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ON THE DEGREE OF APPROXIMATION BY POSITIVE LINEAR OPERATORS USING THE B-SUMMABILITY METHOD.*

by

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ABSTRACT. The aim of this paper is to sharpen the results of censor [3] and Mahapatra [7] given on the degree approximation by positive linear operators.

§1. Introducción. Let $B = \{A^{(n)}\} = \{(a_{pm}^{(n)})\}$ be a sequence of infinite matrices such that $a_{pm}^{(n)} \leq 0$ for $p, m, n = 1, 2, \ldots$ A sequence $\{x_m\}$ of real numbers is said to be *B-summable* to 1 [Bell 1973] if

$$\lim_{p \to \infty} \sum_{m=1}^{\infty} a_{pm}^{(n)} x_m = 1$$

uniformly in $n = 1, 2, \ldots$ If, for some matrix A, $A^{(n)} = A$ for $n = 1, 2, \ldots$, then B-summability is just *matrix summability by A*. If for $n = 1, 2, \ldots$

$$a_{pm}^{(n)} = \begin{cases} \dfrac{1}{p} & \text{for } n+1 \leq m \leq n+p \\ 0 & \text{otherwise} \end{cases}$$

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then B-summability reduces to almost convergence.

Recently some results of Censor [3] and Mohapatra [7] on the rate of convergence of sequence of positive linear operators have been unified by Swetits [10] through the use of the B-summability method. The object of this paper is to sharpen the results of Censor [3] and Mohapatra [7]. Corresponding estimates for some especial operators are also deduced.

Let $\{L_m\}$ be a sequence of positive linear operators on $\mathcal{C}[a,b]$ and let $\{A^{(n)}\} = B$ be a sequence of infinite matrices with non negative entries. For $f \in \mathcal{C}[a,b]$, let $A_p^{(n)}(f;x)$ denote the double sequence

$$A_p^{(n)}(f;x) = \sum_{m=1}^{\infty} a_{pm}^{(n)} L_m(f;x);\ p,n = 1,2,\ldots \tag{1.1}$$

Following Swetits [10], we define $\|A_p f\|$ to be

$$\sup_n \sup_{x \in [a,b]} \left| A_p^{(n)}(f;x) \right|$$

We say that for $f \in \mathcal{C}[a,b]$, $\{L_m f\}$ is *B-summable to f, uniformly on* $[a,b]$ *if and only if* $\|A_p |f| - f\| \to 0$ as $p \to \infty$. The following lemmas are from Anastassiou [1, page 264].

LEMMA 1.1. *For all t, and* $x \in [a,b]$ *and* $\delta > 0$ *one obtains*

$$\int_x^t \left\lceil \frac{|\xi - x|}{\delta} \right\rceil d\xi \le \left\{ \frac{(t-x)^2}{2\delta} + \frac{|t-x|}{2} + \frac{\delta}{8} \right\} \tag{1.2}$$

where $\lceil . \rceil$ *denotes the ceiling of number.*

LEMMA 1.2. *Let f be a convex function in* $C^1[a,b]$, *then*

$$\left| f'(\xi) - f'(x) \right| \le w(f', \delta) \left\lceil \frac{|\xi - x|}{\delta} \right\rceil . \tag{1.3}$$

§2. Main result. Let $\{k_p\}$ be a sequence of positive numbers and $\{L_m\}$ be a sequence of positive linear operators on $\mathcal{C}[a,b]$. Let $f \in \mathcal{C}^1[a,b]$ be such that $|f'(t) - f'(x)|$ is a convex function in t and $w(f'; .)$ is the modulus of continuity of f'. Let $B = \{A^{(n)}\}$ be a sequence of infinite matrices with non-negative real entries such that $\|A_p e_o\| < \infty$, where $e_o(x) = 1$ for all $x \in [a,b]$. Then for each p :

$$\|A_p f - f\| \leq \|f\| \, \|A_p e_0 - 1\| + \|f'\| \, \|A_p(t - x)\|$$

$$+ w(f'; k_p \mu_p)\mu_p \left\{ \frac{1}{2k_p} + \frac{1}{2}\|A_p e_o\|^{1/2} + \frac{k_p}{8} \|A_p e_o\| \right\} \quad (2.1)$$

If an addition $(A_p{}^{(n)} e_0)(x) = 1$ and $(A_p{}^{(n)} t)(x) = x$,

$$\|A_p f - f\| \leq \left(\frac{1}{2k_p} + \frac{1}{2} + \frac{k_p}{8} \right) \mu_p \cdot w\left(f', k_p \mu'_p\right) \quad (2.2)$$

where $\mu_p = \|A_p(t - x)^2\|^{1/2}$ and $\|\cdot\|$ norm being the sup over $[a, b]$ defined in §1.

Proof of main result. We know that

$$f(t) - f(x) = f'(x)(t - x) + \int_x^t \{f'(\xi) - f'(x)\} d\xi \quad (2.3)$$

Using (1.2), (1,3), (2,3) and the inequalities

$$\left| A_p^{(n)} f \right| \leq A_p^{(n)}(|f|), \quad A_p^{(n)}(f . g) \leq \left(A_p^{(n)} f^2\right)^{1/2} \cdot \left(A_p^{(n)} g^2\right)^{1/2}.$$

We get that

$$\left| \left(A_p^{(n)} f\right)(x) - f(x) \left(A_p^{(n)} e_o\right)(x) \right|$$

$$\leq \left| f'(x) \left(A_p^{(n)}(t - x)\right)(x) \right| + \left(A_p^{(n)} \left\{ \int_x^t (f'(\xi) - f'(x)) d\xi \right\} \right)(x)$$

$$\leq |f'(x)| \left| \left(A_p^{(n)}(t-x)\right)(x) \right| + \left(A_p^{(n)}\left\{\int_x^t |f'(\xi) - f'(x) d\xi\right\}\right)(x)$$

$$\leq |f'(x)| \left| \left(A_p^{(n)}(t-x)\right)(x) \right| + w(f', \delta) \left(A_p^{(n)}\left\{\int_x^t \Gamma \frac{|\xi - x|}{\delta} \rceil d\xi\right\}\right)(x)$$

$$\leq |f'(x)| \left| \left(A_p^{(n)}(t-x)\right)(x) \right| +$$

$$w(f', \delta)\left(A_p^{(n)}\left\{\frac{(t-x)^2}{2\delta} + \frac{|t-x|}{2} + \frac{\delta}{8}\right\}(x)\right)$$

$$\leq |f'(x)| \left| \left(A_p^{(n)}(t-x)\right)(x) \right| + w(f', \delta)\left\{\frac{1}{2\delta}\left(A_p^{(n)}(t-x)^2\right)(x) + \frac{1}{2}\left(A_p^{(n)}|t-x|\right)(x) + \frac{\delta}{8}\left(A_p^{(n)} e_o\right)(x)\right\}$$

$$\leq \|f'\| \|A_p(t-x)\| + w(f', \delta)\left\{\frac{1}{2\delta}\left(A_p^{(n)}(t-x)^2\right)(x) + \frac{1}{2}\left(A_p^{(n)}|t-x|\right)(x) + \frac{\delta}{8}\left(A_p^{(n)} e_o\right)(x)\right\} \qquad (2.4)$$

Choosing $\delta = k_p \mu_p$ this reduces to

$$\left| (A_p^{(n)} f)(x) - f(x)\left(A_p^{(n)} e_0\right)(x) \right|$$

$$\leq \|f\| \|A_p(t-x)\| + w(f', k_p \mu_p)\left\{\frac{\mu_p}{2\mu_p} + \frac{1}{2}\mu_p \|A_p e_0\|^{1/2} + \frac{k_p \mu_p}{8}\|A_p e_0\|\right\}$$

$$= \|f\| \|A_p(t-x)\| + w(f', k_p \mu_p)\mu_p\left\{\frac{1}{2\mu_p} + \frac{1}{2}\|A_p e_0\|^{1/2} + \frac{k_p}{8}\|A_p e_0\|\right\} \qquad (2.5)$$

Clearly

$$\left| -f(x) + f(x)\left(A_p^{(n)} e_0\right)(x) \right| \leq \|f\| \|A_p e_0 - 1\| \qquad (2.6)$$

On adding (2.5.) and (2.6) we get (2.1). In case $\mu_p = 0$ then for every $\delta > 0$ we get from (2.5) that

$$\left(A_p^{(n)} f\right)(x) = f(x)\left(A_p^{(n)} e_0\right)(x)$$

so

$$\left|(A_p^{(n)} f)(x) - f(x)\right| = \left|f(x)(A_p^{(n)} e_0)(x) - f(x)\right| \le \|f\| \left\|A_p e_0 - 1\right\|$$

Again, if $(A_p^{(n)} e_0)(x) = 1$ and $(A_p^{(n)} t)(x) = x$, then $(A_p^{(n)}(t-x))(x) = 0$. So from (2.4), we get the rest of the proof.

§3. Applications to almost convergence. By choosing $a_{pm}^{(n)} = 1/p$ for $n+1 \le m \le n+p$ and $a_{pm}^{(n)} = 0$ otherwise, in (2.1) and (2.2), we get an estimate on almost convergence which is sharper than that of Mohapatra [7]. Now we applicate the results to the Bernstein polynomials. For $f \in \mathcal{C}\,[0,1]$ the Bernstein polynomial of m-th order is defined as

$$L_m(f)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k}{m}\right).$$

We know that $L_m(1)(x) = 1$; $L_m(t)(x) = x$ and $L_m(t-x)^2(x) = x(1-x)/m$. So for $p \ge 1$

$$\mu_p^2 = \|A_p(t-x)^2(x)\|$$

$$= \sup_{n \ge 1} \sup_{x} \left(A_p^{(n)}(t-x)^2\right)(x)$$

$$= \sup_{n \ge 1} \sup_{x} \frac{1}{p} \sum_{n+1}^{n+p} \frac{x(1-x)}{m}$$

$$= \frac{1}{4p} \sup_{n \ge 1} \sum_{n+1}^{n+p} \frac{1}{m} \le \frac{1}{4p} \sup_{n \ge 1} \frac{p}{n+1} \le \frac{1}{8}.$$

Therefore $\mu_p \simeq 1/2\sqrt{2}$. Choosing $k_p = 2$ we get from (2.2).

$$\|A_p f - f\| \le \frac{1}{2\sqrt{2}} w\left(f'; \frac{1}{\sqrt{2}}\right) \tag{3.1}$$

which appears to be a new constant in the case of almost convergence on the Bernstein polynomials.

Now we apply the result to the positive linear operators obtained from the inversion of Weierstrass transformations. For a measurable function f defined on $(-\infty, \infty)$ the inversion operators are given by:

$$(L_m f)(x) = \left(\frac{m}{4\pi}\right)^{1/2} \int_{-\infty}^{\infty} exp\left(-(t-x)^2 \frac{m}{4}\right) f(t)dt \quad ; m \geq 1 \quad (3.2)$$

We know that $(L_m f)(x) = 1$, $(L_m t)(x) = x$ and $(L_m (t)^2(x) = x^2 + 2/m$ and consequently $(L_m (t-x)^2)(x) = 2/m$ (see [4]). So for $p \geq 1$,

$$\mu_p^2 = \sup_{n\geq 1} \frac{1}{p} \sum_{m=n+1}^{n+p} \leq \sup_{n\geq 1} \frac{2}{p} \cdot \frac{p}{n+1} \leq 1 .$$

Therefore $\mu_p \simeq 1$. Choosing $k_p = 1$ in (2.2) we get

$$||A_p f - f|| \leq \frac{9}{8} w(f';1) \quad (3.3)$$

wich also appears to be a new constant in the case of Weierstrass transformations.

§4. Aplication on convergence. We have deduced the following estimates by choosing $a_{pm}^{(n)} = \delta_m^p$ in each one of the cases given below:

Case 1. For $f \in \mathfrak{C}[0,1]$ let L_m be the Bernstein operator of order m. So for $m \geq 1$; $\mu_m = 1/2\sqrt{m}$. By choosing $k_m = 2$ in (2.2) one obtains for $f \in \mathfrak{C}^1[0,1]$ and $m \geq 1$.

$$||L_m f - f|| \leq \frac{1}{2\sqrt{m}} w\left(f'; \frac{1}{\sqrt{m}}\right)$$

wich is sharper than the corresponding estimate of Lorentz [5]. Again by choosing $k_m = 2/\sqrt{m}$ in (2.2) we obtain for $f \in \mathcal{C}^1[0,1]$, $m \geq 1$.

$$\|L_m f - f\| \leq \frac{\left(\sqrt{m}+1\right)^2}{8m} w\left(f\,';\, \frac{1}{m}\right) \tag{4.1}$$

This result is due to Schurer [8].

Case 2. For $f \in \mathcal{C}\,[0,\infty]$, let

$$\left(L_m^{\lambda} f\right)(x) = e^{-mx} \operatorname{sech}\left(2\lambda\sqrt{m\,x}\right) \sum_{k=0}^{\infty} \frac{(-1)^k H_{2k}(i\lambda)}{k} (mx)^k f(k/m)$$

where

$$\frac{H_{2k}(i\lambda)}{k} = \sum_{\gamma=0}^{k} \frac{(-1)^{\gamma}(2\,i\,\lambda)^{2k-2\gamma}}{\gamma\,(2k-2\gamma)}, \quad \lambda \text{ real},$$

is the positive linear operator of order m introduced by Meir and Sharma in [6]. We know that

$$(L_m^{\lambda}\, 1)(x) = 1$$

$$\left(L_m^{\lambda}(t-x)\right)(x) = \lambda\sqrt{\frac{x}{m}}\tanh\left(2\lambda\sqrt{m\,x}\right)$$

$$\left(L_m^{\lambda}(t-x)^2\right)(x) = \frac{(\lambda^2+1)}{m}x + \frac{\lambda\sqrt{x}}{2m\sqrt{m}}\tanh\left(2\lambda\sqrt{mx}\right)$$

so for $m \geq 1$, and any $x \in [0,a]$,

$$\mu_m = \sqrt{\frac{(\lambda^2+1)\,a}{m} + \frac{\lambda\sqrt{a}}{2m\sqrt{m}}\tanh\left(2\lambda\sqrt{ma}\right)}$$

$$= \sqrt{\frac{D\,m}{m}} \qquad (say).$$

By choosing $k_m = 1/\sqrt{D_m}$ in (2.2), we get for $f \in \mathcal{C}^1[0, a]$, and $m \geq 1$

$$\|L_m^{\lambda}(f) - f\| \leq \frac{\left(2\sqrt{D_m} + 1\right)^2}{8} \frac{1}{\sqrt{m}} w\left(f'\ ;\ \frac{1}{\sqrt{m}}\right) \qquad (4.2)$$

We note that by choosing $\lambda = 0$ in (4.2) one obtains for Szâsz operators

$$\|L_m^0 f - f\| \leq \frac{\left(2\sqrt{a} + 1\right)^2}{8} \frac{1}{\sqrt{m}} w\left(f\ ;\ \frac{1}{\sqrt{m}}\right),$$

which is sharper than the corresponding estimate of Stancu [9]:

$$\|L_m^0 f - f\| \leq \left(\sqrt{a} + a\right) \frac{1}{\sqrt{m}} w\left(f'\,;\ \frac{1}{\sqrt{m}}\right).$$

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