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The Langlands Program and String Modular K3 Surfaces

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ABSTRACT:

A number theoretic approach to string compactification is developed for Calabi-Yau hypersurfaces in arbitrary dimensions. The motivic strategy involved is illustrated by showing that the Hecke eigenforms derived from Galois group orbits of the holomorphic two-form of a particular type of K3 surfaces can be expressed in terms of modular forms constructed from the worldsheet theory. The process of deriving string physics from spacetime geometry can be reversed, allowing the construction of K3 surface geometry from the string characters of the partition function. A general argument for K3 modularity follows from mirror symmetry, in combination with the proof of the Shimura-Taniyama conjecture.

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1 Introduction

One of the intriguing aspects of string theory is the possibility of understanding the structure of spacetime from first principles in terms of the physics of the worldsheet. In the past a number of different techniques have been used to address this question, e.g. Landau-Ginzburg theories and non-linear sigma models. The aim of the present paper is to continue a different program that uses methods from arithmetic geometry to understand this problem in the context of exactly solvable Calabi-Yau varieties. The structure of higher dimensional Calabi-Yau varieties is quite intricate and there are several aspects of these theories that lend themselves to an arithmetic analysis. The focus here will be on formulating a framework that combines methods from algebraic number theory and arithmetic geometry in the context of Calabi-Yau hypersurfaces of arbitrary dimensions. This general approach is then applied to a special class of K3 surfaces by showing that basic building blocks of the underlying string partition functions can be derived from the geometry of these K3s.

The simplest case in which the idea of using arithmetic geometry to derive worldsheet information from geometry can be tested is provided by the framework of toroidal compactifications. Elliptic curves provide useful examples because of the proof of the Shimura-Taniyama conjecture. This modularity theorem states that all elliptic curves defined over the rational number field are modular in the sense that the Mellin transform of their Hasse-Weil L-series defines a modular form of weight two with respect to some congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ [1, 2, 3, 4]. In the context of exactly solvable elliptic curves the issue then becomes whether the modular forms derived from these curves can be expressed in terms of modular forms derived from the underlying superconformal field theory. It was shown in refs. [5, 6, 7] that this is possible in terms of cusp forms constructed from the string functions associated to the affine Lie algebra $A_1^{(1)}$ of the $N=2$ superconformal minimal models.

The generalization of this result to higher genus curves and higher dimensional varieties is made difficult by the fact that no analog of the elliptic modularity theorem is known, even conjecturally. There is a general expectation, often summarized as part of the Langlands program [8], that many varieties will lead to modular forms, but there is no known systematic procedure which provides guidance how this should be accomplished. There is in particular no known higher dimensional analog of Weil's experimental observation [9] concerning the geometric interpretation of the level of a modular form in the context of elliptic curves, an ingredient that galvanized interest in the Shimura-Taniyama conjecture, culminating in Wiles' breakthrough. For higher genus curves of Brieskorn-Pham type the situation is a little better because one can use a known factorization procedure to decompose the associated Jacobian variety into simple abelian factors, which then can be tested for modularity [6].

In general the L-function of a variety will not be an interesting object in the context of recovering string theoretic modular forms. Except for particularly simple varieties, such as rigid Calabi-Yau threefolds, for which many modular forms have been identified (ref. [10] contains articles with many further references), the L-function itself does not lead to a modular form. It is known, from Grothendieck's proof [11] of part of the Weil conjectures [12], that Artin's congruent zeta function factorizes in a way that is determined by the cohomology groups of the variety. The zeta function at a prime p decomposes into the quotient

$$Z(X/\mathbb{F}_p, t) = \frac{\mathcal{P}_p^1(t)\mathcal{P}_p^3(t)\cdots\mathcal{P}_p^{2n-1}(t)}{\mathcal{P}_p^0(t)\mathcal{P}_p^2(t)\cdots\mathcal{P}_p^{2n}(t)}, \tag{1}$$

where $\dim_{\mathbb{C}} X = n$, and $\mathcal{P}_p^i(t)$ is a polynomial

$$\mathcal{P}_p^i(t) = \sum_{j=0}^{b^i} \beta_j^i(p) t^j \tag{2}$$

associated to the $i^{\rm th}$ cohomology group, with a degree $b^i = \dim\, \mathrm{H}^i(X)$ given by the $i^{\rm th}$ Betti number. This result motivates the introduction of L$-$functions associated to the individual cohomology groups, thereby reducing the complexity of the zeta function. Even though this factorization provides an important simplification, it is not enough for string theory. The individual cohomology groups can be quite complicated because the Betti numbers of Calabi-Yau varieties tend to be large. The idea in this paper is to factorize these polynomials further, and to consider L$-$functions associated to the resulting factors. The problem that arises is that it is unclear a priori which type of factorization leads to a physically meaningful L-function.

In the absence of a clear understanding of what the conditions are in higher dimensions that can lead to string theoretic modular forms on the worldsheet, it is useful to identify selection rules that guide the factorization of the L-functions. There are several ways to think about this problem, and in the present paper the following point of view will be adopted. The first idea is utilitarian in nature, guided by the expectation that the results of [5, 6, 7], or some not too radical modification thereof, will generalize to higher dimensions. It follows from those results that the string theoretic modular forms relevant for the arithmetic approach are forms with coefficients that are rational integers. The primitive factors of the polynomials $\mathcal{P}_p^i(t)$ that arise from the complete factorization

$$\mathcal{P}_p^i(t) = \prod_{j=1}^{b^i}(1-\gamma_j^i(p)t), \tag{3}$$

lead to L-function factors with coefficients $\gamma_j^i(p)$ that are algebraic integers in number fields, defined by extensions of the rational number field $\mathbb{Q}$. This shows that a complete factorization is not useful in the present context, and that from a practical point of view one should be guided by the idea of identifying pieces of the cohomology that lead to forms with coefficients in $\mathbb{Z}$.

A more conceptual way of thinking about this problem is representation theoretic. Associated to a Calabi-Yau hypersurface in a toric variety is a cyclotomic number field, which for Brieskorn-Pham spaces admits an interpretation as the fusion field of the underlying conformal field theory [13]. The Galois group of this field is a finite cyclic group which acts on the cohomology of the variety. The action of this group is reducible in general, and therefore one can use it to decompose the cohomology group into pieces defined by the irreducible representations of the group. A possible strategy therefore is to focus on the L-functions associated to these irreducible representations of the Galois group. A distinguished element in the cohomology ring of any $n-$dimensional Calabi-Yau variety is the holomorphic $n-$form, leading to the concept of an $\Omega-$motive of a Calabi-Yau variety. The notion of such a motive is general, and the question addressed here is whether this motive is string modular in some sense. The approach formulated therefore provides a general method to gain a better understanding of the relation between the geometry of spacetime in string theory and the physics on the worldsheet.

In the present paper the strategy just described is illustrated by considering the class of extremal K3 surfaces of Brieskorn-Pham type, i.e. surfaces S which over the complex field $\mathbb{C}$ have maximal Picard number $\rho(S) = 20$. These surfaces are of the form

$$\begin{aligned} S^4 &= \left\{(z_0 : \cdots : z_3) \in \mathbb{P}_3 \;\middle|\; z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\right\}, \\ S^{6\mathrm{A}} &= \left\{(z_0 : \cdots : z_3) \in \mathbb{P}_{(1,1,1,3)} \;\middle|\; z_0^6 + z_1^6 + z_2^6 + z_3^2 = 0\right\}, \\ S^{6\mathrm{B}} &= \left\{(z_0 : \cdots : z_3) \in \mathbb{P}_{(1,1,2,2)} \;\middle|\; z_0^6 + z_1^6 + z_2^3 + z_3^3 = 0\right\}. \end{aligned} \tag{4}$$

In order to state the results of this analysis some notation is needed. With $q = e^{2\pi i\tau}$, let $f(q) = \sum_n a_n q^n$ be a cusp form, and $\vee^2 f(q) = \sum_n b_n q^n$ be the product given by $b_p = a_p^2 - 2p$. The motivation for this definition will become clear below. Define the Hecke congruence subgroup $\Gamma_0(N)$ as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}, \tag{5}$$

and denote the Galois group of the cyclotomic number field $\mathbb{Q}(\mu_d)$ by $\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})$, were

μ_d is the cyclic group generated by a primitive d^{th} root of unity. The Dedekind eta function is given by $\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, while the theta functions $\Theta^k_{\ell,m}(\tau) = \eta^3(\tau)c^k_{\ell,m}(\tau)$ are Hecke indefinite modular forms associated to the Kac-Peterson string functions $c^k_{\ell,m}(\tau)$ of the affine algebra $A_1^{(1)}$ at level k. Finally, the quadratic characters determined by the Legendre symbol are written as $\chi_n(p) = (\frac{n}{p})$. $\vartheta(q)$ is a modular form of weight one described further below. The following results will be shown.

Theorem 1. *Let $M_\Omega \subset H^2(S^d)$ be the irreducible representation of* $\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})$ *associated to the holomorphic 2−form $\Omega \in H^{2,0}(S^d)$ of the K3 surface S^d, where $d = 4, 6\mathrm{A}, 6\mathrm{B}$. Then the Mellin transforms $f_\Omega(S^d, q)$ of the L-functions $L_\Omega(S^d, s)$ associated to M_Ω are given by*

$$\begin{aligned} f_\Omega(S^4, q) &= \eta^6(q^4) \\ f_\Omega(S^{6\mathrm{A}}, q) &= \vartheta(q^3)\eta^2(q^3)\eta^2(q^9) \\ f_\Omega(S^{6\mathrm{B}}, q) &= \eta^3(q^2)\eta^3(q^6) \otimes \chi_3. \end{aligned} \tag{6}$$

These functions are cusp forms of weight three with respect to $\Gamma_0(N)$ with levels $16, 27$ and 48, respectively. For S^4 and $S^{6\mathrm{A}}$ the L-functions can be written as $L_\Omega(S^d, s) = L(\vee^2 f_d, s)$, where $f_d(q)$ are cusp forms of weight two and levels 64 and 27, respectively, given by

$$\begin{aligned} f_4(\tau) &= \Theta^2_{1,1}(4\tau)^2 \otimes \chi_2 \\ f_{6\mathrm{A}}(\tau) &= \Theta^1_{1,1}(3\tau)\Theta^1_{1,1}(9\tau). \end{aligned} \tag{7}$$

For $S^{6\mathrm{B}}$ the L-series is given by $L_\Omega(S^{6\mathrm{B}}, s) = L(\vee^2 f_{6\mathrm{B}} \otimes \chi_3, s)$ in terms of the cusp form of level 144

$$f_{6\mathrm{B}}(\tau) = \Theta^1_{1,1}(6\tau)^2 \otimes \chi_3. \tag{8}$$

Physically, this result proves a string theoretic interpretation of the motivic L-function associated to the holomorphic Ω−form of K3 surfaces in terms of the affine Lie algebra $A_1^{(1)}$ on the worldsheet. It generalizes results in lower dimensions for the Hasse-Weil L-function of elliptic Brieskorn-Pham curves obtained in [5, 6, 7]. Mathematically, it can be viewed as providing a motivic interpretation of modular forms derived from Kac-Moody algebras, i.e. it provides a string theoretic origin of modular motives for a class of K3s.

Corollary. *The Ω−motivic modular forms of extremal K3 surfaces of Brieskorn-Pham type are twisted products of Hecke indefinite modular forms that arise from Kac-Peterson string functions.*

The modular forms of weight two that appear in the theorem as building blocks of the arithmetic structure of extremal Brieskorn-Pham K3 surfaces are all elliptic, i.e. the Mellin transforms of Hasse-Weil L-series of elliptic curves. Consider the class of elliptic Brieskorn-Pham curves given by

$$
\begin{aligned}
E^3 &= \left\{(z_0 : z_1 : z_2) \in \mathbb{P}_2 \mid z_0^3 + z_1^3 + z_2^3 = 0\right\} \\
E^4 &= \left\{(z_0 : z_1 : z_2) \in \mathbb{P}_{(1,1,2)} \mid z_0^4 + z_1^4 + z_2^2 = 0\right\} \\
E^6 &= \left\{(z_0 : z_1 : z_2) \in \mathbb{P}_{(1,2,3)} \mid z_0^6 + z_1^3 + z_2^2 = 0\right\},
\end{aligned}
\tag{9}
$$

All three curves are string modular in the following sense [7].

Theorem 2. *The Mellin transforms $f_{\mathrm{HW}}(E^d, q)$ of the Hasse-Weil L-functions $L_{\mathrm{HW}}(E^d, s)$ of the curves $E^d, d = 3, 4, 6$ are modular forms $f_{\mathrm{HW}}(E^d, q) \in S_2(\Gamma_0(N))$, with $N \in \{27, 64, 144\}$, respectively. These forms factor into products of Hecke indefinite modular forms as follows*

$$
\begin{aligned}
f_{\mathrm{HW}}(E^3, q) &= \Theta^1_{1,1}(q^3)\Theta^1_{1,1}(q^9) \\
f_{\mathrm{HW}}(E^4, q) &= \Theta^2_{1,1}(q^4)^2 \otimes \chi_2 \\
f_{\mathrm{HW}}(E^6, q) &= \Theta^1_{1,1}(q^6)^2 \otimes \chi_3.
\end{aligned}
\tag{10}
$$

This shows that the string theoretic nature of the modular forms of extremal Brieskorn-Pham K3 surfaces is induced by the string theoretic modularity of elliptic Brieskorn-Pham curves.

Given the central nature of modularity in string theory, it is natural to ask how general modularity is for K3 surfaces. An argument that points to modularity as a common property can be made by combining mirror symmetry with the elliptic modularity theorem of [4]. The original explicit observation of mirror symmetry [14, 15] has been interpreted in a number of different ways, e.g. in terms of toric geometry by Batyrev [16], in a homological context by Kontsevich [17], and in terms of fibrations by Strominger, Yau and Zaslow [18]. It is the latter framework that is most useful for K3 modularity. The idea of the SYZ conjecture is based on a toroidal fibration structure of general Calabi-Yau varieties that is suggested by D-branes. For complex dimension two this conjecture implies that mirror pairs of K3 surfaces are characterized by fibrations in terms of elliptic curves. For elliptic curves defined over $\mathbb{Q}$ the modularity theorem proven by Breuil, Conrad, Diamond and Taylor shows that every elliptic curve is modular in the sense that the Mellin transform of its associated Hasse-Weil L-function is a modular form of weight 2 with respect to a congruence group that is determined by the conductor of the curve. The SYZ conjecture and the modularity theorem therefore imply that mirror pairs of K3 surfaces defined over $\mathbb{Q}$ are modular.

The outline of the paper is as follows. Section 2 contains the arguments for considering irreducible Galois representations as the modular building blocks of Calabi-Yau varieties.

These arguments are general, not restricted to K3 surfaces. Section 3 computes the L-series of the extremal K3 surfaces of Brieskorn-Pham type. Section 4 leads to the identification of the modular forms derived from these K3 surfaces with modular forms derived from the string worldsheet, completing the identifications of the theorem. Section 5 complements the previous computations by constructing the K3 surfaces via the twist map [19]. The philosophy adopted here is similar to the one used in [6] in the context of higher genus curves. Section 6 describes the identification of a singular K3 surface in terms of string theoretic modular forms. Section 7 generalizes to K3 surfaces aspects of complex multiplication and arithmetic moonshine, discussed in [7] in the context of elliptic curves. The appendix proves the identification of the geometric and the modular L-series to all orders. The proof is based on the method of Faltings-Serre-Livné, and uses results from the representation theory of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of the rational numbers.

2 $\Omega-$Motivic L-functions of Calabi-Yau varieties

2.1 Counting and Jacobi sums

L-functions of projective varieties can be computed in several ways. The most direct procedure starts from the rational form (1) of Artin's congruent zeta function

$$Z(X/\mathbb{F}_p, t) = \exp\left(\sum_{r\geq 1} \#(X/\mathbb{F}_{p^r})\frac{t^r}{r}\right), \tag{11}$$

defined in terms of the degree r extensions $\mathbb{F}_{p^r}$ of the finite field $\mathbb{F}_p$ for any prime p. This factorization of the zeta function leads to the definition of cohomological L-functions

$$L^{(i)}(X,s) = \prod_{p \text{ prime}} \frac{1}{\mathcal{P}_p^i(p^{-s})} \tag{12}$$

associated to the i^{th} cohomology group $H^i(X)$.

The simplest way of computing this function is via a direct count of the number of solutions $N_{r,p} = \#(X/\mathbb{F}_{p^r})$. For the case of K3 surfaces the cohomological form of the zeta function simplifies to

$$Z(X/\mathbb{F}_q, t) = \frac{1}{(1-t)\mathcal{P}_p^2(t)(1-p^2t)}. \tag{13}$$

Expanding this quotient as

$$Z(X/\mathbb{F}_p,t) \;=\; 1+(p^2+1-\beta_1)t+\left[p^4+p^2(1-\beta_1)+\beta_1^2-\beta_1-\beta_2+1\right]t^2+\cdots, \quad (14)$$

and comparing it to the expansion of the exponential, leads to

$$\begin{aligned}\beta_1^2(p) &= 1+p^2-N_{1,p} \\ \beta_2^2(p) &= 1+p^2+p^4-3(1+p^2)N_{1,p}+\frac{1}{2}\left(N_{1,p}^2-N_{2,p}\right) \\ &\vdots\end{aligned} \quad (15)$$

for the polynomial $\mathcal{P}_p^2(t)=\sum_i \beta_i^2(p)t^i$.

For weighted Fermat varieties a second method for obtaining both the cardinalities and the L-function was introduced by Weil [12]. This formulation is particularly useful because it allows to disentangle the complicated cohomological structure of higher dimensional varieties in a systematic way, at least for this special type [20].

Theorem 3. *For a smooth weighted projective surface with degree vector* $\underline{n}=(n_0,...,n_s)$

$$X^{\underline{n}}=\{z_0^{n_0}+z_1^{n_1}+\cdots+z_s^{n_s}=0\}\subset \mathbb{P}_{(k_0,k_1,\ldots,k_s)} \quad (16)$$

defined over the finite field $\mathbb{F}_q$ *define the set* $\mathcal{A}_s^{q,\underline{n}}$ *of rational vectors* $\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_s)$ *as*

$$\mathcal{A}_s^{q,\underline{n}}=\left\{\alpha\in\mathbb{Q}^{s+1}\mid 0<\alpha_i<1,\ d_i=(n_i,q-1),\ d_i\alpha_i=0 \bmod 1,\ \sum_{i=0}^{s}\alpha_i=0 \pmod 1\right\}. \quad (17)$$

For each $(s+1)-$*tuple* α *define the Jacobi sum*

$$j_q(\alpha_0,\alpha_1,\ldots,\alpha_s)=\frac{1}{q-1}\sum_{\substack{u_i\in\mathbb{F}_q\\ u_0+u_1+\cdots+u_s=0}}\chi_{\alpha_0}(u_0)\chi_{\alpha_1}(u_1)\cdots\chi_{\alpha_s}(u_s), \quad (18)$$

where $\chi_{\alpha_i}(u_i)=e^{2\pi i\alpha_i m_i}$ *with integers* m_i *determined via* $u_i=g^{m_i}$*, where* $g\in\mathbb{F}_q$ *is a generator. Then the cardinality of* $X^{\underline{n}}/\mathbb{F}_q$ *is given by*

$$\#(X^{\underline{n}}/\mathbb{F}_q)=N_{1,q}(X^{\underline{n}})=1+q+\cdots+q^{s-1}+\sum_{\alpha\in\mathcal{A}_s^{q,\underline{n}}}j_q(\alpha). \quad (19)$$

With these ingredients the L-function associated to $H^{s-1}(X^{\underline{n}})$ takes the form (12) with polynomials $\mathcal{P}_p^{s-1}(t)$ expressed in terms of the Jacobi-sums variables by

$$\mathcal{P}_p^{s-1}(t)=\prod_{\alpha\in\mathcal{A}_s^{\underline{n}}}\left(1-(-1)^{s-1}j_{p^f}(\alpha)t^f\right)^{1/f} \quad (20)$$

where

$$\mathcal{A}_s^{\underline{n}} = \left\{ \alpha \in \mathbb{Q}^{s+1} \;\Big|\; 0 < \alpha_i < 1,\; n_i\alpha_i = 0 \ (\mathrm{mod}\ 1),\; \sum_{i=0}^{s} \alpha_i = 0 \ (\mathrm{mod}\ 1) \right\} \tag{21}$$

and f is determined via

$$(p^f - 1)\alpha_i = 0 \ (\mathrm{mod}\ 1), \quad \forall i. \tag{22}$$

The main point now is to identify an appropriate factor of the L-function of $H^{s-1}(X^{\underline{n}})$.

2.2 Galois representations and $\Omega-$motives of Calabi-Yau varieties

This section defines the notion of $\Omega-$motives for the general class of Calabi-Yau hypersurfaces. The Jacobi sums $j_{p^f}(\alpha)$ are algebraic numbers in the cyclotomic field $\mathbb{Q}(\mu_d)$, generated by primitive d^{th} roots of unity. The factors $(1 - j_{p^f}(\alpha)t^f)^{1/f}$ therefore do not themselves define string theoretic L-series of the type considered in [5, 6, 7] because the latter have integral coefficients. The simplest way to produce real coefficients in the L$-$function is to combine 'dual' pairs of Jacobi sums, i.e. sums parametrized by (α, α') such that $\alpha + \alpha' = 1$. This sometimes leads to success, but in general the L-function of such pairs has coefficients that are elements of the maximal real subfield of the cyclotomic field. The idea in this paper is to achieve the necessary integrality of the coefficients by considering orbits of Jacobi sums defined by the action of the multiplicative group $(\mathbb{Z}/d\mathbb{Z})^\times$, where $d = k_in_i$, $\forall i$, is the degree of the hypersurface. The fact that these orbits lead to integral coefficients in the L$-$function can be derived from a number theoretic result about finite extensions of number fields by noting that the multiplicative groups $(\mathbb{Z}/d\mathbb{Z})^\times$ can be interpreted as Galois groups of cyclotomic fields $\mathbb{Q}(\mu_d)$. This can be seen as follows (the relevant number theoretic concepts can be found in [21] and [22]).

Jacobi sum orbits for the Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q}) = (\mathbb{Z}/d\mathbb{Z})^\times$ are obtained by defining an action

$$(\mathbb{Z}/d\mathbb{Z})^\times \times \mathcal{A}_s^{\underline{n}} \longrightarrow \mathcal{A}_s^{\underline{n}} \tag{23}$$

as

$$(t, \alpha) = t \cdot \alpha (\mathrm{mod}\ 1), \tag{24}$$

where $t \cdot \alpha = (t\alpha_0, ..., t\alpha_s)$. For any $\alpha \in \mathcal{A}_s^{\underline{n}}$ one can therefore consider its orbit

$$\mathcal{O}_\alpha = \left\{ \beta \in \mathcal{A}_s^{\underline{n}} \mid \beta = t \cdot \alpha,\; t \in (\mathbb{Z}/d\mathbb{Z})^\times \right\}, \tag{25}$$

and decompose the set $\mathcal{A}_s^{\frac{n}{}}$ into a set of orbits

$$\mathcal{A}_s^{\frac{n}{}} = \bigcup_\alpha \mathcal{O}_\alpha, \tag{26}$$

where α is a representative in each orbit.

To see that Galois orbits lead to $L-$series with rationally integral coefficients one observes that the Jacobi sums $j_p(\alpha)$ transform under the action of $t \in (\mathbb{Z}/d\mathbb{Z})^\times$ as

$$j_p(t \cdot \alpha) = \sigma_t(j_p(\alpha)), \tag{27}$$

where $\sigma_t \in \mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})$ is defined as $\sigma_t(\xi_d) = \xi_d^t$. Hence the Galois orbit leads to coefficients a_p of the terms with prime exponents that are of the form

$$a_p(\alpha) = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})} \sigma(j_p(\alpha)). \tag{28}$$

The argument can now be completed by noting that the sum $a_p(\alpha)$ is the trace of an algebraic integer and by using a result from the theory of finite extensions. For any element $x \in L$ in a finite extension L/K over a number field K Dedekind defined the map

$$\begin{aligned} \mathrm{T}_x : L \quad &\longrightarrow \quad L \\ y \quad &\mapsto \quad xy, \end{aligned} \tag{29}$$

which can be used to define a trace map

$$\mathrm{Tr}_{L/K}(x) := \mathrm{tr}\ \mathrm{T}_x. \tag{30}$$

It turns out that this map takes values in the base field K, and that when restricted to the ring $\mathcal{O}_L$ of algebraic integers of L it maps integers to integers

$$\mathrm{Tr}_{L/K} : \mathcal{O}_L \longrightarrow \mathcal{O}_K. \tag{31}$$

The link to the above considerations of Jacobi sums is made by noting that for separable extensions L/K one can show that

$$\mathrm{Tr}_{L/K}(x) = \sum_{\sigma : L \to \overline{K}} \sigma(x) \tag{32}$$

where $\overline{K}$ denotes the closure of K. The cyclotomic fields of interest here are finite extensions of the rational field $\mathbb{Q}$, which is a perfect field, hence all its finite extensions are separable. This means that the trace maps the ring of algebraic integers of any cyclotomic field into the rational integers. It follows that the coefficients a_p in the L$-$series are elements in $\mathbb{Z}$. Cyclotomic fields are normal, therefore the embeddings define the Galois group of automorphisms which leave $\mathbb{Q}$ invariant.

The action defined above induces an action on the middle cohomology of any diagonal hypersurface X^d of degree d embedded in a weighted projective space $\mathbb{P}_{(k_0,\ldots,k_s)}$ with weights k_i because the set $\mathcal{A}_s^{\frac{n}{s}}$ parametrizes the (untwisted) cohomology classes in $H^{s-1}(X^d)$. The strategy to compute the L$-$function of the orbits $\mathcal{O}_\alpha$ determined by the factorization

$$\mathcal{P}_p^{s-1}(t) = \prod_{\mathcal{O}_\alpha} \mathcal{P}_p^{s-1}(\mathcal{O}_\alpha, t), \tag{33}$$

where

$$\mathcal{P}_p^{s-1}(\mathcal{O}_\alpha, t) = \prod_{\beta \in \mathcal{O}_\alpha} \left(1 - j_{p^f}(\beta) t^f\right)^{1/f}, \tag{34}$$

therefore implies the computation of the L-function of the irreducible representations of the Galois group of the cyclotomic field associated to the variety on the intermediate cohomology. From now on the superscript of the polynomial $\mathcal{P}_p^{s-1}(t)$, indicating the cohomology group, will be dropped.

Of particular importance for Calabi-Yau spaces is the orbit $\mathcal{O}_\Omega$ of the element $\alpha_\Omega := \left(\frac{k_0}{d}, \ldots, \frac{k_s}{d}\right)$ associated to the holomorphic $(s-1)-$form. This motivates the introduction of the following two definitions.

Definition. *The Galois orbit $\mathcal{O}_\Omega$ of the element $\alpha_\Omega \in \mathcal{A}_s^{\frac{n}{s}}$ with respect to the action of the Galois group* $\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})$ *is called the $\Omega-$representation of the Galois group* $\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})$.

It is possible to associate to an $\Omega-$representation defined by $\mathcal{O}_\Omega$ a projection operator which can be used to define a motive in the sense of Shioda.

Definition. *The motive associated to the $\Omega-$representation defined by the orbit $\mathcal{O}_\Omega$ will be called the $\Omega-$motive M_Ω of the variety.*

In this paper the focus will be on the computation of the L-function $L_\Omega(X,s) := L(M_\Omega, s)$ of the $\Omega-$motive of hypersurfaces X^d

$$L_\Omega(X^d, s) = \prod_p \prod_{\alpha \in \mathcal{O}_\Omega} \frac{1}{\left(1 - (-1)^{s-1} j_{p^f}(\alpha) p^{-fs}\right)^{1/f}}, \tag{35}$$

where $\mathcal{O}_\Omega \subset \mathcal{A}_s^n$ denotes the orbit of the element α that corresponds to the $\Omega-$form. More precisely, the aim is to check whether the $\Omega-$motives of extremal Brieskorn-Pham K3 surfaces admit a modular interpretation and if so, whether the resulting forms admit an affine Lie algebraic construction. Higher-dimensional varieties are considered in [23].

3 Examples

The general framework formulated above is applied in this paper to extremal K3 surfaces of Brieskorn-Pham type. A K3 surface S defined over $\mathbb{C}$ is called extremal if its Picard number $\rho(S) = \mathrm{rk}\ \mathrm{NS}(S)$, defined as the rank of the Néron-Severi group $\mathrm{NS}(S)$, is maximal, i.e. $\rho(S) = 20$. Such surfaces were originally called singular [24], and more recently have been called attractive [25]. The set of extremal Brieskorn-Pham K3 surfaces is given in eq. (4).

3.1 The quartic Fermat K3 surface S^4

A summary of cardinality results $N_{r,p}(S^4)$ for small primes p for the Fermat K3 surface of degree four is contained in Table 1.

Prime p	3	5	7	11	13	17	19	23	29	31	37
$N_{1,p}(S^4)$	16	0	64	144	128	600	400	576	768	1024	1152
$\beta_1(p)$	-6	0	-14	-22	42	-310	-38	-46	74	-62	218

Table 1. *The coefficients $\beta_1(p) = 1 + p^2 - N_{1,p}(S^4)$ of the Hasse-Weil modular form of the quartic Fermat surface S^4 in terms of the cardinalities $N_{1,p}$ for small primes.*

The results in Table 1 lead to the L-series of S^4

$$L(S^4, s) \doteq 1 + \frac{6}{3^s} - \frac{26}{5^s} + \frac{14}{7^s} + \frac{117}{9^s} + \frac{22}{11^s} - \frac{42}{13^s} - \frac{156}{15^s} + \frac{310}{17^s} + \frac{38}{19^s} + \frac{84}{21^s} + \frac{46}{23^s} +, \cdots \quad (36)$$

where the symbol $\doteq$ means that a finite number of Euler factors have been omitted, here the bad prime $p = 2$. Associated to this L-series is the $q-$expansion

$$f(S^4, q) \doteq q + 6q^3 - 26q^5 + 14q^7 + 117q^9 + 22q^{11} - 42q^{13} - 156q^{15} + 310q^{17} + 38q^{19} + 84q^{21} + 46q^{23} + \cdots \quad (37)$$

The product structure of $L(S^4, s)$ can be obtained from the Jacobi-sum formulation by enumerating the set $\mathcal{A}_3^{\underline{n}}$ of the surface S^4. Replacing the degree vector $\underline{n}$ by the degree itself leads to

$$\mathcal{A}_3^4 \;=\; \left\{\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right),\ \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\ \left(\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\right),\ \left(\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4}\right),\ \left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)\right\} + \text{permutations.} \tag{38}$$

The Jacobi sums of the K3 surface S^4 at low primes are collected in Table 2. In this table the permutations and the complex conjugates of the sums listed are suppressed.

Type	q	5	9	13	17	29	37
I	$j_q\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	$-3+4i$	9	$5-12i$	$-15-8i$	$21+20i$	$-35-12i$
II	$j_q\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	5	9	13	17	29	37
III	$j_q\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)$	5	9	13	17	29	37
IV	$j_q\left(\frac{3}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}\right)$	-5	9	-13	17	-29	-37

Table 2. *Jacobi-sums of the quartic K3 surface $S^4 \subset \mathbb{P}_3$.*

The L-function of S^4 therefore factorizes as

$$L(S^4, s) = L_{\mathrm{I}}(S^4, s) \cdot L_{\mathrm{II}}(S^4, s) \cdot L_{\mathrm{III}}(S^4, s) \cdot L_{\mathrm{IV}}(S^4, s), \tag{39}$$

where the individual factors correspond to the orbits of the different Jacobi sums of Table 2. The first factor describes the L-function of the $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})-$orbit of $j_p\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$, which corresponds to the holomorphic 2-form of S^4, and hence is the $\Omega-$motivic L-function of S^4, $L_\Omega(S^4, s) = L_{\mathrm{I}}(S^4, s)$ of S^4,

$$\begin{aligned} L_\Omega(S^4, s) &= \prod_{p\neq 2} \frac{1}{\left(1 - j_{p^f}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\cdot p^{-fs}\right)^{1/f}\left(1 - j_{p^f}\left(\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)\cdot p^{-fs}\right)^{1/f}}\,. \\ &\doteq 1 - \frac{6}{5^s} + \frac{9}{9^s} + \frac{10}{13^s} - \frac{30}{17^s} + \frac{11}{25^s} + \frac{42}{29^s} - \frac{70}{37^s} + \cdots \end{aligned} \tag{40}$$

The factor $L_\Omega(S^4, s)$ of the complete L-function of S^4 is the only one with Jacobi sum characters in an algebraic number field.

3.2 The degree six weighted K3 surface $S^{6\mathrm{A}}$

The L-series of the $\Omega-$motive M_Ω of the double cover $S^{6\mathrm{A}}$ of the projective plane branched over a degree six plane curve is determined by the Jacobi sums

$$j_p\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2}\right), \qquad j_p\left(\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{1}{2}\right), \tag{41}$$

which are complex conjugates. The values for these sums for low primes are collected in Table 3.

Prime p	7	13	19	31	37
$j_p(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2})$	$-\frac{13}{2}-\frac{3}{2}\sqrt{3}i$	$-\frac{1}{2}+\frac{15}{2}\sqrt{3}i$	$\frac{11}{2}-\frac{21}{2}\sqrt{3}i$	$-23+12\sqrt{3}i$	$\frac{47}{2}-\frac{33}{2}\sqrt{3}i$

Table 3. *Jacobi sums for the K3 surface $S^{6\mathrm{A}} \subset \mathbb{P}_{(1,1,1,3)}$ for small primes.*

The resulting L-series of the $\Omega-$orbit is given by

$$L_\Omega(S^{6\mathrm{A}},s) \doteq 1 - \frac{13}{7^s} - \frac{1}{13^s} + \frac{11}{19^s} - \frac{46}{31^s} + \frac{47}{37^s} + \cdots \tag{42}$$

up to a finite number of Euler factors.

3.3 The degree six weighted K3 surface $S^{6\mathrm{B}}$

The $\Omega-$motivic L-series of the degree six Brieskorn-Pham type surface embedded in $\mathbb{P}_{(1,1,2,2)}$ can be computed via the Jacobi sums

$$j_p\left(\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3}\right), \qquad j_p\left(\frac{5}{6},\frac{5}{6},\frac{2}{3},\frac{2}{3}\right) \tag{43}$$

whose values at low primes are collected in Table 4.

p	7	13	19	31	37
$j_p\left(\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3}\right)$	$-1-4\sqrt{-3}$	$-11-4\sqrt{-3}$	$-13-8\sqrt{-3}$	$23-12\sqrt{-3}$	$13+20\sqrt{-3}$

Table 4. *Jacobi-sums for the K3 surface $S^{6\mathrm{B}} \subset \mathbb{P}_{(1,1,2,2)}$.*

The orbit of the $\Omega-$form with respect to the Galois group $\mathbb{Q}(\mu_6)/\mathbb{Q}$ leads to the L-series

$$L_\Omega(S^{6\mathrm{B}}, s) \doteq 1 - \frac{2}{7^s} - \frac{22}{13^s} - \frac{26}{19^s} + \frac{46}{31^s} + \frac{26}{37^s} + \frac{22}{43^s} + \cdots \qquad (44)$$

The question that arises now is whether these expansions can be given a string theoretic meaning, similar to the case of elliptic Brieskorn-Pham curves.

4 CFT Modularity of $\Omega-$motives

4.1 Modular forms

One of the important ingredients in the analysis of modular L-functions is their product structure, hence it is of interest to consider forms which admit Euler products, i.e. Hecke eigenforms.

Definition. *A modular form of weight w, level N, and character χ with respect to $\Gamma_0(N)$ is a map $f : \mathcal{H} \longrightarrow \mathbb{C}$ on the upper half−plane such that for any $\tau \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$*

$$f(\gamma\tau) = \chi(d)(c\tau + d)^w f(\tau). \qquad (45)$$

A cusp form can be characterized by the condition that the general $q-$expansion $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ starts with a_1. It is normalized if $a_1 = 1$.

Hecke defined on the space of cusp forms $S_w(\Gamma_0(N), \chi)$ operators which for prime p take the form

$$T_p^w f(q) = \sum_{n=1}^{\infty} a_{np} q^n + \chi(p) p^{w-1} \sum_{n=1}^{\infty} a_n q^{np}. \qquad (46)$$

Eigenforms of these operators are particularly important in a geometric context.

Theorem 4.(Hecke)
The space $S_w(\Gamma_0(N), \chi)$ of cusp forms of weight w, level N, and character χ is the orthogonal sum of the spaces of equivalent eigenforms. Each space of such forms has a member that is an eigenvector of all Hecke operators $T_w(n)$. A form $f \in S_w(\Gamma_0(N), \chi)$ that is an eigenvector for all $T_w(n)$ can be normalized and its coefficients satisfy

$$a_{p^{n+1}} \;=\; a_{p^n} a_p - \chi(p) p^{w-1} a_{p^{n-1}}, \qquad p \not| \, N,$$

$$
\begin{aligned}
a_{p^n} &= (a_p)^n, \qquad p|N, \\
a_{mn} &= a_m a_n, \qquad (m,n)=1.
\end{aligned} \tag{47}
$$

Moreover, the L-function $L(f,s)$ *has an Euler product of the form*

$$
L(f,s) = \prod_{\substack{p \text{ prime} \\ p|N}} \frac{1}{1-a_p p^{-s}} \prod_{\substack{p \text{ prime} \\ p \not| N}} \frac{1}{1-a_p p^{-s} + p^{w-1-2s}} \tag{48}
$$

which is convergent for $\mathrm{Re}\ s > \frac{w}{2}+1$.

4.2 CFT ingredients

Quantities that have proven useful in the string modularity analysis of elliptic Brieskorn-Pham curves [5, 6, 7] are the theta functions

$$
\Theta^k_{\ell,m}(\tau) = \sum_{\substack{-|x|<y\leq|x| \\ (x,y) \text{ or } (\frac{1}{2}-x,\frac{1}{2}+y) \\ \in \mathbb{Z}^2 + \left(\frac{\ell+1}{2(k+2)},\frac{m}{2k}\right)}} \mathrm{sign}(x) e^{2\pi i\tau((k+2)x^2-ky^2)}, \tag{49}
$$

defined by Kac and Peterson. These are related to the string functions of the affine algebra $A_1^{(1)}$ at level k as

$$
c^k_{\ell,m}(\tau) = \frac{\Theta^k_{\ell,m}(\tau)}{\eta^3(\tau)}. \tag{50}
$$

The string functions, together with the classical theta functions

$$
\theta_{n,m}(\tau,z,u) = e^{-2\pi imu} \sum_{\ell\in\mathbb{Z}+\frac{n}{2m}} e^{2\pi im\ell^2\tau+2\pi i\ell z}, \tag{51}
$$

are the building blocks of the N=2 supersymmetric characters $\chi^k_{\ell,q,s}(\tau)$ of the partition functions of the $N=2$ minimal theories

$$
\begin{aligned}
\chi^k_{\ell,q,s}(\tau,z,u) &= e^{2\pi iu}\mathrm{tr}_{\mathcal{H}^k_{\ell,q,s}} q^{\left(L_0-\frac{c}{24}\right)} e^{2\pi izJ_0} \\
&= e^{2\pi iu} \sum_{Q^k_{\ell,q,s},\Delta^k_{\ell,q,s}} \mathrm{mult}\left(\Delta^k_{\ell,q,s},Q^k_{\ell,q,s}\right) e^{2\pi i\tau\left(\Delta^k_{\ell,q,s}-\frac{c}{24}\right)+2\pi izQ^k_{\ell,q,s}},
\end{aligned} \tag{52}
$$

in terms of the conformal dimensions $\Delta^k_{\ell,q,s}$ and the charges $Q^k_{\ell,q,s}$, leading to [26]

$$
\chi^k_{\ell,q,s}(\tau,z,u) = \sum c^k_{\ell,q+4j-s}(\tau)\theta_{2q+(4j-s)(k+2),2k(k+2)}(\tau,z,u). \tag{53}
$$

It will become clear below that the modular forms that explain the modularity of extremal K3 surfaces of Brieskorn-Pham type in a string theoretic way are given by

$$\begin{aligned}\Theta^1_{1,1}(q) &= q^{1/12}(1-2q-q^2+2q^3+q^4+2q^5-2q^6-2q^8-2q^9+q^{10}+\cdots)\\ \Theta^2_{1,1}(q) &= q^{1/8}(1-q-2q^2+q^3+2q^5+q^6-2q^9+q^{10}+\cdots).\end{aligned} \tag{54}$$

4.3 Modularity of $\Omega-$motives

If an $\Omega-$motive of K3 type is modular the associated modular form is expected to be of weight three. A useful guide in the search for such forms is provided by the speculation that exactly solvable models lead to motives which admit complex multiplication (CM) in the classical sense [27] (see [28] for an alternate discussion of CM). The quartic Fermat surface admits CM with respect to $\mathbb{Q}(\sqrt{-1})$, while the degree six surfaces have CM with respect to the field $\mathbb{Q}(\sqrt{-3})$. The goal therefore is to find modular forms of weight three which admit complex multiplication with respect to these fields.

A further guide is provided by the bad primes of these surfaces. It is expected that the level of a geometrically derived modular form is divisible by the bad primes of the underlying variety. The only bad prime for the quartic Fermat K3 surface is $p=2$, hence the level should be some power of two. For the degree six surfaces the bad primes are $p=2,3$, hence the level should be of the form 2^a3^b for some non-negative integers a,b.

Combining these considerations leads to the candidate cusp forms

$$\begin{aligned}S_3(\Gamma_0(16),\chi_{-1}) &\ni \eta^6(4\tau) = q-6q^5+9q^9+10q^{13}-30q^{17}+11q^{25}+\cdots\\ S_3(\Gamma_0(12),\chi_{-3}) &\ni \eta^3(2\tau)\eta^3(6\tau) = q-3q^3+2q^7+9q^9-22q^{13}+26q^{19}+\cdots\end{aligned} \tag{55}$$

for the K3 surfaces considered above. Comparing the $q-$expansions of these two forms with those of the $\Omega-$motivic L-series of the K3 surfaces shows that while the first form describes the $\mathrm{L}-$function of the quartic K3 surface, the latter describes the corresponding $\mathrm{L}-$function of $S^{6\mathrm{B}}$ only up to sign changes. This indicates that a twist is involved, and it turns out that this twist can be provided by the Legendre character χ_3. This leaves the surface $S^{6\mathrm{A}}$. A candidate modular form can be obtained by lifting a complex multiplication modular form of weight two to weight three via an Eisenstein series associated to the CM field $\mathbb{Q}(\sqrt{-3})$. In the present case a useful modular form turns out to come from a class of theta series considered in [29]. Hecke associates to each element α of an integral ideal $\mathfrak{a}$ in an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ the theta series defined as

$$\vartheta(\tau; \alpha, \mathfrak{a}, Q\sqrt{-D}) = \sum_{z \equiv \alpha (\mathrm{mod} \ \mathfrak{a}Q\sqrt{-D})} q^{\mathrm{N}z/Q|D|\mathrm{N}\mathfrak{a}}, \tag{56}$$

where $\mathrm{N}z$ and $\mathrm{N}\mathfrak{a}$ denote the norms of $z \in \mathcal{O}_K$ and $\mathfrak{a}$ respectively, and $\mathcal{O}_K$ is the ring of integers of K. Relevant for the L-series $L_\Omega(S^{6\mathrm{A}}, s)$ is the special case given by $\alpha = 0$, $Q = 1$, and $\mathfrak{a} = \mathcal{O}_K$ for the Eisenstein field $K = \mathbb{Q}(\sqrt{-3})$, renamed here as

$$\vartheta(q) = \sum_{z \in \mathcal{O}_K} q^{\mathrm{N}z}. \tag{57}$$

In summary, the results above lead to the identification of the respective Mellin transforms $f_\Omega(S^d, q)$ of the motivic L−functions $L_\Omega(S^d, s)$ for $d = 4, 6\mathrm{A}, 6\mathrm{B}$ as noted in eq. (6) of Theorem 1. The proof of these relations to all orders is postponed to the appendix.

The modularity of the Ω−motives of extremal Brieskorn-Pham K3s leaves the question whether the resulting modular forms admit a string theoretic interpretation in terms of forms derived from the associated conformal field theory. The basic idea in the following is to consider the arithmetic building blocks of the K3 surfaces considered here. One way to identify these structures is via the twist map. In this section the focus will remain on the arithmetic aspects, postponing the identification of the basic irreducible geometric structure to section 5.

4.4 Elliptic curves from K3 surfaces

The structure of the quartic K3 surface S^4 is very simple. It is shown below that it can be constructed directly in terms of E^4 via the twist map [19]. Alternatively, one can use the Shioda-Katsura decomposition [30, 31] to reconstruct the cohomology of S^4 from the cohomology of the quartic plane curve

$$C_4 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_2 \mid z_0^4 + z_1^4 + z_2^4 = 0\}. \tag{58}$$

This is a genus three curve whose L-function factors into the triple product of the L-function of the weighted elliptic curve E^4. The L-function of this curve was computed in [6], where it was shown that its Mellin transform is determined by a twist of the Hecke indefinite modular form $\Theta_{1,1}^2(q)$, as described in Theorem 2. Therefore the geometry and arithmetic structure of the quartic Fermat K3 suggests a relation between the modular forms $\eta^6(q)$ and $\Theta_{1,1}^2(q)$ of weight three and two, respectively.

The structure of the degree six K3 surface $S^{6\mathrm{B}}$ can be recovered from the twist map in a way similar to the quartic surface by constructing it from two copies of the elliptic curve E^6, as described below. Alternatively, for both degree six surfaces one can again consider the reduction of the cohomology via Shioda-Katsura. There are two curves to consider, the weighted plane curve

$$C_6 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(1,1,2)} \mid z_0^6 + z_1^6 + z_2^3 = 0\} \tag{59}$$

and the elliptic curve E^3. The latter was analyzed in detail in [5], with the result that its associated modular form is determined by the theta series $\Theta^1_{1,1}(q)$ at conformal level $k = 1$. The Jacobian of C_6 factors into three different types of elliptic curves, E^3 just discussed, the degree six elliptic curve E^6, and a third curve of conductor 432. The curve E^6 has been shown to lead to the string theoretic modular form given in terms of $\Theta^1_{1,1}(q)$. These results then suggest a relation between the modular form $\eta^3(2\tau)\eta^3(6\tau)$ and $\Theta^k_{1,1}(q)$ for levels $k = 1$ or $k = 2$, or both.

4.5 From K3 to CFT forms

For the degree six surface $S^{6\mathrm{A}}$ the relation between the weight three form and the string theoretic form is immediate because this form is the lift of a product of Hecke indefinite modular forms. In general, the expected relation between forms of different weight cannot be established for the forms themselves, but should proceed in terms of their associated L-series. Guidance for such constructions is provided by the theory of convolutions of L-functions. It turns out that for all three modular forms of weight three determined above the relation is of similar type. Denote by

$$f_w(q) = \sum_{n=1}^{\infty} a_n^{(w)} q^n \tag{60}$$

the $q-$expansion of the weight w form. Then the relation between the pairs of weight three and weight two forms for the surfaces

$$\begin{aligned} S^4: \quad (f_3, f_2) &= (\eta^6(q^4), \Theta^2_{1,1}(q^4)^2 \otimes \chi_2), \\ S^{6\mathrm{A}}: \quad (f_3, f_2) &= (\vartheta(q^3)\eta^2(q^3)\eta^2(q^9), \Theta^1_{1,1}(q^3)\Theta^1_{1,1}(q^9)) \end{aligned} \tag{61}$$

is of the form

$$a_p^{(3)} = (a_p^{(2)})^2 - 2p \tag{62}$$

for rational primes p, while for third surface

$$S^{6\mathrm{B}}: \quad (f_3, f_2) = (\eta^3(q^2)\eta^3(q^6), \Theta^1_{1,1}(q^6)^2 \otimes \chi_3), \tag{63}$$

a twist is necessary

$$a_p^{(3)} = \left((a_p^{(2)})^2 - 2p\right)\chi_3(p). \tag{64}$$

The origin of these relations, and the motivation for the various twists, will become clear in the next two sections.

This result shows that the string theoretic modular forms $\Theta^k_{\ell,m}(\tau)$ introduced above suffice to explain the arithmetic structure of spacetime, represented here by the $\Omega-$motive carrying an irreducible representation of the Galois group of the cyclotomic field, determined by the symmetry group of the K3 surfaces. Reading these relations in reverse explains string theoretic modular forms in terms of the arithmetic geometry of the topologically nontrivial part of spacetime.

5 K3 surfaces from elliptic curves

This section describes the elliptic building blocks used in the previous section to identify the string theoretic structure of extremal Brieskorn-Pham K3s. A direct construction of the two surfaces S^4 and $S^{6\mathrm{B}}$ in terms of elliptic curves can be obtained via the twist map [19]. This construction was originally considered in the context of string dualities [32] (see also the later ref. [33]), and will also be useful below for the proof of modularity to all orders for the $\Omega-$motivic L-series of these K3 surfaces.

5.1 The twist map construction

In the notation of [19] consider the map

$$\Phi: \ \mathbb{P}_{(w_0,\ldots,w_m)} \times \mathbb{P}_{(v_0,\ldots,v_n)} \longrightarrow \mathbb{P}_{(v_0w_1,\ldots,v_0w_m,w_0v_1,\ldots,w_0v_n)} \tag{65}$$

defined as

$$\begin{aligned} ((x_0,\ldots,x_m),(y_0,\ldots,y_n)) \quad &\mapsto \quad (y_0^{w_1/w_0}x_1,\ldots,y_0^{w_m/w_0}x_m, x_0^{v_1/v_0}y_1,\ldots,x_0^{v_n/v_0}y_n) \\ &=: (z_1,\ldots,z_m,t_1,\ldots,t_n). \end{aligned} \tag{66}$$

This map restricts on the subvarieties

$$\begin{aligned} X_1 &= \{x_0^\ell + p(x_i) = 0\} \subset \mathbb{P}_{(w_0,\ldots,w_m)} \\ X_2 &= \{y_0^\ell + q(y_j) = 0\} \subset \mathbb{P}_{(v_0,\ldots,v_n)}, \end{aligned} \tag{67}$$

defined by transverse polynomials $p(x_i)$ and $q(y_j)$, to the hypersurface

$$X = \{p(z_i) - q(t_j) = 0\} \subset \mathbb{P}_{(v_0w_1,\ldots,v_0w_m,w_0v_1,\ldots,w_0v_n)} \tag{68}$$

as a finite map. The degrees of the hypersurfaces X_i are given by

$$\deg X_1 = w_0\ell, \qquad \deg X_2 = v_0\ell, \tag{69}$$

leading to the degree $\deg X = v_0w_0\ell$.

5.2 Examples

Applying the twist construction to the quartic elliptic curve $E^4 \subset \mathbb{P}_{(1,1,2)}$ leads first to the map of ambient spaces

$$\Phi : \mathbb{P}_{(2,1,1)} \times \mathbb{P}_{(2,1,1)} \longrightarrow \mathbb{P}_3 \tag{70}$$

defined as

$$((x_0,x_1,x_2),(y_0,y_1,y_2)) \;\mapsto\; (y_0^{1/2}x_1, y_0^{1/2}x_2, x_0^{1/2}y_1, x_0^{1/2}y_2). \tag{71}$$

Restricting the map Φ to the product $E_+^4 \times E_-^4$ of the elliptic curves

$$E_\pm^4 = \{x_0^2 \pm (x_1^4 + x_2^4) = 0\} \subset \mathbb{P}_{(2,1,1)}, \tag{72}$$

leads to the quartic K3 surface $S^4 \subset \mathbb{P}_3$.

The degree six surface $S^{6\mathrm{B}}$ can be constructed similarly by considering the elliptic curve $E^6 \subset \mathbb{P}_{(1,2,3)}$. The ambient space map

$$\Phi : \mathbb{P}_{(3,2,1)} \times \mathbb{P}_{(3,2,1)} \longrightarrow \mathbb{P}_{(1,1,2,2)} \tag{73}$$

is now defined by

$$((x_0,x_1,x_2),(y_0,y_1,y_2)) \;\mapsto\; (y_0^{2/3}x_1, y_0^{1/3}x_2, x_0^{2/3}y_1, x_0^{1/3}y_2), \tag{74}$$

and restricts on the product $E_+^6 \times E_-^6$, where

$$E_\pm^6 = \{x_0^2 \pm (x_1^3 + x_2^6) = 0\} \subset \mathbb{P}_{(3,2,1)}, \tag{75}$$

to the surface $S^{6\mathrm{B}} \subset \mathbb{P}_{(1,1,2,2)}$,

It will become clear below that the surface $S^{6\mathrm{A}}$ is determined by the elliptic curve E^3. Therefore all surfaces considered here can be understood in terms of the arithmetic structure of the three Brieskorn-Pham type elliptic curves described in [7].

5.3 K3 geometry from string theory

One of the fundamental problems in string theory is to derive the structure of spacetime solely from the field theory on the worldsheet. In refs. [5, 7] it was shown that it is possible to construct the elliptic geometry of Gepner models at $c = 3$ directly from the conformal field theory, providing a string theoretic derivation of the extra dimensional geometry for a simple class of exact models. The question arises whether such a derivation can be generalized to the K3 surfaces analyzed here. In contrast to elliptic curves, the cohomology of K3 surfaces is more complicated, and it is not obvious how a direct construction should proceed. The twist map described above does, however, lead directly from the modular forms on the worldsheet to the geometry of spacetime in this higher dimensional case as well. This can be seen as follows.

The first step is the construction, described above, of these K3 surfaces in terms of the elliptic curves of Brieskorn-Pham type via the twist map. This reduces the modular construction from two dimensions to one. The second step is to use the criteria formulated in [7], which uniquely identify the modular forms for these elliptic curves. This leads to the modular forms of weight 2 described in the previous section. This two-step procedure can be reversed by using the Eichler-Shimura construction, which allows to construct elliptic curves from their modular forms. Combining these modularity arguments with the twist map therefore leads to the construction of the K3 surfaces S^4 and $S^{6\mathrm{B}}$ directly from the conformal field theories on the worldsheet. An alternative reduction strategy proceeds via complex multiplication. This approach is less direct, but allows to discuss all three surfaces in a unified manner, as shown below.

6 Modularity of a phase transition

The arithmetic structure of a projective variety is sensitive to deformations, changing as the coefficients of its defining polynomials are varied. It is therefore of interest to consider the behavior of families of varieties and analyze their arithmetic structure as their moduli change. This line of thought has already been followed in recent work [34, 35, 36]. These interesting papers address, among other issues, the question of what happens to the reduced variety at the conifold locus. Reference [36] in particular considers the lower dimensional analog of the conifold transition in the context of the following family of quartic K3 surfaces

$$S^4(\psi) = \left\{ (z_0 : \cdots : z_3) \in \mathbb{P}_3 \;\Big|\; \sum_{i=0}^{3} z_i^4 - 4\psi \prod_i z_i = 0 \right\}. \tag{76}$$

The result is that at the singular locus $\psi = 1$ the congruent zeta functions $Z(S^4(\psi)/\mathbb{F}_p, t)$ degenerate, as expected. It turns out that they become singular in a way that preserves modularity. The question therefore arises whether there is a cohomological L-function whose Mellin transform admits a conformal field theoretic interpretation.

The zeta function computations of ref. [36] indicate that at the singular point a modular form emerges which can be given a string theoretic interpretation as

$$\frac{(\Theta_{1,1}^2(\tau)\Theta_{1,1}^2(4\tau))^2}{\Theta_{1,1}^2(2\tau)} \otimes \chi_2. \tag{77}$$

What is interesting about this form is that it is written in terms of cusp forms determined by precisely the same conformal field theory that leads to the L-function at the Fermat locus in the moduli space. Furthermore, the twist character χ_2 which appears in this expression, is the Legendre character of the field of quantum dimensions $\mathbb{Q}(\sqrt{2})$ of that same theory.

This suggests that at the analog of the conifold point the arithmetic structure points to a conformal field theory at level $k = 2$, precisely the same structure that describes the situation for the Fermat surface.

7 Symmetries

7.1 Complex multiplication

It was conjectured in [27] that exactly solvable Calabi-Yau varieties can be characterized by a complex multiplication symmetry in the sense of [37]. In the context of elliptic curves this coincides with the classical notion of CM, and it was shown in [7] that all elliptic Brieskorn-Pham curves and their associated modular forms admit CM by either the Eisenstein field $\mathbb{Q}(\sqrt{-3})$ or the Gauss field $\mathbb{Q}(\sqrt{-1})$. For higher dimensional Calabi-Yau varieties the concept of complex multiplication must be modified. The idea of ref. [37] is to define the CM property of a general Calabi-Yau variety in terms of the complex multiplication properties of the associated Jacobians of the curves embedded in the variety. This notion can be formulated more generally in terms of the CM properties of motives defined by the Galois representations, as described above.

A natural problem therefore is to determine the CM nature of the forms derived above from the $\Omega-$motives of extremal Brieskorn-Pham K3 surfaces. In terms of the coefficients of the $q-$expansion, $f(q) = \sum_n a_n q^n$, CM implies the existence of quadratic extension fields $K/\mathbb{Q}$ such that the coefficients a_p vanish at all primes p that are inert in K. In [7] this condition was checked for the modular forms associated to elliptic Brieskorn-Pham curves. Similarly one can show that the forms of weight 3 of Theorem 1 admit CM. Table 5 summarizes the results for the CM fields of all the forms encountered so far. The results indicate that the relevance of complex multiplication generalizes from exactly solvable elliptic curves to higher dimensions.

Form	Weight	Level	CM Field	Geometry
$\Theta^1_{1,1}(q^3)\Theta^1_{1,1}(q^9)$	2	27	$\mathbb{Q}(\sqrt{-3})$	Elliptic curve
$(\Theta^1_{1,1}(q^6))^2$	2	36	$\mathbb{Q}(\sqrt{-3})$	Elliptic curve
$(\Theta^2_{1,1}(q^4))^2 \otimes \chi_2$	2	64	$\mathbb{Q}(\sqrt{-1})$	Elliptic curve
$\eta^6(q^4)$	3	16	$\mathbb{Q}(\sqrt{-1})$	K3 $\Omega-$motive
$\vartheta(q^3)\eta^2(q^3)\eta^2(q^9)$	3	27	$\mathbb{Q}(\sqrt{-3})$	K3 $\Omega-$motive
$\eta^3(q^2)\eta^3(q^6) \otimes \chi_3$	3	48	$\mathbb{Q}(\sqrt{-3})$	K3 $\Omega-$motive
$\frac{\left(\Theta^2_{1,1}(q)\Theta^2_{1,1}(q^4)\right)^2}{\Theta^2_{1,1}(q^2)}$	3	8	$\mathbb{Q}(\sqrt{-2})$	Singular quartic K3

Table 5. *CM modular forms with geometric interpretation.*

The complex multiplication property of the K3 motivic modular forms provides an alternative approach to the notion of "dimensional reduction" of K3 surfaces to elliptic curves discussed in §4 and §5. CM modular forms can be obtained from L-series

$$L(\psi, s) = \prod_{\mathfrak{p}\in \mathrm{Spec}\ \mathcal{O}_K} \frac{1}{1 - \frac{\psi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}} = \sum_{\mathfrak{a}\subset \mathcal{I}(O_K)} \frac{\psi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s} \tag{78}$$

associated to Hecke characters ψ of number fields K, where $\mathcal{I}(\mathcal{O}_K)$ (Spec $\mathcal{O}_K$) describes the set of (prime) ideals in the ring of algebraic integers $\mathcal{O}_K$, and $\mathrm{N}\mathfrak{a}$ is the norm of the ideal $\mathfrak{a}$ [38]. In the present case a Hecke characters ψ_4 of $\mathbb{Q}(\sqrt{-1})$, and two characters $\psi_{6\mathrm{A}}$ and $\psi_{6\mathrm{B}}$ associated to $\mathbb{Q}(\sqrt{-3})$, all of weight two, determine the L-series of S^4, $S^{6\mathrm{A}}$ and $S^{6\mathrm{B}}$ by considering the square of the basic characters. More precisely, the arithmetic of the $\Omega-$motive of elliptic Brieskorn-Pham K3s can be obtained as follows. Define the character ψ_4 at the prime ideals $\mathfrak{p}$ of $\mathbb{Q}(\sqrt{-1})$ dividing the rational prime p as

$$\psi_4(\mathfrak{p}) = \alpha_{\mathfrak{p}}, \qquad \text{with} \ \ \alpha_{\mathfrak{p}} \equiv \left(\frac{2}{p}\right) (\mathrm{mod}\ (2+2i)), \tag{79}$$

where $\mathfrak{p} = (\alpha_{\mathfrak{p}})$. Define further the characters $\psi_{6\mathrm{A}}$ and $\psi_{6\mathrm{B}}$ on the prime ideals $\mathfrak{p}|p$ of $\mathbb{Q}(\sqrt{-3})$ dividing the rational primes p as

$$\begin{aligned} \psi_{6\mathrm{A}}(\mathfrak{p}) &= \alpha_{\mathfrak{p}}, \qquad \text{with} \ \ \alpha_{\mathfrak{p}} \equiv 1(\mathrm{mod}\ 3), \\ \psi_{6\mathrm{B}}(\mathfrak{p}) &= \alpha_{\mathfrak{p}}, \qquad \text{with} \ \ \alpha_{\mathfrak{p}} \equiv \left(\frac{3}{p}\right) (\mathrm{mod}\ (2+4\xi_3)). \end{aligned} \tag{80}$$

These characters provide the Hecke theoretic interpretation of the Mellin transforms of the Hasse-Weil L-functions of the elliptic curves E^4, E^3 and E^6 respectively. Renaming E^3 and E^6 as $E^{6\mathrm{A}} := E^3$ and $E^{6\mathrm{B}} := E^6$, one can check that

$$L_{\mathrm{HW}}(E^d, s) = L(\psi_d, s), \qquad d = 4, 6\mathrm{A}, 6\mathrm{B}. \tag{81}$$

The motivic L-series of S^4, $S^{6\mathrm{A}}$ and $S^{6\mathrm{B}}$ can then be described as

$$L_\Omega(S^d, s) = L(\psi_d^2, s), \qquad d = 4, 6\mathrm{A} \tag{82}$$

for the first two surfaces, and as

$$L_\Omega(S^{6\mathrm{B}}, s) = L(\psi_{6\mathrm{B}}^2 \otimes \chi_3, s) \tag{83}$$

for the third example. These results provide an alternative demonstration that the elliptic curves E^4, E^3 and E^6 can be viewed as the modular building blocks of the K3 surfaces considered here. They also explain the relations (62) and (64) by rewriting the coefficients of the Hecke L-series of the weight three forms in terms of the real coefficients of the weight two forms.

The Hecke interpretation of the $\Omega-$motivic L-series of elliptic curves and K3-surfaces provides a way to explicitly prove the modularity of these series to all order of their $q-$expansions by using some results from Hecke and Shimura. Alternatively, it is possible to point to general modularity result for extremal K3 surfaces defined over number fields [24, 39]. It is useful, however, to see modularity explicitly, and a proof is contained in the appendix.

7.2 K3 Arithmetic moonshine

It was pointed out in [7] that the modular forms given by the Mellin transform of the Hasse-Weil L-functions of elliptic Brieskorn-Pham curves can be interpreted as generalized McKay-Thompson series associated to the largest Mathieu group M_{24}, leading to the notion of what might be called arithmetic moonshine. It is therefore natural, as an aside, to ask whether the modular forms that appear in the context of extremal Brieskorn-Pham K3 surfaces can also be interpreted as such series. This is the case if the group considered is enlarged from the Mathieu group to the Conway group, defined as the automorphism group of the Leech lattice.

The specific group elements associated to the forms encountered in the present paper are listed in Table 6, where the notation of ref. [40] is adopted, which in turn follows the atlas [41]. The numerical part of the group element denotes its order while the letter part separates different elements of the same order.

Form	$\Theta^1_{1,1}(q^3)\Theta^1_{1,1}(q^9)$	$\Theta^1_{1,1}(q^6)^2$	$\Theta^1_{1,1}(q^6)^2 \otimes \chi$	$\eta^6(q^4)$	$\eta^3(q^2)\eta^3(q^6) \otimes \chi_3$	$\frac{\left(\Theta^2_{1,1}(q)\Theta^2_{1,1}(q^4)\right)^2}{\Theta^2_{1,1}(q^2)}$
Conway class	$(3D, 3B)$	$6I$	$8F\vert_T$	$4F$	$6G\vert_T$	$8E$

Table 6. *Conway classes for the forms of Table 5.*

It is intriguing to see that all geometrically induced conformal field theoretic modular forms obtained so far admit a sporadic group theoretic interpretation. It is an open question whether this relation can help to provide a conceptual foundation of the relations established so far.

8 Appendix: modularity proof

8.1 Faltings-Serre-Livné strategy

Modularity of the L-series considered here can be shown in several ways, e.g. by using their complex multiplication origin and results of Hecke and Shimura. It is also possible to refer to some general results of Livné. Calabi-Yau modularity is not restricted to manifolds with CM, as illustrated by many examples reviewed in [10]. For completeness, this appendix contains a brief modularity proof of the $\Omega-$motives of extremal K3 surfaces of Brieskorn-Pham type by using techniques essentially developed by Faltings and Serre. It was first observed by Faltings [42] that different Galois representations can be shown to be identical if they agree at a finite number of primes. This observation has been developed by Serre [43] and Livné [44], and Livné in particular made it into a practical tool by specifying precisely the set of primes that has to be tested in order to guarantee agreement of two representations. The varieties discussed here are defined over $\mathbb{Q}$, hence Livné's more general theorem can be reduced to the following form, previously considered by Verrill [45] (see [10] for more references in this direction). Let $\mathbb{Q}_\ell$ be the $\ell-$adic field and denote by F_p the Frobenius element at the prime p.

Theorem 5. *Let S be a finite set of rational primes, and denote by $\mathbb{Q}_S$ the compositum of all quadratic extensions of $\mathbb{Q}$ unramified outside of S. Suppose*

$$\rho_1, \rho_2 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}(2, \mathbb{Q}_2) \tag{84}$$

are $2-$adic continuous representations, unramified outside of S, which further satisfy the following conditions:
1) $\mathrm{tr}\ \rho_1 \equiv \mathrm{tr}\ \rho_2 \equiv 0 \bmod 2$.
2) There exists a finite set $T \neq \emptyset$ of rational primes, disjoint from S, for which
a) the image of the set $\{\mathrm{F}_p\}_{p\in T}$ *in* $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ *is surjective,*
b) $\mathrm{tr}\ \rho_1(\mathrm{F}_p) = \mathrm{tr}\ \rho_2(\mathrm{F}_p)$ *for all $p \in T$,*
c) $\det\ \rho_1(\mathrm{F}_p) = \det\ \rho_2(\mathrm{F}_p)$ *for all $p \in T$.*
Then ρ_1 and ρ_2 have isomorphic semi-simplification.

The idea therefore is to translate the geometric and modular computations into two Galois representations, and to show agreement by satisfying the requirements of Livné's result. The geometric part is given by the representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the $\ell-$adic cohomology group

$$\rho_1 = \rho_\ell^i : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \ \longrightarrow \ \mathrm{Aut}(\mathrm{H}^i_{\acute{e}t}(\bar{X}, \mathbb{Q}_\ell)), \tag{85}$$

where ℓ is a prime different from the prime of reduction, $X/\mathbb{Q}$ is assumed to be smooth projective variety, $\bar{X} = X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, and $\mathrm{H}^i_{\acute{e}t}$ indicates the i^{th} étale cohomology group.

In the absolute Galois group there exists a distinguished element, the geometric Frobenius endomorphism $\overline{\mathrm{F}}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Given the representation ρ_ℓ^i these Frobenius elements act on the étale cohomology, and it turns out that the polynomials $\mathcal{P}_p^i(t)$ that define the geometric L-function associated to these cohomology groups can be determined in terms of these Frobenii as

$$\mathcal{P}_p^i(t) = \det\left(1 - \rho_\ell^i(\overline{\mathrm{F}}_p)|_{\mathrm{H}^i_{\acute{e}t}(\bar{X},\mathbb{Q}_\ell)} t\right). \tag{86}$$

The second representation is the 2−dimensional Galois representation $\rho_2 = \rho_f$ associated by Deligne [46] to Hecke eigenforms of arbitrary weight, where

$$\rho_f : \ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}(2, \mathbb{Q}_\ell) \tag{87}$$

is a representation that is unramified outside of ℓ and the prime divisors of the level N of the modular form. This ℓ−adic representation can be realized in the cohomology of certain ℓ−adic sheaves over a modular curve. These can be defined over $\mathbb{Q}$ and therefore $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the ℓ−adic cohomology. Deligne has shown that these representations have the properties

$$\begin{aligned} \mathrm{tr}\ \rho_f(\overline{\mathrm{F}}_p) &= a_p \\ \det\ \rho_f(\overline{\mathrm{F}}_p) &= p^{w-1}\epsilon(p) \end{aligned} \tag{88}$$

for all primes p different from ℓ. In the present discussion of K3 surfaces the focus is on modular forms of weight three and the goal is to use Theorem 5 to prove the identity of the geometric representation and the modular representation.

8.2 Examples

The application the strategy outlined above to the K3 surface $S^4 \subset \mathbb{P}_3$ starts with the determination of the set T of primes considered in Livné's theorem. This is obtained by considering a representation of the Galois group of the composite field $\mathbb{Q}_S$. For S^4 the bad prime is $p = 2$, which is also the only divisor of the level of modular form $\eta(4\tau)^6 \in S_3(\Gamma_0(16), \chi_{-1})$. Hence $S = \{2\}$, and the composite field is $\mathbb{Q}_{\{2\}} = \mathbb{Q}(i, \sqrt{2}) = \mathbb{Q}(\xi_8)$. Therefore the set T of primes can be chosen as

$$T = \{3, 5, 7, 17\}. \tag{89}$$

This set of primes is a subset of the primes for which agreement of the modular and the geometric L-series is shown by the computations in the previous subsections. It remains to establish the conditions formulated in the theorem.

The fact that $\mathrm{tr}\ \rho_1 = 0 \bmod 2$ follows by noting that the quartic surface can be constructed via the twist map, as explained above. The elliptic curve E^4 involved in the construction of S^4 admits complex multiplication, and it can be shown that its coefficients a_p satisfy $a_p = 0 \bmod 2$ (see e.g. [47], or [7]). Hence the same follows for ρ_1. Alternatively, this follows from the Jacobi-sum formulation of the L-function of the Ω−motive. The same congruence holds for $\eta^6(4\tau)$, which can be seen e.g. via Jacobi's expansion for $\eta^3(\tau)$ [48]. The determinant condition, finally, follows from Weil's result for Jacobi sums in the geometric case, and Deligne's result in the case of the modular form.

The case of the degree six surface $S^{6\mathrm{B}} \subset \mathbb{P}_{(1,1,2,2)}$ is similar to the discussion of the quartic. The set of bad primes is $S = \{2, 3\}$, which gives the set of divisors of the level of the modular form. This leads to the composite field $\mathbb{Q}_{\{2,3\}} = \mathbb{Q}(\xi_{24})$. The set of primes representing the Galois group of this field is given by

$$T = \{5, 7, 11, 13, 17, 19, 23, 73\}. \tag{90}$$

This set of primes is again a subset of the primes considered above for the surface S^6. Comparison between the geometric and modular L-series shows agreement on the set of primes given by T.

The fact that tr $\rho_1 = 0$ mod 2 can again be seen via the twist construction. The curve E^6, the building block of S^6, has complex multiplication, and the coefficients of its Hasse-Weil L-series are zero mod 2. Hence it follows that this holds also for ρ_1. The remaining assumptions of the theorem follow in the same way as in the case of the quartic surface in combination with results proven in [45].

The surface $S^{6\mathrm{A}}$ does not satisfy the conditions of Livné's theorem since the coefficients can be odd. The following result, proven in [49], can be used to complete the proof.

Proposition. *Let ρ_1, ρ_2 be two 2-adic Galois representations with the same determinant and even trace at* Fr_{11} *or* Fr_{13}*, which are unramified outside* $\{2, 3\}$*. Then they have isomorphic semi-simplifications if and only if for any* $p \in \{5, 7, 11, 13, 17, 19, 23, 31, 37\}$ *the traces of the Frobenii are identical,* tr $\rho_1(\mathrm{Fr}_p) =$ tr $\rho_2(\mathrm{Fr}_p)$.

A discussion of the notion of semi-simplification can be found in [50]. Important here is the implication that the two L-functions associated to the Galois representations are identical. It follows from the Jacobi sum representation of the L-series and Deligne's result that the determinants of the geometric representation and the modular representations are identical. The computations collected in Table 1 furthermore show agreement for the traces at the primes determined by the proposition. Hence the proof follows.

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