NONSTANDARD METHODS FOR
NAVIER–STOKES EQUATIONS

NIGEL J. CUTLAND AND MAREK CAPIŃSKI

INTRODUCTION

In this paper we give a survey of recent joint work of the authors in which methods from nonstandard analysis are used to provide a new approach to the solution of the Navier-Stokes equations. These methods provide relatively easy and intuitive proofs of the classical existence results for deterministic Navier--Stokes equations [5], and, as an almost immediate corollary, the construction of statistical solutions, which was originally achieved by Foias using a somewhat lengthy argument [11]. Finally we mention some new results concerning solutions to the stochastic Navier-Stokes equations [6], [7] which solve a problem that has been outstanding for many years.

We will only give sketches of the main ideas of the proofs; for full details, see the original papers or the forthcoming book [10]. In §2 we will provide a very brief introduction to nonstandard analysis, and we indicate references where the reader can find a full exposition of this powerful technique.

§1. THE NAVIER-STOKES EQUATIONS

The classical Navier-Stokes equations describe the evolution in time of the velocity field $u : D \rightarrow \mathbb{R}^n$ of an incompressible fluid in a domain $D \subseteq \mathbb{R}^n$, so we consider a function $u : D \times (0, \infty) \rightarrow \mathbb{R}^n$ given by:

$$\frac{\partial u}{\partial t} - \nu \Delta u + \langle u, \nabla \rangle u + \nabla p = f \qquad (NS)$$

$$div\, u = 0.$$

(where $\langle , \rangle$ denotes the inner product in $\mathbb{R}^n$). There is a variety of possible boundary conditions that can be considered; in this discussion we impose the homogeneous Dirichlet boundary condition $u(x, t) = 0$ for $x \in \partial D$, which we assume to be of class C^2.

For applications the important case is $n = 3$ but mathematically we can

allow $n \leq 4$. In this equation, p denotes the pressure, and f denotes the external forces.

The usual setting for these equations involves the function spaces **H**, **V**, which are obtained by closing the set $\{u \in C_0^\infty(D, \mathbb{R}^n) : div\, u = 0\}$ in the norms $\| \cdot \|$ and $\| \cdot \| + \| \cdot \|$ respectively, where

$$\|u\| = (u, u)^{\frac{1}{2}} \qquad (u, v) = \sum_{j=1}^n \int_D u^j(x) v^j(x) dx$$

$$\|u\| = ((u, u))^{\frac{1}{2}} \qquad ((u, v)) = \sum_{j=1}^n \left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right)$$

H, V are Hilbert spaces, and there is an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ for **H** consisting of eigenvectors of the operator $-\Delta$ suitably extended to an operator A on **H**; i.e. $Au = \sum \lambda_n u_n e_n$ where $u_n = (u, e_n)$ and $\Delta e_n = -\lambda_n e_n$. In the equation (NS) it is usual to take the force $f \in L^2(0, T; \boldsymbol{V}')$ for each $T < \infty$, and then the equation itself is understood as a Bochner integral equation in $\boldsymbol{V}'$. i.e. for each $v \in \boldsymbol{V}$:

$$(u(t), v) - (u(0), v) = \int_0^t -\nu((u(s), v)) - b(u(s), u(s), v) + (f(s), v) ds \qquad (1)$$

where $(\cdot, \cdot)$ denotes the duality between $\boldsymbol{V}'$ and $\boldsymbol{V}$ extending the scalar product in **H**. The pressure p has disappeared from (1) because $(\nabla p, v) = p\, div\, v = 0$ for $v \in \boldsymbol{V}$, so $\nabla p = 0$ in $\boldsymbol{V}'$. The trilinear form b is used to denote the nonlinear term in (NS):

$$b(u, v, w) = \sum_{i,j=1}^n \int_D u^j(x) \frac{\partial v^i}{\partial x_j}(x) w^i(x) dx = ((u, \nabla)v, w).$$

Notice that $b(u, w, v) = -b(u, v, w)$, so $b(u, v, v) = 0$.

The difficulties in solving the Navier-Stokes equations arise from the nonlinear term $b(u, u, v)$ in (1); the construction of solutions (by any method) depends on delicate continuity properties of b in a variety of norms. For our purposes the crucial property is as follows. We write $B(u) = b(u, u, \cdot)$ viewed as an element of $\boldsymbol{V}'$ (i.e. $(B(u), v) = b(u, u, v)$) and for $m \in \boldsymbol{N}$ let K_m be the compact subset of $\boldsymbol{H}$ given by $K_m = \{u : \|u\| \leq m\}$. Then

Lemma 1.1. $B : K_m \to \boldsymbol{V}'$ is continuous with respect to the the $\boldsymbol{H}$-topology on K_m and the weak topology of $\boldsymbol{V}'$.

The second key factor in obtaining solutions for Navier-Stokes is the use of energy estimates. Proceeding informally, if u is a solution to (1) with sufficient regularity then

$$\frac{1}{2} \frac{d\|u(t)\|^2}{dt} = \left(u(t), \frac{du(t)}{dt} \right) = -\nu \|u(t)\|^2 + (f(t), u(t))$$

(since $b(u, u, u) = 0$), from which it is easy to derive the following *energy estimate* :

$$\|u(s)\|^2 + \nu \int_0^s \|u(t)\|^2 dt \leq \|u(0)\|^2 + \frac{1}{\nu} \int_0^s \|f(t)\|_{V'}^2 dt$$

which might be expected of a solution (yet to be constructed). This leads to the following definition :

Definition. Given $u_0 \in H$ and $f \in L^2(0, T; V')$ for all $T < \infty$, a *weak solution* to the Navier-Stokes equations is a function $u : (0, \infty) \to H$ such that equation (1) holds for each $v \in V$ and $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$ for all $T < \infty$.

Nonstandard analysis allows us to formulate a *hyperfinite* version of (1), so that (by transfer) all the techniques of finite dimensional ODE's are available to solve it. In order to explain this we now give a brief introduction to the basic ideas of nonstandard analysis.

§2. What is Nonstandard Analysis?

Standard (real) analysis may be viewed as the study of the structure

$$\underline{R} = ((R, +, \times, <, (f)_{f \in \mathcal{F}}, (S)_{S \in \mathcal{S}})$$

where $\mathcal{F}$ and $\mathcal{S}$ denote the sets of *all* possible real functions $f : \mathbb{R}^m \to R$ and relations $S \subseteq \mathbb{R}^m$ for all m. The basic construction of nonstandard analysis is to extend R to a larger field $^*R$, the *hyperreals* or *nonstandard reals*, which contains both infinitesimal and infinite members. At the same time all functions and relations are extended to $^*R$ so we have the structure

$$^*\underline{R} = (^*R, ^*+, ^*\times, ^*<, (^*f)_{f \in \mathcal{F}}, (^*S)_{S \in \mathcal{S}})$$

with the following properties:

1) $^*R \supset R$;
2) $^*f$ extends f and $S = \mathbb{R}^m \cap ^*S$ for every f and S;
3) $^*R$ and R have exactly the same properties; i.e. properties that can be expressed in the language of first order propositional logic. (For example, $(^*R, ^*+, ^*\times, ^*<)$ is an ordered field since $(R, +, \times, <)$ is.)

The property 3) is known as the *transfer principle*, and is the key to the whole of nonstandard analysis.

There are several ways to construct $^*R$, perhaps the simplest being an explicit ultrapower construction setting

$$^*\underline{R} = \underline{R}^{\mathbb{N}/\mathcal{U}}$$

where $\mathcal{U}$ is an ultrafilter on the natural numbers $\mathbf{N}$.

The following definitions are fundamental:

Definitions. Let $x, y \in {}^*\mathbf{R}$.

(i) x is *infinitesimal* if $\|x\| < \frac{1}{n}$ for all $n \in \mathbf{N}$;

(ii) $x \approx y$ if $x - y$ is infinitesimal;

(iii) x is *finite* if $\|x\| < n$ for some $n \in \mathbf{N}$;

(iv) x is *infinite* if x is not finite.

(Strictly in these definitions we should have used ${}^*\|x\|$, ${}^*<$, etc. but since ${}^*f$ extends f for any function, we can often safely drop the * without ambiguity.)

The next result is crucial to the whole methodology - it allows results in ${}^*\mathbf{R}$ to be interpreted in $\mathbf{R}$. It follows from (and is in fact equivalent to) the completeness of the reals $\mathbf{R}$.

2.1 Standard Part Theorem. *If $x \in {}^*\mathbf{R}$ is finite, there is a unique $r \in \mathbf{R}$ with $r \approx x$. This r is called the standard part of x, and is denoted either $st(x)$ or ${}^{\circ}x$.*

Here is an illustration of nonstandard analysis, showing how the presence of infinitesimals allows a very intuitive (but rigorous) development of analysis.

2.2 Theorem. *Let $f : \mathbf{R} \to \mathbf{R}$ and $c \in \mathbf{R}$. Then f is continuous at c if and only if*

$$x \approx c \;\Rightarrow\; {}^*f(x) \approx {}^*f(c).$$

From this result we get easy derivations of the algebra of continuous functions.

The construction of ${}^*\mathbf{R}$ from $\mathbf{R}$ can be repeated for any given mathematical structure - and in fact there is a way to do it for all structures at once in a way that preserves interrelationships between structures. This results in a *nonstandard universe* ${}^*\mathcal{V}$ extending the standard universe $\mathcal{V}$, such that for every mathematical object $M \in \mathcal{V}$ there is a nonstandard counterpart ${}^*M \in {}^*\mathcal{V}$ having the "same" properties. For this, and a detailed exposition of nonstandard analysis we recommend the reader to consult one of the references [1], first part, [12], [15].

For our purposes, we need the nonstandard versions ${}^*\mathbf{H}$ and ${}^*\mathbf{V}$ of $\mathbf{H}$ and $\mathbf{V}$ etc. By the transfer principle, ${}^*\mathbf{H}$ has an orthonormal basis $\{{}^*e_1, {}^*e_2, \ldots {}^*e_N, {}^*e_{N+1}, \ldots\}$, and taking an infinite natural number $N \in {}^*\mathbf{N}$ (i.e. $N \in {}^*\mathbf{N} \setminus \mathbf{N}$) we have the subspace

$$H_N = span\{{}^*e_1, {}^*e_2, \ldots {}^*e_N\}$$

which is a *hyperfinite* dimensional subspace of ${}^*\mathbf{H}$. For convenience we will write $E_k = {}^*e_k$ for $k \in {}^*\mathbf{N}$. If $U \in H_N$ we have, writing $U_k = (U, E_k)$ for $1 \le k \le N$

$$U = \sum_{k=1}^{N} U_k E_k \quad and \quad \|U\|^2 = \sum_{k=1}^{N} U_k^2.$$

If $\|U\|$ is finite, then U_k is finite for all k, so the following definition makes sense.

Definition. For $U \in \boldsymbol{H}_N$ with $\|U\|$ finite, the *standard part* of U, denoted $u = {}^\circ U$ or $u = \mathrm{st}(U)$, is given by $u_k = {}^\circ U_k$ for each finite k.

Thus we have a *standard part mapping* ${}^\circ = \mathrm{st} : Fin(\boldsymbol{H}_N) \rightarrow \boldsymbol{H}$, where $Fin(\boldsymbol{H}_N)$ denotes $\{U \in \boldsymbol{H}_N : \|U\| \ is \ finite\}$. It is clear that $\|u\| = \|{}^\circ U\| \leq {}^\circ\|U\|$.

One further tool used widely in applications of nonstandard analysis is *Loeb measures*. Briefly, if $(X, \mathcal{A}, \mu)$ is a nonstandard measure space we can define ${}^\circ\mu : \mathcal{A} \rightarrow 0, \infty$ by ${}^\circ\mu(A) = {}^\circ(\mu(A))$ (defining ${}^\circ x = \infty$ if $x \in {}^*\boldsymbol{R}$ is positive infinite). Loeb [16] showed that ${}^\circ\mu$ has a unique extension to a standard measure on the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. This extension is the *Loeb measure* μ_L given by μ, and results in the standard measure space $(X, \sigma(\mathcal{A}), \mu_L)$. Instances of this that we use here are $({}^*(0, \infty), ({}^*m)_L)$, where ${}^*m$ is the nonstandard version of Lebesgue measure on $(0, \infty)$, and (in §5) we use the Loeb measure obtained from a nonstandard Wiener measure on a hyperfinite dimensional space.

§3. NONSTANDARD SOLUTION OF THE NAVIER-STOKES EQUATIONS

In this section we outline a nonstandard proof of the following classical result (see [17] for the standard approach).

Theorem 3.1. *For any $u_0 \in \boldsymbol{H}$ and $f : D \times (0, \infty) \rightarrow \mathbb{R}^n$ with $f \in L^2(0, T; \boldsymbol{V}')$ for all $T < \infty$, there exists a weak solution u to the Navier-Stokes equations with $u(0) = u_0$.*

Sketch of the Proof of Theorem 3.1 Using the nonstandard framework outlined in the previous section, we can write down the following N-dimensional *ODE for the time evolution $U(\tau)$ of an element $U \in \boldsymbol{H}_N$ for $0 \leq \tau \in {}^*\boldsymbol{R}$:

$$\frac{dU_k(\tau)}{d\tau} = -\nu\lambda_k U_k(\tau) - ({}^*B(U(\tau)), E_k) + F_k(\tau) \tag{2}$$

(recall that λ_k is the eigenvalue of $-\Delta$ corresponding to e_k), where $F_k(\tau) = ({}^*f(\tau), E_k)$, and we take as initial condition $U_k(0) = ({}^*u_0, E_k)$.

The transfer of the theory of ODE's immediately gives a unique nonstandard classical solution $U(\tau)$ to the equation (2), and consideration of the energy $\|U(\tau)\|^2$ shows that it satisfies

$$\sup_{\tau \leq T} \|U(\tau)\|^2 + \nu \int_0^T \|U(\tau)\|^2 d\tau \leq \|U(0)\|^2 + \frac{1}{\nu} \int_0^T \|f(t)\|_{\boldsymbol{V}'}^2 dt \tag{3}$$

for each finite T. So, for finite times $t \in \mathbf{R}$ we can *define*

$$u(t) = {}^{\circ}U(t)$$

and it is easy to see that

$$\sup_{t \leq T} \|u(t)\|^2 + \nu \int_0^T \|u(t)\|^2 dt \leq \|u_0\|^2 + \frac{1}{\nu} \int_0^T \|f(t)\|_{\mathbf{V}'}^2 dt$$

so that $u \in L^2(0, T; \mathbf{V}) \cap L^{\infty}(0, T; \mathbf{H})$. It remains to check that $u(t)$ satisfies the equation (1), and the steps are as follows. It is sufficient to check it for $v = e_k$, and we have

$$
\begin{aligned}
(u(t), e_k) - (u(0), e_k) &= {}^{\circ}(U_k(t) - U_k(0)) \\
&= {}^{\circ}\int_0^t \left(-\nu \lambda_k U_k(\tau) - ({}^*B(U(\tau)), E_k) + F_k(\tau) \right) d\tau \\
&= \int_0^t \left(-\nu \lambda_k {}^{\circ}U_k(\tau) - {}^{\circ}({}^*B(U(\tau)), E_k) + {}^{\circ}F_k(\tau) \right) d_L \tau
\end{aligned}
$$

(using Loeb integration theory)

$$= \int_0^t -\nu(\!(u(s), e_k)\!) - b(u(s), u(s), e_k) + (f(s), e_k) ds$$

where the last step uses the continuity property of B (Lemma 1.1).

This method of proof works for dimensions $n \leq 4$ and significantly simplifies traditional existence proofs. The advantage lies in avoiding the use of specialised compactness theorems that have to be formulated carefully in advance in order to ensure convergence of sequences arising from finite dimensional approximations to (1).

In dimension $n \leq 2$ the solution to the Navier-Stokes equations is unique, and this can be established also by our techniques - see [9]. The question of uniqueness is open in dimensions $n \geq 3$, and if indeed this fails a possible explanation in our framework is given by examining perturbations of the initial data $U(0)$. If we take as initial condition $\overline{U}(0)$ with $\|\overline{U}(0) - U(0)\| \approx 0$ and solve (2) to obtain $\overline{U}(\tau)$, this yields another weak solution to the Navier-Stokes equations, $\overline{u}(t) = {}^{\circ}\overline{U}(t)$, with $\overline{u}(0) = u_0$, and we do not know whether $\overline{u} = u$. In [9] it is shown that a suficiently small infinitesimal perturbation of the initial data and the force f does allow a definition of a subclass of the solutions in which we do have uniqueness.

§4. STATISTICAL SOLUTIONS

In the previous section we assumed that the initial condition in equation (1) is a given point $u_0 \in \mathbf{H}$. For *statistical solutions* the initial condition is replaced

by an initial measure μ_0 on $\boldsymbol{H}$ and the idea is to find and solve an equation for a time evolving measure $\mu(t)$ on $\boldsymbol{H}$. The informal idea is that

$$\mu_0(A) = P(u(0) \in A)$$

for some underlying probability measure, and then

$$\mu_t(A) = P(u(t) \in A)$$
$$= \mu_0(S_t^{-1}(A))$$

for $t \geq 0$, where $A \subseteq \boldsymbol{H}$, and where $S_t^{-1}(u)$ is "the solution" of (1) at time t with initial condition $u \in \boldsymbol{H}$. This is problematic because the question of uniqueness for (1) is still open for $n > 2$. Nevertheless, using this informal idea Foias [11] obtained the following equation for the family of measures $(\mu_t)_{t \geq 0}$:

$$\int_{\boldsymbol{H}} \varphi(u) d\mu_t(u) - \int_{\boldsymbol{H}} \varphi(u) d\mu_0(u) =$$
$$\int_0^t \int_{\boldsymbol{H}} - \nu((u, \varphi'(u))) - b(u, u, \varphi'(u)) + (f(s), \varphi'(u)) d\mu_s(u) ds \qquad , (4)$$

where φ is any suitable test function. It is sufficient to consider only test functions of the form $\varphi(u) = \exp(i(u, v))$ with $v \in \boldsymbol{V}$. In [11] Foias solved (4) by a rather involved approximating procedure. The nonstandard approach described in §3 allows an easy proof that makes Foias' heuristic derivation entirely rigorous.

Note first that at the nonstandard level, equation (2) *does* have a unique (nonstandard) solution $S_\tau(U)$ for *any* initial condition $U \in \boldsymbol{H}_N$, which satisfies the energy inequality (3). For U with $\|U\|$ finite we can thus define $^\circ S$ by $^\circ S_t(U) = {}^\circ(S_t(U)) \in \boldsymbol{H}$ and the proof in §3 shows:

Theorem 4.1. *If $\|U\|$ is finite then the function $u(t) = {}^\circ S_t(U)$ is a weak solution to the Navier-Stokes equations with $u(0) = {}^\circ U$.*

From this we can write down a statistical solution in the following way. Suppose that μ is a given Borel probability measure on $\boldsymbol{H}$. This gives a standard probability measure $\hat{\mu}$ on the nonstandard space $\boldsymbol{H}_N$ by

$$\hat{\mu}(X) = (^*\mu^N)_L(X)$$

for $X \subseteq \boldsymbol{H}_N$, where $^*\mu^N$ is the projection of $^*\mu$ onto $\boldsymbol{H}_N$. It is a fundamental fact of Loeb measure theory that

$$\mu(A) = \hat{\mu}(st^{-1}(A))$$

for $A \subseteq \boldsymbol{H}$. It is now routine to establish:

Theorem 4.2. *Suppose that a Borel probability measure μ on $\boldsymbol{H}$ is given, with $\int \|u\|^2 d\mu < \infty$. Then the family of measures μ_t on $\boldsymbol{H}$ defined by*

$$\mu_t(A) = \hat{\mu}({}^\circ S_t^{-1}(A))$$

is a statistical solution to the Navier-Stokes equations, with $\mu_0 = \mu$. i.e. μ_t satisfies equation (4) and for all $T < \infty$ the function $t \mapsto \int_{\boldsymbol{H}} \|u\|^2 d\mu_t(u)$ is $L^\infty(0,T)$, and

$$\int_0^T \int_{\boldsymbol{H}} \|u\|^2 d\mu_t(u) dt < \infty.$$

The proof of this result can be found in [5].

§5. STOCHASTIC NAVIER-STOKES EQUATIONS

The general stochastic Navier-Stokes equations with full feedback take the form

$$du(t) = \nu\Delta u(t) - B(u(t)) + f(t, u(t))dt + g(t, u(t))dw(t) \qquad (5)$$

$$u(0) = u_0$$

where $u_0 \in \boldsymbol{H}$ and w is an infinite dimensional Wiener process on $\boldsymbol{H}$. The noise coefficient $g(t, u(t))$ is given by a suitable linear operator. Notice that we have allowed feedback in the forces f as well as in the new noise coefficient g, and of course this would have been possible in the deterministic equation (1). It is implicit that u given by (5) is now a stochastic process - i.e. $u(t) = u(t, \omega)$ for ω belonging to some underlying probability space. The equation (5) is a nonlinear SPDE, considered as an integral equation in $\boldsymbol{V}'$ using the Bochner integral for the drift terms and the stochastic integral of Ichikawa [13]; it is not susceptible to conventional SPDE methods. The case $g \equiv 1$ with no feedback was considered in [4] and later in [18], and here a pathwise solution is possible in principle. No progress on the case of general noise coefficient was made for 18 years, when the results [3], [6] and [7] were obtained. In the paper [3] Bensoussan deals with dimension $n = 2$, whereas the nonstandard methods developed in [6] and [7] handle all dimensions $n \leq 4$ in full generality. The approach is similar to that used for the deterministic equations in §3, and we now sketch the main points.

The nonstandard universe contains the nonstandard Wiener process $*w$ whose projection onto $\boldsymbol{H}_N$ we denote by W. This lives on a nonstandard probability space (Ω, μ), say. Using W we can write down a nonstandard SDE for a stochastic process $U(\tau, \omega) \in \boldsymbol{H}_N$, which is the stochastic counterpart of (2):

$$dU(\tau) = (-\nu A^N U(\tau) - B^N(U(\tau)) + F(\tau, U(\tau))d\tau + G(\tau, U(\tau))dW(\tau) \qquad (6)$$

$$U(0) = U_0$$

where A^N, B^N, F, G, U_0 are the projections of $^*A$, $^*B$, $^*f$, $^*g$, $^*u_0$ onto H_N. The transfer principle applied to the standard theory of finite dimensional SDE's ensures that (6) has a unique nonstandard solution on the space (Ω, μ), and a careful investigation of the energy evolution shows that (with the conditions on f, g as in the theorem below):

$$E\left(\sup_{\tau \le T} \|U(\tau)\|^2 + \int_0^T \|U(\tau)\|^2\right) < \infty$$

for all finite T.

From this it is routine to define a stochastic process u by

$$u(t, \omega) = {}^\circ U(t, \omega)$$

for $0 \le t \in \mathbf{R}$ and $\omega \in \Omega$, on the probability space (Ω, P), where P is the Loeb measure μ_L. To check that u is a solution to (5), for the drift part the steps are the same as for the deterministic equation. For the stochastic integral, the steps are similar, and follow the same pattern as in the pioneering work on stochastic integrals on Loeb spaces by Anderson [2] and Keisler [14]. Thus we have the following existence theorem [6].

Theorem 5.1. *Suppose that $u_0 \in \mathbf{H}$ and the functions $f : (0, \infty) \times \mathbf{V} \to \mathbf{V}'$ and $g : (0, \infty) \times \mathbf{V} \to \mathcal{L}(\mathbf{H}, \mathbf{H})$ are jointly measurable with the following properties:*

  (i) *$f(t, \cdot) \in C(K_m, \mathbf{V}'_{weak})$ for all m,*
  (ii) *$g(t, \cdot) \in C(K_m, \mathcal{L}(\mathcal{H}, \mathcal{H})_{weak})$ for all m,*
  (iii) *$\|f(t, u)\|_{\mathbf{V}'} + \|g(t, u)\|_{\mathcal{H}, \mathcal{H}} \le a(t)(1 + \|u\|)$ for some a with $a \in L^2(0, T)$ for all T.*

Then equation (5) has a solution u such that for all $T < \infty$

$$E\left(\sup_{t \le T} \|u(t)\|^2 + \int_0^T \|u(t)\|^2\right) < \infty.$$

It is quite straightforward to incorporate a random initial condition into our construction of a solution, and hence we can obtain statistical solutions for the stochastic Navier-Stokes equations also - see [8]. In [6] uniqueness of solutions is established for $n \le 2$ provided the coefficients f, g satisfy appropriate Lipschitz conditions. Similar results are obtained in [7] for a stochastic equation in which the noise term g is given by an unbounded operator of a particular kind. The method employed there is similar to that described above, but it is not covered by the general scheme of Theorem 5.1.

REFERENCES

1. Albeverio, S., Fenstad, J.E., Høegh-Krohn, R. & Lindstrøm, T.L., *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, New York, 1986.

2. Anderson, R.M., *A nonstandard representation for Brownian motion and Ito integration*, Israel J. Math. **25** (1976), 15–46.

3. Bensoussan, A., *A model of stochastic differential equation in Hilbert spaces applicable to Navier Stokes equation in dimension 2*, In : Stochastic Analysis, Liber Amicorum for Moshe Zakai, Eds. E. Mayer-Wolf, E. Merzbach & A. Schwartz (1991), 51–73, Academic Press.

4. Bensoussan A. & Temam, R., *Equations stochastique du type Navier-Stokes*, J. Funct. Anal. **2** (1973), 195–222.

5. Capiński, M. & Cutland, N.J., *A simple proof of existence of weak and statistical solutions of Navier-Stokes equations*, Proc. Royal Soc. Lond. A **436** (1992), 1-11.

6. Capiński, M. & Cutland, N.J., *Stochastic Navier-Stokes equations*, Acta Appl. Mathematicae **25** (1991), 59–85.

7. Capiński, M. & Cutland, N.J., *Navier-Stokes equations with multiplicative noise*, To appear in Nonlinearity.

8. Capiński, M. & Cutland, N.J., *Foias and Hopf statistical solutions for stochastic Navier-Stokes equations*, University of Hull Math. Research Reports.

9. Capiński, M. & Cutland, N.J., *A nonstandard approach to the uniqueness problem for the Navier-Stokes equations*, University of Hull Math. Research Reports.

10. Capiński, M. & Cutland, N.J., *Nonstandard Methods for Stochastic Fluid Mechanics*, World Scientific, to appear.

11. Foias, C., *Statistical study of Navier-Stokes equations I*, Rend. Sem. Mat. Univ. Padova **48** (1973), 219–348.

12. Hurd, A. & Loeb, P.A., *An Introduction to Nonstandard Analysis*, Academic Press, New York, 1985.

13. Ichikawa, A., *Stability of semilinear stochastic evolution equations*, J. Math. Anal. Appl. **90** (1982), 12–44.

14. Keisler, H.J., *An infinitesimal approach to stochastic analysis*, Mem. Amer. Math. Soc. **297** (1984).

15. Lindströ m, T.L., *An invitation to nonstandard analysis*, In: Nonstandard Analysis and its Applications, ed. N.J. Cutland. **LMS Student Texts 10** (1988), 1–105, Cambridge Univ. Press.

16. Loeb, P.A., *Conversion from nonstandard to standard measure spaces and applications in probability theory*, Trans. Amer. Math. Soc. **211** (1975), 113–122.

17. Temam, R., *Navier-Stokes Equations*, North-Holland, Amsterdam, 1979.

18. Vishik, M.I. & Fursikov, A.V., *Mathematical Problems of Statistical Hydromechanics*, Kluwer Academic Publishers, Dordrecht, London, 1988.

FIRST AUTHOR: UNIVERSITY OF HULL – ENGLAND.
SECOND AUTHOR: UNIVERSITY OF KRAKÓW – POLAND