UNIVERSIDAD NACIONAL DE COLOMBIA

# Sustainable Development Schemes: A Complex Network Application to Caldas Region

Esquemas de Desarrollo Sostenible:
Una Aplicación de Redes Complejas a la Región de Caldas

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A mi familia

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# Abstract

This work presents a mathematical model of ordinary differential equations (ODEs), in order to obtain the dynamical description of each one of the sustainability components (economy, social development and environment conservation), together with their dependence with demographic dynamics. Through the work, causal relationships between each of the components were obtained. Once the model was constructed, the steady state of the system was studied. Several dynamical behavior were found, such as codim 1 and 2 bifurcations and chaotic dynamics. Finally, an application of the model is presented for a specific geographical environment (Caldas region), through a complex networks approach.



**Keywords: Differential Equations, Nonlinear Dynamical Systems, Bifurcations, Sustainability, Economical Development, Demographic Dynamics, Complex Systems, Dynamical Networks**.



# Resumen

Este trabajo presenta un modelo matemático de ecuaciones diferenciales ordinarias (EDOs), para obtener la descripción dinámica de cada una de las compenentes de la sostenibilidad (economia, desarrollo social y conservacion del medio ambiente), junto con su dependencia con la dinámica demográfica. A través del trabajo se obtuvieron las relaciones causales entre cada una de las componentes. Una vez construido el modelo, se procedió al estudio del estado estacionario del sistema, en el cual se hallaron ricos comportamientos dinámicos, desde bifurcaciones de codimensión 1 y 2 hasta dinámicas caóticas. Finalmente, se presenta un aplicación del modelo a un entorno geográfico específico (región de Caldas) mediante una aprximación de redes complejas.



**Palabras clave: Ecuaciones Diferenciales, Sistemas Dinámicos No Lineales, Bifurcaciones, Sostenibilidad, Desarrollo Económico, Dinámica Demográfica, Sistemas Complejos, Redes Dinámicas**.

# Declaración

Me permito afirmar que he realizado la presente tesis de manera autónoma y con la única ayuda de los medios permitidos y no diferentes a los mencionados en la propia tesis. Todos los pasajes que se han tomado de manera textual o figurativa de textos publicados y no publicados, los he reconocido en el presente trabajo. Ninguna parte del presente trabajo se ha empleado en ningún otro tipo de tesis.

Manizales, 14.06.2012

David Angulo G.

(David Angulo García)

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# 1. Introduction

## 1.1. Overview

Society we live in is becoming more conscious of the need of preserving the environment. Sustainable Development schemes have grown rapidly as a tool for management, prediction and improvement of the path of growth in different regions and economy sectors. This work attempts to cover the study of a sustainability scheme and its application to a Complex Network system. In this chapter we shall introduce the concepts needed to establish a complete system and define the different tools that will be used through the work, i.e Dynamical Systems Theory and Graph Theory as well as a brief state of the art.

## 1.2. Sustainable Development Background

Over the past few decades (from 1980's) a growing attention regarding environmental issues has been observed. The exact definition of sustainable development is actually a present topic of discussion since it often depends on the field of study of each scientific or political community. Although many concepts about sustainability have been proposed [37], there is an extensive agreement with the definition stated by the so-called Brundtland report *Our common future*. The report, written by the *World Comission Environment and Development* (WCED) established the first formal definition of Sustainable Development (SD) in its meeting in 1987. SD was defined as the obligation to provide the needs of actual generation, without compromising ability of future generations to provide their owns. This implies a dynamical balance between maintenance (sustainability) and transformation (development), as well as harmony between **society, economy and ecosystem**. This definition considers several points:

- Future is not compromised by the present.
- Geographical areas are not compromised by other geographical areas.
- Human needs are provided within biological limits, while natural capital is kept and improved.
- A proactive effort is made in order to keep SD schemes, and eliminate those which are not.

- Sustainability is recognized as a dynamical concept, which can show many faces and cannot be judged by an unique value. [51]

With this aim, scientific communities have devoted important efforts to the study of the viability of sustainability in a mainly capitalist society. Results of mathematical forecast are encouraging in the sense that, under the proper control conditions, it is possible to guarantee environmental conservation as well as acceptable economic growth.
Works devoted to the study of sustainability usually consider two variables of interest. Namely the biomass contained in certain region in a period of time (environmental sustainability), and accumulated capital for manufacturing production concept (economical sustainability). These two variables, which are easily quantifiable, give a partial vision of the regional development yet do not provide a complete description of sustainability. To do so, a third monitoring variable must be introduced which accounts for the social sustainability (education, social welfare etc...). The harmonious coexistence between social, economic and environmental growth is what we will consider from now on, a sustainable scenario.

## 1.3. Mathematical Background

Through this section we will show the mathematical tools used in the course of the work. By using dynamical systems we can describe the time evolution of the variables that one should monitor in a sustainability scheme (economy, human development, resource stock). Likewise, studying equilibrium points, different attractors and bifurcations, it is possible to analyze possible routes or trajectories of development that one society can undergo.

On the other hand, dynamical complex networks formalism will allow us to define a geographical case of study, in our case the municipalities in Caldas interacting through the road network. Through the modeling of the network we should be able to study exchanges of some variables between societies, observing the effect of such exchanges in the sustainability of the interconnected system.

### 1.3.1. Dynamical Systems

A dynamical system is defined by a set of state variables and the evolution law that govern them. Typically, behavior of a continuous-time dynamical system is provided by a set of Ordinary Differential Equation (ODE's) which can be coupled or not. Assume that the state space of a system is $X = \mathbb{R}^n$ with coordinates $(x_1, x_2, ..., x_n)$. Very often, the law of evolution of the system is given implicitly, in terms of the change rates (velocities) $\dot{x}_i$ as a function of the coordinates $(x_1, x_2, ..., x_n)$:

$$\dot{x}_i = f_i(x_1, x_2, ..., x_n) \qquad i = 1, 2, ..., n$$

which can be rewritten in a vector form

$$\dot{x} = f(x) \tag{1-1}$$

where we suppose that the vector field $f(x)$ is smooth (sufficiently differentiable). Notice that in eq. (1-1), time does not appear explicitly but implicitly as $\dot{x}$. A system with these characteristics is called an autonomous system.

### Equilibrium and Stability

Consider a continuous-time dynamical system defined by (1-1). Let $x^*$ be an equilibrium of the system (i.e $f(x^*) = 0$) and $J$ be the Jacobian matrix of the system evaluated at $x^*$.

$$J = \left.\frac{\partial f}{\partial x}\right|_{x^*} \tag{1-2}$$

Eq. (1-2) represents the linearization of the system around the equilibrium. This is a useful tool since we have a local representation of the motion of the system. The eigenvalues ($\lambda_i$) of (1-2) determine the nature of the equilibrium $x^*$ in its neighborhood. Let $n_-$, $n_0$ and $n_+$ be the number of eigenvalues with negative, zero and positive real part respectively. If $n_0 = 0$, $x^*$ is generic and is called an *hyperbolic equilibrium.*

The sign of the real part of the eigenvalue distinguishes the manifolds on which the solutions have divergent or convergent behavior. We shall define two types of manifolds according to this.

- Unstable Manifold: Invariant set defined by the eigenvectors of $\lambda_i$ with $\mathbb{R}(\lambda_i) > 0$, such that the flow of the system ($\phi^t$) tends to the equilibrium as time $t \to -\infty$.

$$W^u(x^*) = \{x : \phi^t x \to x^*, t \to -\infty\}$$

- Stable Manifold: Invariant set defined by the eigenvectors of $\lambda_i$ with $\mathbb{R}(\lambda_i) < 0$, such that the flow of the system ($\phi^t$) tends to the equilibrium as $t \to +\infty$.

$$W^s(x^*) = \{x : \phi^t x \to x^*, t \to +\infty\}$$

According to this we can distinguish three types of hyperbolic equilibria

- **Sink**: an equilibrium is a sink if all of the eigenvalues of $J$ have negative real parts (they are in the left half-part of the complex plane). In this case, any initial value $x(0)$ near $x^*$ will tend to $x^*$.

- **Source**: an equilibrium is a source if all of the eigenvalues of $J$ have positive real parts. Any initial value $x(0)$ near $x^*$ will move away from $x^*$.
- **Saddle**: an equilibrium is a saddle if it is hyperbolic, but not a sink or a source. The initial value $x(0)$ approaches to $x^*$ if $x(0) \in W^s(x^*)$. It will move away otherwise.

A better representation of the behavior of different types of manifolds is depicted in Fig. 1.1(a) and 1.1(b). Both cases correspond to $n_- = 2$ and $n_+ = 1$ which means that the stable manifold $W^s \in \mathbb{R}^2$ and $W^u \in \mathbb{R}$ and all of the trajectories belonging to this manifold approach to the equilibrium. A point of the space outside the stable manifold will move away from the equilibrium point.

### Bifurcations

Now we will consider a dynamical system that depends not only on the state variables, but also on parameters. This situation can be written as follows:

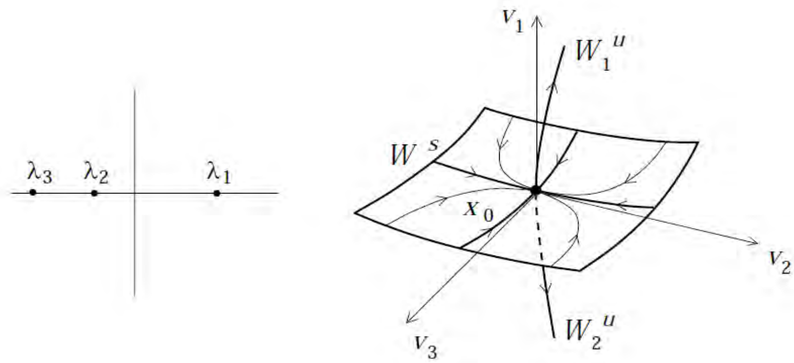
$$\dot{x} = f(x, \alpha) \tag{1-3}$$

where $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^m$ represent state variables and parameters respectively. Variation of parameters leads to a change in the phase portrait. There are two possibilities: either the system remains topologically equivalent to the original, or its topology changes. A bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation value. We can distinguish different bifurcations according to their codimension (codim from now on), this is the number of independent conditions in the parameter space that determine the bifurcation. Through this work we will focus mainly in codim 1 and 2 bifurcations of equilibria, though in some cases local bifurcations of cycles and global bifurcations may arise. Now we will list the different types of codim 1 and 2 bifurcations together with a brief description.

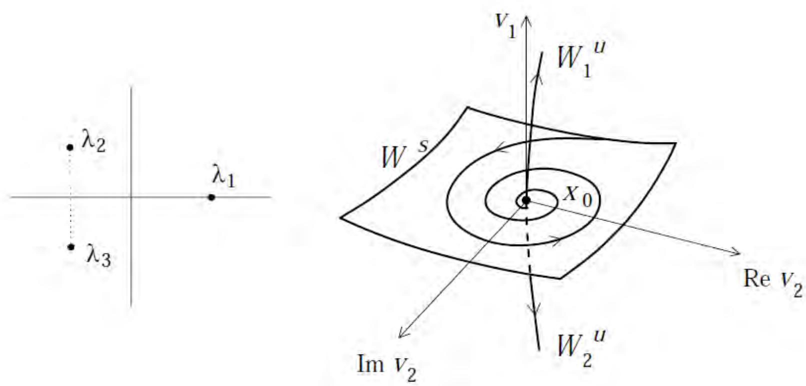
#### codim 1 bifurcations

There are two ways in which the hyperbolicity of an equilibrium can be violated. Either a simple real eigenvalue approaches to zero ($\lambda_1 = 0$) i.e $n_0 = 1$ increasing in one unit the dimension of a new invariant set called the center manifold $W^0(x^*)$; or a pair of simple complex eigenvalues reaches the imaginary axis ($\lambda_{1,2} = \pm i\omega_0$) i.e $n_0 = 2$, increasing the dimension of $W^0(x^*)$ in 2 units. Codim 1 bifurcations can be achieved by varying only one parameter, hence we will suppose that the dimension of the parameter space $m = 1$.

1. **Fold, Limit Point or Saddle Node Bifurcation:** Bifurcation associated with the appearance of $\lambda_1 = 0$. While the parameter $\alpha$ approaches the bifurcation value, two




(a) Saddle equilibrium. Over the stable manifold $W^s$, the equilibrium behaves as a stable node. Outside the stable manifold trajectories are rejected.

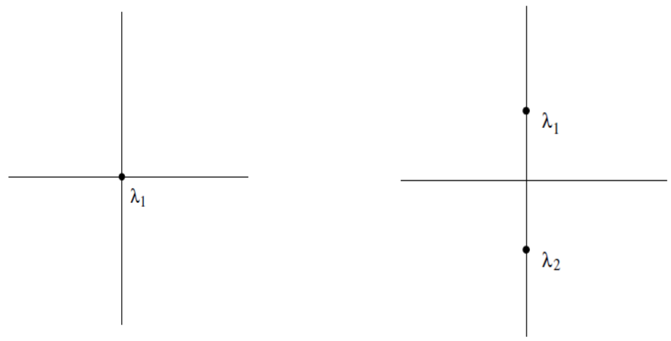



(b) Saddle equilibrium. Over the stable manifold $W^s$, the equilibrium behaves as a stable focus. Outside the stable manifold trajectories are rejected.

**Figure 1-1**.: Stable and Unstable manifolds in the state space. Two different examples for an hyperbolic equilibrium with $n_+ = 1$ and $n_- = 2$. *Figure taken from [31].*

equilibria of the system collide (one stable and one unstable) and disappear (*singularity condition*). Furthermore, two additional conditions must be satisfied: *Nondegeneracy condition* which means that the quadratic term of the Taylor Series of the vector field is nonzero ($f_{xx}(0,0) \neq 0$); and *transversality condition* which guarantees that the parameter $\alpha$ moves the vector field transversal to the singular state ($f_\alpha(0,0) \neq 0$). [31]

2. **Andronov-Hopf Bifurcation:** Bifurcation associated to the appearance of periodic orbits when $\lambda_{1,2} = \pm i\omega_0$ (*singularity condition*).
*Nondegeneracy condition:* $l_1(0) \neq 0$. Where $l_1$ is the first Lyapunov coefficient.
*Transversality condition:* $\mu\prime_\alpha(0) \neq 0$. Where $\mu \equiv \mu(\alpha)$ stands for the real part of the




(a) Fold (b) Hopf

**Figure 1-2.**: Singularity conditions for codim 1 bifurcations. *Figure taken from [31]*

eigenvalues in the neighborhood of $\alpha = 0$. Depending on the sign of $l_1(0)$ the Hopf bifurcation can be either supercritical ($l_1(0) < 0$), where the generated limit cycle is stable and the associated equilibrium becomes unstable; or subcritical ($l_1(0) > 0$) where the limit cycle is repeller and the associated equilibrium gains stability. [31]

**codim 2 bifurcations**

We shall now consider the dimension of parameter space $m = 2$, since we need 2 independent conditions to reach a codim 2 bifurcation. Then we have $\alpha = (\alpha_1, \alpha_2)^T \in \mathbb{R}^2$. Once we have achieved one codim 1 bifurcation by varying one parameter (say $\alpha_1$) we can track the path of this codim 1 point as we move the remaining parameter $\alpha_2$. This lead us to a curve in the $(x, \alpha_1, \alpha_2)$ space where the nonhyperbolic equilibrium (codim 1 point) exists for each pair of parameters. This curve (denoted here as $\Gamma$) is called a continuation of the codim 1 bifurcation. As we compute $\Gamma$, the following events might happen to the monitored nonhyperblic equilibrium at some parameters values. Either extra eigenvalues can approach to the imaginary axis, or some genericity conditions of the codim 1 bifurcation can be violated.

1. **Bogdanov-Takens or Double Zero bifurcation:** $\lambda_1 = 0$ and an additional real eigenvalue $\lambda_2$ approaches to the imaginary axis. [31]

$$\lambda_1 = \lambda_2 = 0$$

2. **Gavrilov-Guckenheimer, Zero-Hopf or Fold-Hopf bifurcation:** Two extra complex eigenvalues $\lambda_{2,3}$ approaches to the imaginary axis, while $\lambda_1$ remains in zero. [31]

$$\lambda_1 = 0,\ \lambda_{2,3} = \pm i\omega_0$$

3. **Hopf-Hopf bifurcation:** As we track a Hopf curve two extra complex-conjugate eigenvalues $\lambda_{3,4}$ approaches to the imaginary axis. [31]

$$\lambda_{1,2} = \pm i\omega_0, \; \lambda_{3,4} = \pm i\omega_1$$

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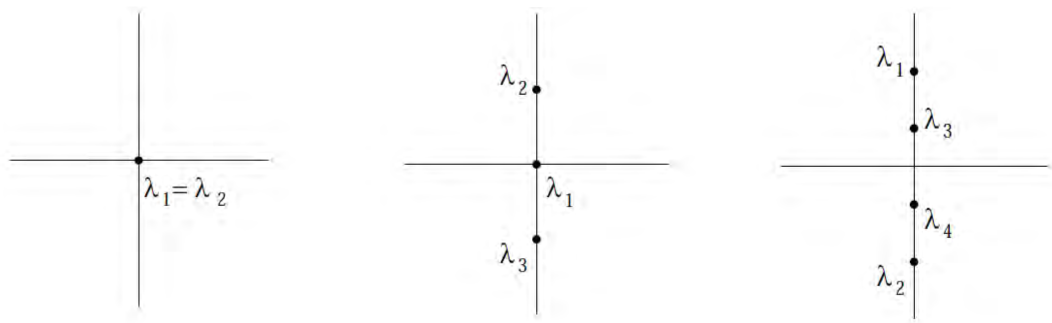



(a) Bogdanov-Takens (b) Gavrilov-Guckenheimer (c) Hopf-Hopf

**Figure 1-3**.: Singularity conditions for the codim 2 bifurcations. *Figure taken from [31]*

4. **Cusp bifurcation:** Result of the degeneration of the fold bifurcation, this is, the quadratic term of the normal form is equal to 0. [31]

5. **Bautin bifurcation** Result of the degeneration of the Hopf bifurcation, this is, the first Lyapunov coefficient is zero. [31]

### 1.3.2. Complex Networks

A graph $G = (\mathfrak{N}, \mathfrak{E})$ consists of two sets $\mathfrak{N}$ and $\mathfrak{E}$, such that $\mathfrak{N} \neq \emptyset$, and $\mathfrak{E}$ is a set of unordered pairs of elements of $\mathfrak{N}$. Elements in $\mathfrak{N}$ are called the nodes or vertices of $G$ while elements in $\mathfrak{E}$ are its links or edges. Two nodes linked by an edge are called adjacent nodes. Since we will focus on the study of directed graphs (digraphs) the direction of the edge must be taken into account.

A graph is said to be weighted if the links carry a numerical value measuring the strength of the connections between nodes. This can be represented as $G = (\mathfrak{N}, \mathfrak{E}, \mathfrak{W})$. Where $\mathfrak{W}$ is called the weights matrix, a $N \times N$ matrix whose entry $w_{ij}$ is the weight of the link connecting node $i$ to node $j$.

Typically, the topology of the graph can be partially determined by some statistical measures. The most common measures are

- *Average Degree ($\langle k \rangle$):* Given a node $i$, the degree $k_i$ of such node is the number of edges incident to the node. The average degree is the mean of the degree of all the nodes of the graph.

$$\langle k \rangle = \sum_k kP(k) \tag{1-4}$$

  Where $P(k)$ is the degree distribution of the graph, i.e the fraction of nodes in the graph having degree $k$.

- *Characteristic path length (L):* It is measure of the typical separation between two nodes in the graph $G$. In order to understand this concept one must first introduce the definition of Geodesic. A geodesic is defined as the shortest path from one node to another. Geodesics are represented by the matrix $\mathfrak{D}$ where the element $d_{ij}$ is the shortest path from node $i$ to node $j$. The maximum entry $d_{ij}$ is called the diameter of $G$. The characteristic path is then defined as the mean of geodesic lengths over all couples of nodes.

$$L = \frac{1}{N(N-1)} \sum_{i,j.i \neq j} d_{ij} \tag{1-5}$$

- *Centrality Measures:* There are several types of centrality measures which allow to determine the relative importance of a node within a graph. We will focus on the Closeness Centrality. The Closeness Centrality of a node $i$ is defined as the inverse of the sum of the distance from $i$ to all other nodes. [8].

The interpretation of these measures is closely related with the nature of the studied problem. In a social network, for example, the mean degree measures the average contacts that one person has. In our specific case where the dynamics of population through the road network in Caldas will be studied, these measures have different interpretations. For example, the diameter will be a measure of the easiness of mobility through the department, while the average degree will be a measure of the connectivity with other municipalities.

Several other measures regarding the topology of the network can be calculated with the Laplacian matrix, or equivalently with the Adjacency matrix. Some of the other measures used are: betweenness, clustering, community structures among others. For example, the spectrum of the graph (defined as the set of eigenvalues of the adjacency matrix), are intimately related with the topological features of the graph such as the diameter, number of triangles in the graph, cycles in the network etcetera [8].

When a graph is given some dynamical or evolving nature, it is called a complex dynamical network. In our particular case, we are interested in giving some dynamical behavior to the nodes, as depicted in Fig. **1-4**. From now on we will refer to this case as a dynamical network. The mathematical representation of a network of this type is given by (1-6).

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} \mathcal{L}_{ij} \left[ h(x_i) - h(x_j) \right], \qquad i = 1, ..., N \tag{1-6}$$

Here, the states of each of the $i^{th}$ node are described by their own vector field plus certain quantity of the states that are exchanged between neighbor nodes $j$. $\sigma$ is called the global coupling parameter, the Laplacian matrix $\mathcal{L}_{ij}$ contains the information of the weights of the edges, while the function $h(x_i) - h(x_j)$ is known as the output function and describes the coupling nature between the nodes. [26]

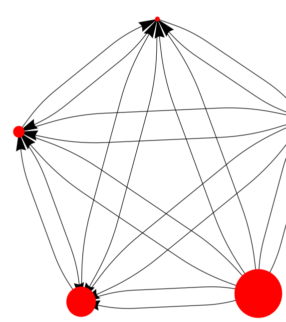


**Figure 1-4**.: Representation of a weighted, directed, dynamical network. Each of the nodes $i$ have a dynamical nature. The value of the states is somehow represented by the size of the red disk, the direction of the link is given by the arrow, and the weight of the link is the opacity of the line.

## 1.4. State of the Art

This section establishes the connection between previous works carried out in the framework of dynamical systems applied to social-economic matters and recent works in complex networks. Although some approaches have been made in order to fill the gap between these two issues, no sustainability schemes have been applied to complex networks, as far as the author knowledge.

The independent diffusion of the works of Lotka and Volterra [33, 50] in the second half of the 20's decade, established the framework in the research of population models through differential equations. The proposed models explained the population dynamics of two species that shared the same location, one of them known as the prey and the other known as the predator. The results predicted three possible scenarios, extinction of either one of the species or periodic oscillations of the population. The coexistence of both species in an oscillatory way brought a very primitive concept of sustainability where no one of the species became extinct.

In spite of the interesting and innovating results of the predator-prey model, biological mathematics didn't get much attention until the last decades of the past century. In the 1990's important contributions were made in population modeling by proposing several modifications to the predator-prey model in order to study the dynamics of man-resources. Particulary, Brander and Taylor proposed a mathematical approach to explain the partial extinction of the population in Eastern Island due to the uncontrolled management of natural resources [9]. This work arouse great interest in economical and political mathematics since it explained the decay of a great civilization when over-exploiting their main natural resource. In the same framework, Reuveney introduced exogenous (time dependent) parameters to Brander-Taylor (BT from now on) model and considered the possibility of introducing the same concepts to modern cases [43].

Several models have been proposed to explain chaos in biological systems such as insects population growth, epidemic propagation [47] and trophic chains [35, 34]. These works remark the importance of the spatial distribution of populations in the dynamics of a system. In the book *Environmental and Ecological Modelling* [30], Jörgensen present a detailed review of about 400 ecology dynamical models.

Important contribution in economic modeling first appeared in the paper *A theory of production*. In this paper Cobb and Douglas proposed a production function which could fit statistical data about the production of many economy sectors [12]. Cobb-Douglas functional calculated the quantity of monetary goods (accumulated capital) as a function of two different inputs, usually invested capital and labor force. The contribution of the Cobb-Douglas production function, together with Solow growth models, allowed much better approaches to the studies of the different micro and macro economic problems [45]. On this basis, many authors studied economic growth as function of the resource stock available, population and technology among others [25, 46, 18]. However, these models approach the problem in a discrete-time way, and very few works have dealt with the problem in a continuous-time approach [39, 40].

Control strategies on ecological and economic systems appear in the first decade of the present century. Particulary [11] suggest that the introduction of toxins in a food chain system decreases chaos in it. Another possible control scheme is proposed in [23], where crops quotas are determined in order to maintain the coexistence of three different types of plant species. Optimal control is proposed in sustainability schemes (contamination, harvesting, hunting, etc...) in [18] under the precautionary principle which aims to environmental management. Finally, studies carried out by the *IDEA-Manizales* (Instituto de Estudios Ambientales), together with the *IBD* (Interamerican Bank of Develpment), have suggested that control strategies can be introduced by government agents. These agents can trace resource exploitation paths, natural disaster prevention and economic development routes [36].

The role of human development in the dynamics of a biological system has been studied by many authors. These studies have shown notable results about the influence of the economic growth in the equilibria of a biodiversity model [5]. On the other hand, Pezzey et al. simulate the influence of critical situations (natural disasters) inside the society, over sustainability of natural resources [38].

An important extended BT model is proposed by D'Alessandro in 2007. In this paper, the author carries out a detailed analysis of the dynamics in a more realistic man-resources model. Such improvements include the introduction of another kind of harvesting sector and the proposal of irreversibility in harvesting processes (Allee effect) [15]. Particulary, the Allee effect predicts the extinction of a specie when its population grows below a defined threshold. Alle effect also has a very important role from the sustainability point of view, since it allows the appearance of more complex behavior in predator-prey (man-resources in this case) systems, such as Hopf bifurcations [52].

The inclusion of a third state variable in a BT extended model is proposed and analyzed in [22]. The third state variable models the accumulation of monetary goods over time as a consequence of manufacturing. This is an important approximation to a SD model since it takes into account both environment and economy and their dependance with population dynamics.

The application of complex networks to dynamical systems, where each node is assigned a vector field, has been widely reported specially in academic examples (e.g a set of $N$ Lorenz oscillators and Chua Oscillators). In such examples, synchronization processes appear via control of the weight of the nodes (see for example [27, 19]). However, no application to sustainability schemes have been used with the network approach.

# 2. 3-Dimensional Model of Development

## 2.1. Overview

We will focus on the development of a set of ODE's which accounts for sustainability. Even though there may exist other types of models, some of them more suitable for explaining economy, we will choose this one so we can study some interesting dynamical behavior, such us Hopf-like bifurcations which allows solutions predicting the existence of all of the states variables in a periodical way (as the definition of sustainability requires). Through this chapter we will discuss the modeling of population, resources and economy by using system dynamics tools. Later, some numerical simulations will be shown so we can capture the main features of the proposed model. Finally continuation diagrams will be shown so topological changes of the system through parameter variations can be observed.

## 2.2. Modeling

### 2.2.1. Brander-Taylor Model for the population dynamics on the Eastern Island

In the paper entitled *The Simple Economics of Easter Island: A Ricardo-Malthus Model of Renewable Resource Use*, Brander and Taylor developed a mathematical approach to the situation lived by people in the Eastern Island some time in their history. According to historians, when Europeans arrived to Eastern Island in 1722, they found a very poor and small population (about 3000). However, there were archeological evidence that shown a much richer and populous past. The evidence consisted of enormous statues carved of volcanic stone called *Moai*. These statues rest in several locations in the island and its weights are of few tens of tons. The largest statue of all (270 tons) lies unfinished near the quarry where it was carved.

The existence of these statues pointed out that the island should have had a large number of people dedicated to handcraft work, however it was not the case. People found on Eastern Island in 1722 seemed too poor and certainly not capable of moving such stones by their owns. Scientific studies showed that 3000 people would not be capable of moving the statues, at least without help of tools such as levers, rollers or poles. However, the island in 1722 had no trees suitable for building such tools. Residents of the island had no knowledge of how to move the statues and they believed that statues walked under influence of spiritual power.

The evidence so far pointed that first Polynesian natives that arrived to the Island, dedicated to wood harvesting in order to construct canoes, fishing rods and basic tools. As population grew, no control over cutting trees down were practiced, and eventually palm trees extinguished according to the Allee effect. The lack of their prime matter produced starving, diseases and probably inner wars which caused a reduction in population of about 90%. The developing crisis inhibited their cultural development and so the artesanal work in *Moai*. Although some authors tried to explain the raise and fall of Eastern Island culture in a more esoterical way [17, 10], Brander and Taylor proposed a mathematical explanation to it in terms of dynamical equations and socio-economical concepts.

By making use of system dynamics we can describe from a qualitative viewpoint the different interactions between the components of the coupled system population-resources. In this sense, the model proposed by Brander and Taylor is depicted in Fig. **2-1**.

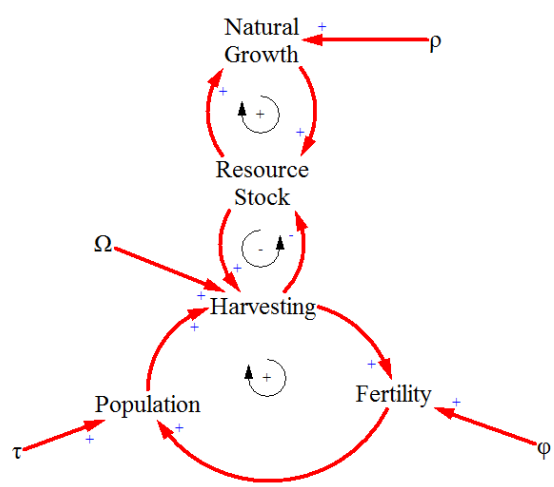



**Figure 2-1**.: Causal Diagram for Brander Taylor Model.

Let $S$ denote the resource stock at time $t$[1]. Change ratio of $S$, which we denote as $\dot{S} \equiv \frac{dS}{dt}$, is given by the natural grow rate $(G(S))$ minus the harvest rate $(H)$

$$\dot{S} = G(S) - H \tag{2-1}$$

[1]Depending on the case of study, stock can be thought as the ecological complex of renewable resources.

As proposed by [15] let $G(S)$ be a logistic function with strong Allee effect as follows[2],

$$G(S) = \rho\left(1 - S/k_1\right)\left(S/k_2 - 1\right)S \tag{2-2}$$

where $\rho$ is the natural regeneration rate, $k_1$ is the maximum quantity of stock that the environment can support (also called carrying capacity), and $k_2$ is the minimum quantity of stock that should exist in order to maintain a positive growth rate. By using a Ricardian approximation of production structure, harvest function can be expressed as follows

$$H = \alpha\beta LS \tag{2-3}$$

Parameter $\alpha$ is the quantity that provides the information of the available technology for resource extraction. Meanwhile, $\beta$ is the proportion of the total population $L$ dedicated to harvesting (labor force). Substituting (2-3) and (2-2) in (2-1) we obtain the dynamic equation

$$\dot{S} = \left(\rho\left(1 - S/k_1\right)\left(S/k_2 - 1\right) - \alpha\beta L\right)S \tag{2-4}$$

Once the variation of $S$ over time has been defined, we now proceed to deduce an expression for the dynamics of population $L$. In order to do that we apply a Malthusian population dynamics. We assume an underlying proportional birth rate $b$ and an underlying proportional death rate $d$. The rate of population increase $(b - d)$ is negative under the assumption that without any forest stock, population will eventually disappear. However, consumption of resource increases fertility, and therefore induces an increment of the population rate change. Thus, in a general form we can describe the dynamics of $L$ as follows

$$\dot{L} = (b - d + F)L \tag{2-5}$$

Where $F$ denotes the fertility function. Assuming that higher per capita consumption of the resource good leads to higher population growth we can assume that $F = \phi H/L$, where $\phi$ is a positive constant expressing the amount of resource necessary to increase the population in 1 unit. Substituting (2-3) in $F$ we can rewrite

$$F = \phi\alpha\beta S \tag{2-6}$$

Thus, introducing (2-6) in (2-5) we obtain the motion equation for population

$$\dot{L} = (b - d + \phi\alpha\beta S)L \tag{2-7}$$

[2]Inclution of Strong Allee effect accounts for irreversibility in the regeneration of certain available resource.

Equation (2-4) and (2-7) can be expressed as a system of equations as follows

$$\begin{cases} \dot{L} = (b - d + \phi\alpha\beta S)L \\ \dot{S} = (\rho\,(1 - S/k_1)\,(S/k_2 - 1) - \alpha\beta L)\,S \end{cases} \tag{2-8}$$

These equations can be thought as a variation of the Lotka-Volterra predator prey model. In this case the "predator" is assumed to be the humans and, the resource stock, the "prey".

### 2.2.2. Cobb-Douglas production function for accumulated capital

The Cobb-Douglas production equation is a functional that has been widely used to express the relationship between an output and its inputs in economy. Formerly it was proposed by Knut Wicksell and tested with statistical evidence of USA economy by Charles Cobb and Paul Douglas in 1928 [12].

Cobb and Douglas considered a simplified view of the economy of United States of America in which the production output was determined by the invested capital and the labor involved in the production. Though it was a very simplified model, it fitted well with the evidence.

The proposed functional had the form

$$\Gamma = \gamma K^{q_1} B^{q_2} \tag{2-9}$$

Where $K$ denotes the capital input, $B$ denotes the labor input (population involved in production), $\gamma$ is a constant called productivity factor and $\Gamma$ denotes the total productivity (monetary value of all goods in a year). Exponents $q_1$ and $q_2$ refers to elasticities in economy. Output elasticity measures the responsiveness of output to a change in levels of either labor or capital used in production. For example if $q_2 = 0.15$, a 1% increase in labor would lead to approximately a 0.15% increase in output. Values of $q_1$ and $q_2$ are important in order to define the type of economy i.e

- If $q_1 + q_2 = 1$ production function has **constant** returns to scale. That is, if we increase both the labor and the capital investment in 10% the total productivity will increase in 10%.

- If $q_1 + q_2 < 1$ production function has **decreasing** returns to scale. That is, output increases by less than the proportional change.

- If $q_1 + q_2 > 1$ production function has **increasing** returns to scale. That is, output increases by more than the proportional change.

Although Cobb-Douglas function has shown interesting results in fitting statistical data, many authors have criticized it. They stand that Cobb-Douglas function was not developed on the basis of any knowledge of engineering, technology, or management of the production process. Moreover, neither Cobb nor Douglas provided any theoretical reason about the constancy over time of exponents $q_1$ and $q_2$, which do not agree with reality of production processes. Finally, dimensional analysis throws out meaningless units of measurement. For instance units of parameter $\gamma$ are $Capital^{q_1+q_2-1}/(Capital^{q_1} Labor^{q_2})$, so it is a simple balancing parameter. A complete description of different difficulties of Cobb-Douglas function is provided in [6]. Nevertheless, different proofs over Cobb-Douglas function have demonstrated that, under the right assumptions, Cobb-Douglas function applies for both Micro and Macroeconomy [7].

In the book [14] a dynamic process to describe the economy growth in a country is proposed. This model considers two production factors, namely capital stock ($K(t)$) and labor force ($L(t)$). In our case, we actually consider the harvesting term, $H(t)$ (see equation 2-3), as the second production factor (instead of labor force), which is a reasonable assumption since we are focusing on a primary sector market [18]. The output $\Gamma(t)$ at the time $t$ is given by equation 2-10

$$\Gamma(t) = f(K(t), H(t)) \tag{2-10}$$

According to the equation above, the capital stock is accumulated through time according to the equation proposed by Solow [45]

$$\dot{K} = \gamma K^{q1} (\alpha\beta LS)^{q2} - \delta K \tag{2-11}$$

Where parameter $\delta > 0$ denotes the rate of capital depreciation and parameter $\gamma$ denotes the fraction of the capital which is saved and invested from one period to the next.
Introducing Eq. (2-11) into the Brander-Taylor system we obtain a 3 dimensional system.

$$\begin{cases} \dot{L} = (b - d + \phi\alpha\beta S)\, L \\ \dot{S} = (\rho\,(1 - S/k_1)\,(S/k_2 - 1) - \alpha\beta L)\, S \\ \dot{K} = \gamma K^{q1} (\alpha\beta LS)^{q2} - \delta K \end{cases} \tag{2-12}$$

Observe that all equations in system (2-12), share a common combination of parameters, the product $\alpha\beta$. Furthermore, the difference $b - d$ which is defined as the net population growth rate, can be reduced to a single parameter. Hence, we will rewrite the system reducing the number of parameters as follows

$$\begin{cases} \dot{L} = (\tau + \phi\Omega S)\, L \\ \dot{S} = (\rho\,(1 - S/k_1)\,(S/k_2 - 1) - \Omega L)\, S \\ \dot{K} = \gamma K^{q1}(\Omega L S)^{q2} - \delta K \end{cases} \tag{2-13}$$

where $\Omega = \alpha\beta$ and $\tau = b - d$. The system dynamics of (2-13) is shown in Fig. **2-2**.

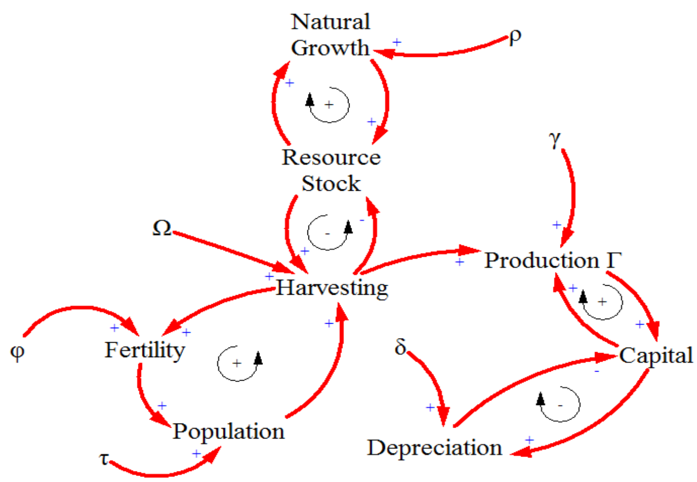



**Figure 2-2**.: Causal Diagram for the 3D model of sustainability. The positive feedback loop between production and invested capital can be guaranteed, because Cobb-Douglas production function obeys the so-called Inada conditions. This is, the marginal product of the capital tends to 0 as the capital goes to infinity.

## 2.3. Steady state analysis

Form the dynamical point of view we can distinguish two different kinds of equilibria: those who are trivial (semi-trivial) where all (some) of the state variables are zero, and those internal. From the sustainability view point, we are only interested in the internal equilibria, which means that there is a coexistence between the state variables, as the definition of sustainable development stated in chapter 1. This thesis will be focused on the study of those internal equilibria (possible bifurcation points, limit cycles appearances etc.), yet taking into account possible non-sustainable scenarios.

Trivial and semi-trivial equilibria ($E_1$ to $E_4$) are shown in Table **2-2**. There exist an internal equilibrium point which has the same open expressions for $L$ and $S$ as in $E_4$. However, no

| | | | |
|---|---|---|---|
| $\Omega$ | $2.5 \times 10^{-5}$ | $\tau$ | -0.1 |
| $\phi$ | 0.55 | $\rho$ | 0.025 |
| $q_1$ | 0.5 | $q_2$ | 0.5 |
| $k_1$ | 700 | $\gamma$ | 0.1 |
| $k_2$ | 12000 | $\delta$ | 0.1 |

**Table 2-1**.: Values of chosen parameters. Some values taken from [9] and [15] for achieving a stable focus.

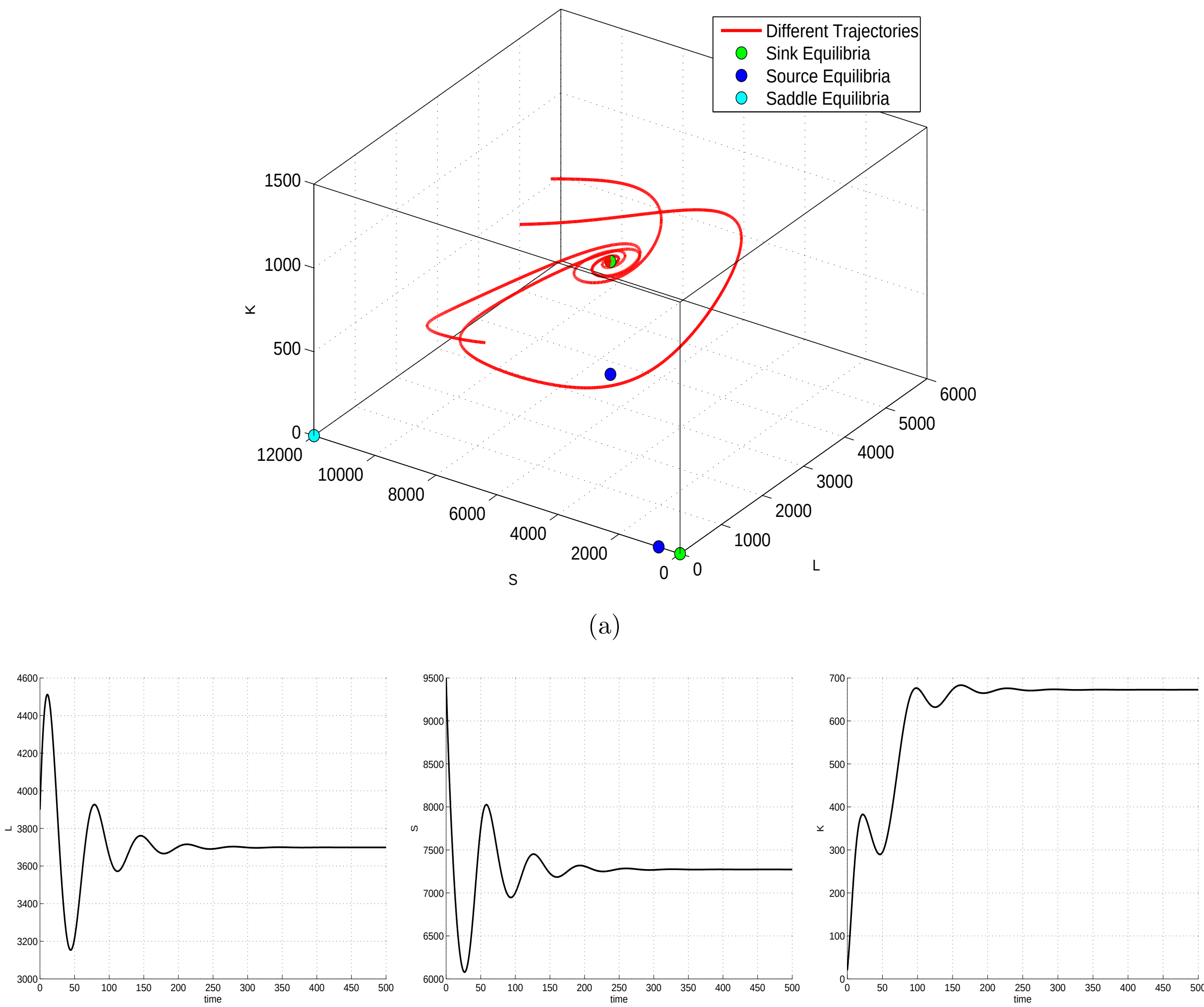
explicit expression for $K^{eq}$ can be constructed due the transcendental nature of the equation $\dot{K} = 0$. Thus, in order to know the equilibrium value for the economy variable, it must be computed by using numerical values of parameters. The criterion used to choose the values for parameters, was to select the numerical data used by Brander and Taylor in their original paper (see Table **2-1**). Thus the system achieves a stable focus. From there on one can proceed to the analysis of the different scenarios that can be obtained from this point by computing the equilibrium paths and bifurcation diagrams. On the other hand, numerical values for economical parameters were selected in such s way that the economy is considered to satisfy constant returns to scale (i.e. $q_1 + q_2 = 1$). Some realistic values for depreciation $\delta$ were chosen as well by considering a 10% depreciation of the goods and adjusting the values of $\gamma$ for a growing economy. Taking the values proposed, we can compute the remaining equilibrium $E_5$.

| *Equilibrium* | Value $(L, S, K)$ | Nature | $(n_+, n_-)$ |
|---|---|---|---|
| $E_1$ | $(0, 0, 0)$ | Sink | (0,3) |
| $E_2$ | $(0, k_1, 0)$ | Saddle | (2,1) |
| $E_3$ | $(0, k_2, 0)$ | Saddle | (1,2) |
| $E_4$ | $(\frac{-\rho\tau^2 - k_1\rho\tau\phi\Omega - k_2\rho\tau\phi\Omega - k_1k_2\rho\phi^2\Omega^2}{k_1k_2\phi^2\Omega^3}, -\frac{\tau}{\phi\Omega}, 0)$ | Source | (3,0) |
| $E_5$ | $(3698.94, 7272.73, 672.53)$ | Sink | (0,3) |

**Table 2-2**.: Equilibria in system (2-13).

$$\begin{pmatrix} \left(-1+\frac{S}{k_1}\right)\left(1-\frac{S}{k_2}\right)\rho+S\left(-\frac{\left(-1+\frac{S}{k_1}\right)\rho}{k_2}+\frac{\left(1-\frac{S}{k_2}\right)\rho}{k_1}\right)-L\Omega & -S\Omega & 0 \\ L\phi\Omega & \tau+S\phi\Omega & 0 \\ -k\phi\Omega+k^{q_1}q_2\gamma\Omega(S\Omega)^{-1+q_2} & 0 & -\delta-\tau-S\phi\Omega+k^{-1+q_1}q_1\gamma(S\Omega)^{q_2} \end{pmatrix} \tag{2-14}$$

We can establish the stability of the equilibria via linearization around the points. By evaluating $E_i$ with $i = 1, .., 5$ in the jacobian matrix given in (2-14), we can find the number of positive real part eigenvalues and negative ones. This characterization is recorded in the last column of Table **2-2**.




(a) (b) (c) (d)

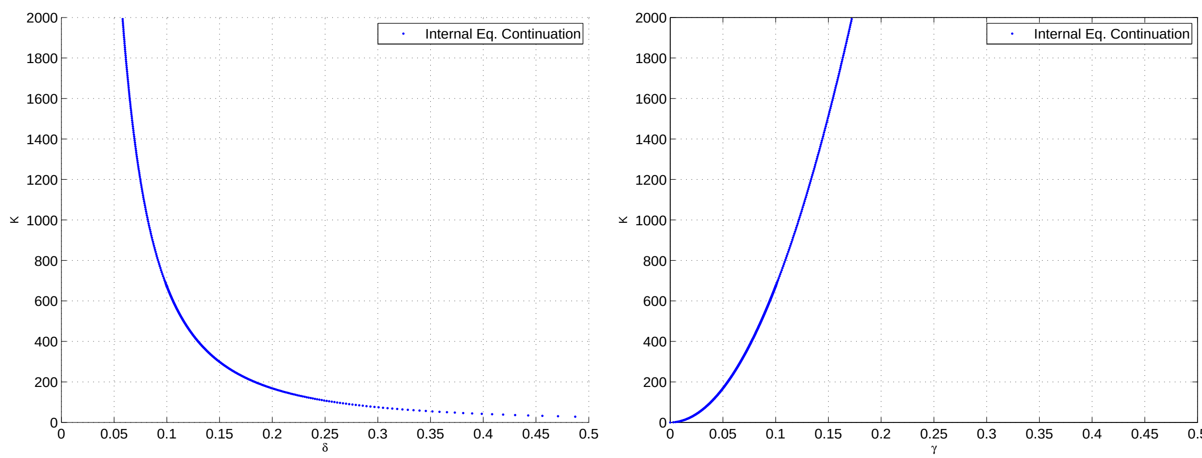
**Figure 2-3.**: (a) Trajectories of the system with different initial conditions in the 3D space. Time series for (b) population, (c) resource stock and (d) accumulated capital for one of the trajectories.

Some trajectories of the system are depicted in Fig. **2-3**. The orbits of the system tend to the internal equilibrium $E_5$ achieving a sustainable state. Nevertheless, the equilibrium $E_1$ is a stable equilibrium as well. Thus in order to get sustainability we must provide an initial condition which belongs to the basin of attraction of the internal equilibrium; otherwise it will cause the extinction of both resources and population, and consequently any economical growth. Time series of the process are shown as well. It can be observed that before reaching complete sustainability, some oscillations may arise. This can be thought as local crisis scenarios inside the society which can be overcome after some period of time.

## 2.3.1. Bifurcations

This section deals with the bifurcation diagrams regarding state variable $K$. A complete description of the bifurcations presented in state variables $L$ and $S$ can be found in [4]. We will focus on important parameters of economy function, i.e depreciation $\delta$ and capital factor $\gamma$. Finally the effect of parameter $\Omega$ will be shown as well. All continuations of equilibria were carried out with the Matlab continuation software package, MatCont [21].

Since $K$ is neither present in motion equations of population nor resources, the continuation of economy parameters only affects the equilibrium of economy. For this reason only pictures regarding this state variable are shown.




(a) (b)

**Figure 2-4.**: Continuation of the accumulated capital internal equilibrium respect (a) Depreciation and (b) Capital factor.

As it is observed in Figs. 2.4(a) and 2.4(b), as $\delta$ and $\gamma$ parameters change, the system does not undergo any qualitative change in behavior (i.e, it does not show any bifurcation). Then, the effect of changing those parameters is to move the internal equilibrium along the $K$ direction, while the other state variables remain invariant. Then we can conclude that the variation of economical parameters has no influence on the topology of (2-13). Now, we should focus on a second type of parameter, say $\Omega$, which can provide conditions for the appearance of some bifurcations.

By computing the continuation of the equilibrium $E_5$ regarding parameter $\Omega$ in the same way we did previously, we obtain the path depicted in Fig. 2.5(a). This continuation shows several codimension 1 points. A Hopf bifurcation arises around $\Omega = 2.86 \times 10^{-5}$ and the the stability of this bifurcation is provided by the first Lyapunov coefficient $l_1(0) < 0$, which means that the bifurcation is supercritical and the limit cycles generated from this point are orbitally stable. This scenario can be thought as a sustainable scenario as long as the period of the limit cycles remain in realistic values. In this sense, the period of the cycles is shown in Fig. 2.5(c) as a function of $\Omega$. Taking into account that a period of integration corresponds to 10 years we can conclude that the generated orbits can take up to 600 years.

Besides a Hopf point, we can also appreciate the existence of two branch points denoted as BP in Fig. 2.5(a). The BP refers to the collision of different equilibrium paths. In this case there exists a collision between the continuation of the equilibrium $E_5$ and the equilibrium $E_4$. This collision takes place in two different points, namely $(0, 700, 0)$ and $(0, 12000, 0)$. Observe that they correspond to the points $E_2$ and $E_3$.

Figure 2.5(b) shows the emergence of limit cycles from the Hopf bifurcation point. Limit cycles continue growing as $\Omega \to 3.5 \times 10^{-5}$ where there is a shift of the basin of attraction and the system tends to the trivial equilibrium $E_1$ (non-sustainable). This result unfolds the importance of the technological parameter in this particular approach, where excessive technological development without any control over harvesting can be harmful not only for resource stock but for all sustainable variables. Some trajectories in the state space after the Hopf bifurcation are shown in Fig. 2.6(a), as well as the time series of such situation (Figs. 2.6(b) to 2.6(d)).

## 2.4. Conclusions and Discussion

It can be concluded form this section that, in a system of this kind, there can appear up to three different behaviors. First, an ideal sustainable scenario, where all the states tend to an internal point in the phase diagram. It is possible to reach this ideal scenario passing through a transient that can be related with small crisis scenarios, where small oscillations can occur. A second level of sustainability is achieved when a periodic orbit appears. This allows solutions predicting consecutive critical periods followed by welfare periods. In the third case, either all or some variables tend to zero, this meaning non-sustainability in both cases. Economical parameters $\delta$ and $\gamma$ do not produce any topological change in system (2-13). This is an important feature of the model because those parameters can be considered as fitting values that can be changed without any critical consequence over the whole system. Technological parameter plays an important role on sustainable development, allowing the appearance of all three different scenarios depending on its value. This is a very important feature also because we can conclude that by increasing technological processes we must guarantee control actions in order to avoid non-sustainability in the long run, where extinction appears to be a consequence in long term scenarios.

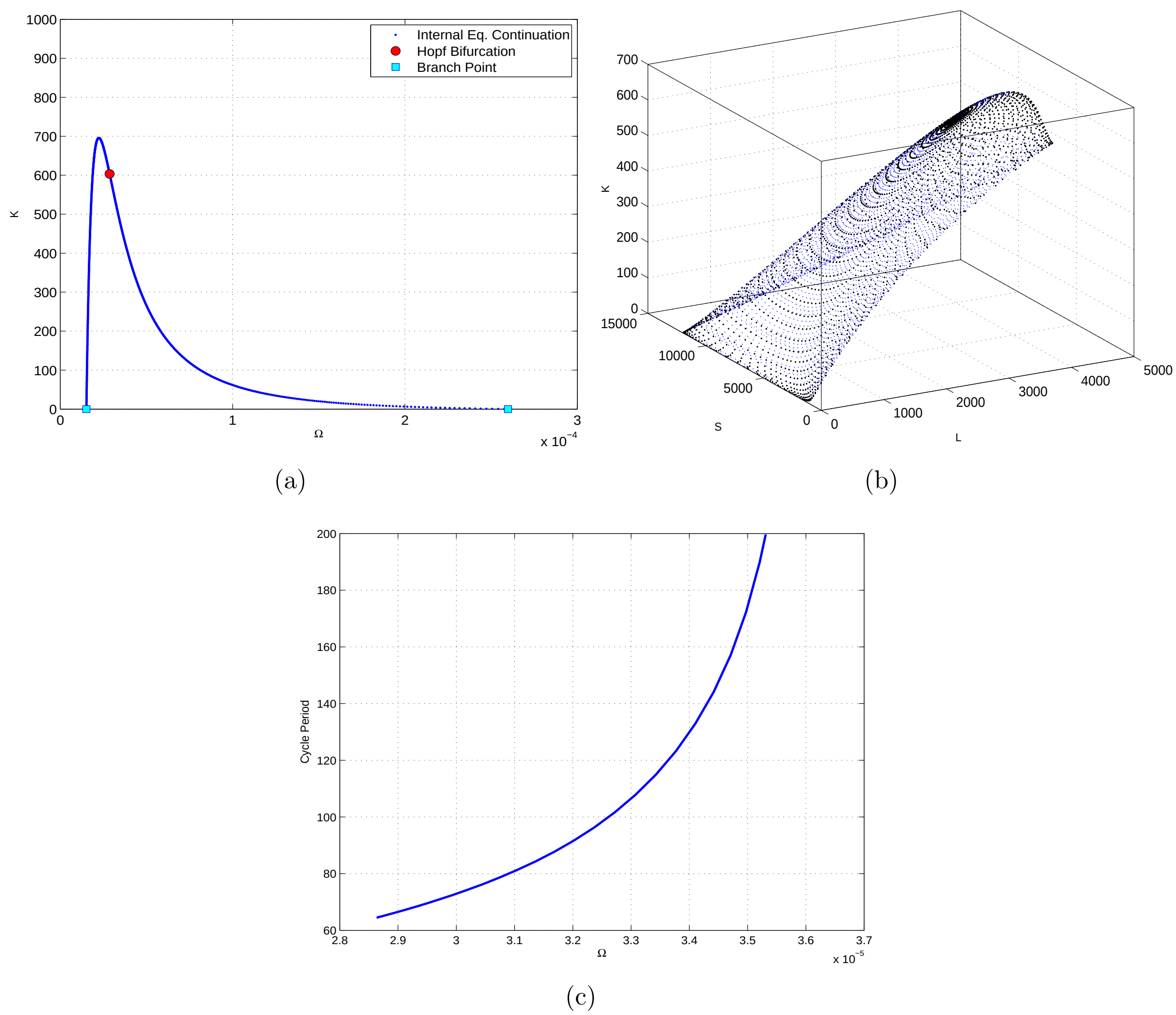
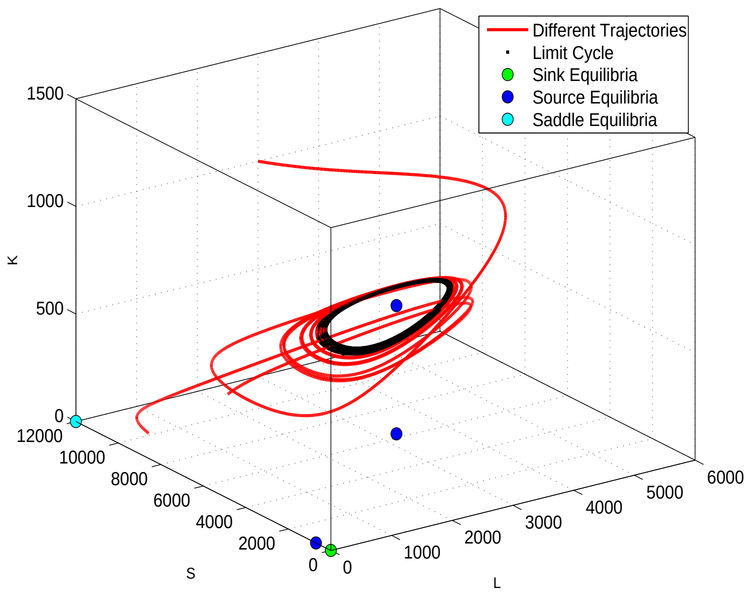
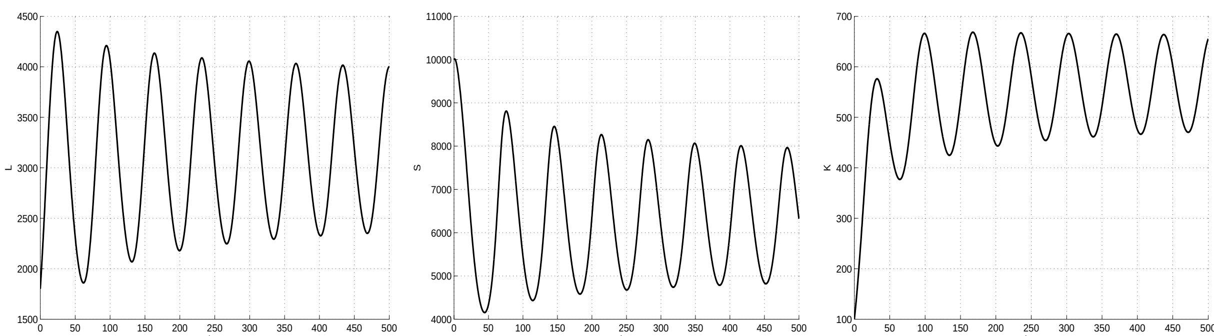



**Figure 2-5.**: (a) Equilibrium path when $\Omega$ varies. A Hopf bifurcation appears at $\Omega \approx 2.86 \times 10^{-5}$. (b) Limit cycles emerging from the Hopf point. (c) Period of the cycles depending on the value of $\Omega$.




(a)




(b) (c) (d)

**Figure 2-6.**: (a) Trajectories of the system after the Hopf bifurcation with $\Omega = 3 \times 10^{-5}$. Time series for (b) population, (c) resource stock and (d) accumulated capital for one of the trajectories (only initial transient is shown).

# 3. 4-Dimensional Model of Development

## 3.1. Overview

The 3D model described in the previous chapter gives an important idea of the economical-demographic development. Nevertheless, processes of economical prosperity are not always linked with social welfare. We will include in our model a new variable, namely social development to show the not-so-trivial influence of economy with human development. The addition of another variable adds complexity to the problem allowing the appearance of chaotic behavior and codim-2 bifurcations among other non linear phenomena.

## 3.2. Modeling

Quantification of social development is not a simple task, since there is not an agreement on which development indicator should be used. Some authors have associated such development with human capital, knowledge and education [39, 40, 48]. However, the most important approaches to the quantification of such variables have been carried out in the system dynamics framework, where Human Development is seen as a result of a feedback loop between Economical Growth and Human Development progress itself [42]. A simplified causal diagram summarizing the model proposed in [41] is depicted in Fig. **3-1**.

Many variables must be taken into account in order to have an accurate forecast model. These approaches can be suitable in the system dynamic frameworks where transitory states are the main goal. However, since we are interested in the study of the steady state and bifurcation analysis, we must find a way to simplify the global problem while keeping the most important features of the model. To do so, we first must point out the conclusions of the analyzed socioeconomic studies. The most common way to quantify the social development is by relating it with a widely used indicator of welfare, namely *Human Development Index* (HDI). This well-known indicator basically measures the life expectancy index, an education index and an income index [28]. The dynamical approach to such indicator is as follows: straightforward analysis says that Social development ($H$) shall increase with economy investment, although in different ways, depending on the distribution of capital. We suppose also that, in order to increase social welfare, a minimum quantity of labor force must be guaranteed. On the other hand, reaching overpopulation values provides unfavor-

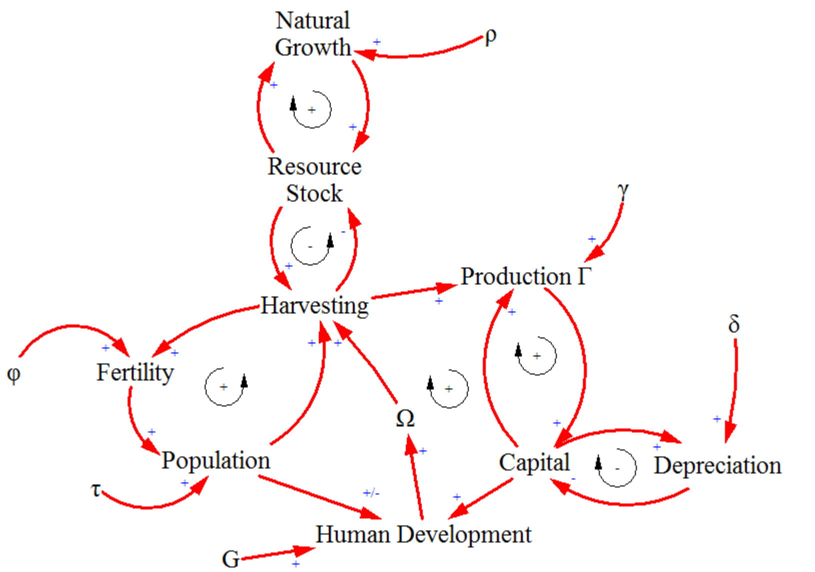



**Figure 3-1**.: Simplified causal diagram describing the dynamical behavior of HD in a feedback loop with the economy through the technological parameter.

able scenarios for social growth. Finally, we must provide a way to bound the indicator to an interest set, i.e. values between 0 and 1.

Different evolution processes of human development have been observed in several social-demographic studies. As proposed in [42], similar Gross Domestic Products in several societies can reach to very different social growth curves. This is, if we consider the HD growth a linear function of $K$ then the different scenarios can be modeled as different slopes of the linear function. On the other hand, a cubic function in the population contribution to HD is suitable for our needs of keeping the HD growth if and only if population is kept between certain bounds.

In this sense social development ($H$) is given by equation

$$\dot{H} = GKH(1-H)\left(1-\frac{L}{L_M}\right)\left(\frac{L}{L_m}-1\right)L \tag{3-1}$$

This expression guarantees that the dynamics that of the system will develop in the interest set of $H$ ($H \in (0,1]$), provided that the initial condition $H(0)$ belongs to this set as well. Demonstration of this fact is given in Appendix A.

We have defined the way in which economy and labor force favor the growth (or decrease) of variable $H$. It is necessary also to define the other chain, i.e., how $H$ favors (or not) the economy. Since $H$ is closely related to education and knowledge, we can model $\Omega$ (which accounts for technology) as a linear function of $H$. Hence, making $\Omega(H) = \Omega_0 - \Delta\Omega(1-2H)$ we obtain the non-linear system (3-2).

$$\begin{cases} \dot{L} = (\tau + \phi\Omega(H)S)\,L \\ \dot{S} = (\rho\,(1 - S/k_1)\,(S/k_2 - 1) - \Omega(H)L)\,S \\ \dot{K} = \gamma K^{q1}(\Omega(H)LS)^{q2} - \delta k \\ \dot{H} = GKH(1-H)\left(1 - \frac{L}{L_M}\right)\left(\frac{L}{L_m} - 1\right)L \end{cases} \tag{3-2}$$

$\Omega_0$ is taken as the original parameter used by Brander and Taylor in order to guarantee a population survivance and resource stock, and $\Delta\Omega$ is defined as the maximum variation of technology, assuming that technological development is bounded. Several parameters arise from this equation, namely overpopulation and underpopulation parameters ($L_M$ and $L_m$), maximum technology variation ($\Delta\Omega$) as well as the growth rate $G$. In particular $G$ can be interpreted as the change ratio between the invested capital and the social growth that in implies. Let us move to the steady state analysis of the system.

## 3.3. Steady state analysis

The proposed model can have up to 8 equilibria (denoted as $E_i$ as in previous cases) plus 4 manifolds of equilibria (denoted as $s_i$ from now on). Equilibria manifolds are found when $K = 0$ and have the form (3-3) to (3-6).

$$s_1 = (0,0,0,H) \tag{3-3}$$
$$s_2 = (0,700,0,H) \tag{3-4}$$
$$s_3 = (0,1200,0,H) \tag{3-5}$$
$$s_4 = (L(H),S(H),0,H) \tag{3-6}$$

Where $L(H)$ and $S(H)$ are functions of the state variable $H$ with the form

$$L(H) = \frac{-\rho - \frac{\rho\tau^2}{k_1k_2\phi^2(-\Delta\Omega+2H\Delta\Omega+\Omega_0)^2} - \frac{\rho\tau}{k_1\phi(-\Delta\Omega+2H\Delta\Omega+\Omega_0)} - \frac{\rho\tau}{k_2\phi(-\Delta\Omega+2H\Delta\Omega+\Omega_0)}}{-\Delta\Omega + 2H\Delta\Omega + \Omega_0}$$
$$S(H) = -\frac{\tau}{\phi\left(-\Delta\Omega + 2H\Delta\Omega + \Omega_0\right)}$$

Even if $s_1$, $s_2$ and $s_3$ represent possible scenarios where population variable extinction is predicted, $s_4$ lacks of physical sense. This is, $s_4$ predicts a coexistence between population and resources without any capital at all (similar to the equilibrium $E_4$ in previous chapter). Then we have a strong similarity with manifolds $s_1$ to $s_3$ and equilibria $E_1$ to $E_3$ from previous chapter. Contrarily to the fact that the state variable $H$ appears as a free variable which can take any value, it is not possible to conceive social development without any capital (or even population at all). In this sense we can conclude that these manifolds represent unsustainable scenarios, since they predict human extinction.

The study of the steady state is then restricted to the 8 remaining equilibria. The first two (and probably the most important) lie in the hyperplanes (from now on referred as planes) $H = 0$ and $H = 1$, as shown in the 3-dimensional projection in Fig. **3-2**. There is a strong numerical evidence showing that both equilibria are joined by an heteroclinic orbit. However the analytical study of this fact is proposed as future work. According to the hypothesis we are using, we can consider an ideal sustainable scenario, where trajectories tend to the plane $H = 1$, which indicates a whole social development. When $H = 1$ trajectories tend to the plane as long as the orbit remains in the region $L_m < L < L_M$, and will move away otherwise. When $H = 0$ the opposite way occurs. Thus, several different results can be obtained, depending on the relative position of $E_1$ and $E_2$ regarding the thresholds $L_M$ and $L_m$.

Covering all possible combinations of different scenarios is a difficult task, since it not only implies taking into account the relative position of $E_1$ and $E_2$, but the shape of the vector field inside the manifolds $H = 0$ and $H = 1$. In these manifolds, the vector fields can point inwards or outwards $E_1$ or $E_2$. In the case where both vector fields point towards the equilibrium, the behavior is straightforward. Nevertheless, when any of them is a repeller (say $E_2$) while the other one is attractive (say $E_1$) there exists the possibility of a continuous feedback due to the continuous transition between region 1 and 2, thus generating periodical orbits. This example is better described in Fig. **3-3**.

Change rate $G$, which involves the total income $K$ with the social development, can be a critical value in the hypothetic scenario suggested previously. This parameter can give rise to periodic orbits which eventually have a transition to chaos while changing $G$ as shown in Poincaré map of Fig. **3-4**.

It is worth to note, that the link of these results in practical applications is not straightforward. The period of some of the periodic trajectories can take up to $10^2$ integration time, means $10^3$ real years. So it is not advisable to assume the model as a quantitative predicting one, yet it is interesting to observe that in the example that we are considering, there is a trend to an ideal scenario followed by a crisis scenario. This unfolds the importance of being careful with a monotonic-like trend of sustainability, since there is no guarantee to remain that way.

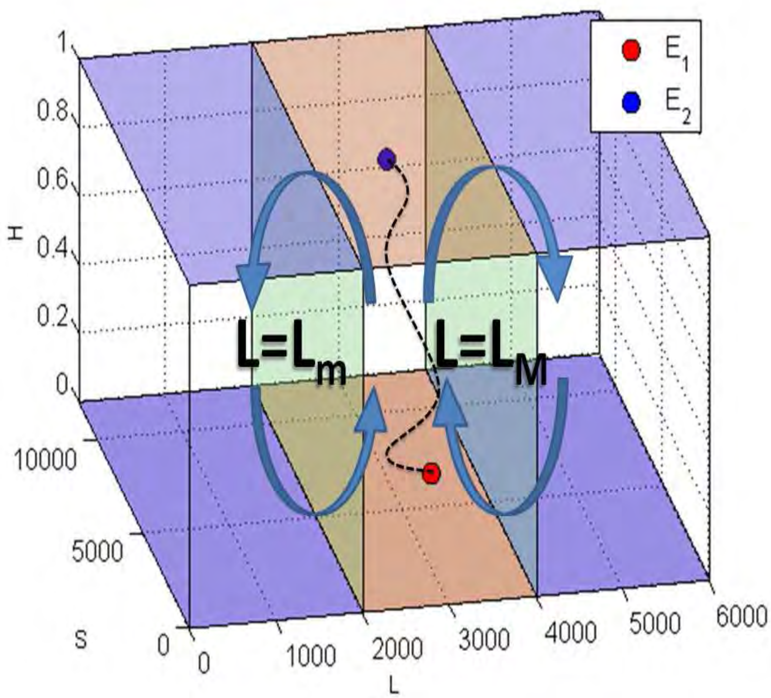



**Figure 3-2**.: Trajectories between the hyperplanes $L = L_m$ and $L = L_M$ tend to the manifold $H = 1$. They move away from it and try to go to $H = 0$ otherwise. A possible heteroclinic orbit is depicted in dashed line.

We have focused our attention to the study of the bound equilibria which lie in the planes $H = 0$ and $H = 1$. However, 6 other equilibria exist in our sustainable development ODEs system. These points lie in the planes $L = L_M$ and $L = L_m$. Under feasible values they will live outside the reachable domain $0 < H < 1$. There are, however, some specific situations where it is actually possible that these points are found inside our interest set and become stable. This situation is depicted in Fig. **3-5**. It is worth to note that the particular behavior previously described guarantees long term sustainability although with some restrictions. For example, the values of $H$ will be rather small and there are also strong limitations in the evolution of technology and the upper population threshold. We can think of this scenario as a difficult one to reach, so we remark that under "realistic" situations, no stable equilibrium in $H \in (0, 1)$ can be found, and sustainability is reached only via saturation ($H = 1$) or periodicity due to global process of stability change around the planes $L = L_M$ and $L = L_m$.

## 3.4. Bifurcations

In the previous section we have seen that the system can evolve through different ways. One can obtain either oscillatory behavior due the existence of the planes $L = L_M$ and $L = L_m$

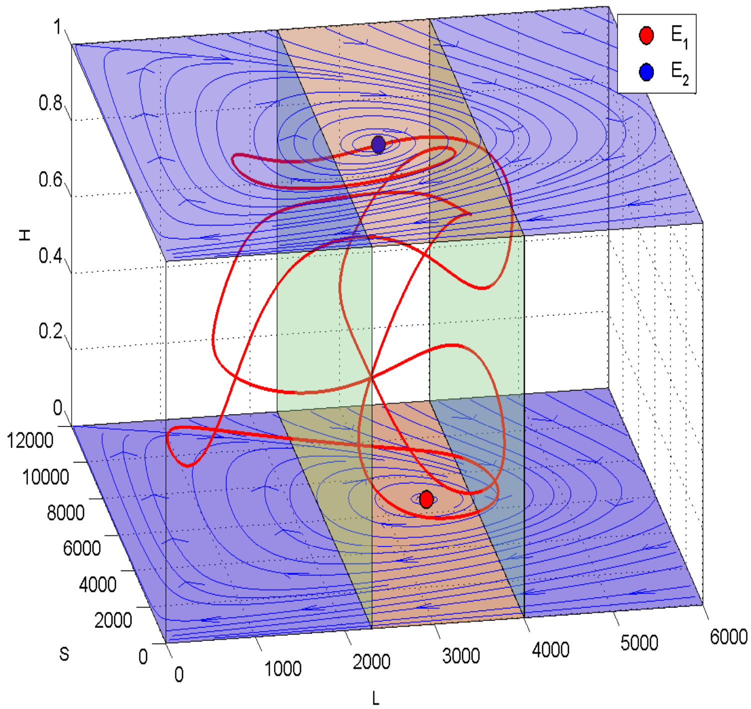



**Figure 3-3**.: Stability transition process produces periodic orbits.

or stable equilibrium. The high number of parameters in the system allows the existence of bifurcation points, both outside and inside the interest set $0 \leq H \leq 1$. Due to the high number of parameters, the study must be focused in a few bifurcation parameters which are considered critical in sustainable schemes. In the previous chapter, the chosen one was the technological parameter $\Omega$. In this case we will choose the parameters related to social development $L_M$ and $L_m$ as well as the fertility constant $\phi$. In this way, we are giving more importance to those parameters related to demographic properties. This is due to the fact that population dynamics plays an important role in the global behavior of the system.
By taking one of the possible equilibria (even though it is unreachable) we can proceed to compute the continuation of the corresponding branch and observe the possible transition to a reachable state. For this matter let us take $L_M$ as a study case. For $L_M = 4000$ no interesting equilibria is observed (negative values of $H$). However, a value for $L_M$ lower than the previously chosen (say 3000) produces a saddle-type internal equilibria appearance, whose continuation is depicted in Fig. **3-6**.

A subcritical Hopf bifurcation ($l_1(0) > 0$) is found around $L_M = 3150$. Generated periodic orbits form Hopf point always obey $H \in (0, 1]$. Since the orbits are orbitally unstable, it is unlikely to observe the periodicity predicted in such scenario, but the existence of a Hopf point implies an organization around the values of the bifurcation. This is, small stable

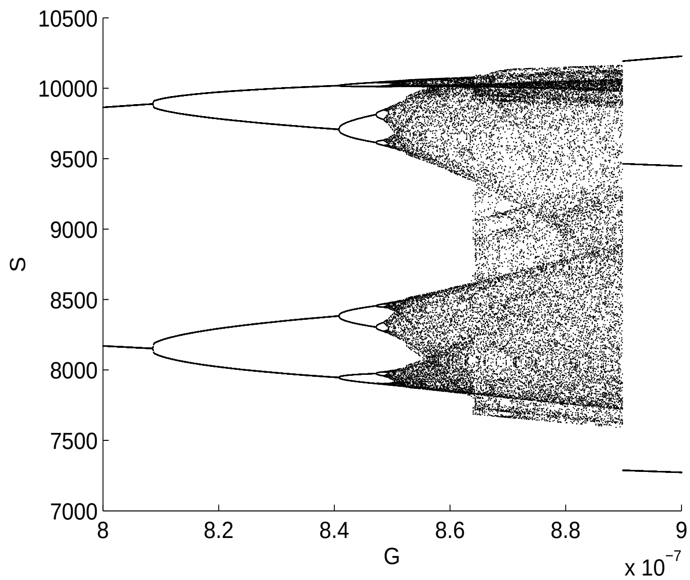
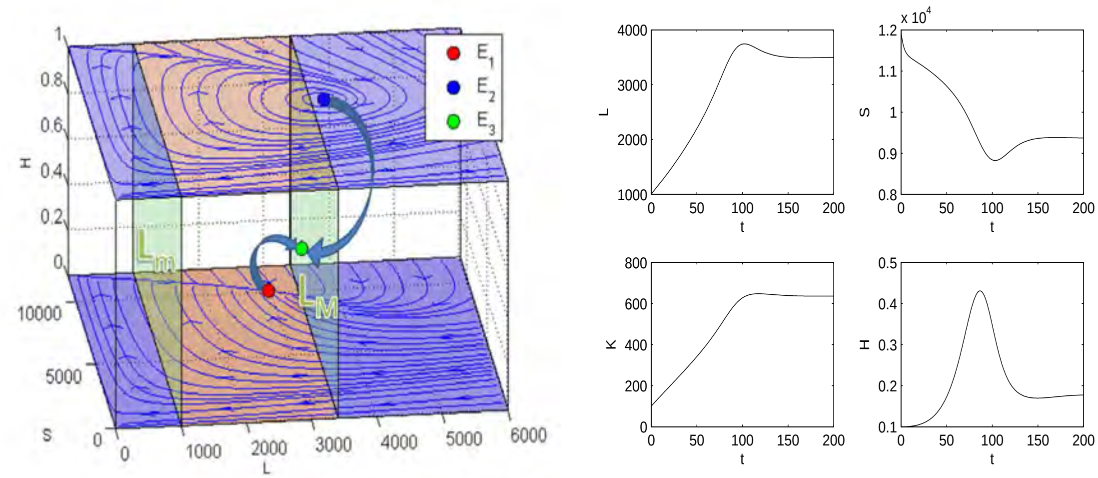



**Figure 3-4.**: Bifurcation diagram at the variation of parameter $G$.

focus-type behavior before moving away from the manifold where the bifurcation occurs, which can be thought as short-term sustainability. A similar analysis was carried out with underpopulation parameter $L_m$ obtaining similar qualitative results for values $L_m > L_M$, which cannot be considered as a possible case.

We will now focus on an important quantity namely fertility parameter $\phi$. We recall the existence of several equilibria living in the threshold planes of population. By taking one of these equilibria and observing the topology of the vector field as we change $\phi$, some interesting codim-1 points will appear as depicted in Fig. 3.7(a). For each one of the equilibria in $L_M$ and $L_m$ we will obtain the expected branch points, when the path crosses the boundaries $H = 0$ and $H = 1$. A limit point bifurcation will also appear for both cases. However the limit point in the plane $L = L_m$ is unreachable and the real implication becomes only a matter of the overpopulation value. In this case, the predicted limit point is meaningless since the involved equilibria are both of saddle type. Moreover, two Hopf bifurcations appear as well, one for each continuation. Limit cycles emerging from each Hopf point, tend to grow towards the boundary $H = 1$. Both bifurcations are supercritical, therefore, the limit cycles are orbitally stable but only on the 2-dimensional manifold where the bifurcation occur. Thus the remaining directions prevent from reaching the orbit.




(a) (b)

**Figure 3-5**.: (a) An internal equilibrium $E_3$ appears in the case where $E_1$ and $E_2$ are close to the threshold $L_M$. (b) Time series for the proposed case.

### 3.4.1. Some codim 2 bifurcation

Now we will observe the behavior of the Limit Points in Fig. 3.7(a) where we perform variations on the population parameters. For this matter we select $L = L_m$ for the sake of simplicity (choosing $L_M$, leads to similar results).

Let us focus on the "lower" Bogdanov-Takens (BT) bifurcation. BT point is found for values of $L_m \approx L_M$. From the BT point both Hopf and Neutral Saddle curves emerge as depicted in Fig. **3-8**. By following the Hopf curve, $L_m$ becomes closer to $L_M$ and in the limit where $L_m = L_M$ a Zero-Hopf point (ZH) is also found. On the other hand, by following the Neutral Saddle curve, another BT bifurcation point arises around $H = 0.98$. After that, another Hopf curve emerges and grows beyond $H = 1$ as $\phi$ grows very fast. We must remak that, even when the codim-2 points theoretically exist, in the case of our application, they appear under limit conditions of the model. For instance, the Zero-Hopf point and one of the Bogdanov-Takens points are born when the thresholds of social growth are very close each other. In a sense where we have a degeneracy in the system. A more accurate analysis regarding the nature of the codim-2 points must be done before concluding anything about these points and thinking about the interpretation about them in real scenarios.

## 3.5. Conclusions and Discussion

A 4-dimensional model was proposed in order to explain the three main components of sustainability. This was achieved by establishing causality relationships between the different

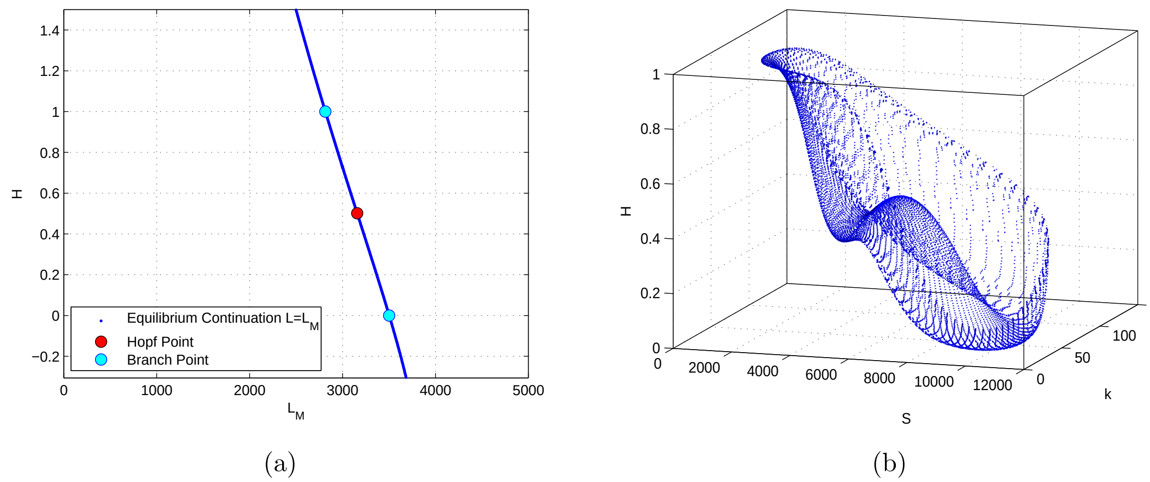
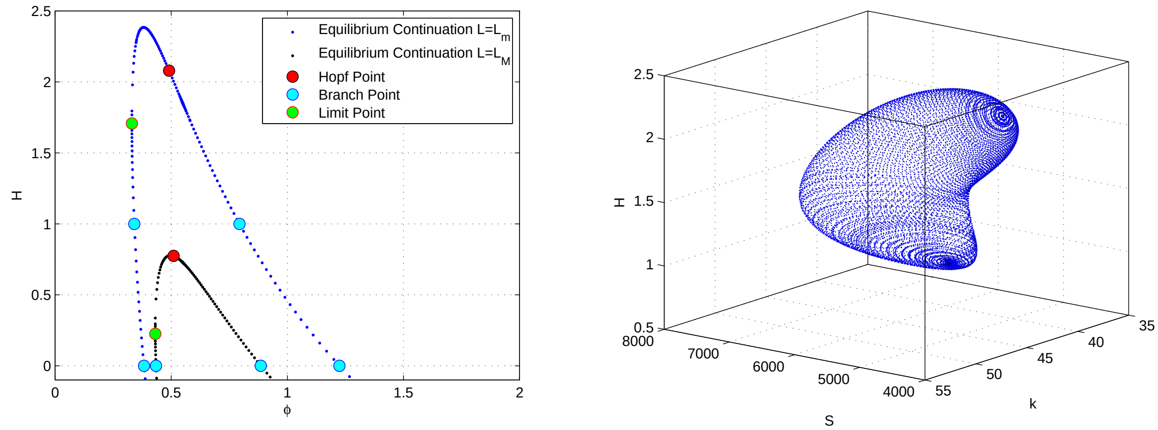


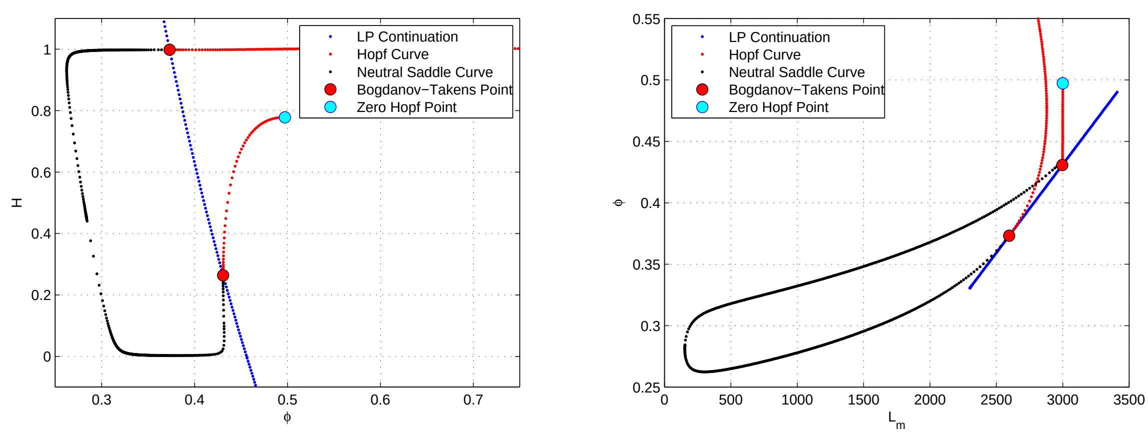

**Figure 3-6**.: (a) Internal equilibria continuation regarding parameter $L_M$. (b) Limit cycles emerging from the Hopf Point.

variables. System has shown a great dynamical richness, where it is possible to find both levels of sustainability, i.e periodicity and stable equilibrium points. Periodic solutions of the system are produced by global processes such as the transition between regions of attraction and repulsion due to population thresholds, which is one of the main features of the model. Some found bifurcations are indeed hard to observe in real life scenarios, e.gg Hopf bifurcations, which allow the appearance of limit cycles of high periodicity. However, it is important to find all of the different processes in a parameter neighborhood where the bifurcation occurred since bifurcation points are organizational centers of not-so-trivial behavior. Both technological ($\Omega_0$ and $\Delta\Omega$) and economical features ($G$) seem to play a crucial role in the dynamics of sustainability. We asuumed a linear function of the technology in our approach $\Omega(H)$; future work is proposed with different technology functions, for instance exogenous technological development $\Omega(t)$ as mentioned in [16].




(a) (b)

**Figure 3-7.**: (a) Internal equilibria continuation regarding parameter $\phi$, following the points $L = L_m$ and $L = L_M$. (b) Projections of Limit Cycles emerging from the "internal" Hopf point in a 3D state space. The limit cycles grow and then die in the "external" Hopf point.




(a) (b)

**Figure 3-8.**: (a) Limit Point and Hopf continuation in $(\phi, H)$ plane. (b) Limit Point and Hopf continuation in $(L_m, \phi)$ plane.

# 4. Complex Network Case Study

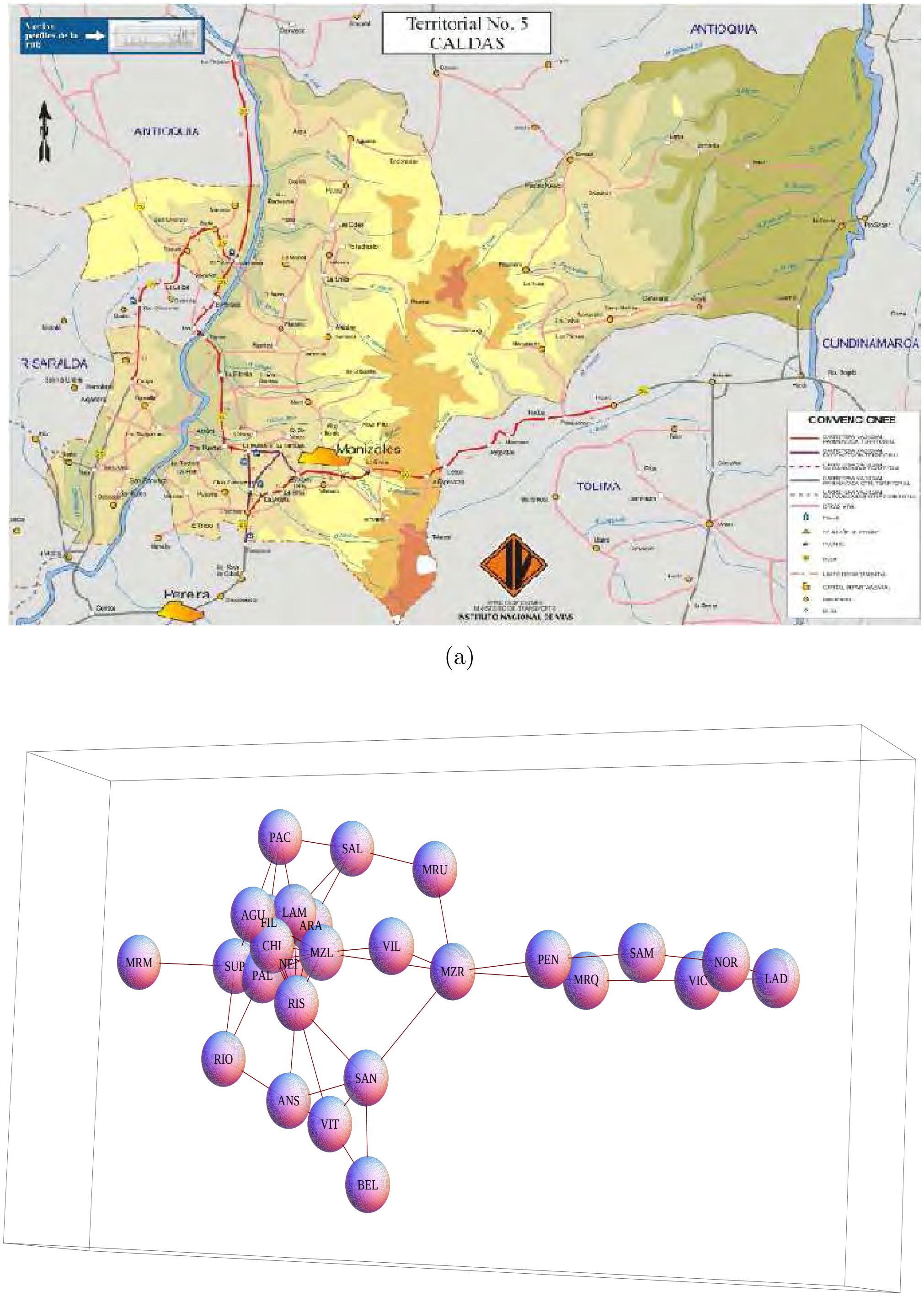
## 4.1. Overview

We have dedicated previous chapters to the development of a dynamical model which includes four dimensions of sustainability. In this chapter we will apply such model in a real network (Caldas department with road network). The implementation of the model through a complex network, will allow us to study the effects over the development trajectories of controlled exchange of variables in a real interconnected system. In a first stage we will consider the unweighted, undirected graph representation of Caldas road network together with some statistical measures graph, that will be related to the geographical situation. Later, we will apply the dynamical model to the different nodes observing the different behavior of state variables when allowing the exchange of population between the nodes of the network (municipalities); as well as the temporal variation of the topological characteristics of the complex network.

## 4.2. Network Design

We will consider Caldas department as a set of nodes (municipalities), each one connected with each other through edges (road network). Information about roads in Caldas is taken from [3] (see Fig. 4.1(a)).

Let us first consider an unweighted network (all of the links have the same weight), so we can calculate some statistical measures of the network before applying any dynamical property. The resulting graph is shown in Fig. 4.1(b); it can be observed that the graph representation coincides approximately with the geographical situation of the different cities in Caldas. See for instance, in the right hand side of the figure the municipalities corresponding to *Caldas Magdalena*; moreover at the bottom we can distinguish cities corresponding to *Low West* district, and so on. We shall proceed to the calculation of the measures of such network. As seen in Fig. **4-2**, we cannot calculate an approximation to the degree distribution, since we have very few samples for a fitting (only 27 nodes). We shall conclude that for such system we cannot obtain an open expression for $P(k)$. However we do can calculate the mean degree of the system




(a)

(b)

**Figure 4-1**.: (a) Caldas Road Network taken from http://www.invias.gov.co/ (b) Graph representation of the road network

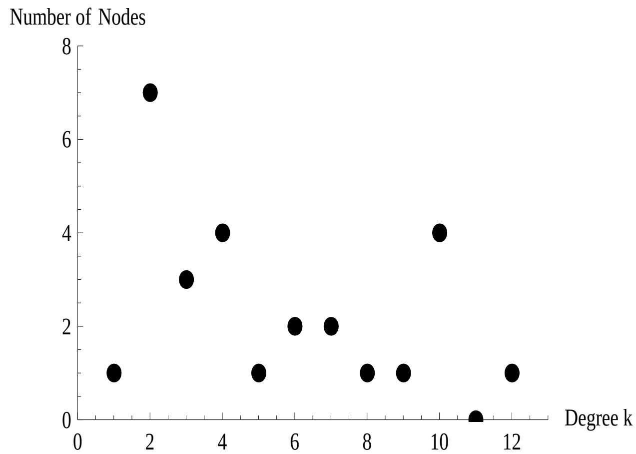



**Figure 4-2.**: Degree distribution of the unweighted, undirected road network.

$$\langle k \rangle = 5.18$$

which means that there exists a mean of 5 roads that arrives (leaves) to (from) a town. The real implication of this simple result is that each city has an average of $\langle k \rangle/2$ neighbor municipalities. Knowing the average degree of our study case, unfolds the connectivity of Caldas network. Connectivity has an impact in the development processes, i.e the more connected the network is, the more exchange of different variables will occur, thus making the development dynamics even more intricate. Connectivity of Caldas network becomes evident as well with the value of diameter

$$\mathfrak{d} = 6$$

which means that the number of municipalities that one must visit to get anywhere in the network is at most 6, for example in order to reach from Viterbo (VIT) in the western part of Caldas to La Dorada (LAD) in the eastern part. As will be observed in later sections, we will focus our attention in migration processes and how these processes affect the paths of development of the involved societies. The knowledge of the diameter is then a measure of the easiness of migration across the department, consequently it is also a measure of how easily sustainability is changed due to the exchange of variables (in our case population).

Closeness centrality was also calculated to observe somehow the most influential cities. This can be observed in Table **4-1**.

See for example the existence of very "central" cities (high closeness centrality) such as Manizales (MZL) the capital of the department; Manzanares (MZR) which establishes the connection between east and west of Caldas; and Risaralda (RIS) which lies in the immediacy of Center South, High West and Low West. On the other hand, corner municipalities

| | | | | | |
|---|---|---|---|---|---|
| VIT | 0.753846 | BEL | 0.6125 | RIS | 0.924528 |
| SAN | 0.875 | ANS | 0.753846 | PAL | 0.859649 |
| RIO | 0.662162 | SUP | 0.875 | MRM | 0.604938 |
| CHI | 0.844828 | VIL | 0.765625 | MZL | 1. |
| NEI | 0.816667 | FIL | 0.875 | ARA | 0.731343 |
| LAM | 0.890909 | SAL | 0.720588 | PAC | 0.653333 |
| AGU | 0.830508 | MRU | 0.7 | MZR | 0.942308 |
| PEN | 0.680556 | MRQ | 0.690141 | SAM | 0.538462 |
| NOR | 0.445455 | VIC | 0.544444 | LAD | 0.4375 |

**Table 4-1**.: Closeness Centrality of unweighted Caldas Road Network.

such as Marmato (MRM) and La Dorada (LAD) own low values of closeness centrality.

Now let us apply some dynamics to the network. First recall the model of sustainability proposed in previous chapters,

$$\begin{cases} \dot{L} = (\tau + \phi\Omega(H)S)\,L \\ \dot{S} = (\rho\,(1 - S/k_1)\,(S/k_2 - 1) - \Omega(H)L)\,S \\ \dot{K} = \gamma K^{q1}(\Omega(H)LS)^{q2} - \delta k \\ \dot{H} = GKH(1-H)\left(1 - \dfrac{L}{L_M}\right)\left(\dfrac{L}{L_m} - 1\right)L \end{cases} \tag{4-1}$$

Each node or municipality will be given certain dynamics according to equation (4-1) with different parametric values. Let us denote the state vector for each municipality $i$ as $x_i = (L_i, S_i, K_i, H_i)$, then let $F(x_i)$ the vector field of the sustainability ODE model for each of the nodes. The global equation of the network is given in (4-2).

$$\dot{x_i} = F(x_i) - \sigma \sum_{j=1}^{N} \mathfrak{L}_{ij}[h(x_j) - h(x_i)] \qquad \forall\, i = 1, ..., N \tag{4-2}$$

Here, the dynamic behavior of the $i^{th}$ node is then provided by its own vector field $F(x_i)$ plus the sum of the incoming nodes (municipalities) contribution. We have turned our graph into a complex dynamical network by allowing exchange of the state variables between neighbor nodes. We will focus this example in the case where only migration is allowed between municipalities, i.e only population state will be affected by the sum. This condition will be guaranteed by the output function $[h(x_j) - h(x_i)]$. Here $\mathfrak{L}_{ij}$ is the Laplacian matrix of the network, which provides the information of the existence of a link between a node $i$ and a node $j$ (whether exists a road or not), as well as the weight of such link. This weight is given by the preference of the population of going from one node to another, and will defined as the difference of the HDI between neighbor nodes, and inverse proportional to the distance that separates them. The whole expression of the population exchange between neighbor nodes is given in the equation (4-3).

$$\mathfrak{L}_{ij}[h(x_j) - h(x_i)] = \begin{cases} -\left[\frac{0.5(H_j - H_i) + 0.5}{D_{ij}}\right] L_j & \text{for } i \neq j \\ deg(i) L_i & \text{for } i = j \end{cases} \tag{4-3}$$

We have turned then our network into a directed, weighted one. Observe that the Laplacian of the network is time-dependant, since it depends on the value of the HDI on each one of the nodes for each integration step. Therefore, the topological characteristics of the network will be time-dependant and must be calculated for each time. $\sigma$, which accounts for the global coupling of the network is constant. Depending on the value of $\sigma$ we can obtain different types of trajectories and it will be considered as a parameter of migration control[1]. From Eq. (4-3), there appears a new matrix, i.e the distance matrix $D_{ij}$. This matrix is symmetric and constant and provides the information of the geographical distance between one municipality and other (measured in kilometers). The distance matrix was calculated from [2] and stored in an excel file *distance.xls*.

In order to have a realistic approach to the problem, we need to find the values of the parameter space, which fits the statistical data available. We find here a first drawback to the problem. Time series of population differentiated with municipalities, are available according to national census [49]. Nevertheless, a suitable quantification of the harvested land (as required by the model) is not available as far as we know. On the other hand GDP is available, for each of the nodes, but only a few samples of the time series (1993, 1997, 2002, 2004). The HDI is a measure that started in 1990, and since then, small sets of samples are available to the user for each of the nodes. Recall, however, the non-statistic nature of the approach, and the data available will be considered enough in order to have an idea of the values of parameters for each one of the municipalities.

## 4.3. Parameter Estimation and Numerical Details

All data used for the model fitting can be found in [29, 44]. Environmental data was calculated based on the national tendency of % forest area [1]. Available data was stored in an excel file *Time Series.xls*. A first gross fitting was achieved by the least squares method in the 2D space $(L, S)$. This gross fitting gives the approximate values of parameters $\phi$, $\rho$ and $\Omega$ $(\Omega_0)$. A second stage of parameter estimation is achieved with the full 4D problem where

[1]In dynamical networks, $\sigma$ provides a way to control the whole system, allowing the appearance of the so-called synchronization

parameters $\gamma$, $q_1$, $L_m$, $L_M$, $G$ and $\Delta\Omega$ are calculated. After this process, all parameters for each node are stored in the **MATLAB** file *parameters.mat*.

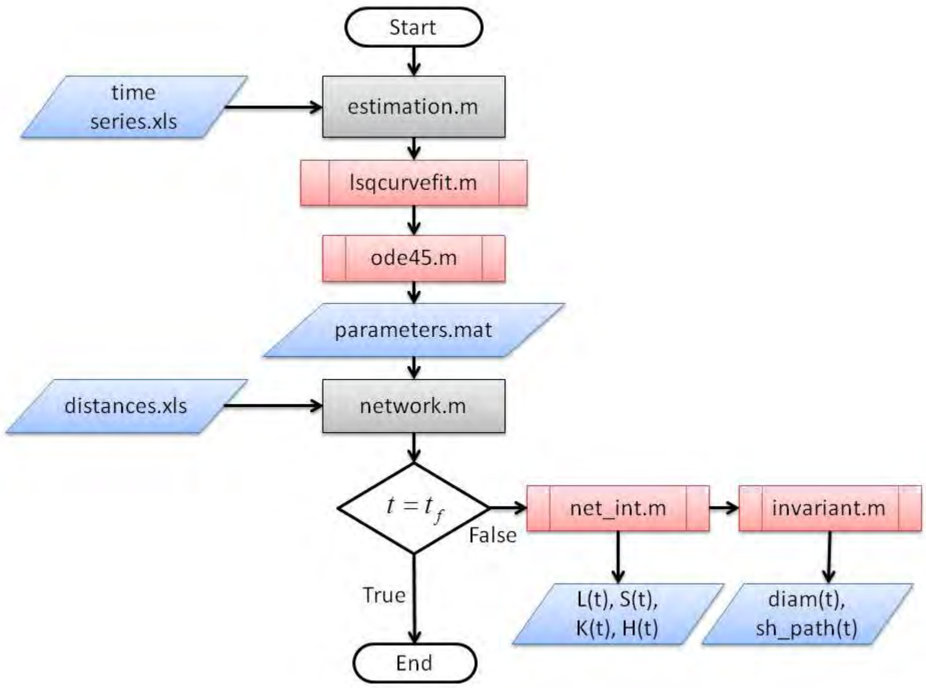



**Figure 4-3.**: Flow diagram of the used algorithm. Two main programs (estimation and network) were constructed.

Once we have calculated (approximately) the values for the parameters we then proceed to the network simulation. This simulation process was carried out through a fixed step Runge-Kutta $O^4$ integrator. In every integration step, both the states and the topological characteristics of the network are calculated. The designed functions to perform these computations are *net_int.m* and *invariant.m*. The process described is shown in an algorithmic way in Fig. **4-3**.

## 4.4. Results

This system was simulated for a 30 years period (3 time steps), taking year 1993 as the initial time of the simulation. This is because all state variables are available in the literature for this year. As a first step we took $\sigma = 0$ ($\sigma_0$ from now on), so we could appreciate the behavior of each node without any interaction at all. Results of such simulations are shown in Fig. **4-4**, where municipalities are differentiated according to its geographical situation (North, High East, High West, Low West, Center South, Caldas Magdalena). It can be observed that some municipalities such as Manizales (MZL) in Center South and La Dorada (LAD) in Magdalena Caldense, show some undesirable behavior. They reach the HDI maximum in a very short time with a subsequent resource extinction. This result is not surprising at all. The model that we have proposed stands for small societies with very simple economic schemes. We know that both Manizales and La Dorada together with some other cities like Villamaria, Chinchiná and Anserma are highly developed cities where the model can fail, since we must take into account several complex dynamics (e.g more elaborated production functions). For this reason we should focus our attention in the cases where the model actually works. Thus, we will study 3 regions which, according to literature, most agricultural cities belong. They are: North district, High East district and High West district. They will be considered as new smaller networks where topological measures like degree distribution and closeness centrality are not relevant due to the reduced number of nodes (4 nodes in North and High East and 5 nodes in High West). Nevertheless we will keep measures like diameter and mean shortest path (m.s.p. from now on) as measures that provide information of the strength of interactions between nodes. The dynamical nature of the network will make such measures, time-dependant.

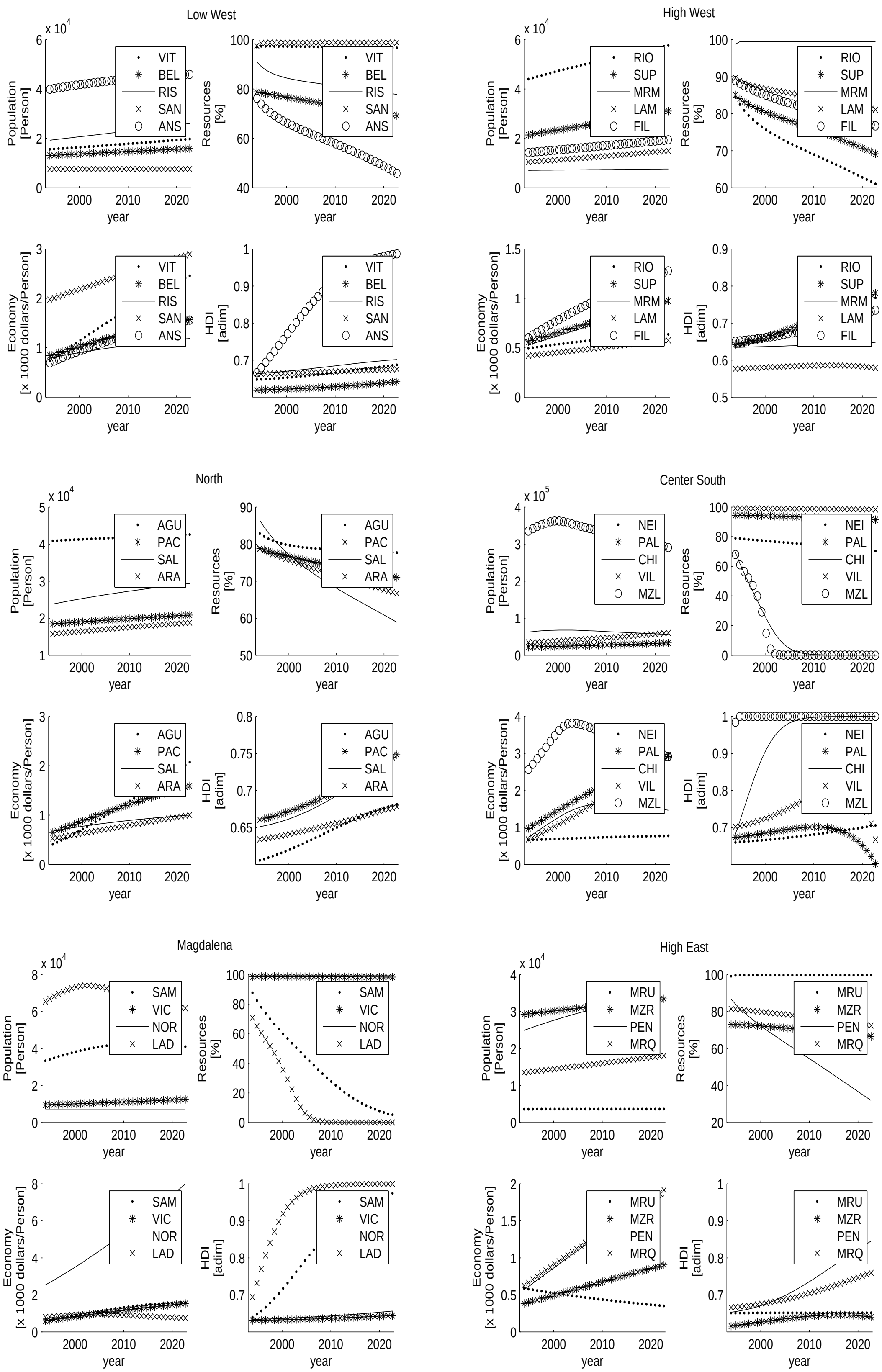
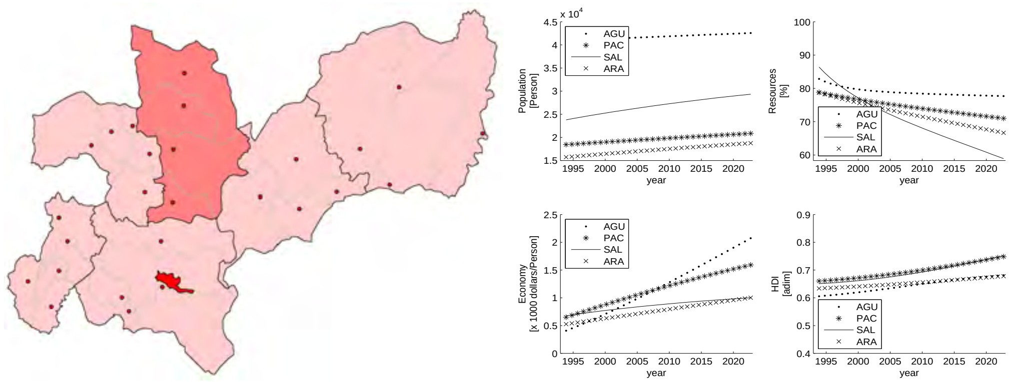



**Figure 4-4.**: Evolution of the different regions of Caldas for $\sigma_0$.

### 4.4.1. North District

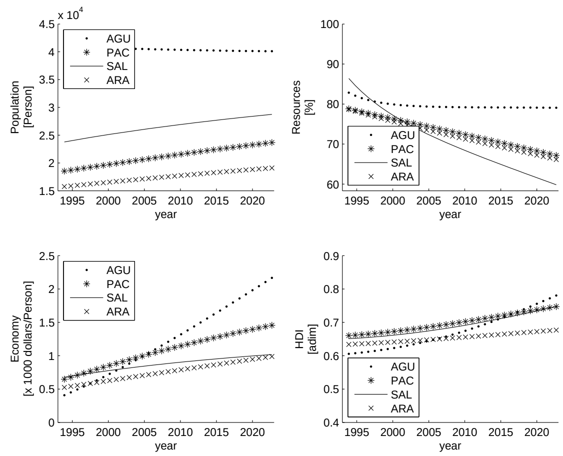



(a) (b)

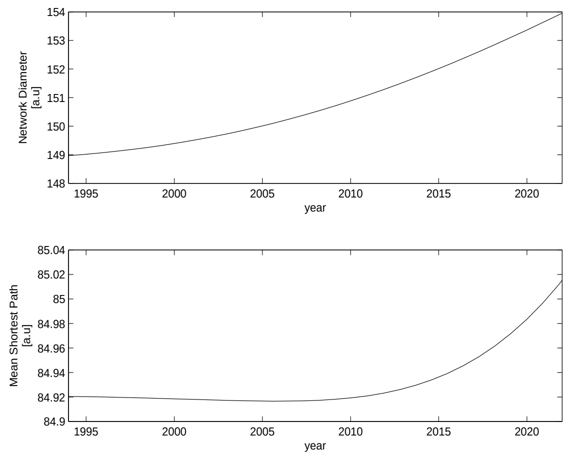
**Figure 4-5**.: (a) Geographical situation of the North district and (b) $\sigma_0$ behavior of the North distirct network. Municipalities which belong to the district: Aguadas, Pacora, Salamina and Aranzazu.

Geographical situation of North district, as well as the behavior of the system without any interaction are shown in Fig. **4-6**. For the sake of simplicity, economy will be presented in per capita income in thousands of dollars, and the quantity of resources will be normalized to the carrying capacity $S/k_2$ (for a better explanation about variable changes for simplifying the model see appendix B). At $\sigma_0$, the most favored cities are those who keep the birth rate index (fertility rate in our case) in low values. Observe that all of the slopes of the panel $L$ vs $t$ in Fig. 4.5(b) are rather small, hence guaranteeing a growth of the HDI since populations are always between the defined thresholds of $L_M$ and $L_m$. Although there exists social improvement for all of the municipalities, cities such as Pacora (PAC) and Salamina (SAL) have a remarkable growth compared with remaining cities of the district. PAC is favored by a higher per capita income compared with Aranzazu (ARA) and SAL. On the other hand SAL is helped by a higher growth in the labor force (population increase) compared with remaining nodes. Increasing labor force does not guarantee higher per capita income though, since economy is strongly related with the resource stock $S$, which is lower in SAL compared with the other cities of the district. In global terms, Aguadas (AGU) shows the most sustainable structure of the North district, because it guarantees stable population (which leads to resource conservation), a quick per capita income increase and a HDI constant growth even though it has the worst initial condition in $H(0)$.

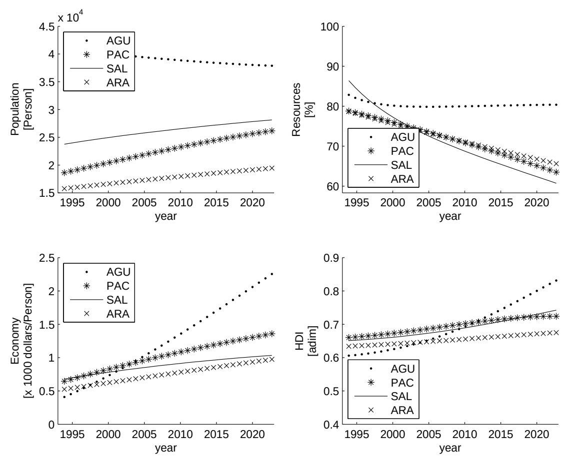
By allowing the interaction between the nodes of the network (e.g $\sigma = 1$) the dynamics changes. Given the interaction law proposed in Eq. (4-3), and recalling that AGU owns the lowest initial condition in Human Development, population of AGU will tend to decrease in the first years because the adjacent node PAC have a remarkably higher life quality. The



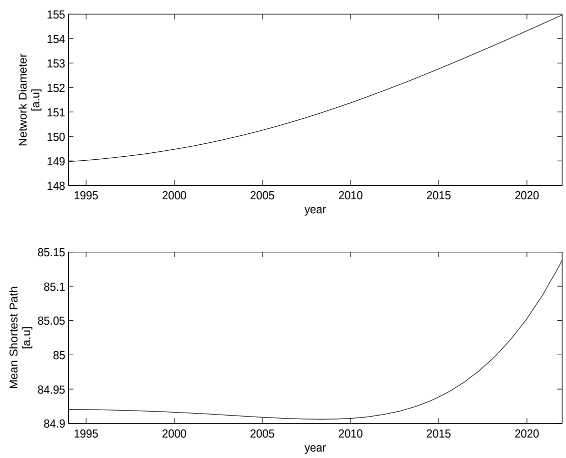

(a) State Variables $\sigma = 1$



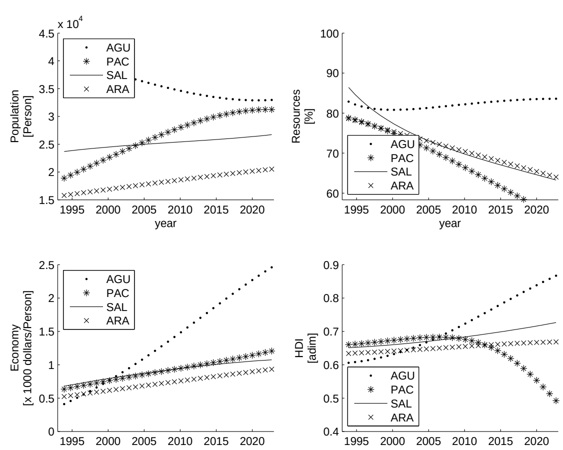

(b) Network Measures $\sigma = 1$



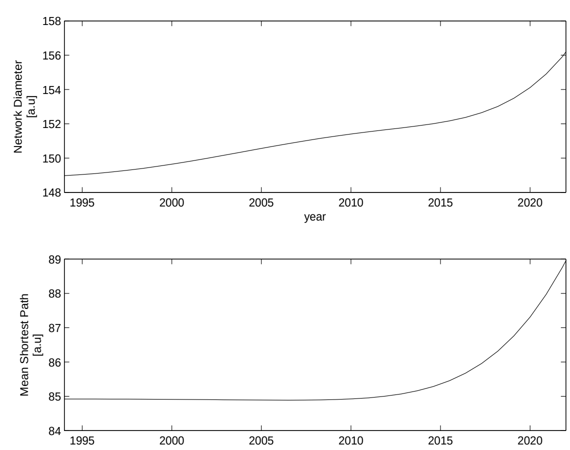

(c) State Variables $\sigma = 2$




(d) Network Measures $\sigma = 2$




(e) State Variables $\sigma = 5$



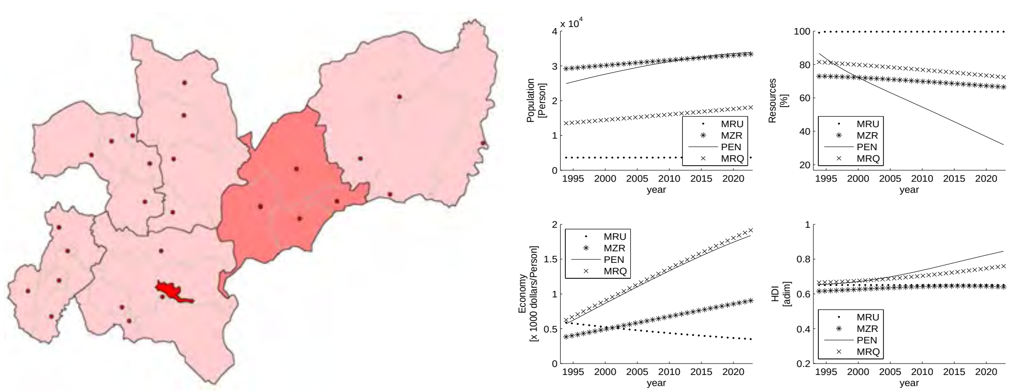

(f) Network Measures $\sigma = 5$

**Figure 4-6.**: North district evolution and topological measures for different values of $\sigma$

subtle decrease in population, together with the environmental sustainability produce a high tendency of growth in human development, going over the HDI of several municipalities in a short period of time. This result is not surprising at all. Time series of the HDI in AGU show that it is a municipality that has reached high indexes of social development. Once again, environmental stability is the main feature of AGU which favors the processes of global growth. The case of PAC for $\sigma = 1$ is remarkable also. It shows an increase of the population faster than for $\sigma = 0$. This is due to the high connectivity which is consequence of both, the central location inside the district and the good performance in HDI through the studied period. On the other hand, the characteristic measures of the network, diameter and m.s.p., show a growing tendency. This can be interpreted as a polarization process in which, as the differences on HDI between municipalities grow, there exist a higher preference to the migration to some specific nodes, while the remaining ones decrease connectivity.

By making the global coupling even higher, i.e $\sigma = 2$ and $\sigma = 5$; exchange processes in population are enhanced. Population of municipalities with higher connectivity such as PAC, grows abruptly (reaching the double of its value in 1993), to the detriment of HDI, Resource Stock and hence the capacity of economical production. It is also worth to note a subtle increase in the network connectivity measures. This means an even higher polarization on migration.

### 4.4.2. High East District




(a) (b)

**Figure 4-7**.: (a) Geographical situation of High East district and (b) $\sigma_0$ behavior of High East district network. Municipalities which belong to the district: Marulanda, Manzanares, Marquetalia and Pensilvania.

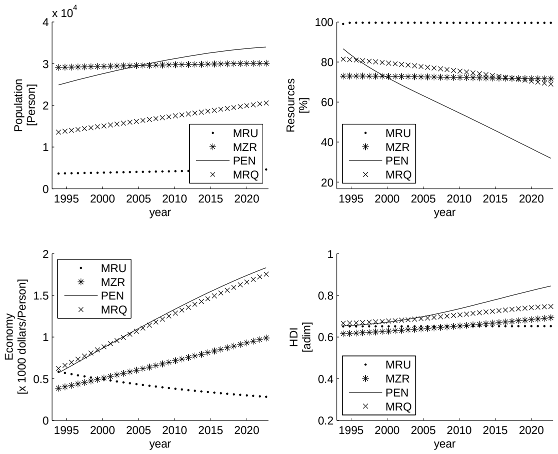
Observe geographical situation of High East district, as well as $\sigma_0$ behavior in Fig. **4-7**.

A very unusual trend appears in this network. Marulanda municipality (MRU), has a decreasing behavior in the per capita distribution of stock. This behavior appears due to the lack of demographic growth which has been stand stilled for many years (probably because of the hard geographic situation together with the forced migration due to violence). Slow population growth prevents labor force generation as mentioned before, hence stopping any social development and finally unfavoring GDP productivity. On the other hand, this municipality is benefitted environmentally. Low values of fertility function mean small harvesting. In this case MRU reaches almost the carrying capacity of its environment[2]. Remaining cities own very similar (and expected) dynamics. There are both social-economic and demographic growth together with an observable resource stock decrease. Pensilvania (PEN) has a more remarkable decrease resource stock and much faster social development than the remaining municipalities. This result, although interesting, can be consequence a of the inherent error of the estimation method. In the final phase of the simulated period, Manzanares (MZR) shows a slight decrease in HDI, which is not unexpected if we take into account the fact that it is the most populated municipality, thus grazing overpopulation threshold.

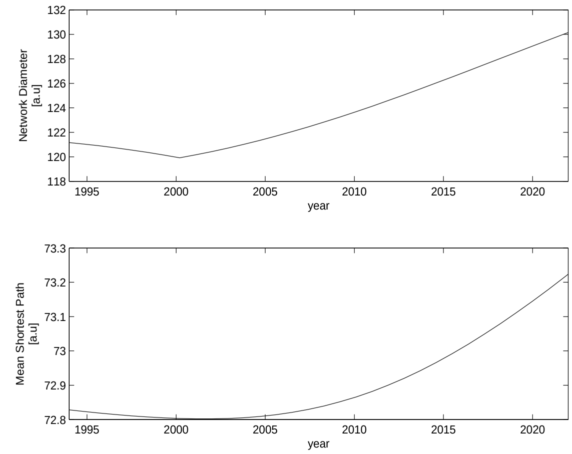
When changing $\sigma$ ($\sigma = 1$ and $\sigma = 2$) cities like MZR and Marquetalia (MRQ) are strongly affected. While population in MZR decreases, in MRQ increases in turn. This is due to the remarkable gap between development indexes in both municipalities. This produces that a part of the population in MZR node is influenced to migration to MRQ. Even if MRU could be a suitable destination for emigrants of MZR (at least in the interval 1993-2003), a migratory transition to MRQ is more probable due to its closeness (14 Km to MRQ vs 40 Km to MRU). Hence, in this case the distance criterion plays an important role in migratory policies. Stronger coupling in the network has a higher influence on the dynamics as seen for $\sigma = 5$. Population exchange is quite similar when $\sigma = 1$ and $\sigma = 2$, although with a higher rate. These high exchange rates allow the appearance of oscillations of population in the short term (different to the oscillations in system (3-2) which occur in long periods of time). These oscillations were actually observed in statistical data, so we can say that we are able to explain fluctuations in the demographical variables with a suitable migration model. Decreasing population in MZR favors per capita income, as well as it allows resource stock recovery, which turns MZR into a mid-term sustainable municipality. On the contrary, overpopulation in MRQ allows the existence of labor for economic growth, yet preventing full education cover and other social issues which produces dramatic decrease in HDI.

Finally, topological measures are quite different with regards to the North district. On one hand, values of diameter and m.s.p. are lower than in the North district. Despite the fact that the number of nodes are equal (4), the edges are different (5 in High East vs 4 in North). This fact provides shorter connections between far nodes. For instance, in order to go from ARA to AGU, one must cover through all nodes; meanwhile in High East in order to go from

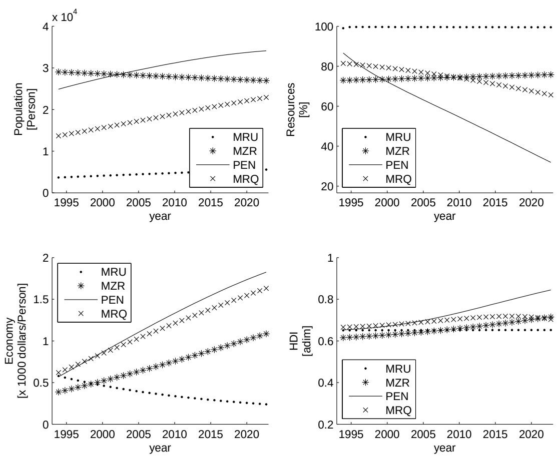
[2] For the dynamical point of view one can say that Marulanda is very close to a stable focus behavior



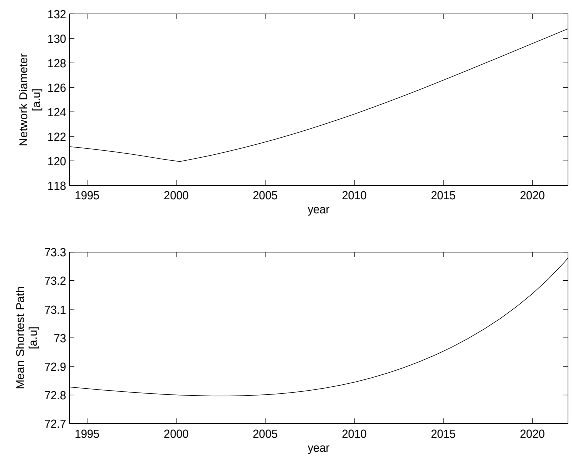

(a) State Variables $\sigma = 1$



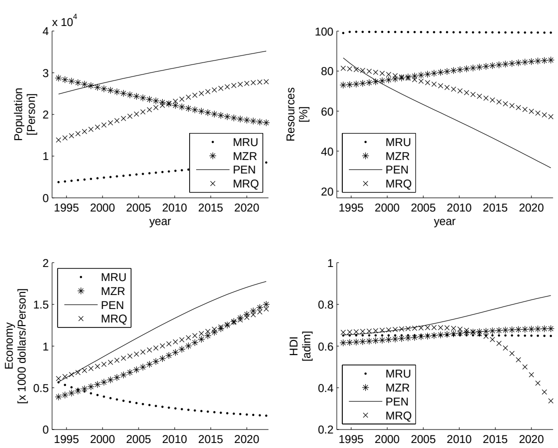

(b) Network Measures $\sigma = 1$



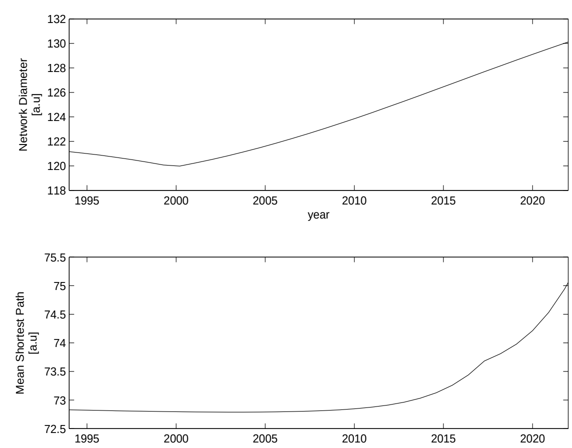

(c) State Variables $\sigma = 2$




(d) Network Measures $\sigma = 2$




(e) State Variables $\sigma = 5$

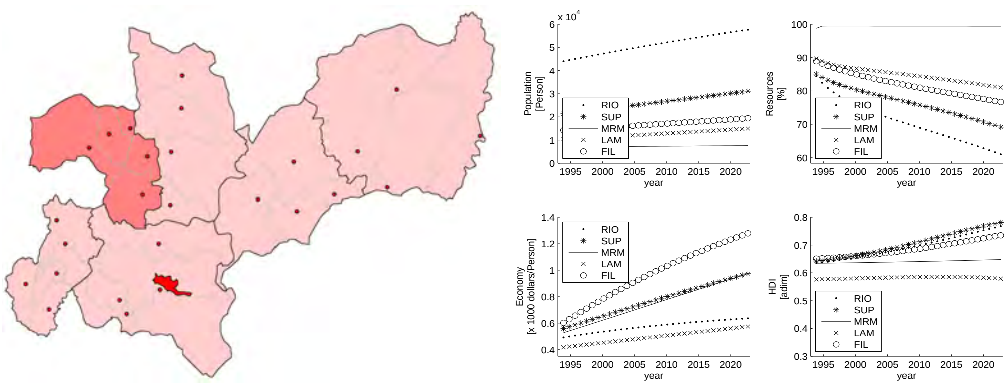



(f) Network Measures $\sigma = 5$

**Figure 4-8.**: High East district evolution and topological measures for different values of $\sigma$.

MRU to MRQ, it is only necessary to pass through MZR. In this sense, MZR is given the status of a high connected node. Another difference in the shape of the statistical measures is that diameter presents a non-smoothness and a local minimum at $\approx 2000$. Non-smoothness appears when there exists a sudden change between the nodes that define the diameter of the network, before 2000 diameter was provided by the distances[3] between PEN and MRQ; after hat, diameter is calculated as the distance between MRU and MRQ.

### 4.4.3. High West District




(a) (b)

**Figure 4-9**.: (a) Geographical situation of High West district and (b) $\sigma_0$ behavior of High West district network. Municipalities which belong to the district: Riosucio, Supia, Marmato, La Merced and Filadelfia.

Figure **4-9** shows geographical situation of our last study case, this is High West district, and its behavior for $\sigma_0$. Let us focus our attention on $\sigma_0$ behavior of municipalities Riosucio (RIO), Supia (SUP) and Filadelfia (FIL). They show an expected behavior with growing population and economy as well as Human Development in agreement with per capita income; furthermore, harvesting is somehow proportional to demographic growth as the model shall predict. Observe, however, the remaining nodes Marmato (MRM) and La Merced (LAM) with some unexpected features. On one hand, at MRM a steady population is predicted together with a maximum of resource stock ($\approx 100\%$). Steady population is due to the fact that in time series of MRM there exist several oscillations in population (see census 1918, 1938, 1951 and 1964 on [49]). These short term fluctuations are probably a consequence of foundation of new municipalities, migration to Antioquia department and other migratory processes that cannot be explained by the model we are using. Small population with high resource stock imply high per capita income which is not the real case of MRM as reported

[3] Recall that here, the expression Distance is not the actual physical distance measured in kilometers, but it is a measure of the proportion of population prone to migration

in [29]. Hence we must remark that in the case of time series with oscillations it is hard to predict correctly the paths of development of the system. On the other hand, La Merced (LAM) appears as a straggled city in human development, mainly due to a lagged economy and low technological development. Moreover, LAM is a young city (it used to belong to Salamina until 1969), then the demographic parameter estimation is highly prone to error.

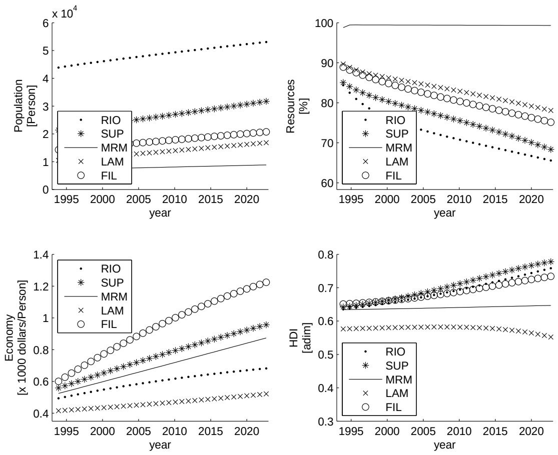
Before analyzing interactions inside the network (i.e values of $\sigma \neq 0$), we shall notice that High West network owns a topology which is slightly more complex than the previous cases since we have 5 nodes and 7 edges. Probably the most affected node by the network interaction is RIO (see slow population growth for $\sigma = 1$ and $\sigma = 2$, and population decrease at $\sigma = 5$). RIO is highly connected to SUP municipality, separated only by 14 Km. According to the development path of SUP this will exceed the human development of the rest of cities by 2005, making the population of RIO prone to migration. SUP and FIL are the nodes with more viable development paths because they guarantee both economical and social growth without inhibiting population growth. FIL has a faster economic growth with lower values of harvesting though, which makes it even more sustainable. This is due to the fact that both saving parameter and elasticity in capital investment are higher than the other nodes. MRM and LAM have again some undesired behavior in social the dimension. LAM presents a fast decrease in HDI after 2015 while MRM shows a steady behavior in it. The reasons for this type of unexpected behavior were pointed out previously, as inherent errors of optimization methods together with the lack of human population sampling before 1970.

Regarding the measures of diameter and m.s.p., it is worth to note that both measures are lower than for North and High East districts (at least for $\sigma = 1$ and $\sigma = 2$). A higher connectivity of the network (higher number of edges) makes the flow through any two nodes easier. In the picture of diameter vs time, there is a non-smoothness in the curve which can be seen between 2012-2018 depending on the value of $\sigma$. After that point diameter grows rapidly; the inflexion point coincides with the abrupt decline of HDI in LAM node, which we have concluded as an error in estimation.

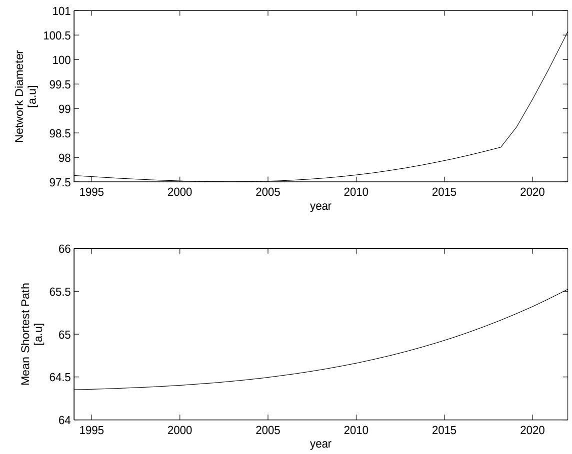
## 4.5. Conclusions and Discussion

The first approach to the network application was considering Caldas as an unweighted, undirected and static graph of municipalities (nodes) connected with roads (links). This allowed us to calculate some department measure which coincide with geographical evidence e.g. high closeness centrality of some important towns and a highly dense road network which allows the towns to be highly connected (low value of diameter and high average degree). Some other measures of the network, such as the degree distribution are not useful since the number of nodes and the edges are rather low (27 nodes and 70 edges) compared with other well-known networks (WWW network with $\sim 2 \times 10^8$ nodes). Hence it is not possible to obtain an open expression $P(k)$. When applying dynamical behavior i.e. system (3-2) to the network, we turned it into not only a weighted and directed one but also time-varying both in the states of the nodes and the topology.

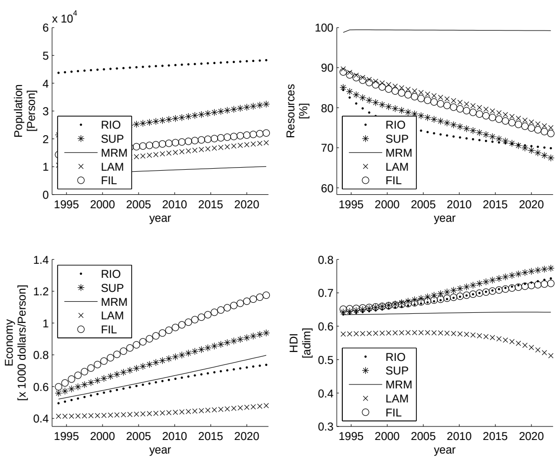
It was not possible to apply the Dynamical Complex Network to the whole Caldas department, mostly because our model failed in the explanation of highly developed economies, such as the cases of Manizales, La Dorada and Chinchiná. However, some useful information could be extracted from the simulation of North, High East and High West districts in Caldas, where we found the model to be applicable. In those cases, different development paths were observed due mainly to the different district topology which allowed them to have also different migration processes. It must be noticed that the aim of this chapter was not to provide exact forecast based on statistical data, but to provide the necessary tools for further implementation. This is because the available data is limited, and probably there are more suitable models for sustainability in cases where the economy consists on several production sectors (the case of some cities in Caldas).



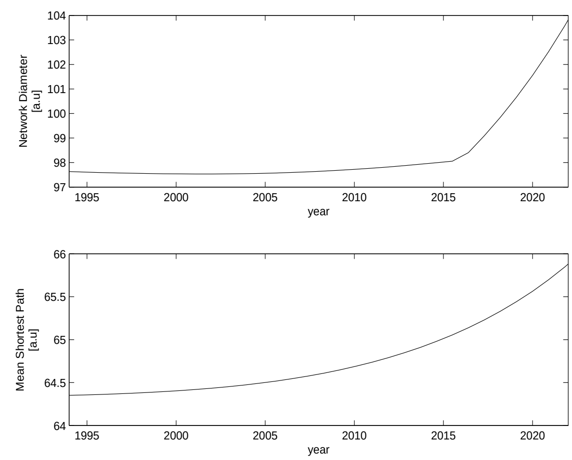

(a) State Variables $\sigma = 1$



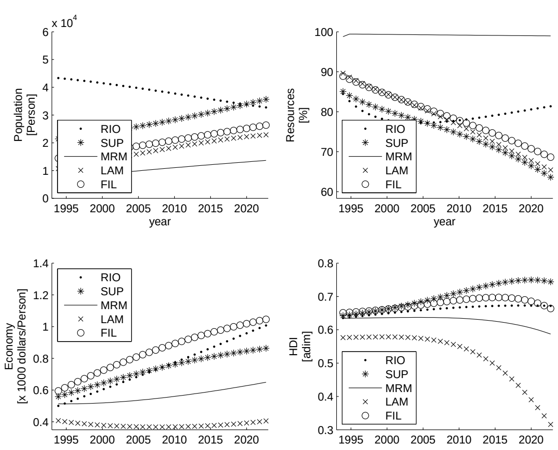

(b) Network Measures $\sigma = 1$



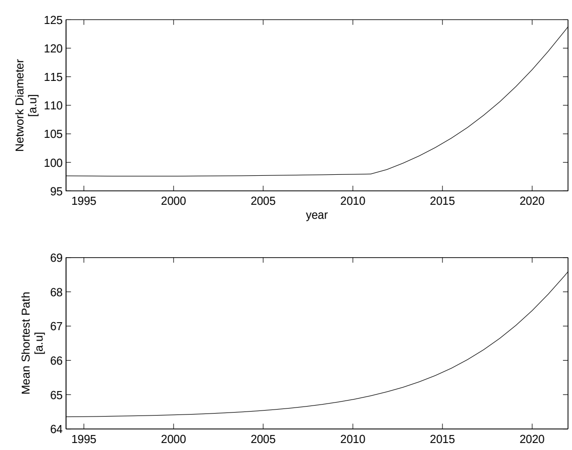

(c) State Variables $\sigma = 2$




(d) Network Measures $\sigma = 2$




(e) State Variables $\sigma = 5$




(f) Network Measures $\sigma = 5$

**Figure 4-10**.: High West district evolution and topological measures for different values of $\sigma$.

# 5. Final Discussion and Future Work

Through the previous chapters we developed a methodology and corresponding software design, in order to apply dynamic complex networks in sustainable development systems. In a first stage, we constructed a continuous time dynamical model which described the behavior of a demographic variable and the dependance of the sustainability variables with the former. A second stage covered the application of such model to a specific example. The chosen example was the case of Caldas municipalities, where we considered each municipality as a node belonging to a network. Each of the nodes where connected by the road network (links). Subsequently, we proceeded to the stage of model calibration, from available statistical data, through optimization methods. Finally, we tried to predict and conclude from the obtained results.

This chapter is devoted to the discussion of the different features of the stages of the thesis, the drawbacks that we found and the possible way to overcome them in following researches, and finally some remarks about future work.

Regarding the complex network application we must mention the innovation of the idea. Although some applied mathematics have been devoted to the study of epidemic issues and fishing in Colombia, there is, as far as we know, very few applications of graph theory and complex networks to some specific applied problem in our geographical situation. The main drawback that we found in the application of the model was the lack of data for calibrating it. In order to apply the model in a right way, one must devote a great amount of time on an statistical study (which was outside the scope of the thesis). Nevertheless, we have provided a methodology for future works through the organization of different phases in a modular way, i.e a first stage of modeling, a second stage of network design, a third phase of model calibration with parameter estimation and finally the whole application with the simulator of evolving networks (see Fig. **5-1**).

The model that we chose to work with, was the Brander-Taylor (BT) system that describes the dynamical behavior of population and resources in communities whose subsistence and economy depend exclusively in the primary economy sector. One of the main differences between the BT system and ours is the inclusion of irreversibility in regeneration process in the intrinsic growth function of the resource stock. This phenomenon is known as Allee effect and allows the existence of some particular behavior (local and global), such as Hopf

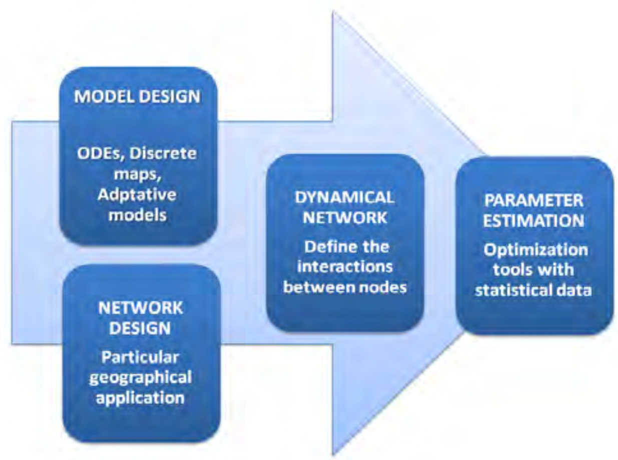



**Figure 5-1**.: Methodology proposed through the development of the thesis.

bifurcations, Branching Points and heteroclinic orbits. By including this so-called effect we have permitted the existence of oscillatory sustainability. Explanation about how Allee effect produces limit cycles can be found in [53].

One of the main features of the model is that it is highly sensitive to the population dynamics (see $H$ dynamics for example). Thus, the most straightforward way to guarantee sustainability in the long term is through the inclusion of birth control policies in the population. The aim of the control strategy is to keep the population inside the limits established by the thresholds $L_M$ and $L_m$. From the modeling viewpoint, we must apply a nonlinear control in the fertility function (2-6), as shown below:

$$F_c = \phi(\Omega S)^{u_i} \tag{5-1}$$

Here, $F_c$ denotes the controlled fertility and $u_i$ is the applied control over the population. This control was formerly proposed by [43] and showed the conservation of both resources and population (a first level of sustainability).

Implementation of such strategy must be carried out by governmental entities, that should establish different policies and campaigns to avoid early age pregnancy (in the case of communities with overpopulation tendencies) or the favoring of immigration policies and support to families with children (in the case of communities with ageing tendencies). Some advances have been made in the modeling of the proposed control, although they were not included inside the core of the thesis.

Another interesting method of population control is implemented in [32]. A predator-prey model is presented in such paper where population of the predator can be modified via hunt seasons opening. Predator population is monitored. When achieving a desired specie population the death rate is increased (by allowing hunting) so that both species are preserved. We can make the analogy to our problem by thinking not in increasing the death rate of the population, but by decreasing the fertility in the way proposed in Eq. (5-1) i.e when population reaches some desired value $L_d$, parameter $u_i$ is changed so that the fertility is decreased; when population is below that value $u_i$ is turned again to the original value.

The mathematical description of this situation can be summarized as follows

$$\begin{cases} \dot{L} = (\tau + \phi(\Omega S)^{u_i})\, L \\ \dot{S} = (\rho\,(1 - S/k_1)\,(S/k_2 - 1) - \Omega L)\, S \end{cases} \tag{5-2}$$

Commutation function will be the plane $L = L_d$, so that

$$L < L_c \Rightarrow u_i = c_1$$

$$L \geq L_c \Rightarrow u_i = c_2$$

Where $c_1 < c_2$.

A system with such discontinuities are called non-smooth systems and were deeply studied by Filippov [24]. In the particular case depicted in the Fig. **5-2**, several trajectories are simulated. When reaching the switching manifold the trajectories can either cross (when vector fields point towards the same direction) or slide (vector fields on both sides of the discontinuity, point in opposite directions). Observe the zoom in Fig. 5.2(b). Some trajectories will cross from one side of the discontinuity to the other until eventually will enter the sliding region and will reach the steady state with a limit cycle that always touches the discontinuity.

Several interesting dynamics arise from Filippov systems, such as a variety of non-smooth bifurcations and even interesting singularities such as the two fold singularity [13]. Some advances in the application of Filippov methods to development models were carried out in [4] where resources are allowed (avoided) to be exchanged between two communities when the resources are above (below) a value $S_d$. This result showed a sustainability enhancement in the communities, compared with their isolated behavior. It is interesting then, to propose as future projects, the way of implementing non-smooth controls over society and to study the efforts that must be made to achieve the desired behavior (control effort) which will result in some capital waste.

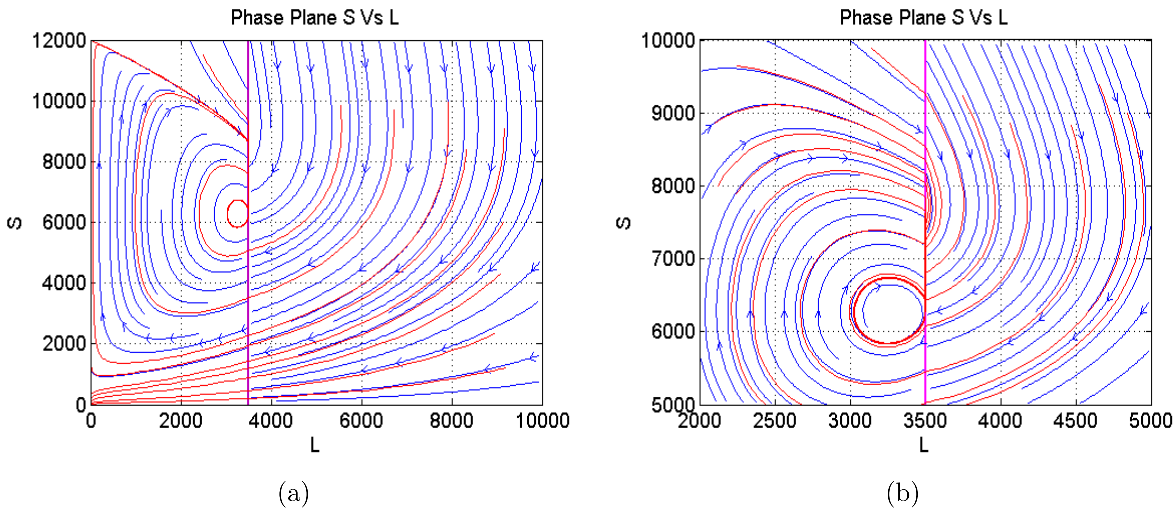
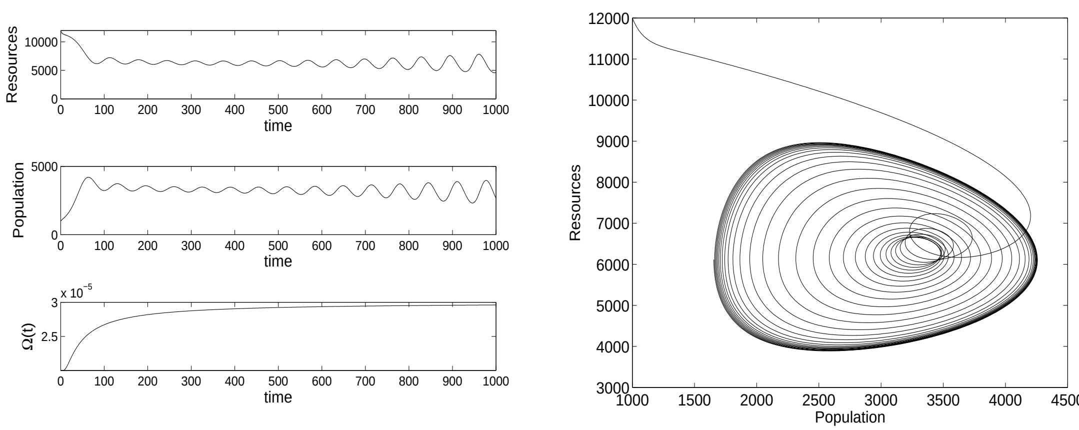



**Figure 5-2.**: Phase plane representation of the non-smooth control in a 2D system of development. $L_d$ is depicted in violet. The vector field (blue) is discontinuous at $L = L_d$. Some trajectories (red) are shown. Some of them will cross and some of them will slide.

In chapter 4,it was evident the difficulty of measuring the available resource stock (biomass of forestal area basically), because the United Nations database does not have complete information about this topic. However, there exists a growing interest in alternative measures such as the carbon footprint and adjusted net saving (closely related with the environmental variable), both with very useful available data. From these data, it is possible to establish correlations with remaining sustainability variables which allows the development of new and more contextualized models.

Through chapters 2, 3 and 4 we could appreciate the importance of the technological advances inside a society. These advances are not always beneficial in the sense that uncontrolled progress can lead to resources extinction. This is not always true, though, because technological progress can also lead to improvements in the resource management, for example the way that the environment regenerates itself can be man-improved as a consequence of education in the agricultural sector ($\rho(H)$). We propose this more accurate modeling as future work. In the same vein technology was modeled as a linear function of the human development which means that we consider the value of $\Omega$ as an endogenous one. Several approaches have mentioned the idea of modeling the technology as an exogenous quantity where it depends exclusively on time $\Omega(t)$. It is actually true that the tendency of $\Omega$ is mostly growing and in this sense we could study it as time-dependant. The reason for considering $\Omega$ as an endogenous function was to capture the implications of crisis scenarios inside society, where usually education is compromised, thus decreasing the chances of economic and social growth. Figure **5-3** shows an exogenous approach technology modeling. Once again we consider the planar system Eq. (5-3) with $\Omega(t) = \Delta\Omega e^{-1/t} + \Omega_0$. Several other types of technology models can be reviewed and studied obtaining different development paths. The study of such systems are proposed here as future work.

$$\begin{cases} \dot{L} = (\tau + \phi\Omega(t)S)\, L \\ \dot{S} = (\rho\,(1 - S/k_1)\,(S/k_2 - 1) - \Omega(t)L)\, S \end{cases} \tag{5-3}$$

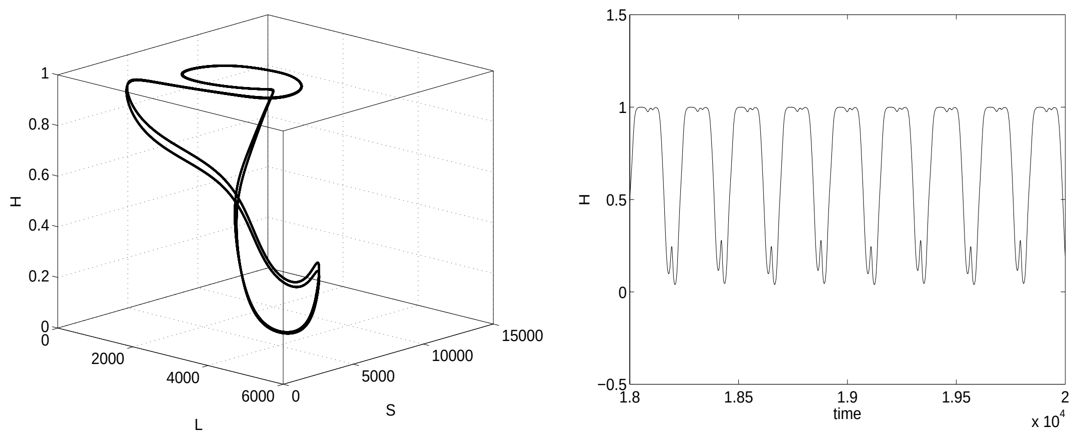



(a) (b)

**Figure 5-3**.: (a) Time series for Population, Resources, and $\Omega(t)$ and (b) Phase Portrait of (5-3)

The state variables in our sustainability model have different time scales. For instance, the HDI change ratio is rather slow compared with economical progress. Similarly, resource stock and population grow slower than economy. The differences between time scales usually lead to orbits so-called canards. Canards are of great interest in the dynamical systems framework that can be studied with singular perturbation theory. The 4-dimensional system presented in this work has the characteristics of a singular perturbed system and some research has demonstrated the existence of *canard induced mixed mode oscillations* or *MMOs* [20]. Work regarding the canard nature of the orbits in the system has being carried out in the late stages of this research. Some numerical results were compared with those in literature showing that our sustainability equations have a singular perturbed system nature which can lead to canard-like orbits (see Fig. **5-4**).

Although the system (3-2) can explain oscillations of population in the long term due to crisis periods, it is unable to predict very short term oscillations like those underwent by




(a) (b)

**Figure 5-4.**: (a) Canard orbit in the steady state of system (3-2). (b) Time series of the slow variable (Human Development in this case) which produces the MMOs

some municipalities which probably occurred due to emigration to new settlements or towns. With the network approach, we can have a better approximation to those processes. However more time must be devoted to the estimation of new parameters like the global coupling, as well as the definition of new exchange rules.

The chosen exchange rule for network evolution was exclusively a demographical rule. The reasons for this were both, to simplify the analysis and because population variable is the one which affects global system mostly. Nevertheless, it is clear that, in a network like Caldas there exist other kinds of exchange such as economical and resource stock ones. Modeling such interactions is reflected in the form of the matrix expression $[h(x_j) - h(x_i)]$. At present, inside the research team some progress has been made where economical variable plays a very important role in the development path of a network with several countries. Another important remark that we must point out is that, the distances between municipalities criterion that we adopted represented in the matrix $D_{ij}$ can be better modeled as a combination of distance and time that takes to get to one city to another. For example $D_{ij}/T_{ij}$ where $T_{ij}$ provides the information of the travel time. This is because the geographical topology in Caldas department is very rough, and even small distances can take a lot of time due to road deterioration.

A very interesting tool that could be applied to this kind of networks, is the control of the whole network through different methods. One of the main methods for controlling the dynamical behavior of a network is the so-called Edge Snapping. One can think of the Edge Snapping as a governmental control, where it can be decided which roads are available for exchange (economy, population, resources) in some periods of time. This can lead to adjust the dynamical behavior of the whole network to a desired form (synchronization), see for example [19].

One of the main drawbacks of the final model was the fact that the theoretical support of such model was made on the basis of very simple societies (e.g Easter Island). This means that, for more complicated cities one must take into account several other phenomena such as urbanity issues, more complex models of development, and so on. As a consequence of this fact we could not be able to apply the whole system into Caldas department because of the existence of a group of big cities where neither Cobb Douglas primary sector economy applies nor the social development equation does (see chapter 4). Such social complexes should be studied from a more statistical point of view, and probably discrete time models could be more suitable for quantities such as Capital Stock and Human Development Index, where information is only available yearly.

# A. Appendix: Demonstration of the bounds of $H$

**Demonstration 1** *Given Eq. 3-1, we can arrange it as follows,*

$$\frac{dH}{H(1-H)} = GKL\left(1-\frac{L}{L_M}\right)\left(\frac{L}{L_m}-1\right)dt$$

*It is straightforward that, as the system evolves, variables L and $K$ will depend on time. In this sense we can simplify the right-hand side of the equation in order to obtain separation of variables:*

$$\int_{H(0)}^{H} \frac{dH}{H(1-H)} = \int_{t_0}^{t} y(t)dt$$

*If we call the solution of the RHS of the previous equation $Y(t)$ we have:*

$$\ln\left(\frac{H}{H-1}\right) - \ln\left(\frac{H(0)}{H(0)-1}\right) = Y(t) - Y(t_0)$$

*Applying logarithm properties we can solve the equation as follows:*

$$\frac{H}{H-1} = \exp(Y(t)-Y(t_0))\left(\frac{H(0)}{H(0)-1}\right)$$

*By assuming an initial condition $H(0) \in (0,1)$ we can guarantee the negativeness of the right hand side of the equation independently of the value of $Y(t)$, in this sense, it only remains to solve the inequality,*

$$\frac{H}{H-1} < 0$$

*which is only true for values of $H \in (0,1)$.*

# B. Appendix: Reduction of the number of parameters through coordinate changes

Let us consider the 4-dimensional sustainability model (3-2). We will apply some coordinate and time scale transformations such that:

$$\bar{L} = L/L_M \qquad \bar{S} = S/k_1 \qquad \bar{K} = K/K_0 \qquad \bar{H} = H/H_0 \qquad \bar{t} = \delta t$$

Making the substitutions we obtain the transformed system and supposing, as we have so far, constant returns to scale, i.e $q_1 + q_2 = 1$ then:

$$\begin{cases} \dot{\bar{L}} = \left(\bar{\tau} + \bar{\phi}\Omega(\bar{H})\bar{S}\right)\bar{L} \\ \dot{\bar{S}} = \left(\bar{\rho}\,(1-S)\,(\bar{r}S-1) - \bar{\omega}\Omega(\bar{H})\bar{L}\right)\bar{S} \\ \dot{\bar{K}} = \bar{K}^{q_1}(\Omega(\bar{H})\bar{L}\bar{S})^{1-q_1} - \bar{K} \\ \dot{\bar{H}} = \bar{\epsilon}\bar{K}\bar{H}(1-\bar{H})\bar{L}(1-\bar{L})(\bar{\lambda}\bar{L}-1) \end{cases} \tag{B-1}$$

where the transformed parameters have the following expressions:

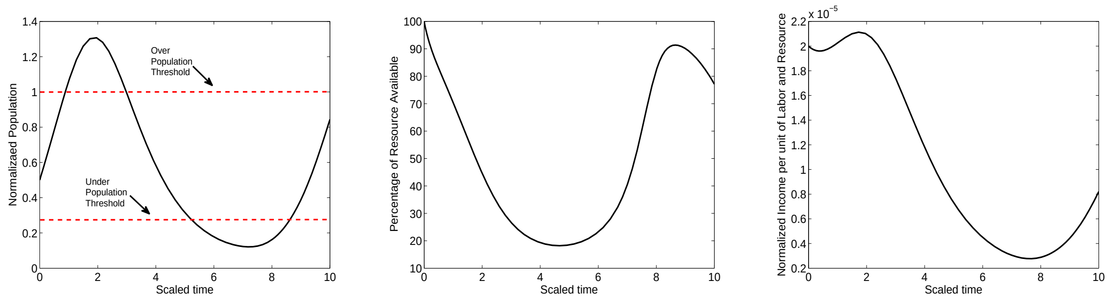
$$\bar{\tau} = \tfrac{\tau}{\delta} \qquad \bar{\phi} = \frac{\phi k_1}{\delta} \tag{B-2}$$

$$\bar{\rho} = \tfrac{\rho}{\delta} \qquad \bar{r} = \frac{k_1}{k_2} \tag{B-3}$$

$$\bar{\omega} = \tfrac{L_M}{\delta} \qquad \bar{\lambda} = \frac{L_M}{L_m} \tag{B-4}$$

$$\bar{\epsilon} = Gk_1\left(\tfrac{\delta^{q_1}}{\gamma}\right)^{1/(1-q_1)} \qquad k_0 = L_M k_1 \left(\frac{\delta}{\gamma}\right)^{1/(1-q_1)} \tag{B-5}$$

The new coordinates have an interesting interpretation. Thus, when $\bar{L} > 1$ we can say that the studied case is under an overpopulation situation. $\bar{S}$ gives the information about the proportion of available resource. $\bar{H}$ is the normalization of the human development; however we have already guaranteed a maximum value of 1, so the value $H_0 = 1$ will not change the quantitative nature of $H$. Finally, $\bar{K}$ gives the information of the quantity of goods obtained per unit of labor force and resource. Figure **B-1** shows some trajectories in the new system coordinates. A better representation of both population and resources can be obtained, as well as a reduction in the number of parameters.




(a) (b) (c)

**Figure B-1**.: Time series of the transformed system B-1 for (a) Normalized population (b) Percentage of resources and (c) Income per unit of labor force and resource.

By substituting the values of the parameters, it can be noticed that the value of $\epsilon$ is very small, thus indicating a different time scale respect to the other variables. This fact is very common in many systems which include so-called canards [20]. Canards are very interesting phenomena, with a wide variety of nonlinearities and are usually studied by singular perturbation theory.

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