_X, we have

$$||(I - P_X)P_Y(I - P_X)h|| \leq \rho_X ||P_Y(I - P_X)h||$$

for any $h \in H$

Therefore,

$$||P_Y(I - P_X)h||^2 \leq \rho_X ||P_Y(I - P_X)h||.||(I - P_X)h||$$

$$||P_Y(I - P_X)h|| \leq \rho_X ||(I - P_X)h|| \tag{VI.2}$$

From equation (VI.1) and (VI.2), we have

$$\begin{aligned} ||(I - P_Y)P_X h||^2 ||P_Y(I - P_X)h||^2 &\leq \rho_Y^2 ||P_X h||^2 + \rho_X^2 ||(I - P_X)h||^2 \\ &\leq (\max\{\rho_X, \rho_Y\})^2.(||P_X h||^2 + ||(I - P_X)h||^2 \\ &\leq (\max\{\rho_X, \rho_Y\})^2.||h||^2 \end{aligned}$$

Thus, $d(X, Y) \leq \max\{\rho_X, \rho_Y\}$ and $d(X, Y) = \max\{\rho_X, \rho_Y\}$ □

Lemma VI.10. If M, E are subspaces of a k-dimensional Hilbert space H and $\dim(M) = n, \dim(E) = k - n$ then $\dim(M + E)^{\perp} = \dim(M \cap E)$.

Proof. Since $(M + E)^{\perp} = M^{\perp} \cap E^{\perp}, (M \cap E)^{\perp} = M^{\perp} + E^{\perp}$, we have $\dim(M + E)^{\perp} = \dim(M^{\perp} \cap E^{\perp}) = \dim(M^{\perp}) + \dim(E^{\perp}) - \dim(M^{\perp} + E^{\perp}) = k - \dim(M^{\perp} + E^{\perp}) = k - \dim(M \cap E)^{\perp} = \dim(M \cap E)$ □

Let $\{y_1, ..., y_l\}$ be an orthonormal basis for $M \cap E$ and $\{z_1, ..., z_l\}$ be an orthonormal basis for $(M + E)^{\perp}$. Let $0 < \epsilon < 1$ and $w_j = y_j + \epsilon z_j$ for $j = 1, ..., l$. Let $s_1, ..., s_{n-l}$ be an orthonormal basis for $(M \cap E)^{\perp}$ in M. So $\{s_1, ..., s_{n-l}, y_1, ..., y_l\}$ is an orthonormal basis for M.

Lemma VI.11. The set $\{s_1, ..., s_{n-l}, y_1 + \epsilon z_1, ..., y_l + \epsilon z_l\}$ are linearly independent.

Proof. Suppose $\sum_{j=1}^{n-l} a_j s_j + \sum_{m=1}^{l} b_m(y_m + \epsilon z_m) = 0$.

Then $\sum_{j=1}^{n-l} a_j s_j + \sum_{m=1}^{l} b_m y_m = -\sum_{m=1}^{l} b_m \epsilon z_m \in M \cap M^{\perp}$.

Therefore, $\sum_{j=1}^{n-l} a_j s_j + \sum_{m=1}^{l} b_m y_m = 0$ and $a_j = b_m = 0$ for all j, m. □

Proposition VI.12. Given $0 < \epsilon < 1$ and let $w_j = y_j + \epsilon z_j$ for $j = 1, ..., l$. Let $\tilde{M}$ be a subspace spanning by $\{s_1, ..., s_{n-l}, w_1, ..., w_l\}$. Then $\tilde{M} \cap E = \{0\}$ and $d(\tilde{M}, M) < 2\epsilon$.

Proof. Suppose $x \in \tilde{M} \cap E$. Then x can be written as

$$x = \sum_{j=1}^{n-l} a_j s_j + \sum_{m=1}^{l} b_m y_m + \sum_{m=1}^{l} b_m \epsilon_m z_m$$

Since $\sum_{m=1}^{l} b_m \epsilon_m z_m \in (M+E)^{\perp} = M^{\perp} \cap E^{\perp}$ and $x \in E$, we have $\langle\, x\,,\, \sum_{m=1}^{l} b_m \epsilon_m z_m \,\rangle = 0$. Since $\sum_{j=1}^{n-l} a_j s_j + \sum_{m=1}^{l} b_m y_m \in M$ and $\sum_{m=1}^{l} b_m \epsilon_m z_m \in M^{\perp} \cap E^{\perp} \subset M^{\perp}$, we have $0 = \langle\, x\,, \sum_{m=1}^{l} b_m \epsilon_m z_m \,\rangle = \sum_{m=1}^{l} |b_m|^2 \epsilon_m^2$ which implies that $b_m = 0$ for all $m = 1, ..., l$. Therefore, $x = \sum_{j=1}^{n-l} a_j s_j \in (M \cap E)^{\perp} \cap (M \cap E)$ and hence $x = 0$. So $\tilde{M} \cap E = \{0\}$.

Consider the Hausdorff distance between two closed unit balls $B_1^M, B_1^{\tilde{M}}$. Let $x \in B_1^{\tilde{M}}$. Then

$$x = \sum_{j=1}^{n-l} a_j s_j + \sum_{m=1}^{l} b_m y_m + \sum_{m=1}^{l} b_m \epsilon z_m$$

and $||x||^2 = \sum_{j=1}^{n-l} |a_j|^2 + \sum_{m=1}^{l} |b_m|^2 + \sum_{m=1}^{l} |b_m|^2 \epsilon^2 \leq 1$ which implies that $\sum_{m=1}^{l} |b_m|^2 < 1$. We have

$$\begin{aligned} d(x, B_1^M) &\leq d(\sum_{j=1}^{n-l} a_j s_j + \sum_{m=1}^{l} b_m y_m + \sum_{m=1}^{l} b_m \epsilon z_m, \sum_{j=1}^{n-l} a_j s_j + \sum_{m=1}^{l} b_m y_m) \\ &= ||\sum_{m=1}^{l} b_m \epsilon z_m|| = (\sum_{m=1}^{l} |b_m|^2 \epsilon^2)^{1/2} < \epsilon \end{aligned}$$

Suppose $x \in B_1^M$. Let $x = \sum_{j=1}^{n-l} a_j s_j + \sum_{m=1}^{l} b_m y_m$. Then $||x||^2 = \sum_{j=1}^{n-l} |a_j|^2 +$

$\sum_{m=1}^{l} |b_m|^2 \leq 1$ which implies that $\sum_{m=1}^{l} |b_m|^2 \leq 1$. Let $d_m = b_m(1-\epsilon^2)$ for $m = 1, ..., l$.

Therefore, $|d_m| < |b_m|$ and $\sum_{m=1}^{l} |d_m|^2 < \sum_{m=1}^{l} |b_m|^2 \leq 1$. We have

$$\sum_{m=1}^{l} |b_m|^2 = \sum_{m=1}^{l} |\frac{d_m}{1-\epsilon^2}|^2 = \frac{1}{(1-\epsilon^2)^2} \sum_{m=1}^{l} |d_m|^2 \geq (1+\epsilon^2) \sum_{m=1}^{l} |d_m|^2 = \sum_{m=1}^{l} d_m^2 + \epsilon^2 \sum_{m=1}^{l} d_m^2$$

Let $y = \sum_{j=1}^{n-l} a_j s_j + \sum_{m=1}^{l} d_m y_m + \sum_{m=1}^{l} d_m \epsilon z_m$. Then

$$||y||^2 = \sum_{j=1}^{n-l} |a_j|^2 + \sum_{m=1}^{l} d_m^2 + \epsilon^2 \sum_{m=1}^{l} d_m^2 \leq \sum_{j=1}^{n-l} |a_j|^2 + \sum_{m=1}^{l} |b_m|^2 \leq 1$$

and so $y \in B_1^{\tilde{M}}$. Then

$$\begin{aligned} d(x, B_1^{\tilde{M}}) \leq d(x,y) &= \sum_{m=1}^{l} |b_m - d_m|^2 + \epsilon^2 \sum_{m=1}^{l} |d_m|^2 \\ &= \epsilon^4 \sum_{m=1}^{l} |b_m|^2 + \epsilon^2 \sum_{m=1}^{l} |d_m|^2 \\ &< \epsilon^4 + \epsilon^2 < 2\epsilon \end{aligned}$$

□

Corollary VI.13. Suppose M is a n-dimensional subspace of a k-dimensional Hilbert space H. Then the set of all $(k-n)$-dimensional subspaces of H that are disjoint from M is open, dense in the set of all $(k-n)$-dimensional subspaces.

Proof. It comes directly from Proposition (VI.8) and Proposition (VI.12). □

Corollary VI.14. The set of all $(k-n)$-dimensional subspaces of H which are disjoint from every diagonal subspace of n-dimension with respect to a fixed orthonormal basis $\{e_j\}_{j=1}^{k}$ is open, dense in the set of all $(k-n)$-dimensional subspaces of H.

Proof. It comes directly from Baire Category Theorem and the fact that the number of diagonal subspace of n-dimension with respect to the basis $\{e_j\}_{j=1}^k$ is finite. $\square$

Lemma VI.15. Suppose E, F are closed subspaces of a Hilbert space H. Let $\Theta_{E,F}$ be the angle between E and F defined by:

$$\cos(\Theta_{E,F}) = \sup\{\cos(\Theta_{l,s})\}$$

where sup is taken over l, s which are a 1-dimensional subspaces of E and F, respectively. Then $\cos(\Theta_{E,F}) = ||PQ||$ where P, Q are the orthogonal projection onto E, F, respectively.

Proof. We have

$$\begin{aligned}
||PQ|| &= \sup\{||PQx|| : x \in H, ||x|| = 1\} \\
&\geq \sup\{||PQx|| : x \in F, ||x|| = 1\} \\
&= \sup\{||Px|| : x \in F, ||x|| = 1\} \\
&= \sup\left\{\frac{\langle Px , Px \rangle}{||Px||} : x \in F, ||x|| = 1\right\} \\
&= \sup\left\{\langle \frac{Px}{||Px||} , x \rangle : x \in F, ||x|| = 1\right\} \\
&= \sup\{|\langle u , v \rangle| : u \in E, v \in F, ||u|| = ||v|| = 1\} \\
&= \cos(\Theta_{E,F})
\end{aligned}$$

Conversely,

$$\begin{aligned}
||PQ|| &= \sup\{||PQx|| : x \in H, ||x|| = 1\} \\
&= \sup\{||Py|| : y \in F, ||y|| \leq 1\} \\
&= \sup\left\{\frac{\langle\, Py\,,\, y\,\rangle}{||Py||} : y \in F, ||y|| \leq 1\right\} \\
&\leq \sup\left\{\frac{\langle\, Py\,,\, y\,\rangle}{||Py||.||y||} : y \in F \setminus \{0\}, ||y|| \leq 1\right\} \\
&\leq \sup\left\{\frac{\langle\, Py\,,\, y\,\rangle}{||Py||.||y||} : y \in F \setminus \{0\}\right\} \\
&\leq \sup\{|\langle\, u\,,\, v\,\rangle| : u \in E, v \in F, ||u|| = ||v|| = 1\} \\
&= \cos(\Theta_{E,F})
\end{aligned}$$

Thus, $||PQ|| = \cos(\Theta_{E,F})$ □

Proposition VI.16. Let H be a real k-dimensional Hilbert space with an orthonormal basis $\mathcal{E} = \{e_j\}_{j=1}^{k}$. Let $n \leq \frac{k}{2}$. Then the set of n-dimensional subspaces which are oblique with respect to $\mathcal{E}$ is dense in the set of n-dimensional subspaces.

Proof. . Let M be any n-dimensional subspaces. Given $1 > \epsilon > 0$. By Corollary (VI.13), there is a n-dimensional subspace N that is disjoint from all diagonal $(k-n)$-dimensional subspaces and $d(N, M) < \epsilon$. Let J be any subset of $\{1, 2, ..., k\}$. If both $|J| > k - n$ and $|J^c| > k - n$, then $k = |J| + |J^c| > 2k - 2n \geq k$, a contradiction. So either $|J| \leq k - n$ or $|J^c| \leq k - n$. If $|J| \leq k - n$. then let I be a subset of $\{1, 2, ..., k\}$ that contains J with $|I| = k - n$. Then $E_J \subseteq E_I$ and $N \cap E_I = \{0\}$ which implies that $N \cap E_J = \{0\}$. Similarly, if $|J^c| \leq k - n$ we can find a subset I that contains J^c with $|I| = k - n$. Then $E_{J^c} \subseteq E_I$ and $N \cap E_I = \{0\}$ which implies that $N \cap E_{J^c} = \{0\}$. Thus, either $N \cap E_J = \{0\}$ or $N \cap E_{J^c} = \{0\}$ and N is oblique with respect to $\mathcal{E}$ by Proposition (VI.4). □

Definition 7. 1) Let $\{x_j\}_{j=1}^k$ be a frame in H_n. We say that $\{x_j\}_{j=1}^k$ has the *n-independent property* if every subset of $\{x_j\}_{j=1}^k$ of cardinality n is linearly independent.

2) Let $\mathcal{E} = \{e_j\}_{j=1}^k$ be an orthonormal basis of a finite dimensional Hilbert space H. A subspace M of dimension n in H is said to have the *n-independent property* relates to $\mathcal{E}$ if every subset of n vectors in $\{P_M e_j\}_{j=1}^k$ is a basis for M.

One simple example of a frame with n-independent property is harmonic frames.

Suppose that $\{x_j\}_{j=1}^k$ is a frame in H_n that does not have the n-independent property. Then there exist $\{x_{j_1}, ..., x_{j_n}\}$ which are linearly dependent. Let M be the range of the analysis operator of $\{x_j\}_{j=1}^k$ and $\{e_j\}_{j=1}^k$ be the standard orthonormal basis for $\mathbb{C}^k$. Then $x_j = \Theta^* P e_j$ where P is the orthogonal projection onto M.

If $\sum_{m=1}^n a_{j_m} x_{j_m} = 0$ for nontrivial coefficients then $\sum_{m=1}^n a_{j_m} \Theta^* P e_{j_m} = 0$ and $\sum_{m=1}^n a_{j_m} P e_{j_m} = 0$ since Θ^* is invertible on M. So $\sum_{m=1}^n a_{j_m} e_{j_m} \in M^\perp$. Let $G = \text{span}\{e_{j_m}\}_{m=1}^n$. Then G is a n-dimensional diagonal subspace such that $G \cap M^\perp \neq \{0\}$.

Proposition VI.17. A frame $\{x_j\}_{j=1}^k$ in H_n has the n-independent property if and only if $M^\perp$ has $\{0\}$ intersection with every diagonal subspace of dimension n.

Proof. The backward direction comes from the discussion above. Assume that $\{x_j\}_{j=1}^k$ has n-independent property and there is a diagonal subspace G of dimension n such that $G \cap M^\perp \neq \{0\}$. Let $0 \neq x \in G \cap M^\perp$ and $G = \text{span}\{e_{j_m}\}_{m=1}^n$. Suppose $x = \sum_{m=1}^n a_{j_m} e_{j_m}$ with a_{j_m} not all zero. Then $P(\sum_{m=1}^n a_{j_m} e_{j_m}) = 0$ and $\sum_{m=1}^n a_{j_m} \Theta^* P e_{j_m} = 0$ which implies that $\sum_{m=1}^n a_{j_m} x_{j_m} = 0$. So $\{x_{j_m}\}_{m=1}^n$ is linearly dependent, a contradiction. □

Corollary VI.18. A subspace M has n-independent property with respect to $\mathcal{E}$ if and only if $M^\perp$ has $\{0\}$ intersection with every n-dimensional diagonal subspace of H.

Proof. It follows from Proposition (VI.17). □

Lemma VI.19. Let $n < k$. Suppose $\mathcal{S}$ is a collection of $(k-n)$-dimensional subspaces of a k-dimensional Hilbert space H which is open and dense in the collection of all $(k-n)$-dimensional subspaces of H. Let $\mathcal{S}^{\perp} = \{S^{\perp} : S \in \mathcal{S}\}$. Then $\mathcal{S}^{\perp}$ is open and dense in the collection of all n-dimensional subspaces of H.

Proof. Suppose $M^{\perp} \in \mathcal{S}^{\perp}$. Then $M \in \mathcal{S}^{\perp}$. Since $\mathcal{S}$ is open, there is an $\epsilon > 0$ such that for any subspace N satisfying $d(N, M) < \epsilon$, we have $N \in \mathcal{S}$.

Since

$$d(N, M) = ||P_N - P_M|| = ||(I - P_N) - (I - P_M)|| = ||P_{N^{\perp}} - P_{M^{\perp}}|| = d(N^{\perp}, M^{\perp})$$

$\mathcal{S}^{\perp}$ is open.

Let M be any n-dimensional subspace and $\epsilon > 0$ given. Since $\mathcal{S}$ is dense, there is a subspace $N \in \mathcal{S}$ such that $d(N, N^{\perp}) < \epsilon$. Thus, $N^{\perp} \in \mathcal{S}^{\perp}$ and $d(N^{\perp}, M) = d(N, M^{\perp}) < \epsilon$. Therefore, $\mathcal{S}$ is dense. □

Corollary VI.20. The collection of n-dimensional subspaces of a k-dimensional Hilbert space H with n-independent property with respect to some orthonormal basis $\mathcal{E}$ of H is an open, dense in the collection of all n-dimensional subspaces of H.

Proof. It follows from Proposition (VI.17), Corolarry (VI.18), Lemma (VI.19). □

Definition 8. Two closed subspaces E, F of a Hilbert space H is said to be *equivalent* $(E \sim F)$ if and only if $\dim(E) = \dim(F), \dim(E^{\perp}) = \dim(F^{\perp})$

Proposition VI.21. Given closed subspaces E, F of a Hilbert space H. Then there is an invertible $T \in \mathcal{B}(H)$ such that $F = TE$ if and only if $E \sim F$ if and only if there is an unitary operator $U : H \to H$ such that $F = UE$

Proof. Suppose that $E \sim F$. Then $\dim(E) = \dim(F), \dim(E^{\perp}) = \dim(F^{\perp})$. We have $H = E \oplus E^{\perp} = F \oplus F^{\perp}$. Since $\dim(E) = \dim(F)$, there is an invertible linear bounded operator $T_1 : E \to F$. Similarly, there is an invertible linear bounded operator $T_2 : E^\perp \to F^\perp$.

Let $T : H \to H$ be difined by $Tx = T_1y_1 + T_2y_2$ where $x = y_1 + y_2, y_1 \in E, y_2 \in E^\perp$. Therefore, $||y_1|| \leq ||x||, ||y_2|| \leq ||x||$. If $x \in E$ then $Tx = T_1x \in F$, so $T(E) \subset F$. Let $y \in F$ then there is $x \in E$ such that $y = T_1(x)$. So $Tx = T_1x = y$ and $T(E) = F$. From the definition of T, we imply that T is a linear bounded map.

If $Tx = 0 = T_1y_1 + T_2y_2$ then $T_1y_1 = -T_2y_2 \in F \cap F^\perp = \{0\}$ and $T_1y_1 = T_2y_2 = 0$. Therefore, $y_1 = y_2 = 0$ and $x = 0$ which implies that T is injective. So T is invertible in $\mathcal{B}(H)$.

Now assume that there is an invertible operator $T \in \mathcal{B}(H)$ such that $F = TE$. Then $\dim(E) = \dim(F)$. If $x \in E^\perp$ then for every $y \in F$, we have $\langle Tx , y \rangle = \langle x , T^*y \rangle = 0$ and so $Tx \in F^\perp$ which implies that $T(E^\perp) \subset F^\perp$. Let $y \in F^\perp$ be arbitrary. There exists $x \in H$ such that $y = Tx$. Let $x = w + t$ where $w \in E, t \in E^\perp$. We have $Tw = Tx - Tt \in F^\perp \cap F$ and so $Tw = 0$. Since T is invertible, $w = 0$ and $x \in E^\perp$. Thus, $T(E^\perp) = F^\perp$.

By polar decomposition theorem, there is a unitary operator $U : E \to F$ such that $T = U|T|$ which makes the proof completed. □

If a frame $\{x_j\}_{j=1}^k$ in H_n has n-independent property then we can find a projection P of rank $0 < l < n$ such that $\{Px_j\}_{j=1}^k$ does not have $(n-l)$-independent property. For example, we choose the range of projection is the orthogonal complement of $\{x_j\}_{j=1}^{n-l}$.

However, "most " of the projections of rank $n - l$ on H_n are projections such that the frame image has $(n - l)$-independent property

Proposition VI.22. Suppose that a frame $\{x_i\}_{i=1}^k$ in H_n has n-independent property and $0 < l < n$. Then the set of projections P with rank $n-l$ on H_n such that $\{Px_i\}_{i=1}^k$

has $(n-l)$-independent property is open and dense in the set of all projections of rank $n-l$ on H_n.

Proof. Since $X = \{x_j\}_{j=1}^k$ has n-independent property, every subset of n elements in X is linearly independent. Therefore, every subset of $n-l$ elements in X is linearly independent. If P_M is a projection of rank $n-l$ on H_n such that $\{Px_j\}_{j=1}^k$ has $(n-l)$-independent property then $\sum_{m=1}^{n-l} a_{j_m} P_M x_{j_m} = 0$ if and only if $a_{j_m} = 0$ for all $m = 1, ..., n-l$ where $\{x_{j_m}\}_{m=1}^{n-l}$ is any subset of $n-l$ elements in X. This is equivavent to $M^\perp \cap \operatorname{span}\{x_{j_m}\}_{m=1}^{n-l} = \{0\}$.

By Corollary (VI.13), the set of all subspaces $M^\perp$ of dimension $k-n+l$ which are disjoint from a subspace spanned by a subset of $n-l$ elements of X is open and dense in the set of all subspaces of dimension $k-n+l$. By Baire Category Theorem and the number of subspaces spanned by a subset of $n-l$ elements of X is finite, the set of all subspaces $M^\perp$ of dimension $k-n+l$ which are disjoint from all subspaces spanned by a subset of $n-l$ elements of X is open and dense in the set of all subspaces of dimension $k-n+l$. Therefore, the set of subspaces M of dimension $n-l$ such that $M^\perp$ is disjoint from subspaces spanned by a subset of $n-l$ elements of X is open and dense in the set of all subspaces of dimension $n-l$. Thus, the set of projections P with rank $n-l$ on H_n such that $\{Px_j\}_{j=1}^k$ has $(n-l)$-independent property is open and dense in the set of all projections of rank $n-l$ on H_n. $\square$

Remark 14. If $\{x_j\}_{j=1}^k$ is a frame in H_n with n-independent property then it is not necessary that a push-out of $\{x_j\}_{j=1}^k$ has $(n+1)$-independent property. For example, $\{x_1 = (1,2)^T, x_2 = (1,3)^T, x_3 = (1,4)^T, x_4 = (4,5)^T\}$ is a frame in $\mathbb{R}^2$ with 2-independent property but a push-out $\{\tilde{x_1} = (1,2,1)^T, \tilde{x_2} = (1,3,1)^T, \tilde{x_3} = (1,4,1)^T, \tilde{x_4} = (4,5,1)^T\}$ is a frame which does not have 3-independent property. Moreover, any push-out of $\{x_j\}_{j=1}^4$ does not have 3-independent property.

Lemma VI.23. If $T \in \mathcal{B}(H)$ is a self-adjoint operator then $\text{Ker}(T) = \text{Ker}T^2)$ and $\overline{\text{ran}(T^2)} = \overline{\text{ran}(T)}$.

Proof. We have $\overline{\text{ran}(T)} = \text{Ker}(T)^\perp, \overline{\text{ran}(T^2)} = \text{Ker}(T^2)^\perp$. Is is obvious that $\text{Ker}(T) \subset \text{Ker}(T^2)$. If $x \in \text{Ker}(T^2)$ then $T^2(x) = 0$ and $0 = \langle T^2(x)\,,\, x\,\rangle = \langle\, Tx\,,\, Tx\,\rangle$. Thus, $Tx = 0$ and $x \in \text{Ker}(T)$. Hence $\text{Ker}(T) = \text{Ker}(T^2)$ and $\overline{\text{ran}(T^2)} = \overline{\text{ran}(T)}$. □

In a finite dimensional space H, $\text{ran}(T)$ and $\text{ran}(T^2)$ are finite dimensional subspaces which are closed. Therefore, $\text{ran}(T) = \text{ran}(T^2)$. However, in an infinite dimensional space, $\text{ran}(T)$ may be different from $\text{ran}(T^2)$. For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & .... & 0 \\ 0 & 1/2 & 0 & 0 & .... & 0 \\ 0 & 0 & 1/3 & 0 & .... & 0 \\ 0 & 0 & 0 & 1/4 & .... & 0 \end{pmatrix}$$

Let $x = (1, 1/2, 1/3, ...)^T$. Then $x \in l^2(\mathbb{N})$. If there is $y = (y_1, y_2, y_3....)^T \in l^2(\mathbb{N})$ such that $T(x) = T^2(y)$ then $T^2(y) = (y_1, \frac{1}{4}y_2, \frac{1}{9}y_3, ...)^T = T(x) = (1, \frac{1}{4}, \frac{1}{9}, ...)^T$. Therefore, $y_1 = y_2 = y_3 = ... = 1$, a contradiction to $y \in l^2(\mathbb{N})$.

Lemma VI.24. Suppose $P, Q \in \mathcal{B}(H)$ are orthogonal projections onto closed subspaces and $||P - Q|| < \delta$ for some $\delta < 1/2$. Let $A = QP + Q^\perp P^\perp$. Then A is invertible.

Proof. Assume that $Ax = 0$. Then $0 = \langle\, QPx + Q^\perp P^\perp x\,,\, QPx\,\rangle = ||QPx||^2$. So $QPx = 0 = Q^\perp P^\perp x$. We have

$$||Qx|| = ||QQx - QPx|| \leq ||Q||.||Q - P||.||x|| < \delta.||x||$$

Similarly,

$$||Q^{\perp}x|| = ||Q^{\perp}Q^{\perp}x - Q^{\perp}P^{\perp}x|| \leq ||Q^{\perp}||.||Q^{\perp} - P^{\perp}||.||x|| < \delta.||x||$$

Therefore,

$$||x||^2 = ||Qx||^2 + ||Q^{\perp}x||^2 < 2\delta^2||x||^2 < ||x||^2$$

Hence, $x = 0$ and A is injective. Similarly we can prove that $PQ + P^{\perp}Q^{\perp}$ is injective.

If there is $y \neq 0$ such that $\langle\, y\,,\, Ax\,\rangle = 0$ for every x then $\langle\, A^*y\,,\, x\,\rangle = 0$ for every x and so $A^*y = 0$. Since $A^* = PQ + P^{\perp}Q^{\perp}$ is injective, $y = 0$, a contradiction. So A is surjective. □

Lemma VI.25. Suppose $P, Q \in \mathcal{B}(H)$ are orthogonal projections onto closed subspaces. Then $||(PQP)^{1/2} - P|| \leq ||PQP - P||$.

Proof. Let $T = (PQP)^{1/2}$. Then $T^2 = PQP = P^2PQP = P^2T^2$ and $PT^2 = T^2P$. Therefore, $PT = TP$ and $(PT)^2 = P^2T^2$ which implies that $T^2 = (PT)^2$. Since T is positive, $PT = T$. Thus, $P(PQP)^{1/2} = (PQP)^{1/2}$. Similarly, $P^{\perp}(P^{\perp}Q^{\perp}P^{\perp})^{1/2} = (P^{\perp}Q^{\perp}P^{\perp})^{1/2}$. We have

$$((PQP)^{1/2} - P)((PQP)^{1/2} + I) = PQP + (PQP)^{1/2} - P(PQP)^{1/2} - P = PQP - P$$

Let $S = (PQP)^{1/2} + I$. Then $S \geq I$ and S is invertible. Since $||(PQP)^{1/2}||^2 = ||PQP|| \leq 1$, $||S|| \leq ||(PQP)^{1/2}|| + 1 \leq 2$. So $I \leq S \leq 2I$ and $\frac{1}{2}I \leq S^{-1} \leq I$. Hence $||S_{-1}|| \leq 1$.

We have

$$||(PQP)^{1/2} - P|| = ||(PQP - P)S^{-1}|| \leq ||PQP - P||.||S^{-1}|| \leq ||PQP - P||$$

□

Lemma VI.26. Suppose $P, Q \in \mathcal{B}(H)$ are orthogonal projections onto closed subspaces. Then

$$(PQP)^{1/2} \leq P, \text{ and } (P^{\perp}Q^{\perp}P^{\perp})^{1/2} \leq P^{\perp}$$

Proof. Let $T = (PQP)^{1/2}$. Then from the proof of Lemma (VI.25), $PT = T$. So $T = T^* = (PT)^* = T^*P^* = TP$. Thus, $PT = TP = T$ which imples that $(PT)P = TP = T$. Therefore, $P(PQP)^{1/2}P = (PQP)^{1/2}$. We have, for every $x \in H$,

$$\begin{aligned} \langle P(PQP)^{1/2}Px\, ,\, x \rangle &= \langle (PQP)^{1/2}Px\, ,\, Px \rangle \\ &\leq ||(PQP)^{1/2}||.||Px||^2 = ||PQP||^{1/2}.||Px||^2 \\ &\leq ||Px||^2 = \langle Px\, ,\, x \rangle \end{aligned}$$

Thus, $P(PQP)^{1/2}P \leq P$ and $(PQP)^{1/2} \leq P$. Similarly, $(P^{\perp}Q^{\perp}P^{\perp})^{1/2} \leq P^{\perp}$. ☐

Lemma VI.27. Suppose $P, Q \in \mathcal{B}(H)$ are orthogonal projections onto closed subspaces and $A = QP + Q^{\perp}P^{\perp}$. Let $A = U|A|$ be the polar decomposition of A. If $||P - Q|| < \delta < 1/2$ then $||U - I|| < \frac{4\delta}{1-2\delta}$

Proof. We have

$$A^*A = (QP+Q^{\perp}P^{\perp})^*(QP+Q^{\perp}P^{\perp}) = (PQ+P^{\perp}Q^{\perp})(QP+Q^{\perp}P^{\perp}) = PQP+P^{\perp}Q^{\perp}P^{\perp}$$

Since

$$\begin{aligned} (PQP)^{1/2}(P^{\perp}Q^{\perp}P^{\perp}) &= ((PQP)^{1/2}(P^{\perp}Q^{\perp}P^{\perp}))^* \\ &= (P^{\perp}Q^{\perp}P^{\perp})(PQP)^{1/2} \end{aligned}$$

which imples that

$$(P^{\perp}Q^{\perp}P^{\perp})^{1/2}(PQP)^{1/2} = (PQP)^{1/2}(P^{\perp}Q^{\perp}P^{\perp})^{1/2}$$

and hence,

$$((P^{\perp}Q^{\perp}P^{\perp})^{1/2}(PQP)^{1/2})^2 = (PQP)(P^{\perp}Q^{\perp}P^{\perp}) = 0$$

Therefore, $(PQP)^{1/2}(P^{\perp}Q^{\perp}P^{\perp})^{1/2} = 0$ and

$$(P^{\perp}Q^{\perp}P^{\perp})^{1/2}(PQP)^{1/2} = ((PQP)^{1/2}(P^{\perp}Q^{\perp}P^{\perp})^{1/2})^* = 0$$

We have

$$((PQP)^{1/2} + (P^{\perp}Q^{\perp}P^{\perp})^{1/2})^2 = PQP + P^{\perp}Q^{\perp}P^{\perp} = A^*A$$

which implies that

$$|A| = (A^*A)^{1/2} = (PQP)^{1/2} + (P^{\perp}Q^{\perp}P^{\perp})^{1/2}$$

It follows that

$$I - |A| = (P - (PQP)^{1/2}) + (P^{\perp} - (P^{\perp}Q^{\perp}P^{\perp})^{1/2}) \geq 0$$

By Lemma (VI.25), we have

$$\begin{aligned} ||I - |A||| &\leq ||P - (PQP)^{1/2}|| + ||P^{\perp} - (P^{\perp}Q^{\perp}P^{\perp})^{1/2}|| \\ &\leq ||PQP - P|| + ||P^{\perp}Q^{\perp}P^{\perp} - P^{\perp}|| \\ &= ||P(Q-P)P|| + ||P^{\perp}(Q^{\perp} - P^{\perp})P^{\perp})|| \\ &\leq ||Q - P|| + ||Q^{\perp} - P^{\perp}|| = 2||Q - P|| \leq 2\delta \end{aligned}$$

Therefore, $\sigma(I - |A|) \subseteq [0, 2\delta]$ and $\sigma(|A|) \subseteq [1 - 2\delta, 1]$ which implies $\sigma(|A|^{-1}) \subseteq [1, \frac{1}{1-2\delta}]$. Thus,

$\sigma(|A|^{-1} - I) \subseteq [0, \frac{1}{1-2\delta} - 1]$ and $|||A|^{-1} - I|| \leq \frac{1}{1-2\delta} - 1 = \frac{2\delta}{1-2\delta}$

Hence,

$$\begin{aligned} ||A - I|| &= ||QP + Q^{\perp}P^{\perp} - (P + P^{\perp})|| \\ &\leq ||QP - P|| + ||Q^{\perp}P^{\perp} - P^{\perp}|| \\ &= ||(Q - P)P|| + ||(Q^{\perp} - P^{\perp})Q^{\perp}|| \\ &\leq ||Q - P|| + ||Q^{\perp} - P^{\perp}|| = 2||Q - P|| \leq 2\delta \end{aligned}$$

From $A = U|A|$ and A is invertible, it follows that $U = A|A|^{-1}$. We have

$$U - I = A|A|^{-1} - I = (A - I)|A|^{-1} + (|A|^{-1} - I)$$

and therefore,

$$||U - I|| \leq ||A - I||.|||A|^{-1}|| + |||A|^{-1} - I|| \leq 2\delta\frac{1}{1 - 2\delta} + \frac{2\delta}{1 - 2\delta} = \frac{4\delta}{1 - 2\delta}$$

So $||U - I|| \leq \frac{4\delta}{1-2\delta}$ □

Lemma VI.28. Suppose P, Q, U are operators in Lemma (VI.27). Then $UPU^* = Q$

Proof. By the polar decomposition theorem, there is a unique partial isometry U_1 such that $QP = U_1|QP|$ with $\mathrm{Ker}(U_1) = \mathrm{Ker}(|QP|), \mathrm{ran}(U_1) = \overline{\mathrm{ran}}(QP)$ and there is a unique partial isometry U_2 such that $Q^{\perp}P^{\perp} = U_2|Q^{\perp}P^{\perp}|$ with $\mathrm{Ker}(U_2) = \mathrm{Ker}(|Q^{\perp}P^{\perp}|), \mathrm{ran}(U_2) = \overline{\mathrm{ran}}(Q^{\perp}P^{\perp})$.

Note that

$$|QP| = ((QP)^*(QP))^{1/2} = (PQQP)^{1/2} = (PQP)^{1/2}$$

and similarly,

$$|Q^{\perp}P^{\perp}| = (P^{\perp}Q^{\perp}P^{\perp})^{1/2}$$

Since $\mathrm{ran}(QP) \subseteq \mathrm{ran}(Q), \mathrm{ran}(Q^{\perp}P^{\perp}) \subseteq \mathrm{ran}(Q^{\perp})$ and $QP + Q^{\perp}P^{\perp}$ is invertible, we have

$$H = \text{ran}(QP + Q^{\perp}P^{\perp}) \subseteq \text{ran}(QP) + \text{ran}(Q^{\perp}P^{\perp}) \subseteq \text{ran}(Q) + \text{ran}(Q^{\perp}) = H$$

which implies that $\text{ran}(QP) = \text{ran}(Q), \text{ran}(Q^{\perp}P^{\perp}) = \text{ran}(Q^{\perp})$. Thus,

$$\text{ran}(U_1) = \overline{\text{ran}}(QP) = \overline{\text{ran}}(Q) = \text{ran}(Q)$$

We have $\text{Ker}(U_1) = \text{Ker}(|QP|) = \text{Ker}((PQP)^{1/2}) = \text{Ker}(PQP)$. Similarly, $\text{Ker}(U_2) = \text{Ker}(P^{\perp}Q^{\perp}P^{\perp})$.

From $P|QP| = |QP|$, it follows that $U_1P|QP| = QP$ which implies $QU_1P|QP| = QP$.

Now we will prove that $\text{Ker}(QU_1P) = \text{Ker}(U_1)$. Since $\text{Ker}(U_1) = \text{Ker}(PQP)$ and $\text{ran}(U_1) = \text{ran}(Q)$, we have

$$\begin{aligned} x \in \text{Ker}(QU_1P) &\iff QU_1Px = 0 \iff U_1Px \in \text{ran}(Q^{\perp}) \cap \text{ran}(Q) \iff U_1Px = 0 \\ &\iff Px \in \text{Ker}(U_1) \iff Px \in \text{Ker}(PQP) \iff PQPPx = 0 \\ &\iff PQPx = 0 \iff x \in \text{Ker}(PQP) \iff x \in \text{Ker}(U_1) \end{aligned}$$

By the uniqueness, $QU_1P = U_1$. Similarly, $Q^{\perp}U_2P^{\perp} = U_2$.

Since $P^{\perp}(P^{\perp}Q^{\perp}P^{\perp})^{1/2} = (P^{\perp}Q^{\perp}P^{\perp})^{1/2}$, and so $\text{ran}((P^{\perp}Q^{\perp}P^{\perp})^{1/2} \subseteq \text{ran}(P^{\perp})$, we have $U_1|Q^{\perp}P^{\perp}| = U_1(P^{\perp}Q^{\perp}P^{\perp})^{1/2} = U_1P(P^{\perp}Q^{\perp}P^{\perp})^{1/2} = 0$. Similarly, $U_2|QP| = 0$.

$$\begin{aligned} (U_1 + U_2)|A| &= (U_1 + U_2)((PQP)^{1/2} + (P^{\perp}Q^{\perp}P^{\perp})^{1/2}) \\ &= (U_1 + U_2)(|QP| + |Q^{\perp}P^{\perp}|) \\ &= QP + Q^{\perp}P^{\perp} = A \end{aligned}$$

So $U_1 + U_2 = A|A|^{-1}$. Then $U = U_1 + U_2 = QU_1P + Q^{\perp}U_2P^{\perp}$ and $UP = QU_1P = QU$.

Hence $UP = QU$ and $UPU^* = Q$. ☐

Theorem VI.29. Suppose $P, Q \in \mathcal{B}(H)$ are orthogonal projections and given $0 < \epsilon < 1/2$. Then there is $0 < \delta < 1/2$ such that whenever $||P - Q|| < \delta$ there exists a unitary operator $U \in \mathcal{B}(H)$ with $||U - I_H|| < \epsilon$ and $Q = UPU^*$.

Proof. Let $\delta = \frac{\epsilon}{4+3\epsilon}$. Then it follows directly from Lemmas (VI.27), (VI.28). ☐

Proposition VI.30. Suppose $X = \{x_j\}_{j=1}^k$ is a Parselval frame for H_n and $\epsilon > 0$ is given. Let $M \subseteq \mathbb{C}^k$ be the range of the analysis operator Θ_X of X. There is a $\delta > 0$ such that whenever E is a subspace in $\mathbb{C}^k$ such that $d(E, M) < \delta$ then there exists a Parselval frame $Z = \{z_j\}_{j=1}^k$ for H_n such that $\text{ran}(\Theta_Z) = E$ and $||x_j - z_j|| < \epsilon$ for all j.

Proof. Let $\tilde{x_j} = \Theta_X(x_j) \in M$. Then $\tilde{x_j} = P_M(e_j)$ where $\{e_j\}_{j=1}^k$ is the standard orthonormal basis for $\mathbb{C}^k$. Then $\tilde{X} = \{\tilde{x_j}\}_{j=1}^k$ is a Parseval frame for M. Let δ be the number satisfying Theorem (VI.29) and $\delta < \frac{\epsilon}{2}$. Assume that E is a subspace in $\mathbb{C}^k$ such that $d(E, M) < \delta$. So $||P_E - P_M|| < \delta$. Let $\tilde{y_j} = P_E(e_j)$. Then $\tilde{Y} = \{\tilde{y_j}\}_{j=1}^k$ is a Parseval frame for E. We have $||\tilde{y_j} - \tilde{x_j}|| = ||P_E(e_j) - P_M(e_j)|| \leq ||P_E - P_M|| < \delta$.

By Theorem (VI.29), there is a unitary operator U in $\mathcal{B}(\mathbb{C}^k)$ such that $||U - I|| < \frac{\epsilon}{2}$ and $P_M = UP_EU^*$. Therefore, $P_MU = UP_E$ and U is a unitary operator from E onto M. Let $\tilde{z_j} = U\tilde{y_j}$. Then $\tilde{Z} = \{\tilde{z_j}\}_{j=1}^k$ is a Parseval frame for M with $\text{ran}(\Theta_{\tilde{Z}}) = \text{ran}(\Theta_{\tilde{Y}}) = E$.

Since $\tilde{Y}$ is a Parseval frame, $||\tilde{y_j}|| \leq 1$, and we have

$$||\tilde{x_j} - \tilde{z_j}|| \leq ||\tilde{x_j} - \tilde{y_j}|| + ||\tilde{y_j} - \tilde{z_j}|| = ||\tilde{x_j} - \tilde{y_j}|| + ||\tilde{y_j} - U\tilde{y_j}|| < \delta + ||I - U||.||\tilde{y_j}|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So $||\tilde{x_j} - \tilde{z_j}|| < \epsilon$.

Note that both $\tilde{X}$ and $\tilde{Z}$ are Parseval frames for M and $W = \Theta_X^*|_M$ is a unitary operator from M onto H_n. Then $W\tilde{x_j} = x_j$. Let $z_j = W\tilde{z}_j$. Thus, $\Theta_X(z_j) = W^*(z_j) = \tilde{z}_j$. We have $Z = \{z_j\}_{j=1}^k$ is a Parseval frame for H_n and $||x_j - z_j|| = ||W\tilde{x_j} - W\tilde{z}_j|| = ||\tilde{x_j} - \tilde{z}_j|| < \epsilon$ for all j.

Now we will prove that $\Theta_Z = U^*\Theta_X$. For any $x \in H_n$, we have

$$\begin{aligned}
\Theta_Z(x) &= \sum_{j=1}^k \langle x , z_j \rangle e_j = \sum_{j=1}^k \langle \Theta_X(x) , \Theta_X(z_j) \rangle e_j \\
&= \sum_{j=1}^k \langle \Theta_X(x) , \tilde{z}_j \rangle e_j = \sum_{j=1}^k \langle \Theta_X(x) , U\tilde{y_j} \rangle e_j \\
&= \sum_{j=1}^k \langle U^*\Theta_X(x) , \tilde{y_j} \rangle e_j = \sum_{j=1}^k \langle U^*\Theta_X(x) , P_E e_j \rangle e_j \\
&= \sum_{j=1}^k \langle P_E U^*\Theta_X(x) , e_j \rangle e_j = P_E U^*\Theta_X(x) \\
&= U^*\Theta_X(x)
\end{aligned}$$

Thus, $\Theta_Z = U^*\Theta_X$ and $\text{ran}(\Theta_Z) = E$ □

Corollary VI.31. Suppose $X = \{x_j\}_{j=1}^k$ is a Parselval frame for H_n and $\epsilon > 0$ is given. Let $M \subseteq \mathbb{C}^k$ be the range of the analysis operator Θ_X of X. There exists a $\delta > 0$ such that whenever E is a subspace in $\mathbb{C}^k$ such that $d(E, M) < \delta$ then there exists a Parselval frame $Z = \{z_j\}_{j=1}^k$ for H_n such that $\text{ran}(\Theta_Z) = E$ and $||\Theta_Z - \Theta_X|| < \epsilon$.

Proof. It follows from the proof of the Proposition (VI.30) and

$$||\Theta_Z - \Theta_X|| = ||U^*\Theta_X - \Theta_X|| \leq ||U^* - I||.||\Theta_X|| = ||U^* - I|| < \epsilon$$

□

Let H be a Hilbert space, $\mathcal{F}_{\mathbb{J}}(H)$ be the set of all frames for H indexed by $\mathbb{J}$. Let $X = \{x_j\}_{j\in\mathbb{J}}$ and $Y = \{y_j\}_{j\in\mathbb{J}}$ in $\mathcal{F}_{\mathbb{J}}(H)$. Define $d(X,Y) = ||\Theta_X - \Theta_Y||, d_\infty(X,Y) =$

$\sup\{||x_j - y_j|| : j \in \mathbb{J}\}, d_2(X,Y) = (\sum_{j\in\mathbb{J}} ||x_j - y_j||^2)^{1/2}$. From Lemma I.4, if X, Y are Parseval frames for H with the same index set $\mathbb{J}$ then X is unitarily equivalent to Y (that means, there is a unitaty operator U such that $UX = Y$) if and only if $\Theta_X\Theta_X^* = P_X = P_Y = \Theta_Y\Theta_Y^*$. So if $\tilde{\mathcal{F}}_{\mathbb{J}}(H)$ is the set of all equivalence classes of Parseval frames for H indexed by $\mathbb{J}$ then we can define a metric on $\tilde{\mathcal{F}}_{\mathbb{J}}(H)$ by $d([X],[Y]) = ||P_X - P_Y||$.

Corollary VI.32. The set of Parseval frames of k vectors in n-dimensional space which has n-independent property is dense in the set of Parseval frames of k vectors in n-dimensional space.

Proof. Assume that $X = \{x_j\}_{j=1}^k$ is a Parseval frame for H_n and $\epsilon > 0$ be given. Let $M = \text{ran}(\Theta_X)$. Then M is a n-dimensional subspace of $\mathbb{C}^k$. Let δ be the number in the proof of Proposition (VI.30). By Corollary (VI.20), there exists a n-dimensional subspace E of $\mathbb{C}^k$ with n-independent property with respect to the orthornomal basis $\mathcal{E} = \{e_j\}_{j=1}^k$ of $\mathbb{C}^k$ such that $d(E, M) < \delta$. So every subset of n vectors in $\{P_E(e_j)\}_{j=1}^k$ is linearly independent. Following the proof of the Proposition (VI.30), $\{\tilde{y}_j\}_{j=1}^k$ has n-independent property. Then $\{\tilde{z}_j\}_{j=1}^k$ and $\{z_j\}_{j=1}^k$ also have this property. Thus, $\{z_j\}_{j=1}^k$ is a Parseval frame with $d(Z, X) < \epsilon$ and has n-independent property. □

Corollary VI.33. Let $n \leq \frac{k}{2}$. The set of Parseval frames of k vectors in a real Hilbert space H_n which has $|\Theta|$-property is dense in the set of all Parseval frames of k vectors in H.

Proof. Let $X = \{x_j\}_{j=1}^k$ is a Parseval frame for H_n and $\epsilon > 0$ be given. Let δ be the number in the proof of the Proposition (VI.30), there exists a n-dimensional subspace E of $\mathbb{C}^k$ which is oblique with respect to the orthornomal basis $\mathcal{E} = \{e_j\}_{j=1}^k$ of $\mathbb{C}^k$ such that $d(E, M) < \delta$ where $M = \text{ran}(\Theta_X)$. By Corollary (VI.31), there is a Parseval frame $Z = \{z_j\}_{j=1}^k$ for H_n such that $\text{ran}(\Theta_Z) = E$ and $d(Z, X) < \epsilon$. By Lemma (VI.1), the frame Z has $|\Theta|$-property. □

Lemma VI.34. If $\{x_j\}_{j=1}^k$ is a frame for $\mathbb{R}_n$ $(k > n)$ with the n-independent property then there are $\lambda_j \in \mathbb{R}$ such that $\{x_j \oplus \lambda_j\}$ is a frame in $\mathbb{R}^{n+1}$ with the $(n+1)$-independent property

Proof. Suppose $x_j = (x_{j1}, x_{j2}, ..., x_{jn})^T$ and A be a $n \times k$ matrix with n column vectors x_j for $j = 1, 2, ..., k$. Let $\mathbb{J} \subset \{1, 2, ..., k\}$ be a set with cardinality n and $A_{\mathbb{J}}$ be a $n \times n$ matrix with n column vectors x_j for $j \in \mathbb{J}$. Since $\{x_j\}_{j=1}^k$ has the n-independent property, each $n \times n$ matrix $A_{\mathbb{J}}$ has determinant different from 0.

A sequence of numbers $\{\lambda_j\}_{j=1}^k$ such that $y_j = (x_{j1}, x_{j2}, ..., x_{jn}, \lambda_j)^T$ for $j = 1, 2, ..., k$ forms a frame in $\mathbb{R}^{n+1}$ with the $(n+1)$-independent property must satisfy that any $(n+1) \times (n+1)$ matrix $B_{\mathbb{I}}$ where $\mathbb{I} \subset \{1, 2, ..., k\}$ is a set with cardinality $(n+1)$ consisting of $(n+1)$ column vectors $y_i, i \in \mathbb{I}$ has determinant different from 0.

We have $\det(B_{\mathbb{I}}) = \sum_{j \in \mathbb{I}} \lambda_j \det(A_{\mathbb{I} \setminus \{j\}}) \neq 0$. So there are finite linear constrained conditions on the sequence $\{\lambda_j\}_{j=1}^k$. Note that $\det(A_{\mathbb{I} \setminus \{j\}}) \neq 0$ for all $j \in \mathbb{I}$. Hence, the existence of such a sequence $\{\lambda_j\}_{j=1}^k$ is obvious. □

Suppose that $\{x_j\}_{j=1}^k$ is a frame with the n-independent property in $\mathbb{R}^n$. A vector $\lambda = (\lambda_j)^T \in \mathbb{R}^k$ is said to be *"good"* if $\{x_j \oplus \lambda_j\}$ is a frame in $\mathbb{R}^{n+1}$ with the $(n+1)$-independent property.

If λ is "good" then $t\lambda$ is "good" for any $t \neq 0$. In fact, every $(n+1) \times (n+1)$ matrix in the proof of the Lemma (VI.34) has determinant different from 0. Then every $(n+1) \times (n+1)$ matrix coming from replacing the last row λ with $t\lambda$ has determinant increasing t times and hence, also has determinant different from 0.

However, if λ_1, λ_2 are "good" then it is not necessary that $\lambda_1 + \lambda_2$ is "good". For example λ and $-\lambda$ are "good" but $\lambda - \lambda = 0$ is not "good".

Lemma VI.35. Suppose that $\{x_j\}_{j=1}^k$ is a frame with the n-independent property in $\mathbb{R}^n$. The set of "good" vectors is open and dense in $\mathbb{R}^k$.

Proof. The proof is similar to the proof of the Lemma (VI.34) and based on the fact that the determinant function is continuous. □

CHAPTER VII

CONCLUSIONS

In this dissertation, we investigated several aspects of frame theory. The topics include the (p, q)-replacement problem for surgery on frames, push-outs frames, frames generated by the action of a group on a single generator vector, a spreading algorithm for finite unit tight frames, and the mathematics involved in the "cocktail party problem". Motivations for this investigation and counter examples were also included. Some topics that are partially treated in this dissertation are worthy of further investigation. Can the spreading algorithm of Chapter III converge to a Grassmannian frame? More work could be done on convergence properties of the algorithm. A computational method for checking whether a given frame is Grassmannian would be needed here.

REFERENCES

[1] R. Balan, P. Casazza and D. Edidin, *On signal reconstruction without phase*, Appl. Comput. Harmon. Anal., 20(2006), pp. 345–356.

[2] J. Benedetto and M. Fickus, *Finite normalized tight frames*, Adv. Comp. Math., 18(2003), pp. 357–385.

[3] P. Casazza, *The art of frame theory*, Taiwanese J. Math., 4(2000), pp. 129–201.

[4] P. Casazza, *Custom building finite frames* Contemp. Math., vol.345, 2004, pp. 61–86.

[5] P. Casazza and J. Kovacevic, *Equal-norm tight frames with erasures*, Adv. Comput. Math., 18(2003), pp. 387–430.

[6] O. Christensen, *An introduction to frames and Riesz bases*, Birkhauser Boston, MA, 2003.

[7] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Memoirs Amer. Math. Soc., 134(1998), no. 640.

[8] I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., 27(1986), pp. 1271–1283.

[9] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., 72(1952), pp. 341–366.

[10] K. Dykema, D. Freeman, K. Kornelson, D. Larson, M. Ordower, and E. Weber, *Ellipsoidal tight frames and projection decomposions of operators*, Illinois J. Math., 48(2004), pp. 477–489.

[11] V. Goyal and J. Kovacevic, *Quantized frame expansions with easures*, Appl. Comput. Harmon. Anal., 10(2001), pp. 203–233.

[12] P. Halmos, *A Hilbert Space Problem Book*, D. Van Nostrand Co., Inc., Princeton, N.J., 1967.

[13] D. Han, and D. Larson, *Frames, bases and group representations*, Memoirs Amer. Math. Soc., 697(2000).

[14] D. Han, K. Kornelson, D. Larson, and E. Weber, *Frames for undergraduates*, Student Mathematical Library, 40(2007).

[15] C. Heil and D. Larson, *Operator theory and modulation spaces*, Contemp. Math., to appear.

[16] R. Holmes and V. Paulsen, *Optimal frames for erasures*, Linear Algebra Appl., 377(2004), pp. 31–51.

[17] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1985.

[18] R. Kadison and J. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Academic Press, New York, vol.1 ,1983.

[19] D. Kalra, *Complex equiangular cyclic frames and erasures*, Linear Algebra Appl., 419(2006), pp. 373–399.

[20] K. Kornelson and D. Larson, *Rank-one decomposition of operators and construction of frames* Contemp. Math., vol.345, 2004, pp. 203–214.

[21] C. Kubrusly, *Elements of Operator Theory*, Birkhauser Boston, MA, 2001.

[22] G. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, Boston, MA, 1990.

[23] M. Naimark and A. Stern, *Theory of Group Representations*, Springer-Verlag, New York, 1982.

[24] T. Strohmer and R.Heath, *Grassmannian frames with applications to coding and communication*, Appl. Comput. Harmon. Anal., 14(2003), pp. 257–275.

[25] M. Sustik, J. Tropp, I. Dhillon and R. Heath, *On the existence of equiangular tight frames*, Linear Algebra Appl., 426(2007), no. 2-3, pp. 619–635.

[26] J. Tropp, I. Dhillon, R. Heath and T. Strohmer, *Designing structured tight frames via an alternating projection method*, IEEE Trans. Inform. Theory, 51 (2005), pp. 188–209.

[27] F. Zhang, *Matrix Theory, Basic Results and Techniques*, Springer-Verlag, New York, 1999.

VITA

Nga Quynh Nguyen was born in Hanoi, Vietnam. She received her B.S. degree in 1991 from the Department of Mathematics-Mechanics-Informatics, Hanoi State University, Vietnam. She joined Texas A&M University in August 2002 as a Ph.D. student in the Department of Mathematics. She may be reached at the following address

Department of Mathematics

Texas A&M University

College Station, TX 77843-3368

The typist for this dissertation was Nga Quynh Nguyen.