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Blow-up for a Nonlocal Nonlinear Diffusion Equation with Source

Explosión para una ecuación no lineal de difusión no local con fuente

Mauricio Bogoya

Universidad Nacional de Colombia, Bogotá, Colombia.

Abstract. We study the initial-value problem prescribing Neumann boundary conditions for a nonlocal nonlinear diffusion operator with source, in a bounded domain in $\mathbb{R}^N$ with a smooth boundary. We prove existence, uniqueness of solutions and we give a comparison principle for its solutions. The blow-up phenomenon is analyzed. Finally, the blow up rate is given for some particular sources.

Key words and phrases. Nonlocal diffusion, Neumann boundary conditions, Blow-up.

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Resumen. Se estudia el problema de valor inicial con condiciones de Neumann para un operador no lineal de difusión no local con fuente, en un dominio acotado en $\mathbb{R}^N$ con frontera suave. Se demuestra la existencia y unicidad de las soluciones y se da un principio de comparación para las soluciones. Se analiza el fenómeno de explosión. La razón de explosión es dada para algunas fuentes particulares.

Palabras y frases clave. Difusión no local, condiciones de Neumann, explosión.

1. Introduction

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a non-negative, smooth, radially symmetric and strictly decreasing function, with $\int_{\mathbb{R}^N} J(x)\, dx = 1$. Assume out that J is supported in the unit ball. Equations of the form

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\, dy - u(x,t), \qquad (1)$$

and variations of it, have been widely used in the last decade to model diffusion processes (see [10, 6]). As stated in [6] if $u(x,t)$ is thought as a density at the point x at time t, and $J(x-y)$ is thought as the probability distribution of jumping from location y to location x, then $(J*u)(x,t)$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}}^N J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external sources, leads immediately to the fact that the density u satisfies equation (1). This equation is called nonlocal diffusion equation since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but also on all the values of u in a neighborhood of x through the convolution term $J*u$. This equation shares many properties with the classical heat equation $u_t = \Delta u$ such as: a maximum principle holds for both of them and, even if J is compactly supported, perturbations propagate with infinite speed.

A classical equation that has been used to model diffusion is the well known porous medium equation, $u_t = \Delta u^m$ with $m > 1$, which shares several properties with the heat equation. However, there exists a fundamental difference: if the initial data $u(\bullet, 0)$ is compactly supported, then $u(\bullet, t)$ has compact support for all $t > 0$. Some properties of solutions for the porous medium equation have been largely studied over the past few years. See for example [2, 11] and the bibliography therein.

Related to the porous medium equation, a simple nonlocal nonlinear model in one dimension where the diffusion at a point depends on the density, was introduced in [4]. In this model if $u(x,t)$ is thought as a density at the point x at time t and the probability distribution of jumping from location y to location x is given by $J\left(\dfrac{x-y}{u(y,t)}\right)\dfrac{1}{u(y,t)}$ when $u(y,t) > 0$ and 0 otherwise, then the rate at which individuals are arriving to position x from all other places is given by $\displaystyle\int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right)dy$ and the rate at which they are leaving location x to travel to all other sites is given by $-u(x,t) = -\displaystyle\int_{\mathbb{R}} J\left(\frac{y-x}{u(x,t)}\right)dy$. As before, in absence of external sources, this leads immediately to the fact that the density u satisfies the equation

$$u_t(x,t) = \int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right)dy - u(x,t). \tag{2}$$

It is proved in [4] that this problem, as well as the porous medium equation, have the finite speed propagation property. Compactly supported initial data develops a free boundary and the support covers the whole $\mathbb{R}$.

Bogoya in [3] extend this model to higher space dimensions. In this model the probability distribution of jumping from location y to location x is given by $J\left(\frac{x-y}{u^{\alpha}(y,t)}\right)\frac{1}{u^{N\alpha}(y,t)}$ for all $0<\alpha\leq\frac{1}{N}$ and $N\geq 1$, when $u(y,t)>0$ and 0 otherwise. In the same way that in previous cases, this consideration absence of external sources leads immediately to the fact that the density u satisfies the equation

$$u_t(x,t)=\int_{\mathbb{R}^N} J\left(\frac{x-y}{u^{\alpha}(y,t)}\right)u^{1-N\alpha}(y,t)\,dy-u(x,t). \tag{3}$$

In the case that the initial data $u(\bullet,0)\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, by technical reasons it is better consider a slightly more general set of initial conditions:

$$\begin{aligned} &u_t(x,t)=\int_{\mathbb{R}^N} J\left(\frac{x-y}{u^{\alpha}(y,t)}\right)u^{1-N\alpha}(y,t)\,dy-u(x,t) \quad \text{in} \quad \mathbb{R}^N\times[0,\infty),\\ &u(x,0)=d+w_0(x) \quad \text{on} \quad \mathbb{R}^N, \end{aligned} \tag{4}$$

where $d\geq 0$, $w_0\in L^1(\mathbb{R}^N)$ and $w_0\geq 0$.

It is proved in [3] that this problem shares with the porous medium equation the finite speed propagation property. Compactly supported initial data develops a free boundary. Furthermore, the Neumann problem is studied as well as the Dirichlet problem for this model in a smooth domain $\Omega\subseteq\mathbb{R}^N$. For the Neumann problem, it is proved that solutions exist globally and stabilize to the mean value of the initial data as $t\to\infty$. On the other hand, for the Dirichlet problem, it is proved the globally existence of solutions. In the same way, it is proved that it stabilizes to zero as $t\to\infty$.

One of the most remarkable properties that can be present in nonlinear evolution problems is the possibility of having solutions that become unbounded in finite time. Such phenomenon is known as Blow-up in the literature, and can be described as follows: there exists a time $0<T<\infty$, called the blow-up time, such that the solution is well defined for all $0<t<T$, while $\sup_{x\in\Omega}u(x,t)\to\infty$ as $t\to T^-$. This means that the solution blows up at finite time T.

For general references on blow-up problems see [5, 8, 9, 1].

The Neumann Problem. We study the problem for $x\in\overline{\Omega}$

$$\begin{aligned} u_t(x,t)=&\int_{\Omega} J\left(\frac{x-y}{u^{\alpha}(y,t)}\right)u^{1-N\alpha}(y,t)\,dy\\ &-\int_{\Omega} J\left(\frac{x-y}{u^{\alpha}(x,t)}\right)u^{1-N\alpha}(x,t)\,dy+f\big(u(x,t)\big)\\ u(x,0)=&\,u_0(x) \end{aligned} \tag{5}$$

where $u_0\in C(\overline{\Omega})$ is a non-negative function and $\Omega\subseteq\mathbb{R}^N$ is a bounded and smooth domain. In this model it is assumed that no individual can jump from the interior towards the outside (and viceversa) of the domain Ω. Therefore the integrals are considered in Ω, and $f(u)$ is a function of u representing reaction (source).

We considered the following hypothesis on f:

(H_1): $f : [0,\infty) \longrightarrow [0,\infty)$, increasing function, convex, $f(0) \geq 0$, $f(s) > 0$ for all $s > 0$.

(H_2): $\displaystyle\int^{\infty} \frac{1}{f(s)}\, ds < \infty$.

We will address in this paper the questions of existence, uniqueness and comparison principles for solutions of (5). Moreover, we study the blow up phenomenon for solutions of (5). We will study in the near future further questions such as the blow up set, the Cauchy problem, the Dirichlet Problem, and the discrete model.

2. The Neumann Problem

2.1. Existence and Uniqueness

In this section, we use the ideas developed in [3]. First, we show existence and uniqueness for $u_0 \in L^1(\Omega)$ and a Lipschitz nonlinearity of f. The existence and uniqueness of solution to (5) it is a consequence of Banach's fixed point theorem. Fix $t_0 > 0$ and we consider the Banach space $X = C\big([0,t_0] : L^1(\Omega)\big)$ with the norm

$$\big|\|w\|\big| = \max_{0\leq t\leq t_0} \|w(\bullet,t)\|_{L^1(\Omega)}.$$

Let $X_{t_0} = \big\{w \in C\big([0,t_0] : L^1(\Omega)\big) : w \geq 0\big\}$ which is a closed subset of $C\big([0,t_0]; L^1(\Omega)\big)$. The solution will be obtained as the fixed point of operator $T_{w_0} : X_{t_0} \longrightarrow X_{t_0}$ defined by

$$\begin{aligned} T_{w_0,f}(w)(x,t) &= \int_0^t \int_\Omega J\left(\frac{x-y}{w(y,s)^\alpha}\right) w(y,s)^{1-N\alpha}\, dy\, ds \\ &- \int_0^t \int_\Omega J\left(\frac{x-y}{w(x,s)^\alpha}\right) w(x,s)^{1-N\alpha}\, dy\, ds + \int_0^t f\big(w(x,s)\big)\, ds + w_0(x), \end{aligned}$$

In what follows, we study the problem (5) for a Lipschitz function f and then, by convergence we extend our results to a function f satisfying H_1.

The following Lemma is very important for our study:

Lemma 1. *Let f be a Lipschitz function with Lipschitz's constant $K > 0, w_0, z_0$ non negative functions such that $w_0, z_0 \in L^1(\Omega)$ and $w, z \in X_{t_0}$. Then, there exists a constant $C = (2+K)t_0 > 0$ such that*

$$\big|\|T_{w_0,f} - T_{z_0,f}\|\big| \leq C\big|\|w - z\|\big| + \|w_0 - z_0\|_{L^1(\Omega)}$$

Proof. Let $w, z \in X_{t_0}$. We have

$$\begin{aligned}
&\int_\Omega \big|T_{w_0,f}(w)(x,t) - T_{z_0,f}(z)(x,t)\big|\,dx \le \\
&\int_0^t \int_\Omega \left| \int_\Omega \left(J\left(\frac{x-y}{w^\alpha(y,s)}\right) w^{1-N\alpha}(y,s) - J\left(\frac{x-y}{z^\alpha(y,s)}\right) z^{1-N\alpha}(y,s) \right) dy \right| dx\,ds \\
&+ \int_0^t \int_\Omega \left| \int_\Omega \left(J\left(\frac{x-y}{w^\alpha(x,s)}\right) w^{1-N\alpha}(x,s) - J\left(\frac{x-y}{z^\alpha(x,s)}\right) z^{1-N\alpha}(x,s) \right) dy \right| dx\,ds \\
&\qquad + \int_0^t \int_\Omega \big|f(w(x,s)) - f(z(x,s))\big|\,dx\,ds + \int_\Omega \big|w_0 - z_0\big|(x)\,dx \\
&\qquad\qquad = I_1 + I_2 + I_3 + \|w_0 - z_0\|_{L^1(\Omega)}.
\end{aligned}$$

For the term I_1, we consider

$$A^+(s) = \big\{y \in \Omega : w(y,s) \ge z(y,s)\big\} \quad \text{and} \quad A^-(s) = \big\{y \in \Omega : w(y,s) < z(y,s)\big\}.$$

We have

$$\begin{aligned}
&\int_\Omega \left| \int_\Omega \left(J\left(\frac{x-y}{w^\alpha(y,s)}\right) w^{1-N\alpha}(y,s) - J\left(\frac{x-y}{z^\alpha(y,s)}\right) z^{1-N\alpha}(y,s) \right) dy \right| dx \\
&\le \int_\Omega \int_{A^+(s)} \left(J\left(\frac{x-y}{w^\alpha(y,s)}\right) w^{1-N\alpha}(y,s) - J\left(\frac{x-y}{z^\alpha(y,s)}\right) z^{1-N\alpha}(y,s) \right) dy\,dx \\
&- \int_\Omega \int_{A^-(s)} \left(J\left(\frac{x-y}{w^\alpha(y,s)}\right) w^{1-N\alpha}(y,s) - J\left(\frac{x-y}{z^\alpha(y,s)}\right) z^{1-N\alpha}(y,s) \right) dy\,dx.
\end{aligned}$$

Since J is a strictly decreasing radial function, the expression under the integrand sign is nonnegative and therefore we can apply the Fubini's theorem to obtain

$$\begin{aligned}
&\int_\Omega \int_{A^+(s)} \left(J\left(\frac{x-y}{w^\alpha(y,s)}\right) w^{1-N\alpha}(y,s) - J\left(\frac{x-y}{z^\alpha(y,s)}\right) z^{1-N\alpha}(y,s) \right) dy\,dx \\
&\qquad\qquad \le \int_{A^+(s)} \big(w(y,s) - z(y,s)\big)\,dy.
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \int_{A^-(s)} \left(J\left(\frac{x-y}{z^\alpha(y,s)}\right) z^{1-N\alpha}(y,s) - J\left(\frac{x-y}{w^\alpha(y,s)}\right) w^{1-N\alpha}(y,s) \right) dy\,dx \\
&\qquad\qquad \le \int_{A^-(s)} \big(z(y,s) - w(y,s)\big)\,dy.
\end{aligned}$$

In summary for I_1, we have that

$$I_1 \leq \int_0^t \int_\Omega \big|w(y,s) - z(y,s)\big|\, dy\, dt.$$

The term I_2 is analysed in a similar way. Therefore we have that

$$I_2 \leq \int_0^t \int_\Omega \big|w(x,s) - z(x,s)\big|\, dx\, dt.$$

For the term I_3, as f is a Lipschitz function with Lipschitz's constant $K > 0$, we have

$$I_3 \leq K \int_0^t \int_\Omega \big|w(x,s) - z(x,s)\big|\, dx\, ds.$$

Finally we have

$$\begin{aligned}\int_\Omega \big|T_{w_0,f}(w)(x,t) - T_{z_0,f}(z)(x,t)\big|\, dx \\ \leq (2+K) \int_0^t \int_\Omega \big|w(y,s) - z(y,s)\big| dy\, ds + \|w_0 - z_0\|_{L^1(\Omega)}.\end{aligned}$$

Therefore, we obtain

$$\big|\|T_{w_0}(w) - T_{z_0}(z)\|\big| \leq C\big|\|w - z\|\big| + \|w_0 - z_0\|_{L^1(\Omega)},$$

with $C = (2+K)t_0$, as desired. ☑

Next, we study a theorem of existence and uniqueness of solutions.

Theorem 2. *If f is a Lipschitz function with Lipschitz's constant $K > 0$, $w_0 \in L^1(\Omega)$ a non-negative function, then there exist a unique solution u to* (5) *such that $u \in C\big([0,t_0] : L^1(\Omega)\big)$.*

Proof. With $z_0 \equiv 0$ and $z \equiv 0$ in Lemma 1 we get $T_{u_0,f} \in C\big([0,t_0] : L^1(\Omega)\big)$. Moreover, if $z_0 \equiv w_0$ in Lemma 1 and $C = (2+K)t_0 < 1$ with t_0 small enough, we obtain that T_{w_0} is a strict contraction in X_{t_0}; therefore there exists a unique fixed point of T_{u_0} in X_{t_0} by the Banach's fixed point theorem. The existence and uniqueness of solution to (5) in $[0,t_0]$ is proved. ☑

***Remark* 3.** The solutions of (5) depend continuously on the initial data. In fact, if u and v are solutions to (5) with initial data u_0 and v_0 respectively, then there exists a constant $\widetilde{C} = \widetilde{C}(t_0, K)$ such that

$$\big|\|u(\bullet,t) - v(\bullet,t)\|\big| \leq \widetilde{C}\|u_0 - v_0\|_{L^1(\Omega)}.$$

***Remark* 4.** The function u is a solution to (5) if and only if

$$
\begin{aligned}
u(x,t) &= \int_0^t \int_\Omega J\left(\frac{x-y}{u(y,s)^\alpha}\right) u(y,s)^{1-N\alpha}\, dy\, ds \\
&\quad - \int_0^t \int_\Omega J\left(\frac{x-y}{u(x,s)^\alpha}\right) u(x,s)^{1-N\alpha} dy\, ds + \int_0^t f\big(u(x,s)\big)\, ds + u_0(x).
\end{aligned}
$$

***Remark* 5.** If u is a solution of (5) with initial data u_0, then the mass verifies

$$
\int_\Omega u(x,t)\, dx = \int_\Omega u_0(x)\, dx + \int_0^t \int_\Omega f\big(u(x,s)\big)\, dx\, ds.
$$

For a continuous initial data we have the following result

Theorem 6. *Let f be a Lipschitz function with Lipschitz's constant $K > 0$, $u_0 \in C\big(\overline{\Omega}\big)$ a non-negative function. Then, there exists an unique solution $u \in C(\overline{\Omega} \times [0,t_0])$ of (5).*

***Proof*.** The proof is similar to the one of Theorem 2 and hence we omit the details. ☑

Next, we study a Comparison Principle.

Theorem 7 (Comparison Principle)**.** *Let u and v be continuous solutions of (5) with initial data u_0 and v_0 respectively. If $u(x,0) \le v(x,0)$ for all $x \in \overline{\Omega}$, then*

$$
u(x,t) \le v(x,t) \qquad \textit{for all} \qquad (x,t) \in \overline{\Omega} \times [0,T).
$$

***Proof*.** First, we assume that $u(x,0) + \delta < v(x,0)$ for all $x \in \overline{\Omega}$, and that $u(x,0)$ and $v(x,0)$ are in C^1. Let us argue by contradiction and suppose that there exists a time $t_0 > 0$ and a point $x_0 \in \overline{\Omega}$ such that $u(x_0,t_0) = v(x_0,t_0)$ and $u(x,t) \le v(x,t)$ for $t < t_0$ for all $x \in \overline{\Omega}$.

Let us consider the set $G = \big\{x \in \overline{\Omega} : u(x,t_0) = v(x,t_0)\big\}$. Clearly G is closed and not empty. Let $x_1 \in G$. Then we have then

$$
\begin{aligned}
0 &\le (u-v)_t(x_1,t_0) = \\
&\int_\Omega \left(J\left(\frac{x_1-y}{u^\alpha(y,t_0)}\right) u^{1-N\alpha}(y,t_0) - J\left(\frac{x_1-y}{v^\alpha(y,t_0)}\right) v^{1-N\alpha}(y,t_0) \right) dy \\
&- \int_\Omega \left(J\left(\frac{x_1-y}{u^\alpha(x_1,t_0)}\right) u^{1-N\alpha}(x_1,t_0) - J\left(\frac{x_1-y}{v^\alpha(x_1,t_0)}\right) v^{1-N\alpha}(x_1,t_0) \right) dy \\
&\qquad + \big(f\big(u(x_1,t_0)\big) - f\big(v(x_1,t_0)\big)\big) \le 0,
\end{aligned}
$$

therefore, there exists $r > 0$ such that $u(y, t_0) = v(y, t_0)$ for all $y \in B(x_1, r)$, so that G is open, and then $G = \overline{\Omega}$. We have obtained a contradiction.

Now, if $u(x, 0)$ and $v(x, 0)$ are continuous functions, we consider the decreasing sequences of functions $u_n(x, 0)$ and $v_n(x, 0)$ in C^1 such that $u_n(x, 0) \searrow u(x, 0)$, $v_n(x, 0) \searrow v(x, 0)$ in $L^1(\Omega)$ as $n \to \infty$, and $u_n(x, 0) \le v_n(x, 0)$. Let u_n and v_n the respective solutions to (5) with initial data $u_n(x, 0)$ and $v_n(x, 0)$ respectively. By previous argument, we have $u_n \le v_n$. We obtain the result letting $n \to \infty$ in view of Remark 4 and the monotone convergence theorem. ☑

***Remark* 8.** The Comparison principle is valid in L^1.

Next, we will prove the local existence and uniqueness of solutions to problem (5) in the case f satisfies (H_1).

Theorem 9. *For all $u_0 \in C(\overline{\Omega})$ non-negative function and f that satisfies (H_1), there exists a time $T > 0$ and a unique solution u of* (5) *such that $u \in C\big(\overline{\Omega} \times [0, T)\big)$.*

Proof. Let $(f_n)_n$ be a increasing sequence of Lipschitz functions such that $f_n \le f_{n+1}$. Assume that $f_n(s) = f(s)$ in $[0, n]$. Let u_n be the unique solution of (5) with source f_n and initial data $u(x, 0)$.

By Comparison Principle (Theorem 7), we have that $u_n(x, t) \le u_{n+1}(x, t)$; hence there exists u, which can be ∞ in some points, such that $\lim_{n\to\infty} u_n = u$. Let $T = \sup\left\{t : \sup_{x\in\Omega} u(x, t) < \|u_0\|_\infty + 1\right\}$. It is easy to prove that $T > 0$.

Like before, if in the integral equation of Remark 4 we let $n \to \infty$, then after an application of the monotone convergence theorem, it follows that u is a unique solution of (5) in $\overline{\Omega} \times [0, T)$ with initial data $u(x, 0)$ and source $f(u)$. This proves the theorem. ☑

In a similar way we obtain the Comparison Principle Theorem for functions f satisfying H_1.

Theorem 10 (Comparison Principle)**.** *Let f be a function that satisfies assumption H_1, u and v be continuous solutions of* (5) *with initial data u_0 and v_0 respectively. If $u(x, 0) \le v(x, 0)$ for all $x \in \overline{\Omega}$, then*

$$u(x, t) \le v(x, t) \qquad \text{for all} \qquad (x, t) \in \overline{\Omega} \times [0, T).$$

2.2. Blow-up Analysis

In this section, we study the blow-up phenomenon for solutions of (5). We use some ideas of [7]. We have the following theorem:

Theorem 11. *Suppose that f satisfies (H_1) and (H_2). Let u be a solution of (5) with initial data $u_0 \in C(\overline{\Omega})$ such that $\int_\Omega u_0(x)\,dx > 0$, then u blows up in finite time.*

Proof. Let u be a solution of (5). We define $M(t)$ for $t > 0$ by

$$M(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx. \tag{6}$$

Taking into account (H_1) we obtain

$$M'(t) = \frac{1}{|\Omega|}\int_\Omega f\big(u(x,t)\big)\,dx \geq f\left(\frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx\right) \geq f\big(M(t)\big),$$

therefore

$$M'(t) \geq f\big(M(t)\big). \tag{7}$$

Since $M(0) > 0$ and $f(u) > 0$ for $u > 0$ we have $M'(t) > 0$, so that $M(t) > 0$ for all $t > 0$.

Integrating (7) on $[0,t]$ we obtain

$$\int_0^t \frac{M'(s)}{f(M(s))}\,ds \geq t,$$

therefore

$$\int_{M(0)}^{M(t)} \frac{ds}{f(s)} \geq t. \tag{8}$$

Let

$$F(u) = \int_u^\infty \frac{ds}{f(s)}. \tag{9}$$

Since f satisfies (H_2), by (8) we have that

$$F(M(0)) - F(M(t)) \geq t.$$

We can conclude that the solution u of (5) blows up in finite time, as desired. ☑

Corollary 12. *If u is a solution to (5) with $f(u) = u^p$, $p > 1$; $f(u) = e^u$; $f(u) = (1+u)\ln^p(1+u)$, $p > 1$, then u blows up in finite time.*

The blow up rate of the solution of (5) for particular cases in that f is given in the previous corollary will be analysed.

With this end, let

$$U(t) = \max_{\overline{\Omega}} u(x,t), \qquad \text{for all} \qquad t \in [0,T),$$

where u is a solution to (5) that blow up in finite time $T > 0$.

Proposition 13. *$U(t)$ is locally Lipschtiz continuous function. Furthermore,*

$$U'(t) \leq f\big(U(t)\big) \qquad a.e. \tag{10}$$

and

$$U'(t) \geq -U(t) + f\big(U(t)\big) \qquad a.e. \tag{11}$$

Proof. Let

$$U(t_1) = \max_{\overline{\Omega}} u(x,t_1) = u(x_1,t_1)$$
$$U(t_2) = \max_{\overline{\Omega}} u(x,t_2) = u(x_2,t_2).$$

Since J is a smooth function, we have that for $h = t_2 - t_1$,

$$U(t_2) - U(t_1) \geq u(x_1,t_2) - u(x_1,t_1) = hu_t(x_1,t_1) + o(h),$$
$$U(t_2) - U(t_1) \leq u(x_2,t_2) - u(x_2,t_1) = hu_t(x_2,t_1) + o(h),$$

from which it follows that $U(t)$ is locally Lipschtiz continuous function.

Next, we show that (10) is true. For $t_2 > t_1$ we have

$$\frac{U(t_2) - U(t_1)}{t_2 - t_1} \leq u_t(x_2,t_2) + o(1).$$

On the other hand,

$$u_t(x_2,t_2) = \int_\Omega J\left(\frac{x_2 - y}{u^\alpha(y,t_2)}\right) u^{1-N\alpha}(y,t_2)\,dy - \int_\Omega J\left(\frac{x_2 - y}{u^\alpha(x_2,t_2)}\right) u^{1-N\alpha}(x_2,t_2)\,dy + f\big(u(x_2,t_2)\big).$$

Since $u(x_2,t_2) \geq u(y,t_2)$ we obtain

$$u_t(x_2,t_2) \leq f\big(u(x_2,t_2)\big)$$

and therefore, we get

$$U'(t) \leq f(U(t)). \qquad a.e.$$

With the aim to show that (11) is true, let $t_2 > t_1$. Then

$$\frac{U(t_2) - U(t_1)}{t_2 - t_1} \geq u_t(x_1,t_1) + o(1).$$

On the other hand

$$u_t(x_1,t_1) = \int_\Omega J\left(\frac{x_1-y}{u^\alpha(y,t_1)}\right)u^{1-N\alpha}(y,t_1)\,dy - u(x_1,t_1) +$$
$$\int_{\mathbb{R}^N\setminus\Omega} J\left(\frac{x_1-y}{u^\alpha(x_1,t_1)}\right)u^{1-N\alpha}(x_1,t_1)\,dy + f\big(u(x_1,t_1)\big) \geq$$
$$-\,u(x_1,t_1) + f\big(u(x_1,t_1)\big).$$

Therefore

$$U'(t) \geq -U(t) + f\big(U(t)\big) \qquad a.e.$$

The proposition is proved. ☑

As a consequence of the previous proposition, from (10) we obtain

$$\frac{U'(t)}{f\big(U(t)\big)} \leq 1,$$

and taking into account (9) we get

$$-\big(F(U)\big)_t \leq 1.$$

Integrating on $[t,T)$ for $t>0$, we obtain

$$F\big(U(t)\big) \leq T-t. \tag{12}$$

Taking into account (H_2), we obtain that $f(s)/s\to\infty$ as $s\to\infty$, and then, from (11), it follows that $U'(t)\geq \frac{1}{2}f\big(U(t)\big)$ for t near T.

As a consequence of previous the analysis, we have the following theorem.

Theorem 14. *Let u be a solution to (5) that bows up in finite time $T>0$, and the source term in given by $f(u)$.*

1) If $f(u)=u^p$ with $p>1$, then

$$\max_{\overline{\Omega}} u(x,t) \sim \frac{1}{(T-t)^{\frac{1}{p-1}}} \qquad \text{for} \qquad t\in(0,T).$$

2) If $f(u)=e^u$, then

$$\max_{\overline{\Omega}} u(x,t) \sim -\ln(T-t) \qquad \text{for} \qquad t\in(0,T).$$

3) If $f(u)=(1+u)\ln^p(1+u)$ with $p>1$, then

$$\max_{\overline{\Omega}} u(x,t) \sim e^{(T-t)^{1/(1-p)}}-1 \qquad \text{for} \qquad t\in(0,T).$$

The notation $f\sim g$ means that there exist finite positive constants c_1 and c_2 such that $c_1 g\leq f\leq c_2 g$.

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Departamento de Matemáticas
Universidad Nacional de Colombia
Carrera 30, Calle 45
Ciudad Universitaria
Bogotá, Colombia
e-mail: mbogoyal@unal.edu.co