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Abstract . The domain decomposition method (DDM) initially designed, with the celebrated paper of Schwarz in 1870 [22] as a theoretical tool for partial differential equations (PDEs) has become, since the advent of the computer and parallel computing techniques, a major tool for the numerical solution of such PDEs, especially for large scale problems. Time harmonic wave problems offer a large spectrum of applications in various domains (acoustics, electromagnetics, geophysics, ...) and occupy a place of their own, that shines for instance through the existence of a natural (possibly small) length scale for the solutions: the wavelength. Numerical DDMs were first invented for elliptic type equations (e.g. the Laplace equation), and even though the governing equations of wave problems (e.g. the Helmholtz equation) look similar, standard approaches do not work in general.

The objective of this work is to make a rapid, but hopefully pedagogical, survey of the research led mainly at INRIA (in the teams ONDES then POEMS and ALPINES) since 1990, on non overlapping domain decomposition methods for time harmonic wave propagation problems, based on the notion of impedance transmission conditions. Our point of view, and we consider that this sets us apart from the rest of the wave DDM community, is theory driven: we proposed and progressively developed a unified framework that guarantees the well-posedness and convergence of the related iterative algorithms in the **most general cases (geometry, variable coefficients, boundary conditions. . .**). This research was punctuated by four Phd theses.

- The PhD thesis of B. Després [10] (1991) is definitely a pioneering work which constitutes a decisive step. It is worthwhile mentioning that P. L. Lions [17], [18] wrote his papers on the theory of DDMs for elliptic problems at the same period.
- With the PhD thesis of S. Ghanemi [15], at CERFACS in 1996, we began to develop our general theoretical framework, proposed the use of second order local transmission conditions and initiated non-local transmission conditions [3].

  Then there was a big pause (about 15 years) in our activity, during which a huge literature was devoted to Optimized Schwarz Methods (OSMs) associated to local impedance operators (see also section 2), up to the opportunity of a contract with CEA (French Nuclear Agency) which started the second phase of our activity.

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- The PhD thesis of M. Lecouvez [16] (2015), in collaboration with CEA, permitted us to develop the notion of non-local transmission operators.
- The PhD thesis of E. Parolin [20] (2020) supported by the ANR Project NonlocalDD which closes a chapter of the story with the notion of elliptic DtN operators, the treatment of Maxwell's equations and a solution to the cross points issue [6].

1 Elliptic equations versus Helmholtz equation

In this section, we expose the general ideas, more formalism will be introduced in Section 2. To emphasize the difference of status between the two types of equations w.r.t. DDM, let us simply consider the equation with constant coefficients

$$-\Delta u + k^2 u = f, \quad \text{in } \mathbb{R}^d, \quad k \in \mathbb{C}, \quad \text{where} \tag{1}$$

- if $k \in \mathbb{R}^+$: in this case (1) is of (strongly) elliptic nature
- if $k = i\omega,\ \omega \in \mathbb{R}^+$ (Helmholtz) : one models waves with frequency ω.

The distinction is important for DDMs : for instance, the classical overlapping Schwarz method converges (**linearly** in most case) in the elliptic case but **does not converge** for the Helmholtz equation. In fact, in the elliptic case, the boundary value problems (BVPs) associated with (1) enjoy many nice properties including the H^1-coercivity of $a(u,v) = \int (\nabla u \cdot \nabla v + k^2\, u\, v)$, the associated bilinear form, and their solutions are often interpreted as the solutions of convex minimization problems. With this point of view, P.L. Lions gave a general proof of convergence of the Schwarz method by interpreting the error at each step of the algorithm as the result of successive orthogonal projections on two (with two subdomains) supplementary subspaces of H^1 [17]. These problems also benefit from the maximum principle, which also provides another way for proving the convergence of the Schwarz method.

On the contrary, if $k = i\omega$, $a(u,v) = \int (\nabla u \cdot \nabla v - \omega^2\, u\, v)$, the natural bilinear form for Helmholtz, is no longer coercive and there is no underlying variational principle for the corresponding BVPs. Also, there is no maximum principle: the (complex valued) solutions naturally oscillate with the wavelength $\lambda = 2\pi/\omega$.

Fortunately, good news comes from the boundary: if u satisfies $-\Delta u - \omega^2 u = 0$ in a bounded domain Ω with boundary Γ and outgoing normal ν then (multiply the equation by $\overline{u}$, integrate over Ω, apply Green's formula and take the imaginary part)

$$\mathit{Im} \int_\Gamma \partial_\nu u\, \overline{u} = 0, \quad \text{i. e.} \quad \mathit{Im} \left\langle \partial_\nu u, u \right\rangle_\Gamma = 0, \tag{2}$$

with $\langle \cdot, \cdot \rangle_\Gamma$ the inner product in $L^2(\Gamma) \equiv L^2(\Gamma; \mathbb{C})$. This leads to the following isometry result, where $\| \cdot \|_\Gamma$ denotes the $L^2(\Gamma)$-norm

$$\| \partial_\nu u + i\omega\, u \|_\Gamma^2 = \| \partial_\nu u - i\omega\, u \|_\Gamma^2, \tag{3}$$

(simply note that the difference of the two sides of (3) is proportional to $\mathit{Im} \langle \partial_\nu u, u \rangle_\Gamma$ which is 0 by (2)). One obtains many other isometry results by playing with identity

(2): introducing a "boundary operator" Λ (understand that it tranforms a function defined on Γ into another function defined on Γ), supposed to be bijective (between appropriate spaces) with (formal) adjoint Λ^*, we remark that

$$\mathcal{I}m\left\langle \partial_\nu u, u\right\rangle_\Gamma = 0 \;\Leftrightarrow\; \mathcal{I}m\left\langle \partial_\nu u, \Lambda^{-1}\Lambda u\right\rangle_\Gamma = 0 \;\Leftrightarrow\; \mathcal{I}m\left\langle (\Lambda^*)^{-1}\partial_\nu u, \Lambda u\right\rangle_\Gamma = 0,$$

from which we deduce the other isometry result

$$\| (\Lambda^*)^{-1}\partial_\nu u + i\omega\, \Lambda u \|_\Gamma^2 = \| (\Lambda^*)^{-1}\partial_\nu u - i\omega\, \Lambda u \|_\Gamma^2. \tag{4}$$

Introducing the positive definite selfadjoint boundary operator $T = \Lambda^*\Lambda$ (called *impedance operator* in the sequel) and the associated norm

$$(\varphi,\psi) := \left\langle \psi, T^{-1}\varphi\right\rangle_\Gamma, \quad \left\|\varphi\right\|^2 := \left\langle \varphi, T^{-1}\varphi\right\rangle_\Gamma, \tag{5}$$

$$\text{so that (4) rewrites} \quad \| \partial_\nu u + i\omega\, Tu \| = \| \partial_\nu u - i\omega\, Tu \|^2. \tag{6}$$

This is one of the reasons which led us, in the context of iterative overlapping DDMs, denoting $\{\Omega_j\}$ the subdomains (with outgoing normals ν_j), to propose

$$\partial_{\nu_j} u_j^n + i\omega\, Tu_j^n = (rhs)_{n-1}, \quad u_j^n = u^n|_{\Omega_j}, \tag{7}$$

as a boundary condition in Ω_j, where $(rhs)_{n-1}$ is a quantity, depending on the previous iteration and the adjacent subdomain, providing the good continuity conditions at convergence (section 2). An important consequence of the properties of T (symmetric positive definite) is that (7) is of absorbing nature so that the **local problem** in Ω_j is automatically **well posed**. Moreover, as we shall see in section 2, the **isometry result** (6) can be exploited to prove the **convergence** of the iterative algorithm.

2 Impedance based transmission conditions and related DDM

Presentation of the method on a simple model. Let (BVP) consist in solving the Hemholtz equation in a $\Omega \subset \mathbb{R}^d$, bounded, with a perfectly reflecting inner boundary Γ_1 and absorbing outer boundary Γ_2.

$$(BVP)\;\; -\Delta u - \omega^2\, u = f, \text{ in } \Omega, \quad u = 0, \text{ on } \Gamma_1, \quad \partial_\nu u + i\,\omega\, u = 0, \text{ on } \Gamma_2,$$

Introducing an interface Σ that splits Ω into two subdomains Ω_1 (interior) and Ω_2 (exterior), see Figure 2 (left picture), (BVP) is equivalent to a transmission problem $(LP)+(TC)$ (local problem + transmission conditions) where, with obvious notation (in particular, ν_j is the unit normal vector to Σ, outgoing w. r. t. Ω_j)

$$(LP)\;\begin{cases} -\Delta u_j - \omega^2\, u_j = f, & \text{in } \Omega_j, j=1,2 \\ u_1 = 0, & \text{on } \Gamma_1, \\ \partial_\nu u_2 + i\,\omega\, u_2 = 0, & \text{on } \Gamma_2, \end{cases} \qquad (TC)\;\begin{cases} (\mathbf{n})\; u_1 = u_2, & \text{on } \Sigma, \\ (\mathbf{d})\; \partial_{\nu_1} u_1 + \partial_{\nu_2} u_2 = 0, & \text{on } \Sigma. \end{cases}$$

Given $s \in [0, 1/2]$, we introduce an impedance operator T with the property that

$$T \in \mathcal{L}\left(H^s(\Sigma), H^{-s}(\Sigma)\right) \text{ is a positive and selfadjoint isomorphism.} \tag{8}$$

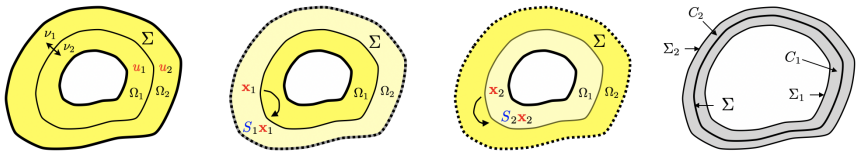

Fig. 1 The subdomains Ω_1 and Ω_2 (left). The scattering operators S_1 ans S_2 (center). The layers C_1 and C_2, cf section 5 (right)

With this choice, the norm defined by (5) (with Γ replaced by Σ, and $\langle\cdot,\cdot\rangle_\Sigma$ understood as a duality bracket) is a **Hilbert space norm** in $H^{-s}(\Sigma)$.

Next, we rewrite (TC) in an equivalent way (thanks to the injectivity of T) by considering the two independent linear combinations $(TC)(\mathbf{n}) \pm i\omega\,(TC)(\mathbf{d})$, i. e.

$$\begin{cases} \partial_{\nu_1} u_1 + i\,\omega\, T u_1 = -\partial_{\nu_2} u_2 + i\,\omega\, T u_2, & (1) \\ \partial_{\nu_2} u_2 + i\,\omega\, T u_2 = -\partial_{\nu_1} u_1 + i\,\omega\, T u_1, & (2) \end{cases} \tag{9}$$

where (9)-(j) is seen here as a boundary condition for u_j. The iterative DDM algorithm consists in applying a **fixed point** procedure (with relaxation) to (9). Precisely, we construct inductively two sequences $u_j^n \in H^1(\Omega_j), j = 1,2$, by imposing, at each step n, the local equations (LP) completed by the following boundary conditions on Σ (where $r \in]0,1[$ is the relaxation parameter)

$$\begin{cases} \partial_{\nu_1} u_1^n + i\,\omega\, T u_1^n = r\left(-\partial_{\nu_2} u_2^{n-1} + i\,\omega\, T u_2^{n-1}\right) + (1-r)\left(\partial_{\nu_1} u_1^{n-1} + i\,\omega\, T u_1^{n-1}\right), \\ \partial_{\nu_2} u_2^n + i\,\omega\, T u_2^n = r\left(-\partial_{\nu_1} u_1^{n-1} + i\,\omega\, T u_1^{n-1}\right) + (1-r)\left(\partial_{\nu_2} u_2^{n-1} + i\,\omega\, T u_2^{n-1}\right). \end{cases} \tag{10}$$

The reader will notice that, by construction, the local problems in (u_1^n, u_2^n) are well posed, and can be solved in parallel.

A functional analytic observation. It is insightful to look at the quantities in (9) for the two extreme values for $s \in [0, 1/2]$. Given $u_j \in H^1(\Omega_j)$ with $\Delta u \in L^2(\Omega_j)$:

- if $s = 0$, for instance $T = I$, the identity : the combination $\partial_\nu u_j \pm i\,\omega\, u_j$ are not well balanced since u naturally belongs to $H^{1/2}(\Sigma)$ while $\partial_\nu u$ only belongs to $H^{-1/2}(\Sigma)$,
- if $s = 1/2$: the presence of $T \in \mathcal{L}\left(H^{1/2}(\Sigma), H^{-1/2}(\Sigma)\right)$ re-equilibrates the combination as a sum of two terms in $H^{-1/2}(\Sigma)$.

In fact, a **misfit** is present as soon as $s \neq 1/2$ and one can thus anticipate that the best option should be $s = 1/2$. This will be confirmed by the analysis (section 3).

A rapid guided tour into the bibliography. A lot of litterature has been devoted to DDMs based on transmission written in impedance form.

- In the original work of B. Desprès [10] (or [11] for Maxwell), $T = \alpha\, I$ where α is a bounded strictly positive function, which fits (8) with $s = 0$.
- Since the mid 90's a huge literature has been devoted to "**local**" operators T as rational functions of the Laplace-Beltrami operator Δ_Σ, e. g. [13, 2], with a great

filiation with local absorbing conditions (Remark 1). These often do not satisfy (8) and a general theory (existence for local problems and convergence) is missing.

- In [4], we promote the use of non-local impedance operators T fitting (8) with $s = \frac{1}{2}$ in particular boundary integral operators issued from potential theory.

Some optimized Schwarz methods, for instance Boubendir-Antoine-Geuzaine's one, perform very well in practice (despite examples of failure, see [4], section 8.2.3). However, they cannot lead to linear convergence (see [4], Thm 4.6).

Remark 1 : There is an ideal choice of transmission conditions with two (not one) operators, $\partial_{\nu_1} u_1 + i\omega T_1 u_1 = -\partial_{\nu_2} u_2 + i\omega T_1 u_2$ and $\partial_{\nu_2} u_2 + i\,\omega T_2 u_2 = -\partial_{\nu_1} u_1 + i\,\omega T_2 u_1$: take T_1 (resp. T_2) as the DtN operator, when it exists, associated to Ω_2 (resp. Ω_1) (see [4] section 1.3.2 and [14]). Then Algorithm (10) with $r = 1$ converges in two iterations. In general, finding T_1 or T_2 is almost as difficult as the original problem. For two homogeneous half-spaces (plane interface), $T_1 = T_2$ with symbol $i\omega\sqrt{1 - |\xi|^2/\omega^2}$, ($\xi$ is the space Fourier variable) whose rational approximations (Taylor, Padé, continued fraction expansions) give local operators, as for ABCs.

3 Convergence analysis

Interface formulation. For both the implementation and the analysis of our method, it is useful to reinterpret the problem and the algorithm on the interface Σ. To do so we introduce the interface auxiliary unknowns (where traces on Σ are implicitly considered), i. e. the outgoing traces x_j and incoming traces y_j:

$$x_j := \partial_{\nu_j} u_j + i\,\omega\, T\, u_j, \quad y_j := -\partial_{\nu_j} u_j + i\,\omega\, T\, u_j, \quad \text{in } H^{-s}(\Sigma) \tag{11}$$

Given x_1 and x_2, u_1 and u_2 can be seen as the solutions of the local problems

$$\begin{cases} -\Delta u_1 - \omega^2 u_1 = f, & \text{in } \Omega_1, \\ u_1 = 0, & \text{on } \Gamma_1, \\ \partial_{\nu_1} u_1 + i\,\omega\, T\, u_1 = x_1, & \text{on } \Sigma, \end{cases} \qquad \begin{cases} -\Delta u_2 - \omega^2 u_2 = f, & \text{in } \Omega_2, \\ \partial_\nu u_2 + i\,\omega u_2 = 0, & \text{on } \Gamma_2, \\ \partial_{\nu_2} u_2 + i\,\omega\, T\, u_2 = x_2, & \text{on } \Sigma, \end{cases} \tag{12}$$

and, exploiting the linearity of (12), the incoming traces y_j can be rewritten as

$$y_1 = S_1\, x_1 + \widetilde{g}_1, \quad y_2 = S_2\, x_2 + \widetilde{g}_2, \tag{13}$$

where, in an obvious manner, the source terms $\widetilde{g}_j$ are due to f (they are issued from (12) with $x_j = 0$) and the scattering operators S_j are constructed from the local problems (12) with $f = 0$. Next, the transmission conditions simply rewrite

$$y_2 = x_1, \quad y_1 = x_2, \tag{14}$$

and the transmission problem (LP, TC) is equivalent to the system (13, 14) in $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$: (13) takes account of local problems and (14) of transmission conditions. Eliminating $\mathbf{y}$ then leads to a problem in $\mathbf{x}$:

$$\text{Find } \mathbf{x} \in \mathbf{V} := H^{-s}(\Sigma) \times H^{-s}(\Sigma) \quad / \quad (\mathbf{I} - \mathbf{A})\,\mathbf{x} = \mathbf{g}, \quad \mathbf{g} = \mathbf{\Pi}\,\widetilde{\mathbf{g}}, \tag{15}$$

with the (T-dependent) **scattering** operator $\mathbf{S}$ and the **exchange** operators $\mathbf{\Pi}$:

$$\mathbf{S} := \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \quad \mathbf{\Pi} := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \text{thus} \quad \mathbf{I} - \mathbf{A} := \begin{pmatrix} I & -S_2 \\ -S_1 & I \end{pmatrix}. \tag{16}$$

Mathematical properties. In the following, we equip the Hilbert space $\mathbf{V}$ with the (T-dependent) norm naturally inherited from the H^{-s}-norm defined by (5), that we still denote $\|\cdot\|$ for simplicity. From (8), it is clear that the operators $\mathbf{\Pi}$ and $\mathbf{S}$ are continuous in $\mathbf{V}$. Obviously, $\mathbf{\Pi}$ is an isometry while, from the identity (6) (applied in Ω_1 and Ω_2), we immediately infer that, for any $(x_1, x_2) \in \mathbf{V}$,

$$(a) \quad \|S_1\, x_1\| = \|x_1\|, \quad (b) \quad \|S_2\, x_2\| \leq \|x_2\|. \tag{17}$$

where the inequality in (17)-(b) is due to the absorbing condition on Γ_2 for u_2 in (12). As a consequence, the operator $\mathbf{S}$, thus the operator $\mathbf{A}$, is **contractant** in $\mathbf{V}$. Concerning the **invertibility** of $\mathbf{I} - \mathbf{A}$, algebraic manipulations show that

$$\mathbf{z} = (\mathbf{I} - \mathbf{A})\,\mathbf{x} \quad \Leftrightarrow \quad x_j = \partial_{\nu_j} w_j + i\,\omega\, T w_j \text{ on } \Sigma,\ j = 1, 2, \tag{18}$$

where, denoting ν the normal to Σ pointing towards Ω_2 and $[\cdot]_\Sigma$ the jump across Σ, $w \in H^1(\Omega_1 \cup \Omega_2)$ satisfies (H) $-\Delta w - \omega^2\, w = f$ in $\Omega_1 \cup \Omega_2$, (BC) : $w = 0$ on Γ_1 and $\partial_\nu w + i\,\omega\, w = 0$ on Γ_2 and the "jump conditions", with $[\mathbf{z}] = z_1 - z_2$, $\{\mathbf{z}\} = \frac{1}{2}(z_1 + z_2)$:

$$[v]_\Sigma = \tfrac{1}{2i\omega}\, T^{-1}\, [\mathbf{z}], \qquad [\partial_n v]_\Sigma = \{\mathbf{z}\} \tag{19}$$

The **injectivity** of $\mathbf{I} - \mathbf{A}$ is due to the **uniqueness** of a solution v of (H, BC, 19): this results from the uniqueness for the original problem. The **surjectivity** is related to the **existence** of v. Trace theorems require $\{\mathbf{z}\} \in H^{-1/2}(\Sigma)$, which holds since $s \leq 1/2$, and $T^{-1}[\mathbf{z}] \in H^{1/2}(\Sigma)$. However, (8) only ensures $T^{-1}[\mathbf{z}] \in H^s(\Sigma)$: we recover the **misfit** mentioned in section 1 unless $s = 1/2$:

Theorem 1 *The operator $\mathbf{I} - \mathbf{A}$ is injective in $\mathbf{V}$ and it is surjective if and only if $s = 1/2$. In this case, by Banach theorem, there exists $\delta > 0$ such that*

$$\forall\, \mathbf{x} \in \mathbf{V}, \quad \|(\mathbf{I} - \mathbf{A})\,\mathbf{x}\| \geq \delta\, \|\mathbf{x}\|, \quad \textit{(with } \delta \leq 2 \textit{ because } \mathbf{A} \textit{ is contractant)} \tag{20}$$

Theorem 1 implies that, when $s = 1/2$, the interface problem (15) is a nice coercive problem in $\mathbf{V}$ (the lack of H^1-coercivity - emphasized in section 1 - is hidden in the definition of $\mathbf{A}$). Indeed, from $\mathbf{A}\mathbf{x} = \mathbf{x} - (\mathbf{I} - \mathbf{A})\,\mathbf{x}$, we get (take the square norms) $\|\mathbf{A}\mathbf{x}\|^2 = \|\mathbf{x}\|^2 + \|(\mathbf{I} - \mathbf{A})\,\mathbf{x}\|^2 - 2\,\mathcal{R}e\,((\mathbf{I} - \mathbf{A})\,\mathbf{x}, \mathbf{x})$. Since $\|\mathbf{A}\mathbf{x}\|^2 \leq \|\mathbf{x}\|^2$, we deduce

$$\forall\, \mathbf{x} \in \mathbf{V}, \quad \mathcal{R}e\,((\mathbf{I} - \mathbf{A})\,\mathbf{x}, \mathbf{x}) \geq (1/2)\, \|(\mathbf{I} - \mathbf{A})\,\mathbf{x}\|^2 \geq (\delta^2/2)\, \|\mathbf{x}\|^2. \tag{21}$$

Convergence. We go back to the iterative method (LP) + (10). If $\mathbf{x}^n := (x_1^n, x_2^n)$ with $x_j^n := \partial_{\nu_j} u_j^n + i\,\omega\, T\, u_j^n$, one easily sees that $\mathbf{x}^n$ satisfies the following Richardson algorithm (or relaxed Jacobi in reference with the block form (16) of $\mathbf{I} - \mathbf{A}$):

$$\mathbf{x}^n = (1 - r)\,\mathbf{x}^{n-1} + r\,\mathbf{A}\,\mathbf{x}^{n-1} + \mathbf{g} \tag{22}$$

The error $\mathbf{e}^n = \mathbf{x}^n - \mathbf{x}$ satisfies $\mathbf{e}^n = (1-r)\,\mathbf{e}^{n-1} + r\,\mathbf{A}\,\mathbf{e}^{n-1}$ $(*)$. From the identity $\|(1-r)\,\mathbf{x} + r\,\mathbf{y}\|^2 = (1-r)\,\|\mathbf{x}\|^2 + r\,\|\mathbf{y}\|^2 - r(1-r)\,\|\mathbf{x}-\mathbf{y}\|^2$, we thus get

$$\left|\begin{array}{l} \|\mathbf{e}^n\|^2 = (1-r)\,\|\mathbf{e}^{n-1}\|^2 + r\,\|\mathbf{A}\mathbf{e}^{n-1}\|^2 - r(1-r)\,\|(\mathbf{I}-\mathbf{A})\,\mathbf{e}^{n-1}\|^2 \\ \qquad\qquad \le \|\mathbf{e}^{n-1}\|^2 - r(1-r)\,\|(\mathbf{I}-\mathbf{A})\,\mathbf{e}^{n-1}\|^2, \quad \text{(contractivity of } \mathbf{A}). \end{array}\right. \tag{23}$$

Thus $\|\mathbf{e}^n\|$ decreases and $\|(\mathbf{I}-\mathbf{A})\,\mathbf{e}^n\| \to 0$. By weak compactness in $\mathbf{V}$, at least for a subsequence, $\mathbf{e}^n \rightharpoonup \mathbf{e}$ (weakly) in $\mathbf{V}$. So $(\mathbf{I}-\mathbf{A})\,\mathbf{e} = 0$ thus (injectivity of $\mathbf{I}-\mathbf{A}$) $\mathbf{e} = 0$. This being true for any such subsequence, the whole sequence $\mathbf{e}^n$ converges and it is easy to infer that $(u_1^n, u_2^n) \to (u_1, u_2)$ in $L^2(\Omega_1) \times L^2(\Omega_2)$.

However, in the case $s = 1/2$, we have better since, using (21) again in (23)

$$\|\mathbf{e}^n\| \le \tau^n\,\|\mathbf{e}^0\|, \quad \tau := \sqrt{1 - r(1-r)\,\delta^2} < 1, \tag{24}$$

i. e. the iterative algorithm converges linearly provided $s = 1/2$ and $0 < r < 1$.

GMRES algorithm. One can of course use more sophisticated algorithms than (22) to update the interface unknowns $\mathbf{x}^n$ (from which (u_1^n, u_2^n) are still reconstructed via the local problems (12)). This includes nonlinear algorithms such as `GMRES` [21], in which $\mathbf{x}^n$ is computed by minimizing $\mathbf{y} \mapsto \|(\mathbf{I}-\mathbf{A})\,\mathbf{y} - \mathbf{g}\|^2$, the square $\mathbf{V}$-norm of the residue, over the Krylov subspace generated by the n first iterates $\mathbf{x}^k, k \le n-1$ [9] . As a consequence, the corresponding error $\mathbf{e}^n$ is such that

$$\left\|(\mathbf{I}-\mathbf{A})\,\mathbf{e}^n\right\| = \min_{p \in \mathbb{P}_n} \left\|(\mathbf{I}-\mathbf{A})\,p(\mathbf{A})\,\mathbf{e}^0\right\|, \qquad \mathbb{P}_n = \{\text{ polynomials of degree} \le n\}$$

Considering the polynomial $P(a) = (1 - r + r\,a)^n$, which corresponds to the Jacobi's algorithm (22), we deduce from Theorem 1, (24) and $\|\mathbf{I}-\mathbf{A}\| \le 2$ that, if $s = 1/2$,

$$\|\mathbf{e}^n\| \le (2/\delta)\left\|(\mathbf{I}-\mathbf{A})\,\mathbf{e}^n\right\| \le (2/\delta)\,\tau^n \quad \text{with } \tau \text{ as in (24)},$$

which means that the convergence rate of the `GMRES` algorithm if necessarily better than with (22). Numerical evidence show that it is strictly better and that it is worthwhile using `GMRES` despite the larger computational cost for each iteration.

4 Construction of appropriate impedance operators

According to what preceeds, the question is to construct an impedance operator T satisfying (8) with $s = 1/2$, i. e. a positive selfadjoint pseudo-differential operator of order 1. A first mathematical fact is that such an operator **cannot be a local operator** in the sense of section 2: this is clearly demonstrated in 2D circular geometries [4] with a Fourier modal expansion in the azimuthal variable θ . On the other hand, there exist many ways to construct good nonlocal operators. Let us describe some of them (see also [16], [4], [20]).

From Sobolev norms (A). The operator T is entirely defined by the scalar product (5), which is used for finite elements. A first choice is the following (if $\Omega \subset \mathbb{R}^3$):

$$\alpha \int_\Sigma \varphi\, \psi \, d\sigma + \frac{\beta}{\omega} \iint_\Sigma \chi\Big(\frac{|x-y|}{L}\Big) \frac{\big(\varphi(x)-\varphi(y)\big)\big(\overline{\psi(x)}-\overline{\psi(y)}\big)}{|x-y|^3}\, d\sigma_x d\sigma_y \qquad (25)$$

with $\alpha, \beta > 0$, $\chi(r) \geq 0$ a C^1 cut off function with support in $[0,1]$ and $\chi(r) = 1$ for $r < 1/2$, and and $L > 0$. If $L = +\infty$, T is fully nonlocal and one recovers the usual Gagliardo-Niremberg norm in $H^{1/2}(\Sigma)$ if $\alpha = \beta = 1$. If not, T only couples points at a distance less than L and the (discretized) impedance condition is less costly.

From potential theory (B). An automatic way to build a good impedance operator is to take $T = \Lambda^*\Lambda$, with Λ an isomorphism from $H^{1/2}(\Sigma)$ in $L^2(\Sigma)$ provided by a Riesz-type potential : given $a, b > 0$, the associated bilinear form is given by

$$a \int_\Sigma \varphi\, \psi \, d\sigma + \frac{b}{\sqrt{\omega}} \iint_\Sigma \chi\Big(\frac{|x-y|}{L}\Big) \frac{\mathrm{rot}_\Sigma \varphi(x) \cdot \mathrm{rot}_\Sigma \overline{\psi}(y)}{|x-y|^{1/2}}\, d\sigma_x d\sigma_y \qquad (26)$$

where rot_Σ denotes the usual tangential curl operator on Σ. Such operators are familiar to specialists of boundary integral equations, except the non standard exponent $1/2$ which ensures that Λ is of order $1/2$. Contrary to **(A)**, Alternative **(B)** can be extended to Maxwell's equations [20]. In separable geometries, the convergence of (22) for **(A)** or **(B)** can be precisely quantified via a modal decomposition. This analysis also permits us to show that a good choice for L is $L \sim \lambda/2$ [4].

From local elliptic DtN operators (C). A more recently investigated option consists in building $T\varphi$ from the solution v^φ of an auxiliary elliptic problem posed in a a layer $C_1 \cup C_2$ surrounding the interface Σ (Figure 2): given $B = I, \partial_\nu$ or $I + \omega^{-1}\,\partial_\nu$ (it can be shown [20] that the Robin operator $I + \omega^{-1}\,\partial_\nu$ is the best choice)

$$\begin{cases} T\varphi := \frac{1}{2}\left(\partial_{\nu_1} v_1^\varphi + \partial_{\nu_2} v_2^\varphi\right) \\ v_1^\varphi = v^\varphi|_{\Omega_1}, v_2^\varphi = v^\varphi|_{\Omega_2} \end{cases} \quad \text{where} \quad \begin{cases} -\Delta v + \omega^2\, v^\varphi = 0, & \text{in } C_1 \cup C_2, \\ v^\varphi = \varphi, & \text{on } \Sigma, \\ B\, v^\varphi = 0 & \text{on } \Sigma_j,\ j = 1,2 \end{cases} \qquad (27)$$

One advantage of such a DtN operator is that it is perfectly adapted to variable coefficients and other types of equations. Moreover it gives very good performances in practice. Let us consider the experiment of the scattering of a plane wave by a circular disk (see Figure 4) : the interface is a circle of radius R and $\omega\, R = 9$. We use P_1 finite elements on a meshstep $h = 2\pi/(40\omega)$ and 0 as the initial guess. In Figure 4, we show the evolution of the relative $H^1(\Omega_1 \cup \Omega_2)$ norm of the error $u_h^n - u_h$, u_h being the solution of the undecomposed discrete problem, as a function of n for $T = I$ and T given by **(B)** or **(C)** with $C_1 = \Omega_1$ (red domain) $C_2 = \Omega_2$ (blue domain). This clearly shows the interest non local versus local and the one of the strategy **(C)** with respect to **(B)**. The picture on the right shows that, with nonlocal operators, the number of iterations needed for reaching a given tolerance is independent of h (this can be proven, see [8] and reflects the linear convergence for the continuous problem) while, if $T = I$ (or more generally any local operator) it increases when one refines the mesh. In figure 4, we show the spatial structure of the error after 80 iterations (be careful the scales are different in the two pictures). With $T = I$, the error concentrates near the interface and highly oscillates (from one mesh point

to the other) along the interface. This is representative of the incapacity of local operators to produce linear convergence at the continuous level and explained in circular geometry by the Fourier azimuthal analysis : the modal convergence rate τ_m for the m^{th} mode in θ tends to 1 for large m. With the DtN operator, the error does not concentrate and oscillates, as explained again by the modal analysis, at the (quasi)-**resonant mode** : observe the $m = 9$ lobes $\Leftrightarrow \omega R = 9$.

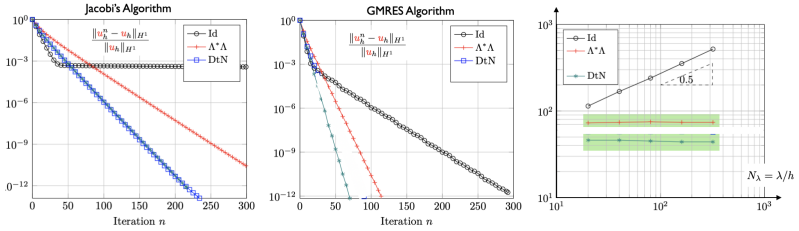

Fig. 2 Convergence histories (left and center). Iteration count versus mesh size (right)

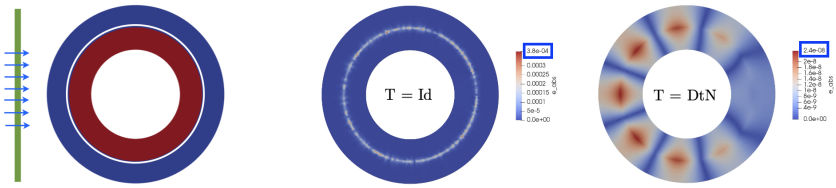

Fig. 3 Left : the experiment. Center, right : the errors after 80 iterations (the color bars differ !)

5 The problem of cross points

Consider now a partition of Ω into $N \geq 2$ subdomains Ω_j, where, for simplicity, Ω_N is an exterior layer, with the possibility that more than 2 boundaries $\partial\Omega_j$ meet at a so called **cross point**. Such points raise theoretical and practical questions for DDMs, that deserve a special treatment [2, 19, 12]. Denoting Σ_{ij} the interface $\partial\Omega_i \cup \partial\Omega_j$ (possibly empty), the most naïve generalization of the transmission condition (9) consists in writing a transmission problem for $\{u_j\}$ with the transmission conditions

$$\begin{cases} \partial_{\nu_i} u_i + i\,\omega\, T_{ij}\, u_i = -\partial_{\nu_j} u_j + i\,\omega\, T_{ij}\, u_j, & \nu_i \text{ outgoing w.r.t. } \Omega_i, \\ \partial_{\nu_j} u_j + i\,\omega\, T_{ij}\, u_j = -\partial_{\nu_i} u_i + i\,\omega\, T_{ij}\, u_i, & \nu_j \text{ outgoing w.r.t. } \Omega_j, \end{cases} \tag{28}$$

where, aiming at achieving linear convergence, T_{ij} would be a **positive definite self-adjoint** operator from $H^{1/2}(\Sigma_{ij})$ in $H^{-1/2}(\Sigma_{ij})$. In this way, defining x_{ij} on Σ_{ij} similarly as (x_1, x_2) in (11) and $\mathbf{x}$ the collection of the $\{x_{ij}\}$, the transmission problem can be rewritten in an abstract form (15) with a natural generalization of the operator $\mathbf{A}$. The convergence of the DDM algorithm (22) is still guaranteed but the **linear convergence** faces the problem of the surjectivity of $\mathbf{I} - \mathbf{A}$ that relies on the existence of a solution to a generalized jump problem in Ω coupling the Helmholtz equation in each Ω_j with the inhomogeneous jump conditions :

$$\left[w\right]_{\Sigma_{ij}} = \frac{1}{2i\omega}\, T^{-1}(z_{ij} - z_{ji}),\ \left[\partial_n w\right]_{\Sigma_{ij}} = z_{ij} - z_{ji},\ \text{given}\ \ (z_{ij}, z_{ji}) \in H^{-1/2}(\Sigma_{ij}). \tag{29}$$

Unfortunately, the inclusion of $\mathcal{T} := \{ \gamma_J v := [v]_{\Sigma_{ij}} / v_i \in H^1(\Omega_i)\}$ in $\Pi\ H^{1/2}(\Sigma_{ij})$ is **strict**, with infinite codimension, if **cross points** exist [23]. This **defect of surjectivity** of the jump operator γ_J is an obstacle to the first condition in (29): we meet again a functional **misfit** as for the two domains case when $s < 1/2$ in (8).

In [6], a new paradigm was proposed, abandoning the interfaces Σ_{ij} to the profit of the boundaries $\Sigma_i = \partial\Omega_i$ ($i < N$) and $\Sigma_N == \partial\Omega_N \setminus \partial\Omega$ and the skeleton $\boldsymbol{\Sigma} = \cup\, \Sigma_i$. This uses the concept of **multi-traces** developed for multi-domain boundary integral equations[5]: let $\Omega_{\boldsymbol{\Sigma}} := \Omega \setminus \boldsymbol{\Sigma}$ and (γ_D, γ_N) the two **surjective** (multi)-trace operators

$$\begin{cases} u \in H^1(\Omega_{\boldsymbol{\Sigma}}) & \mapsto \gamma_D u = \{u_i|_{\Sigma_i}\} \quad \in \mathcal{M}_D(\boldsymbol{\Sigma}) := \Pi\ H^{\frac{1}{2}}(\Sigma_i), \\ \mathbf{v} \in H(\mathrm{div}, \Omega_{\boldsymbol{\Sigma}}) \mapsto \gamma_N \mathbf{v} = \{\mathbf{v}_i \cdot \nu_i|_{\Gamma_i}\} & \in \mathcal{M}_N(\boldsymbol{\Sigma}) := \Pi\ H^{-\frac{1}{2}}(\Sigma_i). \end{cases} \tag{30}$$

Note that $\mathcal{M}_N(\boldsymbol{\Sigma})$ is the dual space of $\mathcal{M}_D(\boldsymbol{\Sigma})$ and we shall denote $\langle \cdot, \cdot \rangle_{\boldsymbol{\Sigma}}$ the natural duality bracket that extends the $L^2(\boldsymbol{\Sigma})$ inner product. As $H^1(\Omega) \subset H^1(\Omega_{\boldsymbol{\Sigma}})$ and $H(\mathrm{div}\,\Omega) \subset H(\mathrm{div}\,\Omega_{\boldsymbol{\Sigma}})$, we can define

$$\mathcal{S}_D(\boldsymbol{\Sigma}) := \gamma_D \left[H^1(\Omega)\right] \subset \mathcal{M}_D(\boldsymbol{\Sigma}), \quad \mathcal{S}_N(\boldsymbol{\Sigma}) := \gamma_N \left[H(\mathrm{div}, \Omega)\right] \subset \mathcal{M}_N(\boldsymbol{\Sigma}).$$

The idea is to reformulate the classical Dirichlet and Neumann transmission conditions for $u = \{u_i\} \in H^1(\Omega_{\boldsymbol{\Sigma}})$, namely $[u]_{\Sigma_{ij}} = 0$ and $[\partial_\nu u]_{\Sigma_{ij}} = 0$, in a non standard form expressed in terms of the traces $\gamma_D u$ and $\gamma_N(\nabla u)$ that writes

$$-\Delta u_i - \omega^2\, u_i = f, \quad (\mathbf{D}) \quad \gamma_D u \in \mathcal{S}_D(\boldsymbol{\Sigma}), \quad (\mathbf{N}) \quad \gamma_N(\nabla u) \in \mathcal{S}_N(\boldsymbol{\Sigma}).$$

To recover the framework of section 3, we first express (**D**) and (**N**) in an impedance form. To do so, we introduce **positive selfadjoint** impedance operators associated to the Σ_i's (and no longer the Σ_{ij}'s), $T_i \in \mathcal{L}(H^{1/2}(\Sigma_i), H^{-1/2}(\Sigma_i))$, where each T_i is an isomorphism, so that, if $\mathbf{T} = \mathrm{diag}\ T_i \in \mathcal{L}(\mathcal{M}_D(\boldsymbol{\Sigma}), \mathcal{M}_N(\boldsymbol{\Sigma}))$,

$$(\boldsymbol{\varphi}, \boldsymbol{\psi}) := \langle \boldsymbol{\varphi}, \mathbf{T}^{-1}\, \boldsymbol{\psi} \rangle_{\boldsymbol{\Sigma}} \quad \text{is an Hilbert inner product in } \mathbf{V} := \mathcal{M}_N(\boldsymbol{\Sigma}). \tag{31}$$

Mimicking (11), we set $(S) : \mathbf{x} := \gamma_N \nabla u + i\,\omega\, \mathbf{T} \gamma_D u$ and $\mathbf{y} := -\gamma_N \nabla u + i\,\omega\, \mathbf{T} \gamma_D u$, the skeleton unknowns in $\mathbf{V}$. Let $\mathbf{S} = \mathrm{diag}\ S_i \in \mathcal{L}(\mathcal{M}_N(\boldsymbol{\Sigma}))$ where each S_i is defined as in (12) (in Ω_i and T_i instead of T). Each S_i is isometric for the T_i-norm - (5) for $T = T_i$ - except $\mathcal{S}_N$ which is contractant. The Helmholtz equations in Ω_i rewrites as (13), namely $\mathbf{y} = \mathbf{S}\mathbf{x} + \widetilde{\mathbf{g}}$. It then remains to account for (**D**) and (**N**). This relies on a key result of [5] characterizing $\mathcal{S}_D(\boldsymbol{\Sigma})$ and $\mathcal{S}_N(\boldsymbol{\Sigma})$ as "orthogonal" to each other:

Lemma 1 *[5] Let $\boldsymbol{\varphi} \in \mathcal{M}_D(\boldsymbol{\Sigma})$ and $\boldsymbol{\psi} \in \mathcal{M}_N(\boldsymbol{\Sigma})$). Then*

$$(i) \quad \boldsymbol{\varphi} \in \mathcal{S}_D(\boldsymbol{\Sigma}) \iff \langle \boldsymbol{\psi}_N, \boldsymbol{\varphi} \rangle_{\boldsymbol{\Sigma}} = 0, \quad \forall\, \boldsymbol{\psi}_N \in \mathcal{S}_N(\boldsymbol{\Sigma}),$$
$$(ii) \quad \boldsymbol{\psi} \in \mathcal{S}_N(\boldsymbol{\Sigma}) \iff \langle \boldsymbol{\psi}, \boldsymbol{\varphi}_N \rangle_{\boldsymbol{\Sigma}} = 0, \quad \forall\, \boldsymbol{\varphi}_D \in \mathcal{S}_D(\boldsymbol{\Sigma}).$$

This lemma is a direct consequence of Green's identity, in which the left hand side vanishes if $u \in H^1(\mathbb{R}^d)$ or $\mathbf{v} \in H(\mathrm{div}, \mathbb{R}^d)$ (below $\mathbb{R}^d_{\boldsymbol{\Sigma}} = \mathbb{R}^d \setminus \boldsymbol{\Sigma}$):

$$\forall\, (u, \mathbf{v}) \in H^1(\mathbb{R}^d_{\boldsymbol{\Sigma}}) \times H(\mathrm{div}, \mathbb{R}^d_{\boldsymbol{\Sigma}}), \quad \sum_i \int_{\Omega_i} (\nabla u_i \cdot \mathbf{v}_i + u_i\, \mathrm{div}\ \mathbf{v}_i) = \langle \gamma_N \mathbf{v}, \gamma_D u \rangle_{\boldsymbol{\Sigma}}.$$

Theorem 2 *[6] Let* $\mathbf{P}_N$ *the orthogonal projector (in* $\mathcal{M}_N(\mathbf{\Sigma})$ *equipped with* (31)*) on* $\mathcal{S}_N(\mathbf{\Sigma})$*. The transmission conditions* (**D**) *and* (**N**) *are satisfied if and only if the unknowns* $\mathbf{x}$ *and* $\mathbf{y}$ *are related by* $\mathbf{y} = \mathbf{\Pi}\,\mathbf{x}$ *where* $\mathbf{\Pi} = \mathbf{I} - 2\,\mathbf{P}_N$*.*

Proof Let $\boldsymbol{\varphi} := \gamma_D u$ and $\boldsymbol{\psi} := \gamma_N u$. By (S), (**N**) is equivalent to $\mathbf{y} - \mathbf{x} \in \mathcal{S}_N(\mathbf{\Sigma})$ while (**D**) is equivalent to $\mathbf{T}^{-1}(\mathbf{x} + \mathbf{y}) \in \mathcal{S}_D(\mathbf{\Sigma})$ that is to say, by Lemma 1 and (31), to $(\mathbf{y} + \mathbf{x}, \boldsymbol{\psi}_N) = 0,\ \forall\, \boldsymbol{\psi}_N \in \mathcal{S}_N(\mathbf{\Sigma})$. Thus, writing $\mathbf{y} + \mathbf{x} = (\mathbf{y} - \mathbf{x}) + 2\mathbf{x}$, this gives

$$\big((\mathbf{y} - \mathbf{x}) + 2\mathbf{x}, \boldsymbol{\psi}_N\big) = 0, \quad \forall\, \boldsymbol{\psi}_N \in \mathcal{S}_N(\mathbf{\Sigma}). \tag{32}$$

Since $\mathbf{y} - \mathbf{x} \in \mathcal{S}_N(\mathbf{\Sigma})$, this is nothing but $\mathbf{y} - \mathbf{x} = \mathbf{P}_N\,(-2\mathbf{x})$. □

Proceeding as in section 3 to eliminate $\mathbf{y}$, the problem in $\mathbf{x}$ rewrites as in (15), with $\mathbf{V} := \mathcal{M}_N(\mathbf{\Sigma})$ and $\mathbf{A} = \mathbf{\Pi}\,\mathbf{S}$, the exchange operator (16) being replaced by $\mathbf{\Pi} = \mathbf{I} - 2\,\mathbf{P}_N$. The reader will notice that, as the exchange operator, $\mathbf{\Pi}$ is **isometric** and **involutive**. As a consequence, $\mathbf{A}$ is contractant. The invertibility of $\mathbf{I} - \mathbf{A}$ is linked to a generalized jump problem across the skeleton (instead of (29)) whose existence of a solution is ensured by the **surjectivity** of γ_D and γ_N (30): the misfit due to the defect of surjectivity of the operator γ_J in the interface approach, has been eliminated. The conditions for linear convergence of (22) are thus satisfied.

It is worthwhile mentioning that the evaluation of $\mathbf{\Pi x}$ amounts to solving the (coercive and $\mathbf{T}$ dependent) variational problem (32) on $\mathbf{\Sigma}$ for $\mathbf{y} - \mathbf{x}$. Even though each T_i is local to Σ_i, being posed in $\mathcal{S}_N(\mathbf{\Sigma})$, the problem is non local over $\mathbf{\Sigma}$. Thus, $\mathbf{\Pi x}$ couples all Σ_i's : rather than an exchange across interfaces, it is a **communication** operator (but without cross point a "natural" choice for T_i gives back the exchange). Working in $\mathbf{V} = \mathcal{M}_N(\mathbf{\Sigma})$ means that the Neumann condition (**N**) is handled in a **strong sense** while the Dirichlet one (**D**) is handled **weakly** via (32). The (dual) opposite choice is possible, see [7]. In our case, the space discretization of the problem uses a finite element space $V_h(\Omega)$ for $H(\mathrm{div}, \Omega)$ and a natural candidate for an approximation space of $\mathcal{S}_N(\mathbf{\Sigma})$ is $\mathcal{S}_N^h(\mathbf{\Sigma}) := \gamma_N[V_h(\Omega)]$. In figure 5, we demonstrate that the developments of this section are not only a question of mathematical beauty. On the model problem of section 4 and a partition of Ω into 10 subdomains with one cross point, we compare Després's condition (a), non local interface operators T_{ij} (b) and finally the **multi-trace** method (c) showing the error after 10 iterations. In case (b), we see that the non local interface operators solve most of the problems with $T = I$ but produce an important error (the big peak) concentrated around the cross point, error which is eliminated with the **multi-trace** strategy !

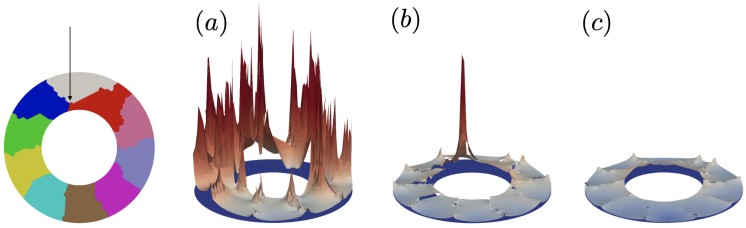

Fig. 4 Left : the 10 subdomains with one cross point (the arrow). Right : the errors after 10 iterations

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