Spectral inequality for an Oseen operator in a two dimensional channel.

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Abstract

We prove a Lebeau-Robbiano spectral inequality for the Oseen operator in a two dimensional channel, that is, the linearized Navier-Stokes operator around a laminar flow, with no-slip boundary conditions. The operator being non-self-adjoint, we place ourself into the abstract setting of [12], and prove the spectral inequaltiy through the derivation of a proper Carleman estimate. In the spirit of [4], we handle the vorticity near the boundary by using the characteristics sets of P_φ or Q_{φ_0} in the different microlocal regions of the cotangent space. As a consequence of the spectral inequality, we derive a new estimate of the cost of the control for the small-time null-controllability.

Keywords : Spectral inequality, Navier-Stokes equation, Null-controllability, Carleman estimate.

Contents

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1 Introduction

In this paper, we consider the spectral inequality for the Navier-Stokes equation, with no-slip condition, in a 2-dimensional channel linearized around a Poiseuille flow. Denote $\Omega = \mathbb{T} \times (0, L)$ the spatial domain, where $L > 0$ and $\mathbb{T}$ is the one dimensional torus. For $T > 0$, the incompressible Navier-Stokes equation with no-slip boundary conditions is

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0, & (x, y, t) \in \Omega \times (0, T), \\ \text{div }(u) = 0, & (x, y, t) \in \Omega \times (0, T), \\ u(x, 0, t) = u(x, L, t) = 0, & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega, \end{cases} \tag{1}$$

where the intial data u_0 belongs to the space H, defined in (8). It is well-known that there exists a corresponding weak Leray solution to (1) that belongs to $C^0([0, T]; H) \cap L^2((0, T); V)$ (see for instance [2], Thoerem V.I.4), where V is also defined in (8) below. The Poiseuille flow is a laminar velocity field satisfying the no-slip condition,

$$\mathbf{U}(y) = \begin{pmatrix} \mathbf{U}_1(y) \\ 0 \end{pmatrix}, \qquad \mathbf{U}_1(0) = \mathbf{U}_1(L) = 0, \tag{2}$$

with $\mathbf{U} \in (C^\infty(0, L))^2$. The linearized Navier-Stokes equation around the 2-dimensional Poiseuille flow writes

$$\begin{cases} \partial_t u - \Delta u + \mathbf{U}_1(y) \partial_x u + \begin{pmatrix} u_2 \partial_y \mathbf{U}_1(y) \\ 0 \end{pmatrix} + \nabla p = 0, & (x, y, t) \in \Omega \times (0, T), \\ \text{div }(u) = 0, & (x, y, t) \in \Omega \times (0, T), \\ u(x, 0, t) = u(x, L, t) = 0, & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega, \end{cases} \tag{3}$$

with $u_0 \in H$. The spatial operator of (3) is the Oseen operator [18]. Note that up to changing u by $e^{-\kappa t} u$, for $\kappa > 0$, one may write

$$\begin{cases} \partial_t u - \Delta u + \mathbf{U}_1(y) \partial_x u + \begin{pmatrix} u_2 \partial_y \mathbf{U}_1(y) \\ 0 \end{pmatrix} + \nabla p + \kappa u = 0, & (x, y, t) \in \Omega \times (0, T), \\ \text{div }(u) = 0, & (x, y, t) \in \Omega \times (0, T), \\ u(x, 0, t) = u(x, L, t) = 0, & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega. \end{cases} \tag{4}$$

From now on, we refer to (4) as the Oseen equation. The shift by κ of the Oseen operator is introduced so as to gain the monotonicity of the Oseen operator (see Section 1.2), and has no consequences on the nature of our control results. Indeed, let us first introduce the null-controllability of the Oseen equation. Let ω be an open subset of Ω, and consider

$$\begin{cases} \partial_t u - \Delta u + \mathbf{U}_1(y) \partial_x u + \begin{pmatrix} u_2 \partial_y \mathbf{U}_1(y) \\ 0 \end{pmatrix} + \nabla p + \kappa u = \chi_\omega f, & (x, y, t) \in \Omega \times (0, T), \\ \text{div }(u) = 0, & (x, y, t) \in \Omega \times (0, T), \\ u(x, 0, t) = u(x, L, t) = 0, & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega, \end{cases} \tag{5}$$

where χ_ω denotes the characteristic function of the open set Ω. The small-time null-controllability can be stated as follows: for any control time $T > 0$ and any initial data $u_0 \in H$, does there exist a control function $f \in L^2((0, T); L^2(\omega))$ such that the solution u of (5) satisfies $u(T) = 0$? In the affirmative case, one is generally led to study the cost of the control, that is the estimation of the constant $C_T > 0$ (depending on $T > 0$ and ω) of the continuity map between the control and the initial data,

$$||f||_{L^2((0,T);L^2(\omega))} \leq C_T ||u_0||_H.$$

Following the Lebeau-Robbiano strategy [13] (see also [10, 15, 17]) for the controllability of parabolic operators using the spectral inequality, we shall prove,

Theorem 1.1 *Let $\varepsilon > 0$. Then, there exists $C > 0$ such that for all $T > 0$, for all initial data $u_0 \in H$, there exists $f \in L^2((0,T); L^2(\omega))$ such that the solution u of (5) satisfies*

$$u(T) = 0,$$

and such that the following estimate holds

$$||f||^2_{L^2((0,T);L^2(\omega))} \leq Ce^{\frac{C}{T^{1+\varepsilon}}} ||u_0||^2_{L^2(\Omega)}. \tag{6}$$

The null-controllability result stated in Theorem 1.1 is not new as it falls within the framework of Fernández-Cara, Guerrero, Yu, Imanuvilov and Puel [7]. Indeed, considering the linearization of the Navier-Stokes equation (1) around smooth solutions $\mathbf{U}$ satisfying $\mathbf{U} \in L^\infty((0,T) \times \Omega))^n$ and $\partial_t \mathbf{U} \in L^2((0,T); L^\sigma(\Omega))^n$ for some $\sigma > 1$, they proved notably, using the strategy of Fursikov and Imanuvilov [8], the small-time null-controllability of the linearized equation. The optimal control cost was however not reached for small time in [7].

Recently, Chaves-Silva and Lebeau have obtained in [4] the optimal cost of controllability $C_T = Ce^{\frac{C}{T}}$ for the Stokes system in dimension greater or equal to 2, that is the Navier-Stokes equation linearized around zero, improving the cost of control of [7] in this specific case. The control cost obtained of [4] is a consequence of the spectral inequality for the Stokes system in combination of the Lebeau-Robbiano strategy [13, 17].

Concerning small-time global controllability to trajectories of the nonlinear Navier-Stokes equation with no-slip boundary condition (1), we refer to the work of Coron, Marbach, Sueur and Zhang [6] (see also the references therein). In [6], the return method [5] was used, with a little help of a phantom force, to drive the solution to zero in arbitrarly small time, by using a trajectory with good transport properties and by analyzing the solution around this trajectory. One of the principal difficulties of [6] is to deal with the boundary layers. One way of removing this little phantom force is to understand the stability (or instability) of such as Poiseuille (or even Couette), and having a precise knowledge of the cost of control. The present paper can be seen as a first step in this direction, although the present work does not allow us to track the dependency of C_T with respect to the size of the potentials.

Our main result is to derive a spectral inequality for the Oseen equation (4) in the spirit of [4]. We refer to Section 1.3 for a precise statement and Section 1.4 for insights on the nature of the result. We begin by considering the equation on the vorticity $v = \partial_x u_2 - \partial_y u_1$ for (4),

$$\begin{cases} \partial_t v - \Delta v + \mathbf{U}_1(y)\partial_x v + (\partial_y^2 \mathbf{U}_1) u_2 + \kappa v = 0, & (x,y,t) \in \Omega \times (0,T), \\ \text{div } (u) = 0, & (x,y,t) \in \Omega \times (0,T), \\ v(x,0,t) = -\partial_y u_1(x,0,t),\ v(x,L,t) = -\partial_y u_1(x,L,t), & (x,t) \in \mathbb{T} \times (0,T), \\ v(x,y,0) = \text{rot } u_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{7}$$

At first sight, the Oseen equation (4) can be thought to be a perturbation of the Stokes equation (or equivalently (7) a perturbation of the equation on the vorticity for the Stokes system), and classical arguments could allow one to absord these lower terms in the Carleman estimates established in [4]. However, the divergence free condition on the velocity field u reveals a strong non-local term linking the vorticity to the velocity for (7) of the form $u_2 = \Delta^{-1}\partial_x v$. Furthermore, this internal coupling does not allow us to prove unique continuation property for the augmented elliptic operator on the vorticity in the s direction as opposed to [4] (see Section 1.4 for the definition of the augmented elliptic operator and Remark 5.6 for the lack of propagation of smallness in the s direction). This lead us to place ourselves in the original framework of [13], where the obtained spectral inequality is expressed with a local integral in the s-variable, as it is the case here (see Theorem 1.8). Since the works of [10, 15] for the heat operator, using propagation of smallness in the s direction, spectral inequalities has been reformulated with an observation from $\{s = 0\} \times \omega$, which would yield, in the present setting

$$\|u\|^2_{L^2(\Omega)} \leq Ce^{K\sqrt{\Lambda}} \int_\omega |u|^2 \, dx, \quad \forall u \in \Pi_\Lambda H.$$

The lack of unique continuation in the s-direction forces us to prove only Theorem 1.8. Finally, the Oseen operator is non self-adjoint, and therefore we shall use the work of [12], where the Lebeau-Robbiano strategy is generalized for a class of non self-adjoint operators. Before stating the spectral inequality, let us introduce the functional framework as well as some spectral properties of the Oseen operator.

1.1 Functional framework

Let us recast equation (4) into a semi-group formalism. First we introduce the following Hilbert spaces,

$$V := \{u \in H_0^1(\Omega)^2,\ \text{div}\ u = 0\}, \quad H := \{u \in L^2(\Omega)^2,\ \text{div}\ u = 0,\ u \cdot \nu_{|\partial\Omega} = 0\}, \tag{8}$$

endowed by their usual norms, and where ν denotes the outward normal vector at the boundary. We define the Stokes operator,

$$A_0 u := -\mathbb{P}\Delta u, \quad A_0 : D(A_0) \subset H \longrightarrow H,$$

where $\mathbb{P} : L^2(\Omega)^2 \to H$ stands for the standard Leray projection (the orthogonal projection of $L^2(\Omega)$ on H), with domain $D(A_0) := V \cap H^2(\Omega)^2$. We moreover define,

$$\tilde{A}_1 u = \mathbf{U}_1(y)\partial_x u + \begin{pmatrix} u_2 \partial_y \mathbf{U}_1(y) \\ 0 \end{pmatrix} + \kappa u,$$

and the operator $A_1 : D(A_1) \subset H \longrightarrow H$ defined by $A_1 u := \mathbb{P}\tilde{A}_1$, of domain, $D(A_1) := V \cap H^1(\Omega)^2$. Finally, the Oseen operator is defined by $A : D(A) = D(A_0) \subset H \to H$,

$$A := A_0 + A_1, \tag{9}$$

and one can write (4) with the semi-group formalism,

$$\begin{cases} \dfrac{d}{dt} u + Au = 0, \\ u_{|t=0} = u_0. \end{cases}$$

1.2 Spectral properties of the Oseen operator

We recall the Weyl asymptotic formula for the two dimensional Stokes operator, obtained in [16]. The operator A_0 being positive, self-adjoint and with compact resolvent, there exists a sequence of positive eigenvalues $(\mu_k)_{k\in\mathbb{N}}$ going to $+\infty$, and a sequence of eigenfunctions $\phi_k \in H$ satisfying $A_0\phi_k = \mu_k\phi_k$.

Theorem 1.2 *Let $N(\mu) = \sharp\{\mu_k < \mu\}$. Then, there exists $C > 0$ such that,*

$$N(\mu) \underset{k\to+\infty}{\sim} C\mu.$$

We recall here [12, Proposition 2.1], in the particular case A_1 is q-subordinate to A_0, with $q = 1/2$, that we shall use in what follows.

Proposition 1.3 *Assume that*

- $\Re\Big(Au, u\Big) \geq 0$*;*
- *A_0 is a positive self-adjoint with compact resolvent and densely defined operator;*
- *There exists $C > 0$ such that for every $u \in D(A_0^{1/2})$ we have,*

$$\Big|\Big(A_1 u, u\Big)\Big| \leq \frac{C}{2} ||A_0^{1/2} u||_H^{1/2} ||u||_H^{1/2}. \tag{10}$$

Then

- *$D(A) = D(A_0) \subset D(A_1) \subset H$ and has a compact resolvent;*
- $\text{Sp}(A) \subset \{z \in \mathbb{C},\ |\Im z| \leq C|z|^{1/2}\}$.

The Oseen operator falls within the hypothesis of Proposition 1.3, provided $\kappa > 0$ large enough to ensure the positivity.

Proposition 1.4 *Let $\mathbf{U} \in (C^\infty(0,L))^2$ defined by (2). Then, there exists $\kappa > 0$ sufficiently large such that the operator A defined by (9) satisfies $\mathrm{Sp}(A^*) \subset \{z \in \mathbb{C},\ |\Im z| \leq C|z|^{1/2}\}$.*

From Proposition 1.4, we have that there exists $C_0 > 0$ such that $\mathrm{Sp}(A) \subset \{z \in \mathbb{C},\ |\Im z| < C_0(\Re z)^{1/2}\}$. We fix such $C_0 > 0$ here and below.

We are now ready to define the spectral projectors on the eigenspaces of A^*, following [12].

Definition 1.5 *Let $\Lambda_1 < \Lambda_2 < \cdots < \Lambda_k < \ldots$ a sequence of real numbers going to infinity and such that $\Lambda_k \notin \Re\,\mathrm{Sp}(A)$ for all $k \in \mathbb{N}^*$. We define the following contours in the complex plane $\gamma_k := \{z \in \mathbb{C},\ \Re z = \Lambda_k,\ |\Im z| < C_0\Lambda_k\} \cup \{z \in \mathbb{C},\ \Re z \leq \Lambda_k^{1/2},\ |\Im z| = C_0(\Re z)^{1/2}\}$. We also define the spectral projectors,*

$$\begin{array}{rcl} \Pi_{\Lambda_k} : H & \longrightarrow & H \\ u & \longmapsto & \dfrac{1}{2i\pi}\displaystyle\int_{\gamma_k} (A^* - z)^{-1} u\, dz. \end{array}$$

We finally recall the following estimate of the resolvent along the contours γ_k (see [12, Theorem 2.5] with $p = 1,\ q = 1/2$). Note that we can take $p = 1$, because of the Weyl formula of Theorem 1.2.

Theorem 1.6 *There exists $C > 0$ such that for all $k \in \mathbb{N}$, for all $z \in \gamma_k$,*

$$||(A^* - z)^{-1}||_{\mathcal{L}(H,H)} \leq Ce^{C\sqrt{\Lambda_k}}$$

As a corollary, we shall use the following result, at several places.

Corollary 1.7 *For every holomorphic function f, for all $u \in \Pi_{\Lambda_k} H$, there exists $C > 0$ such that,*

$$||f(A^*)u||_{L^2(\Omega)} \leq \sup_{z \in \gamma_k} |f(z)| Ce^{C\sqrt{\Lambda_k}} ||u||_{L^2(\Omega)}.$$

Proof. We follow here [12]. One has, by holomorphic functional calculus,

$$\begin{aligned} ||f(A^*)u||_{L^2(\Omega)} = ||f(A^*)\Pi_{\Lambda_k} u||_{L^2(\Omega)} &= \left|\left| \frac{1}{2i\pi}\int_{\gamma_k} \frac{f(z)}{A^* - z} u dz \right|\right|_{L^2(\Omega)} \\ &\leq \sup_{z \in \gamma_k} |f(z)| \mathrm{meas}(\gamma_k) \sup_{z \in \gamma_k} ||(A^* - z)^{-1}||_{\mathcal{L}(H,H)} ||u||_{L^2(\Omega)} \\ &\leq \sup_{z \in \gamma_k} |f(z)| Ce^{C\sqrt{\Lambda_k}} ||u||_{L^2(\Omega)}. \end{aligned}$$

□

1.3 Main result

Our main result is to establish the following spectral inequality for the Oseen operator.

Theorem 1.8 *Let $S_0 > 0$. Let $\omega \subset \Omega$ be a nonempty open set. Then, there exists $C, K > 0$, and $\varphi \in C_0^\infty(0, S_0)$ such that, for every $\Lambda > 1$, we have*

$$\|u\|^2_{L^2(\Omega)} \leq Ce^{K\sqrt{\Lambda}} \iint_{(0,S_0)\times\omega} \left|\varphi(s)(A^*)^{-1/2}\sinh(s(A^*)^{1/2})u\right|^2 dx ds,$$

for all $u \in \Pi_\Lambda H$.

Remark 1.9 *As we believe that Lebeau-Robbiano spectral inequalities are interesting on their own, independently on applications, it is important to note that Theorem 1.8 also hold for the operator A. Indeed this spectral inequality is obtained through a direct application of the Carleman estimates proved below, and a careful inspection of the proofs show that it is also valid for the operator A. Indeed, A^* and A only differs by their lower order terms, that is A_1 and A_1^* respectively. Proofs are written with A_1^* having in mind applications to control, but also work with A_1.*

1.4 The augmented operator

The proof of Theorem 1.8 is an adaptation of the work of Chaves-Silva and Lebeau [4] to our framework. First it is now classical to prove such spectral inequality by deriving a Carleman estimate for an augmented elliptic operator. More precisely, we introduce,

$$\mathcal{X}_\Lambda := \left\{ (A^*)^{-1/2} \sinh(s(A^*)^{1/2})\tilde{u},\ \tilde{u} \in \Pi_\Lambda H \right\}. \tag{11}$$

Then, the augmented operator corresponds to, for any $u \in \mathcal{X}_\Lambda$,

$$\begin{cases} (-\partial_s^2 + A^*)u = 0, & (s,x,y) \in Z, \\ u_{|s=0} = 0, & (x,y) \in \Omega, \\ \partial_s u_{|s=0} = \tilde{u}, & (x,y) \in \Omega, \end{cases} \tag{12}$$

where,

$$Z = (0, S_0) \times \Omega,$$

for some $S_0 > 0$. However, as pointed out in [4], unique continuation property does not hold for system (12), mainly due to the pressure. One of the main ideas of [4], that we shall follow here, is to remove the pressure term by considering instead the equation on the vorticity. But that approach has two major difficulties. The first one is the lack of information of the trace of the vorticity at the boundary. This difficulty is handled, as in [4], with techniques coming from the microlocal analysis for boundary value problems. More specifically, we exploit crucially the fact that the conjugation of the augmented elliptic operator by two different weights yields two different characteristic sets. Hence, we are able to recover the desired estimates using the appropriate conjugated operator in each microlocal region (see Section 3.3). This requires to make sure that these estimates with different weights can be patched together, which is accomplished in Section 4.2.4. The second one is the coupling between the vorticity and the velocity in the interior of the domain due to the low order terms. To overcome this difficulty, we apply successively two Carleman estimates. The drawback however is that we lose the propagation of smallness in the s-direction (as opposed to [4]). That is precisely the reason for using the original strategy of Lebeau-Robbiano [13], developped for non-self-adjoint operators in [12], as it is the case here. We conclude this section with some results on the regularity of the augmented elliptic operator. First, note that by elliptic regularity, one has the following lemma.

Lemma 1.10 *Let $u \in \mathcal{X}_\Lambda$. Then, $\forall k \in \mathbb{N}$, one have $u \in H^k(Z)$.*

Proof. Let $u = (A^*)^{-1/2} \sinh(s(A^*)^{1/2})\tilde{u}$, for some $\tilde{u} \in \Pi_\Lambda H$. By the classical elliptic regularity and the resolvent estimate of Corollary 1.7, there exist $C_1 > 0$, and $C_2 > 0$ (that depends on Λ) such that,

$$||u(s,.)||_{H^2(\Omega)} \leq C||A^*u||_{L^2(\Omega)} \leq C_2||\tilde{u}||_{L^2(\Omega)}.$$

From Definition 1.5 and by holomorphic functionnal calculus, for all $k \in \mathbb{N}$, one has $(A^*)^k u \in D(A^*)$. and consequently, by elliptic regularity, one has

$$||u||_{H^{2k}(\Omega)} \lesssim ||A^*u||_{H^{k-2}(\Omega)} \lesssim ||(A^*)^k u||_{L^2(\Omega)} \lesssim ||\tilde{u}||_{L^2(\Omega)}.$$

from the resolvent estimate of Corollary 1.7. Derivatives in s of u can also be estimated in $L^2((0,S_0);H^k(\Omega))$ by again using the definition of u and the resolvent estimate.

□

We shall also use a different formulation of (12) in the core of the proof of the Carleman estimate, by introducing the pressure term.

Lemma 1.11 *For all $u \in \mathcal{X}_\Lambda$, there exists $q \in C^\infty(Z)$ such that,*

$$\begin{cases} -\partial_s^2 u - \Delta u + \tilde{A}_1^* u + \nabla q = 0, & (s,x,y) \in Z, \\ \operatorname{div}\ u = 0, & (s,x,y) \in Z. \end{cases} \tag{13}$$

Proof. We follow [2]. Let $\phi \in V$ be a test function. One has, for $s \in (0, S_0)$,

$$\begin{aligned} 0 = \left((-\partial_s^2 + A^*)u, \phi\right)_H &= \left(-\partial_s^2 u, \phi\right)_{L^2} + (A^*u, \phi)_{V',V} \\ &= \left(-\partial_s^2 u, \phi\right)_{L^2} + (-\mathbb{P}\Delta u, \phi)_{V',V} + (A_1^* u, \phi)_{V',V} \\ &= \left(-\partial_s^2 u, \phi\right)_{L^2} + (-\Delta u, \phi)_{H^{-1},H_0^1} + (A_1^* u, \phi)_{V',V} \\ &= \left(-\partial_s^2 u - \Delta u + A_1^* u, \phi\right)_{H^{-1},H_0^1}. \end{aligned}$$

As a result, using Theorem IV.2.3 in [2], there exists $q \in L^2$ such that,

$$-\partial_s^2 u - \Delta u + A_1^* u + \nabla q = 0.$$

Moreover, the regularity of q follows from Lemma 1.10.

□

1.5 Notations and semi-classical norms

We conclude the introduction by introducing notations that shall be used throughout the article. We also recall the standard definitions for semi-classical operators as well as semi-classical norms. We also introduce the definition of a particular class of tangential semi-classical operators that shall be used in Section 4.2.2.

1.5.1 Notations

We recall that,

$$Z = (0, S_0) \times \Omega,$$

for some $S_0 > 0$. Let us also define for later conveniences,

$$Y = (\widetilde{S}_0, S_0 - \widetilde{S}_0) \times \Omega,$$

for $\tilde{S}_0 < S_0/2$. Since Y denote a localization in the s variable, we will use the abuse of notation ∂Y to denote $\{\{y = 0\} \cup \{y = L\}\} \cap \overline{Y}$.

Now let $u \in \mathcal{X}_\Lambda$ (hence, u solves (12)), and define here and below,

$$v := \mathrm{rot}(u), \tag{14}$$

which satisfies in Ω,

$$-\partial_s^2 v - \Delta v + \mathrm{rot}\,(A_1^* u) = 0. \tag{15}$$

In what follows,

$$w(s, x, y) = e^{\varphi(s,y)/h} v(s, x, y).$$

where the weight function φ is defined below in (24). We also define the localized functions we shall work with throughout the article, with q defined by Lemma 1.11,

$$\begin{aligned} W(s,x,y) &= \chi(s,x,y) w(s,x,y), & V(s,x,y) &= \chi(s,x,y) v(s,x,y), \\ U(s,x,y) &= \chi(s,x,y) u(s,x,y), & Q(s,x,y) &= \chi(s,x,y) q(s,x,y) \\ \mathfrak{U}(s,x,y) &= \chi_1(s) u(s,x,y), & \mathfrak{V}(s,x,y) &= \chi_1(s) v(s,x,y), \\ \mathfrak{W}(s,x,y) &= \chi_1(s) w(s,x,y), & \mathfrak{Q}(s,x,y) &= \chi_1(s) q(s,x,y), \end{aligned} \tag{16}$$

where $\chi(s, x, y) = \chi_1(s)\chi_2(x)\chi_3(y)$ and where $\chi_1(s), \chi_2(x), \chi_3(y)$ are smooth cut-offs in each variable satisfying near a point $(s_0, x_0, y = 0) \in (0, S_0) \times \mathbb{T} \times \{y = 0\}$:

$$\chi_1(s) = \begin{cases} 1 & \text{if } |s - s_0| < S_1, \\ 0 & \text{if } |s - s_0| > 2S_1, \end{cases} \qquad \chi_3(y) = \begin{cases} 1 & \text{if } y < Y_1, \\ 0 & \text{if } y > Y_1, \end{cases} \tag{17}$$

for some $0 < 2S_1 < \widetilde{S}_0$ (that is, $\mathrm{supp}\,\chi \subset Y$) and some $Y_1 < L/4$. The localization $\chi(s, x, y)$ is defined likewise near a point $(s_0, x_0, y = L) \in (0, S_0) \times \mathbb{T} \times \{y = L\}$.

Finally, $C > 0$ is a generic constant which may change from one line to another.

1.5.2 Pseudo-differential operators and semi-classical norms

We introduce the variable $z = (s, x, y) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}$ and the tangential variable z' to be defined as $z' = (s, x) \in \mathbb{R} \times \mathbb{T}$, as well as their Fourier counterparts $\zeta = (\sigma, \xi, \eta) \in \mathbb{R} \times \mathbb{N} \times \mathbb{R}$ and $\zeta' = (\sigma, \xi)$. Note that $(z, \zeta) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{N} \times \mathbb{R} := T^*(\mathbb{R} \times \mathbb{T} \times \mathbb{R})$ and $(z', \zeta') \in \mathbb{R} \times \mathbb{T} \times \mathbb{R} \times \mathbb{N} := T^*(\mathbb{R} \times \mathbb{T})$. We emphasize that the differential operators ∇, Δ and rot only act on the physical variables x, y. We shall also use the notation $\nabla_z := {}^t(\partial_s, \partial_x, \partial_y)$, as well as $\Delta_z = \nabla_z^2$ when needed.

Semi-classical operators

Here we recall some facts on semi-classical pseudo-differential operators with a small parameter h, say $0 < h \le h_0$. We shall denote by $\mathcal{S}^m_{\rm sc}$ the space of smooth functions $a(z,\zeta,h)$, with $0 < h \le h_0$ as a small parameter, that satisfy the following behavior at infinity: for all multi-indices α, β there exists $C_{\alpha,\beta} > 0$ such that,

$$\left|\partial_z^\alpha \partial_\zeta^\beta a(z,\zeta,h)\right| \le C_{\alpha,\beta}(1+|\zeta|^2)^{(m-|\beta|)/2},$$

for all $(z,\zeta) \in T^*(\mathbb{R}\times\mathbb{T}\times\mathbb{R})$. For $a \in \mathcal{S}^m_{\rm sc}$, we define pseudo-differential operator of order m, denoted by $A = \mathrm{Op}(a)$:

$$Au(z) := \frac{1}{(2h\pi)^3} \sum_{\xi\in\frac{1}{2\pi h}\mathbb{Z}} \int_{\mathbb{R}^2}\int_{\mathbb{R}\times\mathbb{T}\times\mathbb{R}} e^{i\frac{(z-\tilde{z})\cdot\zeta}{h}} a(z,\zeta,h)u(\tilde{z})d\tilde{z}d\sigma d\eta,$$

One says that a is the symbol of A. We shall denote by $\Psi^m_{\rm sc}$ the set of pseudo-differential operators of order m and denote by $\sigma(A)$ (resp. $\sigma(a)$) the principal symbol of the operator A (resp. the symbol a). We refer to [9] to precise definitions of pseudo-differential operators. Thus, define $D = h\partial/i$, then $\sigma(D) = \xi$. We shall also denote by $\mathcal{D}^m_{\rm sc}$ the space of semi-classical differential operators, i.e the case when the symbol $a(x,\xi,h)$ is a polynomial function of order m in ξ. We recall here the composition formula of pseudo-differential operators. Let $a \in \mathcal{S}^m_{\rm sc}$ and $b \in \mathcal{S}^{m'}_{\rm sc}$, $m, m' \in \mathbb{R}$, we have

$$\mathrm{Op}(a)\circ\mathrm{Op}(b) = \mathrm{Op}(c),$$

for some $c \in \mathcal{S}^{m+m'}_{\rm sc}$.

Tangential semi-classical operators

In the section we consider pseudo-differential operators which only acts in the tangential direction z', viewing the variable y as a parameter. We define $\mathcal{S}^m_{T,\rm sc}$ as the set of smooth functions $b(z,\zeta',h)$ defined for h a small parameter, say $0 < h < h_0$, satisfying the following behavior at infinity: for all multi-indices $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^{n-1}$ there exists a constant $C_{\alpha,\beta} > 0$ such that,

$$\left|\partial_x^\alpha \partial_{\zeta'}^\beta b(z,\zeta',h)\right| \le C_{\alpha,\beta}(1+|\zeta'|^2)^{(m-|\beta|)/2},$$

for all $(z,\zeta',h) \in T^*(\mathbb{R}\times\mathbb{T})\times]0,1[$. For $b \in \mathcal{S}^m_{T,\rm sc}$, we define a tangential pseudo-differential operator $B := \mathrm{Op}_T(b)$ of order m by,

$$Bu(x) := \frac{1}{(2h\pi)^2} \sum_{\xi\in h^{-1}\mathbb{Z}} \int_{\mathbb{R}}\int_{\mathbb{R}\times\mathbb{T}} e^{i\frac{(z'-\tilde{z}')\cdot\zeta'}{h}} b(z,\zeta',h)u(\tilde{z}')d\tilde{z}'d\sigma$$

As in the previous section, we define $\Psi^m_{T,\tau}$ as the set of tangential pseudo-differential operators of order m, and $\mathcal{D}^m_{T,\tau}$ the set of tangential differential operators of order m. We shall use this class of pseudo-differential operators in Section 4.2.1, that is the region where $|\xi|$ is small.

Semi-classical norms

We first define the following semi-classical Sobolev tangential norms, for traces of functions on $\mathbb{R}\times\mathbb{T}$ at $\{y = 0\}$ or $\{y = L\}$

$$|u|_{m,\rm sc} := |\,\mathrm{Op}_T((1+|\zeta'|^2)^{\frac{m}{2}})u|_{L^2(\mathbb{R}\times\mathbb{T})}.$$

For $m \in \mathbb{N}$, this semi-classical norm is equivalent to $\sum_{|\alpha|\le m} h^{|\alpha|}|\partial_{z'}^{|\alpha|}u|_{L^2(\mathbb{R}\times\mathbb{T})}$, uniformly for $h \in (0,1)$. We also define the following semi-classical norms, in the interior of the domain $\mathbb{R}\times\Omega$,

$$||u||^2_{m,\rm sc} := \sum_{k=0}^m ||\partial_y^k\,\mathrm{Op}_T((1+|\zeta'|^2)^{\frac{m-k}{2}})u||^2_{L^2(\mathbb{R}\times\mathbb{T}\times[0,L])}.$$

As above, if $m \in \mathbb{N}$, this semi-classical norm is equivalent to $\sum_{|\alpha|\le m} h^{|\alpha|}||\partial_z^{|\alpha|}u||_{L^2(\mathbb{R}\times\mathbb{T}\times[0,L])}$, uniformly for $h \in (0,1)$.

Below, we shall the following semi-classical trace lemma, that can be found in [14, Section 5 (eq. 44)].

Lemma 1.12 *There exists $C > 0$ such that for all $u \in H^1(\mathbb{R}\times\mathbb{T}\times[0,L])$, for $h \in (0,1)$,*

$$|u_{|_{y=L}}|_{0,\rm sc} + |u_{|_{y=0}}|_{0,\rm sc} \le Ch^{-1/2}||u||_{1,\rm sc}. \tag{18}$$

2 From the spectral inequality to the cost of control

In this section, using the works of [12], we show that the spectral inequality of Theorem 1.8 allows one to prove an observability inequality for the Oseen equation, with optimal cost of control. We consider the following controlled system :

$$\begin{cases} \frac{d}{dt}u + Au = \mathbb{P}\chi_\omega f, & (x,y,t) \in \Omega \times (0,T), \\ u_{|_{t=0}} = u_0 \in H, & (x,y) \in \Omega. \end{cases} \tag{19}$$

where χ is the characteristic function of an open subset $\omega \subset \Omega$, and f is the control function.

Theorem 2.1 *Let $\varepsilon > 0$. There exists $C > 0$ such that for all $T > 0$, for all initial data $u_0 \in H$, there exists $f \in L^2(0,T;\Omega)$ such that the solution u of (19) satisfies*

$$u(T) = 0,$$

and such that the following estimate holds

$$||f||^2_{L^2(0,T,L^2(\Omega))} \leq Ce^{\frac{C}{T^{1+\varepsilon}}}||u_0||^2_{L^2(\Omega)} \tag{20}$$

The constant appearing in (20) is referred as the cost of controllability. Controllability of linearized Navier-Stokes equations is already known (see for instance [7], where the author used a global Carleman estimate for the parabolic operator) but the cost of control is not optimal. The work of [4] exhibited for the first time the optimal behavior of this constant as $T \to 0$ for the Stokes equation. The above spectral inequality of Theorem 1.8 allows us to obtain almost the same result for the Oseen operator, using an adaptation of the arguments of [13, 12, 17]. The proof relies on the well-known Lebeau-Robbiano strategy. In [12], in a abstract setting, the author showed that spectral inequality of the form of Theorem 1.8 for non-self-adjoint operators satisfying the properties of Proposition 1.4 implies null-controllability for the parabolic problem (19). However, cost of controllability was not derived in this article. Here, we then adopt the strategy of [17] in Lemma 2.7 and Theorem 2.6 to obtain the cost of control (20). Note that there is a little loss in the power of T in (20), as we may expect to have $\varepsilon = 0$ as in [4].

2.1 Controllability of the low frequencies

We first recast Theorem 1.8 into the formalism of [12].

Corollary 2.2 *Let $S_0 > 0$ and $\omega \subset \Omega$ be a nonempty open set. Then, there exists $C, K > 0$, and $\varphi \in C_0^\infty(0, S_0)$ such that, for every $\Lambda > 1$, we have*

$$\|u\|^2_{L^2(\Omega)} \leq Ce^{K\sqrt{\Lambda}} \iint_{(0,S_0)\times\omega} \left|\varphi(s)\sinh(s(A^*)^{1/2})u\right|^2 dxds,$$

for all $u \in \Pi_\Lambda H$.

Proof. Note that $\sqrt{A^*}$ is an isomorphism on $\Pi_\lambda H$. Using the resolvent estimate of Corollary 1.7, we obtain

$$||\sqrt{A^*}u||_{L^2(\Omega)} = ||\sqrt{A^*}\Pi_\Lambda u||_{L^2(\Omega)} = \left\|\frac{1}{2i\pi}\int_{\gamma_k}(A^* - z)^{-1}\sqrt{A^*}udz\right\|_{L^2(\Omega)} \leq Ce^{C\sqrt{\Lambda}}||u||_{L^2(\Omega)}.$$

Hence, applying Theorem 1.8 to $v = \sqrt{A^*}u$ proves Corollary 2.2. □

Now we consider the parabolic control problem for the low frenquencies,

$$\begin{cases} \frac{d}{dt}u + Au = \mathbb{P}\chi_\omega f, & (x,y,t) \in \Omega \times (0,T), \\ u_{|_{t=0}} = u_0 \in \Pi_\Lambda H. \end{cases} \tag{21}$$

The controllability of parabolic equations (21) of the lower frequencies for non-self-adjoint operators has been already developed in [12] in an abstract setting. From Proposition 1.4 and Corollary 2.2, we obtain Theorem 2.3, as they fall into the hypothesis of [12]. More precisely, Theorem 2.3 is [12, Theorems 4.9 and 4.10] in our setting, with $q = 1/2$.

Theorem 2.3 *Let $\gamma > 1$ and let $T^* > 0$. There exists $C > 0$ such that for all $0 < T < T^*$, for all $u_0 \in \Pi_\Lambda H$, there exists $f \in L^2((0,T); L^2(\Omega))$ such that the solution of (21) satisfies $\Pi_\Lambda u(T) = 0$, with control cost*

$$||f||^2_{L^2((0,T);L^2(\Omega))} \leq Ce^{C\left(\sqrt{\Lambda}+\frac{1}{T^\gamma}\right)}||u_0||^2_{L^2(\Omega)}.$$

Let us consider the dual counterpart of (21) (along with the change of variables $t \to (T-t)$).

$$\begin{cases} \frac{d}{dt}u + A^*u = 0 \\ u_{|t=0} = u_0 \in \Pi_\Lambda H. \end{cases} \tag{22}$$

The following observability inequality then holds, coming from the duality between controllability and observability, noting that the adjoint of the control operator $\mathbb{P}\chi_\omega$ satisfies $(\mathbb{P}\chi_\omega)^*u = u_{|\omega}$.

Corollary 2.4 *Let $\gamma > 1$ and let $T^* > 0$. There exists $C > 0$ such that for all $0 < T < T^*$, and for all $\Lambda > 0$,*

$$||u(T)||^2_H \leq Ce^{C(\sqrt{\Lambda}+\frac{1}{T^\gamma})}\int_0^T\int_\omega |u(t,x)|^2 dtdx,$$

for all u solution of (22).

2.2 Decay of the semigroup

Following the idea of the Lebeau-Robbiano strategy, one shall need the following estimate of the natural dissipation of the higher frequencies (see [12], Proposition 4.12 with $\theta = 1/2$ for a proof : note that one can take $\theta = 1/2$ since in our case, $q = 1/2$ from Proposition 1.4, and $p = 1$ from Theorem 1.2).

Proposition 2.5 *There exists $C > 0$, $\Lambda_0 > 0$, such that for all $\Lambda \geq \Lambda_0$ and for all $t > \frac{1}{\sqrt{\Lambda}}$,*

$$||S_A(t)\,(I - \Pi_\Lambda)\,||^2_{\mathcal{L}(H)} \leq Ce^{C\sqrt{\Lambda}-t\Lambda}.$$

2.3 Derivation of an observability inequality

Let u be solution of the following Oseen equation

$$\begin{cases} \frac{d}{dt}u + A^*u = 0, \\ u_{|t=0} = u_0 \in H. \end{cases} \tag{23}$$

Theorem 2.6 *Let $\gamma > 1$ and $T^* > 0$. There exists $C > 0$ such that for every $0 < T < T^*$,*

$$||u(T)||^2_{L^2(\Omega)} \leq Ce^{\frac{C}{T^\gamma}}\int_0^T\int_\omega |u(t,x)|^2 dtdx,$$

with u solution of (23).

Proof. We only provide a sketch of the proof, as it is classical, and sufficient to apply Lemma 2.7 in combination with Lemmata 4.1 and 4.2 in [4]. The main idea is to consider of proper partition of the interval $[0,T] = \cup_{k=0}^\infty [a_j, a_{j+1}]$, and apply Lemma 2.7 between $t = a_j$ and $t = a_{j+1}$ and sum over j. As the left hand side provides a telescopic sum, one obtains the result. We refer to [17, 4] for more details. □

Lemma 2.7 *There exist $T^* > 0$ and $m \in (0,1)$ such that,*

$$F(t)||u(t)||^2_H - F(mt)||u_0||^2_H \leq \int_0^t\int_\omega |u(t,x)|^2 dtdx,$$

holds for all u solution of (23), for all $t \in [0, T^*]$, with F of the form $F(t) = K_1 e^{-\frac{K_2}{t^\gamma}}$, with $\gamma > 1$.

Proof. One shall denote by $u = v + w$ where $v = \Pi_\Lambda$ and $w = (1 - \Pi_\Lambda)u$. Note that Corollary 2.4 applies for v and Proposition 2.5 applies for w. To this end, one sets,

$$f(t) = C_1 e^{C_1(\sqrt{\Lambda} + \frac{1}{T^\gamma})}, \quad g(t) = C_2 e^{C_2\sqrt{\Lambda} - t\Lambda}.$$

Hence, applying Corollary 2.4 and Proposition 2.5, up to translating time $0 \to T_1$, for $0 < T_1 < T < T^*$, for every $\Lambda > \Lambda_0$, with $\Lambda_0 > 0$ sufficiently large, and $T^* < \Lambda_0^{-\frac{1}{2}}$,

$$\begin{aligned}
||u(T)||_H^2 &\le C\left(||v(T)||_H^2 + ||w(T)||_H^2\right)\\
&\le C\left(f(T-T_1)\int_{T_1}^T\int_\omega |v(t,x)|^2 dtdx + ||w(T)||_H^2\right)\\
&\le C\left(f(T-T_1)\int_{T_1}^T\int_\omega |u(t,x)|^2 dtdx + 2f(T-T_1)\int_{T_1}^T\int_\Omega |w(t,x)|^2 dtdx + ||w(T)||_H^2\right)\\
&\le C\left(f(T-T_1)\int_{T_1}^T\int_\omega |u(t,x)|^2 dtdx + 2f(T-T_1)(T-T_1)g(T-T_1)||w(T_1)||_H^2 + ||w(T)||_H^2\right)\\
&\le C\left(f(T-T_1)\int_{T_1}^T\int_\omega |u(t,x)|^2 dtdx + \left(2f(T-T_1)(T-T_1)g(T-T_1)g(T_1) + g(T)\right)||w(T)||_H^2\right)\\
&\le C\left(f(T-T_1)\int_{T_1}^T\int_\omega |u(t,x)|^2 dtdx + \left(2f(T-T_1)(T-T_1) + 1\right)g(T)||w(T)||_H^2\right).
\end{aligned}$$

Let us set $T_1 = (1-\varepsilon)T$, which implies,

$$(Cf(\varepsilon T))^{-1}||u(T)||_H^2 - (Cf(\varepsilon T))^{-1}\left(2f(\varepsilon T)\varepsilon T + 1\right)g(T)||w(T)||_H^2 \le \int_{(1-\varepsilon)T}^T\int_\omega |u(t,x)|^2 dtdx.$$

Setting $(\varepsilon T)^{1+\gamma}\Lambda = 1$ (note that it is in accordance with hypothesis of Proposition 2.5) yields,

$$(Cf(\varepsilon T))^{-1}\left(2f(\varepsilon T)\varepsilon T + 1\right)g(T) = 2\frac{C_2}{C}(\varepsilon T)e^{\frac{C_2}{(\varepsilon T)^{\frac{\gamma+1}{2}}} - \frac{1}{\varepsilon(\varepsilon T)^\gamma}} + \frac{1}{CC_1}e^{\frac{C_2}{(\varepsilon T)^{\frac{\gamma+1}{2}}} - \frac{1}{\varepsilon(\varepsilon T)^\gamma} - C_1\left(\frac{C_1}{(\varepsilon T)^{\frac{\gamma+1}{2}}} + \frac{1}{(\varepsilon T)^\gamma}\right)}.$$

Hence, there exists $\varepsilon_0 > 0$ and $C_3, C_4, C_5, C_6 > 0$ such that for every $\varepsilon < \varepsilon_0$, one has,

$$(Cf(\varepsilon T))^{-1}\left(2f(\varepsilon T)\varepsilon T + 1\right)g(T) \le C_3 e^{-\frac{C_4}{\varepsilon(\varepsilon T)^\gamma}},$$

and

$$(Cf(\varepsilon T))^{-1} \ge C_5 e^{-\frac{C_6}{(\varepsilon T)^\gamma}}.$$

Taking again ε_0 sufficiently small, there exist $K_1, K_2 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$,

$$F(T)||u(T)||_H^2 - F\left(\frac{T}{2}\right)||w(T)||_H^2 \le \int_0^T\int_\omega |u(t,x)|^2 dtdx,$$

with $F(t) = K_1 e^{-\frac{K_2}{T^\gamma}}$. This ends the proof. □

3 Conjugated operators and weight function properties

In this section, we begin by introducing the different weight functions that shall be used in the Carleman estimates, as well as the conjugated operators. We then set the ground for the analysis of the boundary terms arising in the Carleman estimates (dealt in Section 4.2) by splitting the cotangent space of ∂Y in different regions and by analyzing the roots of the principal symbols of Q_{φ_0} and P_φ.

3.1 Augmented operator conjugated by the weight function

Near the boundaries $\{y = 0\}$ and $\{y = L\}$, we shall work with the following weight function,

$$\varphi(s,y) := e^{\gamma\psi_\epsilon(s,y)}, \tag{24}$$

where $\psi_\epsilon(s,y)$ is smooth and satisfies,

$$\psi_\epsilon(s,y) = \begin{cases} y - \varepsilon(s-s_0)^2 & \text{in a neighborhood of } \{y=0\}, \\ (L-y) - \varepsilon(s-s_0)^2 & \text{in a neighborhood of } \{y=L\}. \end{cases} \tag{25}$$

where $\epsilon > 0$ shall be fixed small and $\gamma > 0$ shall be fixed large in what follows. In order to prove a proper Carleman estimate in Section 4, one shall work with three slightly different conjugated operators.

We shall denote by,

$$\begin{aligned} P_\varphi &:= h^2 e^{\varphi/h}(-\partial_s^2 - \Delta)e^{-\varphi/h} \\ &= -h^2\partial_s^2 - h^2\Delta - ((\partial_y\varphi)^2 + (\partial_s\varphi)^2) + 2h(\partial_y\varphi\partial_y + \partial_s\varphi\partial_s) + h^2(\partial_y^2\varphi + \partial_s^2\varphi), \end{aligned}$$

with semiclassical principal symbol given by,

$$p_\varphi(s,y,\xi,\eta,\sigma) = \sigma^2 + \eta^2 + \xi^2 - ((\partial_y\varphi)^2 + (\partial_s\varphi)^2) + 2i(\partial_y\varphi\eta + \partial_s\varphi\sigma), \tag{26}$$

where (σ,ξ,η) denotes the Fourier variables of the space variables (s,x,y). Defining the weight function φ_0,

$$\varphi_0(s) := \varphi(y=0,s) = \varphi(y=L,s), \tag{27}$$

we shall also use the following conjugated operator,

$$\begin{aligned} Q_{\varphi_0} &:= h^2 e^{\varphi_0/h}(-\partial_s^2 - \Delta)e^{-\varphi_0/h} \\ &= -h^2\partial_s^2 - h^2\Delta - (\partial_s\varphi_0)^2 + 2h(\partial_s\varphi_0)\partial_s + h^2\partial_s^2\varphi_0, \end{aligned} \tag{28}$$

with semiclassical principal symbol given by,

$$q_{\varphi_0}(s,\xi,\eta,\sigma) = \sigma^2 + \eta^2 + \xi^2 - (\partial_s\varphi_0)^2 + 2i(\partial_s\varphi_0)\sigma.$$

Finally, we introduce a third conjugated operator that shall be needed in Section 5,

$$\begin{aligned} -\Delta_\varphi &:= h^2 e^{\varphi/h}(-\Delta)e^{-\varphi/h} \\ &= -h^2\Delta - |\nabla\varphi|^2 + 2h\nabla\varphi\cdot\nabla + h^2\Delta\varphi, \end{aligned} \tag{29}$$

with a semiclassical principal symbol,

$$\delta_\varphi(s,x,y,\xi,\eta) = \eta^2 + \xi^2 - |\nabla\varphi|^2 + 2i\nabla\varphi\cdot\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

3.2 Sub-ellipticity condition

Under the action of the conjugaison, elliptic operators such as P, Q or $-\Delta$ defined in the previous section are not elliptic. To handle characteristics sets, one shall use the sub-ellipticity property, defined in what follows.

Definition 3.1 *Let U be an open set of $\mathbb{R}^3$. We say that (P,φ) satisfies the Hörmander's sub-ellipticity condition in U if $|\nabla_z\varphi| = |(\partial_s\varphi,\nabla\varphi)| \geq C > 0$ on $\overline{U}$ and if*

$$p_\varphi(s,x,y,\sigma,\xi,\eta) = 0 \Longrightarrow \frac{1}{2i}\{\overline{p}_\varphi, p_\varphi\} > 0. \tag{30}$$

In what follows, one shall also need a slightly different sub-ellipticity definition, with the same weight function but for the (not-augmented) - Laplace operator given by (29). Indeed, when considering (15), one observes that the vorticity v is coupled with the velocity u due to the lower-order terms. One shall pay attention that, for the same weight function φ, the Carleman estimates given below applies both for v (associated with the operator P_φ) and for u (associated with the operator $-\Delta$).

Definition 3.2 *We say that $(-\Delta,\varphi)$ satisfies the sub-ellipticity condition in U if $|\nabla\varphi| \geq C > 0$ on $\overline{U}$ and if,*

$$\delta_\varphi(s,x,y,\xi,\eta) = 0 \Longrightarrow \frac{1}{2i}\{\overline{\delta}_\varphi, \delta_\varphi\} > 0. \tag{31}$$

It is now well-known (see [11] for instance) that the convexified weight given by (24) satisfies the sub-ellipticity conditions of definitions 3.1 and 3.2, for $\gamma > 0$ chosen sufficiently large, in a neighbohood of the boundary $\{y=0\}$ or $\{y=L\}$, since one has $|\partial_y\varphi| \neq 0$. The sub-ellipticity condition is a necessary condition for the derivation of a Carleman estimate with loss of a power $h^{1/2}$ in the inequality of Theorem 4.1 and Theorem 4.2.

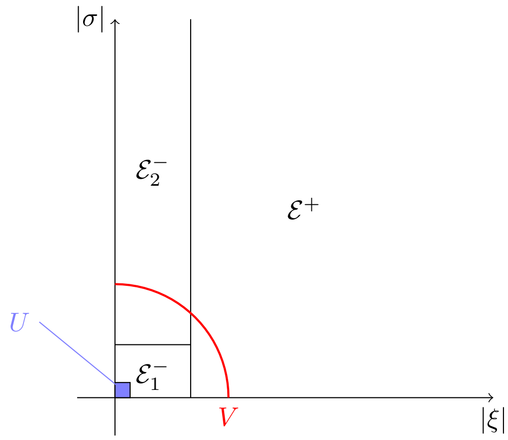

Figure 1: Representation of the microlocal regions of the cotangent space of ∂Y and illustration of the characteristic sets $\operatorname{Char} Q_{\varphi_0} \subset U$ and $\operatorname{Char} P_\varphi \subset V$.

3.3 Microlocal regions

Following the ideas of [4], we introduce three different microlocal regions in the cotangent space of ∂Y. First we define,

$$\mathcal{E}^+ := \{(\sigma, \xi) \in \mathbb{R}^2,\ |\xi|^2 > 2\sup(\partial_s\varphi_0)^2\}, \quad \mathcal{E}^- := \{(\sigma, \xi) \in \mathbb{R}^2,\ |\xi|^2 < 3\sup(\partial_s\varphi_0)^2\},$$

and we shall also split $\mathcal{E}^- = \mathcal{E}_1^- \cup \mathcal{E}_2^-$ where,

$$\mathcal{E}_1^- := \{(\sigma, \xi) \in \mathcal{E}^-,\ |\sigma| < 2\delta\}, \quad \mathcal{E}_2^- := \{(\sigma, \xi) \in \mathcal{E}^-,\ |\sigma| > \delta\},$$

for a small $\delta > 0$ that shall be fixed in what follows. This microlocal decomposition of the phase space are represented in Figure 1, and are motivated by the position of the roots of Q_{φ_0} and P_φ obtained in Section 3.4 and the strategy of the proof detailed in the beginning of Section 4.2.

3.4 Roots properties

It is classical when dealing with boundary value problems to consider the principal symbols of the operators P_φ and Q_{φ_0} as polynomials in the conormal variable η. We first analyse the behavior of the roots of the principal symbol q_{φ_0}.

3.4.1 Analysis of Q_{φ_0}

As the weight function φ_0 does not depend on the x, y variables, the symbol q_{φ_0} can be decomposed as follows,

$$q_{\varphi_0}(s, \sigma, \xi, \eta) = (\eta - r^+(s, \sigma, \xi))(\eta - r^-(s, \sigma, \xi)). \tag{32}$$

Lemma 3.3 *We have in $\mathcal{E}^+$,*

- $-(r^\pm(s, \sigma, \xi))^2 = \sigma^2 + \xi^2 - (\partial_s\varphi_0)^2 + 2i\partial_s\varphi_0\sigma, \quad r^\pm \in \tilde{\mathcal{S}}^1_{T,\mathrm{sc}},$
- $r^+ = -r^-,$
- $|\operatorname{Im} r^\pm| \geq C > 0,$

From the last point, we make the following convention $\pm \operatorname{Im} r^\pm \geq C > 0$.

Proof. The first two points come from the definition of q_{φ_0}. For the last point, notice that, in $\mathcal{E}^+$, we have, $\operatorname{Re}((r^\pm(s, \sigma, \xi))^2) = (\partial_s\varphi_0)^2 - \sigma^2 - \xi^2 \leq -c < 0$, which is sufficient to conclude that $\pm \operatorname{Im} r^\pm \geq C > 0$. □

Note in particular from the third point that in the region $\mathcal{E}^+$, the operator Q_{φ_0} is elliptic, since there is no real characteristic set (characteristics sets are represented in Figure 1).

3.4.2 Analysis of P_φ

Second, we analyze the roots of P_φ. As above, we write,

$$p_\varphi(s,y,\sigma,\xi,\eta) = (\eta - \rho^+(s,y,\sigma,\xi))(\eta - \rho^-(s,y,\sigma,\xi)), \tag{33}$$

where $\rho^\pm(s,y,\sigma,\xi) := -i\partial_y\varphi \pm m(s,y,\sigma,\xi) \in \tilde{\mathcal{S}}^1_{T,\mathrm{sc}}$, with $-m^2 = \xi^2 + \sigma^2 - (\partial_s\varphi)^2 + 2i(\partial_s\varphi)\sigma$, and by convention $\mathrm{Re}(m) \geq 0$. We shall use this polynomial form in the region $\mathcal{E}_1^-$. Hence, we shall need the following lemma, that describes the position of the roots in the complex plane in that microlocal region.

Lemma 3.4 *In $\mathcal{E}_1^-$, we have,*

- *for $y \in (0,L)$ in a neighborhood of $\{y=0\}$, we have $\mathrm{Im}\,\rho^\pm \leq -c < 0$,*
- *for $y \in (0,L)$ in a neighborhood of $\{y=L\}$, we have $\mathrm{Im}\,\rho^\pm \geq c > 0$.*

Proof. On $\mathcal{E}_1^-$, we have $|\sigma| < 2\delta$ and, using the definition of φ, $|\xi|^2 < 3c\epsilon^2$. Hence, $|m^2| \leq c(\delta^2+\epsilon^2)$. Therefore, for δ and ϵ chosen sufficiently small, we have that $\mathrm{Im}\,\rho^\pm$ has the same sign than $-\partial_y\varphi$. Yet, $\partial_y\varphi > 0$ near $y=0$, and $\partial_y\varphi < 0$ (see (25) for an explicit expression of φ in a neighborhood of the boundaries.) □

One note that in fact, $\partial_\nu\varphi_{|_{y\in\{0,L\}}} < 0$, where ν is the unit outward normal vector of the boundary $\{y=0\}\cup\{y=L\}$. From now on, δ is fixed in what follows such that Lemma 3.4 holds true. Note that Lemma 3.4 shows that the operator P_φ is elliptic in the microlocal region $\mathcal{E}_1^-$ (a projection on (ξ,σ) of characteristic set of P_φ is represented in Figure 1).

4 A Carleman estimate near the boundary

4.1 Estimates for the volume terms

We shall use the following result in [4] as a black box. It is worth noticing that their result applies with the weight function defined in (27) since the weight function satisfies the sub-ellipticity condition of Definition 3.1 and moreover satisfies $\partial_\nu\varphi \leq -c < 0$ on the support of W (defined in (16)). We recall that we denoted by ∂Y the boundaries $\{y=0\}$ and $\{y=L\}$.

Theorem 4.1 *There exist $h_0 > 0$, $C > 0$ such that,*

$$h||e^{\varphi/h}V||^2_{L^2(Y)} + h^3||e^{\varphi/h}\nabla_z V||^2_{L^2(Y)} \leq C\Big(h^4||e^{\varphi/h}(\partial_s^2+\Delta)V||^2_{L^2(Y)} + h|e^{\varphi_0/h}V_{|\partial Y}|^2_{L^2(\partial Y)} + h^3|e^{\varphi/h}\nabla_{z'}V_{|\partial Y}|^2_{L^2(\partial Y)}\Big),$$

for all $0 < h \leq h_0$, for all V of the form (16).

Using the particular form of the function V and the divergence free condition, one may recover an estimate on the velocity U.

Theorem 4.2 *There exist $h_0 > 0$, $C > 0$ such that,*

$$\begin{aligned} h||e^{\varphi/h}V||^2_{L^2(Y)} &+ h^3||e^{\varphi/h}\nabla_z V||^2_{L^2(Y)} + ||e^{\varphi/h}U||^2_{L^2(Y)} + h^2||e^{\varphi/h}\nabla U||^2_{L^2(Y)} \\ &\leq C\Big(h^4||e^{\varphi/h}(\partial_s^2+\Delta)V||^2_{L^2(Y)} + h^3||e^{\varphi/h}[\Delta,\chi]u||^2_{L^2(Y)} \\ &+ h^3||e^{\varphi/h}[\nabla,\chi]v||^2_{L^2(Y)} + h|e^{\varphi_0/h}V_{|\partial Y}|^2_{L^2(\partial Y)} + h^3|e^{\varphi/h}\nabla_{z'}V_{|\partial Y}|^2_{L^2(\partial Y)}\Big), \end{aligned} \tag{34}$$

for all $0 < h \leq h_0$, for all U, V of the form (16).

Proof. Using commutation with cut-off functions, and the divergence free condition on u, we obtain,

$$\begin{aligned} h^3||e^{\varphi/h}\Delta U||^2_{L^2(Y)} &\lesssim h^3||e^{\varphi/h}\chi\Delta u||^2_{L^2(Y)} + h^3||e^{\varphi/h}[\Delta,\chi]u||^2_{L^2(Y)} \\ &\lesssim h^3||e^{\varphi/h}\chi\nabla_z v||^2_{L^2(Y)} + h^3||e^{\varphi/h}[\Delta,\chi]u||^2_{L^2(Y)} \\ &\lesssim h^3||e^{\varphi/h}\nabla_z V||^2_{L^2(Y)} + h^3||e^{\varphi/h}[\Delta,\chi]u||^2_{L^2(Y)} + h^3||e^{\varphi/h}[\nabla_z,\chi]v||^2_{L^2(Y)}. \end{aligned} \tag{35}$$

Now observe that classical the Carleman estimate holds for $\Delta_\varphi U$, near the boundary ∂Y, as U satisfies homogeneous Dirichlet boundary condition, and falls into the scope of Definition 3.2. We recall it here, adapted to our setting (see for instance [1, 13]).

Theorem 4.3 *Let $z_0 \in \partial Y$ and let $\varphi \in C^\infty(Y)$ such that the pair $(-\Delta, \varphi)$ satisfies Definition 3.2 in a neighborhood of z_0 in $\overline{Y}$. Then, there exists $h_0 > 0$ such that,*

$$||e^{\varphi/h}U||^2_{L^2(Y)} + h^2||e^{\varphi/h}\nabla U||^2_{L^2(Y)} \lesssim h^3||e^{\varphi/h}\Delta U||^2_{L^2(Y)}, \tag{36}$$

for all U defined in (16).

Using (36) to the right-hand side of (35) (up to shrink the cut-off function χ if necessary), we obtain the sought result. □

4.2 Estimates of the boundary terms

The main difficulty in using estimate (34) lies in the estimation of the trace terms in the right-hand side, since we do not have any straightforward information on the traces of v (or equivalently V). To overcome this difficulty, we shall perform several microlocalizations and use the splitting of the cotangent space of ∂Y : $\mathcal{E}^+, \mathcal{E}_1^-$ and $\mathcal{E}_2^-$. Each region shall have a different treatment.

- $\mathcal{E}_1^-$ is the low-frequency region. It is now well-known (see [14]) that one can estimate the two boundary traces of the solution by the source term, without any prescribed boundary conditions, since from Lemma 3.4, near $\{y = 0\}$, both roots have negative imaginary part and the domain is on the side $\{y > 0\}$, and near $\{y = L\}$ both roots have positive imaginary parts and the domain is on the side $\{y < L\}$. For such discussion, one also refers to [1, 3].
- $\mathcal{E}^+$ is a high-frequency region in the ξ variable. There, we shall perform an elliptic estimate on u (recall the relation rot $u = v$) using the ellipticity of Q_{φ_0} in that region (see Lemma 3.3) and the knowledge of a Dirichlet boundary condition on the vector field u.
- $\mathcal{E}_2^-$ is the remaining region. Using the arguments of [4], we shall prove an analytic estimate in the (s, σ) variable by a Paley-Wiener type theorem.

This section deals with this program. Before doing so, we first recall that, due to the homogeneous boundary condition $U_{|\partial Y} = 0$, we have $W_{|\partial Y} = \partial_y U_1$, where $U = {}^t(U_1, U_2)$ denote the tangential and perpendicular component of U respectively.

4.2.1 Estimates in $\mathcal{E}_1^-$

We start with the low frequency region. Let $\theta_1^-(\xi,\sigma) = \alpha_1(\xi)\beta_1(\sigma)$, with α, β bounded smooth functions of order zero such that supp $\theta_1^- \subset \mathcal{E}_1^-$. From classical elliptic boundary value problems methods (see for instance [3], Proposition 6.1-8), we have the following result.

Proposition 4.4 *There exists $h_0 > 0$ and $C > 0$ such that,*

$$h|\operatorname{Op}(\theta_1^-)W|^2_{L^2(\partial Y)} + h^3|\nabla_{z'}\operatorname{Op}(\theta_1^-)W|^2_{L^2(\partial Y)} \le C\Big(h^4||P_\varphi W||^2_{L^2(Y)} + h^4||\nabla_z W||^2_{L^2(Y)} + h^2||W||^2_{L^2(Y)}\Big), \tag{37}$$

for all $0 \le h \le h_0$ and for all W of the form (16).

4.2.2 Estimates in $\mathcal{E}^+$

We continue with the high frenquency region in the variable ξ. Let $\theta^+(\xi)$ be a bounded smooth function such that supp $\theta^+ \subset \mathcal{E}^+$. In the present section, we shall work globally with respect to the (x, y) variables, but locally with respect to the s variable, in order to avoid truncation coming from $\nabla\chi p$ to appear. To do this, we recall (16)

$$\mathfrak{U} = \chi_1(s)u, \quad \mathfrak{Q} = \chi_1 q,$$

and we shall consider the following system,

$$\begin{cases} -\partial_s^2\mathfrak{U} - \Delta\mathfrak{U} + \nabla\mathfrak{Q} = \mathfrak{F}, & (s,x,y) \in \mathbb{R}\times\Omega, \\ \operatorname{div}\, \mathfrak{U} = 0, & (s,x,y) \in \mathbb{R}\times\Omega, \end{cases} \tag{38}$$

and obtain estimates on $\mathfrak{U}$, since u satisfies (13). Introduce the following functions $\mathcal{U} := e^{\varphi_0/h}\mathfrak{U}$, $\mathcal{F} := h^2 e^{\varphi_0/h}\mathfrak{F}$, $\mathcal{Q} := h e^{\varphi_0/h}\mathfrak{Q}$. As a result, we shall work with the operator Q_{φ_0} (defined in (28)). Hence, $\mathcal{U}$ satisfies the following conjugated system,

$$\begin{cases} Q_{\varphi_0}\mathcal{U} + h\nabla\mathcal{Q} = \mathcal{F}, & (s,x,y) \in \mathbb{R}\times\Omega, \\ \operatorname{div}\, \mathcal{U} = 0, & (s,x,y) \in \mathbb{R}\times\Omega. \end{cases}$$

We emphasize that the weight function φ_0 only depends on the variable s. A key point of this section is that Q_{φ_0} is elliptic on $\mathcal{E}^+$ (see Lemma 3.3). Note also that we have homogeneous boundary condition.

Proposition 4.5 *There exists $h_0 > 0$ and $C > 0$ such that,*

$$|| \operatorname{Op}_T(\theta^+)\mathcal{U}||^2_{2,\mathrm{sc}} + h^3 |e^{\varphi_0/h} \operatorname{Op}_T(\theta^+)(\partial_y u_1)_{|\partial Y}|^2_{1,\mathrm{sc}} \le Ch^4 ||e^{\varphi_0/h}[\partial_s^2, \chi_1]u||^2_{1,\mathrm{sc}}, \tag{39}$$

for all $0 \le h \le h_0$ and for all $\mathcal{U}$ of the form (16).

Remark that the estimate of the boundary term $e^{\varphi_0/h} \operatorname{Op}_T(\theta^+)(\partial_y u_1)_{|\partial Y}$ is precisely the boundary terms appearing in the right-hand side of (34), microlocalized in the $\mathcal{E}^+$ region.

Proof. The proof is based on integration by parts, noting that the computations are valid because of the regularity of the function u (see Lemma 1.10). We recall that $Q_{\varphi_0} = -h^2\Delta_z - (\partial_s\varphi_0)^2 + 2h(\partial_s\varphi_0)\partial_s + h(\partial_s^2\varphi_0)$. Let us denote by $\mathcal{U}_1 = \operatorname{Op}_T(\theta^+)\mathcal{U}$, $\mathcal{Q}_1 = \operatorname{Op}_T(\theta^+)\mathcal{Q}$ and $\mathcal{F}_1 = \operatorname{Op}_T(\theta^+)\mathcal{F}$. As θ^+ is a Fourier multiplier in the x direction, it commutes with all the involved operators and we have the following system,

$$\begin{cases} Q_{\varphi_0}\mathcal{U}_1 + h\nabla\mathcal{Q}_1 = \mathcal{F}_1, & (s,x,y) \in \mathbb{R}\times\Omega, \\ \operatorname{div}\, \mathcal{U}_1 = 0, & (s,x,y) \in \mathbb{R}\times\Omega, \\ \mathcal{U}_{1|_{y=0}} = \mathcal{U}_{1|_{y=L}} = 0, & (s,x) \in \mathbb{R}\times\mathbb{T}, \end{cases} \tag{40}$$

Multiplying the first line of (40) by $\mathcal{U}_1$ and integration by parts yields,

$$h^2 \int_Z |\nabla_z \mathcal{U}_1|^2 - \int_Z (\partial_s\varphi_0)^2 |\mathcal{U}_1|^2 \lesssim ||\mathcal{F}_1||^2_{L^2(Z)}.$$

Note that on the support of θ^+, we have $h|D_x| > 2\sup|\partial_s\varphi_0|^2$, and consequently, there exists $C > 0$ such that,

$$||\mathcal{U}_1||^2_{1,\mathrm{sc}} \le C||\mathcal{F}_1||^2_{L^2(Z)}. \tag{41}$$

Denoting $\mathcal{G}_1 := \mathcal{F}_1 + (\partial_s\varphi_0)^2\mathcal{U}_1 - 2h(\partial_s\varphi_0)\partial_s\mathcal{U}_1 - h(\partial_s^2\varphi_0)\mathcal{U}_1$, (40) reads

$$\begin{cases} -\Delta_z U_1 + h\nabla\mathcal{Q}_1 = \mathcal{G}_1, & (s,x,y) \in \mathbb{R}\times\Omega, \\ \operatorname{div}\, \mathcal{U}_1 = 0, & (s,x,y) \in \mathbb{R}\times\Omega, \\ \mathcal{U}_{1|_{y=0}} = \mathcal{U}_{1|_{y=L}} = 0, & (s,x) \in \mathbb{R}\times\mathbb{T}, \end{cases} \tag{42}$$

and $\mathcal{G}_1$ satisfies, using (41),

$$||\mathcal{G}_1||^2_{L^2(Z)} \le C||\mathcal{F}_1||^2_{L^2(Z)}. \tag{43}$$

Differentiating the second line of (42), we have div $\partial_s^2\mathcal{U}_1 = 0$, and consequently, multiplying the first line by $\partial_s^2\mathcal{U}_1$ and integrating by parts, we have

$$h^2||\partial_s^2\mathcal{U}_1||^2_{L^2(Z)} \le C||\mathcal{F}_1||^2_{L^2(Z)}. \tag{44}$$

Setting $\mathcal{H}_1 := \mathcal{G}_1 + h^2\partial_s^2\mathcal{U}_1$, we have,

$$\begin{cases} -\Delta U_1 + h\nabla\mathcal{Q}_1 = \mathcal{H}_1, & (s,x,y) \in \mathbb{R}\times\Omega, \\ \operatorname{div}\, \mathcal{U}_1 = 0, & (s,x,y) \in \mathbb{R}\times\Omega, \\ \mathcal{U}_{1|_{y=0}} = \mathcal{U}_{1|_{y=L}} = 0, & (s,x) \in \mathbb{R}\times\mathbb{T}, \end{cases} \tag{45}$$

with the estimate

$$||\mathcal{H}_1||^2_{L^2(Z)} \le C||\mathcal{F}_1||^2_{L^2}.$$

Note that we reduce the problem to the derivation of an elliptic estimate of a semi-classical Stokes system. Hence, it is well-known by ellipticity, that there exists $C > 0$ such that

$$||\mathcal{U}_1||^2_{2,\text{sc}} \le C||\mathcal{F}_1||^2_{L^2(Z)}. \tag{46}$$

Hence, by the trace formula (18), we deduce that

$$h^3|(\partial_y\mathcal{U}_1)_{|\partial Y}|^2_{L^2((0,S_0)\times\mathbb{T})} \lesssim h^2||\partial_y\mathcal{U}_1||^2_{1,\text{sc}} \lesssim ||\mathcal{U}_1||^2_{2,\text{sc}} \lesssim ||\mathcal{F}_1||^2_{L^2(Z)}.$$

It remains to estimate $h^5|\nabla_{s,x}(\partial_y\mathcal{U}_1)_{|\partial Y}|^2_{L^2((0,S_0)\times\mathbb{T})}$. To do this, we differentiate (40) in the tangential (s,x) direction, and then

$$\begin{cases} Q_{\varphi_0}\mathcal{U}_2 + h\nabla\mathcal{Q}_2 = \mathcal{F}_2, & (s,x,y) \in \mathbb{R}\times\Omega, \\ \text{div } \mathcal{U}_2 = 0, & (s,x,y) \in \mathbb{R}\times\Omega, \\ \mathcal{U}_{2|_{y=0}} = \mathcal{U}_{2|_{y=L}} = 0, & (s,x) \in \mathbb{R}\times\mathbb{T}, \end{cases} \tag{47}$$

where $\mathcal{U}_2 := \nabla_{s,x}\mathcal{U}_1$, $\mathcal{F}_2 = \nabla_{s,x}\mathcal{F}_1 + [(\partial_s\varphi_0)^2, \nabla_{s,x}]\mathcal{U}_2 + 2h[\partial_s\varphi_0, \nabla_{s,x}]\partial_s\mathcal{U}_1 + h[\partial_s^2\varphi_0, \nabla_{s,x}]\mathcal{U}_2$ and $\mathcal{Q}_2 = \nabla_{s,x}\mathcal{Q}_1$. From (46), we can estimate,

$$||\mathcal{F}_2||^2_{L^2(Z)} \lesssim ||\mathcal{F}_1||^2_{L^2(Z)} + ||\nabla_{s,x}\mathcal{F}_1||^2_{L^2(Z)}.$$

Yet, arguing as above, one can obtain an elliptic estimate on $\mathcal{U}_2$ given by,

$$||\mathcal{U}_2||^2_{2,\text{sc}} \lesssim ||\mathcal{F}_2||^2_{L^2(Z)} \lesssim ||\mathcal{F}_1||^2_{L^2(Z)} + ||\nabla_{s,x}\mathcal{F}_1||^2_{L^2(Z)},$$

which yields, using again (18),

$$\begin{aligned} h^5|\nabla_{s,x}(\partial_y\mathcal{U}_1)_{|\partial Y}|^2_{L^2((0,S_0)\times\mathbb{T})} &= h^5|(\partial_y\mathcal{U}_2)_{|\partial Y}|^2_{L^2((0,S_0)\times\mathbb{T})} \le h^4||\partial_y\mathcal{U}_2||^2_{1,\text{sc}} \lesssim h^2||\mathcal{U}_2||^2_{2,\text{sc}} \lesssim h^2||\mathcal{F}_2||^2_{L^2(Z)} \\ &\lesssim h^2||\mathcal{F}_1||^2_{L^2(Z)} + h^2||\nabla_{s,x}\mathcal{F}_1||^2_{L^2(Z)}. \end{aligned}$$

Summing up, we have proved that there exists $C > 0$ such that,

$$||\mathcal{U}_1||^2_{2,\text{sc}} + h^2||\nabla_{s,x}\mathcal{U}_1||^2_{2,\text{sc}} + h^3|(\partial_y\mathcal{U}_1)_{|\partial Y}|^2_{1,\text{sc}} \le C\left(||\mathcal{F}_1||^2_{L^2(Z)} + h^2||\nabla_{s,x}\mathcal{F}_1||^2_{L^2(Z)}\right).$$

We end the proof by noting that $\mathcal{F}_1 := h^2A_1^*\mathcal{U}_1 + h^2[\partial_s^2, \chi_1]\mathcal{U}$, and the sought result follows by taking $h > 0$ sufficiently small. □

4.2.3 Estimates in $\mathcal{E}_2^-$

In this section, we shall use the particular form of the function U, mainly the analyticity in the s variable, to estimate the boundary traces in the remaining region $\mathcal{E}_2^-$. Analyticity allow us to gain exponential decay as $h \to 0$ here. Let $\theta_2^-(\sigma,\xi) := \alpha_2(\xi)\beta_2(\sigma) \in \tilde{\mathcal{S}}^0_{T,\text{sc}}$ such that $\operatorname{supp}\theta_2^- \subset \mathcal{E}_2^-$.

Proposition 4.6 *There exists h_0, $c_0 > 0$ and C such that,*

$$|\operatorname{Op}(\theta_2^-)W_{|\partial\Omega}|^2_{1,sc} \lesssim Ce^{C\Lambda^{1/2}+2\frac{1-c_0}{h}}||\tilde{u}||^2_{L^2(\Omega)}. \tag{48}$$

for all $h \in (0,h_0]$ and where $\tilde{u}$ and W are defined in (11) and (16).

Proof. The beginning of the proof mimmicks the one in [4]. First note that, one has, for all $n \in \mathbb{N}$,

$$\begin{aligned} \operatorname{Op}_T(\theta_2^-)W_{|\partial\Omega} &= \frac{1}{2\pi h}\int e^{\frac{is\sigma}{h}}\beta_2(\sigma)\left(\int e^{\frac{-it\sigma}{h}}\chi_1(t)e^{\varphi_0(t)/h}\operatorname{Op}_T(\alpha_2)v(t,x,0)dt\right)d\sigma \\ &= \frac{1}{2\pi h}\int e^{\frac{is\sigma}{h}}\beta_2(\sigma)\left(\int \frac{d^n}{(dt)^n}\frac{h^n e^{\frac{-it\sigma}{h}}}{(-i\sigma)^n}\chi_1(t)e^{\varphi_0(t)/h}\operatorname{Op}_T(\alpha_2)v(t,x,0)dt\right)d\sigma \\ &= -\frac{1}{2\pi h}\int e^{\frac{is\sigma}{h}}\beta_2(\sigma)\left(\int \frac{h^n e^{\frac{-it\sigma}{h}}}{(-i\sigma)^n}\frac{d^n}{(dt)^n}\left(\chi_1(t)e^{\varphi_0(t)/h}\operatorname{Op}_T(\alpha_2)v(t,x,0)\right)dt\right)d\sigma \\ &= -\frac{1}{2\pi h}\int e^{\frac{is\sigma}{h}}\beta_2(\sigma)\left(\int \frac{h^n e^{\frac{-it\sigma}{h}}}{(-i\sigma)^n}e^{\frac{\varphi_0(t,\sigma)}{h}}M_n(t,x)dt\right)d\sigma, \end{aligned}$$

where we wrote $M_n(t,x) = e^{-\frac{\varphi_0(t)}{h}} \frac{d^n}{(dt)^n}\left(\chi_1(t)e^{\varphi_0(t)/h}\operatorname{Op}_T(\alpha_2)v(t,x,0)\right)$. In what follows, we shall focus on,

$$I := \int e^{\frac{-it\sigma}{h}} e^{\varphi_0(t)/h} M_n(t,x)dt = \int e^{\frac{i\phi}{h}} M_n(t,x)dt,$$

with $\phi(t,\sigma) := -t\sigma - i\varphi_0(t)$. By definition of the weight function φ_0, we have,

$$\phi(t,\sigma) = -t\sigma - i\left(\varphi_0(s_0) - \varepsilon\gamma(t-s_0)^2 + O(\varepsilon^2)\right). \tag{49}$$

Note that the integrand of I is analytic on the set where $\chi_1 = 1$. We thus introduce the following change of contour in the complex plane,

$$t \mapsto g(t) := t - i\frac{\sigma}{<\sigma>}\kappa(t), \quad \kappa \in C_0^\infty(\mathbb{R}), \tag{50}$$

with $\operatorname{supp}\kappa \subset \{\chi_1 = 1\}$, $\kappa(t) \in [0,\kappa_0]$ and $\kappa(t) = \kappa_0$ on an interval of the type $(-\varepsilon_0 + s_0, \varepsilon_0 + s_0) \subset \{\chi_1 = 1\}$. We recall that $<\sigma> = (1+\sigma^2)^{1/2}$ stands for the Japanese bracket. We have,

$$\operatorname{Im}\phi(g(t),\sigma) = \frac{\sigma^2}{<\sigma>}\kappa(t) - \varphi_0(s_0) + \varepsilon\gamma(t-s_0)^2 - \varepsilon\gamma\frac{\sigma^2}{<\sigma>^2}\kappa(t)^2 + O(\varepsilon^2),$$

and we see that for $\varepsilon > 0$ and $\kappa_0 > 0$ chosen sufficiently small (and fixed from now on), there exists $c_0 > 0$ such that,

$$\operatorname{Im}\phi(g(t),\sigma) \geq c_0 - \varphi_0(s_0),$$

since $|\sigma|$ bounded from below by a strictly positive constant on the support of $\beta_2(\sigma)$. Hence, we shall perform the following change of contour in the integral of I using the analyticity,

$$\begin{aligned}|I| = \left|\int_{\mathbb{R}} e^{\frac{i\phi}{h}} M_n(t,x)dt\right| = \left|\int_{\gamma} e^{\frac{i\phi}{h}} M_n(t,x)dt\right| &= \left|\int_{\mathbb{R}} e^{\frac{i\phi(g(t),\sigma)}{h}} M_n(g(t),x)g'(t)dt\right| \\ &\lesssim e^{(\varphi_0(s_0)-c_0)/h}\int_{\mathbb{R}} |M_n(g(t),x)|\, dt.\end{aligned}$$

As a result, using the fact that $|\xi|$ is uniformly bounded from above in that microlocal region, using trace formulas, and taking $n = 4$ for instance, we have, by definition of the semi-classical norms,

$$\begin{aligned}|\operatorname{Op}_T(\theta_2^-)W_{|\partial\Omega}|^2_{1,sc} &\lesssim |\operatorname{Op}_T(\theta_2^-)W_{|\partial\Omega}|_0^2 + h^2|\operatorname{Op}_T(\theta_2^-)\partial_s W_{|\partial\Omega}|_0^2 \\ &\lesssim \int_{\partial\Omega} h^{2n-2}\left|\int_{\mathbb{R}} e^{i\frac{s\sigma}{h}}\sigma^{-n} I d\sigma\right|^2 dsdx + \int_{\partial\Omega} h^{2n-2}\left|\int_{\mathbb{R}} e^{i\frac{s\sigma}{h}}\sigma^{-n+2} I d\sigma\right|^2 dsdx \\ &\lesssim h^{2n}\int_{\partial\Omega}\left|\sup_{\sigma}|I|\right|^2 dxds \\ &\lesssim e^{2\frac{\varphi_0(s_0)-c_0}{h}}\int_{\partial\Omega}\left(\int_{\mathbb{R}} M_4(t,x)dt\right)^2 dxds \\ &\lesssim e^{2\frac{\varphi_0(s_0)-c_0}{h}}|M_4|_0^2 \lesssim e^{2\frac{\varphi_0(s_0)-c_0}{h}}||M_4||^2_{H^1(Z)},\end{aligned}$$

where we used in the last line a trace formula. Note that there exist $m \in \mathbb{R}$ and $C > 0$ such that,

$$|M_4| \leq Ch^m \sum_{k=0}^{4} |\partial_s^k v(g(s),x,y)|, \quad |\nabla_z M_4| \leq Ch^m \sum_{k=0}^{4} |\partial_s^k \nabla_z v(g(s),x,y)|,$$

Hence,

$$|\operatorname{Op}(\theta_2^-)W_{|\partial\Omega}|^2_{1,sc} \lesssim Ch^m e^{2\frac{1-c_0}{h}} \sum_{k=0}^{4} ||\partial_s^k v(g(.),.,.)||^2_{H^1(Z)}.$$

Using the particular form of v given by (14), one obtains,

$$|\operatorname{Op}(\theta_2^-)W_{|\partial\Omega}|^2_{1,sc} \lesssim Ch^m e^{2\frac{1-c_0}{h}} \sum_{k=0}^{4} ||\partial_s^k u(g(.),.,.)||^2_{H^2(Z)}.$$

Note that by elliptic estimates of the Stokes operator $A_0 = A_0^*$, one has for all $k \in \{0, \dots, 4\}$,

$$\begin{aligned}
&||\partial_s^k u(g(.),.,.)||_{H^2(Z)} \\
&\lesssim ||\partial_s^k A_0^* u(g(.),.,.)||_{L^2(Z)} \\
&\lesssim ||\partial_s^k A^* u(g(.),.,.)||_{L^2(Z)} + ||\partial_s^k A_1^* u(g(.),.,.)||_{L^2(Z)} \\
&\lesssim ||\partial_s^k A^* u(g(.),.,.)||_{L^2(Z)} + ||\partial_s^k u(g(.),.,.)||_{H^1(Z)} \\
&\lesssim ||\partial_s^k A^* u(g(.),.,.)||_{L^2(Z)} + \left(\partial_s^k A_0^* u(g(.),.,.), \partial_s^k u(g(.),.,.)\right) \\
&\lesssim ||\partial_s^k A^* u(g(.),.,.)||_{L^2(Z)} + \left(\partial_s^k A^* u(g(.),.,.), \partial_s^k u(g(.),.,.)\right) - \left(\partial_s^k A_1^* u(g(.),.,.), \partial_s^k u(g(.),.,.)\right) \\
&\lesssim ||\partial_s^k A^* u(g(.),.,.)||_{L^2(Z)} + ||\partial_s^k u(g(.),.,.)||_{L^2(Z)} + ||\partial_s^k A_1^* u(g(.),.,.)||_{L^2(Z)} ||\partial_s^k u(g(.),.,.)||_{L^2(Z)},
\end{aligned}$$

which yields, using the Young inequality,

$$||\partial_s^k u(g(.),.,.)||_{H^2(Z)} \lesssim ||\partial_s^k A^* u(g(.),.,.)||_{L^2(Z)} + ||\partial_s^k u(g(.),.,.)||_{L^2(Z)}.$$

Now we use the particular form of u given by (11), and the resolvent estimate of Corollary 1.7 to obtain for $j = 0, 1$,

$$\begin{aligned}
||\partial_s^k (A^*)^j u(g(.),.,.)||_{L^2} &= \frac{1}{2} ||(A^*)^{j-\frac{1}{2}} \partial_s^k \left(e^{g(s)(A^*)^{1/2}} - e^{-g(s)((A^*)^{1/2}}\right) \tilde{u}||_{L^2(Z)} \\
&\lesssim \Lambda^{\tilde{m}} C e^{C\Lambda^{1/2}} ||\tilde{u}||_{L^2} \lesssim \tilde{C} e^{\tilde{C}\Lambda^{1/2}} ||\tilde{u}||_{L^2},
\end{aligned}$$

for some $\tilde{C} > 0$ independent on Λ. Summing up, up to taking c_0 slightly smaller and h_0 sufficiently small, one has,

$$|\operatorname{Op}(\theta_2^-) W_{|\partial\Omega}|_{1,sc} \lesssim \tilde{C} e^{\tilde{C}\Lambda^{1/2} + \frac{1-c_0}{h}} ||\tilde{u}||_{L^2},$$

which is the sought result. □

4.2.4 Patching boundary estimates together

In this section, we patch all the above microlocal estimates and we absorb the remainder by taking the Carleman parameter $h > 0$ small enough. First, we start by patching the estimates on the boundary trace $W_{|\partial\Omega}$.

We set $Y_\ell = \{z \in Y,\ d(z, \partial Y) > \frac{1}{\ell}\}$. One shall use Y_ℓ for ℓ taken sufficiently large, in order to obtain an observation inequality from the interior up to the boundary. The large parameter ℓ depends on the small parameter $\varepsilon > 0$, and shall be fixed below. We recall that χ_1 is defined in Section 1.5.1.

Proposition 4.7 *There exists $c_0 > 0$, $h_0 > 0$, $\ell > 0$, and $C > 0$ such that,*

$$\begin{aligned}
&||e^{\varphi/h}\mathfrak{U}||^2_{L^2(Y)} + h^2||e^{\varphi/h}\nabla\mathfrak{U}||^2_{L^2(Y)} + h||e^{\varphi/h}\mathfrak{V}||^2_{L^2(Y)} + h^3||e^{\varphi/h}\nabla_z\mathfrak{V}||^2_{L^2(Y)} \\
&\quad \le C\Big(h^2||e^{\varphi_0/h}[\partial_s^2, \chi_1]u||^2_{1,\mathrm{sc}} + h^4||e^{\varphi/h}[\partial_s^2, \chi_1]v||^2_{L^2(Y)} \\
&\qquad + Ce^{C\Lambda^{1/2}} e^{2\frac{1-c_0}{h}} ||\tilde{u}||^2_{L^2(Z)} + h||e^{\varphi/h}\mathfrak{V}||^2_{L^2(Y_\ell)} + h^3||e^{\varphi/h}\nabla\mathfrak{V}||^2_{L^2(Y_\ell)} \\
&\qquad + ||e^{\varphi/h}\mathfrak{U}||^2_{L^2(Y_\ell)} + h^2||e^{\varphi/h}\nabla\mathfrak{U}||^2_{L^2(Y_\ell)}\Big), \qquad (51)
\end{aligned}$$

for all $h \in (0, h_0]$.

Note that the two last terms correspond to an observation from the interior of Z.

Proof. Let $(\Phi_i)_{i\in I}$ be a finite partition of unity of a neighborhood $\mathcal{N}$ of the boundary ∂Y, satisfying $\sum_{i\in I} \Phi_i = 1$ on $\mathcal{N}$ such that Theorem 4.2 holds for every Φ_i. Note that this partition does exist, by compactness of $\mathcal{N}$. Applying Theorem 4.2 for such $\chi = (\chi_1\Phi_i)$ yields,

$$\begin{aligned}
&h||e^{\varphi/h}(\chi_1\Phi_i)v||^2_{L^2(Y)} + h^3||e^{\varphi/h}\nabla_z(\chi_1\Phi_i)v||^2_{L^2(Y)} + ||e^{\varphi/h}(\chi_1\Phi_i)u||^2_{L^2(Y)} + h^2||e^{\varphi/h}\nabla(\chi_1\Phi_i)u||^2_{L^2(Y)} \\
&\quad \le C\Big(h^4||e^{\varphi/h}(\partial_s^2 + \Delta)(\chi_1\Phi_i)v||^2_{L^2(Y)} + h^3||e^{\varphi/h}[\Delta, (\chi_1\Phi_i)]u||^2_{L^2(Y)} \\
&+ h^3||e^{\varphi/h}[\nabla, (\chi_1\Phi_i)]v||^2_{L^2(Y)} + h|e^{\varphi_0/h}((\chi_1\Phi_i)v)_{|\partial Y}|^2_{L^2(\partial Y)} + h^3|e^{\varphi/h}\nabla_{z'}((\chi_1\Phi_i)v)_{|\partial Y}|^2_{L^2(\partial Y)}\Big), \quad (52)
\end{aligned}$$

for all $0 < h \le h_0$, for all $u \in \mathcal{X}_\Lambda$, and $v = \text{rot } u$. Now we combine propositions 4.4, 4.5 and 4.6 to estimate the boundary terms by,

$$\begin{aligned} h|e^{\varphi_0/h}((\chi_1\Phi_i)v)_{|\partial Y}|^2_{L^2(\partial Y)} &+ h^3|e^{\varphi/h}\nabla_{z'}((\chi_1\Phi_i)v)_{|\partial Y}|^2_{L^2(\partial Y)} \\ \lesssim & Ce^{C\Lambda^{1/2}+2\frac{1-c_0}{h}}||\tilde{u}||^2_{L^2(\Omega)} + h^2||e^{\varphi_0/h}[\partial_s^2,\chi_1]u||^2_{1,\text{sc}} \\ &+ h^4||e^{\varphi/h}P(\chi_1\Phi_i)v||^2_{L^2(Y)} + h^4||e^{\varphi/h}\nabla_z(\chi_1\Phi_i)v||^2_{L^2(Y)} + h^2||e^{\varphi/h}(\chi_1\Phi_i)v||^2_{L^2(Y)}. \end{aligned} \quad (53)$$

Note that the power h^2 in front of $||e^{\varphi_0/h}[\partial_s^2,\Phi]u||^2_{1,\text{sc}}$ is natural since the relation between u and v given by $v = \text{rot } v$ is not semi-classical. Combining (52) and (53) implies,

$$\begin{aligned} h||e^{\varphi/h}(\chi_1\Phi_i)v||^2_{L^2(Y)} &+ h^3||e^{\varphi/h}\nabla_z(\chi_1\Phi_i)v||^2_{L^2(Y)} + ||e^{\varphi/h}(\chi_1\Phi_i)u||^2_{L^2(Y)} + h^2||e^{\varphi/h}\nabla(\chi_1\Phi_i)u||^2_{L^2(Y)} \\ \le & C\Big(h^4||e^{\varphi/h}(\partial_s^2+\Delta)(\chi_1\Phi_i)v||^2_{L^2(Y)} + h^3||e^{\varphi/h}[\Delta,(\chi_1\Phi_i)]u||^2_{L^2(Y)} \\ &+ h^3||e^{\varphi/h}[\nabla,(\chi_1\Phi_i)]v||^2_{L^2(Y)} + Ce^{C\Lambda^{1/2}+2\frac{1-c_0}{h}}||\tilde{u}||^2_{L^2(\Omega)} + h^2||e^{\varphi_0/h}[\partial_s^2,\chi_1]u||^2_{1,\text{sc}} \\ &+ h^4||e^{\varphi/h}\nabla_z(\chi_1\Phi_i)v||^2_{L^2(Y)} + h^2||e^{\varphi/h}(\chi_1\Phi_i)v||^2_{L^2(Y)}\Big). \end{aligned} \quad (54)$$

We have,

$$\begin{aligned} ||e^{\varphi/h}&(\partial_s^2+\Delta)(\chi_1\Phi_i)v||^2_{L^2(Y)} \\ &\lesssim ||e^{\varphi/h}(\chi_1\Phi_i)(\partial_s^2+\Delta)v||^2_{L^2(Y)} + ||e^{\varphi/h}[\partial_s^2+\Delta,\chi_1\Phi_i]v||^2_{L^2(Y)} \\ &\lesssim ||e^{\varphi/h}(\chi_1\Phi_i)\,\text{rot }\ A_1^*u||^2_{L^2(Y)} + ||e^{\varphi/h}[\partial_s^2+\Delta,\chi_1\Phi_i]v||^2_{L^2(Y)} \\ &\lesssim ||e^{\varphi/h}(\chi_1\Phi_i)A_1^*v||^2_{L^2(Y)} + ||e^{\varphi/h}(\chi_1\Phi_i)[\text{rot }, A_1^*]u||^2_{L^2(Y)} + ||e^{\varphi/h}[\partial_s^2+\Delta,\chi_1\Phi_i]v||^2_{L^2(Y)}. \end{aligned} \quad (55)$$

Combining (54) and (55), as A_1^* and $[\text{rot }, A_1^*]$ are differential operators of order 1, one obtains, by taking $0 < h < h_0$ sufficiently small,

$$\begin{aligned} h||e^{\varphi/h}(\chi_1\Phi_i)v||^2_{L^2(Y)} &+ h^3||e^{\varphi/h}\nabla_z(\chi_1\Phi_i)v||^2_{L^2(Y)} + ||e^{\varphi/h}(\chi_1\Phi_i)u||^2_{L^2(Y)} + h^2||e^{\varphi/h}\nabla(\chi_1\Phi_i)u||^2_{L^2(Y)} \\ \le & C\Big(h^4||e^{\varphi/h}[\partial_s^2+\Delta,(\chi_1\Phi_i)]v||^2_{L^2(Y)} + h^3||e^{\varphi/h}[\Delta,(\chi_1\Phi_i)]u||^2_{L^2(Y)} \\ &+ h^3||e^{\varphi/h}[\nabla,(\chi_1\Phi_i)]v||^2_{L^2(Y)} + Ce^{C\Lambda^{1/2}+2\frac{1-c_0}{h}}||\tilde{u}||^2_{L^2(\Omega)} + h^2||e^{\varphi_0/h}[\partial_s^2,\chi_1]u||^2_{1,\text{sc}} \\ &+ h^4||e^{\varphi/h}\nabla_z(\chi_1\Phi_i)v||^2_{L^2(Y)} + h^2||e^{\varphi/h}(\chi_1\Phi_i)v||^2_{L^2(Y)}\Big). \end{aligned}$$

Hence, by summing over $i \in I$, we obtain,

$$\begin{aligned} h||e^{\varphi/h}\chi_1 v||^2_{L^2(\mathcal{N})} &+ h^3||e^{\varphi/h}\nabla_z\chi_1 v||^2_{L^2(\mathcal{N})} + ||e^{\varphi/h}\chi_1 u||^2_{L^2(\mathcal{N})} + h^2||e^{\varphi/h}\nabla\chi_1 u||^2_{L^2(\mathcal{N})} \\ \le & C\sum_{i\in I}\Big(h^4||e^{\varphi/h}[\partial_s^2+\Delta,(\chi_1\Phi_i)]v||^2_{L^2(Y)} + h^3||e^{\varphi/h}[\Delta,(\chi_1\Phi_i)]u||^2_{L^2(Y)} \\ &+ h^3||e^{\varphi/h}[\nabla,(\chi_1\Phi_i)]v||^2_{L^2(Y)} + Ce^{C\Lambda^{1/2}+2\frac{1-c_0}{h}}||\tilde{u}||^2_{L^2(\Omega)} + h^2||e^{\varphi_0/h}[\partial_s^2,\chi_1]u||^2_{1,\text{sc}} \\ &+ h^4||e^{\varphi/h}\nabla_z(\chi_1\Phi_i)v||^2_{L^2(Y)} + h^2||e^{\varphi/h}(\chi_1\Phi_i)v||^2_{L^2(Y)}\Big). \end{aligned}$$

Remark that all commutators of of differential operator with Φ_i provides lower order terms, and can be absorbed by the left-hand side when evaluated in the neighborhood $\mathcal{N}$ of the boundary ∂Y, by taking $0 < h < h_0$ sufficiently small. Thus,

$$\begin{aligned} h||e^{\varphi/h}\chi_1 v||^2_{L^2(\mathcal{N})} &+ h^3||e^{\varphi/h}\nabla_z\chi_1 v||^2_{L^2(\mathcal{N})} + ||e^{\varphi/h}\chi_1 u||^2_{L^2(\mathcal{N})} + h^2||e^{\varphi/h}\nabla\chi_1 u||^2_{L^2(\mathcal{N})} \\ \le & C\sum_{i\in I}\Big(h^4||e^{\varphi/h}[\partial_s^2,\chi_1]v||^2_{L^2(Y)} + h^4||e^{\varphi/h}[\Delta,\Phi_i]\chi_1 v||^2_{L^2(Y\setminus\mathcal{N})} + h^3||e^{\varphi/h}[\Delta,\Phi_i]\chi_1 u||^2_{L^2(Y\setminus\mathcal{N})} \\ &+ h^3||e^{\varphi/h}[\nabla,(\chi_1\Phi_i)]v||^2_{L^2(Y\setminus\mathcal{N})} + Ce^{C\Lambda^{1/2}+2\frac{1-c_0}{h}}||\tilde{u}||^2_{L^2(\Omega)} + h^2||e^{\varphi_0/h}[\partial_s^2,\chi_1]u||^2_{1,\text{sc}} \\ &+ h^4||e^{\varphi/h}\nabla_z(\chi_1\Phi_i)v||^2_{L^2(Y\setminus\mathcal{N})} + h^2||e^{\varphi/h}(\chi_1\Phi_i)v||^2_{L^2(Y\setminus\mathcal{N})}\Big). \end{aligned}$$

Now we observe that by definition $\mathfrak{U} = \chi_1$ and $\mathfrak{V} = \chi_1 v$, and moreover there exists $\ell > 0$ sufficiently large satisfying $\mathrm{supp}(\chi_1) \cap (Y \setminus \mathcal{N}) \subset Y_\ell$. As a result we can write,

$$\begin{aligned} h||e^{\varphi/h}\mathfrak{V}||^2_{L^2(\mathcal{N})} &+ h^3||e^{\varphi/h}\nabla_z\mathfrak{V}||^2_{L^2(\mathcal{N})} + ||e^{\varphi/h}\mathfrak{U}||^2_{L^2(\mathcal{N})} + h^2||e^{\varphi/h}\nabla\mathfrak{U}||^2_{L^2(\mathcal{N})} \\ \leq C\Big(& h^4||e^{\varphi/h}[\partial_s^2,\chi_1]v||^2_{L^2(Y)} + h^4||e^{\varphi/h}[\Delta,\Phi_i]\chi_1 v||^2_{L^2(Y_\ell)} + h^3||e^{\varphi/h}[\Delta,\Phi_i]\chi_1 u||^2_{L^2(Y_\ell)} \\ &+ h^3||e^{\varphi/h}[\nabla,(\chi_1\Phi_i)]v||^2_{L^2(Y_\ell)} + Ce^{C\Lambda^{1/2}+2\frac{1-c_0}{h}}||\tilde{u}||^2_{L^2(\Omega)} + h^2||e^{\varphi_0/h}[\partial_s^2,\chi_1]u||^2_{1,\mathrm{sc}} \\ &+ h^4||e^{\varphi/h}\nabla_z(\chi_1\Phi_i)v||^2_{L^2(Y_\ell)} + h^2||e^{\varphi/h}(\chi_1\Phi_i)v||^2_{L^2(Y_\ell)}\Big). \end{aligned}$$

All the commutators in the right-hand side, are at maximum H^1 norms, and as the estimate outside $\mathcal{N}$ is immediate, we finally obtain,

$$\begin{aligned} ||e^{\varphi/h}\mathfrak{U}||^2_{L^2(Y)} + h^2||e^{\varphi/h}\nabla\mathfrak{U}||^2_{L^2(Y)} + h||e^{\varphi/h}\mathfrak{V}||^2_{L^2(Y)} + h^3||e^{\varphi/h}\nabla_z\mathfrak{V}||^2_{L^2(Y)} \\ \leq C\Big(h^2||e^{\varphi_0/h}[\partial_s^2,\chi_1]u||^2_{1,\mathrm{sc}} + h^4||e^{\varphi/h}[\partial_s^2,\chi_1]v||^2_{L^2(Y)} \\ + Ce^{C\Lambda^{1/2}}e^{2\frac{1-c_0}{h}}||\tilde{u}||^2_{L^2(Z)} + h||e^{\varphi/h}\mathfrak{V}||^2_{L^2(Y_\ell)} + h^3||e^{\varphi/h}\nabla\mathfrak{V}||^2_{L^2(Y_\ell)} \\ + ||e^{\varphi/h}\mathfrak{U}||^2_{L^2(Y_\ell)} + h^2||e^{\varphi/h}\nabla\mathfrak{U}||^2_{L^2(Y_\ell)}\Big), \end{aligned}$$

which is the sought result. □

5 Proof of the spectral inequality

The strategy to prove the spectral inequality is divided into two parts. The first one rely on the derivation of an observation of a neighborhood of the boundary from an arbitrary large domain in the interior. The second one is more classical, and rely on the propagation of smallness in the interior, where Carleman estimates are known. However, due to the lack of unique continuation in the s variable, one has to be cautious. The two subsections below are respectively devoted to each part.

5.1 An observation from an arbitrary large interior domain up to the boundary

Lemma 5.1 *For all $\ell > 0$ there exists $C > 0$ such that*

$$||\tilde{u}||_{L^2(\Omega)} \leq Ce^{C\sqrt{\Lambda}}||u||_{L^2(Y_\ell)},$$

for all $u \in \mathcal{X}_\Lambda$ of the form (11).

Proof. We recall that the cut-off functions are defined by $\chi(s,x,y) = \chi_1(s)\chi_2(x)\chi_3(y)$ and where $\chi_1(s),\chi_2(x),\chi_3(y)$ are smooth cut-offs in each variable satisfying near a point $(s_0,x_0,y=0) \in (0,S_0) \times \mathbb{T} \times \{y=0\}$:

$$\chi_1(s) = \begin{cases} 1 & \text{if } |s-s_0| < S_1, \\ 0 & \text{if } |s-s_0| > 2S_1, \end{cases} \qquad \chi_3(y) = \begin{cases} 1 & \text{if } y < Y_1, \\ 0 & \text{if } y > 2Y_1, \end{cases} \tag{56}$$

for some $0 < 2S_1 < \widetilde{S}_0$ (that is, $\mathrm{supp}\,\chi \subset Y$) and some $Y_1 < L/4$. The localization $\chi(s,x,y)$ is defined likewise near a point $(s_0,x_0,y=L) \in (0,S_0) \times (0,L_1) \times \{y=L\}$.

We then define the following regions of the open set Z by,

$$\begin{aligned} O_1 &:= \mathrm{supp}\,\chi_1', \\ O_2 &:= \left\{(s,x,y) \in O_1 \mid d(y,\partial\Omega) < \frac{1}{2}\varepsilon(S_1-s_0)^2\right\}, \\ O_3 &:= \mathrm{supp}\,\chi_1 \cap \left\{(s,x,y) \in O_1 \mid d(y,\partial\Omega) \geq \frac{1}{2}\varepsilon(S_1-s_0)^2\right\}. \end{aligned}$$

and

$$W_0 := \left\{(s,x,y) \in Y \mid \psi_\varepsilon(s,y) \geq -(S_1-s_0)^2\varepsilon\delta\right\},$$

with $\delta \in (0,1)$ that shall be fixed in the proof of the following lemma. Note that $\mathfrak{U} = u$ on W_0.

Lemma 5.2 *We have,*

$$\sup_{O_2} \varphi - \inf_{W_0} \varphi < 0 \qquad \textit{and } 1 - c_0 - \inf_{W_0} \varphi < 0.$$

for δ taken sufficiently small.

Proof. We first estimate,

$$\sup_{\mathcal{O}_2} \psi_\varepsilon = \sup_{\mathcal{O}_2}(-\varepsilon(s-s_0)^2 + d(y,\partial\Omega)) = -\varepsilon(S_1 - s_0)^2 + \sup_{\mathcal{O}_2} d(y,\partial\Omega) < -\frac{1}{2}\varepsilon(S_1-s_0)^2,$$

and

$$\inf_{W_0} \psi_\varepsilon \geq -(S_1-s_0)^2\varepsilon\delta,$$

which proves the first inequality of Lemma 5.2. We also have

$$1 - c_0 - \inf_{W_0}\varphi < 1 - c_0 - e^{-\lambda(S_1-s_0)^2\varepsilon\delta} < 0,$$

for δ taken sufficiently small. □

We analyze the first two terms of the right-hand side of (51). We have on the one hand,

$$h^2||e^{\varphi_0/h}[\partial_s^2,\chi_1]u||^2_{1,\mathrm{sc}} \lesssim h^2 e^{\frac{\sup_{O_1}\varphi_0}{h}}\left(||u||^2_{H^1(Z)} + ||\partial_s u||^2_{H^1(Z)}\right). \tag{57}$$

On the other hand,

$$\begin{aligned} h^4||e^{\varphi/h}[\partial_s^2,\chi_1]v||^2_{L^2(Y)} &= h^4||e^{\varphi/h}[-\partial_s^2,\chi_1]v||^2_{L^2(O_2)} + h^4||e^{\varphi/h}[-\partial_s^2,\chi_1]v||^2_{L^2(O_3)} \\ &\lesssim h^4 e^{\frac{\sup_{O_2}\varphi}{h}}\left(||v||^2_{L^2(Z)} + ||\partial_s v||^2_{L^2(Z)}\right) + h^4||e^{\varphi/h}[-\partial_s^2,\chi_1]v||^2_{L^2(O_3)}. \end{aligned} \tag{58}$$

Note that terms located in O_3 are interior terms, which corresponds to observation terms. Plugging these two estimates in (51) yields,

$$\begin{aligned} ||e^{\varphi/h}\mathfrak{U}||^2_{L^2(Y)} &+ h^2||e^{\varphi/h}\nabla_z\mathfrak{U}||^2_{L^2(Y)} + h^3||e^{\varphi/h}\mathfrak{V}||^2_{L^2(Y)} + h||e^{\varphi/h}\nabla_z\mathfrak{V}||^2_{L^2(Y)} \\ \leq C\Big(&h^2 e^{\frac{\sup_{O_1}\varphi_0}{h}}\left(||u||^2_{H^1(Z)} + ||\partial_s u||^2_{H^1(Z)}\right) + h^4 e^{\frac{\sup_{O_2}\varphi}{h}}\left(||v||^2_{L^2(Z)} + ||\partial_s v||^2_{L^2(Z)}\right) \\ &+ h^4||e^{\varphi/h}[-\partial_s^2,\chi_1]v||^2_{L^2(O_3)} + Ce^{C\Lambda^{1/2}}e^{2\frac{1-c_0}{h}}||\tilde{u}||^2_{L^2(Z)} \\ &+ h||e^{\varphi/h}\mathfrak{V}||_{L^2(Y_\ell)} + h^3||e^{\varphi/h}\nabla_z\mathfrak{V}||_{L^2(Y_\ell)} \\ &+ ||e^{\varphi/h}\mathfrak{U}||_{L^2(Y_\ell)} + h^2||e^{\varphi/h}\nabla_z\mathfrak{U}||_{L^2(Y_\ell)}\Big). \end{aligned} \tag{59}$$

Remark that by definition,

$$\sup_{O_1}\varphi_0 \leq \sup_{O_2}\varphi. \tag{60}$$

Now, we estimate, using the particular form of u given by (11),

$$\begin{aligned} ||u||^2_{H^1(Z)} + ||\partial_s u||^2_{H^1(Z)} &\lesssim (A_0^*u,u) + (A_0^*\partial_s u,\partial_s u) \\ &\lesssim (A^*u,u) + (A^*\partial_s u,\partial_s u) - (A_1^*u,u) - (A_1^*\partial_s u,\partial_s u) \\ &\lesssim (A^*u,u) + (A^*\partial_s u,\partial_s u) + ||u||_{H^1(Z)}||u||_{L^2(Z)} + ||\partial_s u||_{H^1(Z)}||\partial_s u||_{L^2(Z)}, \end{aligned}$$

which finally yields, using the Young inequality,

$$||u||^2_{H^1(Z)} + ||\partial_s u||^2_{H^1(Z)} \lesssim (A^*u,u) + (A^*\partial_s u,\partial_s u) + ||u||_{L^2} + ||\partial_s u||_{L^2}. \tag{61}$$

From the particular form of u given by (11) and from the resolvent estimate of Corollary 1.7, one has,

$$||u||^2_{H^1(Z)} + ||\partial_s u||^2_{H^1(Z)} \lesssim Ce^{C\sqrt{\Lambda}}||\tilde{u}||_{L^2(\Omega)}. \tag{62}$$

Note that we also have,

$$\begin{aligned} ||v||^2_{L^2(Z)} + ||\partial_s v||^2_{L^2(Z)} &\leq ||u||^2_{H^1(Z)} + ||\partial_s u||^2_{H^1(Z)} \\ &\lesssim (A^*u,u) + (A^*\partial_s u,\partial_s u) + ||u||_{L^2} + ||\partial_s u||_{L^2} \\ &\lesssim Ce^{C\sqrt{\Lambda}}||\tilde{u}||_{L^2(\Omega)}. \end{aligned} \tag{63}$$

From (59), (60), (62) and (63), one deduces,

$$\begin{aligned}||e^{\varphi/h}\mathfrak{U}||^2_{L^2(Y)} \lesssim &\left(e^{\frac{\sup_{O_2}\varphi}{h}}Ce^{C\sqrt{\Lambda}} + Ce^{C\Lambda^{1/2}}e^{2\frac{1-c_0}{h}}\right)||\tilde{u}||_{L^2(\Omega)}\\ &+ h||e^{\varphi/h}\mathfrak{V}||_{L^2(Y_\ell)} + h^3||e^{\varphi/h}\nabla_z\mathfrak{V}||_{L^2(Y_\ell)}\\ &+ ||e^{\varphi/h}\mathfrak{U}||_{L^2(Y_\ell)} + h^2||e^{\varphi/h}\nabla_z\mathfrak{U}||_{L^2(Y_\ell)} + ||e^{\varphi/h}[-\partial_s^2,\chi_1]v||^2_{L^2(O_3)}. \end{aligned}\tag{64}$$

Let us obtain a lower bound for the left hand side of (64). We have,

$$\begin{aligned}||u||^2_{L^2(W_0)} \lesssim & e^{-\frac{\inf_{W_0}\varphi}{h}}\left(e^{\frac{\sup_{O_2}\varphi}{h}}Ce^{C\sqrt{\Lambda}} + Ce^{C\Lambda^{1/2}}e^{2\frac{1-c_0}{h}}\right)||\tilde{u}||_{L^2(\Omega)}\\ &+ e^{-\frac{\inf_{W_0}\varphi}{h}}\Big(h||e^{\varphi/h}\mathfrak{V}||_{L^2(Y_\ell)} + h^3||e^{\varphi/h}\nabla_z\mathfrak{V}||_{L^2(Y_\ell)} + ||e^{\varphi/h}\mathfrak{U}||_{L^2(Y_\ell)}\\ &\qquad + h^2||e^{\varphi/h}\nabla_z\mathfrak{U}||_{L^2(Y_\ell)} + ||e^{\varphi/h}[-\partial_s^2,\chi_1]v||^2_{L^2(O_3)}\Big). \end{aligned}\tag{65}$$

Using again the resolvent estimate of Corollary 1.7, one obtains,

$$\begin{aligned}||\tilde{u}||^2_{L^2(\Omega)} \lesssim Ce^{C\sqrt{\Lambda}}\Big(& e^{-\frac{\inf_{W_0}\varphi}{h}}(e^{\frac{\sup_{O_2}\varphi}{h}}Ce^{C\sqrt{\Lambda}} + Ce^{C\Lambda^{1/2}+2\frac{(1-c_0)}{h}})||\tilde{u}||_{L^2(\Omega)}\\ &+ e^{-\frac{\inf_{W_0}\varphi}{h}}\Big(h||e^{\varphi/h}\mathfrak{V}||_{L^2(Y_\ell)} + h^3||e^{\varphi/h}\nabla_z\mathfrak{V}||_{L^2(Y_\ell)}\\ &\qquad + ||e^{\varphi/h}\mathfrak{U}||_{L^2(Y_\ell)} + h^2||e^{\varphi/h}\nabla_z\mathfrak{U}||_{L^2(Y_\ell)} + ||e^{\varphi/h}[-\partial_s^2,\chi_1]v||^2_{L^2(O_3)}\Big)\Big). \end{aligned}$$

From Lemma 5.2, setting $\frac{1}{h} = k\sqrt{\Lambda}$, for $k \in \mathbb{R}_+$ sufficiently large, yields that there exists $C > 0$ such that,

$$\begin{aligned}||\tilde{u}||^2_{L^2(\Omega)} \leq Ce^{C\sqrt{\Lambda}}\Big(&||\mathfrak{V}||^2_{L^2(Y_\ell)} + ||\nabla_z\mathfrak{V}||^2_{L^2(Y_\ell)} + ||\mathfrak{U}||^2_{L^2(Y_\ell)}\\ &+ ||\nabla_z\mathfrak{U}||^2_{L^2(Y_\ell)} + ||[-\partial_s^2,\chi_1]v||^2_{L^2(O_3)}\Big). \end{aligned}\tag{66}$$

Remark that only observation terms remains in the right hand side of (66). Let $Y_\ell \subset \tilde{Y}_\ell \Subset Z$ and let $\Theta \in C_0^\infty(\tilde{Y}_\ell)$ such that $\Theta = 1$ on Y_ℓ. Note that,

$$||[-\partial_s^2,\chi_1]v||^2_{L^2(O_3)} \lesssim ||v||^2_{H^1(Y_\ell)},\tag{67}$$

for a sufficiently large parameter ℓ. Let us finish the proof by writing,

$$\begin{aligned}||\nabla_z\mathfrak{V}||^2_{L^2(Y_\ell)} &\lesssim ||\Theta\nabla_z v||^2_{L^2(\tilde{Y}_\ell)} + ||v||^2_{L^2(Y_\ell)}\\ &\leq \int_{\tilde{Y}_\ell}\Theta\nabla_z v\cdot\overline{\nabla_z v}dz + ||v||^2_{L^2(Y_\ell)}\\ &= -\frac{1}{2}\int_{\tilde{Y}_\ell}\nabla_z\Theta\cdot\nabla_z|v|^2dz - \int_{\tilde{Y}_\ell}\Theta\Delta_z v\bar{v}dz + ||v||^2_{L^2(Y_\ell)}\\ &= \frac{1}{2}\int_{\tilde{Y}_\ell}\Delta\Theta|v|^2dz + \int_{\tilde{Y}_\ell}\Theta\,\mathrm{rot}\ A_1^*u\bar{v}dz + ||v||^2_{L^2(Y_\ell)}\\ &= \frac{1}{2}\int_{\tilde{Y}_\ell}\Delta\Theta|v|^2dz + \int_{\tilde{Y}_\ell}\Theta A_1^*v\bar{v}dz + \int_{\tilde{Y}_\ell}\Theta[A_1^*,\mathrm{rot}\ ]u\bar{v}dz + ||v||^2_{L^2(Y_\ell)}\\ &\lesssim ||v||_{L^2(\tilde{Y}_\ell)} + ||\Theta\nabla_z v||_{L^2(\tilde{Y}_\ell)}||v||_{L^2(\tilde{Y}_\ell)} + ||u||_{H^1(\tilde{Y}_\ell)}||v||_{L^2(\tilde{Y}_\ell)} + ||v||^2_{L^2(Y_\ell)}, \end{aligned}$$

which implies, using the Young inequality,

$$||\nabla_z\mathfrak{V}||^2_{L^2(Y_\ell)} \leq C\left(||v||^2_{L^2(\tilde{Y}_\ell)} + ||u||^2_{H^1(\tilde{Y}_\ell)}\right) \leq 2C||u||^2_{H^1(\tilde{Y}_\ell)}.$$

Using the same arguments than above, in combinaison with (67), allows us to write, up to taking $\hat{Y}_\ell$ slightly larger than $\tilde{Y}_\ell$,

$$||\mathfrak{V}||_{L^2(Y_\ell)} + ||\nabla_z\mathfrak{V}||_{L^2(Y_\ell)} + ||\mathfrak{U}||_{L^2(Y_\ell)} + ||\nabla\mathfrak{U}||_{L^2(Y_\ell)} + ||[-\partial_s^2,\chi_1]v||^2_{L^2(O_3)} \lesssim Ce^{C\sqrt{\Lambda}}||u||_{L^2(\hat{Y}_\ell)}||\tilde{u}||_{L^2(\Omega)}.\tag{68}$$

Indeed, let $\tilde{Y}_\ell \subset \hat{Y}_\ell \Subset Z$ and let $\tilde{\Theta} \in C_0^\infty(\hat{Y}_\ell)$ such that $\tilde{\Theta} = 1$ on $\tilde{Y}_\ell$,

$$\begin{aligned}
||\nabla_z u||^2_{L^2(\tilde{Y}_\ell)} &= ||\nabla u||^2_{L^2(\tilde{Y}_\ell)} + ||\partial_s u||^2_{L^2(\tilde{Y}_\ell)} \\
&= \left(\tilde{\Theta}\nabla u, \nabla u\right)_{L^2(Z)} + \left(\tilde{\Theta}\partial_s u, \partial_s u\right)_{L^2(Z)} \\
&\lesssim \left|\left(\tilde{\Theta}u, \Delta u\right)_{L^2(Z)}\right| + \left|\left(\nabla\tilde{\Theta}\cdot\nabla u, u\right)_{L^2(Z)}\right| + \left|\left(\tilde{\Theta}\partial_s^2 u, u\right)_{L^2(Z)}\right| + \left|\left((\partial_s\tilde{\Theta})\partial_s u, u\right)_{L^2(Z)}\right| \\
&\lesssim ||\tilde{\Theta}u||_{L^2(Y)}||\Delta u||_{L^2(Z)} + \left|\left((\Delta\tilde{\Theta})u, u\right)_{L^2(Z)}\right| + \left|\left(\tilde{\Theta}A^* u, u\right)_{L^2(Z)}\right| + \left|\left((\partial_s^2\tilde{\Theta})u, u\right)_{L^2(Z)}\right| \\
&\lesssim ||u||_{L^2(\hat{Y}_\ell)}||A^* u||_{L^2(Z)},
\end{aligned}$$

where we used in the last inequality the elliptic regularity of the Stokes operator. Thus, by the resolvent estimate of Corollary 1.7, we have

$$||\nabla_z u||^2_{L^2(\tilde{Y}_\ell)} \lesssim Ce^{C\sqrt{\Lambda}}||u||_{L^2(\hat{Y}_\ell)}||\tilde{u}||_{L^2(\Omega)}. \tag{69}$$

Estimates (66) and now implies, by the Young inequality

$$||\tilde{u}||_{L^2(\Omega)} \le Ce^{C\sqrt{\Lambda}}||u||_{L^2(\hat{Y}_\ell)},$$

which is the sought result. □

5.2 Observability in the interior

We construct the weight function by using the following theorem (see for instance [19]).

Theorem 5.3 *There exists $\tilde{\psi} \in C^\infty(\overline{\Omega})$ such that,*

- $\tilde{\psi} > 0$, *in* Ω,
- $\tilde{\psi}_{|\partial\Omega} = 0$,
- $|\nabla\tilde{\psi}| \ge C > 0$ *on* $\Omega \setminus \tilde{\omega}$.

One then defines $\psi_\beta(s, x, y) = \tilde{\psi}(x, y) - \beta|s - s_0|^2$ and $\varphi = e^{\lambda\psi_\beta}$, where β, λ are large parameters, that shall be fixed in what follows. Let $s_1 < s_2 < s_3 < s_4$, such that $s_0 \pm s_4 \in (0, S_0)$, that shall be fixed in what follows. We define (see Figure 2 for a sketch of the different sets),

- $A_1 := ((s_0 - s_3, s_0 - s_2) \cup (s_0 + s_2, s_0 + s_3)) \times \Omega$,
- $A_2 := (s_0 - s_2, s_0 + s_2) \times \omega$,
- $A_3 := (s_0 - s_2, s_0 + s_2) \times \tilde{A}_3$, where $\tilde{A}_3 := \{(x, y) \in \Omega,\ d((x, y), \partial\Omega) < \frac{1}{2\ell}\}$,
- $A_4 := (s_0 - s_1, s_0 + s_1) \times \tilde{A}_4$, where $\tilde{A}_4 := \{(x, y) \in \Omega;\ d((x, y), \partial\Omega) > \frac{3}{4\ell}\} \setminus \omega$,
- $\tilde{A}_5 := \{(x, y) \in \Omega;\ d((x, y), \partial\Omega) > \frac{3}{4\ell}\}$.

Lemma 5.4 *Let $0 < s_1 < s_2 < s_3 < s_4$ as above. There exist $\mu > 0$ and $C > 0$ such that,*

$$||u||_{H^1(A_4)} \le C||u||^{1-\mu}_{H^2(Z)}||u||^{\mu}_{H^2(A_2)},$$

for all u of the form (11).

Proof. Let $\gamma \in (0, 1)$. Let us set $\delta = -\frac{1}{2}\left(\sup_{\tilde{A}_3}\tilde{\psi} - \inf_{\tilde{A}_4}\tilde{\psi}\right)$, and fix $\beta \ge 1$ such that $\frac{\delta}{\beta^{\gamma-1}} > \delta + \sup_\Omega\tilde{\psi} - \inf_{\tilde{A}_4}\tilde{\psi}$ which is always possible by taking $\beta > 1$ sufficiently large. Remark that $\delta > 0$ for $\ell > 0$ taken sufficiently large. One now set s_1 and s_2 such that,

$$|s_1|^2 = \frac{\delta}{\beta}, \quad |s_2|^2 = \frac{\delta}{\beta^\gamma}.$$

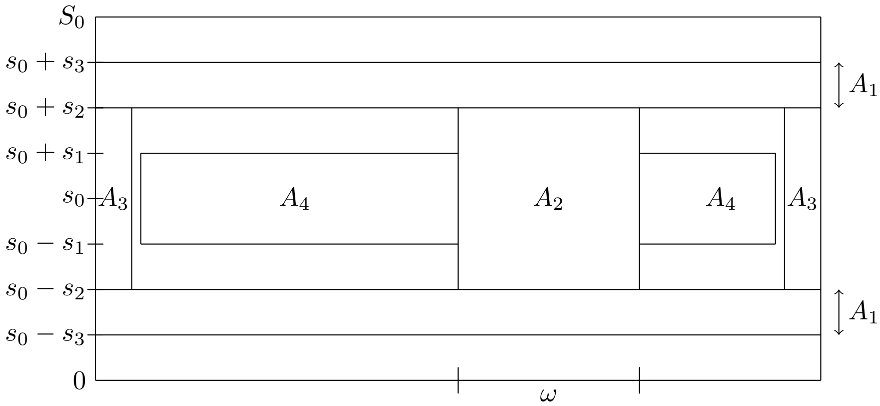

Figure 2: The geometry in the interior of $\Omega \times (0, S_0)$.

Note that the following inequalities hold,

$$\sup_{A_3} \psi_\beta - \inf_{A_4} \psi_\beta = \sup_{\tilde{A}_3} \tilde{\psi} - \inf_{\tilde{A}_4} \tilde{\psi} + \delta < 0, \tag{70}$$

$$\sup_{A_1} \psi_\beta - \inf_{A_4} \psi_\beta = \sup_{A_1} \tilde{\psi} - \inf_{A_4} \tilde{\psi} + \delta(1 - \beta^{1-\gamma}) < 0, \tag{71}$$

$$\sup_{A_2} \psi_\beta - \inf_{A_4} \psi_\beta > 0. \tag{72}$$

Hence, one moreover can remark that $s_3 > 0$ can be chosen as small as we want (and so it is for s_1 and s_2) by taking $\beta > 0$ sufficiently large. From now on, parameters δ, β and γ are fixed, and all the constants involved in the computations below depend on them. Let us define the smooth cut-off function $\Theta(s, x, y) = \Theta_1(s)\Theta_2(x, y) \in C_0^\infty(Z)$ such that

$$\Theta_1 = \begin{cases} 1 & \text{on } (s_0 - s_2, s_0 + s_2), \\ 0 & \text{on } (0, S_0) \setminus (s_0 - s_3, s_0 + s_3), \end{cases} \qquad \Theta_2 = \begin{cases} 1 & \text{on } \{d((x,y), \partial\Omega) \geq \frac{1}{2\ell}\} \setminus \omega, \\ 0 & \text{on } \tilde{\omega} \cup \{d((x,y), \partial\Omega) < \frac{1}{4\ell}\}. \end{cases} \tag{73}$$

Let V be the interior of $\operatorname{supp} \Theta$. One have that there exists λ such that the weight function φ satisfies the two sub-ellipticity conditions of definitions 3.1 and 3.2 on the open set V. The parameter λ is fixed from now on. Thus, the following two Carleman estimates holds on V for such a weight function φ.

Theorem 5.5 *Let V be an open set in Y. Let φ be a weight function satisfying the conditions of Definition 3.1 on $\overline{V}$. Then, there exist $h_0 > 0$ and $C > 0$ such that,*

$$h||e^{\frac{1}{h}\varphi} f||^2_{L^2(V)} + h^3 ||e^{\frac{1}{h}\varphi} \nabla_z f||^2_{L^2(V)} \leq C h^4 ||e^{\frac{1}{h}\varphi}(-\partial_s^2 - \Delta) f||^2_{L^2(V)}, \tag{74}$$

for all $f \in C_0^\infty(V)$, $h \leq h_0$, and such that

$$h||e^{\frac{1}{h}\varphi} f||^2_{L^2(V)} + h^3 ||e^{\frac{1}{h}\varphi} \nabla f||^2_{L^2(V)} \leq C h^4 ||e^{\frac{1}{h}\varphi} \Delta f||^2_{L^2(V)}, \tag{75}$$

for all $f \in C_0^\infty(V)$, $h \leq h_0$.

Remark 5.6 *The key point here is that the two above Carleman estimates will allow us to handle the low order terms. However, to have (74) and (75) at the same time, one needs $|\nabla_z \varphi| \neq 0$ and $|\nabla \varphi| \neq 0$ on $\overline{V}$, which is satisfied only by removing $(0, S_0) \times \tilde{\omega}$ from the support of Θ (see (73)). This is precisely why we will not be able here to propagate the smallness in the s direction.*

Let us apply the estimate (74) to $f = \Theta v := \Theta \operatorname{rot}(u)$, where u is of the form (11). We recall that the function v satisfies,

$$-\partial_s^2 v - \Delta v + \operatorname{rot}\ A_1^* u = 0,$$

which implies that f satisfies,

$$-\partial_s^2 f - \Delta f = -\Theta \operatorname{rot}\ A_1^* u + [\Theta, \partial_s^2 + \Delta] v. \tag{76}$$

Estimate (74) yields, for h sufficiently small,

$$\begin{aligned} h||e^{\frac{1}{h}\varphi}\Theta v||^2_{L^2(V)} + h^3||e^{\frac{1}{h}\varphi}\Theta\nabla_z v||^2_{L^2(V)} &\lesssim h^4||e^{\frac{1}{h}\varphi}\left(-\Theta\,\text{rot}\ A_1^* u + [\Theta, \partial_s^2 + \Delta]v\right)||^2_{L^2(V)} \\ &\lesssim h^4||e^{\frac{1}{h}\varphi}\Theta\,\text{rot}\ A_1^* u||^2_{L^2(V)} + h^4||e^{\frac{1}{h}\varphi}[\Theta, \partial_s^2 + \Delta]v||^2_{L^2(V)} \\ &\lesssim h^4||e^{\frac{1}{h}\varphi}\Theta A_1^* v||^2_{L^2(V)} + h^4||e^{\frac{1}{h}\varphi}\Theta[\text{rot}\ , A_1^*]u||^2_{L^2(V)} + h^4||e^{\frac{1}{h}\varphi}[\Theta, \partial_s^2 + \Delta]v||^2_{L^2(V)} \end{aligned} \quad (77)$$

Now, remark that, using the divergence free condition,

$$||e^{\frac{1}{h}\varphi}\Delta(\Theta u)||^2_{L^2(V)} \lesssim ||e^{\frac{1}{h}\varphi}\Theta\nabla_z v||^2_{L^2(V)} + ||e^{\frac{1}{h}\varphi}[\Delta, \Theta]u||^2_{L^2(V)}. \quad (78)$$

Estimate (75) for $f = \Theta u$ yields, for h sufficiently small,

$$||e^{\frac{1}{h}\varphi}\Theta u||^2_{L^2(V)} + h^2||e^{\frac{1}{h}\varphi}\Theta\nabla u||^2_{L^2(V)} \lesssim h^3||e^{\frac{1}{h}\varphi}\Delta(\Theta u)||^2_{L^2(V)}. \quad (79)$$

Combining (77), (78) and (79), we obtain,

$$\begin{aligned} ||e^{\frac{1}{h}\varphi}\Theta u||^2_{L^2(V)} + h^2||e^{\frac{1}{h}\varphi}\Theta\nabla u||^2_{L^2(V)} + h||e^{\frac{1}{h}\varphi}\Theta v||^2_{L^2(V)} + h^3||e^{\frac{1}{h}\varphi}\Theta\nabla_z v||^2_{L^2(V)} \lesssim h^4||e^{\frac{1}{h}\varphi}\Theta A_1^* v||^2_{L^2(V)} \\ + h^4||e^{\frac{1}{h}\varphi}\Theta[\text{rot}\ , A_1^*]u||^2_{L^2(V)} + h^4||e^{\frac{1}{h}\varphi}[\Theta, \partial_s^2 + \Delta]v||^2_{L^2(V)} + ||e^{\frac{1}{h}\varphi}[\Delta, \Theta]u||^2_{L^2(V)}. \end{aligned}$$

Note that, taking $h > 0$ sufficiently small, one can absorb the low order terms in the right hand side of the above inequality to have,

$$\begin{aligned} ||e^{\frac{1}{h}\varphi}\Theta u||^2_{L^2(V)} + h^2||e^{\frac{1}{h}\varphi}\Theta\nabla u||^2_{L^2(V)} + h||e^{\frac{1}{h}\varphi}\Theta v||^2_{L^2(V)} + h^3||e^{\frac{1}{h}\varphi}\Theta\nabla_z v||^2_{L^2(V)} \\ \lesssim h^4||e^{\frac{1}{h}\varphi}[\Theta, \partial_s^2 + \Delta]v||^2_{L^2(V)} + h^3||e^{\frac{1}{h}\varphi}[\Delta, \Theta]u||^2_{L^2(V)}. \end{aligned} \quad (80)$$

The two remaining terms in the right hand side of the above inequality are localized in regions where Θ varies. Estimate (80) implies,

$$e^{\frac{1}{h}\inf_{A_4}\varphi}\left(||u||^2_{H^1(A_4)} + ||v||^2_{H^1(A_4)}\right) \lesssim \sum_{k=1}^{3} e^{\frac{1}{h}\sup_{A_k}\varphi}\left(||u||^2_{H^1(A_k)} + ||v||^2_{H^1(A_k)}\right),$$

which is equivalent to,

$$\left(||u||^2_{H^1(A_4)} + ||v||^2_{H^1(A_4)}\right) \lesssim \sum_{k=1}^{3} e^{\frac{1}{h}(\sup_{A_k}\varphi - \inf_{A_4}\varphi)}\left(||u||^2_{H^1(A_k)} + ||v||^2_{H^1(A_k)}\right).$$

From (70), (71) and (72), there exist $\nu_1, \nu_2 > 0$ such that,

$$\left(||u||^2_{H^1(A_4)} + ||v||^2_{H^1(A_4)}\right) \lesssim e^{\frac{\nu_1}{h}}\left(||u||^2_{H^1(A_2)} + ||v||^2_{H^1(A_2)}\right) + e^{-\frac{\nu_2}{h}}\left(||u||^2_{H^1(Z)} + ||v||^2_{H^1(Z)}\right).$$

Optimizing this estimate with respect to $h > 0$ implies that there exists $C > 0$ and $\mu \in (0, 1)$ such that

$$||u||_{H^1(A_4)} \leq C||u||^{1-\mu}_{H^2(Z)}||u||^{\mu}_{H^2(A_2)}.$$

The estimate over A_2 being immediate this ends the proof of Lemma 5.4. □

As Lemma 5.4 only provides an estimate on domain A_4 which may be arbitrary small, we use the following lemma to recover the full estimate on Y_ℓ that we need having in mind to use Lemma 5.1.

Lemma 5.7 *There exist and $\tilde{\ell} > \ell$, $\mu > 0$ and $C > 0$ such that,*

$$||u||_{H^1(Y_\ell)} \leq C||u||^{1-\mu}_{H^2(Z)}||u||^{\mu}_{H^2((\frac{1}{\tilde{\ell}}, S_0 - \frac{1}{\tilde{\ell}})\times\omega)},$$

for all u of the form (11).

Proof. Let $A_2 := (s_0^k - s_2^k, s_0 + s_2) \times \omega$ and $A_5^k := (s_0^k - s_1^k, s_0 + s_1) \times \tilde{A}_5$, where $\tilde{A}_5 := \{(x, y) \in \Omega;\ d((x, y), \partial\Omega) > \frac{3}{4\ell}\}$. Remark that there exists a finite number M, and a sequence of $(s_0^k)_{k\in\{1,\dots,M} \subset Y_\ell$, such that,

$$Y_\ell \subset \bigcup_{k=1} A_5^k.$$

Thus, applying Lemma 5.4 for each A_2^k and A_5^k, $k = 1, \dots, M$, yields the result, by noting that $A_2^k \subset (\frac{1}{\tilde{\ell}}, S_0 - \frac{1}{\tilde{\ell}}) \times \omega$, for all $k \in \{1, \dots, M\}$. □

5.3 End of the proof

Now we finish the proof of Theorem 1.8. It is sufficient to combine Lemmata 5.1 and 5.4, to obtain,

$$||\tilde{u}||_{L^2(\Omega)} \leq Ce^{C\sqrt{\Lambda}}||u||_{L^2(Y_\ell)} \leq Ce^{C\sqrt{\Lambda}}||u||^{1-\mu}_{H^2(Z)}||u||^{\mu}_{H^2((\frac{1}{\ell},S_0-\frac{1}{\ell})\times\omega)}.$$

Assuming,

$$||u||_{H^1(Z)} \lesssim \tilde{C}e^{\tilde{C}\sqrt{\Lambda}}||\tilde{u}||_{L^2(\Omega)}, \tag{81}$$

yields immediatly the result by taking $\varphi \in C_0^\infty(0, S_0)$ such that $\varphi = 1$ on $(\frac{1}{\ell}, S_0 - \frac{1}{\ell})$. The estimate (81) follows from the resolvent estimate of Corollary 1.7. □

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