Trazas en categorías simétricas monoidales

Juan Sebastián Jaramillo Díaz

Universidad Nacional de Colombia
Facultad de Ciencias, Escuela de Matemáticas
Medellín, Colombia
2019

Trazas en categorías simétricas monoidales

Juan Sebastián Jaramillo Díaz

Tesis presentada como requisito parcial para optar al título de:
Magister en Ciencias - Matemáticas

Director:
Ph.D. Camilo Arias Abad

Línea de Investigación:
Topología Algebraica
Grupo de Investigación:
Grupo Interinstitucional de Investigación en Geometría y Topología

Universidad Nacional de Colombia
Facultad de Ciencias, Escuela de Matemáticas
Medellín, Colombia
2019

Traces in symmetric monoidal categories

Juan Sebastián Jaramillo Díaz

Universidad Nacional de Colombia
Facultad de Ciencias, Escuela de Matemáticas
Medellín, Colombia
2019

"What would life be if we had no courage to attempt anything? "-Vincent van Gogh

Acknowledgments

First of all, I want to thank to my advisor Professor Camilo Arias, I am constantly impressed by his vision and intuition for mathematics as well as his ability to communicate ideas. I can only hope that some of those skills have rubbed off onto me.

I would also like to thank Professors José Manuel Gómez and Alexander Quintero Vélez for their support at this stage of the journey and for the many talks and lectures in which I have had the opportunity to participate. In addition, I would like to thank them for their time and comments in reviewing this thesis. I apologize for this long thesis, it was longer than I expected.

I would also like to thank my friends who made my stay in Medellín more comfortable. In particular, I would like to mention Luis Fernando Jiménez, Jorge Moreno, Jefferson Zapata, Sebastian Vélez and Manuel Echeverry.

I owe gratitude to Liliana Parra for her help with numerous academic matters during my master's degree.

To my mother and father for their constant support, also to my siblings Camilo and Catalina, and to my niece Sara.

Most importantly, I have no words to express my infinite gratitude to Paulina Hoyos for her help and support throughout these years, from personal to academic, also for her patience. Thanks to her I was able to write this thesis in English.

Resumen

El objetivo de este trabajo es generalizar ideas básicas del álgebra lineal y la topología, tales como trazas y puntos fijos, en un contexto categórico. Cada una de esas generalizaciones tiene detrás objetos e ideas importantes, como el espectro de Thom y homotropía estable. A lo largo de esta tesis, pretendemos conectar esas generalizaciones por medio de la dualidad, para contar una historia desde enfoques tanto categóricos como topológicos. Luego tratamos de ir más allá de esos temas y estudiamos temas distantes pero relacionados que nacieron como ejemplos particulares de la teoría abstracta, por ejemplo, la categoría de homotropía estable (ejemplo de categoría monoidal), la dualidad de Atiyah (ejemplo de dualizabilidad) y el espectro de Thom (ejemplo de objeto dualizable).

Palabras clave: Trazas, puntos fijos, Categorías monoidales, Homotopía estable, Spectro de Thom, Teorema de punto fijo de Lefschetz, Dualidad de Atiyah, Topología de cuerdas.

Abstract

The objective of this work is to generalize basic ideas from linear algebra and topology, such as traces and fixed points, into a categorical context. Each of those generalizations has important objects and ideas behind, such as the Thom spectrum and stable homotopy phenomena. Throughout this thesis, we intend to connect those generalizations by means of dualizability, in order to tell a story from both categorical and topological approaches. We then try to go beyond those topics and we study distant but related topics that were born as particular examples of the abstract theory, for instance, the stable homotopy category (example of monoidal category), Atiyah duality (example of dualizability), and the Thom spectrum (example of a dualizable object).

Keywords: Traces, Fixed points, Monoidal categories, Stable homotopy, Thom Spectra, Lefschetz fixed-point theorem, Atiyah duality, String topology.

Contents

What is this thesis about?

There are two basic ideas involved in this thesis: traces and fixed points. In this work, the author intends to study the connection between them in various topological and categorical aspects. Following that path, interesting things emerge such as categorical dualizability, which aims to extend notions from linear algebra.

Fixed points are solutions of the equation $f(x) = x$ for a map $f : M \to M$. This can be studied from an algebraic, analytical, or topological point of view. In this work, we have decided to take a topological approach. Thus, in Chapter 1 we study intersection theory and how it fits in the study of fixed points. From that perspective, the Lefschetz number of a smooth map $f : M \to M$ emerges, denoted by $L(f)$, which in some way "counts" the number of fixed points. Throughout Chapter 1 we will develop some refinements of the Lefschetz number that allow us to work with that invariant in a suitable way. In addition, Lefschetz number formulas suggest a relation between fixed points and traces.

In linear algebra one defines the trace of a square matrix as the sum of the diagonal elements. One can "describe" the trace in a more abstract way in terms of the categorical structure of the category of vector spaces, namely the tensor product. This will help us, in Chapter 2, to extend the notion of trace to any symmetric monoidal category which, in turn, gives us the notion of dualizable objects in monoidal categories. This notion of general trace in symmetric monoidal categories satisfies many properties, some of which come from linear algebra and others are technical properties. Those properties allow us to characterize and extend the notion of traces as a certain family of maps which satisfies special properties, even, that abstraction does not require a category with dualizable objects. We culminate Chapter 2. by showing how general traces are related to categorical fixed points. That part of the thesis is full of technical concerns which arise from theoretical computer science, to be precise recursion theory.

The first step in this generalization is to extend the context where we will work, that is, symmetric monoidal categories. They can be thought of as a natural generalization of the category of finite-dimensional vector spaces, that is, categories with a product that mimics the features of the tensor product in linear algebra. It is then natural to ask what structures from linear algebra can be extended to symmetric monoidal categories. Dualizability is one of those ideas, this can be described in terms of evaluation and coevaluation maps. The idea of this thesis is to study other examples and their relationship with fixed points. The goal of Chapter 3 is to study Atiyah duality. It is an example of dualizability in the stable homotopy category. Thus, we need to stop for a while to motivate and construct the category of spectra which is a predecessor of the stable homotopy category or **HoSpect**.

In Chapter 3, we also present an alternative proof of the Lefschetz-Hopf theorem. We will show that in the stable homotopy category the trace of a smooth map $f : M \to M$ with M a compact and closed manifold is the fixed point index of f. It requires to extend dualizability in simpler categories such as Alexander duality and Spannier-Whitehead duality in the category of smooth manifolds and spectra, respectively. As a result, we get the Thom spectrum, denoted by M^{-TM}, which is Atiyah-dual of the suspension spectrum $\Sigma^\infty M_+$. Thom spectrum is an important example of spectra and has many connections with other topics like cobordism theory.

Another interesting subject related to Thom spectrum is the string topology, that is, the study of the algebraic structure on the homology of the free loop space of a smooth manifold. In Chapter 4, we start by studying the original ideas of the Chas and Sullivan product. Then, we move to a more general context by using the tools developed in Chapter 3. We will show how we can give to the Thom spectrum the structure of symmetric ring spectrum. That structure is "pulled-back" to the evaluation map $ev : LM \to M$ and it gives to the Thom spectrum LM^{-TM} the structure of ring symmetric spectrum. We can induce a product into $H_*(LM)$, thanks to the product structure in LM^{-TM}, which coincides with the original ideas of Chas and Sullivan.

In the appendix of this work, we have decided to give a brief introduction to string diagrams. It is motivated, in Chapter 2, by studying freely generated monoidal categories, because intuitively one-dimensional bordisms are just strings with certain relations. Following this idea, we can use those diagrams to describe equations in any symmetric monoidal category, that makes this theory more accessible and avoids complicated proofs. Indeed, many of the proofs in Chapter 2 are made by deforming strings.

In conclusion, this thesis has the goal to generalize basic ideas from linear algebra and topology, such as traces and fixed points, into a categorical context. Each of those generalizations has important objects and ideas behind, such as the Thom spectrum and stable homotopy phenomena. Throughout this thesis, we intend to connect those generalizations by means of dualizability, in order to tell a story from both categorical and topological approaches. Then, we try to go beyond those topics and we study distant but related topics which were born as particular examples of the abstract theory, for instance, the stable homotopy category (example of monoidal category), Atiyah duality (example of dualizability), and the Thom spectrum (example of a dualizable object).

Chapter 1

Lefschetz fixed point theorem

In this chapter we will give a brief introduction to fixed point theory for manifolds. We will introduce the necessary tools in order to understand this theory, always with a topological point of view. As a second step we will provide a short introduction to intersection theory, the perspective used to study fixed points. Finally, we will present the main theorem of this chapter, Lefschetz fixed point theorem, which will be proved in Chapter 3 using the tools developed in a more abstract language in Chapter 2.

Let $f : M \to M$ be a smooth map on a compact and oriented manifold M. We want to study the solutions to the equation $f(x) = x$. From a geometric point of view, we can think of the set of fixed points as the pairs $(x, y) \in M \times M$ that belong to the intersection of the graph of f, denoted by G_f, and the diagonal submanifold denoted by Δ. We may use intersection theory to count fixed points. We will do this with an invariant called the Lefschetz number.

We will assume that the reader is familiar with basic concepts in differential topology such as homology and cohomology, Poincaré duality, and tools of integration over manifolds. Good and classical references include [31], [10], [30].

All the manifolds that we will consider in this chapter are smooth and oriented manifolds, unless otherwise stated.

1.1 Preliminaries

Let us start with a review of orientation and transversality. These definitions will be fundamental in our study of intersection theory.

Let V be a vector space of dimension n, $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_n\}$ be two ordered basis for V. Let $A : V \to V$ be the linear map defined by $A(w_i) = v_i$ for $i = 1, \cdots, n$.

We say that these bases are equivalently oriented if $\det(A) > 0$, otherwise they have opposite orientations. This determines an equivalence relation in the set of ordered basis of V. An **orientation** of V is a choice of one oriented class positive or negative.

Definition 1. *Let M be a manifold, we say M is orientable if there is a smooth choice of orientations for all tangent spaces T_xM for $x \in M$.*

Let V_1 and V_2 be two oriented vector spaces, the direct sum $V_1 \oplus V_2$ is also an oriented vector space with an orientation defined as follows. If $\{v_1, \cdots, v_n\}$ is a positively oriented basis for V_1 and $\{w_1, \cdots, w_m\}$ is a positively oriented basis for V_2 then $\{v_1, \cdots, v_n, w_1, \cdots, w_m\}$ is a positive oriented basis for $V_1 \oplus V_2$.

This allows us to induce an orientation in $M \times N$, the product of two oriented manifolds M and N. For $(x, y) \in M \times N$, we have a linear isomorphism $T_{(x,y)}(M \times N) = T_xM \times T_yN \cong T_xM \oplus T_yN$, thus an orientation on $M \times N$ is obtained from the orientation of M and N by taking the direct sum orientation of the oriented tangent spaces T_xM and T_yN.

Example. *Some examples of oriented manifolds include:*
- *The circle is an oriented manifold of dimension 1.*
- *n-dimensional spheres $\mathbb{S}^n$ are n-dimensional orientable manifolds.*
- *Surfaces with holes are examples of oriented manifolds.*

Non-oriented manifolds also are interesting in the theory of differentiable manifolds, but in this work non-orientable manifolds will not be considered. A great introduction to non-orientable manifolds is [10].

Definition 2. *Let N be a submanifold of M. We say that the map $f : X \to M$ is transversal to N, denoted by $f \pitchfork N$, if for each $x \in f^{-1}(N)$ such that $y = f(x)$,*

$$\operatorname{im}(d_xf) + T_yN = T_yM. \tag{1.1}$$

This means that at each point in $im(f) \cap N$, the vectors tangent to $im(f)$ and the vectors tangent to N together span the ambient tangent space.

Let S be a closed submanifold of dimension k of a compact manifold M of dimension n, we can define a functional from the k-cohomology of M to the real line.

$$\begin{aligned} \check{S} : & H^k(M) \to \mathbb{R} \\ & [\alpha] \mapsto \int_S \alpha, \end{aligned}$$

and by Poincaré's duality we get an isomorphism between $(H^k(M))^\vee$ and $H_{n-k}(M)$, the dual of the k-cohomology of M and the complementary homology of M, respectively.

The **Poincaré dual of S in M** is the cohomology class $\eta_S \in H^{n-k}(M)$ uniquely determined by the equality

$$\int_S \alpha = \int_M \alpha \wedge \eta_S, \tag{1.2}$$

where α is any closed form over M and η_S correspond to $\check{S}$ under the Poincaré's duality isomorphism.

A good reference is the famous book of V. Guillemin and A. Pollack, see [18].

Definition 3. *Let $S \subset M$ be a submanifold of M. The **normal bundle** of S in M is the vector bundle over S defined by*

$$N(S) := \frac{(TM)|_S}{TS}.$$

Now let $S \subset M$ be a submanifold of M. Then a natural question emerges: What does M look like "near" S? The famous tubular neighborhood theorem claims that S always admits a "tubular" neighborhood inside M. Moreover, the tubular neighborhood looks like a neighborhood of S inside its "normal bundle".

Theorem 1.1. *Let $S \subset M$ be a smooth submanifold. Then there exists a diffeomorphism from an open neighborhood of S in $N(S)$ onto an open neighborhood of S in M.*

An example of a tubular neighborhood is described in the next figure.

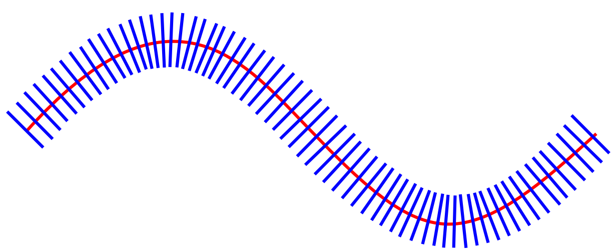


Figure 1.1: Tubular neighborhood

As a special case we will consider a manifold embedded in an euclidean space, that is possible thanks to Whitney's embedding theorems. In this case we can describe in detail tubular neighborhoods which we will call ϵ-neigborhoods.

Theorem 1.2. *Let $\iota : M \to \mathbb{R}^K$ be a smooth compact submanifold. Then there exists a positive number ϵ such that, if we let M_ϵ be the ϵ-**neighborhood** of X,*

$$M_\epsilon := \{y \in \mathbb{R}^n : |y - x| \leq \epsilon \text{ for some } x \in M\}, \tag{1.3}$$

then

1. *For each $y \in M_\epsilon$ possesses a unique closest point $\pi_\epsilon(y) \in M$.*

2. *The map $\pi_\epsilon : M_\epsilon \to M$ is a submersion.*

π_ϵ is defined by $\pi_\epsilon(y) = \pi \circ \varphi^{-1}(y)$ where $\pi : N(M) \to M$ is the vector bundle projection and $\varphi : N(M) \to \mathbb{R}^n$ is given by $\varphi(x, v) = x + v$.

The following figure helps us visualize ϵ-neighborhoods.

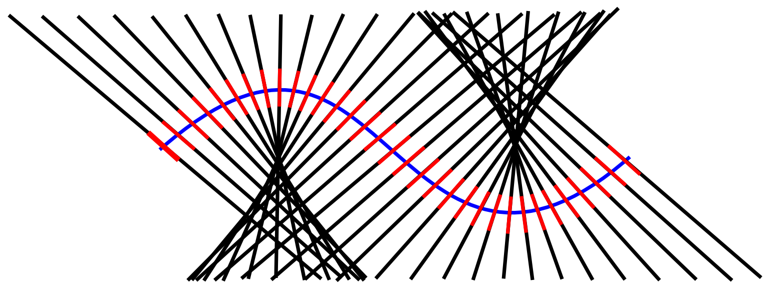


Figure 1.2: ϵ-Tubular neighborhood.

The following definition is of extremely importance in Chapter 3,

Definition 4. *The **Thom space** $Th(V)$ of a real vector bundle $\pi : V \to X$ over a topological space X is the topological space obtained by first forming the disk bundle $D(V)$ of (unit) disks in the fibers of V (with respect to a metric given by any choice of orthogonal structure) and then identifying the boundaries of all the disks to a point, i.e. forming the quotient topological space by the unit sphere bundle $S(V)$*

$$\mathrm{Th(V)} := D(V)/S(V). \tag{1.4}$$

Intuitively, the Thom space are those collection of vectors in V with norm less than 1 and we collapse to a point those vectors with norm equals to 1.

We will also denote the Thom space of a bundle $V \to M$ as M^V.

1.2 Intersection theory

As we mentioned in the previous section, intersection theory will help us to study fixed points. The results presented here are a brief summary of a long theory.

Definition 5. *Let S, L be submanifolds of M such that S, L, M are oriented and compact. If S intersects L transversally, we denote it by $S \pitchfork L$. Moreover, if $\dim(S) + \dim(L) = \dim(M)$ then the **intersection number** is*

$$\mathrm{Int}(S, L) = \int_M \eta_S \wedge \eta_L,$$

where η_S and η_L are the Poincaré's duals of S and L, respectively.

When $S \pitchfork L$ then the Poincaré's duals η_S and η_L obey the following relation

$$\eta_{S\cap L} = \eta_S \wedge \eta_L. \tag{1.5}$$

The above relation does not require a lot of computation, only requires writing the Poincaré's dual in term of the Thom class and its relation with the Whitney sum of vector bundles. For more details, you can check Guillemin's book [18].

Inside $M \times M$ we have two submanifolds: the diagonal Δ and the graph G_f, which have complementary dimension. We can then try to take the intersection number between G_f and Δ. But we warn that $\mathrm{Int}(\Delta, G_f)$ has to be interpreted carefully: it really denotes a fixed point counted with multiplicities.

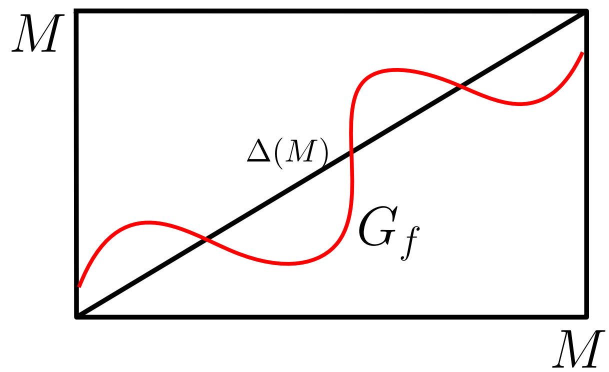


Figure 1.3: Intersection of G_f and Δ.

Definition 6. *Let M be a compact and oriented manifold and $f : M \to M$ a smooth map. Then the **Lefschetz number** of f is*

$$L(f) = \int_{M\times M} \eta_\Delta \wedge \eta_{G_f} \in \mathbb{R}. \tag{1.6}$$

The transversality condition is important in the above definition, because the right-hand side of Equation 1.6 is, in fact, an intersection number. Note that, we can deform the map f in such a way that G_f is always transverse to $\Delta(M)$.

Let us see a particular description of η_Δ the Poincaré's dual of Δ. That will help us to have more control over the formulas that will appear related to intersection theory.

Lemma 1.3. *Let $\{\alpha_1, \cdots, \alpha_n\}$ a graded basis for $H^*(M)$ and $\{\beta_1, \cdots, \beta_n\}$ the respective Poincaré dual basis, i.e. the bases must satisfy the equality $\int_M \beta_i \wedge \alpha_j = \delta_{ij}$. Then*

$$\eta_\Delta = \sum_{i=1}^{n} (-1)^{|\alpha_i|} \pi_1^*(\alpha_i) \wedge \pi_2^*(\beta_i), \tag{1.7}$$

where $\pi_i : M \times M \to M$ for $i = 1, 2$ are the natural projections.

Proof. For any closed n-form ω we want to prove the following formula

$$\int_{\Delta} \omega = \int_{M\times M} \omega \wedge \eta_{\Delta}. \tag{1.8}$$

We can write any closed n-form $\omega \in \Omega(M \times M)$ in terms of the basis as follows

$$\omega = \sum_{k,j=1}^{n} \pi_1^*(\beta_k) \wedge \pi_2^*(\alpha_j), \tag{1.9}$$

For fixed $k, j = 1, \cdots, n$ we can consider the form

$$\omega_{k,j} = \pi_1^*(\beta_k) \wedge \pi_2^*(\alpha_j), \tag{1.10}$$

Let us calculate the left hand side of Equation 1.8

$$\begin{aligned}
\int_{\Delta} \omega_{k,j} &= \int_M \varphi^*(\omega_{k,j}) = \int_M \varphi^*(\pi_1^*(\beta_k) \wedge \pi_2^*(\alpha_j)) \\
&= \int_M (\pi_1 \circ \varphi)^*(\beta_k) \wedge (\pi_2 \circ \varphi)^*(\alpha_j) \\
&= \int_M \beta_k \wedge \alpha_j = (-1)^{|\alpha_j||\beta_k|}\delta_{kj},
\end{aligned}$$

and the right hand side

$$\begin{aligned}
\int_{M\times M} \omega_{k,j} \wedge \eta_{\Delta} &= \sum_i (-1)^{|\alpha_i|} \int_{M\times M} \pi_1^*(\beta_k) \wedge \pi_2^*(\alpha_j) \wedge \pi_1^*(\alpha_i) \wedge \pi_2^*(\beta_i) \\
&= \sum_i (-1)^{|\alpha_i|+|\alpha_i||\alpha_j|} \int_{M\times M} \pi_1^*(\beta_k \wedge \alpha_i) \wedge \pi_2^*(\alpha_j \wedge \beta_i) \\
&= \sum_i (-1)^{|\alpha_i|+|\alpha_i||\alpha_j|} \left(\int_M \beta_k \wedge \alpha_i\right) \left(\int_M \alpha_j \wedge \beta_i\right) \\
&= \sum_i (-1)^{|\alpha_i|+|\alpha_i||\alpha_j|+|\alpha_i||\beta_k|}\delta_{ki}\delta_{ji} \\
&= (-1)^{|\alpha_j|+|\alpha_j||\alpha_j|+|\alpha_j||\beta_k|}\delta_{kj} \\
&= (-1)^{|\alpha_j||\beta_k|}\delta_{kj}.
\end{aligned}$$

Thus, adding over $k, j = 1, \cdots, n$ we get the equality 1.8. $\square$

With the above description of the Poincaré dual of the diagonal, we can calculate the Lefschetz number only from the information provided by its cohomology (homology) regardless of the choice of the coefficients. We get the following theorem.

Theorem 1.4. *Let M be a compact and oriented manifold and $f : M \to M$ a smooth map. Then*

$$L(f) = \sum_{i \geq 0} (-1)^i \operatorname{tr}(f^* : H^i(M) \to H^i(M)). \tag{1.11}$$

Proof. Write the linear map $f^* : H^i(M) \to H^i(M)$ in the form

$$f^*(\beta_j) = \sum_{i=1}^{n} \lambda_{ij} \beta_i, \tag{1.12}$$

where $\{\beta_1, \cdots, \beta_n\}$ is a basis defined as in the Lemma 1.3. By replacing Formula 1.7 in 1.6 we get

$$\begin{aligned}
L(f) &= \int_{M \times M} \eta_\Delta \wedge \eta_{G(f)} = \int_{G(f)} \eta_\Delta = \int_M F^*(\eta_\Delta) \\
&= \sum_{i,j} (-1)^{|\alpha_i|} \int_M F^*(\pi_1^*(\alpha_i) \wedge \pi_2^*(\beta_j)) \\
&= \sum_{i,j} (-1)^{|\alpha_i|} \int_M ((\pi_1 \circ F)^*(\alpha_i) \wedge (\pi_2 \circ F)^*(\beta_j) \\
&= \sum_{i,j} (-1)^{|\alpha_i|} \int_M (\alpha_i \wedge f^*(\beta_j)) \\
&= \sum_{i,j} (-1)^{|\alpha_i|} \int_M \lambda_{ij} \alpha_i \wedge \beta_j \\
&= \sum_{i,j} (-1)^{|\alpha_i|} \lambda_{ij} \beta_i(\alpha_i) \\
&= \sum_{i} (-1)^{|\alpha_i|} \lambda_{ii},
\end{aligned}$$

where $F : M \to G(f)$ is a diffeomorphism defined by $F(x) = (x, f(x))$. □

Lemma 1.5. *Let M be a compact and oriented manifold and η_Δ the Poincaré dual of the diagonal. Then*

$$\chi(M) = \int_M \eta_\Delta \wedge \eta_\Delta. \tag{1.13}$$

Proof. Let us calculate the right hand side. By the Formula 1.7 we get

$$\begin{aligned}
\int_{M\times M}\eta_\Delta\wedge\eta_\Delta &= \int_\Delta \eta_\Delta = \int_M \varphi^*(\eta_\Delta)\\
&= \sum_i(-1)^{|\alpha_i|}\int_M \varphi^*(\pi_1^*(\alpha_i)\wedge\pi_2^*(\beta_i))\\
&= \sum_i(-1)^{|\alpha_i|}\int_M ((\pi_1\circ\varphi)^*(\alpha_i)\wedge(\pi_2\circ\varphi)^*(\beta_i))\\
&= \sum_i(-1)^{|\alpha_i|}\int_M (\alpha_i\wedge\beta_i)\\
&= \sum_i(-1)^{|\alpha_i|}\int_M \alpha_i\wedge\beta_i\\
&= \sum_l(-1)^l\dim(H^l(M)) = \chi(M),
\end{aligned}$$

where $\varphi : M \to \Delta$ is the diffeomorphism defined by $\varphi(x) = (x, x)$. □

A priori from Formulas 1.6 and 1.11 it is not clear why the Lefschetz number is an integer number. The previous lemma tells us that when the self-map is just the identity then we get an important invariant, the Euler's characteristic $\chi(M)$, which is a very famous integer number. A natural question emerges: is the Lefschetz number always an integer number? The answer to that question is affirmative. We will see why this is the case using the following identification.

Definition 7. *Let M be an oriented and compact manifold of dimension 0 i.e. $M = \{p_1, \cdots, p_n\}$ is a finite set of points. An orientation in each point p_i is a choice of $+1$ or -1 and will be denoted by $sgn(p_i)$. The* ***cardinality*** *of M denoted by $\#(M)$ is the number of points counted with sign, that is*

$$\#(M) = \sum_{i=1}^{n} sgn(p_i). \tag{1.14}$$

Lemma 1.6. *Let S and L be compact and oriented submanifolds of M such that their intersection is transversal. If $\dim(S) + \dim(L) = \dim(M)$, then*

$$\mathrm{Int}(S, L) = \#(S \cap L). \tag{1.15}$$

Proof. Because S and L are transverses we get that $S \cap L$ is a compact and oriented submanifolds of zero dimension. Then it is a finite set of points with orientation and must hold

$$\int_M \eta_{S\cap L} = \#(S\cap L). \tag{1.16}$$

By the identification 1.5 and Definition 7 we get

$$\mathrm{Int}(S, L) = \int_M \eta_S \wedge \eta_L = \int_M \eta_{S\cap L} = \#(S \cap L).$$

The last equality is given by the following lemma. □

Lemma 1.7. *Let S be a submanifold of dimension zero of M. Suppose that S and M are compact and oriented. Then*

$$\#(S) = \int_M \eta_S. \tag{1.17}$$

Proof. Let $S = \{p_1, \cdots p_m\}$ be a finite set of points with orientation where the orientation of each point p_i assigns a sign $\mathrm{sign}(p_i) \in \{+1, -1\}$.
There are open sets $\{U_i\}_{i=1}^m$ of M pairwise disjoint such that $p_i \in U_i \cong \mathbb{R}^n$. And Let η_i be differential forms in $\Omega_c^n(U_i)$ for $i = 1, \cdots, m$ such that

$$\int_{U_i} \eta_i = \mathrm{sign}(p_i). \tag{1.18}$$

Then $\eta = \sum_i \eta_i$ is a n-form over M. Thus, we get

$$\int_M \eta = \sum_i \int_{U_i} \eta_i = \sum_i \mathrm{sign}(p_i) = \#(S). \tag{1.19}$$

We affirm that the Poincaré's dual of S is η, indeed we need to prove the following equality

$$\int_S \alpha = \int_M \alpha \wedge \eta, \tag{1.20}$$

for α any closed form of degree zero. The closed forms of degree zero are the locally constant functions, then $\alpha = f$ such that $f|_{U_i}$ is constant for all $i = 1, \cdots, m$. We calculate,

$$\int_M f \wedge \eta = \sum_i \int_M f \wedge \eta_i = \sum_i \int_{U_i} f \wedge \eta_i = \sum_i f(p_i)\mathrm{sign}(p_i) = \int_S f = \int_S \alpha.$$

□

By the previous identification we get that the Lefschetz number is an integer number, because it is the intersection number between Δ and G_f submanifolds of $M \times M$.

The next theorem is very relevant in this theory because it uses the Lefschetz number to say when a self-map has fixed points. It is known as Lefschetz fixed point theorem.

Theorem 1.8. *Let M be a compact and oriented manifold and $f : M \to M$ a continuous map. If $L(f) \neq 0$ then f has at least a fixed point.*

There is a classical proof of this result using simplicial approximation together with Formula 1.11 the result follows. It is also a consequence from the Lefschetz-Hopf theorem, in Chapter 3 we are going to prove that theorem by applying abstracts results.

Example 1. *Let $X = \{p_1, ..., p_n\}$ be a finite set of points and $f : X \to X$ a function. This function induces a $\mathbb{R}$-linear transformation*

$$\begin{aligned} f^* : \mathrm{Maps}(X,\mathbb{R}) &\to \mathrm{Maps}(X,\mathbb{R}) \\ g &\mapsto f^*(g) = g \circ f, \end{aligned}$$

where $\mathrm{Maps}(X,\mathbb{R})$ *is the vector space of functions from X to $\mathbb{R}$, it coincides with the 0th cohomology of X, $H^0(X,\mathbb{R}) = Maps(X,\mathbb{R})$.*
The set $\beta = \{\delta_1, ..., \delta_n\}$ forms a basis for $\mathrm{Maps}(X,\mathbb{R})$*, where $\delta_i : X \to \mathbb{R}$ is defined by $\delta_i(p_j) = \delta_{ij}$, for all $i = 1, ..., n$. Let us consider $A = [f^*]_\beta = [a_{ij}]$, the associated matrix to f^*, where $a_{ij} = \delta_i(f(p_j))$. Henceforth,*

$$\mathrm{tr}(f^*) = \sum_{i=1}^{n} a_{ii}.$$

Since $a_{ii} = \delta_i(f(p_i))$, if p_i is a fixed point of f we have $a_{ii} = 1$, and $a_{ii} = 0$ otherwise. Thus, $\mathrm{tr}(f^*)$ *counts the number of fixed points of f.*

The following examples show how to calculate the Lefschetz number from Formula 1.11.

Example 2. *Let X be the figure eight (see Figure 1.4). Let $f : X \to X$ be a map which is defined by the loops: $f_\#(\alpha) = \alpha^2$ and $f_\#(\beta) = \beta^{-1}$. The function f has two fixed points x_0 and y_0.*

In zero degree we get $\mathrm{tr}(f_{*0}) = 1$ *because the induced linear map $f_{*0} : H_0(X) \to H_0(X)$ is the identity. In dimension one, the integral homology of X is*

$$H_1(X) = \mathbb{Z} \oplus \mathbb{Z} \cong \langle \overline{\alpha}, \overline{\beta} \rangle.$$

Therefore,

$$f_{*1} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},$$

$$L(f) = 1 - (2 - 1) = 0.$$

In the last example the Lefschetz number is zero but it has two fixed points. It is an example where the reciprocal of the Lefschetz fixed point theorem is not true.

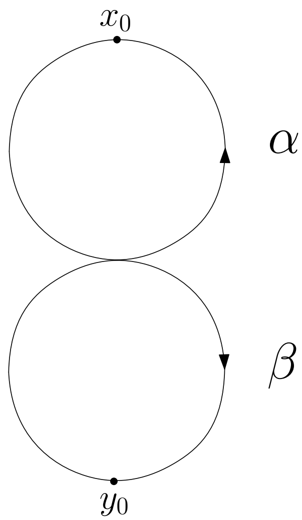


Figure 1.4: Eight figure.

Example 3. *Consider the surface of genus three Σ_3 given by Figure 1.5. Let $f : \Sigma_3 \to \Sigma_3$ be the map defined to be the 180 degree rotation about a vertical axis passing through the central hole. Since f has no fixed points, we should have*

$$L(f) = \sum_{i=0}^{2} (-1)^i tr\left(f_{*i} : H_i\left(\Sigma_3\right) \to H_i\left(\Sigma_3\right)\right) = 0.$$

At degree 0 the induced map f_{*0} is the identity, as always for a path-connected space, so this contributes 1 to $L(f)$. At degree 1, it induces a map $f_{*1} : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ defined by the matrix

$$f_{*1} = \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right),$$

Thus, it contributes -2 in $L(f)$. That is because there is only one way to send vertical loops $\alpha_1 \mapsto \alpha_3$, $\alpha_2 \mapsto \alpha_2$ and $\alpha_3 \mapsto \alpha_1$, in a similar way are sent the horizontal loops.

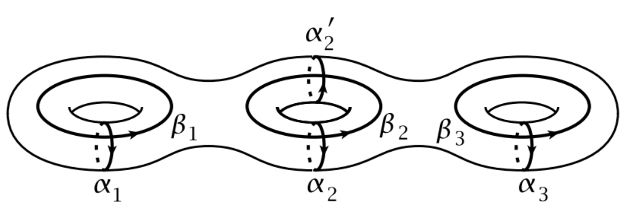


Figure 1.5: Genus 3 surface.

In degree 2, $f_{*2} : \mathbb{Z} \to \mathbb{Z}$ contributes 1 in $L(f)$ because the map f preserve the orientation, it thanks to the right-hand rule, see figure below. Then, the Lefschetz number is:

$$L(f) = 1 - 2 + 1 = 0.$$

The next lemma is a local characterization of transversal intersection.

Lemma 1.9. *Let $f : M \to M$ be a smooth map and $p \in M$ a fixed point. Then $G_f \pitchfork \Delta$ at $p \in M$ if and only if $d_p f - I : T_pM \to T_pM$ is a linear isomorphism.*

Proof. If $G_f \pitchfork \Delta$ at $q = (p, p)$, then

$$\operatorname{im}\left(d_{(p,p)}(id_M \times f)\right) + T_{(p,p)}\Delta = T_{(p,p)}(M \times M), \tag{1.21}$$

that can be written as

$$\operatorname{im}(I \times d_x f) + T_{(p,p)}\Delta = T_pM \times T_pM. \tag{1.22}$$

Note that the subspaces $\operatorname{im}(I \times d_x f)$ and $T_{(p,p)}\Delta$ have the same dimension inside $T_pM \times T_pM$. Therefore, we get the next quality

$$\operatorname{im}(I \times d_x f) \oplus T_{(p,p)}\Delta = T_pM \times T_pM, \tag{1.23}$$

That is equivalent to $\operatorname{im}(I \times d_x f) \cap T_{(p,p)}\Delta = 0$. Then, there are not non-zero vectors such that $d_p f$ has no fixed points. Thus, we get

$$\ker(d_p f - I) = 0.$$

It is $d_p f - I$ is an injective map between vector spaces of the same dimension, and hence it is an isomorphism. □

Proposition 1.10. *With the conditions presented in the previous lemma, the orientation number of $q \in G(f) \pitchfork \Delta$ is in fact the sign of the determinant of the linear transformation $d_p f - I$.*

Proof. Let $\{v_1, \cdots, v_k\}$ be a linear basis for T_pM. Then $\{v_1 \times v_1, \cdots, v_k \times v_k\}$ and $\{v_1 \times d_p f(v_1), \cdots, v_k \times d_p f(v_k)\}$ are bases for $T_{(p,p)}\Delta$ and $im(I \times d_p f)$ respectively. The set

$$C = \{v_1 \times v_1, \cdots, v_k \times v_k, v_1 \times d_p f(v_1), \cdots, v_k \times d_p f(v_k)\}$$

forms a basis for $T_pM \times T_pM \cong T_{(p,p)}(M \times M)$. We can represent the transformation $d_p f - I$ using this basis as the following matrix

$$\left(\begin{array}{ccc|ccc} v_1 & \cdots & 0 & v_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & v_k & 0 & \cdots & v_k \\ \hline v_1 & \cdots & 0 & d_p f(v_1) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & v_k & 0 & \cdots & d_p f(v_k) \end{array}\right),$$

applying elementary matrix operations to the previous matrix we get

$$\left(\begin{array}{ccc|ccc} v_1 & \cdots & 0 & v_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & v_k & 0 & \cdots & v_k \\ \hline & & & d_pf(v_1)-v_1 & \cdots & 0 \\ & 0 & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & d_pf(v_k)-v_k \end{array}\right).$$

Hence, the above basis is positively oriented in the product $M \times M$ if and only if the isomorphism $d_pf - I$ preserves the orientation. □

The following is a local characterization of Lefschetz number in term of the fixed points.

Definition 8. *Let $x \in M$ be a fixed point of the map $f : M \to M$. The local Lefschetz number of f at x, denoted by $L_x(f)$, is defined to be the sign of det$(d_xf - I)$.*

Proposition 1.11. *Let M be a compact and oriented manifold, and $f : M \to M$ such that $G_f \pitchfork \Delta$. Then*

$$L(f) = \sum_{f(x)=x} L_x(f). \tag{1.24}$$

Proof. From definition we know the Lefschetz number is the intersection number between Δ and G_f. Hence, we get

$$L(f) = \sum_{f(p)=p} \operatorname{sgn}(p),$$

where $\operatorname{sgn}(p)$ is the orientation number as in Lemma 1.7. The result follows from Proposition 1.10. □

We can consider another approach to compute the Lefschetz number. This point of view is studying zeros of vector fields. Recall that a vector field is a smooth asignment that correspond to each point p a tangent vector in T_pM.

Definition 9. *Let V be a vector field over M. The image of V is*

$$I(V) = \{v \in T_pM : v = V(p) \text{ for some } p \in M\}, \tag{1.25}$$

which is a submanifold of TM. In fact $I(V) \cong M$.

Lemma 1.12. *Let $N(\Delta)$ be the normal bundle of the diagonal in $M \times M$. Then there is a diffeomorphism of vector bundles $N(\Delta) \cong TM$.*

Proof. Take $(p,p) \in \Delta$ and consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_{(p,p)}\Delta \arrow[r, "\iota"] \arrow[d, "\varphi^{-1}"] & T_pM \times T_pM \arrow[r] \arrow[d, "id"] & N_{(p,p)}(\Delta) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T_pM \arrow[r, "\delta_p"] & T_{(p,p)}(M \times M) \arrow[r] & T_pM \arrow[r] & 0 \end{tikzcd}$$

where $\iota : T_{(p,p)}\Delta \to T_pM \times T_pM$ is defined as $\iota(v,v) = (v,v)$, the map $\delta : T_pM \to T_{(p,p)}(M \times M)$ is defined by $\delta(v) = (v,v)$, and φ is the canonical linear isomorphism between M and its diagonal induced at the tangent space level defined for any vector $v \in T_pM$ as $\varphi(v) = (v,v)$.
Those horizontal exact sequences have the same cokernel. Thus, we can conclude $N_{(p,p)}(\Delta) \cong T_pM$ for all $p \in M$. This previous fact, together with the homomorphism of vector bundles

$$\tau : TM \to N(\Delta)$$

defined by $\tau(x,v) = ((x,x),(-v,v))$ for $(x,v) \in TM$ and $v \in T_xM$, allow us to conclude that the vector bundles TM and $N(\Delta)$ are isomorphic. $\square$

Let U be an open subset of $M \times M$ such that $U \cong N(\Delta)$, i.e. U is a tubular neighborhood. Let $\phi : U \to TM$ be the isomorphism between the tubular neighborhood and the tangent bundle given by the composition between the isomorphism $\psi : U \cong N(\Delta)$ and the isomorphism $\varphi : N(\Delta) \to TM$ in the above lemma. Figure 1.6 helps us understand this situation.

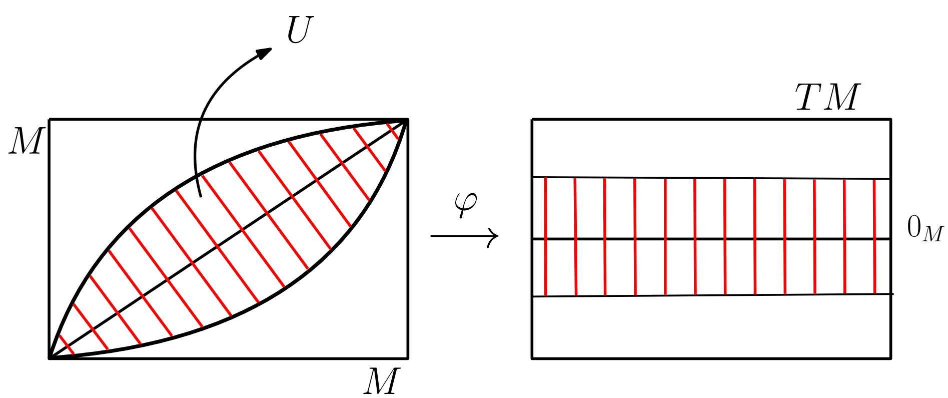


Figure 1.6: Tubular neighborhood of Δ.

Intuitively, we can transfer information of fixed points inside the tubular neighborhood to vector fields over M. Fixed points are in fact intersection points, and zeros of vector fields are also intersection points between sections in the tangent bundle and the zero section. The bridge between them is the tubular neighborhood $\varphi : U \to TM$.

Theorem 1.13. *(**Poincaré-Hopf index theorem**) Let M be an oriented and compact manifold, and V a vector field over M such that $I(V) \pitchfork I(0)$, i.e, the image of the vector field intersects transversally the zero section, then*

$$\chi(M) = \sum_{V(p)=0} ind_p(V). \tag{1.26}$$

Proof. See [18].

□

The following figure shows an idea of the isomorphism φ.

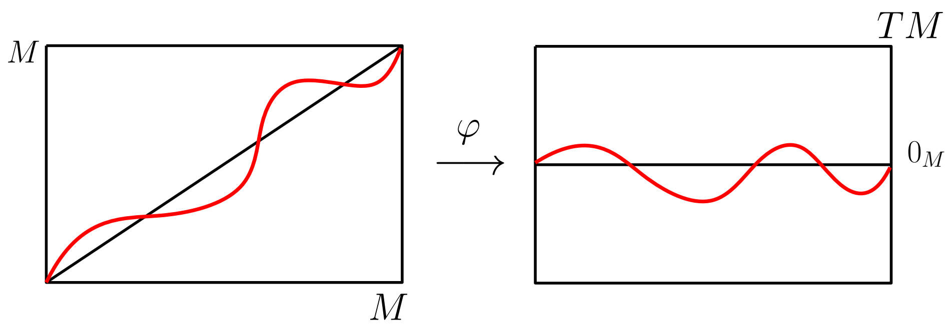


Figure 1.7: Poincaré-Hopf

However, There is a classical version of Formula 1.26. For $f : M \to M$ a self map with M an oriented and compact n-manifold with discrete set of fixed points.

Definition 10. *Let M, N be compact manifolds with the same dimension and $f : M \to N$ a smooth map. If $y \in Y$ is a regular value of f, the degree of f at y is given by*

$$\deg_y(f) = \sum_{x \in f^{-1}(Y)} sgn(\det(d_x f)).$$

It is not hard to see that the degree of a map does not depend on the choice of regular value, thus we can define a global degree denoted by $\deg(f)$. Moreover, this degree is a homotopy invariant; that is, if f is homotopic to g then $\deg(f) = \deg(g)$.

If we consider any self map $f : S^n \to S^n$ of spheres, we can define an equivalent notion of degree as follows: first we can induce an homology homomorphism $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$, which is a map $f_* : \mathbb{Z} \to \mathbb{Z}$. The degree is given by $f_*(\alpha_n) = \deg(f)\alpha_n$, where α_n is the fundamental class of $\tilde{H}_n(S^n)$.

Using this formulation of degree we can give an equivalent notion of index. We will refer to both as index, the context will help us to differentiate each one.

Let $x \in M$ be a fixed point of f, then there exists a $(n-1)$-sphere S_x around x which is approximately mapped to itself by f. Thus, the restriction of f to S_x is $f|_{S_x} : S_x \to S_x$, a self map of spheres. Hence, we get the following definition.

Definition 11. *Let M be a closed smooth n-manifold and $f : M \to M$ a map with a discrete (hence finite) set of fixed points. The **fixed-point index** is the sum*

$$\sum_{f(x)=x} \deg(f|_{S_x}),$$

over all fixed points of f.

1.3 Lefzschet number and trace

In the previous section, we defined an invariant purely in a topological way. That admits many descriptions, but one, Formula 1.11 in Theorem 1.4 tells us that the Lefschetz number of any map $f : M \to M$ is the alternating sum of the trace of the map induced at the cohomology level.

In Example 1, we found that for a map $f : X \to X$ with X a finite set, we can find the Lefschetz number just applying a functor which sends f to the linear map $f^* : \mathrm{Maps}(X, \mathbb{R}) \to \mathrm{Maps}(X, \mathbb{R})$. Now, in the world of linear maps we can calculate the trace of f^* and we get

$$\mathrm{tr}(f^*) = \text{Number of fixed points of } f. \tag{1.27}$$

Following this idea, taking cohomology generalizes the idea of the functor $\mathrm{Maps}(-, \mathbb{R})$. In fact, it is the result of cohomology at the zero dimensional level. Now, If we consider a smooth map $f : M \to M$, with M be a compact and oriented manifold, we can apply the cohomology functor denoted by $H(-)$. Thus, we get a linear map $f^* : H(M) \to H(M)$ and we can calculate its trace. $H(M)$ is a graded vector space, commonly written as $\bigoplus_{i \geq 0} H^i(M)$, and its trace is the sum of the alternating trace in each degree. The sign of each degree is not a convention. The sign comes from the sign in the symmetric condition $V \otimes W \to W \otimes V$ in the vector space level is defined by the formula:

$$\gamma_{V,W} : v \otimes w \mapsto (-1)^{|v||w|} w \otimes v. \tag{1.28}$$

We can conclude that Formula 1.11 relates the trace of $f^* : H(X) \to H(X)$ and the number of fixed points of f counted with multiplicity, in other words trace and number of fixed points (with multiplicity) are the same thing. That is

$$\mathrm{tr}(f^*) = \sum_{i \geq 0} (-1)^i \mathrm{tr}(f_i^*), \tag{1.29}$$

In summary, we have the following:

$$L(f) = \mathrm{tr}(H(f)), \tag{1.30}$$

where we just replaced $f^* = H(f)$, that represents the induced linear map at cohomology.

In the next chapter we will give an introduction to this categorical language. Let us stop for a second and consider the situation in which for a smooth map $f : M \to M$ we may find the trace, denoted by $tr(f)$. This trace has the interesting property of commutativity with the homology functor H. In that situation, the Lefschetz number is

$$L(f) = H(\mathrm{tr}(f)). \tag{1.31}$$

That is the starting point of this work, we will see in the following chapter that the trace can be defined in a more general context. That connection is beyond topological spaces or even from a set-theoretic environment. We are interested in developing a categorical framework where we can conclude some of the results presented in this chapter but from a more abstract point of view.

Chapter 2

General trace and fixed point operator

In this chapter we will introduce some motivations towards the concept of general traces in any symmetric monoidal categories and its relation with fixed points. Thanks to the work of Dold and Puppe, (see [2]), about the theory of dualizable objects in monoidal categories, we can extend the notion of trace to monoidal categories. As an important pedagogical ingredient, we will discuss some examples where traces give us information about fixed points, in special, the stable homotopy category. Historically, it was the category in which Dold and Puppe worked motivating the study of general trace, and it will serve as the starting point of Chapter 3. Finally, we will see a relation between general trace and fixed point operator, independently found by Hyland and Hasegawa, see [20] and [**?**], respectively.

Let us begin by recalling some facts of linear algebra. Let V be a finite dimensional vector space over a characteristic zero field k. We can define the trace of a linear map $f : V \to V$ as the following composition

$$\begin{tikzcd} k \arrow[r, "\eta"] \arrow[rrrr, bend right, "\mathrm{tr}(f)"'] & V \otimes V^* \arrow[r, "f\otimes id_{V^*}"] & V \otimes V^* \arrow[r, "\gamma"] & V^* \otimes V \arrow[r, "\epsilon"] & k, \end{tikzcd}$$

where V^* is the dual vector space of V, η is the co-evaluation map, ϵ is the evaluation map and γ is the canonical isomorphism between $V \otimes V^*$ and $V^* \otimes V$. Let $\{v_1, ..., v_n\}$ be a basis for V and $\{v_1^*, \cdots, v_n^*\}$ its dual basis defined by $v_j^*(v_i) = \delta_{ij}$, with $\delta_{i,j}$ being the Kronecker delta. We can write the evaluation and co-evaluation maps in terms of the bases as follows

$$\eta(1) = \sum_{i=1}^{n} v_i \otimes v_i^*, \qquad \epsilon\,(v_j \otimes v_i^*) = v_i^*(v_j). \tag{2.1}$$

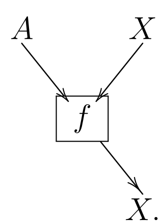
η extends by linearlity to any $r \in k$, so it is only necessary to define what happens to 1. The trace in terms of these bases is the following composition

$$\begin{tikzcd} 1 \arrow[r, mapsto, "\eta"] \arrow[rrrr, bend right, "\mathrm{tr}(f)"'] & \sum_i v_i \otimes v_i^* \arrow[r, mapsto, "f \otimes id_{V^*}"] & \sum_i a_{ij} v_j \otimes v_i^* \arrow[r, mapsto, "\gamma"] & \sum_i a_{ij} v_i^* \otimes v_j \arrow[r, mapsto, "\epsilon"] & \sum_i a_{ii}, \end{tikzcd}$$

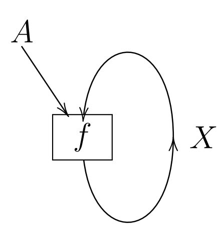
where $a_{ij} \in k$ are the coefficients of the matrix representing f, i.e., $f(v_i) = \sum_{j=1}^{n} a_{ij} v_j$ for all $i = 1, \cdots, n$. This notion of trace allows us to recover the classical notion of trace as the sum of the diagonal elements of a square matrix. Therefore, the previous composition is extremely important because with it in mind we can extend the notion of trace to a more general context, where the classical notion of trace as a sum makes no sense.

An objective of this chapter is to develop the abstract theory behind the general trace and show, with illustrative examples, its relation with fixed points. That was an idea from Hasegawa in [19] and Hyland in [**?**] who independently discovered the abstract relation between traces and fixed points. In particular, they discovered Theorem 2.23 which we will prove at the end of this chapter.

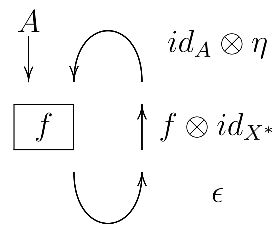
The idea of Hasegawa was to imitate the notion of feedback, such as in the analysis with the use of contractive maps where we can reproduce fixed points with iterative composition. Let us consider $f : A \times X \to X$, the string diagram that represents the map f is given by



This map must satisfies the property that the X input and output have been "fed back into each other" somehow.



We can decompose the above string diagram in short pieces which together bring us the feedback diagram:



Note that the levels of each diagram correspond to the individual functions that we are composing. Using the description presented in the Appendix A, we can write the feed back with a composition of morphism:

$$\epsilon \circ (f \otimes id_{X^*}) \circ (id_A \otimes \eta) : A \to \mathbb{1}. \tag{2.2}$$

We must pay attention to this composite. We will come back to this later in this chapter when the notion of trace is defined. That is because with suitable topological modifications of the above diagram we can recover the notion of trace. Sometimes we will write identity maps as objects, for example Equation (2.2) may be written as

$$\epsilon \circ (f \otimes X^*) \circ (A \otimes \eta).$$

In summary, the notions of trace and fixed point meet thanks to "feedback". Theorem 2.23 describes this relationship and it will be the main objective in this chapter.

2.1 Symmetric monoidal categories and dualizable objects

The first type of categories that we will introduce are monoidal categories. Intuitively, we say that a category $\mathcal{C}$ is **monoidal** if there exists some notion of product, denoted by $\otimes$, and a unit object with respect to the product, denoted by $\mathbb{1}$. This definition extends in a natural way the notion of monoid in a categorical sense. That, is a well defined binary operation, with an identity element which is associative. In addition, we say $\mathcal{C}$ is symmetric if the product is as commutative as possible, i.e., there exists a map γ such that for each pair of objects X, Y in $\mathcal{C}$ we get an isomorphism $\gamma_{X,Y} : X \otimes Y \cong Y \otimes X$.

The next definition might seem bulky, but one only has to keep in mind that it is defined in way such that the tensor product of morphisms is also associative.

Definition 12. *A* ***symmetric monoidal category*** $\mathcal{C}$ *is a category equipped with:*

1. *A functor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *called the* ***tensor product****.*

2. *An object* $\mathbb{1} \in Ob(\mathcal{C})$ *called the* ***unit object***.

3. *A natural isomorphism* $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$ *for* X, Y, Z *objects in* $\mathcal{C}$, *called the* ***associator***.

4. *Natural isomorphisms* $l_X : \mathbb{1} \otimes X \to X$ *and* $r_X : X \otimes \mathbb{1} \to X$ *called the* ***left unitor*** *and* ***right unitor***, *respectively.*

We demand the commutativity of the following diagrams:

Pentagon identity:

$$
\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W\otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \otimes id_Z"'] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X\otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "id_W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd}
$$

Hexagon identity:

$$
\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[r, "\gamma_{XY} \otimes id_Z"] \arrow[dl, "\alpha_{X,Y,Z}"'] & (Y \otimes X) \otimes Z \arrow[dr, "\alpha_{Y,X,Z}"] & \\
X \otimes (Y \otimes Z) \arrow[dr, "\gamma_{Y\otimes Z,X}"'] & & & Y \otimes (X \otimes Z) \arrow[dl, "\gamma_{Y,X\otimes Z}"] \\
 & (Y \otimes Z) \otimes X \arrow[r, "\alpha_{Y,Z,X}"'] & Y \otimes (Z \otimes X) &
\end{tikzcd}
$$

Triangle identity:

$$
\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes id_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "id_X \otimes l_Y"] \\
 & X \otimes Y &
\end{tikzcd}
$$

These are called coherence diagrams or MacLane axioms.

Dualizability of objects in a monoidal category defined by Dold and Puppe, see [2], is a way of saying that an object is "small". For example, in $\mathbf{Vect}_k$ the category of vector spaces, dualizable objects are the finite dimensional vector spaces and their dual is given by $V^* = \mathrm{Hom}_k(V, k)$. The definition presented here of dualizable objects in any monoidal category does not follow immediately from the notion of dual vector space as the set of homomorphisms from the vector spaces to the field k. Therefore, considering the dual of any object M as $Hom(M, \mathbb{1})$ does not make sense, at least in categories which do not have internal homomorphisms.

Definition 13. *An object $M \in ob(\mathcal{C})$ is called **dualizable** if there exists an object $M^* \in \mathrm{ob}(\mathcal{C})$ called its dual, and maps*

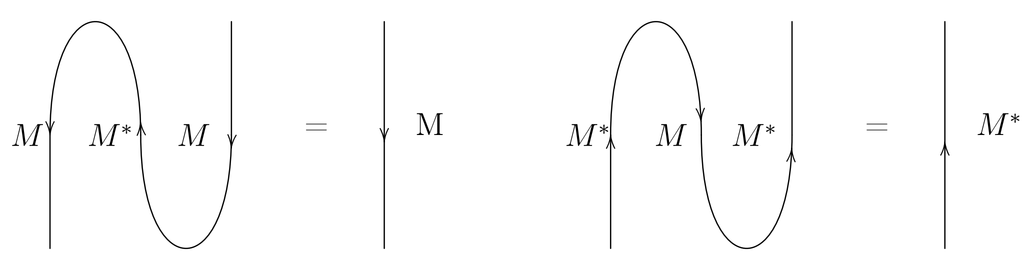
$$\eta : \mathbb{1} \to M \otimes M^*, \qquad \epsilon : M^* \otimes M \to \mathbb{1}, \tag{2.3}$$

satisfying the triangle identities

$$(id_M \otimes \epsilon) \circ (\eta \otimes id_M) = id_M \text{ and } \quad (\epsilon \otimes id_{M*}) \circ (id_{M^*} \otimes \eta) = id_{M^*}.$$

*We call ϵ the **evaluation** and η the **coevaluation** map.*

Most of the proofs for some of the results are inefficient and complicated. The approach that we will use corresponds to drawing morphisms in $\mathcal{C}$ as boxes, and objects in $\mathcal{C}$ as arrows. The images bellow describe the **snakes** identities in terms of these diagrams. These diagrams will be described in depth in Appendix A.



Example 4. *Examples of symmetric monoidal categories and its dualizable objects are:*

(a) ***Vector spaces.*** *Let $Vect_k$ the category of finite dimensional vector spaces over a field k. The usual tensor product and the field k as the unit object give us the structure of a monoidal category. Its symmetric structure is given by the isomorphism of vector spaces $\gamma_{V,W} : V \otimes W \to W \otimes V$ defined by $\gamma(v \otimes w) = w \otimes v$. All finite dimensional vector spaces are dualizables with a canonical dual $V^* = Hom(V, k)$.*

(b) ***R-Modules****. Let R be a commutative ring and $\boldsymbol{Mod}_R$ be the category of R-modules. Then Mod_R is a symmetric monoidal category with the usual tensor product over*

R. The ring R thought of as a module over itself is the unit. The dual of a finitely generated free R-module M is $Hom_R(M, R)$. *This is also a finitely generated free R-module. If M has basis* $\{m_1, m_2, \cdots, m_n\}$ *and dual basis* $\{m'_1, m'_2, \cdots, m'_n\}$ *the coevaluation and evaluation*

$$\eta : R \to M \otimes_R Hom_R(M, R) \text{ and } \epsilon : Hom_R(M, R) \otimes_R M \to R$$

are R-module homomorphisms defined by $\epsilon(\phi, m) = \phi(m)$ *and* $\eta(1) = \sum_i m_i \otimes_R m'_i$.

(c) ***Chain complexes***. *Let R be a commutative ring and consider the category* $\boldsymbol{Ch}_R$ *of chain complexes of R-modules equipped with the structure of monoidal category by taking the monoidal product to be the graded tensor product ⊗ and the monoidal unit to be the module R, viewed as a chain complex with non-trivial degree only in degree 0. Its symmetric structure is given by* $a \otimes b \cong (-1)^{|b||a|} b \otimes a$. *A chain complex over a ring R is dualizable if and only if it is bounded and is a finitely generated projective module in the finitely many degrees where it is non-zero.*

(d) *The categories* ***Set*** *and* ***Top*** *of sets and topological spaces, respectively. Both are symmetric monoidal categories. The tensor product is the set theoretic cartesian product, and any singleton can be fixed as the unit object, uniqueness is given by the fact that they are all canonically isomorphic to one another, and we denote the one point set by *. The only dualizable object is *, and its dual is itself.*

(e) *Let G be a group, the category* $\boldsymbol{Rep}_k(G)$ *of all representations of G over k is a monoidal category, with ⊗ being the tensor product of representations: if for a representation V one denotes by* $\rho(V)$ *the corresponding map* $G \to GL(V)$, *then*

$$\rho(V \otimes W)(g) := \rho(V)(g) \otimes \rho(W)(g).$$

The unit object in this category is the trivial representation $1 = k$. *A similar statement holds for the category* $\boldsymbol{Rep}_k(G)$ *of finite dimensional representations of G.*

(f) ***Cobordism***. *Let* $\boldsymbol{Bord}_n$ *be the category whose objects are closed* $(n-1)$*-manifolds and the morphisms are n-manifolds. Composition of morphisms is given by gluing (see figure below). For each Y the bordism cylinder* $[0, 1] \times Y$ *is* $id_Y : Y \to Y$. *The disjoint union and the empty manifold* $\emptyset_{n-1}$ *defines a symmetric monoidal category.*

(g) *The stable homotopy category* ***HoSpect*** *is an example of symmetric monoidal category. In the Chapter 3, we are going to describe this category, Dold and Puppe used tools developed in this chapter and the category* ***HoSpect*** *to prove the Lefschetz fixed point theorem.*

It is easy to see that in a symmetric monoidal category the dual of the unit object is itself, i.e. $\mathbb{1}^* \cong \mathbb{1}$. It is also obvious from the definition that if M^* is the dual of M,

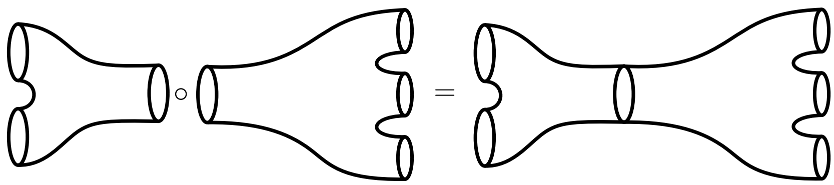


Figure 2.1: Gluing of manifolds

then M is the dual of M^*. Let us see that any two duals of an object M must be isomorphic.

Proposition 2.1. *Let $\mathcal{C}$ be a symmetric monoidal category and $M \in Ob(\mathcal{C})$. Every pair $(M_1^*, \epsilon_1, \eta_1)$ and $(M_2^*, \epsilon_2, \eta_2)$ of duals of M are isomorphic.*

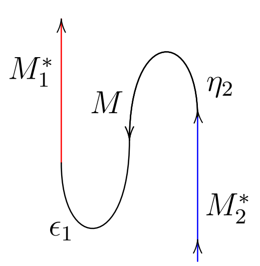
Proof. Let $(M_1^*, \epsilon_1, \eta_1)$ and $(M_2^*, \epsilon_2, \eta_2)$ be duals of M. We can define a morphism $\varphi_1 : M_1^* \to M_2^*$ given by the composite

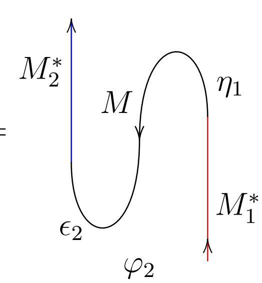
$$M_1^* \xrightarrow{id_{M_1^*} \otimes \eta_2} M_1^* \otimes (M \otimes M_2^*) \xrightarrow{\alpha_{M_1^*, M, M_2^*}} (M_1^* \otimes M) \otimes M_2^* \xrightarrow{\epsilon_1} M_2^*,$$

and $\varphi_2 : M_2^* \to M_1^*$ given by

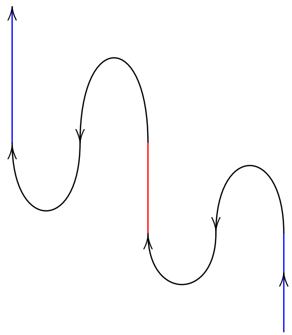
$$M_2^* \xrightarrow{id_{M_2^*} \otimes \eta_1} M_2^* \otimes (M \otimes M_1^*) \xrightarrow{\alpha_{M_2^*, M, M_1^*}} (M_2^* \otimes M) \otimes M_1^* \xrightarrow{\epsilon_2} M_1^*.$$

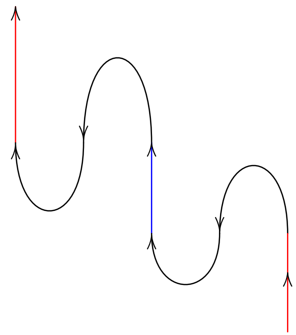
We claim that these morphisms are inverse to each other. In fact, we are going to use string diagrams, as follows. The red string represents the object M_1 and the blue one represents the object M_2. For a detailed description, you can see the Appendix A of this work.

$\varphi_1 =$ 

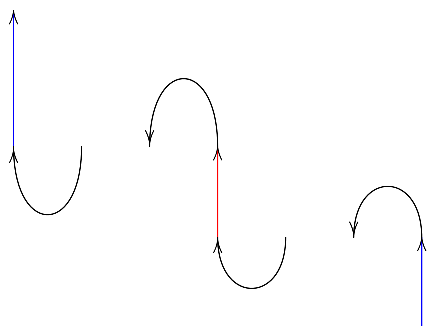
and $\varphi_2 =$ 

The composition $\varphi_1 \circ \varphi_2$ and $\varphi_2 \circ \varphi_1$ can be represented by the following string diagrams

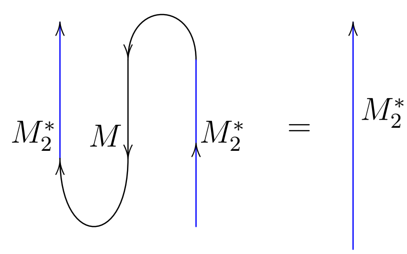
$\varphi_1 \circ \varphi_2 =$ 

$\varphi_2 \circ \varphi_1 =$ 

Let us prove that the above string diagrams are the identity maps, that is, they satisfy $\varphi_1 \circ \varphi_2 = \mathrm{id}_{M_2^*}$ and $\varphi_2 \circ \varphi_1 = \mathrm{id}_{M_1^*}$. We only prove one case, the other one being similar. Let us divide the string diagram in sections, as in the following diagram



The string in the middle is just the identity of M thanks to the snake identities presented in Definition 13. Thus, we get the following diagram



Applying again the snake identities the above equality holds and then we can conclude that $\varphi_2 \circ \varphi_1 = \mathrm{id}_{M_2^*}$. $\square$

In other words, the above proposition tells us that dualizability is a well defined property. The following lemma plays an important role in what follows.

Lemma 2.2. *Let $\mathcal{C}$ be a symmetric monoidal category, and A a dualizable object with dual A^*. Then there is an adjunction between $- \otimes A$ and $- \otimes A^*$, that is, there are* *canonical isomorphisms*

$$\mathrm{Hom}(X \otimes A, Y) \cong \mathrm{Hom}(X, Y \otimes A^*),$$
$$\mathrm{Hom}(A^* \otimes X, Y) \cong \mathrm{Hom}(X, A \otimes Y).$$

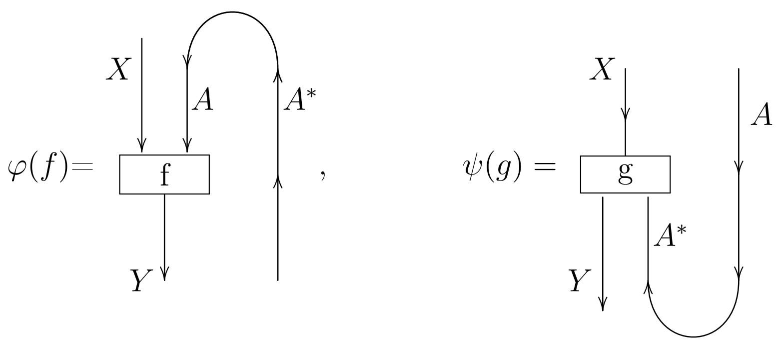
Proof. Let us consider the morphism $\varphi : Hom(X \otimes A, Y) \to Hom(X, Y \otimes A^*)$ which is defined by assigning a morphism $f : X \otimes A \to Y$ to the composite

$$X \xrightarrow{\mathrm{id}_X \otimes \eta} X \otimes A \otimes A^* \xrightarrow{f \otimes \mathrm{id}_{A^*}} Y \otimes A^*.$$

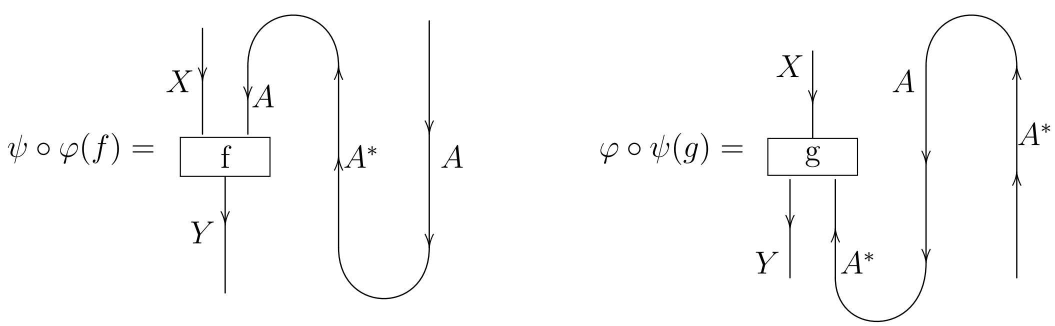
We will show that its inverse is the function $\psi : Hom(X, Y \otimes A^*) \to Hom(X \otimes A, Y)$ which sends each $g : X \to Y \otimes A^*$ is sent to the composite

$$X \otimes A \xrightarrow{g \otimes \mathrm{id}_A} Y \otimes A^* \otimes A \xrightarrow{\mathrm{id}_Y \otimes \epsilon} Y.$$

To prove it, we are going to use string diagrams. First note that



Their composition yields



From the snakes identities the result follows. □

Definition 14. *Let $\mathcal{C}$ be a symmetric monoidal category. We say $\mathcal{C}$ is* ***compact closed*** *if every object is dualizable.*

The example that we must bear in mind is the category of finite dimensional vector spaces denoted by $\mathbf{Vect}_k$, where every object is a finite dimensional vector space V is dualizable and has a dual given by $V^* := \mathrm{Hom}_k(V, k)$.

Definition 15. *Let $\mathcal{C}$ be a symmetric monoidal category. We say that $\mathcal{C}$ is closed if it has an internal hom functor. That is, for each object A in $\mathcal{C}$, there is an internal hom functor $\underline{\mathrm{Hom}}(A, -) : \mathcal{C} \to \mathcal{C}$ that is right adjoint to $A \otimes -$. That is, there is an isomorphism*

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \underline{\mathrm{Hom}}(B, C)), \tag{2.4}$$

where A, B and C are objects in $\mathcal{C}$. In particular when $A = \mathbb{1}$ we get

$$\mathrm{Hom}(B, C) \cong \mathrm{Hom}(\mathbb{1}, \underline{\mathrm{Hom}}(B, C)). \tag{2.5}$$

The above bijection is a relation between sets, and we will call it the universal property of internal Hom functor.

Lemma 2.3. *Let $\mathcal{C}$ be a symmetric monoidal category. If $\mathcal{C}$ is compact closed then is closed.*

Proof. Let A be a dualizable object, with dual A^*. We define an internal hom functor by the formula $\underline{\mathrm{Hom}}(A, B) \cong A^* \otimes B$. Then, we just need to verify that $- \otimes A^*$ is a right adjoint to $- \otimes A$. By Lemma 2.2 the conclusion is obtained. □

The previous lemma tells us that dualizable objects in compact closed categories have the form as dualizable objects in $\mathbf{Vect}_k$. Thus if you take in Definition 15 $B = \mathbb{1}$ we get

$$\underline{\mathrm{Hom}}(A, \mathbb{1}) \cong A^* \otimes \mathbb{1} \cong A^*. \tag{2.6}$$

We will call $\underline{\mathrm{Hom}}(A, \mathbb{1})$ the canonical dual of A. When we use the dual basis theorem to describe the duals in Mod_R and Ch_R we use canonical duals, see Example 4.

Corollary 2.4. *Let $\mathcal{C}$ be a compact closed category. $M \in Ob(\mathcal{C})$ is dualizable if and only if the canonical map*

$$M \otimes \underline{\mathrm{Hom}}(M, \mathbb{1}) \to \underline{\mathrm{Hom}}(M, M), \tag{2.7}$$

is an isomorphism.

Proof. ($\Rightarrow$) Consider a dualizable object M, then by the internal Hom Formula 2.6 we get

$$\underline{\mathrm{Hom}}(M, M) \cong M^* \otimes M \cong \underline{\mathrm{Hom}}(M, \mathbb{1}) \otimes M.$$

($\Leftarrow$) Suppose that the isomorphism 2.7 holds, the object $\underline{\mathrm{Hom}}(M, \mathbb{1})$ will be denoted by M^*. We claim that M^* is the dual of M, so the conclusion holds if we find the evaluation and coevaluation maps. For the coevaluation map consider

$$\mathrm{Hom}(M, M) \cong \mathrm{Hom}(\mathbb{1}, \underline{\mathrm{Hom}}(M, M)) \cong \mathrm{Hom}(\mathbb{1}, M \otimes \underline{\mathrm{Hom}}(M, \mathbb{1})) \cong \mathrm{Hom}(\mathbb{1}, M \otimes M^*).$$

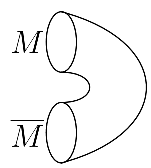
From the above isomorphism, there exists a distinguished map in $\mathrm{Hom}(M, M)$, the identity, whose image under the isomorphism is the coevaluation map. A similar idea allows us to find the evaluation map. It is no hard to see that these maps satisfy the triangle identities. $\square$

The next lemma provides us with a nontrivial example of compact closed category.

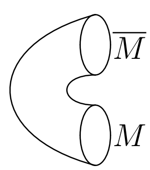
Lemma 2.5. ***Bord**$_n$ is a compact closed category.*

Proof. Let M be an object in $\mathbf{Bord}_n$, that is, an oriented $(n-1)$-manifold. We denote by $\overline{M}$ the same manifold with reverse orientation. Let the evaluation and coevaluation maps be defined as follows:

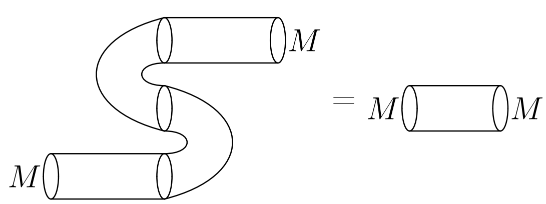
• $\epsilon_M : M \bigsqcup \overline{M} \to \emptyset$ is given by the bordism $M \times [0, 1]$ interpreted as cobordism from $M \bigsqcup \overline{M}$ to $\emptyset$, see figure below



• $\eta_M : \emptyset \to \overline{M} \bigsqcup M$ is given by the bordism $M \times [0, 1]$ interpreted as cobordism from $\emptyset$ to $\overline{M} \bigsqcup M$, see figure below



By diffeomorphism invariance these two maps are related to the identity id_M as follows



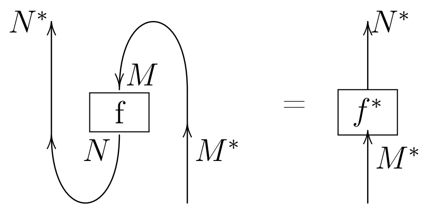
If we interpret the above bordism representation in terms of ϵ_M and η_M we get the same formulas presented in Definition 13. As a result, every object M in $\mathbf{Bord}_n$ is dualizable with dual the same manifold with opposite orientation. $\square$

Definition 16. *If M and N are dualizable objects in a symmetric monoidal category $\mathcal{C}$, the map $f : M \to N$ has a dual map $f^* : N^* \to M^*$ defined by the composite:*

$$N^* \xrightarrow{N^*\otimes\eta} N * \otimes M \otimes M^* \xrightarrow{N^*\otimes f\otimes M^*} N^* \otimes N \otimes M^* \xrightarrow{\epsilon\otimes M^*} M^*.$$

In particular, for any dualizable object M, the endomorphism $f : M \to M$ has a dual map $f^* : M^* \to M^*$.

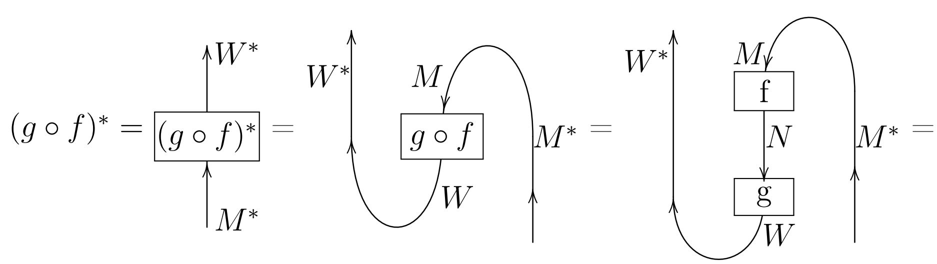
The following diagram represents f^* in string diagram notation.

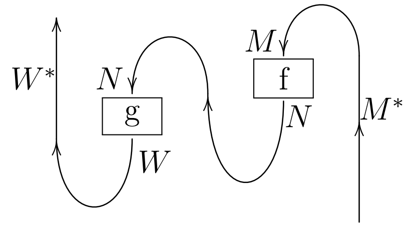


The above definition represents, in a categorical sense, the notion of taking a transpose of a matrix. If we consider $g : V \to W$ a linear transformation, it is not hard to see that the above composition g^* is the transpose of the matrix that represents g.

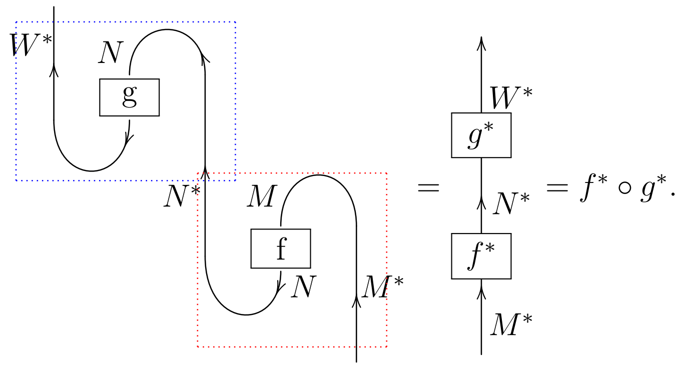
Proposition 2.6. *Let M, N and W be dualizable objects in a symmetric monoidal category $\mathcal{C}$, and $f : M \to N$ and $g : N \to W$ two maps in $\mathcal{C}$. Then $(g \circ f)^* = f^* \circ g^*$.*

Proof. Consider the continuous sequence of deformations of string diagrams:





We can stretch the last diagram and get



If we compare the diagram inside the blue box regarding the above definition we get g^* and the same for the red box we get f^*. Thus, gluing these diagrams we get the last equality and the conclusion holds. □

In the context of linear algebra, the above proposition tells us that the transpose of a product of matrix is the product of the transposes with reverse order.

2.2 Monoidal functors

In category theory, it is important to compare things such as objects, morphisms, and even categories. We compare categories by means of functors. In the categories that interest us, we need to demand functors with more structure that respect the monoidal structure. More specifically, we have the following definition. Again, this definition seems to be bulky but we need to have in mind that this functor must send the unit to the unit and respect the tensor product.

Definition 17. *Let $\mathcal{C}$ and $\mathcal{D}$ be two monoidal categories with unit objects $\mathbb{1}_{\mathcal{C}}$ and $\mathbb{1}_{\mathcal{D}}$ respectively, and tensor product denoted by $\otimes$ in both categories. In addition, let $\alpha_{\mathcal{C}}$ and $\alpha_{\mathcal{D}}$ be the associator maps of the categories $\mathcal{C}$ and $\mathcal{D}$, respectively. A **lax monoidal functor** between $\mathcal{C}$ and $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ together with an isomorphism*

$$\iota : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}}), \tag{2.8}$$

and a natural isomorphism

$$\psi_{A,B} : F(A) \otimes F(B) \to F(A \otimes B), \tag{2.9}$$

for each pair of objects A, B in $\mathcal{C}$, which satisfies the following conditions:

1. ***Associativity**, for all objects $X, Y, Z \in \mathcal{C}$ the following diagram commutes*

$$
\begin{tikzcd}
(F(X)\otimes F(Y))\otimes F(Z) \arrow[r, "\psi_{X,Y}\otimes id_{F(Z)}"] \arrow[d, "\alpha_{\mathcal{D}}"] & F(X\otimes Y)\otimes F(Z) \arrow[d, "\psi_{X\otimes Y,Z}"] \\
F(X)\otimes (F(Y)\otimes F(Z)) \arrow[d, "id_{F(X)}\otimes\psi_{Y,Z}"] & F((X\otimes Y)\otimes Z) \arrow[d, "F(\alpha_{\mathcal{C}})"] \\
F(X)\otimes F(Y\otimes Z) \arrow[r, "\psi_{X,Y\otimes Z}"'] & F(X\otimes (Y\otimes Z))
\end{tikzcd}
.
$$

2. ***Unitality***, *for all objects $X \in \mathcal{C}$ the following diagrams commutes*

$$
\begin{tikzcd}
\mathbb{1}_{\mathcal{D}}\otimes F(X) \arrow[r, "\iota\otimes id_{F(X)}"] \arrow[d, "l_{\mathcal{D}}"] & F(\mathbb{1}_{\mathcal{C}})\otimes F(X) \arrow[d, "\psi"] \\
F(X) \arrow[r, "F(l_{\mathcal{C}})"] & F(\mathbb{1}_{\mathcal{C}}\otimes X)
\end{tikzcd}
.
$$

where $l_{\mathcal{C}}$ and $l_{\mathcal{D}}$ are the left unitors in $\mathcal{C}$ and $\mathcal{D}$, respectively.

In addition, a monoidal functor is symmetric if the following diagram commutes. That is, for all objects X, Y in $\mathcal{C}$

$$
\begin{tikzcd}
F(X)\otimes F(Y) \arrow[r, "\psi_{X,Y}"] \arrow[d, "\gamma_{\mathcal{D}}"] & F(X\otimes Y) \arrow[d, "F(\gamma_{\mathcal{C}})"] \\
F(Y)\otimes F(X) \arrow[r, "\psi_{Y,X}"'] & F(Y\otimes X)
\end{tikzcd}
.
$$

where $\gamma_{\mathcal{C}}$ and $\gamma_{\mathcal{D}}$ are the symmetric isomorphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively.

One of the useful properties of dualizability is its nice behaviour under symmetric monoidal functors:

Proposition 2.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be two symmetric monoidal categories and $F : \mathcal{C} \to \mathcal{D}$ a lax monoidal functor. Let X be a dualizable object of $\mathcal{C}$ with dual object X^*. Then $F(X)$ is dualizable with dual $F(X^*)$. Moreover, the evaluation and coevaluation map in $\mathcal{D}$ satisfy:*

$$\epsilon_{F(X)} = F(\epsilon_X), \qquad \eta_{F(X)} = F(\eta_X).$$

Proof. Let X be an object in $\mathcal{C}$ and X^* its dual. By definition, we know that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(f \otimes g) = F(f) \otimes F(g)$ for any f, g morphisms on $\mathcal{C}$. Then, the triangle identity

$$\begin{tikzcd} X \arrow[rr, "id_X"] \arrow[dr, "id_X \otimes \eta_X"'] & & X \\ & X \otimes X^* \otimes X \arrow[ur, "\epsilon_X \otimes id_X"'] & \end{tikzcd}$$

is sent by F to the diagram

$$\begin{tikzcd} F(X) \arrow[rr, "id_{F(X)}"] \arrow[dr, "id_{F(X)} \otimes \eta_{F(X)}"'] & & F(X) \\ & F(X) \otimes F(X^*) \otimes F(X) \arrow[ur, "F(\epsilon_X) \otimes id_{F(X)}"'] & \end{tikzcd}$$

The other axiom is similar. So we can conclude that $F(X^*)$ is dual to $F(X)$ i.e. $F(X^*) \cong F(X)^*$. $\square$

Example 5. *Examples of monoidal functors are:*

(a) The forgetful functor $U : \boldsymbol{Ab} \to \boldsymbol{Set}$ from the category of Abelian groups to the category of sets.

(b) If R is a (commutative) ring, then the free functor $F : \mathbf{Set} \to \boldsymbol{Mod}_R$ extends to a monoidal functor $F : (\mathbf{Set}, \sqcup, \emptyset) \to (\boldsymbol{Mod}_R, \oplus, 0)$ in a natural way.

(c) Let $\mathcal{C}$ be a compact closed category, and $D : \mathcal{C} \to \mathcal{C}$ a functor such that for any object A in $\mathcal{C}$, $D(A)$ is the dual of A. D is a monoidal functor. This is because for any $A, B \in Ob(\mathcal{C})$, we have the isomorphism $D(A \otimes B) \cong D(A) \otimes D(B)$.

(d) The homology functor $H : \boldsymbol{Ch}_R \to \boldsymbol{Gr}_R$, i.e. the functor from the chain complex over a commutative ring R to the graded R-modules. The Kunneth theorem implies that the natural transformation

$$\tau : H(C^\bullet) \otimes H(D^\bullet) \to H(C^\bullet \otimes D^\bullet), \tag{2.10}$$

is an isomorphism if C^i, D^i are projective for every $i \in \mathbb{Z}$.

*(e) An **n-TQFT** is a symmetric monoidal functor $Z : \boldsymbol{Bord}_n \to \boldsymbol{Vect}_k$ where each $(n-1)$-manifold is sent to a finite dimensional vector space and each n-manifold is sent to a linear map. The $(n-1)$-manifold $M_1 \sqcup M_2$ is sent by means of Z to the vector space $Z(M_1) \otimes Z(M_2)$.*

(f) The free abelian group functor $Z : (\boldsymbol{Set}, \times, \emptyset) \to (\boldsymbol{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ which assigns $\mathbb{Z}[X]$ the free abelian group generated by X to each set X. It is $Z(X) = \mathbb{Z}[X]$, and for each pair of sets X, Y we have $Z(X \times Y) = \mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathbb{Z}[Y]$.

(g) *Consider* $\Sigma^\infty : \boldsymbol{Top}_* \to \boldsymbol{HoSpect}$ *the spectrum functor such that each topological based space* X *is sent to* $\{\Sigma^n X\}_{n\geq 0}$. *In Chapter 3, we are going to describe that functor and the* ***HoSpect*** *category with all the detail. But a difference of a topological suspension (see Chapter 3), the spectrum functor satisfies that for* X *and* Y *based spaces there exist an isomorphism at the homotopic stable level*

$$\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty X \wedge \Sigma^\infty Y.$$

Thanks to the notion of monoidal functor we can move to other categories and extend our framework, such as dualizability. There are many important and intuitive examples, the free abelian functor defined in the above example is a motivation of that; we know the free abelian group $\mathbb{Z}[X]$ is dualizable if and only if X is a finite set of points. This extends dualizability in sets which only one element sets can be dualizables. Lemma 3.8, in Chapter 3, has the same philosophy because Σ^∞ helps us to extend S-duality in spaces to the stable homotopy category.

2.2.1 Freely generated category

String diagrams was an idea introduced first by Turaev with the goal of translating equations into diagrams that we may manipulate in a topological sense. We can think of strings as morphisms in $\mathbf{Bord}_1$, the category of 1-dimensional bordisms. In this thesis we are interested in the relationship between bordism theory and traces. Therefore, it is convenient to stop for a while, and discuss $\mathbf{Bord}_1$ as a freely generated category.

Recall that the notion of *topological quantum field theories* was borned from the work of Witten in [41]. The motivation comes from physics studying a quantum field theory that does not depend on the Riemannian metric of the underlying space-time manifold. Shortly after, Atiyah in [6] proposed a set of axioms which were supposed to lay a rigorous foundation for a mathematical treatment of TQFTs. The Atiyah's idea was to consider an n-dimensional TQFT as a rule which assigns finite-dimensional vector spaces to closed oriented $(n-1)$-manifolds and linear maps to n-dimensional oriented cobordisms (up to diffeomorphism preserving the boundary) between two such $(n-1)$-manifolds. Using the modern language of category theory we say that an n-dimensional TQFT is a functor from the cobordism category $n\mathbf{Cob}$ to the category $\mathbf{Vect}_k$ of finite dimensional vector spaces which obeys certain additional properties.

First let us make precise what a freely generated category means. Intuitively it is a decomposition in objects morphisms and relations that morphisms must satisfy. Formally, we require a set of objects G_0, morphisms G_1 in which there exist source and target maps denoted by s, t which are defined by follows: For $f : x \to y$ a morphism in G_1 and $x, y \in G_0$ such that $s(f) = x$ and $t(f) = y$. In addition, we require a set of relations G_2 which are tuple of morphisms in G_1 that can be identified.

To make the above more formal in a categorical sense a **freely generated category** by sets G_0, G_1 and G_2 is a symmetric monoidal category with the following universal property:

Definition 18. *Let $\mathcal{C}$ be a symmetric monoidal category. A symmetric monoidal functor $\mathcal{F}(G_0, G_1, G_2) \to \mathcal{C}$ is characterized uniquely (up to monoidal isomorphism) by choosing an object in $\mathcal{C}$ for each element in G_0 and a morphism in $\mathcal{C}$ for each element in G_1 (with correct source and target) such that the relations in G_2 are satisfied. We will say that $\mathcal{F}(G_0, G_1, G_2)$ is the symmetric monoidal category* ***freely generated*** *by G_0, G_1 and G_2.*

We will construct this freely generated category in the following sequence of steps below until we get a category that satisfies the above universal property. It is important to point out that it is unique up to natural monoidal isomorphism.

Objects: We start with the free symmetric monoidal category $\mathcal{F}(G_0)$ whose objects are freely generated by G_0. First, let us consider G_0 as a category with objects the elements of G_0 and morphism the identity map for any object. We can induce a monoidal structure in $\mathcal{F}(G_0)$ by a map $I : G_0 \to \mathcal{F}(G_0)$ which sends each $x \in G_0$ to the one-element list denoted by (x). The tensor product is given on objects by concatenation of lists, and the tensor unit is the empty list. Morphisms between lists are defined only in lists of equal length and are all the permutations of the length of the lists. Such pair $(\mathcal{F}(G_0), I)$ must obey the following universal property: For all symmetric monoidal category $\mathcal{C}$,

$$Fun(\mathcal{F}(G_0), \mathcal{C}) \to F^{G_0}(\mathcal{C}) \tag{2.11}$$

induces an equivalence of categories, where $F^{G_0}(\mathcal{C})$ is the category of functors from G_0 to $\mathcal{C}$. In our particular situation, with G_0 the set of objects represented in Figure 2.2

$$\bullet_+ , \quad \bullet_-$$

Figure 2.2: Set of objects.

We get that $\mathcal{F}(G_0)$ is a monoidal category whose objects are disjoint union of points with orientation and morphisms, as we said before, are defined between sets of points of the same length connected by oriented strings, Figure 2.3 gives us an intuitive idea.

In our particular case, giving a symmetric monoidal functor $\mathcal{F}(G_0) \to \mathbf{Vect}_k$, by the universal property 2.11, is equivalent to a choice of a vector space V for $\bullet+$ and U for $\bullet-$.

Objects and morphisms: We can try to add a set G_1 of extra morphisms to $\mathcal{F}(G_0)$. First, given a symmetric monoidal category $\mathcal{C}$, let us consider $F^{G_0.G_1}(\mathcal{C})$ the category with:

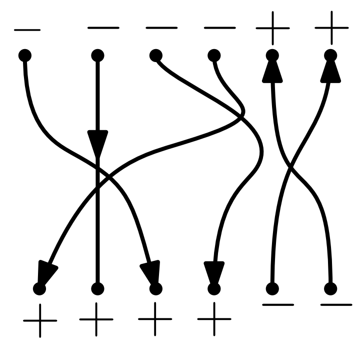


Figure 2.3: Morphism in $\mathcal{F}(G_0)$

• Objects are pairs (Φ, H), where Φ is an object in $Fun(\mathcal{F}(G_0), \mathcal{C})$ and $H : G_1 \to Mor(\mathcal{C})$ is a map such that for all $f \in G_1$, $H(f)$ has source $\Phi(s(f))$ and target $\Phi(t(f))$, where $s, t : G_1 \to Ob(\mathcal{F}(G_0))$ are the source and target maps, respectively.

• Morphisms are transformations $\varphi : (\Phi, H) \to (\Phi', H')$ which make the following diagram commute

$$\begin{tikzcd} \Phi(s(f)) \arrow[r, "H(f)"] \arrow[d, "\varphi_{s(f)}"] & \Phi(t(f)) \arrow[d, "\varphi_{t(f)}"] \\ \Phi'(s(f)) \arrow[r, "H'(f)"'] & \Phi'(t(f)) \end{tikzcd}$$

for each $f \in G_1$.

Then, the free symmetric category generated by G_0 and G_1 is the category $\mathcal{F}(G_0, G_1)$ and a pair of symmetric monoical functors (J, j) where $j : G_1 \to Mor(\mathcal{F}(G_0, G_1))$ and $J : \mathcal{F}(G_0) \to \mathcal{F}(G_0, G_1)$, which makes

$$Fun(\mathcal{F}(G_0, G_1), \mathcal{C}) \to F^{G_0.G_1}(\mathcal{C}) \tag{2.12}$$
$$\Phi \mapsto (\Phi \circ J, \Phi \circ j) \tag{2.13}$$

an equivalence of categories.

In our example G_1 is the set of morphisms represented by Figure 2.4

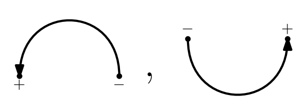
$G_1 = \{$  $\}$

Figure 2.4: Morphisms.

The above set of morphisms G_1 are what we called the evaluation and coevaluation map, respectively.

By the above universal property, giving a monoidal functor $\mathcal{F}(G_0, G_1) \to \mathbf{Vect}_k$ amounts to picking V and U for the objects in G_1 and linear maps $d : V \otimes U \to k$ and $b : k \to V \otimes U$, given by Figure 2.4. Note that the vector spaces U and V do not necessarily have to be finite dimensional.

Objects, morphisms and relations: We already have $\mathcal{F}(G_0, G_1)$ at our disposal. Then, the relations G_2 are a set of diagrams in $\mathcal{F}(G_0, G_1)$ which we would like to commute. An element in G_2 is a pair (f_1, f_2) where f_1, f_2 are morphisms in $\mathcal{F}(G_0, G_1)$ with the same source and target.

Let us define $F^{G_0,G_1,G_2}(\mathcal{C})$ to be the subcategory of $Fun(\mathcal{F}(G_0, G_1), \mathcal{C})$ with $\mathcal{C}$ be an arbitrary monoidal category, and whose objects are symmetric monoidal functors $F \in Fun(\mathcal{F}(G_0, G_1), \mathcal{C})$ which satisfy $F(f_1) = F(f_2)$ for $(f_1, f_2) \in G_2$.

The free category generated by G_0, G_1 and G_2 is a symmetric monoidal category $\mathcal{F}(G_0, G_1, G_2)$ together with with a symmetric functor $S : \mathcal{F}(G_0, G_1) \to \mathcal{F}(G_0, G_1, G_2)$ such that $S(f_1) = S(f_2)$ for each pair $(f_1, f_2) \in G_2$, that satisfy the following universal property: The functor

$$Fun(\mathcal{F}(G_0, G_1, G_2), \mathcal{C}) \to F^{G_0,G_1,G_2}(\mathcal{C}) \tag{2.14}$$

is an equivalence of categories.

In our particular example, G_2 is the set of two elements represented by the following figure.

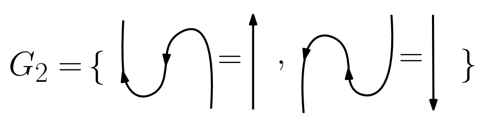


Figure 2.5: Relations.

To give a symmetric monoidal functor $\mathcal{F}(G_0, G_1, G_2) \to \mathbf{Vect}_k$, we need to know how to assign objects and morphisms (U, V, b, d) as we did above, but now subject to the relations.

$$(d \otimes id_U) \circ (id_U \otimes b) = id_U, \qquad (id_V \otimes d) \circ (b \otimes id_V) = id_V. \tag{2.15}$$

From all this detailed description, we can conclude that $\mathrm{Bord}_1 \cong \mathcal{F}(G_0, G_1, G_2)$. That is because Bord_1 satisfies the universal property 2.14 in a natural way defined by the descomposition in the objects and morphisms presented in G_0, G_1 and G_2.

More concretely, we can define a monoidal functor $Z : \mathrm{Bord}_1 \to \mathbf{Vect}_k$ which satisfies the universal property 2.14, that functor is a 1-TQFT. From the Equation 2.15, we can conclude that U and V are finite dimensional vector spaces and $U \cong V^*$, equivalently $U^* \cong V$.

We can move towards 2-dimensional bordisms. As in the 1-dimensional case, we start with the generators of the bordism category. There is only one compact connected oriented 1-manifold up to diffeomorphism, a circle, that has orientation-reversing diffeomorphisms. Then an object in Bord_2 is orientation-preserving diffeomorphic to a finite disjoint union of (oriented) $\mathbb{S}^1$'s. Thus, we have $G_0 = \{\mathbb{S}^1\}$.

Morphisms in $\mathbf{Bord}_2$ are oriented surfaces. It is a classical result, which may be proven using Morse theory, that surfaces can be classified by their genuses, that is the number of holes.

Another classical result, also from Morse theory, is that the surfaces can be decomposed by elementary surfaces:

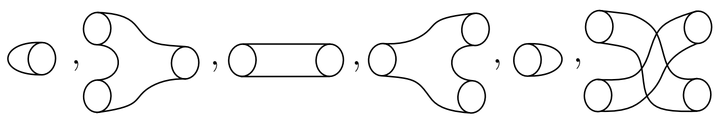


Figure 2.6: elementary bordisms.

However, we can reduce the above list and dispense of the cylinder and the braiding morphism from the set of morphism generators. Hence, we set:

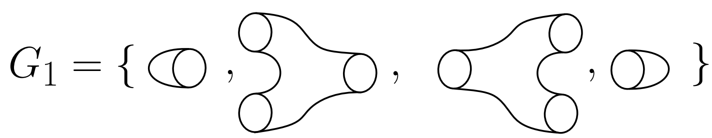


Figure 2.7: Set G_1 of generators.

The set of relations that the set of morphisms G_1 must obey is:

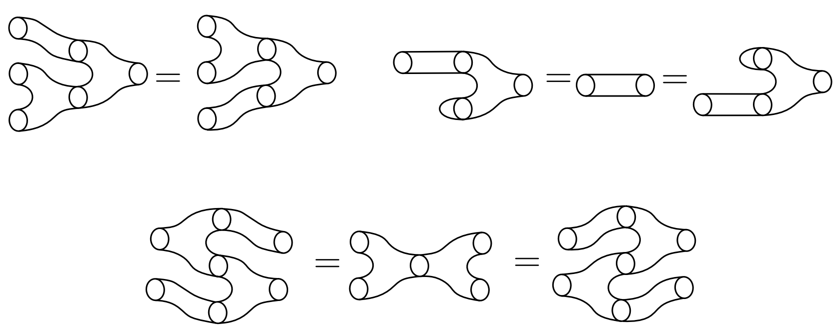


Figure 2.8: Set G_2 of relations.

There are many other relations that we did not draw in this work. In [24] the author presents a good description of the above relations. Thus, we get the following theorem.

Theorem 2.8. *Let $G_0 = \{S^1\}$, G_1 and G_2 given by the sets in Figures 2.7 and 2.8, respectively. Then, $\boldsymbol{Bord}_2$ is freely generated as a symmetric monoidal category by G_0, G_1, G_2, that is, $\boldsymbol{Bord}_2 \cong \mathcal{F}(G_0, G_1, G_2)$.*

In the same way we can define, by abuse of notation, a monoidal functor $Z : \mathbf{Bord}_2 \to \mathbf{Vect}_k$ that satisfies the Property 2.14. Under the functor Z the image of S^1 is a vector space $Z(S^1) = A$, and together the relations in the Figure 2.6 we can conclude that there is an equivalence of categories $2 - \mathbf{TQFT} \sim \mathbf{coFrob}_k$, where $\mathbf{coFrob}_k$ is the category of commutative frobenious algebras. In [24] the author presents a detailed description of this equivalence.

In dimension more than 3, we do not have a generalization of the previous theorem. That is, we can not reduce the category $\mathbf{Bord}_n$ as a gluing of finite bordisms. J. Baez and J. Doland in [7] conjectured the following:

Theorem 2.9. ***Cobordism hypothesis***
The $Bord_n$ category is the free symmetric monoidal n-category with duality on one object.

Recently there has been substantial progress towards these results due to the work of Hopkins and Lurie. In [26], Lurie outlines a sophisticated program which reformulates and proves the Baez-Dolan Cobordism Hypothesis. Both the proof sketch and the reformulation of the Cobordism Hypothesis use the language of (∞, n)-categories in an essential way. In particular they exploit the relationship between (∞, n)-categories and homotopy theory to a great extend. Their solution is consequently cast in the same language. All these developments are beyond the scope of this thesis

2.3 Cartesian categories

In computer science, there are processes of duplication and deletion of data. We can try to mimic this idea and, for example, in set theory we can define a map in which any $a \in A$ is sent to the pair (a, a) in the Cartesian product $A \times A$. This map is denoted by Δ_A and called the A-diagonal map, and also we can define a map $e_A : A \to *$ in which all $a \in A$ in sent to the one-point set. Such maps do not necessarily exist for any object in any arbitrary category. We want to restrict our attention to those categories in which there is a diagonal map with certain properties.

Diagonal maps are the most interesting because, as we saw in chapter one, this allows us to study fixed points from a geometric point of view. In this section we want to establish that condition in a more abstract sense. This type of categories with diagonal and deleting maps motivate the following definition.

Definition 19. *A **cartesian monodial category** is a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \gamma)$ equipped with monoidal natural transformations*

$$\Delta_X : X \to X \otimes X \quad \textit{and} \quad e_X : X \to \mathbb{1}, \tag{2.16}$$

such that the following composites are the identity morphisms

$$X \xrightarrow{\Delta_X} X \otimes X \xrightarrow{e_X \otimes \mathbb{1}} \mathbb{1} \otimes X \xrightarrow{l_X} X,$$

$$X \xrightarrow{\Delta_X} X \otimes X \xrightarrow{\mathbb{1} \otimes e_X} X \otimes \mathbb{1} \xrightarrow{r_X} X,$$

where l and r and the right and left unitors.

Example 6. *Important examples of cartesian monoidal categories include:*

1. ***Sets**, together the empty set and the cartesian product is a cartesian monoidal category.*
2. ***Top**, the category of topological spaces together with the cartesian product of topological spaces and the discrete topological space of one element define a cartesian monoidal category.*
3. ***Cat**, the bicategory of small categories with the product category, where the category with one object and only its identity map is the unit.*
4. *The category of finite dimensional k-vector spaces $\textbf{\textit{Vect}}_k$ is a cartesian monoidal category with the cartesian product of vector spaces.*

Remark 1. We will denote the cartesian product as $\times$ with the aim to recall that we are in a cartesian category.

We can think of the above conditions as: Duplicating a piece of data and then deleting one copy is the same as not doing anything. Note that any object of the form $X \times Y$ in a cartesian category comes equipped with morphisms $\pi_{X,Y} : X \times Y \to X$ and $\pi'_{X,Y} : X \times Y \to Y$ the projections to the first and second component, respectively. These maps are given by the following composites

$$X \times Y \xrightarrow{id_X \times e_Y} X \times \mathbb{1} \xrightarrow{r_X} X,$$

$$X \times Y \xrightarrow{e_X \times id_Y} \mathbb{1} \times Y \xrightarrow{l_Y} Y.$$

In Appendix A, we will represent these new maps in terms of string diagrams, they will be useful in the proof of the relation between traces and fixed point operators.

The next proposition describes the dualizable objects in monoidal cartesian categories.

Proposition 2.10. *The dualizable objects in any cartesian monoidal category are the final objects.*

Proof. Let $\mathcal{C}$ be a cartesian monoidal category, A a dualizable object and A^* its dual. By Lemma 2.2 the functor $- \otimes A^*$ is right adjoint to $- \otimes A$. Then for X, Y objects in $\mathcal{C}$, we have:

$$\mathrm{Hom}(X \otimes A, Y) \cong \mathrm{Hom}(X, Y \otimes A^*). \tag{2.17}$$

In particular for $X = Y = \mathbb{1}$, we have:

$$\mathrm{Hom}(A, \mathbb{1}) \cong \mathrm{Hom}(\mathbb{1}, A^*). \tag{2.18}$$

We know $\mathbb{1}$ is a final object. Thus, we can conclude that $\mathbb{1} \cong \mathrm{Hom}(A, \mathbb{1})$. Therefore, $\mathrm{Hom}(\mathbb{1}, A^*) \cong \mathbb{1}$. Thus, for any object W in $\mathcal{C}$ there is a unique map from W to $\mathbb{1}$, and there exist a unique map from $\mathbb{1}$ to A^*, the composite gives us a unique map from W to A^*. In conclusion A^* is a final object, and since final objects are unique up to isomorphism, it must satisfy $A^* \cong \mathbb{1}$. $\square$

The above theorem tells us that there is not an interesting notion of dualizability in cartesian categories. **Sets** is an example of cartesian monoidal category, its dualizable objects are one-element sets or singletons, denoted by $\{*\}$. In addition, the category of topological spaces (**Top**) is a cartesian monoidal category as well. Also, their dualizable objects are discrete topological spaces of one elemenent.

Monoidal functors allow us to define diagonal maps in monoidal categories that are not necessarily cartesian. That is important because diagonal maps will allow us to get more information about fixed points. The following example shows how we can construct diagonal maps in the image of a cartesian category under a monoidal functor.

Example 7. *Consider $Z : \textbf{Sets} \to \textbf{Ab}$ the monoidal functor defined in the Example 5. For each set X, we have a diagonal map $\Delta_X : X \to X \times X$ defined by $\Delta_X(x) = (x, x)$ for each $x \in X$. The image of this map under this functor is: $\mathbb{Z}(\Delta_X) : \mathbb{Z}[X] \to \mathbb{Z}[X \times X]$ a map in $\textbf{Ab}$. The monoidal structure gives us an isomorphism $\mathbb{Z}[X \times X] \cong \mathbb{Z}[X] \otimes \mathbb{Z}[X]$. Thus, under the previous isomorphism we can define a diagonal map for $\mathbb{Z}[X]$ as the following composite:*

$$\begin{tikzcd} \mathbb{Z}[X] \arrow[r, "\mathbb{Z}(\Delta_X)"] \arrow[rr, bend right, "\Delta_{\mathbb{Z}[X]}"'] & \mathbb{Z}[X \otimes X] \arrow[r, "\psi"] & \mathbb{Z}[X] \otimes \mathbb{Z}[X] \end{tikzcd}$$

where ψ is the natural isomorphism as in 2.9.

The previous example allow us to consider **Ab** as a cartesian category, at least in the image of the functor $\mathbb{Z}(-)$. A similar construction you get if you replace the functor $\mathbb{Z}(-)$ by other monoidal functor.

Example 8. *Consider the suspension functor Σ^∞, defined also in Example 5, as we did in the previous example we can transfer diagonal maps Δ_X in **Top** to diagonal maps in **HoSpect**. Thus, for X a based topological space (also based CW complex) we can define a map, as in the previous example, given by the image under the functor Σ^∞*

$$\Delta_{\Sigma^\infty(X)} : \Sigma^\infty(X) \to \Sigma^\infty(X) \wedge \Sigma^\infty(X). \tag{2.19}$$

In Chapter 3, we will describe the constructions presented in the above example. Another important characteristic of the connection between cartesian categories and monoidal functors, with codomain a noncatesian category, is that it helps us to extend dualizability. The baby example that we can have in mind is the functor $\mathbb{Z}[-]$, which sends a finite set X which is not dualizable in **Sets** to the freely generated group $\mathbb{Z}[X]$ which is dualizable in **Ab** with itself as a dual. We can define the dual maps by mean of the formulas

$$\eta : \mathbb{Z} \to \mathbb{Z}[X] \otimes \mathbb{Z}[X] \qquad \epsilon : \mathbb{Z}[X] \otimes \mathbb{Z}[X] \to \mathbb{Z}, \tag{2.20}$$

$$1 \mapsto \sum_{x \in X} x \otimes x \qquad \sum x \otimes x' \mapsto \sum_{x = x'} 1. \tag{2.21}$$

In a more abstract setting, we will show that for X a based topological space, with good topological conditions, then the object $\Sigma^\infty(X)$ is dualizable in **HoSpect**.

2.4 General trace

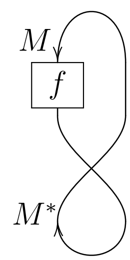
In this section we are going to define the abstract notion of trace. This was historically defined by Dold and Puppe in their famous work [2] and also defined by Ponto and Schulman in [34]. As we mentioned at the beginning of this chapter, the trace in symmetric monoidal categories is nothing more than the reinterpretation of the definitions linear algebra framework at a categorical level. In recent years, many authors have found applications of categorical traces in other branches of science such as algebraic topology, knot theory, computer science and quantum mechanics. In this part of this work we are interested in studying the relation between traces and fixed points.

Definition 20. *Let $\mathcal{C}$ be a symmetric monoidal category, M a dualizable object in $\mathcal{C}$ with dual M^* and $f : M \to M$ an endomorphism of M. The **canonical trace** of f, denoted by $tr(f)$, is the following composite map*

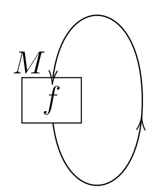
$$\mathbb{1} \xrightarrow{\eta} M \otimes M^* \xrightarrow{f \otimes M^*} M \otimes M^* \xrightarrow{\cong} M^* \otimes M \xrightarrow{\epsilon} \mathbb{1},$$

*In particular, the **Euler characteristic** is the trace of the identity map.*

Some authors refer to the trace of the identity as the dimension of the object. We can represent the trace in string diagram notation as



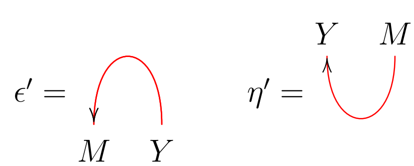
For a trace, we are going to consider a topological switch that allows us to simplify the above diagram (see figure below). A fun fact: It shows a relationship between trace and feedback. We will come back to that in the future.



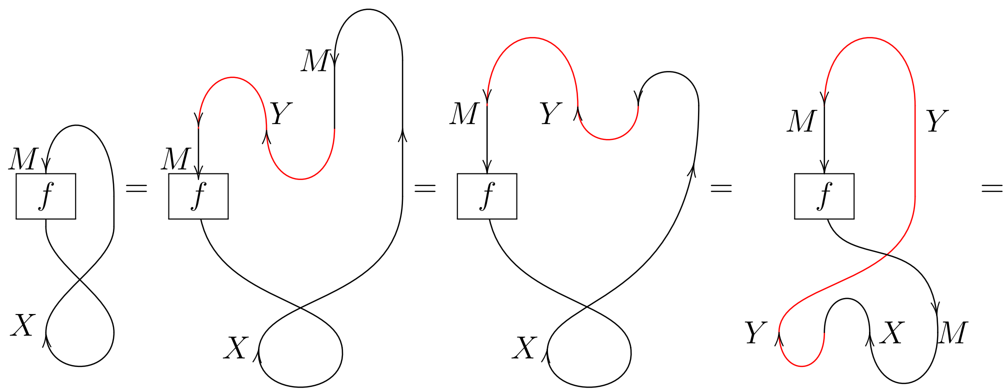
First, let us prove that the trace is well defined.

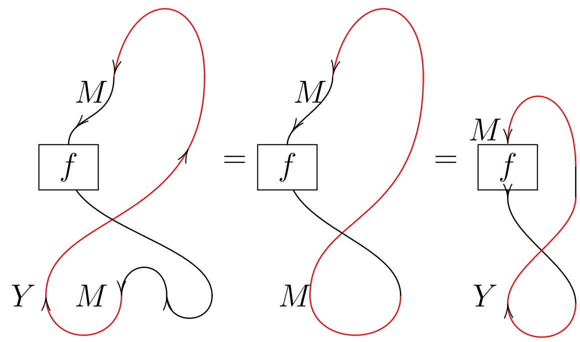
Lemma 2.11. *The trace of a morphism is well-defined.*

Proof. We must show that the same value is obtained whichever choice we make of dual, evaluation and co-evaluation maps. Suppose then that we have dualities (M, X, η, ϵ) and (M, Y, η', ϵ'), where we draw the first duality using the conventions of Definition 13. We draw η' and ϵ' as η and ϵ but the difference is that the prime maps will be colored red:



We then make the following argument:





Therefore, the trace does not depend on the choice of the "color" and then we can conclude that the trace does not depend on the evaluation and coevaluation maps. □

There are some familiar properties of traces coming from linear algebra that we can generalize to a categorical level. For example, the trace of the product of two matrices does not depend on the order of the product. Thus, in a general context there is a symmetry condition with respect to the composition of morphisms.

Lemma 2.12. *(**Cyclicity**) For any maps $f : M \to N$ and $g : N \to M$ in a symmetric monoidal category $\mathcal{C}$, with M, N both dualizable objects, we have*

$$\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f).$$

Proof. Applying the snake identities from Definition 13, and the dual mate of f defined by the composite in Definition 16, we get the following formulas

$$
\begin{aligned}
\mathrm{tr}(f \circ g) &= \epsilon\gamma(f \circ g \otimes \mathrm{id}_{N^*})\eta \\
&= \epsilon\gamma(f \otimes \mathrm{id}_{N^*})(g \otimes \mathrm{id}_{N^*})\eta \\
&= \epsilon\gamma(f \otimes \mathrm{id}_{N^*})(\mathrm{id}_M \otimes \epsilon \otimes \mathrm{id}_{N^*})(\eta \otimes \mathrm{id}_M \otimes \mathrm{id}_{N^*})(g \otimes \mathrm{id}_{N^*})\eta \\
&= \epsilon\gamma(\mathrm{id}_N \otimes \epsilon \otimes \mathrm{id}_{N^*})(f \otimes \mathrm{id}_{M^*} \otimes \mathrm{id}_M \otimes \mathrm{id}_{N^*})(\mathrm{id}_M \otimes \mathrm{id}_{M^*} \otimes g \otimes \mathrm{id}_{N^*})(\eta \otimes \eta) \\
&= (\epsilon \otimes \epsilon)\gamma(\mathrm{id}_N \otimes \mathrm{id}_{M^*} \otimes g \otimes \mathrm{id}_{N^*})(f \otimes \mathrm{id}_{M^*} \otimes \mathrm{id}_N \otimes \mathrm{id}_{N^*})(\eta \otimes \eta) \\
&= (\epsilon \otimes \epsilon)\gamma(\mathrm{id}_N \otimes \mathrm{id}_{M^*} \otimes g \otimes \mathrm{id}_{N^*})(\mathrm{id}_N \otimes \mathrm{id}_{M^*} \otimes \eta)(f \otimes \mathrm{id}_{M^*})\eta \\
&= \epsilon\gamma(\mathrm{id}_M \otimes \epsilon \otimes \mathrm{id}_{M^*})(g \otimes \mathrm{id}_{N^*} \otimes \mathrm{id}_N \otimes \mathrm{id}_{M^*})(\eta \otimes \mathrm{id}_N \otimes \mathrm{id}_{M^*})(f \otimes \mathrm{id}_{M^*})\eta \\
&= \epsilon\gamma(g \otimes \mathrm{id}_{M^*})(\mathrm{id}_N \otimes \epsilon \otimes \mathrm{id}_{M^*})(\eta \otimes \mathrm{id}_N \otimes \mathrm{id}_{M^*})(f \otimes \mathrm{id}_{M^*})\eta \\
&= \epsilon\gamma(g \otimes \mathrm{id}_{M^*})(f \otimes \mathrm{id}_{M^*})\eta \\
&= \epsilon\gamma(g \circ f \otimes \mathrm{id}_{M^*})\eta \\
&= \mathrm{tr}(g \circ f).
\end{aligned}
$$

□

In the above lemma, it is important to note that the trace of $f \circ g$ depends on N while the trace of $g \circ f$ depends on M. Some authors may denote this as $\mathrm{tr}^N(f \circ g)$ and $tr^M(g \circ f)$ respectively. Even though it has a great advantage when we are working with many dualizable objects in the same equation, we have decided in this work to save this notation for a special type of trace gotten from a fixed point operator.

The above notion of trace can be extended to maps with more general sources and targets. For simplicity, we will describe properties for morphisms of the form $f : P \otimes M \to M \otimes Q$. We only require the same duailazable object M in the source and target. In that case $\mathrm{tr}(f) \in \mathrm{End}(Q, P)$

Proposition 2.13. *Let M be a dualizable object and $f : Q \otimes M \to P \otimes M$ a morphism in $\mathcal{C}$. Then*

$$\mathrm{tr}(f) = \mathrm{tr}(\gamma f^* \gamma), \tag{2.22}$$

if $P = Q = \mathbb{1}$ the formula is $tr(f) = tr(f^*)$.

Proof. We can give a similar proof using similar formulas than the previous lemma. However, we have decided to leave this proof until the Appendix A where we use string diagram deformations. □

The above property, in the linear algebra world, tells us that the trace of any matrix is equal to the trace of its transpose. The complicated process to follow these equations and the simplicity of the string diagrams is why the author of this work prefers to work and develop the proofs using string diagrams in most of the cases.

Example 9. *Some important examples of traces in monoidal categories include:*

*(a) In **Vect**$_k$ the trace of an endomorphism is an element of $End(k) \cong k$, it is an element of the field k. Fixing a basis we recover the sum of the diagonal elements of the matrix that represents the endomorphism. The Euler characteristic is the dimension of the vector space.*

*(b) In **Mod**$_R$ the trace of any endomorphism of a projective and finitely generated module is an element of $End(R) \cong R$. The Euler characteristic is the rank of the module.*

*(c) In **Ch**$_R$ the trace of any endomorphism is an element of $End(R) \cong R$, and it is the alternating sum of its degreewise traces. The Euler characteristic is the alternating sum of its degreewise dimensions. Some authors call the trace of a map in this category as the Lefschetz number.*

*(d) In **Set** and **Top** the dualizable objects are sets with only one element and one-point spaces, respectively. In this particular case there is not an interesting notion of trace for a self map $f : X \to X$ with X a set or topological space with more than one element.*

*(e) In **Bord**$_n$ for a morphism $\Sigma : M \Rightarrow M$ given the by the figure 2.9.*

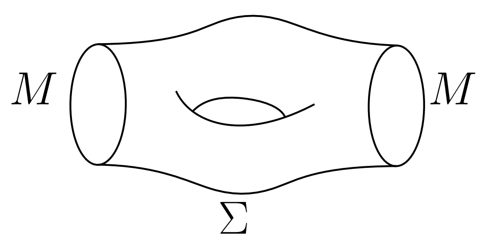


Figure 2.9: Bordism Σ

The trace of Σ is the composite of the following piece of manifolds given by Definition 20

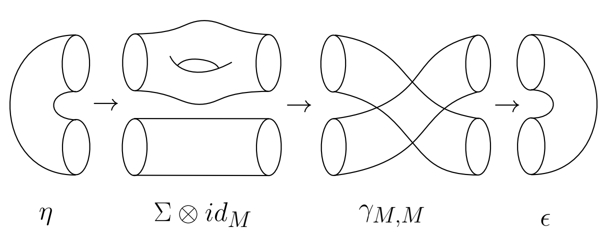


Figure 2.10: $tr(\Sigma)$

If we glue all these pieces, and if we omit the twisted diagram we get that the trace is the following manifold:

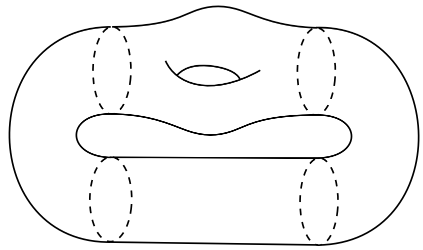


Figure 2.11: $tr(\Sigma)$

The Euler characteristic is diffeomorphic to $M \times S^1$.

We know **Set** and **Top** are cartesian categories, so there is not an interesting notion of dualizable objects, see Example 4-d. Hence, the notion of trace is not interesting at this level. However, we saw that we can extend the dualizability in **Set** by means of the free Abelian functor Z. Thus, the following example helps us extend the trace to any finite set X.

Example 10. *Let $X = \{p_1, \cdots, p_m\}$ be a finite set and $f : X \to X$ a map. If we apply the free abelian functor Z : $\boldsymbol{Set} \to \boldsymbol{Ab}$ which sends each X to the free generated Abelian group. The trace of $Z(f)$ computed by the formula given in the Definition 20 is the following composite*

$$\mathbb{Z} \xrightarrow{\ \eta\ } \mathbb{Z}[X] \otimes \mathbb{Z}[X]^* \xrightarrow{Z(f)\otimes id} \mathbb{Z}[X] \otimes \mathbb{Z}[X]^* \xrightarrow{\ \gamma\ } \mathbb{Z}[X]^* \otimes \mathbb{Z}[X] \xrightarrow{\ \epsilon\ } \mathbb{Z}$$

$$1 \longmapsto \sum_{x\in X} x \otimes x \longmapsto \sum_{x\in X} f(x) \otimes x \longmapsto \sum_{x\in X} x \otimes f(x) \longmapsto \sum_{f(x)=x} 1.$$

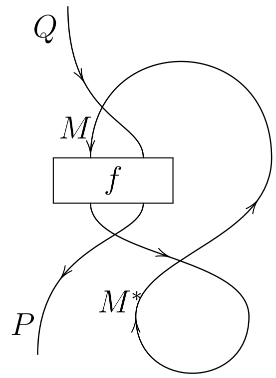
Observation. Note that the trace calculated in the previous example is the same as the Lefschetz number calculated in the Example 1 in Chapter 1. It is a natural extension to a categorical framework.

The key example for us is the category of topological spaces, which is once again a cartesian category and there is not an interesting notion of dualizability or trace. Then, we apply the suspension functor Σ^∞_+ : **Top** $\to$ **HoSpect** and reproduce the same idea as in the previous example. In Chapter 3 we will discuss this particular construction and its relation with Lefschetz fixed point theorem.

We can extend the original definition of trace, presented in Definition 20 to any map $f : Q \otimes M \to M \otimes P$ in $\mathcal{C}$, we only require the dualizable object to appear as a factor of the source and target. Thus, for a dualizable object M and any pair of objects P, Q in $\mathcal{C}$ we can define the twisted trace for f as the following composite

$$Q \xrightarrow{\ \eta\ } Q \otimes M \otimes M^* \xrightarrow{f\otimes M^*} M \otimes P \otimes M^* \xrightarrow{\ \cong\ } M^* \otimes M \otimes P \xrightarrow{\ \epsilon\ } P,$$

This notion of trace may seem more complicated since more objects are included in the input and output data; actually, its string diagram gets more complicated, see figure below. However, it also has its advantages as we will see in the following example.



Twisted trace

Example 11. *Let $f : Q \times X \to X$ be a map in **Set**. Applying the free abelian functor Z : **Set** $\to$ **Ab**, we get the map $Z(f) : \mathbb{Z}[Q] \otimes \mathbb{Z}[X] \to \mathbb{Z}[X]$. Its twisted trace is given by the following composite*

$$\begin{array}{ccccccccc} \mathbb{Z}[Q] & \xrightarrow{\eta} & \mathbb{Z}[Q] \otimes \mathbb{Z}[X] \otimes \mathbb{Z}[X]^* & \xrightarrow{Z(f)\otimes id} & \mathbb{Z}[X] \otimes \mathbb{Z}[X]^* & \xrightarrow{\cong} & \mathbb{Z}[X]^* \otimes \mathbb{Z}[X] & \xrightarrow{\quad \epsilon \quad} & \mathbb{Z} \\ \\ q & \longmapsto & \sum_{x \in X} q \otimes x \otimes x & \longmapsto & \sum_{x \in X} f(q,x) \otimes x & \mapsto & \sum_{x \in X} x \otimes f(q,x) & \mapsto & \sum_{f(q,x)=x} 1. \end{array}$$

The previous example shows us that the trace for this particular map recollects the number of fixed points of $f(q,-)$ for all $q \in Q$. In general, these suitable modifications allow us to get more information about fixed points.

Observation. If we consider $P = Q = \mathbb{1}$, we recover the trace in Definition 20, and we call it the **canonical trace**. In the future we will consider the twisted trace of any map $f : Q \otimes M \to P \otimes M$, with M dualizable, only as the trace of f.

Observation. This light change in the definition of traces for more general morphisms where its input and output are tensor products of arbitrary and different objects preserves Lemma 2.12 and Proposition 2.13 with suitable changes.

Thanks to this general notion of trace, we can consider a special case where diagonal maps are involved.

Definition 21. *Let $M \in Ob(\mathcal{C})$ be a dualizable object joint with a diagonal morphism $\Delta : M \to M \otimes M$, and $f : M \to M$ an endomorphism in $\mathcal{C}$. The **trace of f respect to** Δ is the trace of $\Delta \circ f$. Additionally, the trace of the identity map id_M respect to the diagonal is called the **transfer**.*

Remark 2. Let us consider M and $f : M \to M$ as in the above definition. The trace of $\Delta \circ f$ is the following composite

$$\mathbb{1} \xrightarrow{\eta} M \otimes M^* \xrightarrow{\Delta\circ f\otimes M^*} M \otimes M \otimes M^* \xrightarrow{M\otimes\gamma} M \otimes M^* \otimes M \xrightarrow{\epsilon\otimes M} M \qquad .$$

Thus, the trace of f with respect to the diagonal is a map taking values in M.

In general, the following corollary tells us that the trace of f respect to the diagonal recollects fixed points. It is a direct consequence of the properties of traces presented in the following section, we leave its proof to the reader.

Corollary 2.14. *If M is dualizable, $\Delta_M : M \to M \otimes M$ is a diagonal map and $f : M \to M$ is an endomorphism, then*

$$f \circ (\mathrm{tr}(\Delta_M \circ f)) = \mathrm{tr}(\Delta_M \circ f).$$

Example 12. *Consider the functor Z : **Set** $\to$ **Ab**. In Example 7 we saw that for any set X there exists an induced diagonal map $\Delta_{\mathbb{Z}[X]} : \mathbb{Z}[X] \to \mathbb{Z}[X] \otimes \mathbb{Z}[X]$ which sends each generator x to $x \otimes x$. $\mathbb{Z}[X]$ is dualizable for a finite set X. Then, the trace of $Z(f)$ with respect to the diagonal $\Delta_{\mathbb{Z}[X]}$ is given by the following composite*

$$\mathbb{Z} \xrightarrow{\eta} \mathbb{Z}[X] \otimes \mathbb{Z}[X]^* \xrightarrow{Z(f)\otimes id} \mathbb{Z}[X] \otimes \mathbb{Z}[X] \otimes \mathbb{Z}[X]^* \xrightarrow{\gamma} \mathbb{Z}[X] \otimes \mathbb{Z}[X]^* \otimes \mathbb{Z}[X] \xrightarrow{\epsilon} \mathbb{Z}[X]$$

$$1 \longmapsto \sum_{x\in X} x \otimes x \longmapsto \sum_{x\in X} f(x) \otimes f(x) \otimes x \longmapsto \sum_{x\in X} f(x) \otimes x \otimes f(x) \longmapsto \sum_{x=f(x)} x.$$

Example 13. *In the same way, for M a compact ENR, its suspension spectrum $\Sigma^\infty_+(M)$ is dualizable in the homotopy stable category, see Chapter 3. Here we also have a diagonal morphism $\Sigma^\infty_+(M) \to \Sigma^\infty_+(M) \wedge \Sigma^\infty_+(M)$ induced by the diagonal map $\Delta : M \to M \times M$ in Example 8, and hence the trace $tr(\Sigma^\infty_+(\Delta \circ f)) : \mathbb{S} \to \Sigma^\infty_+(M)$. This can again be regarded as the "formal sum" of all the fixed points of f, as in the previous example.*

In general, if we consider a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ with being $\mathcal{C}$ a cartesian category, then for each object A in $\mathcal{C}$ we have a diagonal morphism $\Delta_A : A \to A \times A$. And its image under F is the morphism $F(\Delta_A) : F(A) \to F(A\times A)$, by the compatibility of F with the monoidal structure. Thus we can define the following diagonal map:

$$\Delta_{F(A)} : F(A) \to F(A) \otimes F(A) \tag{2.23}$$

If we consider the image of $\mathcal{C}$ under F then each object in $(Fhrm)$ has a diagonal map and thus we can find the trace of any morphism with respect to the diagonal as in Definition 21.

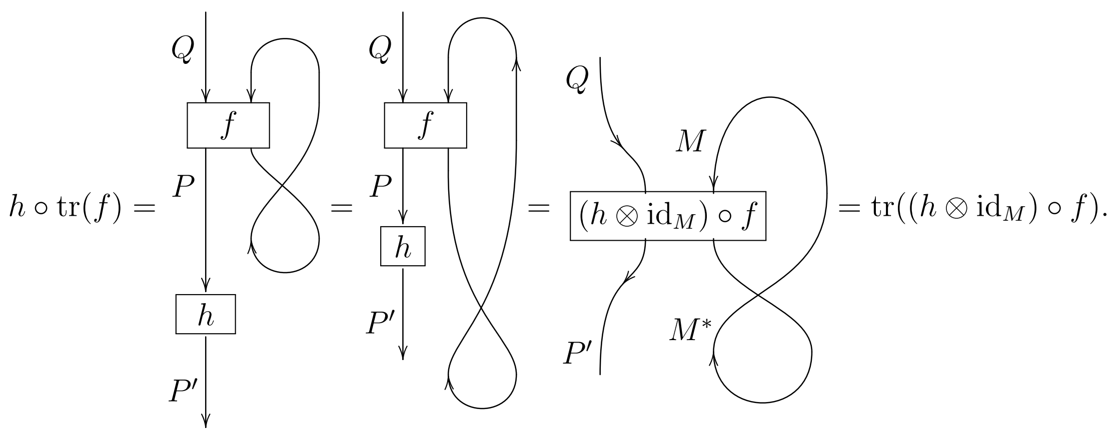
2.5 Properties of traces

Let us see some properties of traces. The idea is to give an axiomatic description of what a trace must satisfy. In other words, we would like to describe traces as a functional which satisfies some properties. In this section we will consider $\mathcal{C}$ as a symmetric monoidal category.

Proposition 2.15. ***(Binaturality)***
Let M be a dualizable object, $f : Q \otimes M \to P \otimes M$ a map, $g : Q' \to Q$ and $h : P \to P'$ maps in $\mathcal{C}$. Then

$$h \circ \operatorname{tr}(f) = \operatorname{tr}((h \otimes \mathrm{id}_M) \circ f), \tag{2.24}$$
$$\operatorname{tr}(f) \circ g = \operatorname{tr}(f \circ (g \otimes \mathrm{id}_M)). \tag{2.25}$$

Proof. Let us prove only one of the conclusions, the other ones being similar.



The first equality holds by stretching the cross and the second equality is gotten from simplifying the diagram. □

The previous proposition tells us that the trace is natural with respect to the objects P and Q. Regarding the string diagram notation we can think of the above equations as stretching the maps g and h. Some literature call these properties as **right tightening** and **left tightening** respectively, for example Hasegawa's thesis [19]. In this work we are reading the diagrams vertically so we will know them as top and bottom **tightening**, respectivelly. There is also a condition about the naturality of the trace with respect to the dualizable object. Thus, we have the following proposition.

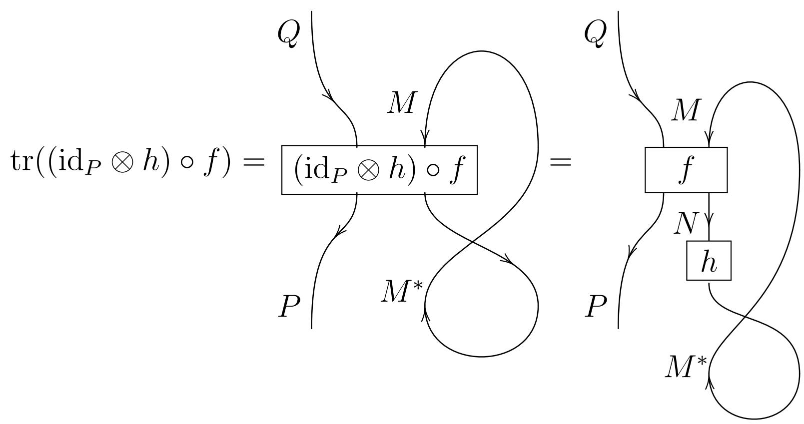
Remark 3. Note that with a suitable modification of the bottom tightening formula, we can prove the fixed point property, Corollary 2.14.

Proposition 2.16. ***(Naturality with respect to** M**)***
Let M and N be dualizable objects in $\mathcal{C}$ and $f : Q \otimes M \to P \otimes N$ and $h : N \to M$ two

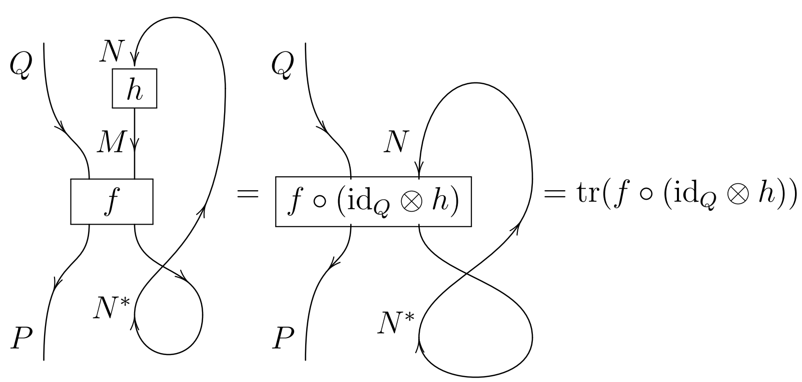
maps. Then,

$$\operatorname{tr}((\mathrm{id}_P \otimes h) \circ f) = \operatorname{tr}(f \circ (\mathrm{id}_Q \otimes h)). \tag{2.26}$$

Proof. Let us consider the following sequence of strings diagrams,



Now we can apply the cyclicity property, Lemma 2.12. Then, we get that the above string diagram can be deformed into:

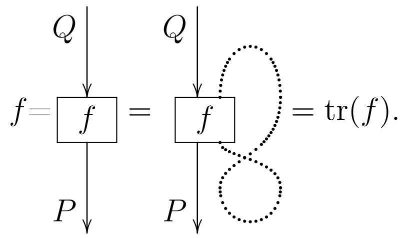


∎

The following list of properties of the trace are more technical and we will use them for the axiomatic characterization of abstract trace.

Proposition 2.17. ***(Nullarity)*** *If $f : Q \otimes \mathbb{1} \to P \otimes \mathbb{1}$ is a morphism in $\mathcal{C}$. Then, $f = \operatorname{tr}(f)$.*

Proof. Note that $\mathbb{1}$ is a dualizable object with $\mathbb{1}^* \cong \mathbb{1}$. In that case we consider the following sequence of string diagrams.

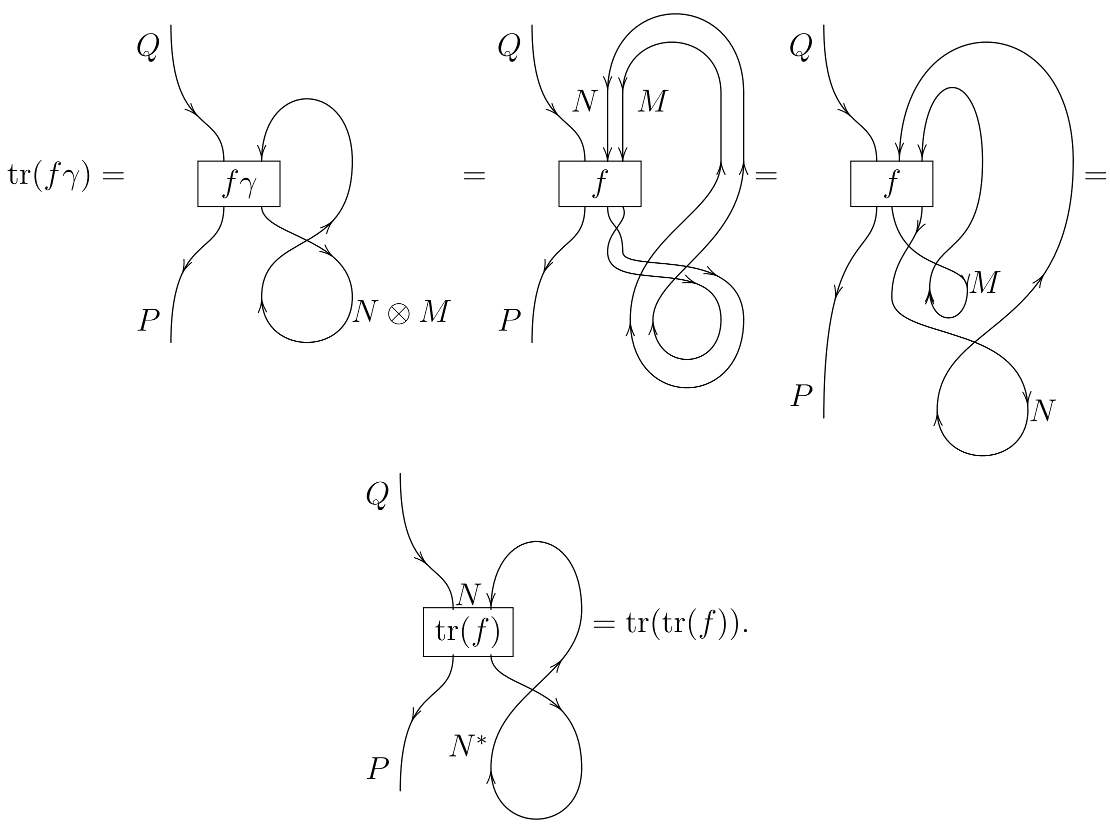


□

Proposition 2.18. *(**Binary**) Let $f : Q \otimes N \otimes M \to M \otimes N \otimes P$ be a morphism in $\mathcal{C}$ with M and N dualizable objects, then we have*

$$\operatorname{tr}(f\gamma_{M,N}) = \operatorname{tr}(\operatorname{tr}(f)). \tag{2.27}$$

Proof. Note that $M \otimes N$ is a dualizable object with dual $M^* \otimes N^*$ and we consider the following deformation sequence of string diagrams:



□

Note that, for $f : Q \otimes M \otimes N \to P \otimes M \otimes N$, the previous proposition can be simplified as $\operatorname{tr}(f) = \operatorname{tr}(\operatorname{tr}(f))$ where the left hand side is the trace of f respect the dualizable object $M \otimes N$ and the right hand side is the trace calculated twice first of f respect the dualizable object N and then respect M. In the linear algebra world the above property is just the trace of the Kronecker product of matrices which for A an $m \times n$ matrix and B a $p \times q$ matrix, then $A \otimes B$ is the $mp \times nq$ block matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}.$$

Thanks to that, we can generalize the above property and get the following proposition, in which for A and B square matrices, in the linear algebra world, the trace of $A \otimes B$ becomes the product of the traces of A and B.

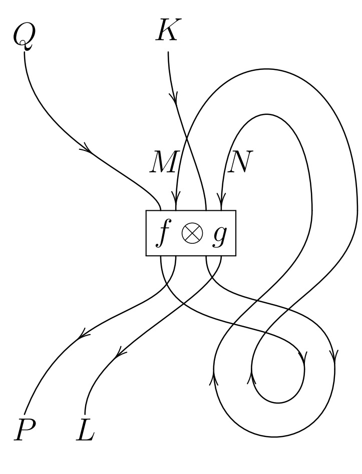
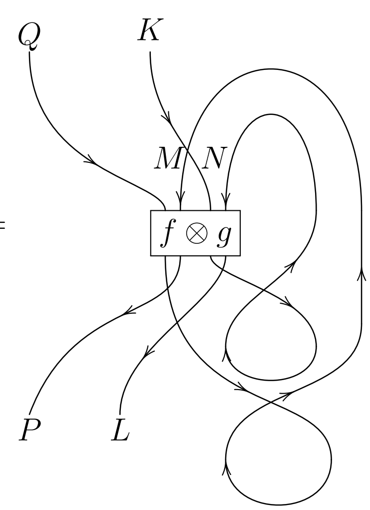
Proposition 2.19. *Let $f : Q \otimes M \to M \otimes P$ and $g : K \otimes N \to N \otimes L$, with M and N dualizable objects. Then, we have*

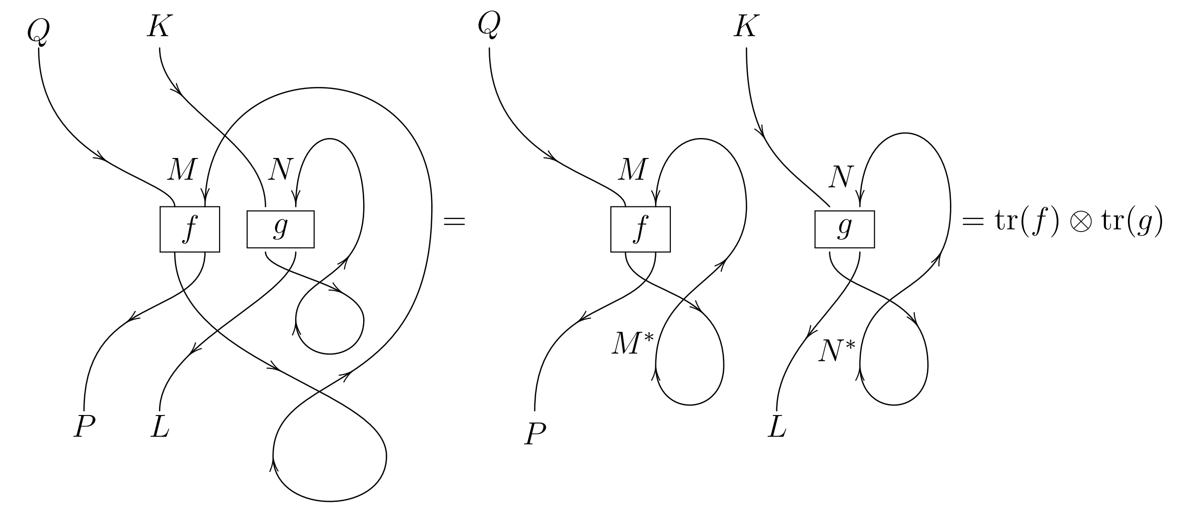
$$\operatorname{tr}((\gamma_{P,M} \otimes \gamma_{N,L}) \circ (f \otimes g) \circ (\gamma_{K,M})) = \operatorname{tr}(f) \otimes \operatorname{tr}(g). \tag{2.28}$$

If we consider $Q = P = K = L = \mathbb{1}$, the above formula becomes

$$\operatorname{tr}(f \otimes g) = \operatorname{tr}(f) \otimes tr(g).$$

Proof. Consider the following deformation sequence of string diagrams

$\operatorname{tr}(f \otimes g) =$  $=$  $=$



The second and third equality hold by the description presented in the Appendix A of maps that come from tensor products of simpler maps. The last diagram consists of two disconnected traces of f and g, respectively. Thus, the proposition holds. $\square$

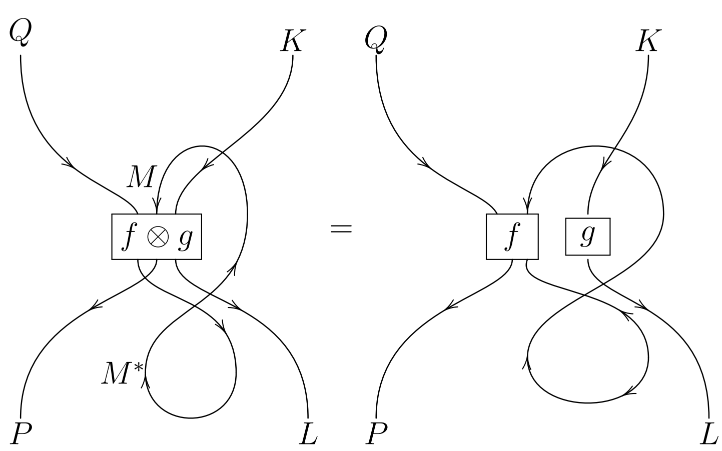
If we take $N = \mathbb{1}$ in the above proposition, we get the superposing condition.

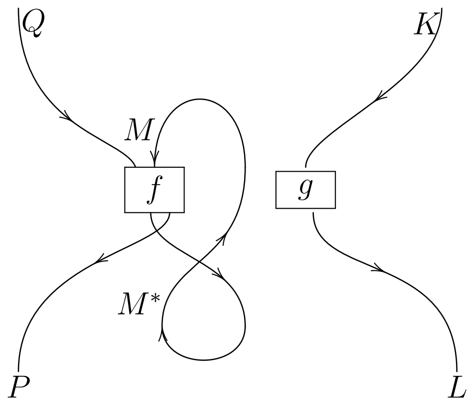
Corollary 2.20. *(**Superposing**) If M is dualizable, $f : Q \otimes M \to M \otimes P$ and $g : K \to L$ are maps, then*

$$\operatorname{tr}(\gamma(f \otimes g)) = \operatorname{tr}(f) \otimes g.$$

Proof. Consider the following deformation of string diagrams

$\operatorname{tr}(f \otimes g) =$



$=$ 

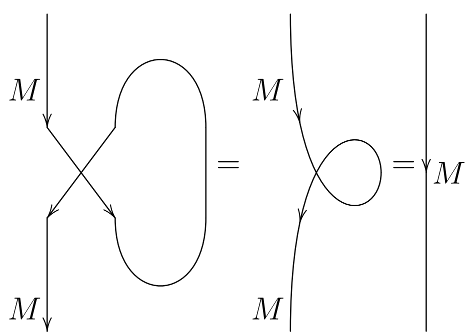
$= \mathrm{tr}(f) \otimes g.$

The last diagram is a juxtaposition of two string diagrams: $tr(f)$ and g. Hence the result follows. □

Proposition 2.21. *(**Yanking**) Let M be a dualizable object in $\mathcal{C}$, then*

$$\mathrm{tr}(\gamma_{M\otimes M}) = \mathrm{id}_M. \tag{2.29}$$

Proof. Consider the following deformation of string diagrams,

$\mathrm{tr}(\gamma_{M,M}) =$ 

The last step was yanking the loop. □

We have learned so far that traces requires a dualizable object and it is characterized by the canonical trace given by the composite in Definition 22. However, traces can be extended without dualizability. Joyal, Ross and Verity conceived a category with an additional structure that give us a notion of feedback.

Definition 22. *A **categorical trace** in a symmetric monoidal category $\mathcal{C}$ is a family of functions*

$$\mathrm{tr} : Hom_{\mathcal{C}}(Q \otimes M, M \otimes P) \to Hom_{\mathcal{C}}(Q, P) \tag{2.30}$$

satisfying the properties of top and bottom tightening (Prop. 2.15), naturality (Prop. 2.16), nullarity (Prop. 2.17), binary (Prop. 2.18), superposing (Prop. 2.20) and yanking (Prop. 2.21).

We will use this abstract notion of trace to study the relation with categorical fixed points in Section 2.7.

Definition 23. *A symmetric monoidal category $\mathcal{C}$ is* ***traced*** *if it is equipped with a trace given by Definition 22.*

Example 14. *A trivial example is a compact closed category, we know every object is duailazable by the composition given in Definition 20 we can define a trace which satisfies the mentioned properties.*

The next step is to study how traces behave with monoidal functors and a natural question emerges: What are the conditions such that a lax monoidal functor $H : \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories commutes with the trace? The following proposition answers that question.

Proposition 2.22. ***(Preservation of the trace)***
Let $\mathcal{C}$, $\mathcal{D}$ be symmetric monoidal categories and M dualizable in $\mathcal{C}$. If $H : \mathcal{C} \to \mathcal{D}$ is a monoidal functor, i.e. it preserves $\otimes$ and $\mathbb{1}$ up to isomorphism, then $H(M)$ is dualizable in $\mathcal{D}$, and for any map $f : M \to M$ we have $\mathrm{tr}(H(f)) = H(\mathrm{tr}(f))$.

Proof. First, let us denote $\mathbb{1}_{\mathcal{C}}$ and $\mathbb{1}_{\mathcal{D}}$ as the identity objects in $\mathcal{C}$ and $\mathcal{D}$, respectively. We know the trace of f is the following composite morphism

$$\mathbb{1}_{\mathcal{C}} \xrightarrow{\eta} M \otimes M^* \xrightarrow{f \otimes id_{M^*}} M \otimes M^* \xrightarrow{\gamma} M^* \otimes M \xrightarrow{\epsilon} \mathbb{1}_{\mathcal{C}}.$$

Then, we apply the monoidal functor and we get a composite morphism in $\mathcal{D}$

$$H(\mathbb{1}_{\mathcal{C}}) \xrightarrow{H(\eta)} H(M \otimes M^*) \xrightarrow{H(f \otimes id_{M^*})} H(M \otimes M^*) \xrightarrow{\gamma} H(M^* \otimes M) \xrightarrow{\epsilon} H(\mathbb{1}_{\mathcal{C}}).$$

On the other hand, the trace of $H(f) : H(M) \to H(M)$ is the following composition

$$\mathbb{1}_{\mathcal{D}} \xrightarrow{\eta} H(M) \otimes H(M^*) \xrightarrow{H(f) \otimes id_{H(M^*)}} H(M) \otimes H(M^*) \xrightarrow{\gamma} H(M^*) \otimes H(M) \xrightarrow{\epsilon} \mathbb{1}_{\mathcal{D}}.$$

Since H preserves the unit and product, we have the commutativity of each block in the following diagram

$$\begin{tikzcd}
H(\mathbb{1}_{\mathcal{C}}) \arrow[r, "\eta"] \arrow[d, "\cong"] & H(M \otimes M^*) \arrow[r, "H(f\otimes id)"] \arrow[d, "\cong"] & H(M \otimes M^*) \arrow[r, "\gamma"] \arrow[d, "\cong"] & H(M^* \otimes M) \arrow[r, "\epsilon"] \arrow[d, "\cong"] & H(\mathbb{1}_{\mathcal{C}}) \arrow[d, "\cong"] \\
\mathbb{1}_{\mathcal{D}} \arrow[r, "\eta"'] & H(M) \otimes H(M^*) \arrow[r, "H(f)\otimes id"'] & H(M) \otimes H(M^*) \arrow[r, "\gamma"'] & H(M^*) \otimes H(M) \arrow[r, "\epsilon"'] & \mathbb{1}_{\mathcal{D}}.
\end{tikzcd}$$

Then, the conclusion holds. $\square$

The key example and motivation of this thesis is the functor $H : \mathbf{Mfd} \to \mathbf{GrVect}_k$ from the category of smooth manifolds to the category of graded vector spaces, which is defined as $H(M) = H_*(M, \mathbb{Q})$, the rational homology. By the Kunneth theorem, the homology satisfies the above proposition. Then, the Lefschetz number can be calculated using the following formula

$$L(f) = \mathrm{tr}(H(f) = H(\mathrm{tr}(f)).$$

2.6 Fixed point operator

As we mentioned in the introduction of this chapter and our examples suggest, there is a strong relation between traces and fixed points. In this section we are going to describe the notion of fixed point in an abstract sense.

The bigest issue that we have in the abstract context, is that we do not have elements to describe the functions, so the equation $f(x) = x$ in an arbitrary category does not make much sense. Fixed points can be written abstractly as follows: For a set X, we say that $x \in X$ is a fixed point of f if there exist a map $i : * \to X$ (with source an indistinguished point) defined by $i(*) = x$. Then the maps must satisfy the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & X \\ * \arrow[u, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

With this idea we get the next definition.

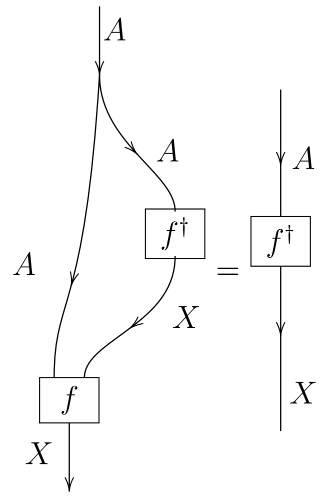
Definition 24. *Let $f : A \times X \to X$ be a map in a cartesian category $\mathcal{C}$. We say that a* ***parametrized fixed point of*** *f, and call it A-fixed point, is a morphism $f^\dagger : A \to X$ that obeys the relation*

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[rrr, bend right, "f^\dagger"'] & A \times A \arrow[r, "id_A \times f^\dagger"] & A \times X \arrow[r, "f"] & X \end{tikzcd}$$

i.e. $f \circ (id_A \times f^\dagger) \circ \Delta_A = f^\dagger$.

Remark 4. Let $\mathcal{C}$ be a category, and $X \in \mathrm{Ob}(\mathcal{C})$, we say that a fixed point of $f : X \to X$ is a map $f^\dagger : \mathbb{1} \to X$ which satisfies $f \circ f^\dagger = f^\dagger$.

In string diagrams parametrized fixed points can be represented as:



From Definition 24 we can try to produce fixed points in a coherent way. For each automorphism $f : X \to X$ we get a map $f^\dagger : \mathbb{1} \to X$. It will be called the fixed point operator. In general, for parametrized maps, we get the following definition.

Definition 25. *Let us fix a Cartesian category $\mathcal{C}$. A* ***parametrized fixed point operator*** *is a family of maps*

$$(.)^\dagger : \mathrm{Hom}_{\mathcal{C}}(A \times X, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$
$$f \mapsto f^\dagger,$$

satisfying:

1. ***Fixed point property****: For any map $f : A \times X \to X$, we have*
$$f \circ (\mathrm{id}_A \times f^\dagger) \circ \Delta_A = f^\dagger. \tag{2.31}$$
2. ***Natural in A****: For any $g : A \to B$ and $f : B \times X \to X$, we have*
$$f^\dagger \circ g = f \circ (g \times \mathrm{id}_X)^\dagger. \tag{2.32}$$
3. ***Natural in X****: For $f : A \times X \to Y$ and $g : Y \to X$, then*
$$(g \circ f)^\dagger = g \circ (f \circ (\mathrm{id}_A \times g))^\dagger. \tag{2.33}$$
4. ***Diagonal property****: For any $f : A \times X \times X \to X$, then*
$$(f \circ (\mathrm{id}_A \times \Delta_X))^\dagger = (f^\dagger)^\dagger. \tag{2.34}$$

There are several equivalent formulations of the fixed point operator. The formulation presented here was introduced by Hyland in [?]. This axiomatization is the same as that of "Conway cartesian categories" in [9]. On the other hand, Hasegawa also introduced fixed point operator in [19], where he considers the same axioms, but instead of the diagonal property he takes the Bekic's property, Lemma A.4, which allows simultaneous fixed points to be reduced to sequential ones.

Those operators appear first in the study of iteration theory in computer science. The axiomatization was used by Hyland and Haswegawa in certain categories where the tensor product is also cartesian in order to stablish a relation with the notion of trace due to Joyal, street and Verity in [1]. A good reference of fixed point operator in the context of computer science is [37]. However, that topic is out of scope of this work.

2.7 Fixed point operator and general trace

We finish this chapter showing the relation between traces and fixed point operator in cartesian monoidal categories.

At the introduction of this chapter we showed that the idea behind fixed points is to mimic the feedback that is represented in string diagram notation as a loop. However, traces can be represented as a loop with a cross, thanks to the symmetry condition. We can deform the trace diagram as a loop, so the notion of feedback is similar to the trace in some way.

Hyland and Hasegawa have independently observed the following.

Theorem 2.23. *A cartesian monoidal category $\mathcal{C}$ is traced if and only if it has a fixed point operator.*

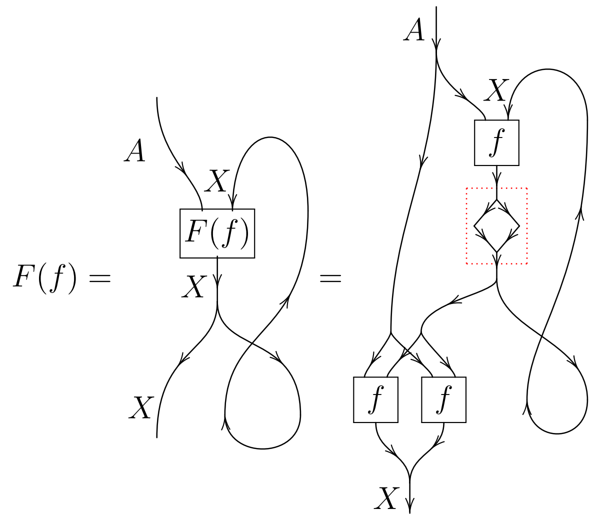
Proof. First, let us consider a traced cartesian category $\mathcal{C}$, that is, a cartesian category with a trace operator which satisfies the conclusions of Definition 22. Let us denote the trace of $\mathcal{C}$ as tr, and we consider the operator $F : \mathrm{Hom}_{\mathcal{C}}(A \times X, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$ defined by

$$F(f) = \mathrm{tr}(\Delta_X \circ f) : A \to X, \tag{2.35}$$

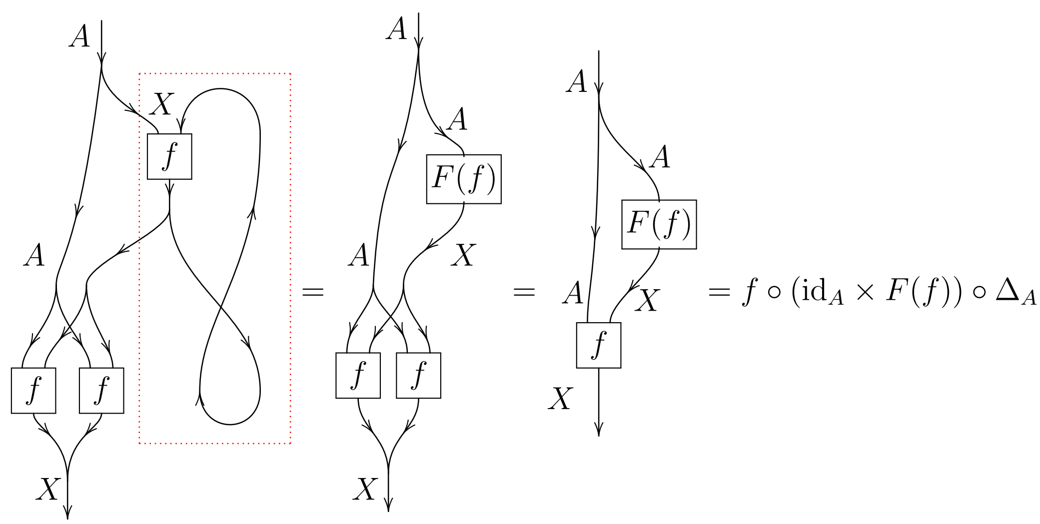
where $f : A \times X \to X$ is a map in $\mathcal{C}$. We need to check that the above operator is a fixed point operator, that is, F must satisfy all the conditions of Definition 25.

Note that through this proof we will require identities or lemmas which make our life easier. Those helpful lemmas will be written at the end of Appendix A with its respective proof. However, the simplicity of string diagrams to represent equations into manipulable topological strings is not very convenient in some parts of the proof, so we have decided to use them only when the profit is clear, otherwise we will use equations.

- **Condition 1:** Let $f : A \times X \to X$ be any map in $\mathcal{C}$

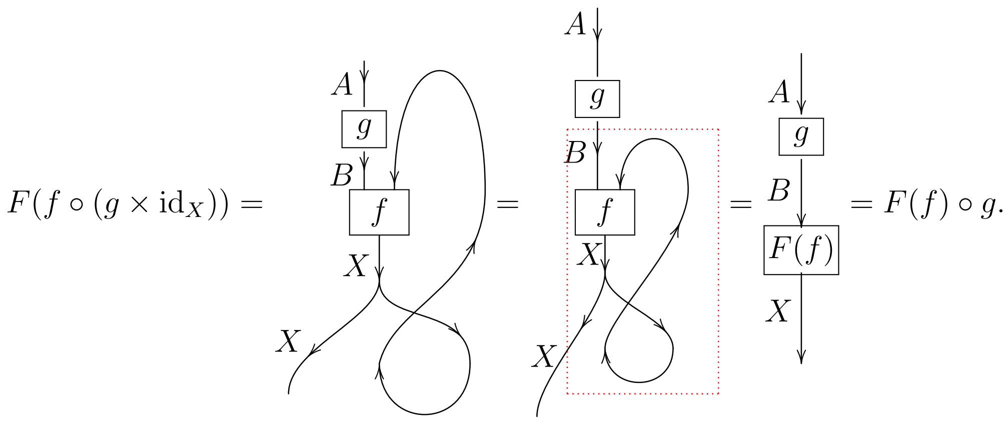


The diagram inside the red rectangle can be transformed into a line; thus, the above diagram can be deformed into



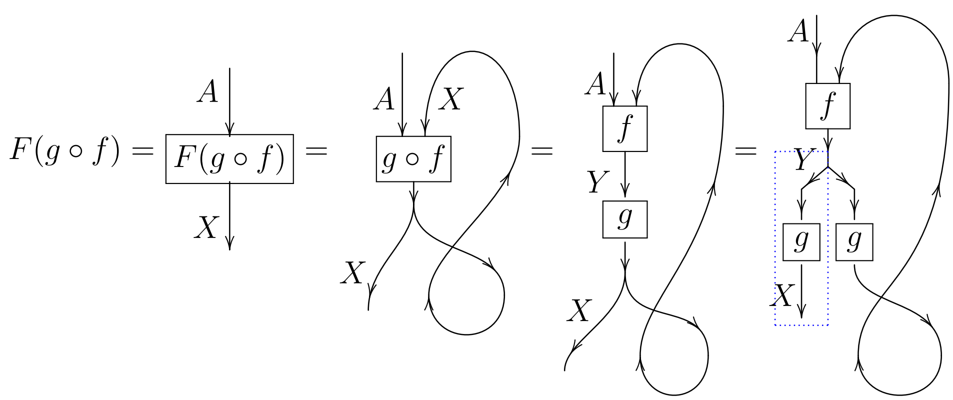
Inside the red rectangle above we have $F(f)$. The third equality holds by Lemma A.2.

- **Condition 2:** Let $f : B \times X \to X$ and $g : A \to B$ be maps in $\mathcal{C}$, then

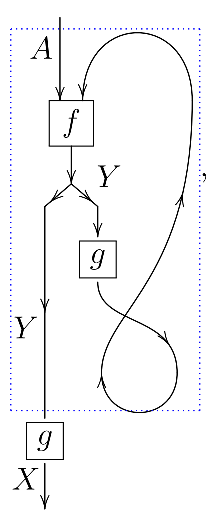


The second equality holds by stretching the diagram, and the third equality holds because the diagram inside the red rectangle is $F(f)$.

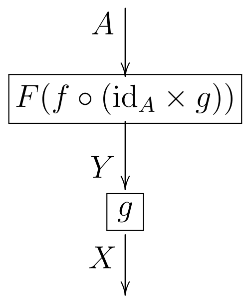
- **Condition 3:** Let $f : A \times X \to Y$ and $g : Y \to X$ be maps in $\mathcal{C}$, then



We can stretch the map g inside the blue rectangle and get

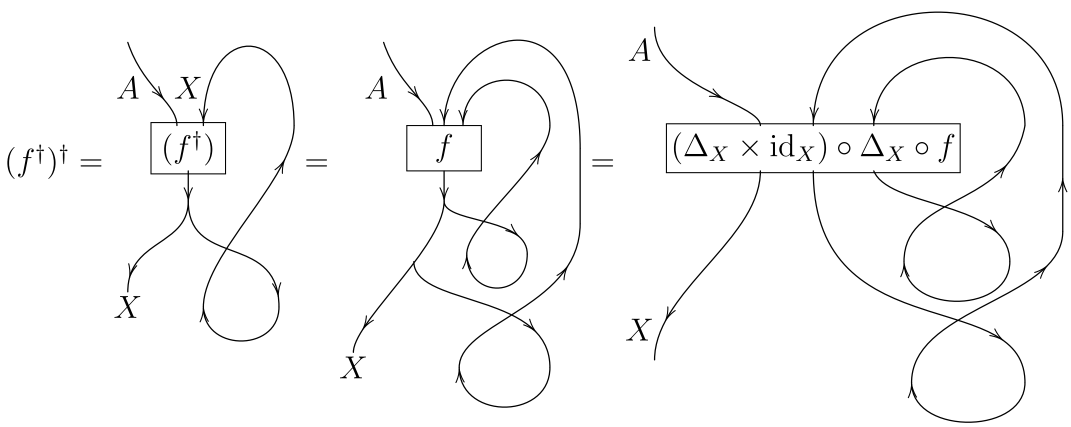


Next, we can simplify the diagram inside the blue rectangle as $F(f \circ (\mathrm{id}_A \times g))$. Thus, we get

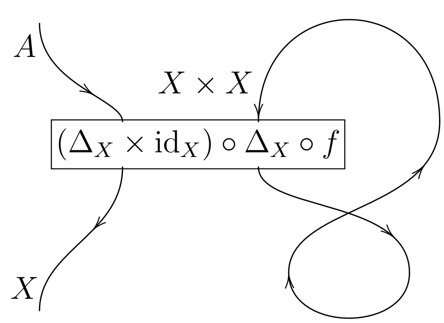


$$= g \circ F(f \circ (\mathrm{id}_A \times g))).$$

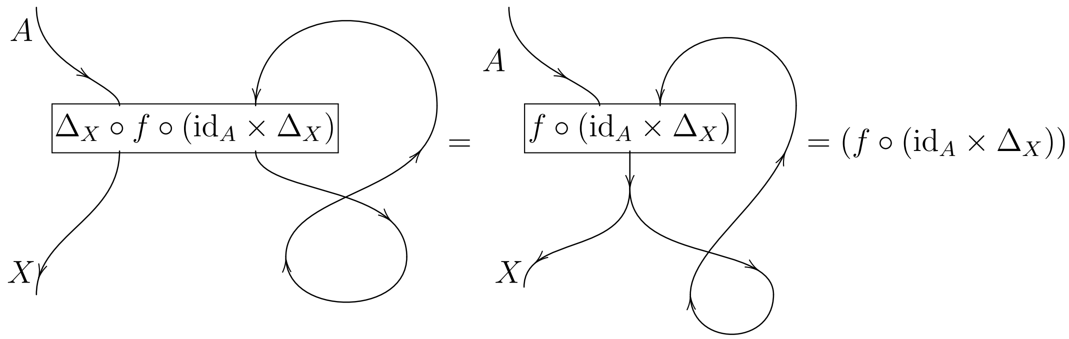
- **Condition 4:** Let $f : A \times X \times X \to X$ be a map in $\mathcal{C}$, then



We can apply the vanishing axiom which states that these two traces over X can be joined as a trace over $X \times X$ and then we get



Applying the ciclicity axiom yields



Conversely, suppose $(-)^\dagger : \mathrm{Hom}_{\mathcal{C}}(A \times X, B \times X) \to \mathrm{Hom}_{\mathcal{C}}(A, B)$ is any fixed point operator on $\mathcal{C}$ which satisfies conditions $1-4$ of Definition 25. We can define a trace operator in the following way: Let $g : A \times X \to B \times X$ be a morphism in $\mathcal{C}$, then

$$\mathrm{tr}^X(g) = \pi_{B,X} \circ (g \circ (\mathrm{id}_A \times \pi'_{B,X}))^\dagger, \tag{2.36}$$

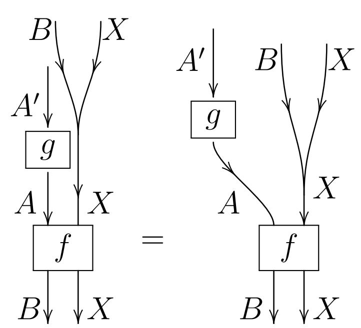
where $\pi_{B,X}$ and $\pi'_{B,X}$ are the projections on the first and second components, respectively. We need to verify that Formula 2.36 defines a trace operator, that is, it satisfies the conclusions of Definition 22.

• **Binaturality:** We need to prove that the trace given by Formula 2.36 satisfies both conditions in Lemma 2.15.

1. **Top tightening**: Let us consider $f : A \times X \to B \times X$ and $g : A' \to A$, then

$$\mathrm{tr}^X(f \circ (g \times \mathrm{id}_X)) = \pi_{B,X} \circ (f \circ (g \times \mathrm{id}_X) \circ (\mathrm{id}_{A'} \times \pi'_{B,X}))^\dagger.$$

Note that we can deform the map inside the operator $(-)^\dagger$



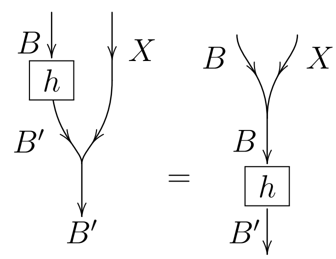
Thus, we get

$$\begin{aligned}
\mathrm{tr}^X(f \circ (g \times \mathrm{id}_X)) &= \pi_{B,X} \circ (f \circ (\mathrm{id}_A \times \pi'_{B,X}) \circ (g \times \mathrm{id}_{B \times X}))^\dagger \\
&= \pi_{B,X} \circ (f \circ (\mathrm{id}_A \times \pi'_{B,X})))^\dagger \circ g, \qquad \text{by property 2} \\
&= \mathrm{tr}^X(f) \circ g.
\end{aligned}$$

2. **Bottom tightening**: Let us consider $f : A \times X \to B \times X$ and $h : B \to B'$, then

$$\begin{aligned}\mathrm{tr}^X((h \times \mathrm{id}_X) \circ f) &= \pi_{B',X} \circ ((h \times \mathrm{id}_X) \circ f \circ (\mathrm{id}_A \times \pi'_{B',X}))^\dagger \\ &= \pi_{B',X} \circ (h \times \mathrm{id}_X) \circ (f \circ (\mathrm{id}_A \times \pi'_{B',X}) \circ (\mathrm{id}_A \times (h \times \mathrm{id}_X))^\dagger\end{aligned}$$

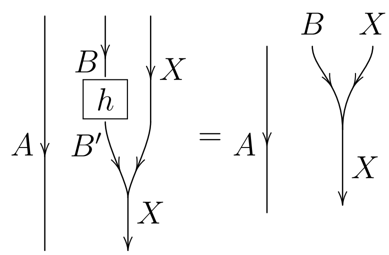
where the second equality holds by Property 3. Note that the map $\pi_{B',X} \circ (h \times \mathrm{id}_X)$ can be transformed into:



With this in mind, plus the Property (3), the above formula equals

$$h \circ \pi_{B,X} \circ f \circ ((\mathrm{id}_A \times \pi'_{B',X}) \circ (\mathrm{id}_A \times h \times \mathrm{id}_X) \circ (\mathrm{id}_A \times f))^\dagger.$$

Similarly, the map $(\mathrm{id}_A \times \pi'_{B',X}) \circ (\mathrm{id}_A \times h \times \mathrm{id}_X)$ can be transformed into:

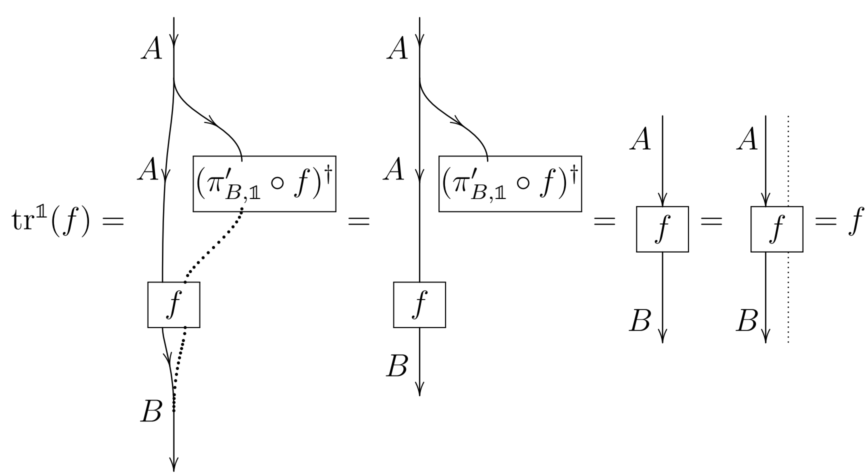


Therefore, we get

$$h \circ \pi_{B,X} \circ f \circ ((\mathrm{id}_A \times \pi'_{B,X}) \circ (\mathrm{id}_A \times f))^\dagger = h \circ \pi_{B,X} \circ (f \circ (\mathrm{id}_A \times \pi'_{B,X}))^\dagger,$$

where the last equality is given by applying Property (3). Definition 2.36 tells us that the right hand side is just $h \circ tr^X(f)$, and then the result follows.

- **Nullarity**: Let $f : A \times \mathbb{1} \to B \times \mathbb{1}$, then



• **Binary**: Consider $f : A \times X \times Y \to B \times X \times Y$, then

$$\begin{aligned}\mathrm{tr}^{X\times Y}(f) &= \pi_{B,X\times Y} \circ f \circ (\mathrm{id}_A \times (\pi'_{B,X\times Y} \circ f)^\dagger) \circ \Delta_A \\ &= \pi_{B,X\times Y} \circ f \circ (\mathrm{id}_A \times (((\pi'_{B,X} \circ \pi_{B\times X,Y} \circ f) \times (\pi'_{B\times X,Y} \circ f)) \circ \Delta_{A\times X\times Y})^\dagger) \circ \Delta_A.\end{aligned}$$

In the last equality we apply Lemma A.5 which allows us to discompose $\pi'_{B,X\times Y} \circ f$ in a particular way. Now, we apply Bekic's lemma (Lemma A.4) and the above equality becomes

$$\begin{aligned}&= \pi_{B,X\times Y} \circ f \circ \big(\mathrm{id}_A \times \big[(\pi'_{A\times X} \times (\pi'_{B\times X,Y} \circ f)^\dagger) \circ \Delta_{A\times X} \circ (\mathrm{id}_A \times (h)^\dagger) \circ \Delta_A\big]\big) \circ \Delta_A \\ &= \pi_{B,X\times Y} \circ f \circ (\mathrm{id}_{A\times X} \times (\pi'_{B\times X,Y} \circ f)^\dagger) \circ \Delta_{A\times X} \circ (\mathrm{id}_A \times (h)^\dagger) \circ \Delta_A.\end{aligned}$$

where $h = \pi'_{B,X} \circ \pi_{B\times X,Y} \circ f \circ (id_{A\times X} \times (\pi'_{B\times X,Y} \circ f)^\dagger) \circ \Delta_{A\times X}$, which by Formula 2.36, is the same as $h = \pi'_{B,X} \circ tr^Y(f)$. Thus, we get so far:

$$\begin{aligned}\mathrm{tr}^{X\times Y}(f) &= \pi_{B,X\times Y} \circ f \circ (\mathrm{id}_{A\times X} \times (\pi'_{B\times X,Y} \circ f)^\dagger) \circ \Delta_{A\times X} \\ &\quad \circ (\mathrm{id}_A \times (\pi'_{B,X} \circ \mathrm{tr}^Y(f))^\dagger) \circ \Delta_A \\ &= \pi_{B,X} \circ \pi_{B\times X,Y} \circ f \circ (\mathrm{id}_{A\times X} \times (\pi'_{B\times X,Y} \circ f)^\dagger) \circ \Delta_{A\times X} \\ &\quad \circ (\mathrm{id}_A \times (\pi'_{B,X} \circ \mathrm{tr}^Y(f))^\dagger) \circ \Delta_A\end{aligned}$$

where the last equality holds by the following identity

$$\pi_{B,X\times Y} = \pi_{B,X} \circ \pi_{B\times X,Y}.$$

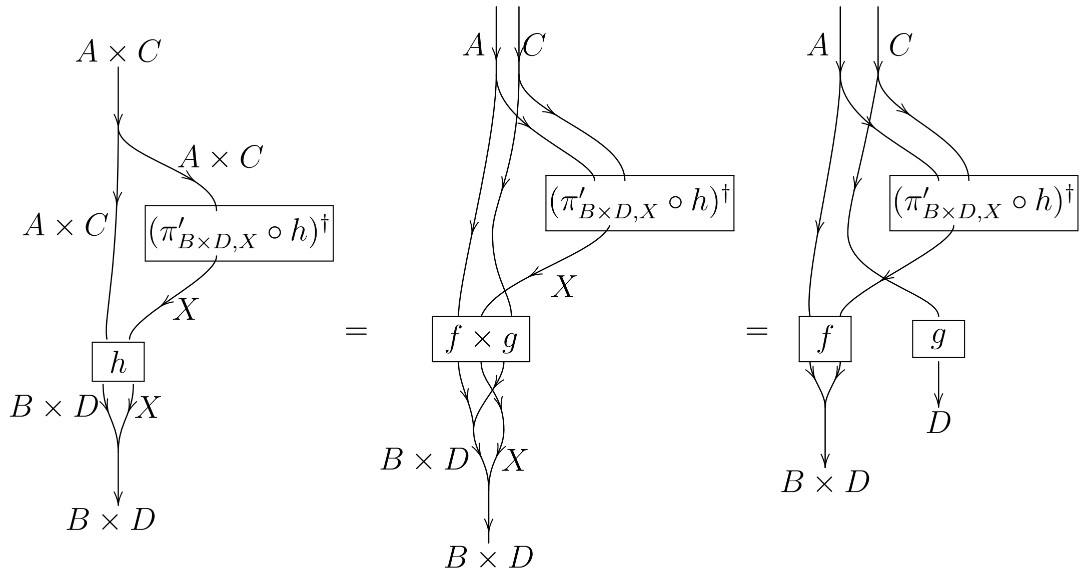
Again, applying Formula 2.36 twice, we get

$$\begin{aligned}\mathrm{tr}^{X\times Y}(f) &= \pi_{B,X} \circ \mathrm{tr}^Y(f) \circ (\mathrm{id}_A \times (\pi'_{B,X} \circ \mathrm{tr}^Y(f))^\dagger) \circ \Delta_A \\ &= \mathrm{tr}^X(\mathrm{tr}^Y(f)).\end{aligned}$$

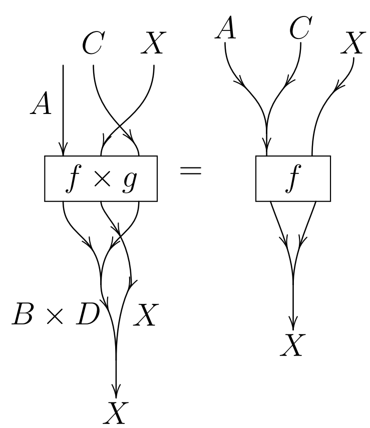
• **Superposing:** Consider $f : A \times X \to B \times X$ and $g : C \to D$ morphisms in $\mathcal{C}$. Let us call $h = (\mathrm{id}_B \times \gamma_{X,D}) \circ (f \times g) \circ (\mathrm{id}_A \times \gamma_{X,C}) : A \times C \times X \to B \times D \times X$ to make the diagrams easy to understand. By Lemma A.3 in the appendix, we can write the trace as the following composite

$$\mathrm{tr}^X(h) = \pi_{B\times D,X} \circ h \circ (\mathrm{id}_{A\times C} \times (\pi'_{B\times D,X} \circ h)^\dagger) \circ \Delta_{A\times C}.$$

It can be represented as the following string diagrams



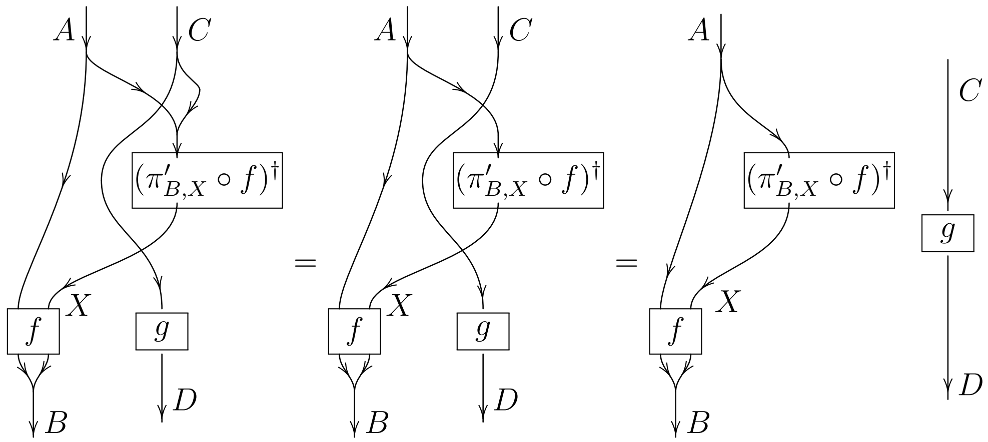
Note that $\pi'_{B\times D,X} \circ h = \pi'_{B,X} \circ f \circ (\pi_{A,C} \times \mathrm{id}_X)$, this is clear by the following diagram:



Continuing with the trace, we can apply Property (2) and together with the above identity we get:

$$\mathrm{tr}^X(h) = ((\pi_{B,X} \circ f) \times g) \circ (\mathrm{id}_A \times \gamma_{X,C}) \circ (\mathrm{id}_{A\times C} \times ((\pi'_{B,X} \circ f)^\dagger \circ \pi_{A,C})) \circ \Delta_{A\times C}.$$

Again, we can represent it into string diagrams



which is $\mathrm{tr}^X(f) \times g$, then the result holds.

•**Naturality respect X**: Let $f : A \times X \to B \times Y$ and $g : Y \to X$ be morphisms in $\mathcal{C}$. Then,

$$\begin{aligned}\mathrm{tr}^X((id_B \times g) \circ f) &= \pi_{B,X} \circ ((\mathrm{id}_B \times g) \circ f \circ (\mathrm{id}_A \times \pi'_{B,X}))^\dagger \\ &= \pi_{B,X} \circ (\mathrm{id}_B \times g) \circ (f \circ (\mathrm{id}_A \times \pi'_{B,X}) \circ (\mathrm{id}_A \times \mathrm{id}_B \times g))^\dagger \\ &= \pi_{B,Y} \circ (f \circ (\mathrm{id}_A \times \pi'_{B,X}) \circ (\mathrm{id}_A \times \mathrm{id}_B \times g))^\dagger.\end{aligned}$$

The last equality is given by Lemma A.6 in the appendix. In addition, Lemma A.7 allows us to replace the equation inside $(-)^\dagger$ as follows

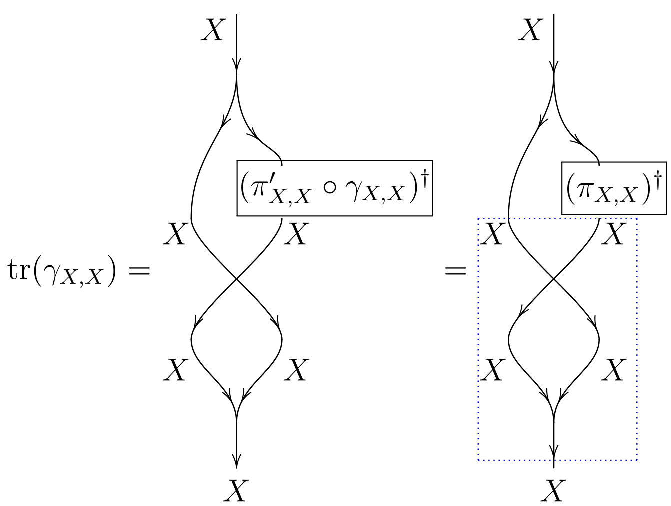
$$f \circ (\mathrm{id}_A \times \pi'_{B,X}) \circ (\mathrm{id}_A \times \mathrm{id}_B \times g) = f \circ (\mathrm{id}_A \times g) \times (\mathrm{id}_A \times \pi'_{B,Y}). \tag{2.37}$$

Thus, we get

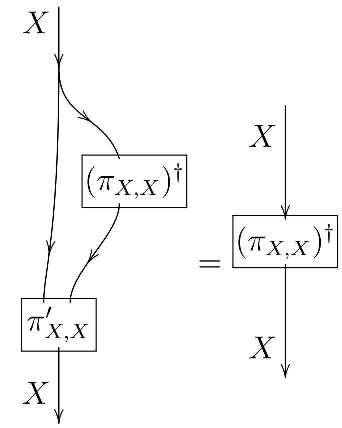
$$\mathrm{tr}^X((\mathrm{id}_B \times g) \circ f) = \pi_{B,Y} \circ (f \circ (\mathrm{id}_A \times g) \times (\mathrm{id}_A \times \pi'_{B,Y}))^\dagger = \mathrm{tr}^Y(f \circ (\mathrm{id}_A \times g)).$$

Then, the result holds.

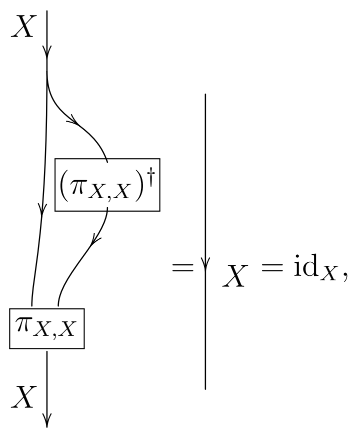
• **Yanking:** Let X be an object of $\mathcal{C}$ and $\gamma_{X,X}$ the symmetry isomorphism, then



Note that $\pi'_{X,X} \circ \gamma_{X,X}$ is just $\pi_{X,X}$ and $\pi_{X,X} \circ \gamma_{X,X} = \pi'_{X,X}$, see Appendix A, then the second equality holds. The diagram inside the blue box is $\pi_{X,X} \circ \gamma_{X,X}$. Hence, the last diagram can be transformed into



Now, we can apply the fixed point property and then we get



which is projected to the first component, so we get the identity map.

$\square$

Chapter 3

Atiyah duality and Spectra

In this chapter we are going to study the theory necessary to understand $\mathrm{tr}(f)$ for a smooth map $f : M \to M$. The first step is to study dualizability for manifolds. We will assume the manifold M has an Euclidean neighborhood retract (or a ENR space), that is, there is an open set V in $\mathbb{R}^n$ such that M is a retract of B. In the same way, we will assume M is a compact manifold in order to avoid technical problems.

In 1915, Alexander conceived a relation between the homology of a manifold and the cohomology of its complement. It is what is now called Alexander duality. More formally, we get the following definition.

Definition 26. *Let X be a compact subspace of the sphere S^n of dimension n. Let $Y \cong S^n - X$ be an equivalence of the complement of X in S^n. Then if $\tilde{H}$ stands for reduced homology or reduced cohomology, there is an isomorphism*

$$\tilde{H}_q(Y) \cong \tilde{H}_q(S^n - X) \cong \tilde{H}^{n-q-1}(X) \tag{3.1}$$

for all $q \geq 0$. We say that X and Y are ***Alexander duals*** *of each other.*

Let K be a knot in S^3, i.e. the image of an embedding of $S^1 \hookrightarrow S^3$. If, K and $S^3 - K$ are **Alexander duals** then there is an isomorphism

$$\tilde{H}^{n-i}(K, \mathbb{Z}) \cong \tilde{H}_i(S^{n+1} - K, \mathbb{Z}).$$

It satisfies $H_1(S^3 - K) \cong H^1(S^1) \cong \mathbb{Z}$, and it does not depend on the choice of the knot. On the other hand, for different knots K and K' the homotopy type of $S^3 - K$ and $S^3 - K'$ are not necessarily the same; they depend on the embedding.

However, Spainner and Whitehead proved that after suspending many times they become homotopically equivalent. Thus, in general we get the following theorem.

Theorem 3.1. *Let X be a compact simplicial complex. Let $f, g : X \to S^n$ be two simplicial embeddings. For some $N >> 0$ the N-fold suspensions $\Sigma^N(S^n - f(X))$ and $\Sigma^N(S^n - g(X))$ are homotopy equivalent.*

Proof. The proof follows directly from two facts:

1. For k big enough, any two embeddings of X into S^{n+k} are isotopic.
2. Giving an embedding $X \hookrightarrow S^n$ for $n >> 0$ that $\Sigma(S^n - X)$ is homotopy equivalent to $S^{n+1} - X$, where X is embedded into S^{n+1} via the equatorial inclusion $S^n \hookrightarrow S^{n+1}$.

□

Motivated by this issue, Spanier defined the S-category in the 50's as a natural frame work to the Alexander duality.

Definition 27. *The **S-category** is the category whose objects are the objects in $\textbf{Top}_*$ and its morphisms are colimits over homotopy classes of continuous functions between their arbitrary high suspensions.*

$$[X, Y]_S := \operatorname{colim}_{N\to\infty}[\Sigma^N X, \Sigma^N Y].$$

Remark 5. The **S-category** has some issues: it is neither complete nor co-complete. We like gluing stuff together, so this is unfortunate. As a consequence, Brown representability theorem does not hold in the **S-category**.

At this level, we can start to talk about *desuspension*. For each finite CW-complex X and each $n \geq 1$, we define the n-th formal desuspension of X, denoted by $\Sigma^{-n}X$, by

$$\mathrm{Hom}_S[\Sigma^{-n}X, \Sigma^{-n}Y] = [\Sigma^{N-n}X, \Sigma^{N-n}Y] \tag{3.2}$$

for large enough $N \in \mathbb{N}$. Shortly after introducing the S-category, Spanier and Whitehead (1955) developed their duality theory.

Definition 28. *Let A, B two based spaces (or based CW-complex). We say A, B are n-dual if there is an embedding $\Sigma^k A \hookrightarrow S^{k+n+1}$ and homotopy equivalence $\Sigma^l B \simeq \Sigma^l(S^{k+n+1} - \Sigma^k A)$.*

In other words, we say A and B are n-dual, if $A \cong S^n - B$ in the **S-category**.

Milnor and Spanier (1960) clarified the relation between Spanier–Whitehead duality and Poincaré duality on a closed differentiable manifold. Thom (1952) had introduced what is now called the Thom space M^α of a vector bundle α over M. The Spainer-Milnor theorem asserts that for a closed manifold M embedded in a euclidean space $\mathbb{R}^k$ with normal bundle η over M then

$$M^\eta \text{ is k-dual to } M_+,$$

where M_+ is the disjoint union of M with a point. Later in 1961 Atiyah extended the Spanier-Milnor theorem to manifolds with boundary.

So, the problem is to find a category with good properties for stable phenomena. Highly motivated by the work of Atiyah, this attempt was made by Lima in his Ph.D. thesis and it is called *spectra*. Nevertheless, this category does not have good properties; for example there are models of spectra that represents the same reduced cohomology theory but are not homotopy equivalent. This issue was solved by Boardman in his Ph.D. thesis by considering the category that we know as stable homotopy category.

We will finish this chapter by studying how Atiyah's duality formulas are used to prove the Lefschetz fixed point theorem following the ideas and results presented in the previous Chapter.

3.1 Preliminaries

Let us start this section with some concepts and theorems necessary to understand the stable homotopy category denoted by **HoSpect**. Good references include [21].

Let us start this section with an important definition in topology that we presented in the above section. There is a process that to each based-topological space X you can construct a space of one higher dimension, it is called **the suspension** of X and denoted by ΣX, it is defined by the quotient space

$$\Sigma X = \frac{X \times I}{X \vee I} \tag{3.3}$$

where $X \vee I$ is the subespace $(X \times *) \cup (x_0 \times I) \subset X \times I$ with base points $x_0 \times *$. It can be proved that the suspension of X is homeomorphic to $X \wedge S^1$, that is again a based topological space. In a categorical language we say that there is a functor $\Sigma : \mathbf{Top}_* \to \mathbf{Top}_*$.

There is a right adjoint functor to Σ called the loop space functor $\Omega : \mathbf{Top}_* \to \mathbf{Top}_*$, which is defined by the topological base space $\Omega X = \mathrm{Maps}(S^1, X)$. For a map $f : \Sigma X \to Y$, we can get a map $g : X \to \mathrm{Map}(S^1, Y) = \Omega Y$ in the following way:

$$\begin{tikzcd} X \times I \arrow[d, "p"'] \arrow[dr, "F"] & \\ \Sigma X \arrow[r, "f"'] & Y \end{tikzcd}$$

There is a function $F : X \times I \to Y$ that factors through ΣX and satisfies $F(x_0, 0) = F(x_0, 1)$. Then, we define $g : X \to \mathrm{Map}(I, Y)$ by $g(x) = F(x, -) : I \to Y$, for $t = 0, 1$ we get the same value for g. So, $g(x)$ is a map which takes values in S^1.

Definition 29. *A continuous function $f : A \to B$ is called a* ***cofibration*** *if given a continuous function $g : B \to C$ and a homotopy $H_t : A \to C$ such that $H_0 = g \circ f$, then there is a homotopy $\tilde{H}_t : B \to C$ such that $\tilde{H}_t \circ f = H_t$.*

Definition 30. *Let $f : A \to B$ be a function, we define the mapping cone of f denoted as $Cone(f)$ as the topological space*

$$Cone(f) := A \times I \cup B / \sim, \tag{3.4}$$

where $\sim$ is the equivalence relation defined by $(a, 1) \sim f(a)$ and $(a, 0) \sim (a', 0)$, for $a, a' \in A$.

The following theorem helps us study homotopy groups. We know that finding the homotopy groups of the sphere is a hard problem. However, the groups seems to (eventually) stabilize.

Theorem 3.2. ***(Freudenthal suspension theorem)*** *Let X be an $(n-1)$-connected based space. Then the map $j : \pi_i(X, x_0) \to \pi_{i+1}(\Sigma X, x_0)$ is an isomorphism when $i < 2n - 1$ and is a surjection when $i = 2n - 1$.*

Proof. The result follows from the homotopy excision theorem. For a detailed description see [**?**]. □

In the previous theorem, if we replace X by a sphere, we get the stable homotopy groups of spheres. The groups $\pi_{n+k}(S^n)$ with $n > k + 1$ are called the stable homotopy groups of spheres, and are denoted by π_k^s: They are finite abelian groups for $k \neq 0$, and have been computed in numerous cases, although the general pattern is still elusive.

If X is n-connected, then ΣX is $(n+1)$-connected and so the Freudenthal theorem implies that, if X is a CW-complex of any connection and $i \in \mathbb{N}$, the morphisms in the sequence

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \cdots \to \pi_{i+k}(\Sigma^k X) \to \cdots$$

will become isomorphisms from a certain point. The i-th stable homotopy group $\pi_i^s(X)$ of X is

$$\pi_i^s(X) = \mathrm{colim}_k \pi_{i+k}(\Sigma^k X). \tag{3.5}$$

Here is where the word **stable** appears, in other words, we say that a phenomenom is stable if it occurs independent of the dimension or can occur in any dimension. That is the starting point of this theory: to study a context that generalizes the spaces and where stable phenomena would be easier.

3.2 Spectra

There are many approaches to spectra. In this work we have decided to start with the discussion of generalized homology and cohomology theories. There are many other interesting approaches to define spectra. For example, the desuspension, that is, inverting in some way the suspension functor Σ, thanks to the suspension condition in cohomology theories. That has as a result the notion of "negative" spheres. A good introduction is the famous book by Adams [4].

Definition 31. *(**Eilenberg-MacLane Spaces**)*
*Let G be any Abelian group, and $n \in \mathbb{N}$. An **Eilenberg-MacLane space** of type (G, n) is a space X of the homotopy type of a based CW-complex such that:*

$$\pi_k(X) = \begin{cases} G & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

We denote such a space by $K(G, n)$.

Proposition 3.3. *For any abelian group G and $n \geq 1$, we have the homotopy equivalence $\Omega(K(G, n)) \cong K(G, n-1)$.*

Proof. The result follows directly from the isomorphism $\pi_m(X) \cong \pi_{m-1}(\Omega X)$, which is a direct implication from the adjunction between Σ and Ω. $\square$

Now, let us recall the axioms for a generalized cohomology theory due to Eilenberg and Steenrod.

Definition 32. *A **reduced cohomology theory** is a functor $\tilde{\mathbb{E}}^* : (\boldsymbol{Top_*})^{op} \to \boldsymbol{Ab}^{\mathbb{Z}}$ from the opposite of pointed topological spaces to $\mathbb{Z}$-graded abelian groups ("cohomology groups"); for maps we have*

$$\tilde{\mathbb{E}} : (f : X \to Y) \mapsto (f^* : \tilde{\mathbb{E}}(Y) \to \tilde{\mathbb{E}}(X))$$

equipped with a natural isomorphism of degree $+1$, to be called the suspension isomorphism, of the form

$$\sigma_n : \tilde{\mathbb{E}}^{*+1}(\Sigma -) \to \tilde{\mathbb{E}}^*(-).$$

$\tilde{\mathbb{E}}^*$ satisfies:

1. ***Homotopy invariance**: If $f_1, f_2 : X \to Y$ are two morphisms of pointed topological spaces such that, if there is a (base point preserving) homotopy $f_1 \simeq f_2$ between them, then the induced homomorphisms of abelian groups are equal $f_1^* = f_2^*$.*

2. ***Exactness***: *For $i : A \hookrightarrow X$ an inclusion of based topological spaces, with $j : X \to Cone(i)$ the induced mapping cone, then this gives an exact sequence of graded Abelian groups*

$$\tilde{\mathbb{E}}^*(Cone(i)) \xrightarrow{j^*} \tilde{\mathbb{E}}^*(X) \xrightarrow{i^*} \tilde{\mathbb{E}}^*(A).$$

3. ***Additivity axiom***: *For $\{X_i\}_{i\in I}$ any set of based CW-complexes, then the canonical comparison morphism*

$$\tilde{\mathbb{E}}^*\left(\bigvee_{i\in I} X_i\right) \to \prod_{i\in I} \tilde{\mathbb{E}}^*(X_i)$$

is an isomorphism.

Similarly, we say that a ***reduced homology theory*** *is the same thing but contravariant, written as $\tilde{\mathbb{E}}_*$. We say that a cohomology theory is an ordinary cohomology theory if in addition it satisfies the "dimension axiom" of the Eilenberg–Steenrod axioms, that is, the homology of a point vanishes in dimension other than* 0.

Remark 6. Homology and cohomology theories also satisfies the Mayer-Vietoris axiom. It is not necessary to add it to the above list of axioms, since it follows directly from the same one.

Definition 33. *A homomorphism of reduced cohomology theories $\eta : \tilde{E}^* \to \tilde{F}^*$ is a natural transformation between the underlying functors which is compatible with the suspension isomorphisms in that the following square commutes*

$$\begin{tikzcd}
\tilde{E}^*(X) \arrow[r, "\eta_X"] \arrow[d, "\Sigma"] & \tilde{F}^*(X) \arrow[d, "\Sigma"] \\
\tilde{E}^{*+1}(\Sigma X) \arrow[r, "\eta_{\Sigma X}"'] & \tilde{F}^{*+1}(\Sigma X).
\end{tikzcd}$$

Thanks to the above definitions we say that the reduced cohomology theories form a category denoted by **CohomThy**, respectively there is a category of reduced homology theory.

The most important property of Eilenberg-MacLane spaces is that they represent the singular cohomology, i.e., there is an isomorphism

$$\tilde{H}^n(X, G) = [X, K(G, n)]. \tag{3.6}$$

Also, we can define the singular homology by the formula

$$\tilde{H}_n(X, G) = \mathrm{colim}_k \pi_{n+k}(X \wedge K(G, k)). \tag{3.7}$$

In [21],the author proves that the above formulas satisfy the axioms in Definition 32. By Proposition 3.3, we have an homotopy equivalence

$$\tilde{\sigma}_n : K(G,n) \to \Omega K(G,n+1). \tag{3.8}$$

This map is adjoint to the map

$$\sigma_n : \Sigma K(G,n) \to K(G,n+1). \tag{3.9}$$

In a more general sense, in order to construct a generalized homology or cohomology theory, one can consider a family of based spaces endowed with relations among the reduced suspensions of these spaces. It will turn out to be sufficient to consider only a sequence $E_0, E_1, E_2, \cdots$ of based spaces, together with maps $\Sigma E_n \to E_{n+1}$. Such a sequence is called a **spectrum**. Throughout the literature, one finds various definitions of spectrum that do not necessarly agree. Without going through the details, the reason for these different definitions is to find a convenient category to work with. Here is where the notion of stable homotopy category plays a key role because in all those definitions of spectra, if we invert the weak equivalences, or we localize it using the Quillen's model theory, then we get the stable homotopy category.

The next theorem is due to Edgar H. Brown, see [12]. It states that any generalized cohomology (equiv. homology) theory defined on the category **Top** of compactly generated weak Hausdorff spaces gives rise to a sequence of spaces $\{E_n\}_{n\in\mathbb{Z}}$ such that the functor $\tilde{\mathbb{E}}^n(-)$ is naturally isomorphic to the functor $[-, E_n]$. That is, E_n represents the functor $\tilde{\mathbb{E}}^n(-)$. In a more general and precise language:

Theorem 3.4. ***(Brown's representability theorem)***
Let F be a contravariant functor from the category of topological spaces weakly equivalent to CW complexes to the category of point sets and maps. Suppose F satisfies the Mayer–Vietoris axiom, and the wedge axiom in Definition 32. Then there exists a CW complex Y, unique up to homotopy, such that $F(-)$ is naturally isomorphic to $[-, Y]$. Furthermore, there is an element $u \in F(Y)$ such that this natural isomorphism is given by $[X, Y] \cong F(X)$ via $f \mapsto f^*u$.

Proof. See Theorem 4E.2 in [21]. □

Thus we get the following definition.

Definition 34. *A **spectrum** is a sequence $E = \{E_n\}_{n\in\mathbb{Z}}$ of based spaces together with maps $\sigma_n : \Sigma E_n \to E_{n+1}$ called the structure maps. The space E_n is called the n-th term of the spectrum E. If E and F are spectra, a map of spectra $f : E \to F$ is family of maps $\{f_n : E_n \to F_n\}_{n\in\mathbb{Z}}$ such that the following square commutes*

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma f"] \arrow[d, "\sigma_n"] & E_{n+1} \arrow[d, "\sigma_{n+1}"] \\ \Sigma F_n \arrow[r, "f_{n+1}"'] & F_{n+1} \end{tikzcd}$$

We will denote the category of spectra by **Spect**. Some examples of spectra include:

Example 15. ***The suspension spectrum***: *To each based space X one can associate a spectrum. We define $\Sigma^\infty X$ the suspension spectrum of X, where the n-th term is given by the n-th reduced suspension of X, namely $\Sigma^n X$, and its structure maps $\sigma_n : \Sigma(\Sigma^n X) \to \Sigma^{n+1} X$ are the obvious identity maps. For each based map $f : X \to Y$ we define $\Sigma^\infty f : \Sigma^\infty X \to \Sigma^\infty Y$ as the family of maps $\{\Sigma^n f : \Sigma^n X \to \Sigma^n Y : n \geq 0\}$. Obviously, we have the required commutativity of the following diagram for each $n \geq 0$*

$$\begin{array}{ccc} \Sigma(\Sigma^n X) & \xrightarrow{\Sigma(\Sigma^n f)} & \Sigma(\Sigma^n Y) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ \Sigma^{n+1} X & \xrightarrow[\Sigma^{n+1} f]{} & \Sigma^{n+1} Y \end{array}$$

*making $\Sigma^\infty f$ a map of spectra. Since compositions and identies are obviously preserved, we have defined a functor $\Sigma^\infty f$: **Top**$_*$ $\to$ **Spect**.*

The sphere spectrum $\mathbb{S}$: A particularly nice example is given by the 0-sphere S^0. Applying the suspension spectrum functor Σ^∞ we get that its n-th term is homeomorphic to S^n. Subsequently, we denote the sphere spectrum $\Sigma^\infty S^0$ by $\mathbb{S}$.

It is clear that the sequence of Eilenberg-MacLane spaces form a spectrum. Eilenberg-MacLane spectrum enjoys a special property: The n-th space $(HG)^n$ and the loop space of $(HG)^{n+1}$ are both Eilenberg-MacLane spaces of type $K(G, n)$ and in fact the map $\tilde{\sigma}_n : (HG)^n \to \Omega(HG)^{n+1}$ adjoint to the structure map is a weak equivalence for all $n \geq 0$. Spectra with this property play an important role in stable homotopy theory, and they deserve a special name.

Definition 35. *An* **Ω*-spectrum*** *E is a spectrum such that the adjoint map $\tilde{\sigma}_n : E_n \to \Omega E_{n+1}$ of the structure map σ_n is a weak equivalence for each $n \geq 0$. If E and F are Ω-spectra, then a map of Ω-spectra $f : E \to F$ is a map of the underlying spectra.*

In [3] Adams introduced his representability theorem that relates generalized cohomolgy theories with Ω-spectra.

Theorem 3.5. *Let H^* be a generalized cohomology theory defined on finited CW-complexes. Then H^* is the generalized cohomology theory corresponding to a Ω-spectrum E represented by the formula*

$$H^n(X) = [X, E_n]. \tag{3.10}$$

The sphere spectrum does not represent a cohomology theory given by Formula 3.10, because it is not a Ω-spectrum. However, it is not hard to verify that the stable homotopy groups π_n^s defines a reduced homology theory, equivalently, the stable cohomotopy groups defines a reduced cohomology theory.

Formula 3.5 can be generalized at the level of spectra by the formula

$$\pi_n^s(E) = \mathrm{colim}_k \pi_{n+k}(E_{n+k}) \tag{3.11}$$

for E a spectrum.

In general, given a spectrum E and a based space X the following formulas

$$\tilde{E}^n(X) = [\Sigma^\infty X, \Sigma^n E], \tag{3.12}$$

$$\tilde{E}_n(X) = \pi_n^s(X \wedge E), \tag{3.13}$$

where $(X \wedge E)_n = X \wedge E_n$, define generalized reduced cohomology and homology theories, respectively. One can also define homology and cohomology for spectra, and the same results hold once we translate the Eilenberg-Steenrod axioms into the world of spectra.

What gets confusing is that there are dozens of models of spectra, nearly all of which give the same homotopy category, but which are pretty different on the point-set level.

We can extend the smash product $E \wedge F$ of spectra E, F of the above spectra $X \wedge E$ where X is an space and E an spectrum. The definition (construction) of the smash product of spectra can be found in the book by Adams [4].

Theorem 3.6. *There is a construction which assigns to spectra E and F a certain spectrum denoted by $E \wedge F$. This construction is called the smash product of spectra and has the following properties:*

1. *It is a covariant functor of each of its arguments.*
2. *There are natural equivalences:*
$$\begin{aligned} &a : (E \wedge F) \wedge G \to E \wedge (F \wedge G), \\ &\tau : E \wedge F \to F \wedge G, \\ &l : \mathbb{S} \wedge E \to E, \\ &r : E \wedge \mathbb{S} \to E, \\ &\Sigma : \Sigma E \wedge F \to \Sigma(E \wedge F). \end{aligned}$$
3. *For every spectrum E and CW-complex X, there is a natural equivalence $e : E \wedge X \to E \wedge \Sigma^\infty X$. In particular, $\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty X \wedge \Sigma^\infty Y$ for every pair of CW-complexes X, Y.*
4. *If $f : E \to F$ is an equivalence then $f \wedge id_G : E \wedge G \to F \wedge G$ is.*
5. *Let $\{E_\lambda\}$ be a family of spectra, and let $i_\lambda : E_\lambda \to \bigvee_\lambda E_\lambda$ be the inclusions. Then the morphism*
$$\{i\lambda \wedge id\} : \bigvee_\lambda (E_\lambda \wedge F) \to (\bigvee_\lambda E_\lambda) \wedge F.$$
is an equivalence.

6. if $A \to B \to C$ is a cofiber sequence of spectra, then so is the sequence

$$A \wedge E \to B \wedge E \to C \wedge E. \tag{3.14}$$

for every spectrum E.

Proof. See [4]. □

However, there is not a natural definition of the smash product between spectra. Thus, thanks to the previous theorem we can expect desirable properties for the smash product.There is an obvious first generalization: For E and spectrum, X a space and $\Sigma^\infty X$ its suspension spectrum. Then, we may define $E \wedge \Sigma^\infty X$ by

$$(E \wedge \Sigma^\infty X)_n = E_n \wedge (\Sigma^\infty X)_0 = E_n \wedge X \tag{3.15}$$

and the obvious structure maps, so we get $E \wedge \Sigma^\infty X = E \wedge X$. However, that construction has a problem: If we suspend $\Sigma^\infty X$ we get $(\Sigma^\infty X)_0 = *$ and then the smash product gives $*$, which is not the suspension of $E \wedge X$.

There is a very intuitive construction called *naive smash product*: Let E and F be spectra we can define a wedge product as the spectra $E \wedge F$ defined by the formula

$$(E \wedge F)_n = \begin{cases} E_k \wedge F_k & \text{if } n = 2k, \\ \Sigma(E_k \wedge F_k) & \text{if } n = 2k+1. \end{cases}$$

Denoting the structure maps of E and F by $e : \Sigma E_n \to E_{n+1}$ and $f : \Sigma F_n \to F_{n+1}$, then we define $\Sigma(E \wedge F)_n \to (E \wedge F)_{n+1}$ by $e \wedge \mathrm{id} : S^1 \wedge E_n \wedge F_n \to E_{n+1} \wedge F_n$ for even n; for odd n we define it by $S^1 \wedge E_n \wedge F_n \xrightarrow{\tau \wedge \mathrm{id}} E_n \wedge S^1 \wedge F_n \xrightarrow{\mathrm{id} \wedge f} E_n \wedge F_{n+1}$. This defines a spectrum

We are ready now to give the following definition:

Definition 36. *A **ring spectrum** is a triple (E, μ, ι) where E is a spectrum, $\mu : E \wedge E \to E$ and $\iota : \mathbb{S} \to E$ are morphisms such that the following diagrams commute up to homotopy:*

- *Associativity:*

$$\begin{tikzcd}
(E \wedge E) \wedge E \arrow[r, "\mu \wedge id"] \arrow[dd, "\alpha"'] & E \wedge E \arrow[dr, "\mu"] & \\
 & & E \\
E \wedge (E \wedge E) \arrow[r, "id \wedge \mu"'] & E \wedge E \arrow[ur, "\mu"'] &
\end{tikzcd}$$

where a is a natural equivalence given by definition of the smash product of spectra.

- *Unitary:*

$$\begin{tikzcd} \mathbb{S}\wedge E \arrow[r] \arrow[d, "l"'] & E\wedge E \arrow[d, "\mu"'] & E\wedge \mathbb{S} \arrow[l] \arrow[d, "r"] \\ E \arrow[r, "id"'] & E & E \arrow[l, "id"] \end{tikzcd}$$

where l, r are the natural equivalences given by definition of the smash product of spectra.

In addition, we say that the ring spectrum is commutative if the following diagram commutes up to homotopy

$$\begin{tikzcd} E\wedge E \arrow[r, "\tau"] \arrow[d, "\mu"'] & E\wedge E \arrow[d, "\mu"] \\ E \arrow[r, "\mathrm{id}"'] & E \end{tikzcd}$$

where τ twists the factors of the smash product.

A morphism of ring spectra $\varphi : (E, \mu, \iota) \to (E', \mu', \iota')$ is a morphism $\varphi : E \to E'$ such that the following diagrams commute up to homotopy:

$$\begin{tikzcd} E\wedge E \arrow[r, "\varphi\wedge\varphi"] \arrow[d, "\mu"'] & E'\wedge E' \arrow[d, "\mu"] \\ E \arrow[r, "\varphi"'] & E' \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S} \arrow[r, "\iota"] \arrow[d] & E \arrow[d, "\varphi"] \\ \mathbb{S} \arrow[r, "\iota'"'] & E'. \end{tikzcd}$$

3.3 Symmetric Spectrum

The concept of spectrum was introduced by Lima in his Ph.D. thesis [25] and later generalized by Whitehead [40]. Later, different categories of spectra were constructed. We use the category suggested by Adams. Each construction was motivated by desirable properties that a spectrum should have. However, the problem of which is a category with good properties that allows us to capture the essence of stability was solved by Boardman in his Ph.D. thesis and it was called the homotopy category of spectra or the stable homotopy category. Therefore, a question emerges: What is the correct notion of spectra?

One of the answers that got a remarkable attention was recently provided by Mark Hovey, Brooke shipley and Jeff Smith in the famous work [22] and corresponds to the category of symmetric spectra. For a detailed description of symmetric spectra the reader can consult the above article.

A symmetric spectrum is a spectrum equipped with an action of the symmetric group on each component space, such that the structure maps intertwine these actions combined with the canonical permutation action on the n-spheres.

Definition 37. *A **Symmetric spectrum** consists of:*

1. *A sequence of $\{X_n\}_{n\in\mathbb{N}}$ of pointed simplicial sets.*
2. *A basepoint preserving left action of the symmetric group Σ_n on X_n.*
3. *A sequence of morphisms of pointed simplicial sets $\sigma_n : X_n \wedge S^1 \to X_{n+1}$.*

We require that for all $n, m \geq 0$, the following composite is $\Sigma_n \times \Sigma_m$-equivariant:

$$\sigma_{n,m} : X_n \wedge S^m \to X_{n+m}. \tag{3.16}$$

We should always think of Σ_n as acting on S^n by permuting sphere coordinates. And the action in X_{n+m} is given by the injection of $\Sigma_n \times \Sigma_m \hookrightarrow \Sigma_{n+m}$.

A morphism $f : X \to Y$ of symmetric spectra consists of Σ_n-equivariant based maps $f_n : X_n \to Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that the following commutes for all $n \geq 0$:

Symmetric spectra with the defined morphisms form a category usually denoted Sp^Σ.

Let us study a particular spectrum which can be gotten from the two previous constructions.

Definition 38. *A **symmetric ring spectrum** R is*

1. *A sequence of pointed spaces $\{R_n\}$.*
2. *A basepoint preserving continuous left action of the symmetric group Σ_n on R_n for each $n \geq 0$.*
3. *For all $m, k \in \mathbb{N}$ a multiplication map $\mu_{m,k} : R_k \wedge R_m \to R_{k+m}$.*
4. *Two unit maps $i_0 : S^0 \to R_0$ and $i_1 : S^1 \to R^1$.*

such that the associativity, unit as in the previous definition plus the following diagram commute:

$$\begin{tikzcd} R_n \wedge S^1 \arrow[r, "id\wedge i_1"] \arrow[d] & R_n \wedge R_1 \arrow[r, "\mu_{n,1}"] & R_{n+1} \arrow[d, "\gamma_{n,1}"] \\ S^1 \wedge R_n \arrow[r, "i_1\wedge id"'] & R_1 \wedge R_n \arrow[r, "\mu_{1,n}"'] & R_{1+n} \end{tikzcd}$$

where $\gamma_{n,1} \in \Sigma_{n,1}$ is the shuffle permutation, which $\gamma_{n,m}$ moves the block of n things past the other m. In addition, we say that R is commutative if the following diagram commutes

$$\begin{tikzcd} R_n \wedge R_m \arrow[r, "twist"] \arrow[d, "\mu_{n,m}"] & R_m \wedge R_n \arrow[d, "\mu_{m,n}"] \\ R_{n+m} \arrow[r, "\gamma_{n,m}"'] & R_{m+n} \end{tikzcd}$$

We can define a tensor product in the category of symmetric spectra, as follows. Let E and F be symmetric spectra, then we can define a new spectrum given by

$$(E \otimes F)_n = \bigvee_{p+q=n} (\Sigma_n)^+ \bigwedge_{\Sigma_p \times \Sigma_q} (E_p \wedge F_q) \tag{3.17}$$

Remark 7. It is clear that the above spectrum is equivariant under the action of the symmetric group. In addition, it is the coequalizer of certain maps, for a detailed description see [22].

Remark 8. The sphere spectrum is an example of a symmetric ring spectrum.

Remark 9. Given the symmetric spectrum category Sp^Σ, a tensor product given by Formula 3.17 and the sphere spectrum we can verify that $(\mathrm{Sp}^\Sigma, \otimes, \mathbb{S})$ is a closed symmetric monoidal category.

3.4 Stable homotopy category

We are going to list some properties that we would like to have in the stable homotopy category to be defined, denoted by **HoSpect**. Those properties are imposed in order to study stable phenomena. Formally, we can get the stable homotopy category inverting weak equivalences in the sense of Quillen.

In addition, we will prove from the next list of axioms that objects in **HoSpect** can define generalized homology and cohomology theories.

1. **HoSpect** is a closed symmetric monoidal category, i.e., **HoSpect** together with a monoidal product denoted by $\wedge$, an identity object denoted by $\mathbb{S}$ and a internal

Hom which will be denoted by F. Thus, if $X, Y,$ and Z are objects in **HoSpect**, there are natural coherent isomorphisms in **HoSpect** such that

$$\begin{aligned}
&\mathbb{S} \wedge X \cong X,\\
&X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z,\\
&X \wedge Y \cong Y \wedge X,\\
&[X \wedge Y, Z] \cong [X, F(Y, Z)],\\
&F(\mathbb{S}, X) \cong X,\\
&F(X \wedge Y, Z) \cong F(X, F(Y, Z)).
\end{aligned}$$

The last two properties follow from the first four properties plus Yonneda's lemma. Indeed

$$[\mathbb{S}, F(\mathbb{S}, X)] \cong [\mathbb{S} \wedge \mathbb{S}, X] \cong [\mathbb{S}, X].$$

By Yonneda's lemma the result follows. For the last property, note that

$$[\mathbb{S}, F(X \wedge Y, Z)] \cong [\mathbb{S} \wedge (X \wedge Y), Z] \cong [X \wedge Y, Z] \cong [X, F(Y, Z)] \cong [\mathbb{S}, F(X, F(Y, Z)]$$

and again by Yonneda's the result follows. Here, the bracket $[-, -]$ denotes the morphisms in **HoSpect**.

2. There is a faithfull monoidal functor $\Sigma^\infty : \mathbf{Top}_* \to \mathbf{HoSpect}$. That is, some objects in **HoSpect** come from based spaces. In addition, this functor must respect product and coproduct, that is, $\Sigma^\infty(X \vee Y) \cong \Sigma^\infty X \vee \Sigma^\infty Y$ for X, Y based spaces.

3. There is a suspension functor $\Sigma : \mathbf{HoSpect} \to \mathbf{HoSpect}$ that comes from the classical suspension in topological spaces. That is the following square commutes

$$\begin{tikzcd}
\mathbf{Top}_* \arrow[r, "\Sigma"] \arrow[d, "\Sigma^\infty"] & \mathbf{Top}_* \arrow[d, "\Sigma^\infty"] \\
\mathbf{HoSpect} \arrow[r, "\Sigma"'] & \mathbf{HoSpect}.
\end{tikzcd}$$

Even more, in **HoSpect** the functor Σ defines an equivalence of categories.

4. There is a right adjoint functor $\Omega^\infty : \mathbf{HoSpect} \to \mathbf{Top}_*$, that is, an isomorphism

$$[\Sigma^\infty K, X] \cong [K, \Omega^\infty X].$$

5. There is a loop functor $\Omega : \mathbf{HoSpect} \to \mathbf{HoSpect}$ that comes from the usual based loop space. That is, the following square commutes

$$\begin{tikzcd} \mathbf{Top}_* & \mathbf{Top}_* \arrow[l, "\Omega"'] \\ \mathbf{HoSpect} \arrow[u, "\Omega^\infty"] & \mathbf{HoSpect}. \arrow[l, "\Omega"'] \arrow[u, "\Omega^\infty"'] \end{tikzcd}$$

Here in **HoSpect** the functors Σ and Ω are adjoints and the compositions $\Sigma \circ \Omega$ and $\Omega \circ \Sigma$ are naturally isomorphic to the identity. In simple words, we can write every object in **HoSpect** as a suspension or a loop of other object.

6. **HoSpect** must be an additive category. That is the set of morphisms is an abelian group and composition of morphisms $[X, Y] \times [Y, Z] \to [X, Z]$ is a bilinear map that induces an abelian group homomorphism $[X, Y] \otimes [Y, Z] \to [X, Z]$. In addition, we require the existence of finite products and coproducts that we will denote by $X \times Y$ and $X \vee Y$, respectively. We can equip $[X, Y]_*$ with the structure of an Abelian group in the following way:

$$[X, Y]_n = [\Sigma^n X, Y].$$

Here the order of the suspensions does not matter if we put the suspension on the left, we call the above convention the homological grading. Otherwise $[X, Y]_n = [X, \Sigma^n Y]$ is called the cohomological grading. You should note that $[X, Y]$ contributes to the 0-th level in the graded abelian group.

7. There is a zero object in **HoSpect** denoted by $*$, coming from the one-point based space $*$ in $\mathbf{Top}_*$. That is, there are unique maps $* \to E \to *$ in **HoSpect**. Therefore $*$ is the unit for both product and coproduct then $E \vee * \cong E$ and $E \times * \cong E$. In addition, in **HoSpect** the product and coproduct as above are the same. For based spaces it is not true.

8. For an object X in **HoSpect** the stable homotopy group is defined by

$$\pi_n(X) = [\mathbb{S}, X]_n = [\Sigma^n \mathbb{S}, X]. \tag{3.18}$$

Let us consider now a topological based space K, then there is an isomorphism

$$[\Sigma^\infty K, X] \cong [K, \Omega^\infty(X)]$$

in **HoSpect**.

9. We will require a Whithehead's theorem: If $f : X \to Y$ is a map in **HoSpect** that induces an isomorphism $f_* : \pi_*(X) \to \pi_*(Y)$, then f is an isomorphism. Thus, stable groups help us to identify isomorphisms in **HoSpect**.

10. We require a triangulated structure in **HoSpect**. Before talking about triangulated categories, let us start by describing triangles and then we will describe a **triangulated category**. We can form a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \tag{3.19}$$

and we say that (X, Y, Z, f, g, h) is a triangle. In **HoSpect**, there are triangles that we will call distinguished triangles, that satisfy the following properties:

For each distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X, \tag{3.20}$$

and each object W in **HoSpect** there are long exact sequences of abelian groups

$$\cdots \longrightarrow [W, X]_n \longrightarrow [W, Y]_n \longrightarrow [W, Z]_n \longrightarrow [W, Y]_{n-1} \longrightarrow \cdots$$

$$\cdots \longleftarrow [X, W]_n \longleftarrow [Y, W]_n \longleftarrow [Z, W]_n \longleftarrow [X, W]_{n+1} \longleftarrow \cdots \tag{3.21}$$

If we take $W = \mathbb{S}$, then the previous sequence gives rise to a long exact sequence

$$\cdots \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Z) \longrightarrow \pi_{n-1}(X) \longrightarrow \cdots \tag{3.22}$$

In addition, we require that Σ^∞ takes cofiber sequences in $\mathbf{Top}_*$ to distinguished triangles in **HoSpect**. Also, Ω^∞ takes distinguished triangles in **HoSpect** to fiber sequences in $\mathbf{Top}_*$.

In the stable homotopy level we can say that a generalized reduced cohomology (equivalently homology) theory can be represented whitout distinction with the spectra case, we will call it the Adams-Brown representability theorem.

Theorem 3.7. ***(Adams-Brown representability theorem)***. *Every cohomology (resp. homology) theory is represented (resp. corepresented) by an object in **HoSpect**. That is, there is a functor $\tilde{\mathbb{E}}$: **HoSpect** $\to$ **CohomThy** which is full and faithfull. The functor is defined by*

$$\tilde{\mathbb{E}}^n(Y) = [\Sigma^\infty Y, \Sigma^n E] \tag{3.23}$$

*where Y is a based space and E an object in **HoSpect**.*

Proof. To see that the functor defined by Formula 3.23 is a generalized reduced cohomology theory we need to check that our construction satisfies the conditions of Definition 32.

By definition $\tilde{E}^n(X)$ is a set of morphisms in **Hospect** which are abelian groups. Now we need to verify that $\tilde{E}$ satisfies all the the axioms of Definition 32.

1. *Suspension condition*: Let E be an object in **Hospect** and X a based space. Then we have the following sequence of isomorphisms

   $$\tilde{E}^n(X) = [\Sigma^\infty X, \Sigma^n E] \to [\Sigma(\Sigma^\infty X), \Sigma(\Sigma^n E)] \cong [\Sigma^\infty(\Sigma X), \Sigma^{n+1} E] = \tilde{E}^{n+1}(\Sigma X).$$

   Here we used that $\Sigma^\infty \circ \Sigma = \Sigma \circ \Sigma^\infty$ and that the suspension is an equivalence in **HoSpect**.

2. *Homotopy invariance*: Let $f, g : X \to Y$ be two maps in $\mathbf{Top}_*$ such that $f \sim g$ are homotopy equivalent. We know the functor Σ^∞ factors through $\mathbf{HoTop}_*$, that is, the following composite represents the functor Σ^∞

$$\begin{tikzcd} \mathbf{Top}_* \arrow[rr, "\Sigma^\infty"] \arrow[dr, "W"'] & & \mathbf{HoSpect} \\ & \mathbf{HoTop}_* \arrow[ur, "\Sigma^\infty"'] & \end{tikzcd}$$

By abuse of notation we will call Σ^∞ the functor from $\mathbf{HoTop}_*$ to $\mathbf{HoSpect}$.

3. *Exactness axiom*: Let $f : X \to Y$ be a map in $\mathbf{Top}_*$, then there is a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{j} Cone(f) \xrightarrow{h} \Sigma X \longrightarrow \cdots$$

again in $\mathbf{Top}_*$. We can apply the functor Σ^∞ and then we get a distingished triangle in **Hospect**

$$\Sigma^\infty(X) \xrightarrow{\Sigma^\infty(f)} \Sigma^\infty Y \xrightarrow{\Sigma^\infty(j)} \Sigma^\infty \mathrm{Cone}(f).$$

Applying the functor $[-, \Sigma^n E]$ we get

$$\cdots \longrightarrow [\Sigma^\infty(\mathrm{Cone}(f)), \Sigma^n E] \xrightarrow{j_*} [\Sigma^\infty Y, \Sigma^n E] \xrightarrow{f_*} [\Sigma^\infty X, \Sigma^n E] \longrightarrow \cdots$$

Since **Hospect** is a triangulated category, then the functor $[-, \Sigma^n E]$ induces a long exact squence of abelian groups. Then, we can conclude that there exists an exact sequence

$$\tilde{E}^n(\mathrm{Cone}(f)) \xrightarrow{j_*} \tilde{E}^n(Y) \xrightarrow{f_*} \tilde{E}^n(X).$$

4. *Additivity axiom*: Let $\{X_i\}_{i\in I}$ be a collection of based spaces, then

$$\tilde{E}^n\left(\bigvee_{i\in I} X_i\right) = \left[\Sigma^\infty(\bigvee_{i\in I} X_i), \Sigma^n E\right] \cong \left[\bigvee_{i\in I} \Sigma^\infty X_i, \Sigma^n E\right].$$

A map $\bigvee_{i\in I}(\Sigma^\infty X_i) \to Z$ in **Hospect** is the same as a collection of maps $\{\Sigma^\infty X_i \to Z\}$ for any object Z in **Hospect**. Then,

$$\left[\bigvee_{i\in I}(\Sigma^\infty X_i), \Sigma^n E\right] \cong \bigvee_{i\in I} [\Sigma^\infty X_i, \Sigma^n E] \cong \prod_{i\in I} [\Sigma^\infty X_i, \Sigma^n E].$$

The last isomorphism is due to the fact that in **Hospect** products and coproducts are the same.

□

We can extend the previous construction and equivalently show that $E^n(X) = \tilde{E}^n(X_+)$ form an unreduced cohomology theory, following in a similar way the axioms that describe the unreduced cohomology; see these axioms in [21].

Remark 10. The abelian groups $\tilde{E}_n(X) = [\mathbb{S}, ((\Sigma^\infty X) \wedge E]_n \cong \pi_n(\Sigma^\infty X) \wedge E)$ define a reduced homology theory. The proof is very similar to the one presented from reduced cohomology theory.

We can extend the notion of generalized reduced homology and cohomology theories for objects in **Hospect** with the formulas

$$\tilde{E}_n(Y) = [\mathbb{S}, Y \wedge E]_n \cong \pi_n(Y \wedge E), \tag{3.24}$$

$$\tilde{E}^n(Y) = [Y, E]_{-n} \cong \pi_{-n}(F(Y, E)), \tag{3.25}$$

where E, Y are objects in **Hospect**.

We have shown a list of axioms that a stable homotopy category should have. However, we do not know so far if we are constructing an empty category. Historically, we have many attempts: **S-category**, category of **Spectra** but as we said before those categories do not satisfy properties that we would like to have. Thus, let us stop for a while to study a construction made by Adams and convince ourselves that this theory which satisfies all those axioms is not empty. A historical reference is Adams [4], but a good and clear reference is the book by Margolis, see [28].

First of all, the constructions of **HoSpect** require the category of spectra motivated in the previous sections. Then, let us consider a spectrum $E = \{E_n, e_n\}$ with the following properties: The sequence of based spaces E_n is a CW-complex, and it gives to the bases space ΣE_n the a structure of CW-complex by the obvious one on $E_n \wedge S^1$, where S^1 is thought as a CW-complex with a 0-cell and one 1-cell. Thus, ΣE_n has a 0-cell and one cell ΣC_α for C_α cell of E_n. In addition, we require that the map $\Sigma E_n \hookrightarrow E_{n+1}$ be an inclusion of subcomplex. This is the definition of a **CW-spectrum**.

Note that every k-cell in E_n becomes a $(k+1)$-cell in E_{n+1}, a $(k+2)$-cell in E_{n+2} and so on. This is an stable phenomenon and we will call it a **stable (k-n)-cell**, at this point negative dimensional cells are allowed. That is an intuitive idea of a CW-spectrum like a CW-complex where we allow negative dimensional cells.

We define the homotopy stable category to be the category **Ad**, by Adams, whose objects are CW-spectra and morphisms are homotopy class of maps of CW-spectra, where homotopy, as in the case of ordinary topology, is defined from a cylinder. More precisely, we can define the cylinder spectrum $\mathrm{Cyl}(E)$ by

$$(\mathrm{Cyl}(E))_n := I_+ \wedge E_n$$

with structure maps given by

$$(I_+ \wedge E_n) \wedge S^1 \xrightarrow{id \wedge e_n} I_+ \wedge E_{n+1}$$

Thus, we say that two maps of spectra $f_0, f_1 : E \to F$ are **homotopic** if there exists a map $h : \mathrm{Cyl}(E) \to F$ such that $f_0 = hi_0$ and $f_1 = hi_1$ with the inclusions $i_0, i_1 : E \to \mathrm{Cyl}(E)$.

In addition, we can define a functor $\Sigma^\infty : \mathbf{HoTop}_* \to \mathbf{Ad}$ whose $n-th$-term is $\Sigma^n X$ and with structure maps the identity. That functor allows us to get two familiar objects such as the zero object defined by $* := \Sigma^\infty(\{pt\})$, i.e. the image of a point, and the sphere spectrum $\mathbb{S} := \Sigma^\infty(S^0)$. Those objects where examples of spectra, but they also have structure of CW-spectra.

In [4], Adams shows that this category satisfies the following properties:

- Arbitrary coproducts.
- **Ad** is a triangulated category.
- Whitehead theorem.

However, the big issue is the smash product: How we can get a "good" symmetric monoidal structure? We know that the category of symmetric spectra and orthogonal spectra. The idea of Adams was the following. Given two CW-spectra $X = \{X_n, e_n\}$ and $Y = \{Y_n, e'_n\}$ there are $\mathbb{Z} \times \mathbb{Z}$ collection of spaces $X_m \wedge Y_l$ with obvious structure maps. That is the reason why there are many notion of smash product and by the stability conditions all of them are equivalent. The idea is then consider a sequence of pairs of nonnegative integers $\{(i_n, j_n) : n \geq 0, i_n + j_n = n, \text{and } \{i_n\}, \{j_n\} \text{ are monotone unbounded sequences}\}$. Let $X \wedge Y$ be the spectrum with $(X \wedge Y)_n = X_{i_n} \wedge Y_{j_n}$, the structure maps are induced from X and Y.

3.5 Atiyah duality

As we did in the previous chapter, we are interested in studying dualizability in symmetric monoidal categories. By the axiomatic construction made of **HoSpect**, we know that the tuple $(\mathbf{HoSpect}, \wedge, \mathbb{S})$ is a symmetric monoidal category. Thus, two natural questions emerge: What does dualizability in the homotopy stable category represent? Given a dualizable object A in **HoSpect**, what does its dual represent?

To recap, A, B are n-dual if $A \cong S^{n+1} \setminus B$ in the **S-category**; some authors also refer to them as strongly n-duals. By Alexander duality, we get isomorphisms:

$$\tilde{H}_q(A) \cong \tilde{H}^{n-q}(B),$$
$$\tilde{H}^q(A) \cong \tilde{H}_{n-q}(B).$$

Then, we can say that Spanier-Whitehead duality generalizes the Alexander duality in a broder context.

Remark 11. The sphere S^n is dualizable in the **S-category** with dual S^{-n}.

If two spaces A, B are strongly n-dual we can define a map

$$u : \Sigma^{k+l}(A \wedge B) \to \Sigma^{k+l} S^n. \tag{3.26}$$

We call this map the n-duality map. Applying the top-dimensional cohomology functor we get a map

$$u^* : H^{n+k+l}(\Sigma^{k+l} S^n) \to H^{n+k+l}(\Sigma^{k+l}(A \wedge B)). \tag{3.27}$$

Hence

$$u^* : H^n(S^n) \to H^n(A \wedge B). \tag{3.28}$$

Let γ_n be a generator of $H^n(S^n)$, then the slant product $u^*(\gamma_n)/- : H_q(A) \to H^{n-q}(B)$ realizes the Alexander duality isomorphism. Thus, we can form a weaker definition due to Spanier.

Definition 39. *Two spaces A and B are n-dual if there is a map $A \wedge B \to S^n$ which gives the isomorphisms*

$$\tilde{H}_q(A) \cong \tilde{H}^{n-q}(B),$$
$$\tilde{H}^q(A) \cong \tilde{H}_{n-q}(B).$$

By abuse of notation, we will say that duality in the above definition and definition 28 are both called Spanier-Whitehead duality or S-duality.

Remark 12. A geometric $(n+1)$–dual A^* gives rise to an n–duality map $A \wedge A^* \to S^n$. Spanier suggested that it would be more natural to call A^* an n–dual of A, which is the terminology that is now used.

Thanks to the theory developed in Chapter 2, we say that two objects A, B in **HoSpect** are dual if there are maps

$$A \wedge B \to \mathbb{S} \text{ and } \mathbb{S} \to B \wedge A,$$

which satisfy the snake identities. However, **HoSpect** is a closed category, then dualizable objects in that category have a particular description, see Formula 2.6.

Therefore, we say that two objects A, B in **HoSpect** are n-dual if A and $\Sigma^{-n}B$ are dual, or equivalently B and $\Sigma^{-n}A$ are dual

$$\Sigma^{-n}A \cong F(B, \mathbb{S}) \text{ and } \Sigma^{-n}B \cong F(A, \mathbb{S}). \tag{3.29}$$

Then we get the induced isomorphisms

$$A = DB \cong F(B, \Sigma^n \mathbb{S})$$
$$B = DA \cong F(A, \Sigma^n \mathbb{S}),$$

where DB amd DA are the duals of B and A, respectively.

Given two n-dual spaces A and B, then we can consider the class of maps

$$[\Sigma^{k+l}(A \wedge B), \Sigma^{k+l} S^n].$$

Taking the colimit when $k + l \to \infty$ we get a map in the **S-category**. Moreover, we get a map in **HoSpect**

$$\Sigma^\infty A \wedge \Sigma^\infty B \to \mathbb{S}. \tag{3.30}$$

Thus a natural question emerges: Does the suspenstion functor Σ^∞ preserve duality? That is, for A and B n-dual spaces, are the objects $\Sigma^\infty A$ and $\Sigma^\infty B$ $n-$duals as well?

Theorem 3.8. *If A and B are n-dual spaces, then $\Sigma^\infty A$ and $\Sigma^\infty B$ are n-dual spectra.*

Proof. We must verify the isomorphism $\Sigma^\infty A \cong F(\Sigma^\infty B, \mathbb{S}^n)$, or equivalently $\Sigma^\infty B \cong F(\Sigma^\infty A, \mathbb{S}^n)$. The idea here is to prove that a map $f : \Sigma^\infty A \to F(\Sigma^\infty B, \mathbb{S}^n)$ is a weak equivalence in the homotopy stable world. Hence, by the Whitehead theorem in **HoSpect**, see [14], the conclusion holds if we verify that there is an isomorphism

$$\tilde{H}_*(F(\Sigma^\infty B, \mathbb{S}^n)) \cong \tilde{H}_*(\Sigma^\infty A). \tag{3.31}$$

Indeed, let $H\mathbb{Z}$ be the Eillenberg-MacLane spectrum given by the spaces $(H\mathbb{Z})_n = K(\mathbb{Z}, n)$. Then by Brown's representability theorem, Theorem 3.5, we get the following sequence of isomorphisms

$$\begin{aligned}
\tilde{H}_m(F(B, \mathbb{S}^n)) &\cong \pi_m(F(B, \mathbb{S}^n) \wedge H\mathbb{Z}) \cong \pi_m(F(B, K(\mathbb{Z}, n))) \cong [\Sigma^\infty B, K(\mathbb{Z}, n)] \\
&\cong [\Sigma^n \Sigma^\infty B, H\mathbb{Z}] \cong H^{-m}(\Sigma^n B) \cong \tilde{H}^{n-m}(B) \cong \tilde{H}_m(A) \\
&\cong \tilde{H}_m(\Sigma^\infty A).
\end{aligned}$$

All the steps above are clear except that

$$F(B, \mathbb{S}) \wedge H\mathbb{Z} \cong F(B, H\mathbb{Z}). \tag{3.32}$$

More generally, for X, W and Z objects in **HoSpect** and X a dualizable object there is an isomorphism

$$[W, Z \wedge X^*] \cong [W \wedge X, Z] \cong [W, F(X, Z)].$$

The first isomorphism is given by the adjuntion between $\wedge X$ and $\wedge X^*$, and the second isomorphism is given by the adjuntion between $\wedge X$ and $F(X, \cdot)$. We know **HoSpect** is a closed category then we get the isomorphism $X^* \cong F(X, \mathbb{S})$. Thus,

$$[W, Z \wedge F(X, \mathbb{S})] \cong [W, F(X, Z)]$$

and by Yonneda's lemma the conclusion holds. □

It is a technical fact that if two based spaces A and B are $(n-1)$-dual, then taking the suspension of whichever A or B makes them n-dual. Thus, we get the following lemma.

Lemma 3.9. *Let A and B be two based spaces which are n-duals, then the spaces ΣA and B are $(n+1)$-dual, or equivalently A and ΣB are $(n+1)$-dual.*

Proof. Let us prove one of the results and the other will follow by the symmetric conditions. Let $u : A \wedge B \to S^n$ be the n-duality map, then we get

$$\Sigma(A) \wedge B \longrightarrow \Sigma(A \wedge B) \xrightarrow{\Sigma u} \Sigma S^n \cong S^{n+1}.$$

Let us call the above composite $v : \Sigma(A) \wedge B \to S^{n+1}$. Then we can induce a map at top degree cohomology

$$v^* : \tilde{H}^{n+1}(S^{n+1}) \to \tilde{H}^{n+1}(\Sigma(A) \wedge B) \cong \tilde{H}^{n+1}(\Sigma(A \wedge B)) \cong \tilde{H}^n(A \wedge B).$$

Similarly, let γ_n be a generator of $H^n(S^n)$. Thus we can get a map $v^*(\gamma_n)/ : \tilde{H}_q(A) \to \tilde{H}^{n-q}(B)$ which by hypothesis is an isomorphism. Then we conclude that ΣA and B are $(n+1)$-dual. $\square$

It is time to give a geometric interpretation of the definitions presented in this section. By Whitney's theorem we know there exists a number $n >> 0$ such that $e : M \hookrightarrow \mathbb{R}^n$ is an embedding, for a smooth manifold M. We can assume M is compact then we can get an embedding $e_+ : M_+ \hookrightarrow S^n$ with $M_+ = M \bigsqcup *$.

Let $\nu_e \to M$ be a normal bundle of M in $\mathbb{R}^n$ associated to the embedding $e : M \hookrightarrow \mathbb{R}^n$, and $\varphi : \nu_e \cong M_\epsilon$ the ϵ-neighborhood. Let us also consider D the disk bundle and we will denote the image of the disk bundle as $W := \varphi(D) \subset \mathbb{R}^n$.

Lemma 3.10. *Let M be a compact manifold of dimension m, and $e : M \hookrightarrow \mathbb{R}^n$ an embedding, then M_+ and $\mathbb{R}^n - M$ are strongly $(n-1)$-dual.*

Proof. The embedding $e : M \hookrightarrow \mathbb{R}^n$ can be compactified to get $e_+ : M_+ \hookrightarrow S^n$ an embedding of M_+ into the n-dimensional sphere. On the other hand, it is clear that

$$S^n - M_+ \cong \mathbb{R}^n - M \tag{3.33}$$

is a homotopy equivalence given by the identity map. Thus, the conditions of Definition 28 hold. Hence we can conclude the lemma. $\square$

By Lemma 3.9, we can conclude that M_+ and $\Sigma(\mathbb{R}^n - M)$ are strongly n-duals. The following two technical lemmas allow us to conclude Atiyah's duality.

Lemma 3.11. *Let $f : X \to \mathbb{R}^n$ be a continuous function with cone* Cone(f). *Then, there is an homotopy equivalence*

$$\mathrm{Cone}(f) \cong \Sigma(X). \tag{3.34}$$

Proof. Let us define a function $g : \mathrm{Cone}(f) \to \Sigma(X)$ by

$$g(p) = \begin{cases} (x,t) & \text{if } p = (x,t), \\ (x,1) & \text{if } p \in \mathbb{R}^n. \end{cases}$$

By Tietze extension theorem we can conclude that g is a continuous function. This map collapses $\mathbb{R}^n$ to a point and, as result, we get a space which is homotopically equivalent to $\Sigma(X)$. On the other hand, consider $r : \Sigma(X) \to \mathrm{Cone}(f)$ given by

$$r(x,t) = \begin{cases} (x,2t) & \text{if } t \leq 1/2, \\ H_{2t-1}(f(x)) & \text{if } t \geq 1/2 \end{cases}$$

where H_t is a retraction of $\mathbb{R}^n$ to $q = f(p)$. Intuitively this map is collapsing the second half cone to the point q. It is clear that the compositions $r \circ g$ and $g \circ r$ are homotopic to the identity. Hence, there is a homotopy equivalence between $\mathrm{Cone}(f)$ and $\Sigma(X)$. □

Lemma 3.12. *Let $e : M \hookrightarrow \mathbb{R}^n$ be an embedding with tubular neighborhood $\varphi : \nu \to U$ and with disk bundle D. Let $X = \mathbb{R}^n - \varphi(D^o)$, and $j : X \hookrightarrow \mathbb{R}^n$ given by the inclusion. Then, the projection map $\rho : Cone(j) \to \mathbb{R}^n/X$ is an homotopy equivalence.*

Proof. Let us define a function $\xi : \mathbb{R}^n/X \to Cone(j)$ given by:

$$\xi(p) = \begin{cases} (p,0) & \text{if } p \notin \varphi(D^0), \\ (\varphi(\frac{v}{|v|}, 2-2t) & \text{if } p = \varphi(v) \text{ with } |v| \geq 1/2, \\ \varphi(2v) & \text{if } p = \varphi(v) \text{ with } |v| \leq 1/2. \end{cases}$$

By Tietze extension theorem we can conclude that ξ is a continuous function. It is not hard to see that $\rho \circ \varphi$ and $\varphi \circ \rho$ are homotopic equivalent to the identity. □

The following lemma states that the suspension of $\mathbb{R}^n - M$ has the same homotopy type of the Thom space of the normal bundle.

Lemma 3.13. *Let M be a manifold and $e : M \hookrightarrow \mathbb{R}^n$ and embedding with normal bundle $\nu_e \to M$. The Thom space* $\mathrm{Th}(\nu_e) = M^{\nu_e}$ *is homotopy equivalent to the suspension of $\mathbb{R}^n - M$.*

Proof. Let us fix a metric in the normal bundle ν_e. Let $\varphi : \nu \xrightarrow{\cong} M_\epsilon$ be a tubular neighborhood and D the disk bundle of ν_e, and $W = \varphi(D)$. Then,

$$\mathrm{Th}(\nu) \cong D/\partial D \cong W/\partial W \cong \mathbb{R}^n/(\mathbb{R}^n - W^0).$$

Hence, if we prove that $\mathbb{R}^n/(\mathbb{R}^n - W^o)$ is homotopy equivalent to $\Sigma(\mathbb{R}^n - M)$, the result follows. On the other hand, due to the deformation retract

$$H_t(v) = \frac{|v| + t(1-|v|)}{|v|}v,$$

we get that $D - D^0$ is a deformation retract of $D - M$, then $\mathbb{R}^n - W^0$ is a deformation retract of $\mathbb{R}^n - M$. Therefore, it is enough to prove that $\mathbb{R}^n/(\mathbb{R}^n - W^o)$ is homotopic equivalent to $\Sigma(\mathbb{R}^n - W^o)$.

Let us call $X = \mathbb{R}^n - W^o$ and j the inclusion of X in $\mathbb{R}^n$. By Lemma 3.11 we know that $Cone(j)$ is homotopy equivalent to $\Sigma(X)$, and by Lemma 3.12 we get that $Cone(j)$ is homotopy equivalent to $\mathbb{R}^n/X$. Then, we can conclude:

$$\mathbb{R}^n/(\mathbb{R}^n - W^o) \cong \mathbb{R}^n/X \cong \Sigma(X) = \Sigma(\mathbb{R}^n - W^o).$$

□

Therefore, M_+ and $Th(\nu)$ are strongly n-dual. Lemma 3.8 allows us to transfer by means of the suspension functor Σ^∞ this dualizability to the stable homotopy category

$$\Sigma^\infty(M_+) \cong F(\Sigma^\infty(\mathrm{Th}(\nu)), \mathbb{S}^n),$$
$$\Sigma^\infty(\mathrm{Th}(\nu)) \cong F(\Sigma^\infty(M_+), \mathbb{S}^n).$$

That is $\Sigma^\infty(M_+)$ and $\Sigma^\infty(Th(\nu))$ are n-dual. That notion of duality was classically called **Atiyah duality**. Then in **HoSpect** we get a particular object defined by:

$$M^{-TM} := \Sigma^{-n}\Sigma^\infty(\mathrm{Th}(\nu)) \cong F(\Sigma^\infty(M_+), \mathbb{S}).$$

and known as **Thom spectra**. That means that M^{-TM} is dual to $\Sigma^\infty(M_+)$ in **HoSpect**.

This result can be viewed as a refinement of Poincare duality. It implies, for instance, the Poincare duality theorem in generalized cohomology theories, assuming that M is an orientable manifold.

Let E be an object in **HoSpect**, then we get isomorphisms

$$\tilde{E}_q(\mathrm{Th}(\nu)) \cong E^{n-q}(M) \cong \tilde{E}^{n-q}(M_+),$$
$$\tilde{E}^q(\mathrm{Th}(\nu)) \cong E_{n-q}(M) \cong \tilde{E}_{n-q}(M_+).$$

There is a generalization of the Thom-Dold isomorphism to generalized cohomology and homology theories, so we get the isomorphism:

$$\tilde{E}^{\bullet+k}(\mathrm{Th}(\nu)) \cong E_\bullet(M) \tag{3.35}$$

where k is the rank of ν, that is, $k = n - m$. Then, we get

$$E^{\bullet}(M) \cong E_{n-(\bullet+k)}(M) \cong E_{m-\bullet}(M)$$

which is the Poincaré duality at the level of generalized cohomology theory E.

Let us study the idea of Atiyah. Recall that for $e : M \hookrightarrow \mathbb{R}^n$ an embedding with tubular neithgborhood $M_\epsilon(e)$, then for small enough ϵ

$$\mathrm{Th}(\nu) \cong \mathbb{R}^n/(\mathbb{R}^n - M_\epsilon(e)).$$

$B_\epsilon(0)$ will denote the ball of radius ϵ centred in the origin of $\mathbb{R}^n$. In the 30's, Alexander considered the map

$$(\mathbb{R}^n - M_\epsilon(e)) \times M \to \mathbb{R}^n - B_\epsilon(0) \cong S^{n-1}. \tag{3.36}$$

That induces a map in homology

$$H_{n-q-1}(\mathbb{R}^n - M_\epsilon(e)) \otimes H_q(M) \to H_{n-1}(S^{n-1}) \cong \mathbb{Z}, \tag{3.37}$$

that is adjoint to the map

$$H_q(M) \to H^{n-q-1}(\mathbb{R}^n - M_\epsilon(e)).$$

This gives us the *Alexander duality* isomorphism. In the 60's, Atiyah in [5] considerd the map

$$\mathrm{Th}(\nu) \wedge M_+ \to S^n \tag{3.38}$$
$$(v, y) \to v - e(y) \tag{3.39}$$

This produces a map

$$\mathrm{Th}(\nu) \to \mathrm{Maps}(M_+, S^n). \tag{3.40}$$

Then, at the spectrum level we get

$$M^{-TM} \to F(\Sigma^\infty M_+, \mathbb{S}). \tag{3.41}$$

We showed that the previous map is an equivalence in **HoSpect**.

We can rewrite the above characterization of Atiyah's duality in terms of theory developed in Chapter 2. We are able to construct morphisms ϵ and η in **HoSpect** which will turn out to be the evaluation and coevaluation for the duality between M_+ and M^{-TM}.

Let us consider first in **HoSpect**

$$\epsilon : \Sigma^{-n} \frac{\mathbb{R}^n}{(\mathbb{R}^n - M)} \wedge M_+ \to S^0, \tag{3.42}$$

but instead of defining the above morphism, let us define $\Sigma^n \epsilon$ which can be gotten from

$$\begin{tikzcd}
\frac{\mathbb{R}^n}{(\mathbb{R}^n-M)} \wedge M_+ \arrow[r, dotted, "\Sigma^n\epsilon"] \arrow[d, "p"'] & \Sigma^n S^0 \\
\frac{\mathbb{R}^n}{(\mathbb{R}^n-M)} \times M_+ \arrow[d, "id"'] & S^n \arrow[u, "\cong"] \\
\frac{\mathbb{R}^n\times M}{(\mathbb{R}^n-M)\times M} \arrow[r, "\rho"] & \frac{\mathbb{R}^n}{\mathbb{R}^n-0} \arrow[u, "\cong"]
\end{tikzcd}$$

where ρ is the diffeomorphism defined by the formula

$$\rho(x,k) = x - k. \tag{3.43}$$

Now we consider again in **HoSpect** the morphism

$$\eta : S^0 \to M_+ \wedge \Sigma^{-n}\frac{\mathbb{R}^n}{(\mathbb{R}^n - M)}. \tag{3.44}$$

Similarly, we can define $\Sigma^n\eta$ by

$$\begin{tikzcd}
S^n \arrow[r, dotted, "\Sigma^n\eta"] \arrow[d, dash, "\cong"'] & M_+ \wedge \frac{\mathbb{R}^n}{\mathbb{R}^n-M_+} \arrow[d, "p"] \\
\frac{\mathbb{R}^n}{\mathbb{R}^n-0} & M_+ \times \frac{\mathbb{R}^n}{\mathbb{R}^n-M_+} \arrow[d, dash] \\
\frac{\mathbb{R}^n}{\mathbb{R}^n-B} \arrow[u, "\subset"] \arrow[d, dash, "\cong"'] & \frac{M_+\times\mathbb{R}^n}{M\times(\mathbb{R}^n-M)} \\
\frac{V}{V-M} \arrow[r, "\Delta"] & \frac{V\times\mathbb{R}^n}{V\times(\mathbb{R}^n-M)} \arrow[u, "r\times id"']
\end{tikzcd}$$

where B is a closed ball centred in 0, $r : V \to M$ is a retraction of V an open subset in $\mathbb{R}^n$ and Δ is the diagonal morphism defined by

$$\Delta(v) = (v,v). \tag{3.45}$$

Thanks to the above descriptions of the evaluation and coevaluation morphisms we can define a geometric construction that we will use in order to find $\mathrm{tr}(f)$. First, the map $\Sigma^n\eta$ can be easily constructed in the following way: We identify S^n with the one point compactification of $\mathbb{R}^n$. Then, points outside the tubular neighborhood go to the basepoint. On the other hand, points inside the tubular neighborhood go to themselves, paired with their projection to M.

$$\Sigma^n\eta(x) = \begin{cases} (m,v) & \text{if } x \in M_\epsilon \\ \infty & \text{if } x \notin M_\epsilon \end{cases}$$

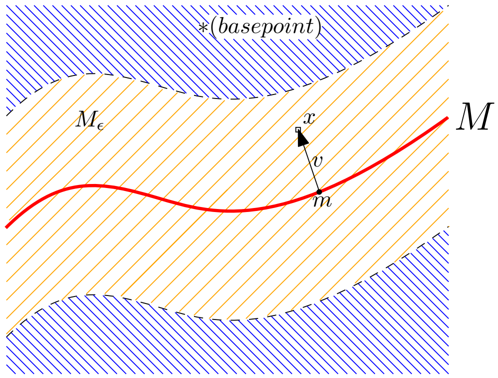


Figure 3.1: $\eta(x) = (v, m)$

The following figure represents intuitively the morphism $\Sigma^n\eta$

The evaluation map ϵ can be constructed in the following way:

First of all, let (v, m) be a pair in $Th(\nu) \wedge M_+$. If m and v are far apart, then (v, m) goes to the infinite point, see the following figure

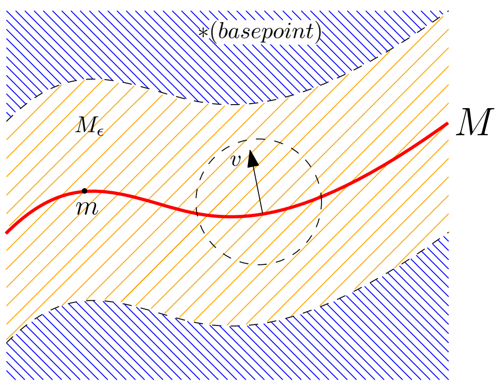


Figure 3.2: m and v distant.

On the other hand, if m and v are close, we can add them and we apply the formula used by Alexander and Atiyah. Thus we get:

$$\Sigma^n\epsilon(v, m) = \begin{cases} v - e(m) & \text{if } v, m \text{ are close} \\ \infty & \text{in other case.} \end{cases}$$

Remark 13. Maps $\Sigma^n\eta$ and $\Sigma^n\epsilon$ will be denoted just by η and ϵ in order to alleviate the notation; the context will allow us to recognize them.

The following figure represents intuitively what the map ϵ does for m and v close together.

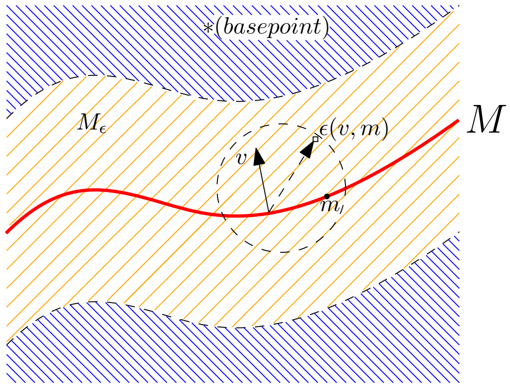


Figure 3.3: evaluation map-ϵ

In [2] Dold and Puppe show that M_+ and $Th(\nu)$ are strongly n-duals proving that ϵ and η satisfy the snake identities in the stable context.

3.6 Lefschetz fixed point theorem

Using the formulations given in the last section plus theory developed in Chapter 2 we are going to prove the Lefschetz fixed point theorem. There are many proofs of this theorem, the common one is due to algebraic topology using simplicial approximation.

In Chapter 1 we introduced the fixed point index of a self map, see Definition 11. In [17], Dold describes the fixed-point index using the maps presented in the above section as follows: Let F be the set of fixed points of $f : M \to M$. We can describe the trace with the following diagram

$$\begin{tikzcd}
S^n \arrow[rr, dotted, "\mathrm{tr}(f)"] \arrow[d, dash, "\cong"'] & & S^n \arrow[d, dash, "\cong"'] \\
\frac{\mathbb{R}^n}{\mathbb{R}^n - 0} & & \frac{\mathbb{R}^n}{\mathbb{R}^n - 0} \\
\frac{\mathbb{R}^n}{\mathbb{R}^n - B} \arrow[u, "\subset"] \arrow[d, dash, "\subset"] & & \\
\frac{\mathbb{R}^n}{\mathbb{R}^n - M} \arrow[r, dash, "\cong"'] & \frac{V}{V - M} \arrow[r, "\subset"'] & \frac{V}{V - F} \arrow[uu, "\psi"']
\end{tikzcd}$$

where $\psi(v) = v - f(r(v))$ for $r : V \to M$ a retraction. Applying the homology functor $H_n(-, \mathbb{Z})$ one gets a homomorphism which is multiplication by a number. That number is the fixed-point index.

Now, we can try to extend the above construction to a more general framework. Thanks to the notion of trace we get the following definition.

Definition 40. *Let M be a compact manifold embedded in $\mathbb{R}^n$ for some $n >> 0$ and $f : M \to M$ a smooth map, then the* ***fixed-point index*** *of f is the degree of the following composite*

$$S^n \xrightarrow{\Sigma^n \eta} M_+ \wedge \mathrm{Th}(\nu) \xrightarrow{f \wedge id} M_+ \wedge \mathrm{Th}(\nu) \xrightarrow{\cong} \mathrm{Th}(\nu) \wedge M_+ \xrightarrow{\Sigma^n \epsilon} S^n.$$

We will denote the previous composite as $\mathrm{tr}(f)$.

Remark 14. We must note that the above composite is defined at the level of based spaces. However, it is not a trace in the sense of Chapter 2. Denoting it as a trace is an abuse of notation. However, we can recover the notion of trace by translating it to **HoSpect**. If we apply the suspension functor Σ^∞ to the previous composite we get

$$\mathbb{S}^n \xrightarrow{\eta} \Sigma^\infty(M)_+ \wedge \Sigma^\infty \mathrm{Th}(\nu) \xrightarrow{f \wedge id} \Sigma^\infty(M_+) \wedge \Sigma^\infty \mathrm{Th}(\nu) \xrightarrow{\cong} \Sigma^\infty \mathrm{Th}(\nu) \wedge \Sigma^\infty(M_+) \xrightarrow{\epsilon} \mathbb{S}^n.$$

If we "desuspend" n-times, i.e., apply the functor Σ^{-n} we get in **HoSpect** the following composite

$$\mathbb{S} \xrightarrow{\Sigma^\infty \eta} \Sigma^\infty(M)_+ \wedge \Sigma^{\infty - n} \mathrm{Th}(\nu) \xrightarrow{\Sigma^\infty f \wedge id} \Sigma^\infty(M_+) \wedge \Sigma^{\infty - n} \mathrm{Th}(\nu) \xrightarrow{\Sigma^\infty \epsilon} \mathbb{S},$$

which is $\mathrm{tr}(\Sigma^{\infty-n}(f))$ in **HoSpect**. We must also note that the class of homotopy maps from the sphere spectrum $\mathbb{S}$ to itself is $\mathbb{Z}$. In addition, if we apply the functor $\Sigma^{\infty-n}$ to $\mathrm{tr}(f)$ we get

$$\mathrm{tr}(\Sigma^{\infty-n}(f)) = \Sigma^{\infty-n}(\mathrm{tr}(f)).$$

It is also important to note that we are not applying Proposition 2.22. Actually, we can not do it. Thus, the above equality means that applying the suspension functor Σ^∞ and then calculating the trace in **HoSpect** is the same as applying the composite in Definition 40 and then applying the suspension functor.

So, we can conclude that $\mathrm{tr}(\Sigma^{\infty-n} f)$ is a number, and we will call it **the categorical fixed-point** of f. For a map $f : M \to M$, we can explicitly calculate the above composite from a geometric point of view and check that it counts fixed points.

Let us find the above trace $\mathrm{tr}(f)(x)$ for $x \in S^n$. Consider the compactification $S^n \cong \mathbb{R}^n \cup \infty$. Then, there are two cases:

- If $x \notin M_\epsilon$, the above composition sends x to ∞.

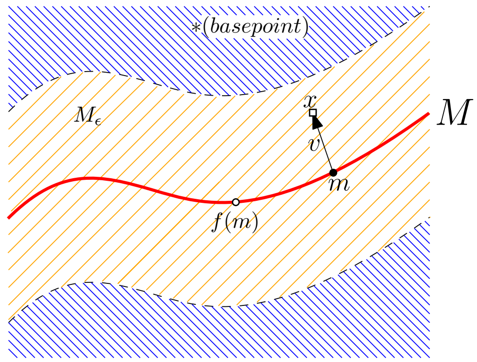


Figure 3.4

- If $x \in M_\epsilon$, we first apply $\Sigma^n \eta$ and we get (m, v) as in Figure 3.1. Then we apply $f \wedge \mathrm{id}$ and we get $(f(m), v)$ as in the above figure

Therefore we have two cases:

- $f(m)$ is far from m, we get $\mathrm{tr}(f)(x) = \infty$, see Figure 3.5.
- $f(m)$ is close to m, we get $\mathrm{tr}(f)(x)$ as in the Figure 3.6.

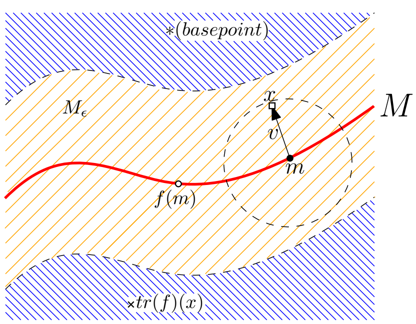


Figure 3.5: $f(m)$ is far from m

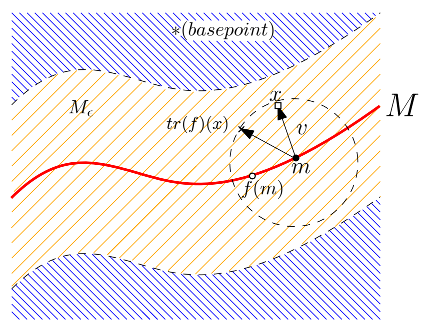


Figure 3.6: $f(m)$ is close to m

Whichever the case, the trace as the previous composite is a map from a sphere to a sphere. As m varies near a fixed point, $\mathrm{tr}(f)(x)$ has some degree. Everywhere else the degree is zero. Hence, we have

$$\mathrm{ind}(f) = H(\mathrm{tr}(f), \mathbb{Z}). \tag{3.46}$$

Remark 15. We say that m and $f(m)$ are close together if the vector $v - e(f(m))$ is inside M_ϵ, in other case we say that they are far apart. The following figure represents intuitively that

We have two important consequences of the above result. First, the Lefschetz-Hopf theorem:

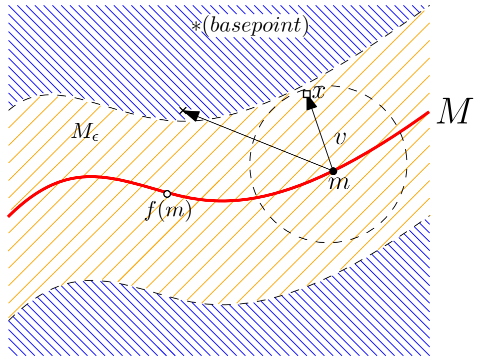


Figure 3.7

Corollary 3.14. ***(Lefschetz–Hopf theorem)***
If $f : M \to M$ is a map of a compact manifold into itself then the index of f is equal to the Lefschetz number of f.

Proof. We must note that the rational homology functor from the category of based spaces to the category of graded vector spaces, denoted by $\tilde{H}_* : \mathbf{Top}_* \to \mathbf{Grvect}$, factors through **HoSpect**. That is, the following diagram commutes

$$\begin{tikzcd} \mathbf{Top}_* \arrow[r, "\tilde{H}_*"] \arrow[d, "\Sigma^{\infty-n}"'] & \mathbf{GrVect}_k \\ \mathbf{HoSpect} \arrow[ur, "H_*"'] & \end{tikzcd}$$

Hence,

$$\begin{aligned} \mathrm{ind}(f) &= H_*(\mathrm{tr}(f), \mathbb{Z}) \cong \tilde{H}_*(\mathrm{tr}(f_+), \mathbb{Q}) = H_*(\Sigma^{\infty-n}\mathrm{tr}(f)) = H_*(\mathrm{tr}(\Sigma^{\infty-n}(f))) \\ &= \mathrm{tr}(H_*(\Sigma^{\infty-n}(f))) = \mathrm{tr}(\tilde{H}_*(f, \mathbb{Q})) = L(f), \end{aligned}$$

where $f_+ : M_+ \to M_+$ is a based map which sends ∞ to itself. In addition, the last equality is given by Proposition 2.22. □

Corollary 3.15. ***(Lefschetz–fixed point theorem)***
*Let M be a closed smooth manifold and $f : M \to M$ a smooth map, and let H : **Mfd** $\to$ **GrVeck** denote rational homology, $H(M) = H_*(M, Q)$. If*

$$L(f) = \mathrm{tr}(H(f)) = \sum_{n \neq 0} (-1)^n \mathrm{tr}(H_n(f)) \tag{3.47}$$

is nonzero, then f has a fixed point.

Proof. Thanks to the above corollary, if $L(f) \neq 0$, then the degree of the map $\mathrm{tr}(f)$ is not zero. Thus, by the composite at the beginning of this section we can conclude that there exist a fixed point for f.

□

Chapter 4

String topology

In [13] Chas and Sullivan started the study of spaces of loops and path in a manifold. For a smooth closed manifold M of dimension d, the space of smooth free loops on M is $LM := \mathrm{Maps}(S^1, M)$. The **string topology** is the study of the all possible structures in the homology of LM. In that work, Chas and Sullivan define a product in $H_*(LM)$ of degree $-\dim(M)$, that is a map

$$\circ : H_i(LM) \otimes H_j(LM) \to H_{i+j-d}(LM). \tag{4.1}$$

The idea of the authors was to consider and combine two constructions from algebraic topology: the intersection product and the concatenation of paths.

Intersection product: We know there is not a ring structure in the homology of M an oriented manifold of dimension d. However, due to the Poincare duality and assuming that M is also compact we can define a ring structure $\cap$ in $H_*(M)$ by the following composite

$$\begin{tikzcd}
H_P(M) \otimes H_q(M) \arrow[r, dashed, "\cap"] \arrow[d, "P.D."'] & H_{p+q-n}(M) \\
H^{n-p}(M) \otimes H^{n-q}(M) \arrow[r, "\sqcup"'] & H^{2n-p-q}(M) \arrow[u, "P.D."']
\end{tikzcd}$$

where $\sqcup : H^{n-p}(M) \otimes H^{n-q}(M) \to H^{2n-p-q}(M)$ is the smash product of classes of forms in $H^*(M)$. It is called the intersection product. This product can be interpreted in terms of intersection theory, let α and β be two representative classes of $H_*(M)$ with transverse intersection, then the intersection product $\cap$ is defined by

$$[\alpha] \cap [\beta] := [\alpha \cap \beta], \tag{4.2}$$

it is of degree $-d$, associative, graded commutative and it can be defined as the dual of the cup product in cohomology. It can be interpreted geometrically as follows: A

p-homology class can be represented by a p-dimensional submanifold P of dimension p. Similarly, a q-homology class can be represented by a submanifold Q of dimension q. We can perturb P, Q so as to make their intersection transverse. Then, $P \cap Q$ is $p+q-n$ dimension submanifold of M, hence determines a chain. Passing to homology this all is well defined and reproduces the above product.

Remark 16. We say that a homology class $\sigma \in H_p(M)$ can be represented by submanifold if there exists a p-dimensional closed oriented submanifold P, such that σ is the push forward of the fundamental class of P under the inclusion map.

Let LM be the space of piecewise smooth maps from S^1 to M, the free loop space. We can try to mimic the above idea of the intersection product but the great issue is that LM is an infinite dimensional manifold and we do not have Poincare duality.

However, we do not expect to have a non-trivial ring structure on the homology. In despite of that, Chas and Sullivan defined a "loop product" in the homology of the free loop space of a closed oriented d-dimensional manifold using the intersection product and the **concatenation** of based loops.

In $C_*(LM)$, each loop has a marked point, namely the image of $0 \in S^1$, there is a map $ev : LM \to M$. Consider $\alpha \in C_p(LM)$ and $\beta \in C_q(LM)$, the set of marked points are submanifolds of dimension p and q respectively, i.e. p, q-chains on M. Their intersection give us a $(p+q-n)$-chain γ on M. Then, when the marked points of α and β coincide then we can form a loop by the concatenation of loops of α then β. This defines a $(p+q-n)$-chain in LM. Passing to homology we get the Chass-Sullivan product 4.1.

However, Cohen and Jones in [16] got the same formulas of string topology product thanks to Atiyah duality. The goal of this chapter is understand the constructions made by the authors and apply the tools developed in the previous chapters.

4.1 Pontrjagyn-Thom collapse map

Let us start by recalling the famous Thom-Pontrjagyn map and study some generalizations of this construction. For $e : P \hookrightarrow N$ an embedding with ν_e tubular neighborhood of the image of P under the embedding. Let $\tau : N \to P^{\nu_e} \cong \nu_e \cup \infty$ be defined by

$$\tau(x) = \begin{cases} x & \text{if } x \in \nu_e \\ \infty & \text{if } x \notin \nu_e. \end{cases}$$

The map τ is the Thom-Pontrjagyn collapse map. If we consider an oriented context we get by the Thom isomorphism, the following:

$$e_! : H_q(N) \xrightarrow{\tau_*} H_q(P^{\nu_e}) \xrightarrow{Thom} H_{q-n}(P).$$

In particular, if we consider the diagonal embedding $\Delta_M : M \to M \times M$ of a oriented, closed manifold M of dimension d. Then the Thom-Pontrjagin map is a map

$$\tau : M \times M \to M^{TM}, \tag{4.3}$$

In the same way we can define the push forward

$$\Delta_! : H_*(M \times M) \xrightarrow{\tau_*} H_*(M^{TM}) \xrightarrow{\cong} H_{*-d}(M).$$

which is simply the intersection product.

We can extend the Thom-Pontrjagin collapse map for any vector bundle (or virtual vector bundles). Let us consider the case of an embedding $e : P \hookrightarrow N$ and any vector bundle $\pi : \xi \to N$, then we define the pull back bundle $e^*(\xi) \to P$ such that the following diagram commutes

$$\begin{tikzcd} e^*(\xi) \arrow[r] \arrow[d] & \xi \arrow[d, "\pi"] \\ P \arrow[r, "e"'] & N. \end{tikzcd}$$

The embedding $e^*(\xi) \hookrightarrow \xi$ gives us a tubular neighborhood $\nu(e^*(\xi))$, then we can get the Thom-Pontrjagin map

$$\tau : \xi \cup \infty \to \nu(e^*(\xi)) \cup \infty. \tag{4.4}$$

It can also be written as

$$\tau : N^{\xi} \to P^{\nu(e^*(\xi))} \tag{4.5}$$

with the identification of the Thom spaces as: $N^{\xi} \cong \xi \cup \infty$ and $P^{\nu(e^*(\xi))} \cong \nu(e^*(\xi)) \cup \infty$ is the Thom space of $\nu(e^*(\xi))$. Note that $\nu(e^*(\xi)) \cong e^*(\xi) \oplus \nu_e$ it can be checked from the following exact sequence of vector bundles over P:

$$0 \to \nu_e \xrightarrow{d\sigma} \nu(e^*(\xi)) \xrightarrow{d\pi} e^*(\xi) \to 0 \tag{4.6}$$

where $\sigma : N \to \xi$ is the zero section of the bundle $\pi : \xi \to N$. Thus, we get the map

$$\tau : N^{\xi} \to P^{e^*(\xi) \oplus \nu_e}.$$

The above construction also applies for virtual vector bundles.

Thus, for the diagonal embedding $\Delta : M \to M \times M$ and using the virtual vector bundle $-TM \times -TM$ over $M \times M$, we get the Thom-Pontrjagin map

$$\tau : (M \times M)^{-TM \times -TM} \to M^{TM \oplus \Delta^*(-TM \times -TM)}. \tag{4.7}$$

which can be simplified as

$$\tau : M^{-TM} \wedge M^{-TM} \to M^{-TM}. \tag{4.8}$$

The above map defines a ring structure for M^{-TM}, we will see that tangentially in the following section.

For the Thom space of a normal bundle we will require the following technical fact: given two embeddings $e_1 : M \hookrightarrow \mathbb{R}^k$ and $e_2 : M \hookrightarrow \mathbb{R}^r$ with tubular neighborhoods $\nu_{\epsilon_1}(e_1)$ and $\nu_{\epsilon_2}(e_2)$, respectively. There is a projection map

$$\frac{\mathbb{R}^m}{(\mathbb{R}^m - \nu_{\epsilon_1(e_1)})} \wedge \frac{\mathbb{R}^k}{(\mathbb{R}^k - \nu_{\epsilon_2(e_2)})} \to \frac{\mathbb{R}^{n+k}}{\mathbb{R}^{m+k} - \nu_{\epsilon_{1,2}(e_1 \times e_2)}} \tag{4.9}$$

Thus, for $x \in M^{\nu_{\epsilon_1}(e_1)}$ and $y \in M^{\nu_{\epsilon_2}(e_2)}$ its smash product $x \wedge y$ lives in the Thom space $M^{\nu_{\epsilon_{1,2}(e_1 \times e_2)}}$.

4.2 Multiplicative structure of Thom spectrum

In this section we want to show how to get the structure of symmetric ring spectrum by considering all the possible choices that were not considered before. First, let us recall, the evaluation map, defined by the Formula 3.42, is a map

$$\epsilon : M^{-\nu_\epsilon} \wedge M_+ \to S^n$$

that allow us to define an adjoint map in the homotopy stable category

$$\alpha : M^{-TM} \cong F(\Sigma^\infty M_+, \mathbb{S}). \tag{4.10}$$

In category of symmetric spectra the spectrum of maps $F(\Sigma^\infty M_+, \mathbb{S})$ has the structure of commutative ring spectrum. Thus, emerges a natural question that comes from the above isomorphism in **HoSpect**: Is the Thom spectrum a symmetric ring spectrum that makes the adjoint map α a map of symmetric ring spectra?

In [16], R. Cohen and Jones, proved that by constructing a new spectrum denoted by $M^{-\tau}$, stable equivalent to the Thom spectrum, that does not depend of the choice of the embedding nor tubular neighboorhod. In this section we want to study that construction and try to understand the ideas of that work before we start with the goal of this chapter.

The central issue of Atiyah duality and the construction of Thom spectrum is the arbitrary choice of the embedding, tubular neighboorhod and the dimension of the Euclidean space. Cohen and Jones solved that problem by considering all the possible embedding and possible tubular neighborhoods. First, let us denote the space of embedding of M in some euclidean space by $\mathrm{Emb}(M, \mathbb{R}^k)$. We can consider the following set of embeddings and tubular neighborhood

$$\tilde{M}_k^{-\tau} := \begin{cases} * & \text{if } \mathrm{Emb}(M, \mathbb{R}^k) = \emptyset, \\ (e, \epsilon, x) & \text{if } e : M \hookrightarrow \mathbb{R}^k \text{ is an embedding,} \\ & 0 < \epsilon < L_\epsilon, \text{ and } x \in \mathbb{R}^k/(\mathbb{R}^k - \nu_e) \end{cases} \tag{4.11}$$

where $L_\epsilon := \min\{1, \epsilon\}$ and $\epsilon > 0$ is the least upper bound of those which satisfying the tubular neighborhood theorem. That set has a structure of topological space given by the topology given by the fiber bundle

$$p : \tilde{M}_k^{-\tau} \to \varepsilon_k \tag{4.12}$$

where the space $\varepsilon_k = \{(e, \epsilon) : e \in \mathrm{Emb}(M, \mathbb{R}^k), \text{ and } \epsilon \in (0, L_\epsilon)\}$.

Note that bundle has a lot of points at infinite, in fact to each fibre (e, ϵ) which is the Thom space associated to the normal bundle ν_e we get a point at infinite, we will call them without distinction as ∞. Then, we can consider the natural section of this bundle $\sigma_\infty^k : \epsilon_k \to \tilde{M}_k^{-\tau}$ given by $\sigma_\infty^k(e, \epsilon) = (e, \epsilon, \infty)$. We can take the quotient of that section and get a new space

$$M_k^{-\tau} := \tilde{M}_k^{-\tau}/\sigma_\infty^k(\varepsilon_k). \tag{4.13}$$

It is, we collapse all the infinite point in each fiber Thom space to a single point and we get a space with one infinite point.

Fixing an embedding $e : M \hookrightarrow \mathbb{R}^k$ and $\epsilon > 0$ as above; the tubular neighborhood theorem give us an homeomorphism $\phi : M^{\nu_e} \to \mathbb{R}^k/(\mathbb{R}^k - \nu_e)$. This defines an inclusion $j_e : M^{\nu_e} \hookrightarrow M_k^{-\tau}$ given by $j_e(x) = (e, \epsilon, x)$.

Remark 17. In the last chapter we identified the Thom space as: $Th(\nu_e) \cong \mathbb{R}^k/(\mathbb{R}^k - M_\epsilon)$, with $M_\epsilon \cong \nu_e$ the tubular neighborhood. In this chapter, we have decided to use the identification $Th(\nu_e) \cong \mathbb{R}^k/(\mathbb{R}^k - \nu_\epsilon(e))$ in order to have in mind that it depends on the embedding and the tubular neighbohood.

There is a fun fact: we know the symmetry group Σ_k acts on $\mathbb{R}^k$ by permuted of the coordinates, then it induces an action on ε_k, and on $M_k^{-\tau}$. In addition, there is an action of Σ^m in the unit sphere S^m given by the identification $S^m = \mathbb{R}^r/(\mathbb{R}^m - B_1(0))$. Thus, we can define the maps

$$\sigma_{m,k} : S^m \wedge M_k^{-\tau} \to M_{m+k}^{-\tau} \tag{4.14}$$
$$t \wedge (e, \epsilon, x) \to (0 \times e, \epsilon, t \wedge x). \tag{4.15}$$

where $M \hookrightarrow S^k \times M \xrightarrow{0 \times e} \mathbb{R}^{m+k}$, and $t \wedge x \in \mathbb{R}^{k+m}/(\mathbb{R}^{k+m} - \nu_\epsilon(0 \times e))$ is the image under the following projection

$$\frac{\mathbb{R}^k}{(\mathbb{R}^k - B_1(0))} \wedge \frac{\mathbb{R}^m}{(\mathbb{R}^m - \nu_\epsilon(e))} \to \frac{\mathbb{R}^{k+m}}{(\mathbb{R}^{k+m} - \nu_\epsilon(0 \times e))}$$

It is clear that the following diagram commutes

$$\begin{array}{ccc} M_k^{-\tau} \wedge S^m & \xrightarrow{\sigma_{m,k}} & M_{k+m}^{-\tau} \\ {\scriptstyle \pi_k \times \pi_m}\big\downarrow & & \big\downarrow{\scriptstyle \pi_k \times \pi_m} \\ M_k^{-\tau} \wedge S^m & \xrightarrow[\sigma_{m,k}]{} & M_{k+m}^{-\tau} \end{array}$$

where $\pi_k \in \Sigma_k$ and $\pi_m \in \Sigma_m$. Then, we can conclude that the maps $\sigma_{k,m}$ are $\Sigma_k \times \Sigma_m$-equivariant. Thus, we can conclude that $\{M_k^{-\tau}, \sigma_{m,k} : S^m \wedge M_k^{-\tau} \to M_{m+k}^{-\tau}\}$ is a symmetric spectrum.

So this is true for the spectrum $M^{-\tau}$, but what about with the Thom spectrum defined in the previous chapter? The following proposition clarifies that.

Proposition 4.1. *The inclusion $j_e : M^{\nu_e} \to M_k^{-\tau}$ induces an isomorphism in homotopy groups trough dimension $\frac{k}{2} - n - 2$.*

Proof. Let us consider the fibration

$$M^{\nu_e} \to M_k^{-\tau} \to \varepsilon_k \tag{4.16}$$

This fibration induces a long exact sequence in homotopy groups

$$\cdots \to \pi_q(M^{\nu_e}) \to \pi_q(M_k^{-\tau}) \to \pi_q(\varepsilon_k) \to \pi_{q-1}(M^{\nu_e}) \to \cdots$$

It is a technical fact that the space $\mathrm{Emb}(M, \mathbb{R}^k)$ is $(\frac{k}{2} - n - 2)$-connected. Hence, the base space ε_k also has the same connectedness number. Therefore, the long exact sequence

$$\cdots \to \pi_{\frac{k}{2}-n-1}(\varepsilon_k) \to \pi_{\frac{k}{2}-n-2}(M^{\nu_e}) \to \pi_{\frac{k}{2}-n-2}(M^{-\tau}) \to \pi_{\frac{k}{2}-n-2}(\varepsilon_k) \to \cdots \tag{4.17}$$

satisfies that $\pi_{\frac{k}{2}-n-1}(\varepsilon_k) = 0$ and $\pi_{\frac{k}{2}-n-2}(\varepsilon_k) = 0$. Hence, we can conclude

$$\pi_{\frac{k}{2}-n-2}(M^{\nu_e}) \cong \pi_{\frac{k}{2}-n-2}(M_k^{-\tau}). \tag{4.18}$$

□

Remark 18. The above result states that $M_k^{-\tau}$ has the right homotopy type, that is, in a homotopy category of spaces **HoTop**, $M_k^{-\tau}$ is just M^{ν_e}.

Remark 19. We will denote the spectrum $\{M_k^{-\tau}\}$ as $M^{-\tau}$.

Then is intuitively natural to consider this new spectrum $M^{-\tau}$ in order to study properties of the Thom spectrum.

We are ready with the main theorem of this section.

Theorem 4.2. *$M^{-\tau}$ is a symmetry ring spectrum without unit.*

Proof. We can define maps

$$\mu_{m,k} : M_m^{-\tau} \wedge M_k^{-\tau} \to M_{m+k}^{-\tau} \tag{4.19}$$

by the formula

$$\mu_{m,k}((e_1, \epsilon_1, x_1) \wedge (e_2, \epsilon_2, x_2)) = ((e_1 \times e_2) \circ \Delta_M, \epsilon_{1,2}, x_1 \wedge x_2) \tag{4.20}$$

where $\epsilon_{1,2} = \min\{\epsilon_1, \epsilon_2, L_{e_1 \times e_2}\}$ and $x_1 \wedge x_2$ is in the image of the following map

$$\frac{\mathbb{R}^m}{(\mathbb{R}^m - \nu_{\epsilon_1(e_1)})} \wedge \frac{\mathbb{R}^k}{(\mathbb{R}^k - \nu_{\epsilon_2(e_2)})} \to \frac{\mathbb{R}^{n+k}}{\mathbb{R}^{m+k} - \nu_{\epsilon_{1,2}(e_1 \times e_2)}}.$$

There are basic properties that the map $\mu_{m,k}$ satisfies. For example, it is $\Sigma_m \times \Sigma_k$-equivariant and for k, r, m the following diagram commutes

$$\begin{tikzcd}
(M_k^{-\tau} \wedge M_r^{-\tau}) \wedge M_m^{-\tau} \arrow[r] \arrow[d, "\mu_{k,r}\wedge id"'] & M_k^{-\tau} \wedge (M_r^{-\tau} \wedge M_m^{-\tau}) \arrow[d, "id\wedge\mu_{r,m}"] \\
M_{k+r}^{-\tau} \wedge M_m^{-\tau} \arrow[d, "\mu_{k+r,m}"'] & M_k^{-\tau} \wedge M_{r+m}^{-\tau} \arrow[d, "\mu_{k,r+m}"] \\
M_{r+k+m}^{-\tau} \arrow[r, no head, "="'] & M_{r+k+m}^{-\tau},
\end{tikzcd}$$

that is, the collection of maps $\{\mu_{m,r}\}$ is associative. Those maps also define a map in spectra

$$\mu : M^{-\tau} \otimes M^{-\tau} \to M^{-\tau}. \tag{4.21}$$

It is clearly commutative, given by the following commutative diagrams

$$\begin{tikzcd}
M_p^{-\tau} \wedge M_q^{-\tau} \arrow[r, "\mu_{p,q}"] \arrow[d] & M_{p+q}^{-\tau} \arrow[d, "\tau_{p,q}"] \\
M_q^{-\tau} \wedge M_p^{-\tau} \arrow[r, "\mu_{q,p}"'] & M_{p+q}^{-\tau}.
\end{tikzcd}$$

∎

We can also give the sphere spectrum the structure of symmetric ring spectrum, also without unit, in a similar way, we can define

$$(\tilde{S}_M)_k = \{(e, \epsilon, t) : (e, \epsilon) \in \varepsilon_k \text{ and } t \in \mathbb{R}^k/(\mathbb{R}^k - B_\epsilon(0))\},$$

in the same way, it can be topologized by the bundle $p : (\tilde{S}_M)_k \to \varepsilon_k$ and there is a canonical section $\sigma_\infty(e, \epsilon) = (e, \epsilon, \infty)$. Then we consider the symmetric spectrum

$$(S_M)_k = (\tilde{S}_M)_k / \sigma_\infty(\varepsilon_k). \tag{4.22}$$

Similarly as before, it can be shown that the maps

$$\beta_{r,k} : S^r \wedge (S_M)_k \to (S_M)_{k+r} \tag{4.23}$$

by $s \wedge (e, \epsilon, t) \to (\varphi_e, \epsilon, s \wedge t)$, where φ_e is the embedding $y \to (0, e(y)) \in \mathbb{R}^r \times \mathbb{R}^k$. And also the maps

$$m_{r,s} : (S_M)_r \wedge (S_M)_s \to (S_M)_{r+s} \tag{4.24}$$

defined by $(e_1, \epsilon_1, t_1) \wedge (e_2, \epsilon_2, t_2) \to (e_1 \times e_2, \min(\epsilon_1, \epsilon_2,), t_1 \wedge t_2)$. Those maps provide S_M the structure of symmetric spectrum without unit. Furthermore, it is a clear map $S_M \to \mathbb{S}$, which is indeed an equivalence of symmetric ring spectra.

The above construction allows us to define the symmetric ring spectrum $F(\Sigma^\infty M, S_M)$, induced by (S_M). If we write Atiyah's ideas in this new context we have a map

$$M_k^{-\tau} \wedge M_+ \to (S_M) \tag{4.25}$$

$$(e, \epsilon, x) \wedge y \to (e, \epsilon, x - e(y)). \tag{4.26}$$

This defines an adjoint map of symmetric ring spectra

$$\alpha : M^{-\tau} \to F(\Sigma^\infty M, S_M),$$

the above defines a homotopy equivalence.

Unfortunately there is not a good choice for the unit in the above symmetric ring spectra construction. However, the problem was solved in [15] by modifying the constructions presented above to restrict to embeddings of the form $\varphi \circ e$, with $\varphi : \mathbb{R}^k \to \mathbb{R}^{kn}$ a linear, isometric embedding. For a detailed description, the reader can consult [15].

4.3 Multiplicative structure of LM^{-TM}

In this section we will finish discussing the ring structre of LM^{-TM} from the ring structure gotten from the Thom spectrum M^{-TM}, from the previous section. Moreover, we are interested in proving the following theorem.

Theorem 4.3. *Let M be a smooth, closed manifold of dimension d. The spectrum LM^{-TM} is a ring spectrum with unit and multiplication*

$$\mu : LM^{-TM} \wedge LM^{-TM} \to LM^{-TM}.$$

In addition, if M is also orientable the ring structure is compatible with the Chas-Sullivan homology product:

$$\begin{tikzcd}
H_q(LM^{-TM}) \otimes H_r(LM^{-TM}) \arrow[r] \arrow[d, "\cong"', "Thom"] & H_{q+r}(LM^{-TM} \wedge LM^{-TM}) \arrow[r, "\mu_*"] & H_{q+r}(LM^{-TM}) \arrow[d, "\cong"', "Thom"] \\
H_{q+d}(LM) \otimes H_{r+d}(LM) \arrow[rr, "\circ"'] & & H_{q+r+d}(LM)
\end{tikzcd}$$

Before we study a proof of the above theorem, let us study for a second a formal construction made by Chass and Sullivan in order to develop intuition in the case of spectra. We can define $LM \times_M LM$ to be the pull back

$$\begin{tikzcd}
\mathrm{Map}(8, M) = LM \times_M LM \arrow[r, "\tilde{\Delta}"] \arrow[d, "ev"'] & LM \times LM \arrow[d, "ev \times ev"] \\
M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

which coincides with $\mathrm{Map}(8, M)$ the space of piecewise smooth maps from the figure 8 to M. And it can be thought as

$$LM \times_M LM = \{(\alpha, \beta) \in LM \times LM : \alpha(0) = \beta(0)\}.$$

The evaluation map $ev : LM \to M$ can be considered as a fiber bundle (of infinite dimensional manifolds), then the map $\mathrm{Map}(8, M) \to LM \times LM$ is an embedding of finite codimension d and its tubular neighborhood is the pull back of the normal bundle of the diagonal embedding, which is TM, thus $\nu(\tilde{\Delta}) = ev^*(TM)$. Then, the Thom-Pontrjagin construction gives us a map

$$\tau : LM \times LM \to \mathrm{Map}(8, M)^{ev^*(TM)}. \tag{4.27}$$

In the oriented case, we have

$$H_*(LM \times LM) \xrightarrow{\tau_*} H_*(Map(8, M)^{ev^*(TM)}) \xrightarrow[\cong]{Thom} H_{*-d}(\mathrm{Map}(8, M)).$$

There is also a map $\gamma : \mathrm{Map}(8, M) \to LM$ gived by the concatenation of loops, that is, for α and β loops in M with the same base points, i.e. $\alpha(0) = \beta(0)$, γ is defined by

$$\gamma(\alpha, \beta) = \alpha * \beta$$

where

$$\alpha * \beta(t) = \begin{cases} \alpha(2t), \\ \beta(2t-1). \end{cases}$$

Then we have

$$\begin{tikzcd}
H_p(LM)\otimes H_q(LM) \arrow[rr, "\circ"] \arrow[d, "Kunneth"'] & & H_{p+q-n}(LM) \\
H_{p+q}(LM\times LM) \arrow[r, "\tau_*"'] & H_{p+q}(\mathrm{Map}(8,M)^{ev^*(TM)}) \arrow[r, "Thom"'] & H_{p+q-n}(\mathrm{Map}(8,M)) \arrow[u, "\gamma_*"']
\end{tikzcd}$$

which by Kunneth's theorem we can take a cycle in $LM \times LM$ and intersecting it with the d-submanifold $\mathrm{Map}(8, M)$ and the resulting is mapped via γ to LM. This is the exactly idea of the Chass and Sullivan product in $H_*(LM)$.

As we did before, we can construct the Thom-Pontrjagin map with the virtual vector bunlde $-TM \times -TM$ over $LM \times LM$. Thus we get the map

$$\tau : (LM \times LM)^{(ev\times ev)^*(-TM\times -TM)} \to \mathrm{Map}(8, M)^{ev^*(TM)\oplus ev^*(\Delta^*(-TM\times -TM))}$$

but note that $\Delta^*(-TM \times -TM) = -2TM$. Thus we get

$$\tau : LM^{ev^*(-TM)} \wedge LM^{ev^*(-TM)} \to \mathrm{Map}(8, M)^{TM\oplus -2TM} = \mathrm{Map}(8, M)^{-TM} \tag{4.28}$$

In the same way the concatenation map $\gamma : \mathrm{Map}(8, M) \to LM$ defines a map at the level of Thom spectra

$$\gamma : \mathrm{Map}(8, M)^{ev^*(-TM)} \to LM^{ev^*(-TM)}. \tag{4.29}$$

Thus, from Equations 4.28 and 4.29 we get

$$\mu : LM^{ev^*(-TM)} \wedge LM^{ev^*(-TM)} \xrightarrow{\tau} \mathrm{Map}(8, M)^{ev^*(-TM)} \xrightarrow{\gamma} LM^{ev^*(-TM)} \tag{4.30}$$

that is associative up to homotopy.

Let $\sigma : M \to LM$ be a section of the evaluation map, it is, points in M can be interpreted as constant loops. Then, it can induce a map, denoted also by σ, at the level of Thom spectra

$$\sigma : M^{-TM} \to LM^{-TM}$$

and join with the unit of the ring spectrum of M^{-TM} defined in the previous section by $j : S^0 \to M^{-TM}$, we can define a unit for LM^{-TM} by the composition

$$i : S^0 \xrightarrow{i} M^{-TM} \xrightarrow{\sigma} LM^{-TM}.$$

Thus, we conclude Theorem 4.3. In addition, for M an oriented manifold we have following commutative diagram

$$\begin{tikzcd}
H_{q-2d}(LM^{-TM}\wedge LM^{-TM}) \arrow[r, "\mu_*"] \arrow[d, "\cong"'] & H_{q-2d}(LM^{-TM}) \arrow[d, "\cong"] \\
H_q(LM\times LM) \arrow[r, "\circ"'] & H_{q-d}(LM)
\end{tikzcd}$$

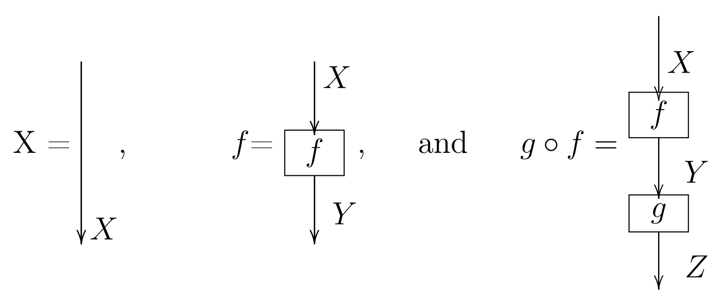
which again realizes the string topology.

Appendix A

String diagrams

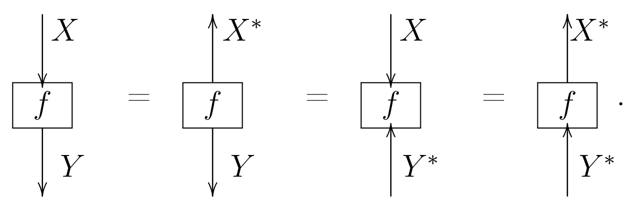
An important tool used in monoidal categories is called *string diagrams*, it is a way to represent morphisms and objects as union of arrows and nodes. These diagrams, which are sometimes also called Penrose diagrams, have their origins in the work of Roger Penrose in physics. Later Turaev started to use those diagrams to study braid monoidal categories, that is monoidal categories with "weaker" symmetry condition. String diagrams give us intuition and make some arguments obvious as we have seen in Chapter 2. In this appendix we are going to give a brief overview of string diagrams, for a detailed treatment the reader can consult Turaev [39].

First of all, let us fix a convention for this work, there are many approaches and there is not a common agreement of which is the correct one, all of them are correct and equivalent to each other. The author of this thesis has chosen to draw the diagrams vertically from top to bottom, that is, the source is in the top and the target is in the bottom. Thus, given a category $\mathcal{C}$, the identity id_X of an object X of $\mathcal{C}$, a morphism $f : X \to Y$ in $\mathcal{C}$, and the composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ may be graphically represented as follows:

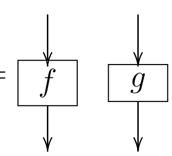


where X, Y, Z are objects in $\mathcal{C}$.
We sometimes omit the object labels (e.g. X and Y above) when they are clear or unimportant.

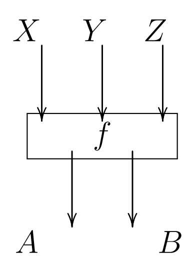
We will consider oriented diagrams, that is, with a given an orientation in the arrows: downwards arrows represent the duals of the upwards arrows. Thus, we will denote the morphism $f : X \to Y$ by the following equivalent diagrams:



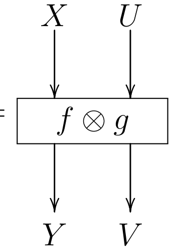
The monoidal product of two morphisms $f : X \to Y$ and $g : U \to V$ in $\mathcal{C}$ is represented by juxtaposition:

$f \otimes g =$ 

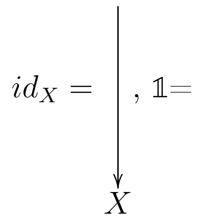
We can also use boxes with several strands attached to their horizontal sides. For example, a morphism $f : X \otimes Y \otimes Z \to A \otimes B$ with $X, Y, Z, A, B \in Ob(C)$ may be represented in various ways, such as:



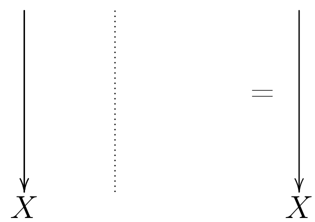
With the above convention the tensor product also can be represented as

$f \otimes g =$ 

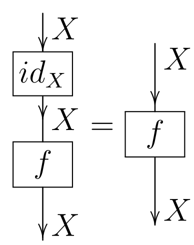
Some objects and morphisms are special; we draw them more simply. The identity morphism of an object X we will draw just as arrow without any vertex.



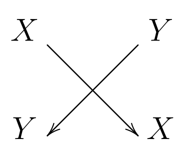
the unit object will be denoted as the empty arrow. The product with unit $X \otimes \mathbb{1} \cong X$ is denoted by:



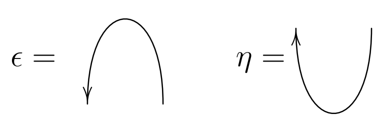
The composition of $f : X \to Y$ with identity $id_X : X \to X$, $f \circ id_X = f$ can be simplified as:



The symmetry condition $X \otimes Y \cong Y \otimes X$ can be represented by



We will omit the unit object, so the evaluation and co-evaluation morphisms can be noted as follows:

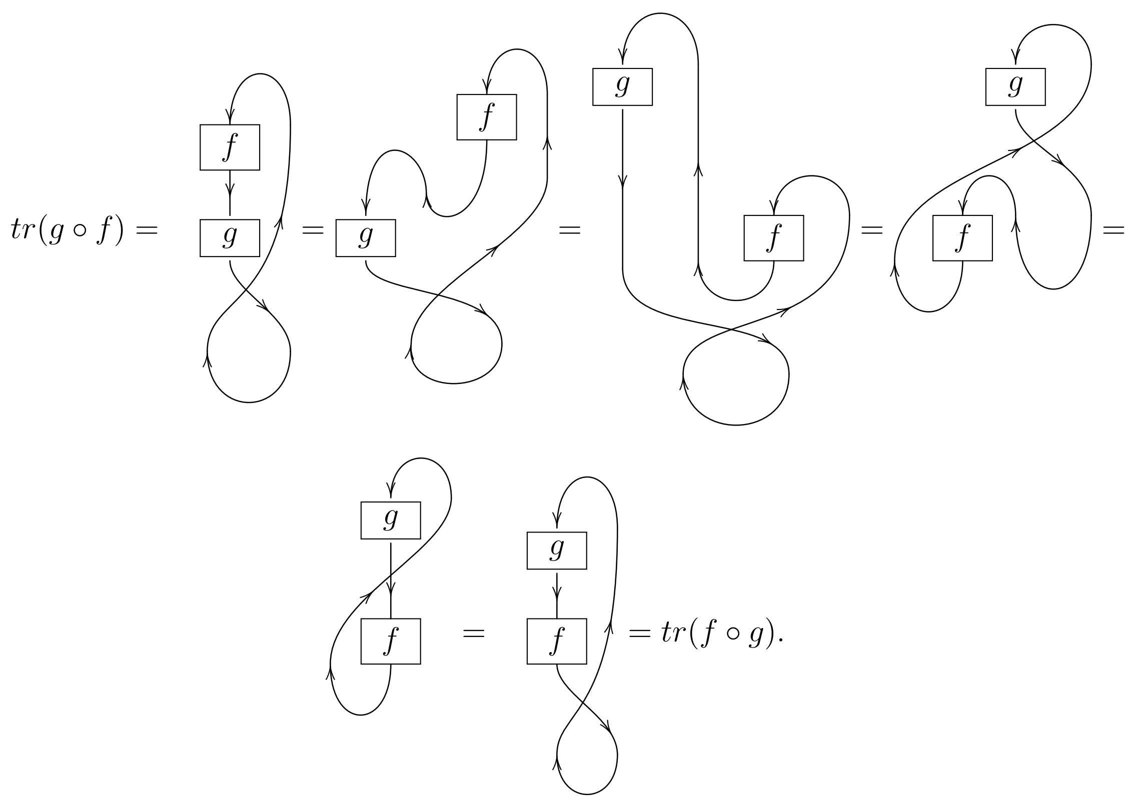


Theorem A.1. ***(Coherence theorem)***
Any two string diagrams which are topologically equivalent represent equal morphisms in a symmetric monoidal category.

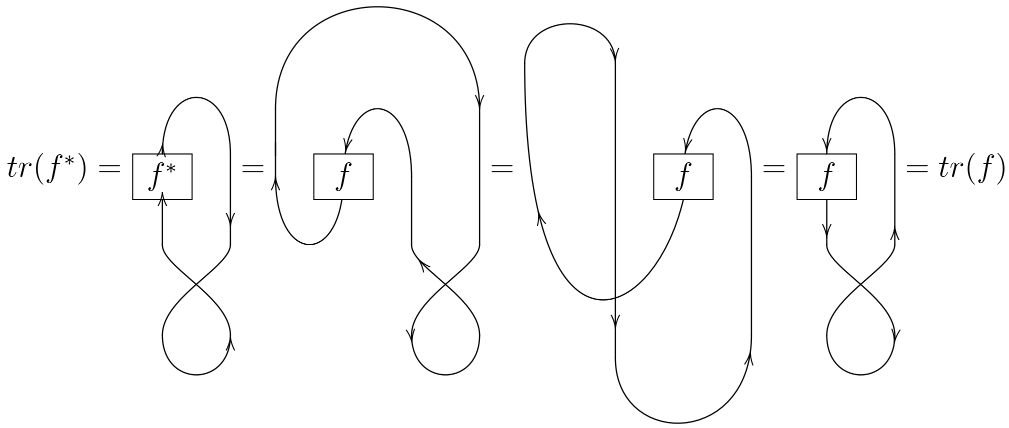
By "topologically equivalent", we mean that two diagrams, drawn in a rectangle in the plane with incoming and outgoing wires attached to the boundaries of the rectangle, are equivalent if it is possible to transform one to the other by continuously moving, without allowing boxes or wires to cross each other or to be detached from the boundary of the rectangle during the move. For precise definitions and a proof of the coherence theorem, see Joyal and Street [23].

The formulas presented in Chapter 2 were obtained from manipulating string diagrams. Each of these equations can be described in terms of diagrams. Let us see a different proof of Lemmas 2.12 and 2.13

Observation. The following sequence of diagrams represent an alternative proof of Lemma 2.12.

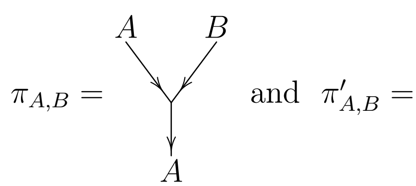


Observation. The following sequence of diagrams represents an alternative proof of the Proposition 2.13.

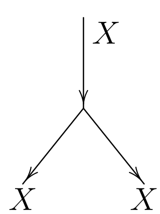


Let us describe some different kinds of diagrams useful for cartesian categories, their description is fundamental for many proofs presented in Chapter 2. For this category, we will represent maps of the form $f : A \times X \to X \times B$ in the same way that for maps in a symmetric monoidal category we exchange $\times$ by $\otimes$.

In cartesian categories we have two kinds of special maps the diagonal map $\Delta_X : X \to X \times X$ and the projection maps which will be denoted by $\pi_{A,B} : A \times B \to A$ and $\pi'_{A,B} : A \times B \to B$ the projections to the first and second component, respectively. Then



And the diagonal map can be represented as



A.1 Important identities

We have seen a description of basic string diagrams and relations. Through this work we presented proofs only using string diagrams, those proofs required identities that were not proved in the respective sections. So, we have dedicated this section to explain how to prove those identities using string diagrams.

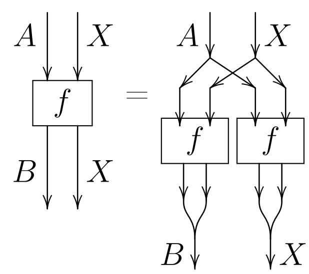
Consider the following lemmas, they are used in Chapter 2 to prove the bijective relation given by Hasegawa.

Lemma A.2. *Let $\mathcal{C}$ be a traced monoidal category, for $f : A \times X \to B \times X$ we have*

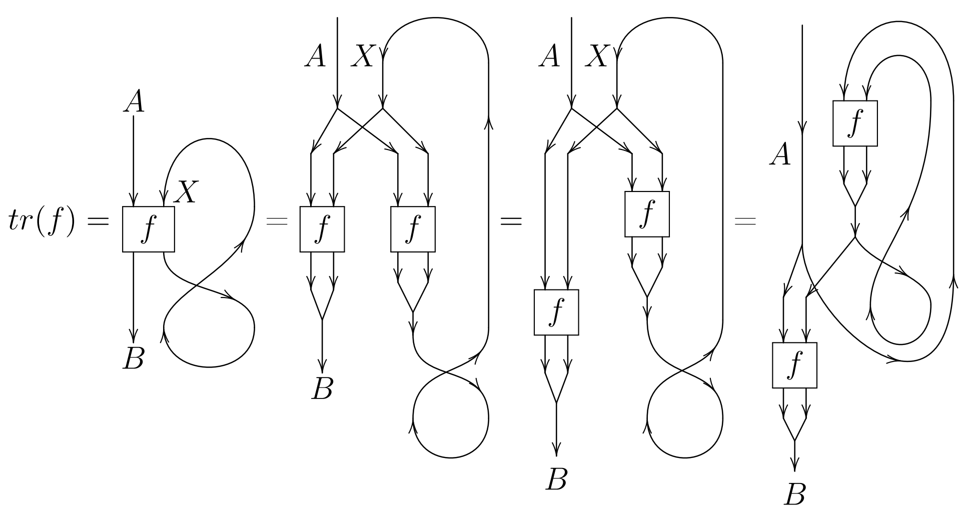
$$tr(f) = \pi_{B,X} \circ f \circ (id_A \times tr(\Delta_X \circ \pi'_{B,X} \circ f)) \circ \Delta_A : A \to B \tag{A.1}$$

where $\pi_{B,X} : B \times X \to B$ and $\pi'_{B,X} : B \times X \to X$ are the projections to the first and second component respectively.

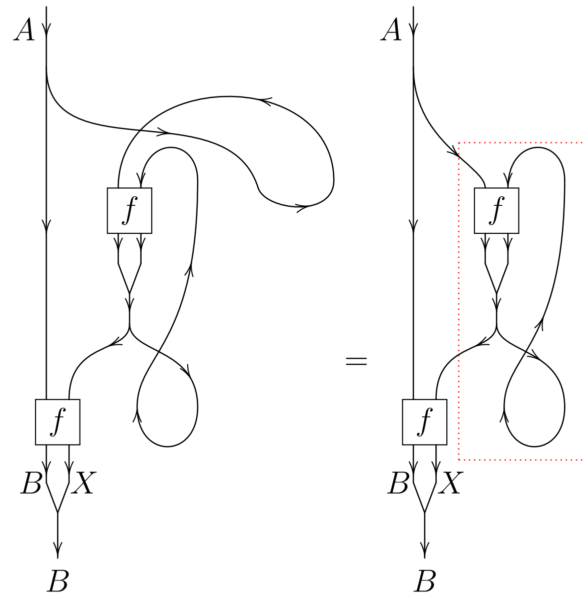
Proof. First of all, we need to note that f can be represented as



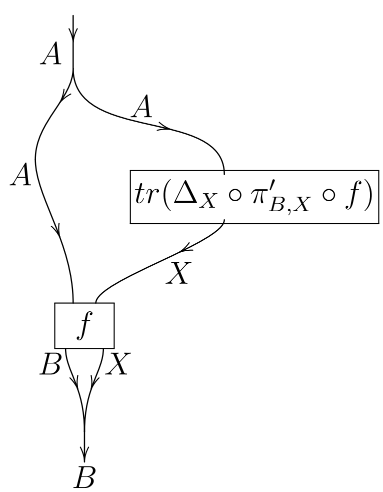
Using the above description of f we can rewrite the trace as follows



The second equality follows by the above description. The third equality is given by Proposition 2.15, and the last equality holds by naturality in X, see Proposition 2.16. If we continue transforming the diagram we get



The first equality holds by yanking the loop on the right corner. Now we can simplify the diagram inside the red rectangle as $tr(\Delta_X \circ \pi'_{B,X} \circ f) : A \to X$ then we get the following diagram



where the last diagram is $\pi_{B,X} \circ f \circ (id_A \times tr(\Delta_X \circ \pi'_{B,X} \circ f)) \circ \Delta_A : A \to B$ then we can conclude the lemma. $\square$

Lemma A.3. *Let* $()^\dagger : \mathcal{C}(A \times X, B \times X) \to \mathcal{C}(A, B)$ *be a fixed point operator which satisfies properties* $1 \sim 4$ *of Definition 25. Then, for* $f : A \times X \to B \times X$,

$$tr^X(f) = \pi_{B,X} \circ f \circ (id_A \times (\pi'_{B,X} \circ f)^\dagger) \circ \Delta_A. \tag{A.2}$$

Proof. For this lemma it is most convenient to work with equations. Thus, by property

1 we get

$$
\begin{aligned}
tr^x(f) &= \pi_{B,X} \circ (f \circ (id_A \times \pi'_{B,X}))^\dagger \\
&= \pi_{B,X} \circ f \circ (id_A \times \pi'_{B,X}) \circ (id_A \times (f \circ (id_A \times \pi'_{B,X}))^\dagger) \circ \Delta_A \\
&= \pi_{B,X} \circ f \circ (id_A \times (\pi'_{B,X} \circ (f \circ (id_A \times \pi'_{B,X}))^\dagger)) \circ \Delta_A
\end{aligned}
$$

by Property 3 we can conclude the result. □

Lemma A.4. ***(Bekic Lemma)***
Let A, X and Y be objects in $\mathcal{C}$, a cartesian category, and $f : A \times X \times Y \to X$ and $g : A \times X \times Y \to Y$, then the maps

$$((f \times g) \circ \Delta_{X \times Y})^\dagger : A \to X \times Y \tag{A.3}$$

$$(\pi'_{x,A} \times g^\dagger) \circ \Delta_{A \times X} \circ (id_A \times (f \circ (id_{A \times X} \times g^\dagger) \circ \Delta_{A \times X})^\dagger) \circ \Delta_A : A \to X \times Y. \tag{A.4}$$

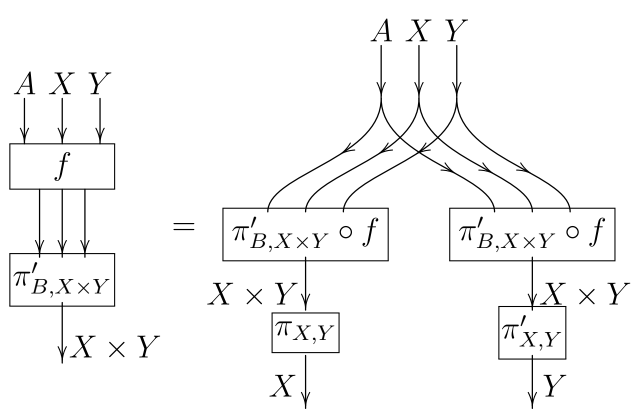
are the same.

Proof. It can be proved with the properties $1 \sim 4$ of Definition 25. The reader can consult the proof in [**?**]. □

Lemma A.5. *Let $f : A \times X \times Y \to B \times X \times Y$ be a map in a cartesian category $\mathcal{C}$. Then,*

$$\pi'_{B,X \times Y} \circ f = ((\pi'_{B,X} \circ \pi_{B \times X,Y} \circ f) \times (\pi'_{B \times X,Y} \circ f)) \circ \Delta_{A \times X \times Y}.$$

Proof. Using the description of a function given in the proof of Lemma A.2 we get



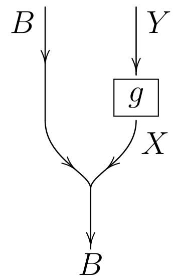
In addition, note that $\pi_{X,Y} \circ \pi'_{B,X \times Y} = \pi'_{B,X} \circ \pi_{B \times X,Y}$ and $\pi'_{X,Y} \circ \pi'_{B,X \times Y} = \pi'_{B \times X,Y}$, then the conclusion holds. □

Lemma A.6. *Let $g : Y \to X$ be a map and B an object in $\mathcal{C}$, a cartesian category, then:*

$$\pi_{B,X} \circ (id_B \times g) = \pi_{B,Y}. \tag{A.5}$$

Proof. Let us consider the following string diagram deformation

$$\pi_{B,X} \circ (id_B \times g) =$$



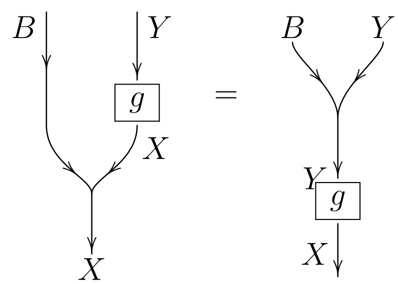
$$= \pi_{B,Y}$$

It is due to the fact that projection $\pi_{B,X}$ deletes the information of the second component, and both maps have the same input and output. □

Lemma A.7. *Consider the conditions as in the previous lemma (A.6), then*

$$\pi'_{B,X} \circ (id_B \times g) = g \circ \pi'_{B,Y}. \tag{A.6}$$

Proof. Let us consider the following string diagram deformation



It is due thanks to the second projection because the information given by B is deleted. □

Bibliography

[1] R Street A, royal and D Verity. Traced monoidal categories.

[2] D. Puppe A. Dold. Duality, trace, and transfer. 1980.

[3] J.F. Adams. A variant of e. h. brown's representability theorem. *Topology*, 10(3):185 – 198, 1971.

[4] J.F. Adams. *Stable Homotopy and Generalised Homology*. Chicago Lectures in Mathematics. University of Chicago Press, 1974.

[5] M. F. Atiyah. Thom complexes. *Proc. London Math. Soc.*, 11(5):291–310, 1961.

[6] Michael F. Atiyah. Topological quantum field theory. *Publications Mathématiques de l'IHÉS*, 68:175–186, 1988.

[7] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36(11):6073–6105, Nov 1995.

[8] Nick Benton and Martin Hyland. Traced premonoidal categories. *ITA*, 37(4):273–299, 2003.

[9] Stephen L. Bloom and Zoltán Ésik. Fixed-point operations on ccc's. part i. *Theoretical Computer Science*, 155(1):1 – 38, 1996.

[10] Raoul Bott and Loring W. Tu. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer New York, 1990.

[11] G.E. Bredon. *Topology and Geometry*. Graduate Texts in Mathematics. Springer, 1993.

[12] Edgar H. Brown. Cohomology theories. *Annals of Mathematics*, 75(3):467–484, 1962.

[13] M. Chas and D. Sullivan. String Topology. *arXiv Mathematics e-prints*, November 1999.

[14] Joel M. Cohen. Stable homotopy. 1970.

[15] Ralph L. Cohen. Multiplicative properties of atiyah duality. *Homology Homotopy Appl*, pages 269–281.

[16] Ralph L. Cohen and John D. S. Jones. A homotopy theoretic realization of string topology. *arXiv Mathematics e-prints*, page math/0107187, Jul 2001.

[17] A. Dold. *Lectures on Algebraic Topology*. Classics in Mathematics. Springer Berlin Heidelberg, 1995.

[18] Victor Guillemin and Alain Pollack. *Differential Topology*. Graduate Texts in Mathematics. Springer New York, 1990.

[19] M. Hasegawa. Models of sharing graphs: A categorical semantics of let and letrec.

[20] M. Hasegawa. Recursion from cyclic sharing: traced monoidal categories and models of cyclic lambda calculi. 1997.

[21] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[22] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *arXiv Mathematics e-prints*, page math/9801077, Jan 1998.

[23] André Joyal and Ross Street. The geometry of tensor calculus, i. *Advances in Mathematics*, 88(1):55 – 112, 1991.

[24] J. Kock. *Frobenius Algebras and 2-D Topological Quantum Field Theories*. Frobenius Algebras and 2D Topological Quantum Field Theories. Cambridge University Press, 2004.

[25] E.L. Lima. *Duality and Postnikov Invariants*. University of Chicago, 1994.

[26] Jacob Lurie. On the classification of topological field theories. *Current Developments in Mathematics*, 2008:129–280, 2009.

[27] Cary Malkiewich. The stable homotopy category. `http://people.math.binghamton.edu/malkiewich//stable.pdf`.

[28] Harvey Margolis. Spectra and the steenrod algebra. 1983.

[29] W.S. Massey. *Algebraic Topology: An Introduction*. Graduate Texts in Mathematics. Springer New York, 1990.

[30] W.S. Massey. *Differential forms in Algebraic Topology*. Graduate Texts in Mathematics. Springer New York, 1990.

[31] J.R. Munkres. *Elements of algebraic topology.* Advanced book program. Addison-Wesley, 1984.

[32] J.R. Munkres. *Topology.* Featured Titles for Topology Series. Prentice Hall, Incorporated, 2000.

[33] Kate Ponto and Michael Shulman. Shadows and traces in bicategories. *Journal of Homotopy and Related Structures*, 8(2):151–200, Oct 2013.

[34] Kate Ponto and Michael Shulman. Traces in symmetric monoidal categories. *Journal of Homotopy and Related Structures*, 8(2):151–200, Oct 2013.

[35] Daniel G. Quillen. *Homotopical algebra.* Lecture Notes in Mathematics. Springer, 1 edition, 1967.

[36] Graeme Segal. *The definition of conformal field theory*, page 421–422. London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.

[37] Alex Simpson and Gordon Plotkin. Complete axioms for categorical fixed-point operators. In *Proceedings of the 15th Annual IEEE Symposium on Logic in Computer Science*, LICS '00, pages 30–, Washington, DC, USA, 2000. IEEE Computer Society.

[38] R.M. Switzer. *Algebraic Topology: Homotopy and Homology.* Classics in Mathematics. Springer, 2002.

[39] V. Turaev and A. Virelizier. *Monoidal Categories and Topological Field Theory.* Progress in Mathematics. Springer International Publishing, 2017.

[40] George W. Whitehead. Generalized homology theories. *Trans. Amer. Math. Soc.*, 102:227–283, feb 1962.

[41] Edward Witten. Topological quantum field theory. *Comm. Math. Phys.*, 117(3):353–386, 1988.