Topological and algebraic characterization of coverings sets obtained in rough sets discretization and attribute reduction algorithms.

Mauricio Restrepo López
Lic. en Matemáticas, M. Sc. Matemáticas
Code: 299951

Universidad Nacional de Colombia
Facultad de Ingeniería
Departamento de Ingeniería de Sistemas e Industrial
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Code: 299951

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Advisor
Jonatan Goméz Perdomo, Ph.D.
Ph. D. in Computer Sciences

Coadvisor
Elizabeth León, Ph.D.
Ph. D. in Computer Sciences

Universidad Nacional de Colombia
Facultad de Ingeniería
Departamento de Ingeniería de Sistemas e Industrial
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Title in English

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Título en español

Caracterización topológica y algebraica de cubrimientos obtenidos mediante conjuntos aproximados, en algoritmos de discretización y reducción de atributos.

Abstract: A systematic study on approximation operators in covering based rough sets and some relations with relation based rough sets are presented. Two different frameworks of approximation operators in covering based rough sets were unified in a general framework of dual pairs. This work establishes some relationships between the most important generalization of rough set theory: Covering based and relation based rough sets. A structured genetic algorithm to discretize, to find reducts and to select approximation operators for classification problems is presented.

Resumen: Se presenta un estudio sistemático de los diferentes operadores de aproximación en conjuntos aproximados basados en cubrimientos y operadores de aproximación basados en relaciones binarias. Se unifican dos marcos de referencia sobre operadores de aproximación basados en cubrimientos en un único marco de referencia con pares duales. Se establecen algunas relaciones entre operadores de aproximación de dos de las más importantes generalizaciones de la teoría de conjuntos aproximados. Finalmente, se presenta un algoritmo genético estructurado, para discretizar, reducir atributos y seleccionar operadores de aproximación, en problemas de clasificación.

Keywords: approximation operators, covering and relation based rough sets, discretization, attributes reduction, evolutionary approach.

Palabras clave: operadores de aproximación, conjuntos aproximados basados en cubrimientos y en relaciones, discretización, reducción de atributos, aproximación evolutiva.

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Introduction

Rough set theory is a non-statistical approach to the analysis of imprecision, uncertainty and vagueness of databases in information systems. Since the proposal of the rough set theory, a big number of papers related to its theoretical foundations and to its applications have appeared. Rough set theory was introduced by Z. Pawlak in 1982 [46]. This theory extends the classical set theory in which every set is expressed by means of two approximations.

Rough sets have been used in combination with intelligent systems, fuzzy sets, genetic algorithms, neural networks, and other metaheuristics, in the development of algorithms for knowledge discovery in databases. In conjunction with fuzzy sets there are two concepts called *Fuzzy Rough Sets* and *Rough Fuzzy Sets*, which have been used to propose discretization and attribute reduction algorithms [15, 22, 25, 34, 80, 89]. Genetic algorithms and rough sets have been used in problems of rules generation and clustering methods [28]. Metaheuristics as Tabu search [27], Ant colony and Bee swarm are used for attribute reduction problems [5, 27]. Rough sets have also been used in pattern recognition, as a technique for pre-processing discretized data, reducing the number of attributes, and as a rule generation technique [2, 51].

The original definition given by Pawlak is based on equivalence relation. In order to expand the application fields, many generalizations of rough sets have been given. A first generalization of rough sets is to replace the equivalence relation by a general binary relation. In this case, the binary relation determines collections of sets that no longer form partitions. This generalization has been used in applications with incomplete information systems and tables with continuous attributes [21, 24, 82, 83]. A second generalization is to replace the partition obtained by the equivalence relation with a covering [80, 87, 101, 103, 105]. Some connections between them have been established [82, 90, 102, 100].

Unlike in the classical rough set theory, in covering based rough sets there is no a unique way to define lower and upper approximation operators. In fact, different equivalent definitions of the classical approximations cease to be equivalent when the partition is generalized by a covering. Based on this observation, Yao and Yao in [87], consider twenty pairs of lower and upper approximation operators, where each pair is governed by a duality constraint. Other operators outside this framework appear for example in [80], where Yang and Li present a summary of seven non-dual pairs of approximation operators that were used by Żakowski [88], Pomykala [50], Tsang [70], Zhu [102], Zhu and Wang [104] and Xu and Zhang [76].

From the theoretical point of view, the rough sets definition involves some topological concepts, for example interior and closure operators. Different algebraic and geometric structures, have been used for their representation [52, 79, 101].

Some works [16, 32, 81, 101, 97] show topological and algebraic relationships with rough sets. Each set can be represented as a pair of approximations, which can be used to define structures of Boolean algebras, Heyting algebras, double Stone algebras, among others. In [52] some relationships of rough sets with Grothendieck topologies, which can be viewed as a generalization of topological spaces from coverings, were studied.

Every covering of a non-empty set U defines at least one lower and an upper approximation. W. Zhu, presents different types of coverings and studies the conditions under which the approximations satisfy the original properties defined by Pawlak's approximations [95, 96, 97, 98]. Attributes reduction has been studied from reducing coverings, i.e. minimal coverings that generate the same approximations of original covering. Covering based rough sets is an area with an important development. This generalization produces an important number of different approximation operators.

The goal of this dissertation is to study the properties of approximation operators in covering based rough sets and other generalizations based on general binary relations and the topological properties of different types of coverings that are obtained in discretization and attribute reduction methods. As a framework, the work focus on generalized models of covering based rough sets, relation based rough sets and possible relationships based on the idea of adjointness given in [1, 105, 102]. Adjointness has played an important role in the theory and recent applications [7, 77].

A detailed review of the theories and fundamentals of rough sets, and its generalizations, is necessary for studying the properties of coverings from a topological and algebraic perspective. It will be necessary to study in detail the most representative covering based approximation operators and then to classify the different types of coverings obtained in these algorithms. From the perspective of rough sets, it is necessary to consider the basic notions of definable set, the positive region, the quality of classification and other associated measures. The properties of approximation operators in generalized rough sets and the relationship with attribute reduction and discretization, will be studied.

Chapter 1 presents some preliminaries of rough sets, approximation operators, covering based and relation based rough sets and basic properties of these operators. Chapter 2 shows a characterization of approximation operators using concepts of duality, conjugacy and adjointness. The main results were published in International Journal of Approximate Reasoning 55, 2014 [53]. Chapter 3 presents a partial order relation for the approximation operators in an unified framework. This was published in Information Sciences 284, 2014 [54]. Chapter 4 establishes some connections between covering and relation based rough sets. A preliminary version of this work was published in Lecture Notes on Artificial Intelligence 8537, 2014 [55]. An extended version was submitted to Information Sciences. Finally, Appendix A presents a structured genetic algorithm to discretize attributes, find reducts and to select appropriate approximation operators in classification models.

CHAPTER 1

Preliminaries

The main concept of rough set theory is the indiscernibility between objects given by an equivalence relation in a non-empty set U, called *Universe*. Many generalizations of rough set theory have been proposed. Three different definitions of approximation operators called element, granule and subsystem were presented in a general framework for the study of covering based rough sets by Yao and Yao in [87]. This framework enables us to reproduce many existing approximation operators and introduce new ones. For the element based definition, Yao and Yao consider four different neighborhood operators, in the granule based definition, they consider six new coverings defined from a covering $\mathbb{C}$. In subsystem based definition two new coverings are defined. From these neighborhoods operators, new coverings and systems, it is possible to obtain twenty pair of dual approximation operators. But, as Yao and Yao noted [87], there are other approximations out of this framework. Järvinen presents in [30] some lattice concepts for rough sets, for example, order preserving (reversing) functions, meet and join morphims, conjugates, duals and Galois connections.

This chapter presents the basic concepts related to rough set theory, their new developments and generalizations. Also, some discretization and attribute reduction algorithms, used on rough sets are presented.

1.1 Rough Set Theory

A rough set is a mathematical model to deal with approximate classification of objects.

1.1.1 Rough sets

In rough set theory information is organized in a decision table, in which the rows correspond to objects and the columns to a set of attributes that define those objects. One of the attributes is used as classification and it is called attribute decision.

Definition 1.1. *An information system is a quadruple $S = \langle U, \mathcal{A}, V, f\rangle$, where $U = \{x_1, x_2, \dots, x_n\}$ is a finite set, $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$ is a finite set of attributes, V_a is the domain of each attribute $a \in \mathcal{A}$ and V is equals to union of all domains:*

$$V = \bigcup_{a\in\mathcal{A}} V_a \tag{1.1}$$

and $f : U \times \mathcal{A} \to V$ is a function such that $f(x,a) \in V_a$, for $x \in U$. Function f is called information function.

A decision system is an information system with a decision attribute d.

Any information system can be considered as a pair $(U, \mathcal{A})$, where each U is the finite set of objects called universe and $\mathcal{A}$ the set of attributes.

Let us consider the following example, adapted from [26]. Table 1.1 shows an information system for six (6) students, with three conditional attributes (math, physics, language), which represent the performance in each of three subjects and one decision attribute (class) that expresses the student's status at the end of the first year.

Table 1.1: An information system.

Objects	**Attributes**			
Students	Math	Physics	Language	Class
1	good	good	bad	normal
2	medium	bad	bad	desertor
3	medium	bad	bad	normal
4	bad	bad	bad	desertor
5	medium	good	good	desertor
6	bad	bad	good	normal

If $P \subseteq \mathcal{A}$ is a non-empty subset of attributes, it is possible to define a relation I_P among the elements of $U = \{1,2,3,4,5,6\}$ as follows: if $x, y \in U$ we will say x is related with y and written xI_Py if and only if $f(x,a) = f(y,a)$ for all attribute $a \in P$. I_P relates object such that all the conditional attributes have the same information and therefore they can be considered like identical or indiscernible, it means, it not possible to distinguish them. The relation can be expressed like a set of ordered pairs:

$$I_P = \{(x,y) \in U \times U : f(x,a) = f(y,a), \ \forall a \in P\} \tag{1.2}$$

where, $P \subseteq \mathcal{A}$ and $P \neq \emptyset$.

I_P is an equivalence relation and it defines a partition of U in equivalence classes. The set $[x]_P$ represents the equivalence class of x.

Any union of equivalence classes is called a *definable set* and it represents the subsets of U that can be described accurately from the set of attributes P.

Figure 1.1 shows the equivalence classes for dataset in Table 1.1, obtained from the attribute set $P = \{\text{math}, \text{physics}, \text{language}\}$ and the set $A = \{1,3,6\}$ of students in status "normal".

Definition 1.2. *An approximation space, is an ordered pair (U,E), where U is a set called universe and E is an equivalence relation*[1] *on U.*

1.1.2 Approximation operators

For a set $A \subseteq U$ and an element $x \in A$, is possible that the equivalence class $I_P(x)$ is a subset of A or it has elements outside A, therefore we need to distinguish these elements, with two sets: $\underline{apr}(A)$

[1] An equivalence is a reflexive, symmetric and transitive relation.

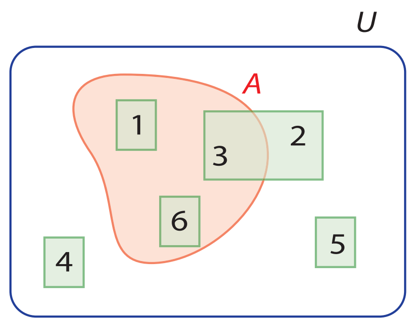

Figure 1.1: Partition of U from the equivalence relation and the set $A = \{1,3,6\}$.

and $\overline{apr}(A)$.

If S is a decision table, A a non empty subset of U and $P \subseteq \mathcal{A}$, $P \neq \emptyset$ is a subset of attributes, the lower and upper approximation of A in U are defined by means of:

$$\underline{apr}(A) = \{x \in U : [x]_P \subseteq A\} \tag{1.3}$$

$$\overline{apr}(A) = \{x \in U : [x]_P \cap A \neq \emptyset\}. \tag{1.4}$$

The elements of $\underline{apr}(A)$ are elements $x \in U$ whose equivalence class $[x]_P$, is totally contained in A and $\overline{apr}(A)$ contains elements such that A and its equivalence class have common elements.

Definition 1.3. *The boundary of a set A is defined as the difference between the upper and lower approximation:* $B_P(A) = \overline{apr}(A) - \underline{apr}(A)$.

Definition 1.4. *A set* $A \subseteq U$*, A is called a rough set if* $B_P(A) \neq \emptyset$, .

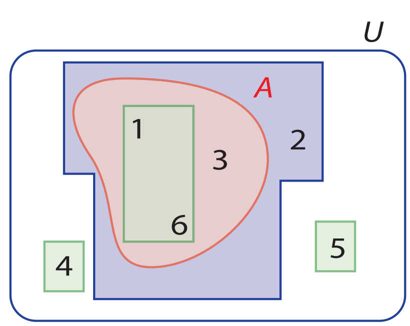

Figure 1.2: Upper and lower approximation of $A = \{1,3,6\}$.

A rough set has elements which cannot be classified as a member of a set or its complement. So, a rough set can be represent as a pair of definable sets, called lower and upper approximation.

Figure 1.2 shows the lower and upper approximations of $A = \{1,3,6,\}$: $\underline{apr}(A) = \{1,6\}$ and $\overline{apr}(A) = \{1,2,3,6\}$, respectively. In this case the boundary of A is $B_P(A) = \{2,3\}$.

Let $\emptyset$ be the empty set, A and B two subsets of U and $\sim A$ the complement of A in U. The following properties for lower and upper approximation operators have been established [82, 101].

Table 1.2: Properties of approximation operators in classical rough sets.

Name	**Property**
Duality	$\underline{apr}(\sim A) = \sim \overline{apr}(A)$
	$\overline{apr}(\sim A) = \sim \underline{apr}(A)$
Adjointness	$\overline{apr}(A) \subseteq B \Leftrightarrow A \subseteq \underline{apr}(B)$
Inclusion	$A \subseteq \overline{apr}(A)$
	$\underline{apr}(A) \subseteq A$
Monotonicity	$A \subseteq B \Rightarrow \underline{apr}(A) \subseteq \underline{apr}(B)$
	$A \subseteq B \Rightarrow \overline{apr}(A) \subseteq \overline{apr}(B)$
Meet/join-morphism	$\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$
	$\overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B)$
Idempotence	$\underline{apr}(\underline{apr}(A)) = \underline{apr}(A)$
	$\overline{apr}(\overline{apr}(A)) = \overline{apr}(A)$
Normality	$\underline{apr}(U) = U$
	$\overline{apr}(\emptyset) = \emptyset$

1.1.3 Discernibility

The relation I_P defined from an information function $f(x,y)$ as above, is called discernibility relation. This function can be used to define the discernibility matrix.

1.1.3.1 Discernibility matrix

The discernibility matrix of a decision table is a symmetric matrix $n \times n$, with entries given by:

$$c_{ij} = \{a \in \mathcal{A} : f(x_i, a) \neq f(x_j, a)\}. \tag{1.5}$$

Each c_{ij} has the attributes $a \in \mathcal{A}$, where x_i and x_j are different.

For the decision system of the Table 1.1, the discernibility matrix is shown in Table 1.3.

Table 1.3: Discernibility matrix.

Object	**1**	2	**3**	4	5	**6**
1	∅					
2	{M,P}	∅				
3	{M,P}	∅	∅			
4	{M,P}	{M}	{M}	∅		
5	{M,L}	{P,L}	{P,L}	{M,P,L}	∅	
6	{P,L}	{M,L}	{M,L}	{M,L}	{M,P}	∅

1.1.3.2 Discernibility function

The discernibility function is defined from a matrix discernibility, using a boolean function with m variables (number of attributes):

$$f_D(a_1^*, a_2^*, \ldots, a_m^*) = \wedge\{\vee c_{ij}^*, c_{ij} \neq \emptyset\}, \tag{1.6}$$

where a_i^* are boolean variables, representing attributes a_i and $c_{ij}^* = \{a^* | a \in c_{ij}\}$.

The discernibility function for the matrix of the information system is: $f_D(M, P, L) = (M \vee P) \wedge (M \vee L) \wedge M \wedge (P \vee L)$, and it is equivalent with:

$$f_D(M, P, L) = (M \wedge P) \vee (M \wedge L) \tag{1.7}$$

The expression (1.7) is the disjunctive normal form of f_D and it means that the sets of attributes $\{M, P\}$ and $\{M, L\}$ are reducts of $\{M, P, L\}$.

Let $P \subseteq \mathcal{A}$ and $a \in P$. It is said that attribute a is superfluous in P if $I_P = I_{P-\{a\}}$. Otherwise, a is indispensable in P. The set P is independent if all its attributes are indispensable.

The subset P' of P is a reduct of P, if P' is independent and $I_P = I_{P'}$.

Accuracy and quality. The accuracy of an approximation of $A \subseteq U$ with the attributes set P is defined as the ratio:

$$\alpha_P(A) = \frac{|\underline{apr}(A)|}{|\overline{apr}(A)|}, \tag{1.8}$$

where $|A|$ is the cardinal of A. If $\alpha_P(A) < 1$, A is rough respect to attributes P.

The quality of classification of $A \subseteq U$ for the attributes P is defined by means of:

$$\gamma_P(A) = \frac{|\underline{apr}(A)|}{|A|}. \tag{1.9}$$

The quality $\gamma_P(A)$ represents the ratio between the number of elements of A correctly classified by the attribute set P.

The quality of classification can be extended to a partition of U. If $\{Y_1, Y_2, \ldots, Y_k\}$ is a partition of U, i.e., if $U = \bigcup_i Y_i$, with $Y_i \cap Y_j = \emptyset$ for $i \neq j$ and $Y_i \neq \emptyset$, the quality of classification is given by:

$$\gamma_P(U) = \frac{\sum_i |\underline{apr}(Y_i)|}{|U|}. \tag{1.10}$$

This ratio represents the quality of classification of U, using the attributes set P.

1.2 Topological Spaces

The relationship between rough set theory and theory of topological spaces was recognized by many authors already in the early days of rough set theory [40, 49].

Definition 1.5. *A topology for U is a collection τ of subsets of U satisfying the following conditions:*

1. *The empty set and U belong to τ.*

2. *The union of the members of each sub-collection of τ is a member of τ.*

3. *The intersection of the members of each finite sub collection of τ is a member of τ.*

The Pair (U,τ) is called a topological space. The elements in τ are called open sets. The complement of a open set is called a closed set.

A family $\mathcal{B}$ is called a base for (U,τ) if for every non-empty open subset O of U and each $x \in O$, there exists a set $B \in \mathcal{B}$ such that $x \in O$. Equivalently, a family $\mathcal{B}$ is called a base for (U,τ) if every non-empty open subset O of U can be represented as union of a subfamily of $\mathcal{B}$.

The interior of a subset A of a topological space U is the union of the members of the family of all open sets contained in A. The interior operator on U is an operator which assigns to each subset A of U a subset A° such that the following statements (Kuratowski axioms) are true.

1. $U^\circ = U$
2. $(A^\circ)^\circ = A^\circ$
3. $A^\circ \subseteq A$
4. $(A \cap B)^\circ = A^\circ \cap B^\circ$

The closure of a subset A of a topological space U is the intersection of the members of the family of all closed sets containing A. The closure operator on U is an operator which assigns to each subset A of U a subset $\overline{A}$ such that:

1. $\overline{\emptyset} = \emptyset$
2. $\overline{\overline{A}} = \overline{A}$
3. $A \subseteq \overline{A}$
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Let $\mathbb{P}$ be a partition of U. It can easily be seen that the collection of all sets that can be written as unions of some members of $\mathbb{P}$ together with the empty set is a topology for U. This topology is called the partition topology generated by $\mathbb{P}$. The partition topologies are characterized by the fact that every open set is also closed, and vice versa. Furthermore, $\underline{apr}$ and $\overline{apr}$ are interior and closure operators, according to properties in Table 1.2. We will consider topological properties of approximation operators in generalized rough sets.

1.3 Order relation

We present some concepts about ordered structures, according to Blyth [8].

Definition 1.6. *If E is a non-empty set then by an order on E we mean a binary relation that is reflexive, anti-symmetric and transitive.*

Usually the order is denoted by the symbol $\leq$. An ordered set $(E, \leq)$ is a set E on which there is defined an order $\leq$.

In an ordered set $(E, \leq)$ we say that x is covered by y (or y covers x) if $x < y$ and there is no $z \in E$ such that $x < z < y$. We denote this by using the notation $x \prec y$.

Many ordered sets can be representing by means of a **Hasse Diagram**. In such a diagram we represent elements by points and interpret $x \prec y$, joining the points by an increasing line segment.

Definition 1.7. *If E and F are ordered sets, then the set $Map(E, F)$ of all mappings $f : E \to F$ can be ordered by defining:*

$$f \leq g \Leftrightarrow (\forall x \in E)\ f(x) \leq g(x) \tag{1.11}$$

1.3.1 Order preserving mappings

Definition 1.8. *If $(E, \leq_1)$ and $(F, \leq_2)$ are ordered sets then we say that a mapping $f : E \to F$ is isotone (or order preserving) if:*

$$(\forall x, y \in E)\ x \leq_1 y \Rightarrow\ f(x) \leq_2 f(y) \tag{1.12}$$

and is antitone (or order inverting) if:

$$(\forall x, y \in E)\ x \leq_1 y \Rightarrow\ f(x) \geq_2 f(y) \tag{1.13}$$

1.4 Three definitions of Pawlak's approximations

In Pawlak's rough set model an approximation space is an ordered pair (U, E), where U is a non-empty set and E is an equivalence relation on U. According to [87], there are three different, but equivalent ways to define lower and upper approximation operators: element based definition, granule based definition and subsystem based definition. For each $A \subseteq U$, the lower and upper approximations are defined as:

Element based definition

$$\underline{apr}(A) = \{x \in U : [x]_E \subseteq A\} \tag{1.14}$$

$$\overline{apr}(A) = \{x \in U : [x]_E \cap A \neq \emptyset\} \tag{1.15}$$

Granule based definition

$$\underline{apr}(A) = \bigcup\{[x]_E \in U/E : [x]_E \subseteq A\} \tag{1.16}$$

$$\overline{apr}(A) = \bigcup\{[x]_E \in U/E : [x]_E \cap A \neq \emptyset\} \tag{1.17}$$

Subsystem based definition

$$\underline{apr}(A) = \bigcup \{X \in \sigma(U/E) : X \subseteq A\} \tag{1.18}$$

$$\overline{apr}(A) = \bigcap \{X \in \sigma(U/E) : X \supseteq A\} \tag{1.19}$$

where $\sigma(U/E)$ is the σ algebra that is obtained from the equivalence classes U/E, by adding the empty set and making it closed under set unions.

1.5 Covering based rough sets

Many authors have investigated some generalized rough set models obtained by changing the condition that E is an equivalence relation, or equivalently, that U/E forms a partition. In a first approach the case where the partition is replaced by a covering of U is considered.

Figure 1.3 shows the evolution of two of most important generalization of rough sets: covering based rough sets and relation based rough sets.

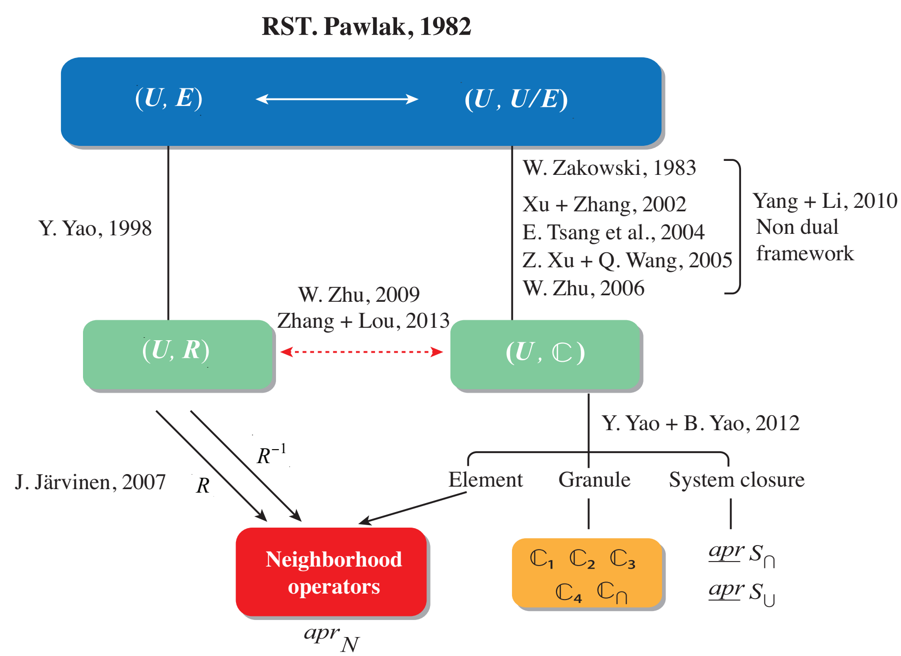

Figure 1.3: Evolution of generalization of rough sets theory.

In 1983, W. Żakwoski gave the first generalization from coverings, while Y. Yao in 1998, presented a generalization from a general binary relations, studied later by Järvinen. Yang and Li (2010) presented a framework of seven pairs of approximation operators. In 2012 Y. Yao and B. Yao presented a framework of twenty pairs of dual approximation operators. Some connections between these generalization were presented by W. Zhu (2009) and Zhang and Lou (2013).

Definition 1.9. *Let $\mathbb{C} = \{K_i\}$ be a family of nonempty subsets of U. $\mathbb{C}$ is called a covering of U if $\cup K_i = U$. The ordered pair $(U, \mathbb{C})$ is called a covering approximation space.*

It is clear that a partition generated by an equivalence relation is a special case of a covering of U, so the concept of covering is a generalization of a partition.

1.5.1 Framework of dual approximation operators

In [87] Yao and Yao proposed a general framework for the study of covering based rough sets. It is based on the observation: if the partition U/E is generalized to a covering, the different definitions of lower and upper approximations in Section 1.4 are no longer equivalent. A distinguishing characteristic of their framework is the requirement that the obtained lower and upper approximation operators form a dual pair, that is, for $A \subseteq U$, $\underline{apr}(\sim A) = \sim \overline{apr}(A)$, where $\sim A$ represents the complement of A, i.e., $\sim A = U \setminus A$.

Below, a brief review of the generalizations of the element, granule and subsystem based definitions is done.

In the element based definition, equivalence classes are replaced by neighborhood operators.

Definition 1.10. *A neighborhood operator is a mapping $N : U \to \mathcal{P}(U)$. If $N(x) \neq \emptyset$ for all $x \in U$, N is called a serial neighborhood operator. If $x \in N(x)$ for all $x \in U$, N is called a reflexive neighborhood operator.*

Each neighborhood operator defines a pair $(\underline{apr}_N, \overline{apr}_N)$ of dual approximation operators, similar to Equations 1.14 and 1.15.

$$\underline{apr}_N(A) = \{x \in U : N(x) \subseteq A\} \tag{1.20}$$

$$\overline{apr}_N(A) = \{x \in U : N(x) \cap A \neq \emptyset\} \tag{1.21}$$

Different neighborhood operators, and hence different element based definitions of covering based rough sets, can be obtained from a covering $\mathbb{C}$. For an element $x \in U$, the sets K in $\mathbb{C}$ such that $x \in K$ are interesting to define its neighborhood.

Definition 1.11. *If $\mathbb{C}$ is a covering of U and $x \in U$, a neighborhood system $C(\mathbb{C}, x)$ is defined by:*

$$C(\mathbb{C}, x) = \{K \in \mathbb{C} : x \in K\} \tag{1.22}$$

In a neighborhood system $C(\mathbb{C}, x)$, the minimal and maximal sets containing an element $x \in U$ are particularly important.

Definition 1.12. *Let $(U, \mathbb{C})$ be a covering approximation space and x in U. The set*

$$md(\mathbb{C}, x) = \{K \in C(\mathbb{C}, x) : (\forall S \in C(\mathbb{C}, x)), (S \subseteq K \Rightarrow K = S)\} \tag{1.23}$$

is called the minimal description of x [9]. On the other hand, the set

$$MD(\mathbb{C}, x) = \{K \in C(\mathbb{C}, x) : (\forall S \in C(\mathbb{C}, x)), (S \supseteq K \Rightarrow K = S)\} \tag{1.24}$$

is called the maximal description of x [105].

The sets $md(\mathbb{C},x)$ and $MD(\mathbb{C},x)$ represent extreme points of $C(\mathbb{C},x)$: for any $K \in C(\mathbb{C},x)$, there exist neighborhoods $K_1 \in md(\mathbb{C},x)$ and $K_2 \in MD(\mathbb{C},x)$ such that $K_1 \subseteq K \subseteq K_2$. From $md(\mathbb{C},x)$ and $MD(\mathbb{C},x)$, Yao and Yao [87] defined the following neighborhood operators:

1. $N_1(x) = \bigcap\{K : K \in md(\mathbb{C},x)\}$
2. $N_2(x) = \bigcup\{K : K \in md(\mathbb{C},x)\}$
3. $N_3(x) = \bigcap\{K : K \in MD(\mathbb{C},x)\}$
4. $N_4(x) = \bigcup\{K : K \in MD(\mathbb{C},x)\}$

The set $N_1(x)$ for each $x \in U$, is called the minimal neighborhood of x, and it satisfies some important properties as is shown in the following proposition [80]:

Proposition 1. *Let $\mathbb{C}$ be a covering of U and $K \in \mathbb{C}$, then*

- $K = \bigcup_{x \in K} N_1(x)$
- *If $y \in N_1(x)$ then $N_1(y) \subseteq N_1(x)$.*

Example 1. *For simplicity, a special notation for sets and collections is used. For example, the set $\{1,2,3\}$ will be denoted by 123 and the collection $\{\{1,2,3\},\{2,3,5\}\}$ will be written as $\{123,\ 235\}$. Let us consider the covering $\mathbb{C} = \{1,5,6,14,16,123,456,2345,12346,235,23456,2356,12345\}$ of $U = 123456$. The neighborhood system $C(\mathbb{C},x)$, the minimal description $md(\mathbb{C},x)$ and the maximal description $MD(\mathbb{C},x)$ are listed in Table 1.4.*

Table 1.4: Illustration of neighborhood system, minimal and maximal description.

x	$C(\mathbb{C},x)$	$md(\mathbb{C},x)$	$MD(\mathbb{C},x)$
1	{1, 14, 16, 123, 12346, 12345}	{1}	{12346, 12345}
2	{123, 2345, 12346, 235, 23456, 2356, 12345}	{123, 235}	{12346, 12345, 23456}
3	{123, 2345, 12346, 235, 23456, 2356, 12345}	{123, 235}	{12346, 12345, 23456}
4	{14, 456, 2345, 12346, 23456, 12345}	{14, 456, 2345}	{12346, 12345, 23456}
5	{5, 456, 2345, 235, 2356, 12345}	{5}	{23456, 12345}
6	{6, 16, 456, 12346, 23456, 2356}	{6}	{12346, 23456}

The four neighborhood operators obtained from $C(\mathbb{C},x)$ are listed in Table 1.5.

Table 1.5: Illustration of neighborhood operators.

x	$N_1(x)$	$N_2(x)$	$N_3(x)$	$N_4(x)$
1	1	1	1234	123456
2	23	1235	234	123456
3	23	1235	234	123456
4	4	123456	234	123456
5	5	5	2345	123456
6	6	6	2346	123456

For the set $A = 246$*, we have that* $\underline{apr}_{N_1}(A) = 46$*, because* $N_1(x) \subseteq A$ *only for* $x = 4$ *and* $x = 6$. $\underline{apr}_{N_2}(A) = 6$ *and* $\underline{apr}_{N_3}(A) = \underline{apr}_{N_4}(A) = \emptyset$*. The upper approximations are:* $\overline{apr}_{N_1}(A) = \overline{apr}_{N_2}(A) = 2346$*, and* $\overline{apr}_{N_3}(A) = \overline{apr}_{N_4}(A) = 123456$.

The following dual pairs of approximation operators based on a covering $\mathbb{C}$ are considered in [87], by generalizing the granule based in Equations (1.16) and (1.17):

$$\begin{aligned}
\underline{apr}'_{\mathbb{C}}(A) &= \bigcup\{K \in \mathbb{C} : K \subseteq A\} = \{x \in U : (\exists K \in \mathbb{C})(x \in K \wedge K \subseteq A)\} && (1.25)\\
\overline{apr}'_{\mathbb{C}}(A) &= \sim \underline{apr}'_{\mathbb{C}}(\sim A) = \{x \in U : (\forall K \in \mathbb{C})(x \in K \Rightarrow K \cap A \neq \emptyset)\} && (1.26)\\
\underline{apr}''_{\mathbb{C}}(A) &= \sim \overline{apr}''_{\mathbb{C}}(\sim A) = \{x \in U : (\forall K \in \mathbb{C})(x \in K \Rightarrow K \subseteq A)\} && (1.27)\\
\overline{apr}''_{\mathbb{C}}(A) &= \bigcup\{K \in \mathbb{C} : K \cap A \neq \emptyset\} = \{x \in U : (\exists K \in \mathbb{C})(x \in K \wedge K \cap A \neq \emptyset)\} && (1.28)
\end{aligned}$$

$(\overline{apr}'_{\mathbb{C}}, \underline{apr}'_{\mathbb{C}})$ and $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}})$ are referred to as the tight and loose pair of approximations, reflecting the fact that $\underline{apr}''_{\mathbb{C}}(A) \subseteq \underline{apr}'_{\mathbb{C}}(A) \subseteq A \subseteq \overline{apr}'_{\mathbb{C}}(A) \subseteq \overline{apr}''_{\mathbb{C}}(A)$, for all $A \subseteq U$.

Furthermore, Yao and Yao introduced four new coverings derived from a covering $\mathbb{C}$ in [87]:

1. $\mathbb{C}_1 = \cup\{md(\mathbb{C}, x) : x \in U\}$
2. $\mathbb{C}_2 = \cup\{MD(\mathbb{C}, x) : x \in U\}$
3. $\mathbb{C}_3 = \{\cap(md(\mathbb{C}, x)) : x \in U\} = \{\cap(C(\mathbb{C}, x)) : x \in U\}$
4. $\mathbb{C}_4 = \{\cup(MD(\mathbb{C}, x)) : x \in U\} = \{\cup(C(\mathbb{C}, x)) : x \in U\}$

For example, the covering $\mathbb{C}_1$ is the collection of all sets in the minimal description of each $x \in U$, while $\mathbb{C}_3$ is the collection of the intersections of minimal descriptions for each $x \in U$, i.e., $\{N_1(x) : x \in U\}$. Additionally, they considered the so-called intersection reduct $\mathbb{C}_\cap$ and union reduct $\mathbb{C}_\cup$ of a covering $\mathbb{C}$:

$$\begin{aligned}
\mathbb{C}_\cap &= \mathbb{C} \setminus \left\{K \in \mathbb{C} : (\exists \mathbb{K} \subseteq \mathbb{C} \setminus \{K\})\left(K = \bigcap \mathbb{K}\right)\right\} && (1.29)\\
\mathbb{C}_\cup &= \mathbb{C} \setminus \left\{K \in \mathbb{C} : (\exists \mathbb{K} \subseteq \mathbb{C} \setminus \{K\})\left(K = \bigcup \mathbb{K}\right)\right\} && (1.30)
\end{aligned}$$

These reducts eliminate intersection (respectively, union) reducible elements from the covering, and it can be proven that they also form a covering of U.

Each of the above six coverings determines two pairs of dual approximations given by equations (1.25) and (1.26) and equations (1.27) and (1.28), respectively.

Example 2. *The six coverings obtained from the covering* $\mathbb{C}$ *in Example 1 are:*

1. $\mathbb{C}_1 = \{1, 123, 235, 14, 456, 2345, 5, 6\}$

2. $\mathbb{C}_2 = \{12346, 12345, 23456\}$

3. $\mathbb{C}_3 = \{1, 23, 4, 5, 6\}$

4. $\mathbb{C}_4 = \{123456\}$

5. $\mathbb{C}_\cup = \{123, 456, 2345, 235, 5, 1, 14, 6\}$

6. $\mathbb{C}_\cap = \{123, 456, 2345, 12346, 5, 23456, 14, 16, 2356, 12345\}$

The lower and upper approximations of $A = 246$ *using the operators discussed above are shown in Table 1.6.*

Table 1.6: Illustration of granule-based definitions of approximation operations.

Covering	$\underline{apr}''_{\mathbb{C}}$	$\underline{apr}'_{\mathbb{C}}$	$\overline{apr}'_{\mathbb{C}}$	$\overline{apr}''_{\mathbb{C}}$
$\mathbb{C}$	$\emptyset$	6	2346	123456
$\mathbb{C}_1$	$\emptyset$	6	2346	12346
$\mathbb{C}_2$	$\emptyset$	$\emptyset$	123456	123456
$\mathbb{C}_3$	46	46	2346	2346
$\mathbb{C}_4$	$\emptyset$	$\emptyset$	123456	123456
$\mathbb{C}_\cap$	$\emptyset$	6	2346	123456
$\mathbb{C}_\cup$	$\emptyset$	$\emptyset$	12346	123456

Finally, to generalize the subsystem based definitions (1.18) and (1.19), Yao and Yao use the notion of a closure system over U, i.e., a family of subsets of U that contains U and is closed under set intersection. Given a closure system $\mathbb{S}$ over U, it is possible to construct its dual system $\mathbb{S}'$, containing the complements of each K in $\mathbb{S}$, as follows:

$$\mathbb{S}' = \{\sim K : K \in \mathbb{S}\} \tag{1.31}$$

The system $\mathbb{S}'$ contains $\emptyset$ and it is closed under set union. Given $S = (\mathbb{S}', \mathbb{S})$, a pair of dual lower and upper approximations can be defined as follows:

$$\underline{apr}_S(A) = \bigcup\{K \in \mathbb{S}' : K \subseteq A\} \tag{1.32}$$
$$\overline{apr}_S(A) = \bigcap\{K \in \mathbb{S} : K \supseteq A\} \tag{1.33}$$

As a particular example of a closure system, [87] considered the so-called intersection closure $S_{\cap,\mathbb{C}}$ of a covering $\mathbb{C}$, i.e., the minimal subset of $\mathcal{P}(U)$ that contains $\mathbb{C}$, $\emptyset$ and U, and is closed under set intersection. On the other hand, the union closure of $\mathbb{C}$, denoted by $S_{\cup,\mathbb{C}}$, is the minimal subset of $\mathcal{P}(U)$ that contains $\mathbb{C}$, $\emptyset$ and U, and is closed under set union. It can be shown that the dual system $S'_{\cup,\mathbb{C}}$ defines a closure system. Both $S_\cap = ((S_{\cap,\mathbb{C}})', S_{\cap,\mathbb{C}})$ and $S_\cup = (S_{\cup,\mathbb{C}}, (S_{\cup,\mathbb{C}})')$ along with Eqs. (1.32) and (1.33), can be used to obtain pairs of dual approximation operations.

Example 3. *For the covering* $\mathbb{C} = \{12, 124, 25, 256, 345, 26, 6\}$, *the intersection and union closure can be obtained as follows.*

- $S_{\cap,\mathbb{C}} = \mathbb{C} \cup \{\emptyset, 2, 4, 5, 123456\}$
- $S_{\cup,\mathbb{C}} = \mathbb{C} \cup \{\emptyset, 125, 1256, 1245, 126, 1245, 12456, 1246, 2345, 23456, 3456, 123456\}$

The corresponding lower approximations of $A = 246$ *are:* $\underline{apr}_{S_\cap}(A) = 246$ *and* $\underline{apr}_{S_\cup}(A) = 26$. *The upper approximations are:* $\overline{apr}_{S_\cap}(A) = 123456$ *and* $\overline{apr}_{S_\cup}(A) = 246$.

In summary, twenty pairs of dual approximation operators are defined in this framework: four from the element based definition based on neighborhood operators, fourteen from the granule based definition (based on the covering $\mathbb{C}$ and six derived coverings), and two from the subsystem based definition (using intersection and union closure). All pairs are listed in Table 1.7.

Table 1.7: List of dual pairs of approximation operators considered in [87].

#	Dual pair		#	Dual pair	
1	$\overline{apr}_{N_1}$	$\underline{apr}_{N_1}$	2	$\overline{apr}_{N_2}$	$\underline{apr}_{N_2}$
3	$\overline{apr}_{N_3}$	$\underline{apr}_{N_3}$	4	$\overline{apr}_{N_4}$	$\underline{apr}_{N_4}$
5	$\overline{apr}'_{\mathbb{C}}$	$\underline{apr}'_{\mathbb{C}}$	6	$\overline{apr}''_{\mathbb{C}}$	$\underline{apr}''_{\mathbb{C}}$
7	$\overline{apr}'_{\mathbb{C}_1}$	$\underline{apr}'_{\mathbb{C}_1}$	8	$\overline{apr}''_{\mathbb{C}_1}$	$\underline{apr}''_{\mathbb{C}_1}$
9	$\overline{apr}'_{\mathbb{C}_2}$	$\underline{apr}'_{\mathbb{C}_2}$	10	$\overline{apr}''_{\mathbb{C}_2}$	$\underline{apr}''_{\mathbb{C}_2}$
11	$\overline{apr}'_{\mathbb{C}_3}$	$\underline{apr}'_{\mathbb{C}_3}$	12	$\overline{apr}''_{\mathbb{C}_3}$	$\underline{apr}''_{\mathbb{C}_3}$
13	$\overline{apr}'_{\mathbb{C}_4}$	$\underline{apr}'_{\mathbb{C}_4}$	14	$\overline{apr}''_{\mathbb{C}_4}$	$\underline{apr}''_{\mathbb{C}_4}$
15	$\overline{apr}'_{\mathbb{C}_\cap}$	$\underline{apr}'_{\mathbb{C}_\cap}$	16	$\overline{apr}''_{\mathbb{C}_\cap}$	$\underline{apr}''_{\mathbb{C}_\cap}$
17	$\overline{apr}'_{\mathbb{C}_\cup}$	$\underline{apr}'_{\mathbb{C}_\cup}$	18	$\overline{apr}''_{\mathbb{C}_\cup}$	$\underline{apr}''_{\mathbb{C}_\cup}$
19	$\overline{apr}_{S_\cap}$	$\underline{apr}_{S_\cap}$	20	$\overline{apr}_{S_\cup}$	$\underline{apr}_{S_\cup}$

1.5.2 Framework of non-dual approximation operators

Another important line of research on covering-based rough sets has focused on pairs of approximation operators that are not necessarily dual. The first of such proposals goes back to Żakowski [88], who was in fact the first to generalize Pawlak's rough set theory from a partition to a covering. In recent papers [78, 80], a total of two lower approximation operators and seven upper approximation are summarized, which are listed below.[2]

For a covering $\mathbb{C}$ of U, the principal lower approximations for $A \subseteq U$ are:

- $L_1^{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} : K \subseteq A\} = \underline{apr}'_{\mathbb{C}}(A)$
- $L_2^{\mathbb{C}}(A) = \bigcup\{N_1(x) : x \in U \wedge N_1(x) \subseteq A\}$

It can be seen that $L_2^{\mathbb{C}}$ is the particular case of $L_1^{\mathbb{C}}$ when we use $\mathbb{C}_3$ instead of $\mathbb{C}$, so $L_2^{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_3}$. The seven upper approximations are listed as follows:

- $H_1^{\mathbb{C}}(A) = L_1^{\mathbb{C}}(A) \cup (\bigcup\{md(\mathbb{C}, x) : x \in A - L_1^{\mathbb{C}}(A)\})$
- $H_2^{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} : K \cap A \neq \emptyset\} = \overline{apr}''_{\mathbb{C}}(A)$
- $H_3^{\mathbb{C}}(A) = \bigcup\{md(\mathbb{C}, x) : x \in A\}$
- $H_4^{\mathbb{C}}(A) = L_1^{\mathbb{C}}(A) \cup (\bigcup\{K : K \cap (A - L_1^{\mathbb{C}}(A)) \neq \emptyset\})$

[2] There is no uniform notation for these approximation operators in literature. For example in [89], SH refers to the sixth upper approximation, while in [97] SH refers to the second upper approximation. For ease of reference, here we use numerical subscripts in the definitions.

- $H_5^{\mathbb{C}}(A) = \bigcup\{N_1(x) : x \in A\}$
- $H_6^{\mathbb{C}}(A) = \{x : N_1(x) \cap A \neq \emptyset\} = \overline{apr}_{N_1}(A)$
- $H_7^{\mathbb{C}}(A) = \bigcup\{N_1(x) : N_1(x) \cap A \neq \emptyset\}$

The couple $(H_1^{\mathbb{C}}, L_1^{\mathbb{C}})$ is proposed by Żakowski in [88]. Pomykala [50] considers the pair $(H_2^{\mathbb{C}}, L_1^{\mathbb{C}})$, while Tsang et al. study $(H_3^{\mathbb{C}}, L_1^{\mathbb{C}})$ in [70]. Zhu and Wang define $(H_4^{\mathbb{C}}, L_1^{\mathbb{C}})$ in [104], while $(H_5^{\mathbb{C}}, L_1^{\mathbb{C}})$ is considered by Zhu in [102]. Xu and Wang propose $(H_6^{\mathbb{C}}, L_2^{\mathbb{C}})$, and finally $(H_7^{\mathbb{C}}, L_2^{\mathbb{C}})$ is discussed by Xu and Zhang in [76].

The definitions of $L_2^{\mathbb{C}}$, $H_5^{\mathbb{C}}$, $H_6^{\mathbb{C}}$ and $H_7^{\mathbb{C}}$ can generate new approximation operators if N_1 is replaced by another neighborhood operator. The operators $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$, $H_4^{\mathbb{C}}$ and $H_5^{\mathbb{C}}$ do not appear explicitly in Yao's framework, although $H_1^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$ can be expressed as union of a lower and an upper approximation operator. For example, $H_1^{\mathbb{C}}$ can be written as:

$$H_1^{\mathbb{C}}(A) = \underline{apr}'_{\mathbb{C}}(A) \cup \overline{apr}''_{\mathbb{C}}(A - \underline{apr}'_{\mathbb{C}}(A)). \tag{1.34}$$

Example 4. *Table 1.8 presents the upper approximations for some subsets of U and the covering* $\mathbb{C} = \{12, 124, 25, 256, 345, 26, 6\}$. *Since there are no two identical columns, we can conclude that all seven upper approximations are different.*

Table 1.8: Illustration of upper approximations $H_1^{\mathbb{C}}$-$H_7^{\mathbb{C}}$.

A	$H_1^{\mathbb{C}}$	$H_2^{\mathbb{C}}$	$H_3^{\mathbb{C}}$	$H_4^{\mathbb{C}}$	$H_5^{\mathbb{C}}$	$H_6^{\mathbb{C}}$	$H_7^{\mathbb{C}}$
16	126	12456	126	1246	126	16	126
25	25	123456	123456	25	25	1235	12345
123	12345	123456	123456	12345	12345	123	12345
34	12345	12345	12345	12345	345	34	345

1.6 Relation based rough sets

Relation based rough sets is another generalization of rough set theory, where the equivalence relation is replaced by a general binary relation. In applications of data sets with missing data general binary relations are defined. Some works for incomplete information are [11, 12, 20, 23, 45]. In datasets with numerical attributes, is possible to define similarity relations from data [6, 31].

If R is a binary relation on U and $x \in U$, the sets:

$$S(x) = \{y \in U : xRy\} \text{ and } P(x) = \{y \in U : yRx\} \tag{1.35}$$

are called successor and predecessor neighborhood, respectively.

The lower and upper approximation of $A \subseteq U$, from a binary relation R on U, is given by:

$$\underline{apr}_R(A) = \{x \in U : R(x) \subseteq A\} \tag{1.36}$$

$$\overline{apr}_R(A) = \{x \in U : R(x) \cap A \neq \emptyset\} \tag{1.37}$$

where $R(x)$ can be replaced by $S(x)$ or $P(x)$. The pair (U, R) is called a relation approximation space.

Therefore, each binary relation defines two pairs of approximation operators $(\underline{apr}_S, \overline{apr}_S)$ and $(\underline{apr}_P, \overline{apr}_P)$.

Järvinen shows in [30] that the ordered pairs $(\underline{apr}_S, \overline{apr}_S)$ and $(\underline{apr}_P, \overline{apr}_P)$, defined using the element based definitions, with $S(x)$ or $P(x)$, are dual pairs.

Example 5. *The lower and upper approximations for $S(x)$ and $P(x)$ are different. Let R be the relation $R = \{(1,1),(1,2),(2,3),(2,4),(4,4)\}$ defined on U. The values for $S(x)$ are: $S(1) = \{1,2\}$, $S(2) = \{3,4\}$, $S(3) = \{\}$, $S(4) = \{4\}$, while $P(1) = \{1\}$, $P(2) = \{1\}$, $P(3) = \{2\}$, $P(4) = \{2,4\}$. If $A = \{2,3\}$, it is easy to see that $\underline{apr}_S(A) = \{\}$ and $\underline{apr}_P(A) = \{2\}$, therefore $\underline{apr}_S \neq \underline{apr}_P$. Similarly, for $\overline{apr}_S$ and $\overline{apr}_P$.*

The following proposition about relation based rough sets is presented by W. Zhu in [102].

Proposition 2. *The operators defined in Equations (1.36) and (1.37) satisfy:*

a. $\overline{apr}_R(\emptyset) = \emptyset$

b. $\underline{apr}_R(U) = U$

c. $\overline{apr}_R(A \cup B) = \overline{apr}_R(A) \cup \overline{apr}_R(B)$

d. $\underline{apr}_R(A \cap B) = \underline{apr}_R(A) \cap \underline{apr}_R(B)$

e. $\underline{apr}_R(\sim A) = \sim \overline{apr}_R(A)$

If R is a reflexive and transitive binary relation on U, then upper approximation operator is a closure operator characterized by Kuratowski axioms on U. Conversely, if $C : \mathcal{P}(U) \to \mathcal{P}(U)$ is a closure operator then there exists a reflexive and transitive relation on U such that $\overline{apr}_R(A) = C(A)$, for all A subset of U. Therefore, a closure in a topological space is also interpreted as an upper approximation operator [86].

Let L be an ordered set and $S \subseteq P$. An element $x \in L$ is an upper bound of S if $a \leqslant x$ for all $a \in S$. If there is a least element in the set of all upper bounds of S, it is called the *supremum* of S and is denoted by $\sup S$ or $\bigwedge S$. An infimum of S is defined dually.

Definition 1.13. *If L is an ordered set and and $x \wedge y$ and $x \vee y$ exists for all $x, y \in L$, then L is called a lattice. If $\bigwedge S$ and $\bigwedge S$ exist for all $S \subseteq L$, then L is called a complete lattice. Clearly, every finite lattice is complete.*

1.7 Approximation operators

This section recalls the basic relations among dual, conjugate and adjoint operators, following the ideas introduced by Järvinen in [30] in the context of lattices.

Let L be a bounded lattice with a least element 0 and greatest element 1. For $a \in L$, we say that $b \in L$ is a complement of a if $a \vee b = 1$ and $a \wedge b = 0$. A distributive and bounded lattice with complement for all $a \in L$ is called a Boolean lattice. In particular, the collection $\mathcal{P}(U)$ of subsets of a set U, with least element $\emptyset$, greatest element U and intersection, union and complement operations is a Boolean lattice.

1.7.1 Meet and join morphisms

If L, K are lattices, a map $f : L \to K$ is a complete join morphism if whenever $S \subseteq L$ and $\vee S$ exists in L, then $\vee f(S)$ exists in K and $f(\vee S) = \vee f(S)$. Analogously, a map $f : L \to K$ is a complete meet morphism if whenever $S \subseteq L$ and $\wedge S$ exists in L, then $\wedge f(S)$ exists in K and $f(\wedge S) = \wedge f(S)$.

A finite lattice is always complete, i.e. $\vee S$ and $\wedge S$ exist for all $S \subseteq L$. In this case a meet morphism f (a morphism that satisfies $f(a \wedge b) = f(a) \wedge f(b)$ for a and b in L) is a complete meet morphism, and dually, a join morphism f (a morphism that satisfies $f(a \vee b) = f(a) \vee f(b)$ for a and b in L) is a complete join morphism. Since this work, assumes that U is a finite universe, for the approximation operators will be sufficient to establish that they are meet (join, respectively) morphisms.

Some known results about the approximation operators from Section 1.5.2 are: Zhu shows that $H_2^{\mathbb{C}}$, $H_3^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$ are join morphisms in [96, 97, 105], while Wu et al. in [79] show this for $H_5^{\mathbb{C}}$.

Moreover, in [95] it was shown that the upper approximation $H_1^{\mathbb{C}}$ is a join morphism and the lower approximation $L_1^{\mathbb{C}}$ is a meet morphism if and only if $\mathbb{C}$ is an unary covering. Recall that a covering $\mathbb{C}$ is unary if for all $x \in U$, $md(\mathbb{C}, x)$ is a singleton, or equivalently if $\forall K_1, K_2 \in \mathbb{C}$, $K_1 \cap K_2$ is a union of elements of $\mathbb{C}$ [105]. As a particular example, the covering $\mathbb{C}_3$ obtained from any covering $\mathbb{C}$ is an unary covering.

1.7.2 Duality

Definition 1.14. *[30] Let $f, g : B \to B$ be two self-maps on a complete Boolean lattice B. We say that g is the dual of f, if for all $x \in B$,*

$$g(\sim x) = \sim f(x),$$

where $\sim x$ represents the complement of $x \in B$.

For any f, we denote by f^{∂} the dual of f. If $g = f^{\partial}$ then $f = g^{\partial}$.

1.7.3 Conjugacy

Definition 1.15. *Let f and g be two self-maps on a complete Boolean lattice B. We say that g is a conjugate of f, if for all $x, y \in B$,*

$$x \wedge f(y) = 0 \text{ if and only if } y \wedge g(x) = 0.$$

If g is a conjugate of f, then f is a conjugate of g. If a map f is the conjugate of itself, then f is called self-conjugate. The conjugate of f will be denoted as f^c.

Proposition 3. *Let f be a self-map on a complete Boolean lattice B. Then f has a conjugate if and only if f is a complete join morphism on B.*

From the proof given in [30], the conjugate of f is defined, for $y \in B$, by:

$$g(y) = \sim (\vee\{x : f(x) \leq \sim y\}) = \wedge\{\sim x : f(x) \wedge y = 0\} \tag{1.38}$$

In the context of rough sets, the concept of conjugate is related to upper approximation operators. It is possible to define a dual notion of co-conjugate for lower approximations, as follows:

Definition 1.16. *Let f and g be two self-maps on a complete Boolean lattice B. We say that g is a co-conjugate of f, if for all $x, y \in B$,*

$$x \vee f(y) = 1 \text{ if and only if } y \vee g(x) = 1.$$

If g is a co-conjugate of f, then f is a co-conjugate of g. If a map f is the co-conjugate of itself, then f is called self co-conjugate. The co-conjugate of f will be denoted as f_c.

Proposition 4. *If (f_1, g_1) and (f_2, g_2) are pairs of dual self-maps on a complete Boolean lattice B, then f_1 and f_2 are conjugate if and only if g_1 and g_2 are co-conjugate.*

Proof. We show just one part of the equivalence. The other part is similar.

If $f_1^c = f_2$ then, for $y \in B$:

$$\begin{aligned} x \vee g_1(y) = 1 &\Leftrightarrow \sim x \wedge \sim g_1(y) = 0 \\ &\Leftrightarrow \sim x \wedge f_1(\sim y) = 0 \\ &\Leftrightarrow \sim y \wedge f_2(\sim x) = 0 \\ &\Leftrightarrow y \vee \sim f_2(\sim x) = 1 \\ &\Leftrightarrow y \vee g_2(x) = 1 \end{aligned}$$

So, $(g_1)_c = g_2$. □

1.7.4 Adjointness

The idea of adjoint can be found in various settings in mathematics and theoretical computer science. We consider the particular case of adjoints defined on ordered sets, known as Galois connections.

Definition 1.17. *Let P and Q be two ordered sets; an ordered pair (f, g) of maps $f : P \to Q$ and $g : Q \to P$ is called a Galois connection if for all $p \in P$ and $q \in Q$,*

$$f(p) \leq q \text{ if and only if } p \leq g(q). \tag{1.39}$$

The map g is called the *adjoint* of f and will be denoted as f^a. The map f is called the *co-adjoint* of g and will be denoted as g_a. It is easy to show that the maps are order-preserving, i.e., if $p \leq p'$ then $f(p) \leq f(p')$ and if $q \leq q'$, then $g(q) \leq g(q')$, and that the adjointness condition (1.39) is equivalent to the condition that f and g are order-preserving and that for all p and q, $p \leq g(f(p))$ and $f(g(q)) \leq q$.

Conditions about the existence of adjoints and co-adjoints of a morphism between complete lattices are given in the following proposition.

Proposition 5. *Let K and L be complete lattices.*

1. *A map $f : L \to K$ has an adjoint if and only if f is a complete join morphism.*

2. *A map $g : K \to L$ has a co-adjoint if and only if g is a complete meet morphism.*

In this case, the adjoint of f is given by:

$$f^a(y) = \vee\{x \in L : f(x) \leq y\} \tag{1.40}$$

and the co-adjoint of g is obtained as:

$$g_a(y) = \wedge\{x \in K : y \leq g(x)\}. \tag{1.41}$$

The following important proposition establishes the relationship between duality, conjugacy and adjointness, and will be used frequently in the next section. [30]

Proposition 6. *Let B be a complete Boolean lattice. For any complete join morphism f on B, its adjoint is the dual of the conjugate of f. On the other hand, for any complete meet morphism g on B, its co-adjoint is the conjugate of the dual of g.*

In classical rough set theory, the lower and upper approximations $(\overline{apr}, \underline{apr})$ form a Galois connection on $\mathcal{P}(U)$. Järvinen shows in [30] that Galois connections also there exist in generalized rough set based on a binary relation. In particular, if R is a binary relation on U, $x \in U$ and $S(x)$ and $P(x)$ are the successor and predecessor neighborhoods, respectively, the ordered pairs $(\overline{apr}_P, \underline{apr}_S)$ and $(\overline{apr}_S, \underline{apr}_P)$, defined using the element based definitions, form adjoint pairs.

On the other hand, Yao [85] establishes the following important proposition which relates dual pairs of approximation operators with the relation-based generalized rough set model considered by Järvinen.

Proposition 7. *Suppose $(\overline{apr}, \underline{apr}) : \mathcal{P}(U) \to \mathcal{P}(U)$ is a dual pair of approximation operators, such that $\overline{apr}$ is a join morphism and $\overline{apr}(\emptyset) = \emptyset$. There exists a symmetric relation R on U, such that $\underline{apr}(A) = \underline{apr}_R(A)$ and $\overline{apr}(A) = \overline{apr}_R(A)$ for all $A \subseteq U$ if and only if the pair $(\overline{apr}, \underline{apr})$ satisfies: $A \subseteq \underline{apr}(\overline{apr}(A))$.*

By duality, we know that $\overline{apr}$ is join morphism if and only if $\underline{apr}$ is a meet morphism and $\overline{apr}(\emptyset) = \emptyset$ if and only if $\underline{apr}(U) = U$. According to the proof, the symmetric relation R is defined by, for x, y in U,

$$xRy \Leftrightarrow x \in \overline{apr}(\{y\}). \tag{1.42}$$

1.8 Discretization

Discretization of real value attributes is an important task in data mining, particularly for the classification problem. Empirical results show that the quality of classification methods depends on the discretization algorithm used in preprocessing step. In general, discretization is a process of searching for partition of attribute domains into intervals and unifying the values over each interval. Hence discretization problem can be defined as a problem of searching for a suitable set of cuts (i.e. boundary points of intervals) on attribute domains. [42].

1.8.1 Discretization in rough sets

The acquisition of knowledge from databases, such as rule induction or generation of decision trees, requires numerical attributes become categorical attributes. This process of converting a numerical attribute into intervals, is called discretization [4].

Discretization is based on searching for cuts that determine intervals. All values that lie within each interval are then treated as indiscernible, then cuts are placed in the middle of each interval.

Discretization methods have been developed along different lines due to different needs: Supervised vs. unsupervised, dynamic vs. static, global vs. local, splitting vs. merging. A typical process of discretization is a four-stage process, according to [36] and it is as follows.

1. The continuous values of an attribute are sorted in either descending or ascending order. If sorting is done once and for all at the beginning it is called global. If sorting is done at each iteration of a process, it is called local.
2. Evaluating a cutoff to divide (splitting) an interval or to group two intervals in one (merging). In this case evaluation functions are used to determine the correlation of a separation or union regarding the class.
3. According to some criterion, two continuous intervals can be attached or detached depending on the chosen method .
4. Stopping criterion specifies when to stop the discretization process. They usually use a maximum number of intervals or an evaluation function.

Consider the decision table $\mathbf{A} = (U, A \cup \{d\})$ where $d : U \to \{1, 2, \ldots, r(d)\}$, the attribute decision d has the values $\{1, 2, \ldots, r(d)\}$. Then suppose that attribute $a \in A$ has a domain $V_a = [l_a, r_a) \subset \mathbb{R}$. An ordered pair (a, c) with $c \in \mathbb{R}$ is a cut of V_a. A set of pairs $\{(a, c_1), (a, c_2), \ldots, (a, c_k)\}$ define a partition P_a of V_a

$$l_a = c_0 < c_1 < c_2 \cdots < c_k < c_{k+1} = r_a. \tag{1.43}$$

$P = \bigcup_{a \in A} P_a$ define a new table $A^P = (U, A^P \cup \{d\})$ where $A^P(x) = i$ if and only if $a(x) \in [c_i, c_{i+1})$, for $x \in U$ and $i \in \{0, \ldots, k\}$. Two partitions P y Q are equivalents if and only if $A^P = A^Q$. A partition P is consistent with A, if quality classification is the same in both cases: $\gamma_A(U) = \gamma_{A^P}(U)$.

A set of cuts P_{ir} is irreducible in A if P is consistent with A for all $P \subset P_{ir}$. The set of cuts P_{op} is optimum if $|P_{op}| \leq |P|$ for any P consistent with A. Find the optimum set of cuts is a NP-hard problem [29].

Of course, there exist algorithms which do not require discretizing continuous domains, neither as a pre learning step nor during the learning process; the conditional part of the rule is not expressed as a conjunction of elementary conditions. Many of these algorithms are based on the use of a relation of similarity that allows constructing similarity classes of objects, as IRBASIR algorithm and its modification presented in [6]. In this particular case, two binary relations are defined. The first one $F_1(x, y)$ is based on conditional attributes and the second one $F_2(x, y)$ on the decision attribute. The comparison functions can be defined as:

$$F_1(x, y) = \sum_{i=1}^{n} w_i \delta_i(x_i, y_i) \tag{1.44}$$

where n is the number of attributes, w_i is the weight of attribute i with $\sum w_i = 1$ and δ_i is the comparison function for attribute i.

A classical comparison function is defined as:

$$\delta_i(x_i, y_i) = \begin{cases} 1 - \frac{|x_i - y_1|}{\max(n_i) - \min(n_i)} & \text{if } i \text{ is continuous} \\ 1 & \text{if } i \text{ is discrete and } x_i = y_i \\ 0 & \text{if } i \text{ is discrete and } x_i \neq y_i \end{cases} \tag{1.45}$$

From a pair of threshold θ_1, θ_2, two relations R_i are can be defined as follows:

$$R_i(x, y) \text{ if and only if } F_i(x, y) \geq \theta_i \tag{1.46}$$

R_i are reflexive and symmetric relations. The respective neighborhood are defined by means of:

$$N_i(x) = \{y \in U : xR_iy\} \tag{1.47}$$

The problem in this point is to find functions F_1 and F_2 such that $N_1(x)$ and $N_2(x)$ have the greatest similarity and it means to calculate the weights w_i [6].

1.9 Attributes reduction

W. Zhu in [103] discussed the reduction of a covering approximation space and the conditions in which two coverings generate the same lower or upper approximations for any subset $A \subseteq U$.

The number of attributes in a data set constitutes a serious obstacle in the efficiency of most data mining algorithms. This obstacle is some times known as the curse of dimensionality. There exist many algorithms designed specifically to address the problem of attribute reduction [3].

The attribute selection problem can be presented as finding a small subset of m attributes taken a total of n of attributes of a data table, with no significant loss of performance, the total of attributes. In general, it is determined by a measure that assesses the strength of the subset of selected attributes. The main purpose is to identify significant attributes, eliminate irrelevant and to build a good model of learning.

Genetic algorithms are a good alternative to the selection of attributes in high dimensionality problems and have been used as reduction technique [12, 75].

1.9.1 Attribute reduction in rough sets

One fundamental aspect of rough set theory involves a search for particular subsets of condition attributes. Such subsets are called attribute reductions. Many types of attribute reductions have been proposed, each of the reductions aimed at some basic requirements. In [63], Skowron introduced the notion of discernibility matrix which became a major tool for searching for reductions in information systems. Using the similar idea, Zhang et al. discussed approaches to attribute reduction in inconsistent and incomplete information systems [92].

1.9.2 Reducts

It is the process of reducing an information system such that the set of attributes of reduced information system is independent and no attributes can be eliminated further, without losing some information [33]. One of the main contributions of rough sets is the ability to use their ideas to the problem of attribute reduction. Get a reduct is the process of finding subsets of attributes with the same quality in the classification of the original set. A subset B of the set of attributes A is a relative reduct if and only if B is a minimal set with respect to the property to preserve the quality.

Calculate reducts can be made from discernibility function, as shown in the previous section. The reducts can be obtained from the minimal elements (prime implicants) of the disjunctive normal form of discernibility function, for example, the reducts are given by the attributes sets: $\{M,P\}$ and $\{M,L\}$.

Given the fact that exhaustive search over the attributes space has exponential time in the number of attributes, it might not always be computational feasible to search for the minimum size reduct of attributes. The lower and upper approximations for a data set with m attributes can be obtained from the discernibility matrix in a time $O(mn^2)$, although in [44] there is an algorithm with time $O(mn \log n)$, using a space of order $O(n)$.

Moreover, the decision problem of the existence of a reduct of length k is NP-complete, while the problem of finding a reduct of minimum length is NP-hard [44]. For large databases, metaheuristics and genetic algorithms are used for finding reducts.

1.10 Decision Rules

One of the most important goals of the rough set theory is the ability to generate a model based on rules, in a classification problem. The set of rules generated can be considered as a *representation* of the knowledge acquired on all objects contained in a data table. This representation is used to classify new individuals based on their attributes.

A decision rule can be expressed as a logical implication of the form $P\ toQ$ which relates the condition attributes to the decision attribute. In general there are two types of rules. The rules that are derived from lower approximations are called exact rules, while the rules obtained from the boundary are called approximated rules.

1.10.1 Rules quantification

For a rule $P \to Q$ the concepts of support, confidence and cover can be defined, as follows:

Support:
$supp(P \to Q)$ is the number of elements which satisfy P and Q.

Confidence:
The ratio among the support and the number of elements which satisfy P.

$$Conf(P \to Q) = \frac{supp(P \to Q)}{supp(P)} \tag{1.48}$$

According to conditional probability,we can see:

$$Conf(P \to Q) = Pr(Q|P) = \frac{Pr(P \wedge Q)}{Pr(P)} \tag{1.49}$$

Cover:
The cover of the rule $P \to Q$ is the confidence of the rule $Q \to P$: $Cov(P \to Q) = conf(Q \to P)$

Also we have:

$$Cov(P \to Q) = \frac{supp(Q \to P)}{supp(Q)} \tag{1.50}$$

1.10.2 Exact and approximate rules

The exact rules can be obtained from the positive region: $Pos(U)$. In the example of Table 1.1 we have $Pos(U) = \{1,4,5,6\}$, therefore, the exact rules are:

```
R1:  (M = good) => (d = good) {1,6}
R2:  (M = bad) => (d = bad) {4}
R3:  (M = medium) & (P = good) => (d =good) {5}.
```

Approximate rules can be obtained from the boundary of each subset. Since $B(U) = \{2,3\}$, we have:

```
R4:  (M = medium) & (P = bad) => (d = good o d = bad).
```

CHAPTER 2

Approximation operators

This chapter relates the two groups of approximation operators discussed in Sections 1.5.1 and 1.5.2, and presents a new framework of dual pairs of approximation operators. Additionally, a characterization of operators that satisfy both the duality and adjointness condition is derived.

2.1 Relationship among approximation operators

The section starts with a proposition that allows to compute the adjoint of an upper approximation operator in a computationally efficient way. According to Proposition 5, an upper approximation operator $H : \mathcal{P}(U) \to \mathcal{P}(U)$ has an adjoint if and only if H is a join morphism. This adjoint is given by:

$$H^a(A) = \bigcup \{B \subseteq U : H(B) \subseteq A\}. \tag{2.1}$$

Hence, the adjoint must be calculated on subsets of U. However, the following proposition provides a less complex alternative.

Proposition 8. *If $H : \mathcal{P}(U) \to \mathcal{P}(U)$ is a join morphism, then the adjoint of H can be calculated by, for $A \subseteq U$:*

$$H^a(A) = \{x \in A : H(\{x\}) \subseteq A\}. \tag{2.2}$$

Proof. We will show that the following equality holds:

$$\{x \in A : H(\{x\}) \subseteq A\} = \cup\{B \subseteq A : H(B) \subseteq A\}.$$

If $x \in A$ and $H(\{x\}) \subseteq A$ then $\{x\} \subseteq \cup\{B \subseteq A : H(\{x\}) \subseteq A\}$ and $x \in \cup\{B \subseteq A : H(\{x\}) \subseteq A\}$. On the other hand, if $x \in \cup\{B \subseteq A : H(B) \subseteq A\}$, there exists $B = \{y_1, \ldots, y_n\}$, such that $H(B) \subseteq A$ and $x \in B$. Because H is a join morphism, $H(B) = \cup_{i=1}^n H(\{y_i\})$, from which follows that $H(\{x\}) \subseteq A$. □

This form of the adjoint approximation operator is actually the same as that of the Wybraniec-Skardowska lower approximation operator [73]. Recall that the Wybraniec-Skardowska approximation operator pair $(\overline{apr}_h, \underline{apr}_h)$ is defined as:

$$\underline{apr}_h(A) = \{x \in U : \emptyset \neq h(x) \subseteq A\} \tag{2.3}$$

$$\overline{apr}_h(A) = \bigcup_{x \in A} h(x) \tag{2.4}$$

where h is an upper approximation distribution, i.e., an $U \to \mathcal{P}(U)$ mapping that satisfies $\bigcup_{x \in U} h(x) = U$ [73]. In this case, for all upper approximations H such that $H(\{x\}) \neq \emptyset$, Equations (2.2) and (2.3) are the same. Function h can be considered as a restriction of H to the singletons. The pair of approximation operators given by Equations. (2.3) and (2.4) are not dual operators, but they are an adjoint pair, by definition.

2.1.1 Dual framework of approximation operators

This section examines the twenty pairs of approximation operators considered by Yao and Yao in [87]. First, we establish an important proposition that follows from the duality of these operators.

Proposition 9. *If $(\overline{apr}, \underline{apr})$ is a dual pair of approximation operators, then $\underline{apr}$ is a meet morphism, if and only if $\overline{apr}$ is a join morphism.*

Proof. If $\underline{apr}$ is a meet morphism, then $\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$ for $A, B \subseteq U$ and so, using the duality of $\underline{apr}$ and $\overline{apr}$, we have:

$$\begin{aligned}
\overline{apr}(A \cup B) &= \sim \underline{apr}(\sim (A \cup B)) \\
&= \sim \underline{apr}\,[(\sim A) \cap (\sim B)] \\
&= \sim \left[\underline{apr}(\sim A) \cap \underline{apr}(\sim B)\right] \\
&= \sim \underline{apr}(\sim A) \cup \sim \underline{apr}(\sim B) \\
&= \overline{apr}(A) \cup \overline{apr}(B).
\end{aligned}$$

The other part of the equivalence is similar. □

2.1.1.1 Element based definitions

Propositions 10 and 11 establish that upper (resp., lower) approximation element based definitions have adjoints (resp., co-adjoints).

Proposition 10. *For any neighborhood operator N, $\underline{apr}_N$ is a meet morphism.*

Proof. Since $\underline{apr}_N(A) = \{x \in U : N(x) \subseteq A\}$, we have $x \in \underline{apr}_N(A \cap B)$ iff $N(x) \subseteq A \cap B$ iff $N(x) \subseteq A$ and $N(x) \subseteq B$ iff $x \in \underline{apr}_N(A)$ and $x \in \underline{apr}_N(B)$ iff $x \in \underline{apr}_N(A) \cap \underline{apr}_N(B)$. □

Corollary 2.1. *For any neighborhood operator N, $\overline{apr}_N$ is a join morphism.*

Corollary 2.2. *For any neighborhood operator N, $\underline{apr}_N$ has a co-adjoint and it is equal to the conjugate of $\overline{apr}_N$.*

Proof. By Proposition 6 and the duality of $\overline{apr}_N$ and $\underline{apr}_N$, $\left(\underline{apr}_N\right)_a = \left(\underline{apr}^{\partial}_N\right)^c = (\overline{apr}_N)^c = (G_6^N)^c = G_5^N$. □

Corollary 2.3. *For any neighborhood operator N, $\overline{apr}_N$ has an adjoint and it is equal to the dual of G_5^N.*

Proof. Indeed, by Proposition 6, we find $(\overline{apr}_N)^a = ((\overline{apr}_N)^c)^\partial = \left(G_5^N\right)^\partial$. □

The remaining question now is whether $(\overline{apr}_N, \underline{apr}_N)$ can ever form an adjoint pair. For this to hold, based on the above we need to have that $\left(\underline{apr}_N\right)_a = G_5^N = G_6^N = \overline{apr}_N$.

Proposition 11. *$(\overline{apr}_N, \underline{apr}_N)$ is an adjoint pair if and only if N satisfies $G_5^N = G_6^N$.*

The following proposition characterizes the neighborhood operators N that satisfy $G_5^N = G_6^N$, and establishes the link with the generalized rough set model based on a binary relation.

Theorem 1. *Let N be a neighborhood operator. The following are equivalent:*

(i) For all x, y in U, N satisfies

$$y \in N(x) \Rightarrow x \in N(y) \tag{2.5}$$

(ii) $G_5^N = G_6^N$

(iii) There exists a symmetric binary relation R on U such that $N(x) = \{y \in U : xRy\}$.

Proof. We first prove (i) ⇒ (ii). Let $A \subseteq U$. If $w \in G_5^N(A)$, then $w \in \cup\{N(x) : x \in A\}$. This means that $w \in N(x)$ for some $x \in A$, and by (2.5) $x \in N(w)$, so $N(w) \cap A \neq \emptyset$. Hence $w \in G_6^N(A)$.

If $w \in G_6^N(A)$, then $N(w) \cap A \neq \emptyset$. In other words, there exists $x \in U$ with $x \in A$ and $x \in N(w)$. By (2.5), $w \in N(x)$ and thus $w \in \cup\{N(x) : x \in A\} = G_5^N(A)$.

On the other hand, to prove (ii) ⇒ (i), by the definition of G_5^N, we have $G_5^N(\{x\}) = N(x)$. If $G_5^N(A) = G_6^N(A)$, for all $A \subseteq U$ and $y \in N(x)$ then $N(x) \cap \{y\} \neq \emptyset$, so $x \in G_6^N(\{y\}) = G_5^N(\{y\}) = N(y)$.

Finally, the equivalence (i) ⇔ (iii) is immediate, with R defined by $xRy \Leftrightarrow x \in N(y)$ for x, y in U. □

The proposition thus shows that the only adjoint pairs among element-based definitions are those for which the neighborhood is defined by Eq. (1.35), with symmetric R. The following example shows that for none of the neighborhood operators considered in Section 1.5.1, the adjointness holds.

Example 6. *We illustrate the fact that $(\overline{apr}_{N_i}, \underline{apr}_{N_i})$ is not an adjoint pair for $i = 1, \ldots, 4$, by showing that the property $f(g(x)) \leq x$, satisfied by any Galois connection (f, g), does not hold for them.*

For the covering $\mathbb{C} = \{12, 124, 25, 256, 345, 26, 6\}$ of $U = 123456$, the neighborhoods N_i for the elements of U are shown in Table 2.1.

Table 2.1: Neighborhood operators for the covering in Example 6.

x	$N_1(x)$	$N_2(x)$	$N_3(x)$	$N_4(x)$
1	12	12	124	124
2	2	1256	2	12456
3	345	345	345	345
4	4	12345	4	12345
5	5	23456	5	23456
6	6	6	256	256

We have that:

- $\overline{apr}_{N_1}(\underline{apr}_{N_1}(45)) = \overline{apr}_{N_1}(45) = 345 \not\subseteq 45$.
- $\overline{apr}_{N_2}(\underline{apr}_{N_2}(12)) = \overline{apr}_{N_2}(1) = 124 \not\subseteq 12$.
- $\overline{apr}_{N_3}(\underline{apr}_{N_3}(12)) = \overline{apr}_{N_3}(2) = 126 \not\subseteq 2$.
- $\overline{apr}_{N_4}(\underline{apr}_{N_4}(124)) = \overline{apr}_{N_4}(124) = 12345 \not\subseteq 124$.

2.1.1.2 Granule based definitions

Propositions 12 to 19 prove some properties that provide a relationship between different granule based definitions, and between particular element and granule based definitions.

Proposition 12. $\underline{apr'}_{\mathbb{C}} = \underline{apr'}_{\mathbb{C}_1}$.

Proof. Let $A \subseteq U$ be a subset of U. For all $K \in \mathbb{C}$ with $K \subseteq A$ there exists a $K' \in md(\mathbb{C}, x)$ for some $x \in U$, such that $K' \subseteq K \subseteq A$, so $\cup\{K \in md(\mathbb{C}, x) : x \in U\} \subseteq \cup\{K \subseteq \mathbb{C} : K \subseteq A\}$, then $\underline{apr}_{\mathbb{C}_1} \leq \underline{apr}_{\mathbb{C}}$. On the other hand, if $w \in K \subseteq A$, there exists $K' \in md(\mathbb{C}, w)$ such that $w \in K' \subseteq K \subseteq A$, so $w \in \cup\{K \in \mathbb{C}_1 : K \subseteq A\}$. □

Proposition 13. $\underline{apr'}_{\mathbb{C}} = \underline{apr'}_{\mathbb{C}_\cup}$.

Proof. We will show that for each $x \in U$, $md(\mathbb{C}, x) = md(\mathbb{C}_\cup, x)$ and so, by Proposition 12, we have $\underline{apr'}_{\mathbb{C}} = \underline{apr'}_{\mathbb{C}_\cup}$.

From the definition of $\mathbb{C}_\cup$, we know that $\mathbb{C}_\cup$ is the $\cup$-reduct and $\mathbb{C}_\cup \subseteq \mathbb{C}$ and $md(\mathbb{C}, x) \subseteq md(\mathbb{C}_\cup, x)$. If $K \in md(\mathbb{C}_\cup, x)$ and let us suppose that $K \notin md(\mathbb{C}, x)$ there exists K' $md(\mathbb{C}, x)$ such that $x \in K' \subseteq K$. □

Proposition 14. $\overline{apr''}_{\mathbb{C}} = \overline{apr''}_{\mathbb{C}_2}$

Proof. Clearly, we have $\mathbb{C}_2 \subseteq \mathbb{C}$, and therefore $\overline{apr''}_{\mathbb{C}_2}(A) \subseteq \overline{apr''}_{\mathbb{C}}(A)$, for $A \subseteq U$.

On the other hand, for each $K \in \mathbb{C}$ there exists $K' \in \mathbb{C}_2$ such that $K \subseteq K'$, so $K \cap A \neq \emptyset$ implies $K' \cap A \neq \emptyset$, thus $\overline{apr''}_{\mathbb{C}}(A) \subseteq \overline{apr''}_{\mathbb{C}_2}(A)$. □

Proposition 15. $\overline{apr''}_{\mathbb{C}} = \overline{apr''}_{\mathbb{C}_\cap}$

Proof. From the relation $\mathbb{C}_\cap \subseteq \mathbb{C}$, we have $\overline{apr}''_{\mathbb{C}_\cap}(A) \subseteq \overline{apr}''_{\mathbb{C}}(A)$, for all $A \subseteq U$.

If $K \in \mathbb{C} - \mathbb{C}_\cap$ there exists $K' \subseteq \mathbb{C} - \{K\}$ such that $K = \bigcap K'$, $K \subseteq L$ for all $L \in K'$, thus if $K \cap A \neq \emptyset$ then $L \cap A \neq \emptyset$, thus $\overline{apr}''_{\mathbb{C}}(A) \subseteq \overline{apr}''_{\mathbb{C}_\cap}(A)$. □

Proposition 16. $\underline{apr}'_{\mathbb{C}_3} = \underline{apr}_{N_1}$.

Proof.

$$\begin{aligned}\underline{apr}_{N_1}(A) &= \{x \in U : N_1(x) \subseteq A\}\\ &= \cup\{N_1(x) : N_1(x) \subseteq A\} = \underline{apr}'_{\mathbb{C}_3}(A)\end{aligned}$$

For the second equality, if $w \in \underline{apr}'_{\mathbb{C}_3}(A)$, there exists x such that $w \in N_1(x) \subseteq A$. By Proposition (1), $N_1(w) \subseteq N_1(x) \subseteq A$, so $w \in \underline{apr}_{N_1}(A)$. □

From propositions 16 and 10, the following corollary can be established.

Corollary 2.4. *The approximation operator $\underline{apr}'_{\mathbb{C}_3}$ is a meet morphism, but $(\overline{apr}'_{\mathbb{C}_3}, \underline{apr}'_{\mathbb{C}_3})$ is not an adjoint pair.*

In general, the dual pairs $(\overline{apr}'_{\mathbb{C}}, \underline{apr}'_{\mathbb{C}})$ are not adjoint, because $\underline{apr}'_{\mathbb{C}} = L_1^{\mathbb{C}}$ is not a meet morphism when the covering $\mathbb{C}$ is not unary. Next, the case of the approximation operators $\overline{apr}''_{\mathbb{C}}$ and $\underline{apr}''_{\mathbb{C}}$ is studied.

Proposition 17. *$\overline{apr}''_{\mathbb{C}}$ is a self-conjugate join morphism.*

Proof. By definition $\overline{apr}''_{\mathbb{C}} = H_2^{\mathbb{C}}$, so it is a join morphism and by Proposition 42 it is self-conjugate. □

Using Proposition 17, it is possible to establish that $\underline{apr}''_{\mathbb{C}}$ is a meet morphism, which allows us to prove the following result.

Proposition 18. *The pair $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}})$ is an adjoint pair.*

Proof. From propositions 6 and 17, we have: $(\overline{apr}''_{\mathbb{C}})^a = [(\overline{apr}''_{\mathbb{C}})^c]^\partial = [\overline{apr}''_{\mathbb{C}}]^\partial = \underline{apr}''_{\mathbb{C}}$ and $(\underline{apr}''_{\mathbb{C}})_a = [(\underline{apr}''_{\mathbb{C}})^\partial]^c = [\overline{apr}''_{\mathbb{C}}]^c = \overline{apr}''_{\mathbb{C}}$. □

Moreover, the following proposition shows that this adjoint pair can also be seen as a particular case of an element-based definition.

Proposition 19. $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}}) = (\overline{apr}_N, \underline{apr}_N)$*, where N is defined by*

$$N(x) = \{y \in U : (\exists K \in \mathbb{C})(x \in K \wedge y \in K)\} \tag{2.6}$$

Proof. By Proposition 7, there exists a symmetric relation R on U such that $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}}) = (\overline{apr}_R, \underline{apr}_R)$, where $xRy \Leftrightarrow x \in \overline{apr}''_{\mathbb{C}}(\{y\}) \Leftrightarrow x \in \cup\{K \in \mathbb{C} : K \cap \{y\} \neq \emptyset\} \Leftrightarrow x \in \cup\{K \in \mathbb{C} : y \in K\}$.

Putting $N(x) = R(x)$, we find that $y \in N(x)$ if and only if there exists $K \in \mathbb{C}$ such that $x \in K$ and $y \in K$, or in other words $N(x) = \{y \in U : (\exists K \in \mathbb{C})(x \in K \wedge y \in K)\}$. □

Summarizing, only the loose pair of granule based approximation operators in Yao and Yao's framework is an adjoint pair, and moreover this pair coincides with a particular element-based definition.

To conclude this section, I point out an error in [87]: it is stated there on page 104 that $\underline{apr}''_{\mathbb{C}} = \underline{apr}_{N_4}$ and $\overline{apr}''_{\mathbb{C}} = \overline{apr}_{N_4}$, however this incorrect; in particular, knowing that $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}})$ is an adjoint pair and $(\overline{apr}_{N_4}, \underline{apr}_{N_4})$ is not, this equality cannot hold.

2.1.1.3 Subsystem based definitions

First, looking at the definitions it is possible to show that $\underline{apr}_{S_\cap} = L_1^{S_{\cap,\mathbb{C}}}$ and $\underline{apr}_{S_\cup} = L_1^{(S_{\cup,\mathbb{C}})'}$. Furthermore, the following relationship between $\underline{apr}_{S_\cup}$ and the granule-based model can be established.

Proposition 20. $\underline{apr}_{S_\cup} = \underline{apr}'_{\mathbb{C}}$

Proof. Clearly, $\mathbb{C} \subseteq S_{\cup,\mathbb{C}}$, and therefore $\underline{apr}'_{\mathbb{C}}(A) \subseteq \underline{apr}_{S_\cup}(A)$, for $A \subseteq U$.

On the other hand, for each $X \in S_{\cup,\mathbb{C}}$, there exists $\mathbb{K} \subseteq \mathbb{C}$ such that $X = \bigcup \mathbb{K}$, thus if $X \subseteq A$ then $L \subseteq A$, for all $L \in \mathbb{K}$. Hence $\cup\{X \in S_{\cup,\mathbb{C}} : X \subseteq A\} \subseteq \cup\{K \in \mathbb{C} : K \subseteq A\}$, i.e., $\underline{apr}_{S_\cup}(A) \subseteq \underline{apr}'_{\mathbb{C}}(A)$. □

This equality can be also be understood by the fact that adding unions of elements to a covering does not refine the covering.

The following example shows that the approximation operators $\underline{apr}_{S_\cap}$ and $\underline{apr}_{S_\cup}$ are not meet morphisms, and neither $\overline{apr}_{S_\cap}$ nor $\overline{apr}_{S_\cup}$ are join morphisms, so they cannot form adjoint pairs.

Example 7. *Consider the subsystems $S_\cap$ and $S_\cup$ from Example 9.*

If $A = 123$, $B = 2456$, then $A \cap B = 2$, $\underline{apr}_{S_\cap}(A) = 123$, $\underline{apr}_{S_\cap}(B) = 2456$ and $\underline{apr}_{S_\cap}(A \cap B) = \emptyset$, then $\underline{apr}_{S_\cap}(A \cap B) \neq \underline{apr}_{S_\cap}(A) \cap \underline{apr}_{S_\cap}(B) = 2$.

On the other hand, If $A = 1236$, $B = 1235$, then $A \cap B = 23$, $\underline{apr}_{S_\cup}(A) = 1236$, $\underline{apr}_{S_\cup}(B) = 1235$ and $\underline{apr}_{S_\cup}(A \cap B) = \emptyset$, then $\underline{apr}_{S_\cup}(A \cap B) \neq \underline{apr}_{S_\cup}(A) \cap \underline{apr}_{S_\cup}(B) = 123$.

Analogously, it can be verified that $\overline{apr}_{S_\cap}$ and $\overline{apr}_{S_\cup}$ are not join morphisms.

2.1.2 Non-dual framework of approximation operators

This section establishes an important conjugacy relation between the upper approximation operators $H_5^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$. This relationship holds regardless of the neighborhood operator N which is used in the definition, so it begins by proving the following more general proposition.

Proposition 21. *Let N be a neighborhood operator and $G_5^N(A) = \cup\{N(x) : x \in A\}$, $G_6^N(A) = \{x \in U : N(x) \cap A \neq \emptyset\}$ operators defined for N, then G_5^N is the conjugate of G_6^N.*

Proof. We show that $A \cap G_5^N(B) \neq \emptyset$ if and only if $B \cap G_6^N(A) \neq \emptyset$, for $A, B \subseteq U$.

If $A \cap G_5^N(B) \neq \emptyset$, then there exists $w \in U$ such that $w \in A$ and $w \in G_5^N(B)$. Since $w \in G_5^N(B)$, there exists $x_0 \in B$ such that $w \in N(x_0)$. Then $N(x_0) \cap A \neq \emptyset$, with $x_0 \in G_6^N(B)$. Since $x_0 \in B$, then $B \cap G_6^N(A) \neq \emptyset$.

If $B \cap G_6^N(A) \neq \emptyset$, then there exists $w \in U$ such that $w \in B$ and $w \in G_6^N(A)$, i.e., $w \in B$ and $N(w) \cap A \neq \emptyset$. Then there exists z such that $z \in N(w)$ and $z \in A$. Since $z \in N(w)$ and $w \in B$, then $z \in G_5^N(B)$. So, $z \in A \cap G_5^N(B)$, with $A \cap G_5^N(B) \neq \emptyset$. □

Corollary 2.5. *$H_5^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$ are conjugates.*

Proof. In this case, the operators $H_5^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$ correspond to G_5^N and G_6^N, when neighborhood operator N_1 is used. □

From the following lemma it is possible to prove that $L_2^{\mathbb{C}}$ is the adjoint of $H_5^{\mathbb{C}}$.

Lemma 1. *For all $w \in U$, $H_5^{\mathbb{C}}(N_1(w)) = N_1(w)$.*

Proof. By Proposition 1, from $x \in N_1(w)$ follows $N_1(x) \subseteq N_1(w)$, hence $H_5^{\mathbb{C}}(N_1(w)) \subseteq N_1(w)$. On the other hand, it is clear that $N_1(w) \subseteq H_5^{\mathbb{C}}(N_1(w))$, since $w \in N_1(w)$. □

Proposition 22. $L_2^{\mathbb{C}} = (H_5^{\mathbb{C}})^a$.

Proof. We will show that $L_2^{\mathbb{C}}(A) \subseteq (H_5^{\mathbb{C}})^a(A)$ and $(H_5^{\mathbb{C}})^a(A) \subseteq L_2^{\mathbb{C}}(A)$, for $A \subseteq U$. If $w \in L_2^{\mathbb{C}}(A)$, there exists $x \in U$ such that $w \in N_1(x)$ with $N_1(x) \subseteq A$. The upper approximation $H_5^{\mathbb{C}}$ of $N_1(x)$ is equal to $N_1(x)$, by Lemma 1; i.e., $H_5^{\mathbb{C}}(N_1(x)) = N_1(x)$. Hence, $w \in \cup\{Y \subseteq U : H_5^{\mathbb{C}}(Y) \subseteq A\}$, so $w \in (H_5^{\mathbb{C}})^a(A)$. On the other hand, if $w \in (H_5^{\mathbb{C}})^a(A)$, then there exists $Y \subseteq U$, such that $w \in Y$ and $H_5^{\mathbb{C}}(Y) \subseteq A$; i.e., $\cup\{N_1(x) : x \in Y\} \subseteq A$; in particular, $w \in N_1(w) \subseteq H_5^{\mathbb{C}}(Y) \subseteq A$, so $w \in L_2^{\mathbb{C}}(A)$. □

Corollary 2.6. *The dual of $H_6^{\mathbb{C}}$ is equal to $L_2^{\mathbb{C}}$.*

Proof. According to Propositions (21), (22) and Corollary (2.5), we have: $L_2^{\mathbb{C}} = (H_5^{\mathbb{C}})^a = ((H_5^{\mathbb{C}})^c)^\partial = (H_6^{\mathbb{C}})^\partial$. □

The upper approximation operators $H_2^{\mathbb{C}}$ and $H_7^{\mathbb{C}}$ are closely related; they are discussed in the next two propositions.

Proposition 23. *$H_2^{\mathbb{C}}$ is self-conjugate.*

Proof. According to Proposition 6 and the fact that $H_2^{\mathbb{C}}$ is a join morphism, $(H_2^{\mathbb{C}})^a = ((H_2^{\mathbb{C}})^c)^\partial$, so $H_2^{\mathbb{C}}$ is self-conjugate if and only if $(H_2^{\mathbb{C}})^a = (H_2^{\mathbb{C}})^\partial$, that is $(H_2^{\mathbb{C}})^a(\sim A) =\sim H_2^{\mathbb{C}}(A)$. We show that $(H_2^{\mathbb{C}})^a(\sim A) =\sim H_2^{\mathbb{C}}(A)$ for any $A \subseteq U$. $x \notin H_2^{\mathbb{C}}(A)$ if and only if $N_1(x) \cap A = \emptyset$ if and only if $N_1(x) \subseteq\sim A$ if and only if $x \in (H_2^{\mathbb{C}})^a(\sim A)$. □

Proposition 24. $H_7^{\mathbb{C}} = H_2^{\mathbb{C}_3}$

Proof. From the definition of $H_7^{\mathbb{C}}$ and $\mathbb{C}_3$, we can see that, for all $A \subseteq U$:

$$\begin{aligned} H_7^{\mathbb{C}}(A) &= \{N_1(x) : N_1(x) \cap A \neq \emptyset\} \\ &= \{K \in \mathbb{C}_3 : K \cap A \neq \emptyset\} \\ &= H_2^{\mathbb{C}_3}(A). \end{aligned}$$

□

Corollary 2.7. *$H_7^{\mathbb{C}}$ is self-conjugate and its adjoint is equal to $(H_2^{\mathbb{C}_3})^{\partial} = \underline{apr}''_{\mathbb{C}_3}$.*

Proof. Using $H_7^{\mathbb{C}} = H_2^{\mathbb{C}_3}$ and proposition 23, we have that $H_7^{\mathbb{C}}$ is self-conjugate. By proposition 39, we have: $(H_7^{\mathbb{C}})^a = ((H_7^{\mathbb{C}})^c)^{\partial} = (H_7^{\mathbb{C}})^{\partial} = (\overline{apr}''_{\mathbb{C}_3})^{\partial} = \underline{apr}''_{\mathbb{C}_3}$. □

Finally, we investigate whether any of the remaining upper approximation operators $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$ forms an adjoint pair with $L_1^{\mathbb{C}}$.

Example 8. *In Table 2.2, we compare lower approximations for some subsets obtained with $L_1^{\mathbb{C}}$, and with the adjoints of $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$. Since none of the final three columns is identical to the first one, we conclude that none of $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$ or $H_4^{\mathbb{C}}$ forms an adjoint pair with $L_1^{\mathbb{C}}$.*

Table 2.2: Comparison of the adjoints of $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$ and $H_4^{\mathbb{C}}$ with $L_1^{\mathbb{C}}$.

Set	$L_1^{\mathbb{C}}$	$(H_1^{\mathbb{C}})^a$	$(H_3^{\mathbb{C}})^a$	$(H_4^{\mathbb{C}})^a$
246	6	6	64	6
145	145	15	$\emptyset$	15
123	123	1	1	1

From the above results, the following conclusion can be drawn: none of the pairs $(H_1^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_2^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_3^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_4^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_5^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_6^{\mathbb{C}}, L_2^{\mathbb{C}})$ and $(H_7^{\mathbb{C}}, L_2^{\mathbb{C}})$ which have previously been considered in the literature (see Section 1.5.2) forms an adjoint pair; on the other hand, $(H_5^{\mathbb{C}}, L_2^{\mathbb{C}})$ does, but this is not a dual pair.

2.1.3 Summary of relationships and properties

Table 2.3 summarizes the results established in the previous subsections. In particular, we rearrange the group of 20 dual pairs considered by Yao and Yao [87] into 14 groups of equivalent operators, showing in each case their equivalence with members of the non-dual framework considered in [80]. Furthermore, it indicates whether the operators form an adjoint pair and whether their members are join/meet morphisms.

For instance, group A consists of the dual pairs 1 and 11 from Table 1.7 which are equal due to Proposition 16. They are equivalent to the pair $(H_6^{\mathbb{C}}, L_2^{\mathbb{C}})$, because $\underline{apr}_{\mathbb{C}_3} = L_2^{\mathbb{C}}$ by definition, and the dual of $L_2^{\mathbb{C}}$ is $H_6^{\mathbb{C}}$ by Corollary 2.6. They are join and meet morphisms, but not an adjoint pair as shown in Example 6.

It is interesting to note that all pairs of approximation operators in Yao and Yao's framework can be described from approximation operators $L_1^{\mathbb{C}}$, $L_2^{\mathbb{C}}$, $H_2^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$, their duals and/or their conjugates.

Group	#	Dual pair		Equivalent pair		Adjoint pair	Meet/Join
A	1, 11	$\overline{apr}_{N_1}$ $\overline{apr}_{\mathbb{C}_3}$	$\underline{apr}_{N_1}$ $\underline{apr}_{\mathbb{C}_3}$	$H_6^{\mathbb{C}}$	$L_2^{\mathbb{C}}$	No	Yes
B	2	$\overline{apr}_{N_2}$	$\underline{apr}_{N_2}$	$G_6^{N_2}$	$(G_6^{N_2})^\partial$	No	Yes
C	3	$\overline{apr}_{N_3}$	$\underline{apr}_{N_3}$	$G_6^{N_3}$	$(G_6^{N_3})^\partial$	No	Yes
D	4	$\overline{apr}_{N_4}$	$\underline{apr}_{N_4}$	$G_6^{N_4}$	$(G_6^{N_4})^\partial$	No	Yes
E	5, 7, 17, 20	$\overline{apr}'_{\mathbb{C}}$ $\overline{apr}'_{\mathbb{C}_1}$ $\overline{apr}'_{\mathbb{C}_\cup}$ $\overline{apr}_{S_\cup}$	$\underline{apr}'_{\mathbb{C}}$ $\underline{apr}'_{\mathbb{C}_1}$ $\underline{apr}'_{\mathbb{C}_\cup}$ $\underline{apr}_{S_\cup}$	$(L_1^{\mathbb{C}})^\partial$	$L_1^{\mathbb{C}}$	No	No
F	9	$\overline{apr}'_{\mathbb{C}_2}$	$\underline{apr}'_{\mathbb{C}_2}$	$(L_1^{\mathbb{C}_2})^\partial$	$L_1^{\mathbb{C}_2}$	No	No
G	13	$\overline{apr}'_{\mathbb{C}_4}$	$\underline{apr}'_{\mathbb{C}_4}$	$(L_1^{\mathbb{C}_4})^\partial$	$L_1^{\mathbb{C}_4}$	No	No
H	15	$\overline{apr}'_{\mathbb{C}_\cap}$	$\underline{apr}'_{\mathbb{C}_\cap}$	$(L_1^{\mathbb{C}_\cap})^\partial$	$L_1^{\mathbb{C}_\cap}$	No	No
I	6, 10, 18	$\overline{apr}''_{\mathbb{C}}$ $\overline{apr}''_{\mathbb{C}_2}$ $\overline{apr}''_{\mathbb{C}_\cup}$	$\underline{apr}''_{\mathbb{C}}$ $\underline{apr}''_{\mathbb{C}_2}$ $\underline{apr}''_{\mathbb{C}_\cup}$	$H_2^{\mathbb{C}}$	$(H_2^{\mathbb{C}})^\partial$	Yes	Yes
J	8	$\overline{apr}''_{\mathbb{C}_1}$	$\underline{apr}''_{\mathbb{C}_1}$	$H_2^{\mathbb{C}_1}$	$(H_2^{\mathbb{C}_1})^\partial$	Yes	Yes
K	12	$\overline{apr}''_{\mathbb{C}_3}$	$\underline{apr}''_{\mathbb{C}_3}$	$H_2^{\mathbb{C}_3}$	$(H_2^{\mathbb{C}_3})^\partial$	Yes	Yes
L	14	$\overline{apr}''_{\mathbb{C}_4}$	$\underline{apr}''_{\mathbb{C}_4}$	$H_2^{\mathbb{C}_4}$	$(H_2^{\mathbb{C}_4})^\partial$	Yes	Yes
M	16	$\overline{apr}''_{\mathbb{C}_\cap}$	$\underline{apr}''_{\mathbb{C}_\cap}$	$H_2^{\mathbb{C}_\cap}$	$(H_2^{\mathbb{C}_\cap})^\partial$	Yes	Yes
N	19	$\overline{apr}_{S_\cap}$	$\underline{apr}_{S_\cap}$	$(L_1^{S_\cap})^\partial$	$L_1^{S_\cap}$	No	No

Table 2.3: Summary of relationships and properties of the approximation operators.

2.2 Characterization of dual adjoint pairs

This section characterizes pairs of dual and adjoint approximation operators in the covering-based rough set framework.

First, the left hand side of Figure 2.1 illustrates schematically the relations among duality, conjugacy and adjointness. The arrow ∂ represents a dual transformation and the arrows c and co represent transformations of conjugate and co-conjugates, respectively. The pairs (f_i, g_i) are dual, f_1 and f_2 are conjugate and g_1 and g_2 are co-conjugate. The adjoint and co-adjoint can be obtained after two consecutive transformations, so the adjoint of f_1 is g_2, and the co-adjoint of g_1 is f_2. The middle diagram represents the specific situation for the pair $(\overline{apr}''_{\mathbb{C}}, \underline{apr}''_{\mathbb{C}})$, while the right hand side represents the case of approximation operators defined from a binary relation as considered by Järvinen [30].

The following important proposition establishes the relationship between duality, conjugacy and adjointness in the general case of a complete Boolean lattice.

Proposition 25. *Let (f, g) be a dual pair on a complete Boolean lattice B. The pair (f, g) is a Galois connection if and only if f is self-conjugate.*

$$
\begin{tikzcd}
f_1 \arrow[r, leftrightarrow, "\partial"] \arrow[d, leftrightarrow, "c"'] & g_1 \arrow[d, leftrightarrow, "co"] \\
f_2 \arrow[r, leftrightarrow, "\partial"'] & g_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{apr}''_{\mathbb{C}} \arrow[r, leftrightarrow, "\partial"] \arrow[d, leftrightarrow, "c"'] & \underline{apr}''_{\mathbb{C}} \arrow[d, leftrightarrow, "co"] \\
\overline{apr}''_{\mathbb{C}} \arrow[r, leftrightarrow, "\partial"'] & \underline{apr}''_{\mathbb{C}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{apr}_R \arrow[r, leftrightarrow, "\partial"] \arrow[d, leftrightarrow, "c"'] & \underline{apr}_R \arrow[d, leftrightarrow, "co"] \\
\overline{apr}_{R^{-1}} \arrow[r, leftrightarrow, "\partial"'] & \underline{apr}_{R^{-1}}
\end{tikzcd}
$$

Figure 2.1: Arrow diagram for approximation operators.

Proof. If (f,g) is a Galois connection, then $g = f^a = (f^c)^\partial$. By duality $g = f^\partial$, so $f^a = (f^c)^\partial = f^\partial$, hence $f = f^c$.

On the other hand, if $f = f^c$, then $f^a = (f^c)^\partial = (f)^\partial = g$ and $g_a = (g^\partial)^c = f^c = f$. $\square$

In general, if DP represents dual pairs, GC Galois connections and SC pairs for which the upper approximation is self-conjugate, then we have the following implications:

$$GC + SC \to DP \tag{2.7}$$

$$DP + SC \to GC \tag{2.8}$$

To summarize, Figure 2.2 contains a set diagram which depicts pairs of approximation operators. **P** is the set of pairs of approximation operators (H,L). **D** contains all the dual pairs of Yao and Yao's framework and **A** the adjoint pairs. The pairs in the intersection are precisely those pairs of approximation operators for which the upper approximation is self conjugate. Outside of $\mathbf{D} \cup \mathbf{A}$ there are other pairs which are neither dual nor adjoint, such as $(H_7^{\mathbb{C}}, L_2^{\mathbb{C}})$. The pairs of approximations in Yao and Yao's framework are represented with the letters from A to N and correspond to the groups in Table 2.3. Finally, the pair $(\overline{apr}_h, \underline{apr}_h)$, defined from an upper approximation distribution h (Wybraniec-Skardowska) is also an adjoint, but not dual pair.

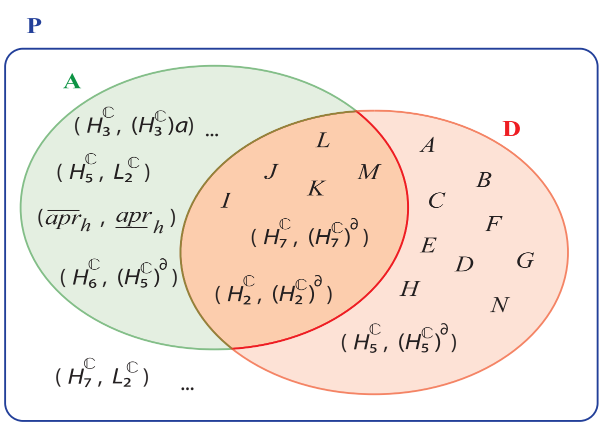

Figure 2.2: Set diagram of pairs of approximation operators.

2.3 Summary

In this chapter some relationships between pairs of lower and upper approximation operators within the covering-based rough set model have been studied. In particular for the framework of twenty dual pairs of approximation operators proposed by Yao and Yao in [87], only fourteen of them are different, and of these only five pairs are adjoint. On the other hand, it has been demonstrated that none of the pairs of approximation operators $(H_1^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_2^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_3^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_4^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_5^{\mathbb{C}}, L_1^{\mathbb{C}})$, $(H_6^{\mathbb{C}}, L_2^{\mathbb{C}})$ and $(H_7^{\mathbb{C}}, L_2^{\mathbb{C}})$ considered in e.g. [78, 80] is adjoint; on the other hand, $(H_5^{\mathbb{C}}, L_2^{\mathbb{C}})$ is an adjoint, non-dual pair. Furthermore, it has been established that all operators in Yao and Yao's framework can be equivalently expressed in terms of $L_1^{\mathbb{C}}$, $L_2^{\mathbb{C}}$, $H_2^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$.

Also a characterization of dual and adjoint pairs in terms of the self-conjugacy of the upper approximation operator has been derived, and related this equivalence to previous results established for generalized rough sets, based on a symmetric binary relation.

CHAPTER 3

Order relations

3.1 Introduction

In this chapter a partial order relation for the most commonly used covering based approximation operators, providing an exhaustive evaluation of their pairwise comparability is established. The number of operators can be reduced by proving some equivalences between them, and on the other hand, we consider some new ones which emerge as duals of the approximation operators considered by Yang and Li. This gives us sixteen pairs of dual and distinct approximation operators. We also list their most important theoretical properties. Then we evaluate the fineness order, first for each group of operators separately, and then for all the operators jointly. The central result of our analysis is a Hasse diagram positioning the 16 lower (resp., upper) approximation operators according to the fineness order. We also show the orders for subsets of operators satisfying particular properties, like adjointness and being a meet/join-morphism.

Based on the operators discussed in Section 1.5 a list of dual pairs of approximation operators to be considered in this study on order relations is compiled. On one hand, this list includes the dual pairs proposed by Yao and Yao in [87] and discussed in Section (1.5.1). As we will see below, some of them are equivalent, hence the total number of pairs can be reduced. On the other hand, we add to the list those pairs obtained by coupling $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$, $H_4^{\mathbb{C}}$ and $H_5^{\mathbb{C}}$ from Section (1.5.2) with their respective dual lower approximations.

3.2 Dual pairs of approximation operators and their properties

Some equivalences about granule based definition, were already established in Section 2.1.1.2, and can be summarized as follows:

a. $\underline{apr}'_{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_1} = \underline{apr}'_{\mathbb{C}_\cup} = \underline{apr}'_{S_\cup}$
$\overline{apr}'_{\mathbb{C}} = \overline{apr}'_{\mathbb{C}_1} = \overline{apr}'_{\mathbb{C}_\cup} = \overline{apr}'_{S_\cup}$.

b. $\underline{apr}''_{\mathbb{C}} = \underline{apr}''_{\mathbb{C}_2} = \underline{apr}''_{\mathbb{C}_\cap}$
$\overline{apr}''_{\mathbb{C}} = \overline{apr}''_{\mathbb{C}_2} = \overline{apr}''_{\mathbb{C}_\cap}$.

c. $\underline{apr}'_{\mathbb{C}_3} = \underline{apr}_{N_1}$
$\overline{apr}'_{\mathbb{C}_3} = \overline{apr}_{N_1}$.

This already reduces the twenty dual pairs considered by Yao and Yao to fourteen. The following propositions show two further equivalences.

Proposition 26. $\mathbb{C}_1 = \mathbb{C}_\cup$.

Proof. We will show that $\mathbb{C}_1 \subseteq \mathbb{C}_\cup$ and $\mathbb{C}_\cup \subseteq \mathbb{C}_1$.

Let us suppose that $K \in \mathbb{C}_1$ and that K is union reducible, that is, $K \in md(\mathbb{C}, x)$ for some $x \in U$ and $K = K_1 \cup K_2 \cup \cdots \cup K_l$ with $K_i \in \mathbb{C}$ and $K_i \neq K$, for $i = 1, \ldots, l$. We have $x \in K$, therefore there exists a $j \in \{1, 2, \ldots, l\}$ such that $x \in K_j \subset K$. Hence, $K \notin md(\mathbb{C}, x)$. This is a contradiction, so K must be union irreducible, so $K \in \mathbb{C}_\cup$.

On the other hand, if $K \notin \mathbb{C}_1$, then for all $x \in U$, $K \notin md(\mathbb{C}, x)$. In particular, let $x \in K$. Since $K \notin md(\mathbb{C}, x)$, there exists $K_0^x \in md(\mathbb{C}, x)$ such that $x \in K_0^x \subset K$. So, we have $K = \cup_{x \in K}\{x\} \subseteq \cup_{x \in K} K_0^x \subseteq K$, so $K = \cup K_0^x$, hence K is reducible and $K \notin \mathbb{C}_\cup$. □

Corollary 3.1. $\underline{apr}''_{\mathbb{C}_1} = \underline{apr}''_{\mathbb{C}_\cup}$.

Proposition 27. $\underline{apr}''_{\mathbb{C}_2} = \underline{apr}_{N_4}$.

Proof. Let $x \in U$ and $A \in \mathcal{P}(U)$. It holds that $x \in \underline{apr}_{N_4}(A) \Leftrightarrow (\forall K \in MD(\mathbb{C}, x))(K \subseteq A)$ and $x \in \underline{apr}''_{\mathbb{C}_2}(A) \Leftrightarrow (\forall K \in \cup\{MD(\mathbb{C}, y) : y \in U\})(x \in K \Rightarrow K \subseteq A)$.

Clearly, if $x \in \underline{apr}''_{\mathbb{C}_2}(A)$, then $x \in \underline{apr}_{N_4}(A)$, so $\underline{apr}''_{\mathbb{C}_2}(A) \subseteq \underline{apr}_{N_4}(A)$.

On the other hand, suppose $x \in \underline{apr}_{N_4}(A)$ and $x \notin \underline{apr}''_{\mathbb{C}_2}(A)$. Let $y \in U$ and $K \in MD(\mathbb{C}, y)$ such that $x \in K$ and $K \nsubseteq A$. Then $K \notin MD(\mathbb{C}, x)$, so there exists $S \in MD(\mathbb{C}, x)$ such that $K \subset S$ and $S \subseteq A$. But then $K \subseteq A$ as well, which is a contradiction. In other words, $\underline{apr}_{N_4}(A) \subseteq \underline{apr}''_{\mathbb{C}_2}(A)$. □

It can be checked that no further identities hold among the approximation operators considered by Yao and Yao [87] and those considered by Yang and Li [80]. Hence, there are sixteen groups of different dual pairs of approximations operators, which are listed in Table 3.1.

As mentioned in the introduction, the main objective of this chapter will be to establish a pointwise partial order for the lower and upper approximation operators in Table 3.1, comparing them according to the fineness of their approximations. At the same time, it is possible to differentiate between the approximation operators according to the theoretical properties they satisfy. Table 3.2 lists five important properties, all of which hold in an approximation space in Pawlak's sense, and points out which of the groups in Table 3.1 satisfy them. The proofs of most of these properties can be reconstructed from literature, see e.g. [53, 95, 96, 97, 98, 102, 99], taking into account that fourteen out of the sixteen dual pairs of operators can be expressed by means of $L_1^{\mathbb{C}}$, $L_2^{\mathbb{C}}$, $H_i^{\mathbb{C}}$ and their respective dual operators. The remaining proofs can be established by simple verification, and counterexamples are easy to find for the negative results. It is interesting to note that none of the currently considered groups satisfies all properties; in particular, the properties of adjointness and idempotence are never simultaneously satisfied.

Table 3.1: List of different dual pairs of lower and upper approximations

Number	Lower approximation	Upper approximation
1	$\underline{apr}_{N_1} = \underline{apr}'_{\mathbb{C}_3} = (H_6^{\mathbb{C}})^{\partial} = L_2^{\mathbb{C}}$	$\overline{apr}_{N_1} = \overline{apr}'_{\mathbb{C}_3} = H_6^{\mathbb{C}}$
2	$\underline{apr}_{N_2}$	$\overline{apr}_{N_2}$
3	$\underline{apr}_{N_3}$	$\overline{apr}_{N_3}$
4	$\underline{apr}_{N_4} = \underline{apr}''_{\mathbb{C}} = \underline{apr}''_{\mathbb{C}_2} = \underline{apr}''_{\mathbb{C}_\cap} = (H_2^{\mathbb{C}})^{\partial}$	$\overline{apr}_{N_4} = \overline{apr}''_{\mathbb{C}} = \overline{apr}''_{\mathbb{C}_2} = \overline{apr}''_{\mathbb{C}_\cap} = H_2^{\mathbb{C}}$
5	$\underline{apr}'_{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_1} = \underline{apr}'_{\mathbb{C}_\cup} = \underline{apr}_{S_\cup} = L_1^{\mathbb{C}}$	$\overline{apr}'_{\mathbb{C}} = \overline{apr}'_{\mathbb{C}_1} = \overline{apr}'_{\mathbb{C}_\cup} = \overline{apr}_{S_\cup}$
6	$\underline{apr}'_{\mathbb{C}_2}$	$\overline{apr}'_{\mathbb{C}_2}$
7	$\underline{apr}'_{\mathbb{C}_4}$	$\overline{apr}'_{\mathbb{C}_4}$
8	$\underline{apr}'_{\mathbb{C}_\cap}$	$\overline{apr}'_{\mathbb{C}_\cap}$
9	$\underline{apr}''_{\mathbb{C}_1} = \underline{apr}''_{\mathbb{C}_\cup}$	$\overline{apr}''_{\mathbb{C}_1} = \overline{apr}''_{\mathbb{C}_\cup}$
10	$\underline{apr}''_{\mathbb{C}_3} = (H_7^{\mathbb{C}})^{\partial}$	$\overline{apr}''_{\mathbb{C}_3} = H_7^{\mathbb{C}}$
11	$\underline{apr}''_{\mathbb{C}_4}$	$\overline{apr}''_{\mathbb{C}_4}$
12	$\underline{apr}_{S_\cap}$	$\overline{apr}_{S_\cap}$
13	$(H_1^{\mathbb{C}})^{\partial}$	$H_1^{\mathbb{C}}$
14	$(H_3^{\mathbb{C}})^{\partial}$	$H_3^{\mathbb{C}}$
15	$(H_4^{\mathbb{C}})^{\partial}$	$H_4^{\mathbb{C}}$
16	$(H_5^{\mathbb{C}})^{\partial}$	$H_5^{\mathbb{C}}$

Table 3.2: Evaluation of properties of covering based rough sets.

Name	Property	Satisfied by
Adjointness	$\overline{apr}(A) \subseteq B \Leftrightarrow A \subseteq \underline{apr}(B)$	$4, 9, 10, 11$
Monotonicity	$A \subseteq B \Rightarrow \underline{apr}(A) \subseteq \underline{apr}(B)$ $A \subseteq B \Rightarrow \overline{apr}(A) \subseteq \overline{apr}(B)$	All groups, except 13 and 15
Meet/join-morphism	$\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$ $\overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B)$	$1, 2, 3, 4, 9, 10, 11, 14, 16$
Idempotence	$\underline{apr}(\underline{apr}(A)) = \underline{apr}(A)$ $\overline{apr}(\overline{apr}(A)) = \overline{apr}(A)$	$1, 5, 6, 7, 8, 13, 15, 16$
$\emptyset$ and U	$\underline{apr}(U) = U$ $\overline{apr}(\emptyset) = \emptyset$	All groups

3.3 Partial order relation for approximation operators

This section, systematically investigates a point-wise partial order relation among pairs of lower and upper approximation operators. This partial order is defined as follows:

Definition 3.1. *Let $\underline{apr}_1$ and $\underline{apr}_2$ be two lower approximation operators, and $\overline{apr}_1$ and $\overline{apr}_2$ two upper approximation operators. Since these operators are defined over parts of U and it is ordered by inclusion relation, it is possible to define an order relation among approximation operators. We write:*

- $\underline{apr}_1 \leq \underline{apr}_2$, *if and only if* $\underline{apr}_1(A) \subseteq \underline{apr}_2(A)$, *for all* $A \subseteq U$. *And similarly for upper approximations.*

- $(\underline{apr}_1, \overline{apr}_1) \leq (\underline{apr}_2, \overline{apr}_2)$, *if and only if* $\underline{apr}_1 \geq \underline{apr}_2$ *and* $\overline{apr}_1 \leq \overline{apr}_2$.

It is easy to see that $\leq$ is indeed reflexive, anti-symmetric and transitive. Moreover, it is easy to verify that the partial order $\leq$, which may be read as "is finer than" forms a bounded lattice on the set $\mathcal{APR}$ of pairs of approximation operators, with smallest element $(\underline{apr}, \overline{apr})$ where $\underline{apr}(A) = \overline{apr}(A) = A$, and largest element $(\underline{apr}, \overline{apr})$ where $\underline{apr}(A) = \emptyset$ and $\overline{apr}(A) = U$, for any $A \subseteq U$. The lattice meet operation is defined as: $(\underline{apr}_1, \overline{apr}_1) \sqcap (\underline{apr}_2, \overline{apr}_2) = (\underline{apr}, \overline{apr})$ where $\underline{apr}(A) = \underline{apr}_1(A) \cup \underline{apr}_2(A)$ and $\overline{apr}(A) = \overline{apr}_1(A) \cap \overline{apr}_2(A)$, with $A \subseteq U$. The lattice join operation is defined dually.

Remark 1. *In a natural way, it is possible to consider a second partial order on* $\mathcal{APR}$, *by defining* $(\underline{apr}_1, \overline{apr}_1) \leq' (\underline{apr}_2, \overline{apr}_2)$ *if and only if* $\underline{apr}_1 \leq_l \underline{apr}_2$ *and* $\overline{apr}_1 \leq_u \overline{apr}$. *The partial order* $\leq'$ *forms a bounded lattice on* $\mathcal{APR}$, *with smallest element* $(\underline{apr}, \overline{apr})$ *where* $\underline{apr}(A) = \emptyset$ *and* $\overline{apr}(A) = A$, *and largest element* $(\underline{apr}, \overline{apr})$ *where* $\underline{apr}(A) = A$ *and* $\overline{apr}(A) = U$, *for all* $A \subseteq U$. *The lattice meet operation is defined as:* $(\underline{apr}_1, \overline{apr}_1) \sqcap' (\underline{apr}_2, \overline{apr}_2) = (\underline{apr}, \overline{apr})$ *where* $\underline{apr}(A) = \underline{apr}_1(A) \cup \underline{apr}_2(A)$ *and* $\overline{apr}(A) = \overline{apr}_1(A) \cup \overline{apr}_2(A)$, *with* $A \subseteq U$. *Again, the join operation is defined dually. In the context of bilattice theory [17],* $\leq$ *is called the knowledge order on* $\mathcal{APR}$ *and* $\leq'$ *is called its truth order.*

For practical purposes, the partial order $\leq$ is particularly relevant, since it allows us to compare pairs of approximation operators in terms of their suitability for data analysis. In particular, the definitions of accuracy and quality of classification provided for Pawlak's rough sets [47] can be generalized to covering based rough sets.

Definition 3.2. *If* $(\underline{apr}, \overline{apr})$ *is a pair of a lower and an upper approximation operator, the accuracy of* $A \subseteq U$ *is defined as:*

$$\alpha_{\underline{apr}}^{\overline{apr}}(A) = \frac{|\underline{apr}(A)|}{|\overline{apr}(A)|} \tag{3.1}$$

On the other hand, the quality of classification of $A \subseteq U$, *by means of* $\underline{apr}$, *is defined as:*

$$\gamma_{\underline{apr}}(A) = \frac{|\underline{apr}(A)|}{|A|} \tag{3.2}$$

The quality of classification of a subset $A \subseteq U$ can be extended to a partition $Y = \{Y_1, \ldots, Y_n\}$ of U:

$$\gamma_{\underline{apr}}(Y) = \sum \frac{|\underline{apr}(Y_i)|}{|U|} \tag{3.3}$$

$\gamma_{\underline{apr}}(Y)$ can be seen as the ratio of elements of U that can be classified with certainty into one of the classes of Y. Clearly, it is desirable to have $\gamma_{\underline{apr}}(Y)$ as high as possible. The following proposition shows the relationship with the partial order $\leq$.

Proposition 28. *If* $(\underline{apr}_1, \overline{apr}_1)$ *and* $(\underline{apr}_2, \overline{apr}_2)$ *are two pairs of approximation operators such that* $(\underline{apr}_1, \overline{apr}_1) \leq (\underline{apr}_2, \overline{apr}_2)$, *then* $\alpha_{\underline{apr}_2}^{\overline{apr}_2}(A) \leq \alpha_{\underline{apr}_1}^{\overline{apr}_1}(A)$ *and* $\gamma_{\underline{apr}_2}(A) \leq \gamma_{\underline{apr}_1}(A)$ *for all* $A \subseteq U$.

Proof. Let $A \subseteq U$. If $(\underline{apr}_1, \overline{apr}_1) \leq (\underline{apr}_2, \overline{apr}_2)$ then $\underline{apr}_2(A) \subseteq \underline{apr}_1(A)$ and $\overline{apr}_1(A) \subseteq \overline{apr}_2(A)$. Therefore $|\underline{apr}_2(A)| \leq |\underline{apr}_1(A)|$ and $|\overline{apr}_1(A)| \leq |\overline{apr}_2(A)|$. So $\alpha_{\underline{apr}_2}^{\overline{apr}_2}(A) = \dfrac{|\underline{apr}_2(A)|}{|\overline{apr}_2(A)|} \leq \dfrac{|\underline{apr}_1(A)|}{|\overline{apr}_1(A)|} = \alpha_{\underline{apr}_1}^{\overline{apr}_1}(A)$. The inequality $\gamma_{\underline{apr}_2}(A) \leq \gamma_{\underline{apr}_1}(A)$ can be established similarly.

In the remainder of this section, the following result can be established.

Proposition 29. *Let $(\underline{apr}_1, \overline{apr}_1)$ and $(\underline{apr}_2, \overline{apr}_2)$ be two dual pairs of approximation operators. It holds that*

$$(\underline{apr}_1, \overline{apr}_1) \leq (\underline{apr}_2, \overline{apr}_2) \iff \underline{apr}_1 \geq_l \underline{apr}_2 \iff \overline{apr}_2 \leq_u \overline{apr}_1. \tag{3.4}$$

Proof. Direct from the definition of duality and the partial orders. □

In other words, in order to establish the partial order for dual pairs of approximation operators, it suffices to know the partial order $\leq_l$ for lower approximation operators, as the partial order $\leq_u$ for upper approximation operators can be obtained with the reverse partial order of its duals. From now on, to simplify the notation, we will refer to both $\leq_l$ and $\leq_u$ by $\leq$.

The following subsections evaluate the order relationships that hold between elements of different groups of approximation operators: element based, granule based and system based definitions of Yao and Yao [87], and upper approximation operators of Yang and Li [80]. Afterwards, it is possible combine these results to construct an integrated Hasse diagram for all the operators considered in Table 3.1.

3.3.1 Partial order for element based definitions

The following propositions establish the relationship among element based approximation operators, defined in equations (1.20) and (1.21) using neighborhood operators.

Proposition 30. *If N and N' are neighborhood operators such that $N(x) \subseteq N'(x)$ for all $x \in U$, then $\underline{apr}_{N'} \leq \underline{apr}_N$.*

Proof. We will show that $\underline{apr}_{N'}(A) \subseteq \underline{apr}_N(A)$, for any $A \subseteq U$. If $x \in \underline{apr}_{N'}(A)$, $N'(x) \subseteq A$, hence $N(x) \subseteq N'(x) \subseteq A$ for all $x \in U$, so $x \in \underline{apr}_N(A)$. □

Proposition 31. *For $x \in U$, it holds that $N_1(x) \subseteq N_2(x)$, $N_3(x) \subseteq N_4(x)$, $N_1(x) \subseteq N_3(x)$ and $N_2(x) \subseteq N_4(x)$.*

Proof. The first two inclusions follow directly from the definition of neighborhood systems. For the third one, we can see that for each $K \in N_1(x)$ there exists $K' \in N_3(x)$ such that $K \subseteq K'$. So, $\cap\{K \in md(\mathbb{C}, x)\} \subseteq \cap\{K' \in MD(\mathbb{C}, x)\}$, from which follows $N_1(x) \subseteq N_3(x)$. The final inclusion can be proved similarly. □

Proposition 32.

a. $\underline{apr}_{N_4} \leq \underline{apr}_{N_2} \leq \underline{apr}_{N_1}$.

b. $\underline{apr}_{N_4} \leq \underline{apr}_{N_3} \leq \underline{apr}_{N_1}$.

Proof. Direct from Propositions 30 and 62. ◻

Moreover, $\underline{apr}_{N_2}$ and $\underline{apr}_{N_3}$ are not comparable, as we can see in Example 9 below.

Example 9. *Let us consider the covering* $\mathbb{C} = \{1, 23, 123, 34\}$ *of* $U = 1234$. *The neighborhood system* $C(\mathbb{C}, x)$*, the minimal description* $md(\mathbb{C}, x)$*, the maximal description* $MD(\mathbb{C}, x)$ *and the four neighborhood operators obtained from* $C(\mathbb{C}, x)$ *are listed in Table 3.3.*

Table 3.3: Minimal and maximal descriptions, and neighborhood operators for Example 9.

x	$C(\mathbb{C}, x)$	$md(\mathbb{C}, x)$	$MD(\mathbb{C}, x)$	$N_1(x)$	$N_2(x)$	$N_3(x)$	$N_4(x)$
1	{1,123}	{1}	{123}	1	1	123	123
2	{23,123}	{23}	{123}	23	23	123	123
3	{23,123, 34}	{23,34}	{123,34}	3	234	3	1234
4	{34}	{34}	{34}	4	34	4	34

From the neighborhoods in Table 3.3, the lower approximations of $A = 23$ *are:* $\underline{apr}_{N_1}(A) = 23$, $\underline{apr}_{N_2}(A) = 2$, $\underline{apr}_{N_3}(A) = 3$ *and* $\underline{apr}_{N_4}(A) = \emptyset$. *In this example, we can see that* $\underline{apr}_{N_2} \nleq \underline{apr}_{N_3}$ *and* $\underline{apr}_{N_3} \nleq \underline{apr}_{N_2}$*, so these operators are not comparable.*

Using Propositions 30 and 62 and Example 9, we can establish the partial order for the lower approximation operators in this section; the partial order for the upper approximations follows from Proposition 29. The corresponding Hasse diagrams are shown in Figure 3.1. The order relation $apr_{N_i} \leq apr_{N_j}$ is represented by means of an arrow from apr_{N_i} to apr_{N_j}.

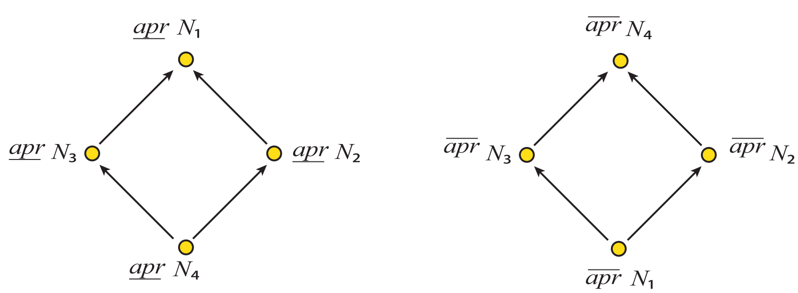

Figure 3.1: Partial order for element based approximation operators.

3.3.2 Partial order for granule based definitions

The granule based approximation operators definitions were presented in equations (1.26) to (1.28). This section will evaluate the order relation for approximation operators related with the coverings $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_3$, $\mathbb{C}_4$ and $\mathbb{C}_\cap$ (recall that $\underline{apr}'_{\mathbb{C}} = \underline{apr}'_{\mathbb{C}_1}$ and $\underline{apr}''_{\mathbb{C}} = \underline{apr}''_{\mathbb{C}_\cap}$, and that by Proposition 26, $\mathbb{C}_\cup = \mathbb{C}_1$). First of all, Propositions 33 and 34 establish a general order relation for granule based lower approximation operators $\underline{apr}'$.

Proposition 33. *If* $\mathbb{C}$ *and* $\mathbb{C}'$ *are coverings of* U *such that* $\mathbb{C} \subseteq \mathbb{C}'$*, then* $\underline{apr}'_{\mathbb{C}} \leq \underline{apr}'_{\mathbb{C}'}$.

Proof. Since $\underline{apr'}_{\mathbb{C}}(A) = \cup\{K \in \mathbb{C} : K \subseteq A\}$ and $\mathbb{C} \subseteq \mathbb{C}'$, we have $\cup\{K \in \mathbb{C} : K \subseteq A\} \subseteq \cup\{K \in \mathbb{C}' : K \subseteq A\}$. Then $\underline{apr'}_{\mathbb{C}}(A) \subseteq \underline{apr'}_{\mathbb{C}'}(A)$, for all $A \subseteq U$ and $\underline{apr'}_{\mathbb{C}} \leq \underline{apr'}_{\mathbb{C}'}$. □

Proposition 34. *If $\mathbb{C}$ and $\mathbb{C}'$ are coverings of U such that, for all $K \in \mathbb{C}$, $K = \bigcup_{\alpha\in I} L_\alpha$ for $(L_\alpha)_{\alpha\in I} \subseteq \mathbb{C}'$, then $\underline{apr'}_{\mathbb{C}} \leq \underline{apr'}_{\mathbb{C}'}$.*

Proof. If $x \in \underline{apr'}_{\mathbb{C}}(A)$, then there exists a $K_0 \in \mathbb{C}$ such that $x \in K_0 \subseteq A$. But $x \in K_0 = \bigcup_{\alpha\in I} L_\alpha \subseteq A$, with $(L_\alpha)_{\alpha\in I} \subseteq \mathbb{C}'$. Hence $x \in L_\alpha \subseteq A$, for some α in I, therefore $x \in \underline{apr'}_{\mathbb{C}'}(A)$. □

Proposition 35. *Let $\mathbb{C}$ be a covering of U. It holds that:*

$$\underline{apr'}_{\mathbb{C}_4} \leq \underline{apr'}_{\mathbb{C}_2} \leq \underline{apr'}_{\mathbb{C}_\cap} \leq \underline{apr'}_{\mathbb{C}_1} \leq \underline{apr'}_{\mathbb{C}_3}.$$

Proof. It is easy to verify that the pairs of coverings $\mathbb{C}_4$ - $\mathbb{C}_2$, $\mathbb{C}_2$ - $\mathbb{C}_1$ and $\mathbb{C}_1$ - $\mathbb{C}_3$ satisfy the conditions of Proposition 34. For example, for coverings $\mathbb{C}_4$ - $\mathbb{C}_2$, we have: $\mathbb{C}_4 = \{\cup MD(\mathbb{C}, x) : x \in U\}$ and $\mathbb{C}_2 = \cup\{MD(\mathbb{C}, x) : x \in U\}$, clearly they satisfy the conditions of Proposition 34. Hence, $\underline{apr'}_{\mathbb{C}_4} \leq \underline{apr'}_{\mathbb{C}_2} \leq \underline{apr'}_{\mathbb{C}_1} \leq \underline{apr'}_{\mathbb{C}_3}$.

To see that $\underline{apr'}_{\mathbb{C}_2} \leq \underline{apr}_{\mathbb{C}_\cap}$, the relation $\mathbb{C}_2 \subseteq \mathbb{C}_\cap$ is established. The result then follows from Proposition 33. If $K \in \mathbb{C}_2$, $K \in MD(\mathbb{C}, x_0)$ for some $x_0 \in U$. If $K = \bigcap_{i\in I} K_i$ for $(K_i)_{i\in I} \subseteq \mathbb{C} - \{K\}$, then $K \subseteq K_i$ for all i in I, so $K \notin MD(\mathbb{C}, x_0)$ which is a contradiction. Hence, $K \in \mathbb{C}_\cap$.

Finally, the order relation $\underline{apr'}_{\mathbb{C}_\cap} \leq \underline{apr'}_{\mathbb{C}_1}$ is proven. First of all, it is easy to see that $\mathbb{C}_\cap \subseteq \mathbb{C}$, so by Proposition 33, we have $\underline{apr'}_{\mathbb{C}_\cap} \leq \underline{apr'}_{\mathbb{C}}$. From Propositions in Section 2.1.1.2, we know that $\underline{apr'}_{\mathbb{C}_1} = \underline{apr'}_{\mathbb{C}}$, so $\underline{apr'}_{\mathbb{C}_\cap} \leq \underline{apr'}_{\mathbb{C}_1}$. □

The following proposition establishes a general order relation for granule based upper approximation operators $\overline{apr}''_{\mathbb{C}}$.

Proposition 36. *If $\mathbb{C}$ and $\mathbb{C}'$ are coverings of U such that, for all $K \in \mathbb{C}$, there exists $L \in \mathbb{C}'$ such that $K \subseteq L$, then $\overline{apr}''_{\mathbb{C}} \leq \overline{apr}''_{\mathbb{C}'}$.*

Proof. If $x \in \overline{apr}''_{\mathbb{C}}(A)$, then there exists a $K_0 \in \mathbb{C}$ such that $x \in K_0 \cap A \neq \emptyset$. By the assumption, there exists $L_0 \in \mathbb{C}'$ such that $K_0 \subseteq L_0$, so $x \in L_0$ and $L_0 \cap A \neq \emptyset$. Hence, $x \in \overline{apr}''_{\mathbb{C}'}(A)$. □

Proposition 37. *Let $\mathbb{C}$ be a covering of U. It holds that:*

$$\overline{apr}''_{\mathbb{C}_3} \leq \overline{apr}''_{\mathbb{C}_1} \leq \overline{apr}''_{\mathbb{C}_2} = \overline{apr}''_{\mathbb{C}_\cap} \leq \overline{apr}''_{\mathbb{C}_4}.$$

Proof. Clearly, $\cap md(\mathbb{C}, x) \subseteq K$ for each $K \in md(\mathbb{C}, x)$ and for all $K \in md(\mathbb{C}, x)$ there exists $L \in MD(\mathbb{C}, x)$ such that $K \subseteq L$ and finally for all $L \in MD(\mathbb{C}, x)$ we have $L \subseteq MD(\mathbb{C}, x)$. Therefore, the result follows as a consequence of Propositions in Section 2.1.1.2 and 36. □

To relating the lower approximation operators $\underline{apr'}$ and $\underline{apr''}$, the following proposition follows easily from the definitions of granule based approximation operators.

Proposition 38. *Let $\mathbb{C}$ be a covering of U. It holds that $\underline{apr''}_{\mathbb{C}} \leq \underline{apr'}_{\mathbb{C}}$.*

Proof. Direct from equations (1.26) and (1.28). □

Apart from this, we also have the following result.

Proposition 39. $\underline{apr''}_{\mathbb{C}_2} \leq \underline{apr'}_{\mathbb{C}_4}$.

Proof. Let $x \in \underline{apr''}_{\mathbb{C}_2}(A)$. Clearly, for all $K \in MD(\mathbb{C}, x)$, it holds that $x \in K$. Therefore, $K \subseteq A$ for all $K \in MD(\mathbb{C}, x)$. From this follows that $\cup MD(\mathbb{C}, x) \subseteq A$ and since $\cup MD(\mathbb{C}, x) \in \mathbb{C}_4$, therefore $x \in \underline{apr'}_{\mathbb{C}_4}(A)$. □

The remaining covering based lower approximation operators are not comparable, as the following examples show.

Example 10. *For the covering* $\mathbb{C} = \{1, 3, 13, 24, 34, 14, 234\}$ *of* $U = 1234$*, we have:*

1. $\mathbb{C}_1 = \{1, 3, 24, 14, 34\}$
2. $\mathbb{C}_2 = \{13, 14, 234\}$
3. $\mathbb{C}_3 = \{1, 24, 3, 4\}$
4. $\mathbb{C}_4 = \{134, 234, 1234\}$
5. $\mathbb{C}_\cap = \{13, 24, 34, 14, 234\}$

Table 3.4: Granule based lower approximations of Example 10.

A	$\underline{apr'}_{\mathbb{C}_1}$	$\underline{apr'}_{\mathbb{C}_2}$	$\underline{apr'}_{\mathbb{C}_3}$	$\underline{apr'}_{\mathbb{C}_4}$	$\underline{apr'}_{\mathbb{C}_\cap}$	$\underline{apr''}_{\mathbb{C}_1}$	$\underline{apr''}_{\mathbb{C}_2}$	$\underline{apr''}_{\mathbb{C}_3}$	$\underline{apr''}_{\mathbb{C}_4}$
1	1	∅	1	∅	∅	∅	∅	1	∅
2	∅	∅	∅	∅	∅	∅	∅	∅	∅
3	3	∅	3	∅	∅	∅	∅	3	∅
4	∅	∅	4	∅	∅	∅	∅	∅	∅
12	1	∅	1	∅	∅	∅	∅	1	∅
13	13	13	13	∅	13	∅	∅	13	∅
14	14	14	14	∅	14	1	∅	1	∅
23	3	∅	3	∅	∅	∅	∅	3	∅
24	24	∅	24	∅	24	2	∅	24	∅
34	34	∅	34	∅	34	3	∅	3	∅
123	13	13	13	∅	13	∅	∅	13	∅
124	124	14	124	∅	124	12	∅	124	∅
134	134	134	134	134	134	13	1	13	∅
234	234	234	234	234	234	23	2	234	∅
1234	1234	1234	1234	1234	1234	1234	1234	1234	1234

The lower approximations of all non-empty subsets of U are shown in Table 3.4. From these results, we can conclude that the pairs $\underline{apr''}_{\mathbb{C}_1}$ - $\underline{apr'}_{\mathbb{C}_4}$, $\underline{apr''}_{\mathbb{C}_1}$ - $\underline{apr'}_{\mathbb{C}_2}$, $\underline{apr''}_{\mathbb{C}_3}$ - $\underline{apr'}_{\mathbb{C}_2}$, $\underline{apr''}_{\mathbb{C}_3}$ - $\underline{apr'}_{\mathbb{C}_4}$ *and* $\underline{apr''}_{\mathbb{C}_3}$ - $\underline{apr'}_{\mathbb{C}_\cap}$ *are not comparable. This example does not allow us to conclude anything about the incomparability of* $\underline{apr''}_{\mathbb{C}_3}$ - $\underline{apr'}_{\mathbb{C}_1}$*, neither about* $\underline{apr''}_{\mathbb{C}_1}$ - $\underline{apr'}_{\mathbb{C}_\cap}$.

Example 11. *For the covering* $\mathbb{C} = \{1,12,123,24,23,234\}$ *of* $U = 1234$*, we have that* $\mathbb{C}_1 = \{1,12,23,24\}$ *and* $\mathbb{C}_\cap = \{1,12,123,24,234\}$*. We can see that:* $\underline{apr}'_{\mathbb{C}_\cap}(12) = 12 \supset 1 = \underline{apr}''_{\mathbb{C}_1}(12)$*, while* $\underline{apr}'_{\mathbb{C}_\cap}(23) = \emptyset \subset 3 = \underline{apr}''_{\mathbb{C}_1}(23)$*. Therefore,* $\underline{apr}'_{\mathbb{C}_\cap}$ *and* $\underline{apr}''_{\mathbb{C}_1}$ *are not comparable.*

Example 12. *For the covering* $\mathbb{C} = \{13,14,23,24,34,234\}$ *of* $U = 1234$*, we have that* $\mathbb{C}_1 = \{13,14,23,24,34\}$*,* $\mathbb{C}_3 = \{1,2,3,4\}$*. We can see that:* $\underline{apr}'_{\mathbb{C}_1}(12) = \emptyset \subset 12 = \underline{apr}''_{\mathbb{C}_3}(12)$*. On the other hand, in Example 10, we have* $\underline{apr}'_{\mathbb{C}_1}(14) = 14 \supset 1 = \underline{apr}''_{\mathbb{C}_3}(14)$*. Therefore,* $\underline{apr}''_{\mathbb{C}_3}$ *and* $\underline{apr}'_{\mathbb{C}_1}$ *are not comparable.*

Order relations for upper approximation operators $\overline{apr}'_{\mathbb{C}}$ and $\overline{apr}''_{\mathbb{C}}$ can be established as a consequence of duality.

To conclude this subsection, the partial order relations for granule based approximation operators are shown in Figure 3.2.

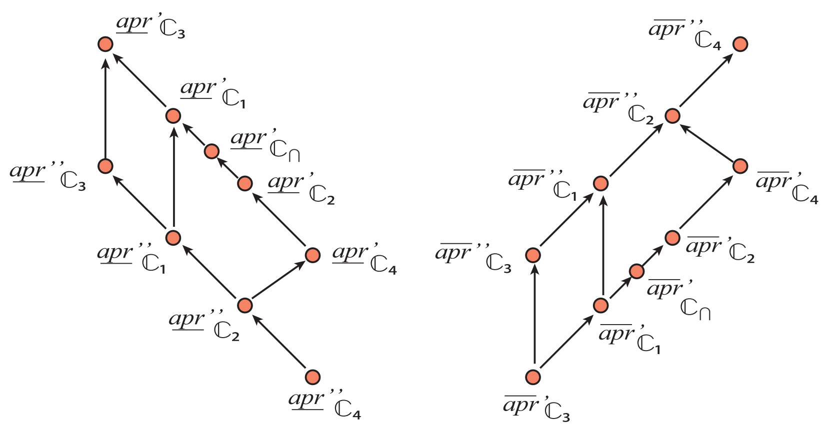

Figure 3.2: Partial order for granule based approximation operators.

3.3.3 Partial order for system based definitions

The following example shows that the two system-based lower approximation operators defined in equations (1.32) and (1.33) are not comparable. By duality, therefore, the corresponding upper approximations are not comparable, either.

Example 13. *Consider the covering* $\mathbb{C} = \{2,12,23,14,124\}$*. The corresponding* $S_\cap$ *and* $S_\cup$ *can be obtained from the following closure systems and their duals.*

- $\cap$*-closure* $= \{\emptyset, U, 2, 12, 23, 14, 124, 1\}$
- $\cup$*-closure* $= \{\emptyset, U, 2, 12, 23, 14, 124, 123\}$

The lower approximations of all non-empty subsets of U are shown in Table 3.5. From these values, we can see that $\underline{apr}_{S_\cap}$ *and* $\underline{apr}_{S_\cup}$ *are not comparable.*

Table 3.5: System based lower approximations for Example 13.

A	$\underline{apr}_{S_\cup}$	$\underline{apr}_{S_\cap}$
1	$\emptyset$	$\emptyset$
2	$\emptyset$	2
3	3	$\emptyset$
4	$\emptyset$	$\emptyset$
12	$\emptyset$	12
13	3	$\emptyset$
14	14	14
23	23	23
24	$\emptyset$	2
34	34	$\emptyset$
123	23	123
124	14	124
134	134	14
234	234	23
1234	1234	1234

3.3.4 Partial order for the operators from the non-dual framework

A partial order relations among the first six upper approximation operators $H_1^{\mathbb{C}}$–$H_6^{\mathbb{C}}$ have already been established in [78] and it is presented in the next proposition [78].

Proposition 40. *a.* $H_1^{\mathbb{C}} \leq H_4^{\mathbb{C}} \leq H_2^{\mathbb{C}}$.

b. $H_1^{\mathbb{C}} \leq H_3^{\mathbb{C}} \leq H_2^{\mathbb{C}}$.

c. $H_5^{\mathbb{C}} \leq H_1^{\mathbb{C}}$.

d. $H_6^{\mathbb{C}} \leq H_2^{\mathbb{C}}$.

This section completes the partial order for this framework, considering also $H_7^{\mathbb{C}}$, by means of the following propositions and example.

Proposition 41. $H_6^{\mathbb{C}} \leq H_7^{\mathbb{C}}$.

Proof. Direct from their definitions. □

Proposition 42. $H_7^{\mathbb{C}} \leq H_2^{\mathbb{C}}$.

Proof. If $x \in H_7^{\mathbb{C}}(A)$ then $x \in N_1(w)$ for some $w \in U$ and $N_1(w) \cap A \neq \emptyset$. But $N_1(w) \subseteq K \in \mathbb{C}$ for some K, therefore $x \in H_2^{\mathbb{C}}(A)$. □

Proposition 43. $H_5^{\mathbb{C}} \leq H_7^{\mathbb{C}}$.

Proof. If $x \in H_5^{\mathbb{C}}(A)$ then $x \in N_1(w)$ for some $w \in A$. Since $w \in N_1(w)$, $N_1(w) \cap A \neq \emptyset$ and $x \in H_7^{\mathbb{C}}(A)$. □

Example 14. *Let us consider the covering $\mathbb{C} = \{1, 12, 34, 123\}$ of $U = 1234$. The upper approximations $H_i^{\mathbb{C}}(A)$ of all non-empty subsets A of U are shown in Table 3.6. From these results, we can see that the pairs $H_7^{\mathbb{C}}$ - $H_1^{\mathbb{C}}$, $H_7^{\mathbb{C}}$ - $H_3^{\mathbb{C}}$ and $H_7^{\mathbb{C}}$ - $H_4^{\mathbb{C}}$ are not comparable.*

Table 3.6: Upper approximations $H_i^{\mathbb{C}}(A)$ for Example 14.

A	$H_1^{\mathbb{C}}$	$H_2^{\mathbb{C}}$	$H_3^{\mathbb{C}}$	$H_4^{\mathbb{C}}$	$H_5^{\mathbb{C}}$	$H_6^{\mathbb{C}}$	$H_7^{\mathbb{C}}$
1	1	123	1	1	1	12	12
2	12	123	12	123	12	2	12
3	1234	1234	1234	1234	3	34	34
4	34	34	34	34	34	4	34
12	12	123	12	12	12	12	12
13	1234	1234	1234	1234	13	1234	1234
14	134	1234	134	134	134	124	1234
23	1234	1234	1234	1234	123	234	1234
24	1234	1234	1234	1234	1234	24	1234
34	34	1234	1234	34	34	34	34
123	123	1234	1234	123	123	1234	1234
124	1234	1234	1234	1234	1234	124	1234
134	134	1234	1234	134	134	1234	1234
234	1234	1234	1234	1234	1234	234	1234
1234	1234	1234	1234	1234	1234	1234	1234

To summarize, the order relations for the upper approximation operators $H_i^{\mathbb{C}}$ are shown in Figure 3.3. Again, by Proposition 29, the reverse ordering for the lower approximation operators $(H_i^{\mathbb{C}})^{\partial}$ can be considered.

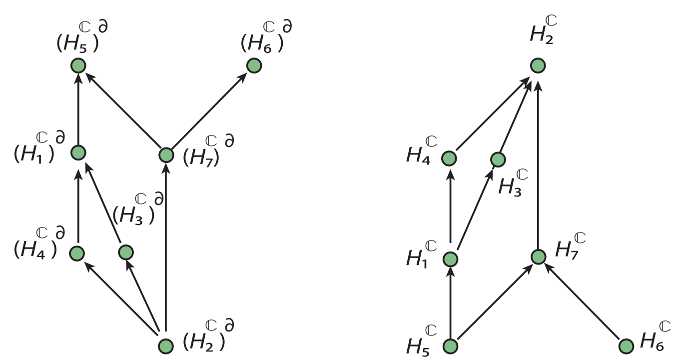

Figure 3.3: Partial order relation for lower and upper approximations of the non-dual framework.

3.3.5 Partial order for all approximation operators

To establish the partial order relation among all lower approximation operators considered in Table 3.1, the following proposition which relates operators in different groups, is proved.

Proposition 44. *a.* $\underline{apr'}_{\mathbb{C}_2} \leq \underline{apr}_{N_3}$.

b. $\underline{apr}_{N_4} \leq \underline{apr}'_{\mathbb{C}_4}$.

c. $\underline{apr}_{N_4} \leq \underline{apr}'_{\mathbb{C}_2}$.

d. $\underline{apr}_{N_2} \leq \underline{apr}'_{\mathbb{C}_1}$.

e. $\underline{apr}''_{\mathbb{C}_1} \leq H_3^{\mathbb{C}}$.

f. $\underline{apr}''_{\mathbb{C}_1} \leq \underline{apr}_{N_2}$.

g. $\underline{apr}_{S_\cap} \leq (H_5^{\mathbb{C}})^{\partial}$.

Proof. a. if $x \in \underline{apr}'_{\mathbb{C}_2}(A)$, $x \in K$ for some $K \in \mathbb{C}_2$, and $K \subseteq A$. $N_3(x) = \cap MD(\mathbb{C}, x) \subseteq K \subseteq A$, hence $x \in \underline{apr}_{N_3}(A)$.

b. If $x \in \underline{apr}_{N_4}(A)$, $N_4(x) \subseteq A$. But $N_4(x) \in \mathbb{C}_4$, and $x \in N_4(x)$, hence $x \in \underline{apr}'_{\mathbb{C}_4}(A)$.

c. If $x \in \underline{apr}_{N_4}(A)$, $N_4(x) \subseteq A$. Therefore, for all $K \in \mathbb{C}_2$ with $x \in K$, we have $K \subseteq \cup MD(\mathbb{C}, x) = N_4(x) \subseteq A$, hence $x \in \underline{apr}'_{\mathbb{C}_2}(A)$.

d. If $x \in \underline{apr}_{N_2}(A)$, $N_2(x) = \cup md(\mathbb{C}, x) \subseteq A$. Therefore, for all $K \in \mathbb{C}_1$ with $x \in K$, we have $K \subseteq \cup md(\mathbb{C}, x) = N_2(x) \subseteq A$, hence $x \in \underline{apr}'_{\mathbb{C}_1}(A)$.

e. Remark that, for $A \in \mathcal{P}(U)$, $\underline{apr}''_{\mathbb{C}_1}(A) = \{x \in U : \forall K \in \{md(\mathbb{C}, y) : y \in U\}(x \in K \Rightarrow K \subseteq A)\}$. If $x \in \underline{apr}''_{\mathbb{C}_1}(A)$, then $x \in A$. Since $x \in K$ for all K in $md(\mathbb{C}, x)$, it holds that $x \in H_3^{\mathbb{C}}(A)$.

f. We will show that $\overline{apr}_{N_2} \leq \overline{apr}''_{\mathbb{C}_1}$. If $x \in \overline{apr}_{N_2}(A)$, then $N_2(x) \cap A \neq \emptyset$, but $N_2(x) = \cup\{K : K \in md(\mathbb{C}, x)\}$, so $\cup\{K : K \in md(\mathbb{C}, x)\} \cap A = \cup\{K \cap A : K \in md(\mathbb{C}, x)\} \neq \emptyset$. Thus there exists $K_0 \in md(\mathbb{C}, x)$ such that $K_0 \cap A \neq \emptyset$. Therefore $x \in \cup\{K \in \mathbb{C}_1 : K \cap A \neq \emptyset\} = \overline{apr}''_{\mathbb{C}_1}(A)$ and so, $\overline{apr}_{N_2} \leq \overline{apr}''_{\mathbb{C}_1}$. The result [f.] is a consequence of duality.

g. We will see that $H_5^{\mathbb{C}} \leq \overline{apr}_{S_\cap}$. For this, let us suppose $w \in H_5^{\mathbb{C}}(A)$, then $w \in N_1(x)$ for some $x \in A$. From Proposition 1.a, we have $N_1(w) \subseteq N_1(x)$. We will show that $w \in X$, for all $X \in$ ($\cap$-closure ($\mathbb{C}$)) with $X \supseteq A$. Let X be a set in ($\cap$-closure ($\mathbb{C}$)) with $X \supseteq A$, then $x \in A \subseteq X$ and $X = K_1 \cap \cdots \cap K_l$ with $K_j \in \mathbb{C}$. So $x \in K_j$ for all $j = 1, 2, 3, \ldots, l$. Again, from Proposition 1.b, each K_j can be expressed as $K_j = \cup_{x_j \in K_j} N_1(x_j)$, therefore $x \in N_1(x_{j_0})$, for some $x_{j_0} \in K_j$ and $N_1(x) \subseteq K_j$ for all $j = 1, 2, 3, \ldots, l$. Thus we have $w \in N_1(w) \subseteq N_1(x) \subseteq X$. This shows that $w \in \overline{apr}_{S_\cap}$ and that $H_5^{\mathbb{C}}(A) \leq \overline{apr}_{S_\cap}(A)$.

□

Next, the following examples show that the operator $\underline{apr}_{S_\cap}$ is not comparable with any of the other ones. This is partially a consequence of the fact that when $\mathbb{C}$ is a partition, $\underline{apr}_{S_\cap}$ does not coincide with Pawlak's lower approximation operator.

Example 15. *Since the covering $\mathbb{C} = \{1, 2, 3, 4\}$ of $U = 1234$ is a partition, we have that $\underline{apr}(A) = A$, for all $A \subseteq U$ with $\underline{apr}$ any of the lower approximation operators in Table 3.1 different from $\underline{apr}_{S_\cap}$. On the other hand, $\cap$-closure ($\mathbb{C}$) $= \{\emptyset, 1234, 1, 2, 3, 4\}$ and ($\cap$-closure ($\mathbb{C}$))$' = \{\emptyset, 1234, 234, 134, 124, 123\}$, so $\underline{apr}_{S_\cap}(A) = \emptyset$, if $|A| < 3$ and $\underline{apr}_{S_\cap}(A) = A$, if $|A| \geq 3$. Then $\underline{apr} \nleq \underline{apr}_{S_\cap}$.*

Example 16. *Consider the covering* $\mathbb{C} = \{1,12,123,24,23,234\}$ *of* $U = 1234$ *in Example 3.*

The lower approximations of all non-empty subsets of U for the approximation operators $\underline{apr}_{S_\cap}$, $\underline{apr}_{N_1}$ *and* $(H_1^{\mathbb{C}})^\partial$ *are shown in Table 3.7. From these values, we can see that* $\underline{apr}_{S_\cap}$ *is comparable with neither* $\underline{apr}_{N_1}$ *nor* $(H_1^{\mathbb{C}})^\partial$.

Table 3.7: System based lower approximations for Example 16.

A	$\underline{apr}_{S_\cap}$	$\underline{apr}_{N_1}$	$(H_1^{\mathbb{C}})^\partial$
1	1	1	1
2	$\emptyset$	2	$\emptyset$
3	$\emptyset$	$\emptyset$	3
4	4	$\emptyset$	4
12	1	12	1
13	13	1	13
14	14	1	14
23	$\emptyset$	23	3
24	4	24	4
34	34	$\emptyset$	34
123	13	123	13
124	14	124	14
134	134	1	$\emptyset$
234	234	234	234
1234	1234	1234	1234

From Examples 15 and 16, and the Figure 3.4 below, we can see that the operator $\underline{apr}_{S_\cap}$ is not comparable with any of the other ones, different from $(H_5^{\mathbb{C}})^\partial$.

An integrated Hasse diagram of the partial order relation among the different lower approximation operators can be seen in Figure 3.4. A completely analogous diagram can be constructed for the upper approximation operators. Each group of operators is represented in Table 3.1 with a circled number. The green circles represent operators which form an adjoint pair, with their dual; the yellow circles represent meet morphisms which do not form an adjoint pair with their dual; and the red circles represent operators which do not form an adjoint pair with their dual, neither are meet morphisms. The label *P.n* is the number of proposition where the order relation is established.

There are two maximal elements: the group (16), represented by $(H_5^{\mathbb{C}})^\partial$ and the group (1), represented by $\underline{apr}_{N_1}$. The minimal elements are the groups (11) represented by $\underline{apr}''_{\mathbb{C}_4}$ and (12) represented by $\underline{apr}_{S_\cap}$. Let us recall that maximal elements represent those lower approximation operators for which the quality of classification in Equation (3.3) is highest.

It is interesting to note that while both top elements of the partial order are meet-morphisms, they do not form an adjoint pair with their duals. This can be seen as a disadvantage of these operators, because the adjointness property guarantees for a dual pair $(\underline{apr}, \overline{apr})$ that the fix points of $\underline{apr}$ and $\overline{apr}$ coincide, in other words, $\underline{apr}(A) = A$ iff $\overline{apr}(A) = A$.

The subset of lower approximation operators that satisfy adjointness form a chain, with group (10), represented by $\underline{apr}''_{\mathbb{C}_3}$, as the top element.

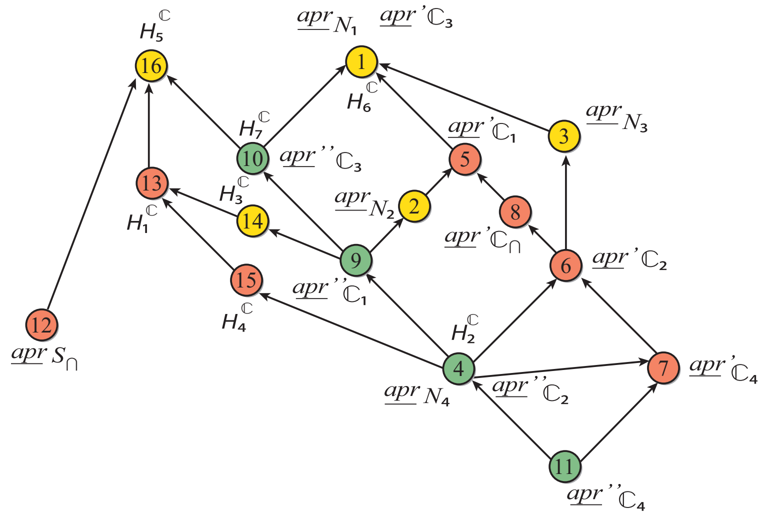

Figure 3.4: Partial order relation for groups of lower approximation operators in Table 3.1.

Finally, the diagram also suggests some additional approximation operators to be considered. For example, the order relation between the groups (10)-(1), (9)-(5), (4)-(6) and (11)-(7) corresponds to the relation: $\underline{apr}'_{\mathbb{C}} \leq \underline{apr}''_{\mathbb{C}}$. The group (2) is between (9) and (5) and it is defined from $\underline{apr}_{N_2}$. If we consider the neighborhood operators $N_2^{\mathbb{C}}(x) = \cup\{K : K \in md(\mathbb{C}, x)\}$ for different coverings, we obtain new lower approximation operators: $\underline{apr}_{N_2^{\mathbb{C}_1}}$, $\underline{apr}_{N_2^{\mathbb{C}_2}}$, $\underline{apr}_{N_2^{\mathbb{C}_3}}$, $\underline{apr}_{N_2^{\mathbb{C}_4}}$ and $\underline{apr}_{N_2^{\mathbb{C}_\cap}}$. Following the proof in Proposition 44.d and f, order relations with the new operators can easily be established:

1. $\underline{apr}''_{\mathbb{C}_3} \leq \underline{apr}_{N_2^{\mathbb{C}_3}} \leq \underline{apr}'_{\mathbb{C}_3}$.
2. $\underline{apr}''_{\mathbb{C}_1} \leq \underline{apr}_{N_2^{\mathbb{C}_1}} \leq \underline{apr}'_{\mathbb{C}_1}$.
3. $\underline{apr}''_{\mathbb{C}_2} \leq \underline{apr}_{N_2^{\mathbb{C}_\cap}} \leq \underline{apr}'_{\mathbb{C}_\cap}$.
4. $\underline{apr}''_{\mathbb{C}_2} \leq \underline{apr}_{N_2^{\mathbb{C}_2}} \leq \underline{apr}'_{\mathbb{C}_2}$.
5. $\underline{apr}''_{\mathbb{C}_4} \leq \underline{apr}_{N_2^{\mathbb{C}_4}} \leq \underline{apr}'_{\mathbb{C}_4}$.

Example 17 below shows that some of these new approximations operators are different. In particular we will see that $\underline{apr}''_{\mathbb{C}_2} \neq \underline{apr}_{N_2^{\mathbb{C}_\cap}} \neq \underline{apr}'_{\mathbb{C}_\cap}$. Similar results can be established for other coverings.

Example 17. *From the covering $\mathbb{C}$ in Example 10, we have: $\mathbb{C}_2 = \{13, 14, 234\}$ and $\mathbb{C}_\cap = \{13, 24, 34, 14, 234\}$. The minimal description $md(\mathbb{C}, x)$ for these coverings and the neighborhood operators $N_2^{\mathbb{C}}$, are shown in Table 3.8.*

Table 3.8: Minimal descriptions, and neighborhood operator N_2 for the coverings $\mathbb{C}_2$ and $\mathbb{C}_\cap$.

x	$md(\mathbb{C}_2, x)$	$md(\mathbb{C}_\cap, x)$	$N_2^{\mathbb{C}_2}(x)$	$N_2^{\mathbb{C}_\cap}(x)$
1	{13,14}	{13,14}	134	134
2	{234}	{24}	234	24
3	{13,234}	{13,34}	1234	134
4	{14,234}	{14,24,34}	1234	1234

From the neighborhoods in Table 3.8, the lower approximations of $A = 24$ *are:* $\underline{apr}_{N_2^{\mathbb{C}_2}}(A) = \emptyset$ *and* $\underline{apr}_{N_2^{\mathbb{C}_\cap}}(A) = 2$*, so* $\underline{apr}_{N_2^{\mathbb{C}_2}} \neq \underline{apr}_{N_2^{\mathbb{C}_\cap}}$*. Also, we can see that* $\underline{apr}''_{\mathbb{C}_2}(A) = \emptyset$, $\underline{apr}_{N_2^{\mathbb{C}_\cap}}(A) = 2$ *and* $\underline{apr}'_{\mathbb{C}_\cap}(A) = 24$*, therefore* $\underline{apr}''_{\mathbb{C}_2} \neq \underline{apr}_{N_2^{\mathbb{C}_\cap}} \neq \underline{apr}'_{\mathbb{C}_\cap}$.

In general, additional approximation operators can be defined combining the different coverings with the neighborhood based lower approximation operators as well as with $(H_i^{\mathbb{C}})^\partial$ ($i = 1, \ldots, 7$). All of them may be included in the Hasse diagram in Figure 3.4, but in order not to complicate the visual representation of the partial order, we refrain from doing so here.

3.4 Summary

In this chapter a study of order relation between lower and upper approximation operators proposed in the literature for covering-based rough sets was done. Among the sixteen dual pairs that we have considered in our study, we have identified $((H_5^{\mathbb{C}})^\partial, H_5^{\mathbb{C}})$ and $(\underline{apr}_{N_1}, \overline{apr}_{N_1}) = (\underline{apr}'_{\mathbb{C}_3}, \overline{apr}'_{\mathbb{C}_3})$ as the ones that produce the finest approximations. If additionally adjointness is required, then the finest pair is $(\underline{apr}''_{\mathbb{C}_3}, \overline{apr}''_{\mathbb{C}_3})$. These results may guide practitioners who are faced with an ample collection of approximation operators to choose from.

CHAPTER 4

Relations and Coverings

Relation based rough sets and covering based rough sets are two important extensions of the classical rough sets. This chapter investigates relationships between relation based rough sets and the covering based rough sets, presents a new group of approximation operators obtained by combining coverings and neighborhood operators and establishes some relationships between covering based rough sets and relation based rough sets.

4.1 Introduction

W. Zhu, established an equivalence between a type of covering-based rough sets and a type of binary relation based rough sets [102]. Y. L. Zhang and M. K. Luo established the equivalence between four types of covering-based rough sets and a type of relation-based rough sets, respectively [90]. Covering based rough sets and tolerance relation based rough sets are used in information systems with missing and numerical data [11, 12, 31].

In this chapter, we will extend the connections established by Y. L. Zhang and M. K. Luo to the Yao and Yao framework. In section 4.2, we present preliminary concepts about rough set theory, covering based and relation based rough sets. In section 4.3, we review some equivalences between covering and relation based rough sets. We also present a new group of approximation operators, combining some coverings with neighborhood operators. Finally, we establish some equivalences of these operators and relation based rough sets. Section 4.4, presents some conclusions.

4.2 Coverings and Relations

This section establishes an equivalence between relation based rough sets and some types of covering based rough sets from Yao and Yao's framework.

4.2.1 Coverings and Neighborhood

Definition 4.1. *Let $\mathbb{C}$ be a covering of U, for each neighborhood operator $N_i^{\mathbb{C}}$ with $i = 1,2,3,4$, two relations on U are defined by means of:*

$$xS_i^{\mathbb{C}}y \Leftrightarrow y \in N_i^{\mathbb{C}}(x) \tag{4.1}$$
$$xP_i^{\mathbb{C}}y \Leftrightarrow x \in N_i^{\mathbb{C}}(y) \tag{4.2}$$

It is easy to see that $S_i^{\mathbb{C}}$ and $P_i^{\mathbb{C}}$ for $i = 1,2,3,4$ are reflexive relations and $S_1^{\mathbb{C}}$ and $P_1^{\mathbb{C}}$ are transitive relations.

The next propositions establish a connection between element based definition operators and relation based rough sets. Proposition (45) can be seen as a generalization of Theorem 6 in [102].

Proposition 45. *If $\mathbb{C}$ is a covering of U, $N_i^{\mathbb{C}}$ is any neighborhood operator and $\overline{apr}_{P_i}$ the upper approximation operator defined by the relation in Equation 4.2 then $\overline{apr}_{P_i} = G_5^{N_i}$.*

Proof. We can see that $x \in \overline{apr}_{P_i}(A)$ if and only if $P_i(x) \cap A \neq \emptyset$. So, there exist $w \in U$, such that xP_iw with $w \in A$, then $x \in N_i(w)$ and $w \in A$ if and only if $x \in G_5^{N_i}(A)$. $\square$

Proposition 46. *Let $\mathbb{C}$ be a covering of U and $N_i^{\mathbb{C}}$ a neighborhood operator. If S_i is the relation defined in Equation (4.1), then $\underline{apr}_{S_i} = \underline{apr}_{N_i}^{\mathbb{C}}$ and $\overline{apr}_{S_i} = \overline{apr}_{N_i}^{\mathbb{C}}$.*

Proof. We can see that $w \in N_i(x)$ if and only if xS_iw, if and only if $w \in S_i(x)$. So, for a covering $\mathbb{C}$, $N_i(x) = S_i(x)$ and therefore $\underline{apr}_{S_i} = \underline{apr}_{N_i}^{\mathbb{C}}$ and $\overline{apr}_{S_i} = \overline{apr}_{N_i}^{\mathbb{C}}$. $\square$

From Proposition (46), each pair of covering-based approximation operators in element based definition, $(\underline{apr}_{N_i}, \overline{apr}_{N_i})$ can be treated as the relation based approximation operators, generated by a reflexive relation. Also: $G_6^{N_i} = \overline{apr}_{N_i}^{\mathbb{C}}$.

Proposition 47. *The operators $\overline{apr}_{S_i}$ and $\overline{apr}_{P_i}$ are conjugate.*

Proof. The conjugate relation between $G_5^{N_i}$ and $G_6^{N_i}$ was established in Section (2.1.2). From Propositions (45) and (46), we have that $\overline{apr}_{S_i}$ and $\overline{apr}_{P_i}$ are conjugate. $\square$

In particular, we have:

- $\overline{apr}_{N_1} = H_6^{\mathbb{C}}$
- $\overline{apr}_{M_1} = H_5^{\mathbb{C}}$
- $\overline{apr}_{M_2} = G_5^{N_2} = H_3^{\mathbb{C}}$
- $\overline{apr}_{N_4} = \overline{apr}_{M_4} = G_5^{N_4} = \overline{apr}''_{\mathbb{C}} = H_2^{\mathbb{C}}$

In the arrow diagram of Figure 4.1, the arrow N_i represents the neighborhood operators given in Section 1.5.1, S and P arrows represent the successor and the predecessor neighborhood and apr the operator defining the dual pair of approximations, obtained from a neighborhood operator. The notation ∂ is used for representing the dual operator.

$$\begin{tikzcd}
& & S_i \arrow[r, "apr"] & ((G_6^{N_i})^{\partial}, G_6^{N_i}) \\
\mathbb{C} \arrow[r, "N_i"] & N_i^{\mathbb{C}} \arrow[ur, "S"] \arrow[dr, "P"'] & & \\
& & P_i \arrow[r, "apr"] & ((G_5^{N_i})^{\partial}, G_5^{N_i})
\end{tikzcd}$$

Figure 4.1: Arrow diagram for element based definition approximation operators.

Proposition 48. *Let $\mathbb{C}$ be a covering of U and $N_4^{\mathbb{C}}$ the neighborhood system associated with the covering $\mathbb{C}$. If $S_4^{\mathbb{C}}$ is the relation defined in Equation (4.1) using $N_4^{\mathbb{C}}$, then $\underline{apr}''_{\mathbb{C}} = \underline{apr}^{\mathbb{C}}_{S_4}$ and $\overline{apr}''_{\mathbb{C}} = \overline{apr}^{\mathbb{C}}_{S_4}$.*

Proof. We will see that $\underline{apr}''_{\mathbb{C}} = \underline{apr}_{S_4}$. In Proposition the equivalence $\underline{apr}''_{\mathbb{C}} = \underline{apr}^{\mathbb{C}}_{N_4}$ was established. According to Proposition (46), $\underline{apr}^{\mathbb{C}}_{N_4} = \underline{apr}_{S_4}$ then $\underline{apr}''_{\mathbb{C}} = \underline{apr}^{\mathbb{C}}_{S_4}$. □

According to Proposition (48), any approximation operator $\underline{apr}''_{\mathbb{C}}$ from Yao and Yao's framework, can be seen as a relation based approximation.

The approximation operators $\underline{apr}'_{\mathbb{C}}$, $\underline{apr}_{S_\cap}$ and $\underline{apr}_{S_\cup}$ do not have a relation based equivalence, because they do not satisfy the property (d) in Proposition 2.

So, the four operators $\overline{apr}_{N_i}$, in the Yao and Yao's framework and the operators $\overline{apr}''_{\mathbb{C}} = H_2^{\mathbb{C}}$, $H_3^{\mathbb{C}}$, $H_5^{\mathbb{C}}$ and $H_6^{\mathbb{C}}$ can be defined as a relation based operator. The operators $\overline{apr}'_{\mathbb{C}}$ can not be defined as relation based definition, because they are not join morphisms.

4.2.2 New Approximation Operators

It is interesting to note that we only use the neighborhood system $N_4^{\mathbb{C}}$ to describe all the approximation operators $\underline{apr}''_{\mathbb{C}}$, then it seems reasonable to ask about other approximation operators related with neighborhood systems N_1, N_2 and N_3. For each covering $\mathbb{C}$ and each neighborhood operator $N_i^{\mathbb{C}}$, we have the operators $\underline{apr}^{\mathbb{C}}_{N_i}$.

Combining each covering with the four neighborhood operators, new approximation operators can be obtained, for example $\underline{apr}^{\mathbb{C}_3}_{N_2}$ is the lower approximation for covering $\mathbb{C}_3$ using the neighborhood operator N_2.

Example 18. *For the covering $\mathbb{C} = \{1, 2, 12, 12, 34, 123, 24, 234\}$ of $U = 1234$ the Table (4.1) shows the upper approximation operators $\underline{apr}^{\mathbb{C}_j}_{N_i}$ for each singletons:* $1, 2, 3$ *and* 4.

The approximations for $A \subseteq U$ can be calculated using the property $\overline{apr}(A) = \cup_{x \in A} \overline{apr}(x)$.

The results in Table (4.1) can be used to establish the difference between some operators. For example, since $\overline{apr}^{\mathbb{C}}_{N_2}(3) \neq \overline{apr}^{\mathbb{C}}_{N_4}(3)$ then $\overline{apr}^{\mathbb{C}}_{N_2} \neq \overline{apr}^{\mathbb{C}}_{N_4}$.

Table 4.1: Upper approximation for singletons of U.

Operator	1	2	3	4	Operator	1	2	3	4
$\overline{apr}^{\mathbb{C}}_{N_1}$	1	2	3	4	$\overline{apr}^{\mathbb{C}_1}_{N_1}$	1	2	3	4
$\overline{apr}^{\mathbb{C}}_{N_2}$	123	24	34	234	$\overline{apr}^{\mathbb{C}_1}_{N_2}$	123	24	34	234
$\overline{apr}^{\mathbb{C}}_{N_3}$	1	1234	1234	4	$\overline{apr}^{\mathbb{C}_1}_{N_3}$	1	2	3	4
$\overline{apr}^{\mathbb{C}}_{N_4}$	123	1234	1234	234	$\overline{apr}^{\mathbb{C}_1}_{N_4}$	123	124	134	234
$\overline{apr}^{\mathbb{C}_2}_{N_1}$	1	1234	1234	4	$\overline{apr}^{\mathbb{C}_3}_{N_1}$	1	2	3	4
$\overline{apr}^{\mathbb{C}_2}_{N_2}$	123	1234	1234	234	$\overline{apr}^{\mathbb{C}_3}_{N_2}$	1	2	3	4
$\overline{apr}^{\mathbb{C}_2}_{N_3}$	1	1234	1234	4	$\overline{apr}^{\mathbb{C}_3}_{N_3}$	1	2	3	4
$\overline{apr}^{\mathbb{C}_2}_{N_4}$	123	1234	1234	234	$\overline{apr}^{\mathbb{C}_3}_{N_4}$	1	2	3	4
$\overline{apr}^{\mathbb{C}_4}_{N_1}$	1	1234	1234	4	$\overline{apr}^{\mathbb{C}_\cap}_{N_1}$	1	2	3	4
$\overline{apr}^{\mathbb{C}_4}_{N_2}$	123	1234	1234	234	$\overline{apr}^{\mathbb{C}_\cap}_{N_2}$	123	124	134	234
$\overline{apr}^{\mathbb{C}_4}_{N_3}$	1234	1234	1234	1234	$\overline{apr}^{\mathbb{C}_\cap}_{N_3}$	1	1234	1234	4
$\overline{apr}^{\mathbb{C}_4}_{N_4}$	1234	1234	1234	1234	$\overline{apr}^{\mathbb{C}_\cap}_{N_4}$	123	1234	1234	234

4.3 Neighborhood Operators based on Coverings

There are four different types of neighborhood operators N and six types of coverings $\mathbb{C}$. We want to explore if it is necessary to consider all 24 combinations $N_i^{\mathbb{C}_j}$, or if there are some equivalent definitions.

With the following proposition is possible to establish connection between neighborhood operators and approximation operators.

Proposition 49. *If $N_i \neq N_j$ then $\overline{apr}^{\mathbb{C}}_{N_i} \neq \overline{apr}^{\mathbb{C}}_{N_j}$*

Proof. We will see that if $\overline{apr}^{\mathbb{C}}_{N_i} = \overline{apr}^{\mathbb{C}}_{N_j}$ then $N_i = N_j$, showing that $N_i(x) \subseteq N_j(x)$ and $N_j(x) \subseteq N_i(x)$.

For $N_i(x) \subseteq N_j(x)$, let us suppose that $w \in N_i(x)$, then $N_i(x) \cap \{w\} \neq \emptyset$, i.e., $x \in \overline{apr}_{N_i}(w)$ and $x \in \overline{apr}_{N_j}(w)$. So, $x \in \overline{apr}_{N_j}(w)$, and thus $N_j(x) \cap \{w\} \neq \emptyset$ then $w \in N_j(x)$. $N_i(x) \supseteq N_j(x)$ is similar. □

If we look at Example (18), then we see that there are six groups of possibly equivalent operators.

1. $N_1^{\mathbb{C}}, N_1^{\mathbb{C}_1}, N_1^{\mathbb{C}_3}, N_1^{\mathbb{C}_\cap}, N_2^{\mathbb{C}_3}, N_3^{\mathbb{C}_1}, N_3^{\mathbb{C}_3}, N_4^{\mathbb{C}_3}$
2. $N_2^{\mathbb{C}}, N_2^{\mathbb{C}_1}$
3. $N_1^{\mathbb{C}_2}, N_1^{\mathbb{C}_4}, N_3^{\mathbb{C}}, N_3^{\mathbb{C}_2}, N_3^{\mathbb{C}_\cap}$
4. $N_2^{\mathbb{C}_2}, N_2^{\mathbb{C}_4}, N_4^{\mathbb{C}}, N_4^{\mathbb{C}_2}, N_4^{\mathbb{C}_\cap}$
5. $N_3^{\mathbb{C}_4}, N_4^{\mathbb{C}_4}$
6. $N_2^{\mathbb{C}_\cap}, N_4^{\mathbb{C}_1}$

A interesting question in this point is about if all the neighborhood operators in one group are indeed equivalent to each other.

4.3.1 Group 1

Proposition 50. $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_1}$

Proof. For all $x \in U$ it holds that $md(\mathbb{C}, x) = md(\mathbb{C}_1, x)$. □

Proposition 51. $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_3}$

Proof. We can write $\mathbb{C}_3$ as follows: $\mathbb{C}_3 = \{N_1^{\mathbb{C}}(x) \mid x \in U\}$. Since $\mathbb{C}_3$ is unary, there exists a $y \in U$ such that $N_1^{\mathbb{C}_3}(x) = N_1^{\mathbb{C}}(y)$. Because $x \in N_1^{\mathbb{C}_3}(x)$, $x \in N_1^{\mathbb{C}}(y)$ and $N_1^{\mathbb{C}}(x) \subseteq N_1^{\mathbb{C}}(y) = N_1^{\mathbb{C}_3}(x)$.

On the other hand, $N_1^{\mathbb{C}_3}(x) = \cap\{N_1^{\mathbb{C}}(z) \mid x \in N_1^{\mathbb{C}}(z)\} \subseteq N_1^{\mathbb{C}}(x)$. □

Proposition 52. $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_\cap}$

Proof. We will prove that $\cap C(\mathbb{C}_\cap, x) = \cap C(\mathbb{C}, x)$. Since $\mathbb{C}_\cap \subseteq \mathbb{C}$, we always have that $\cap C(\mathbb{C}_\cap, x) \supseteq \cap C(\mathbb{C}, x)$.

For the other inclusion, let $K \in C(\mathbb{C}, x) \setminus C(\mathbb{C}_\cap, x)$. Since $x \in K$,

$$K \in \{K' \in \mathbb{C} \mid (\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K'\})(K' = \cap \mathbb{C}'\}.$$

So, there exist $K_1, \ldots, K_n \in \mathbb{C}$ such that $K_i \neq K$, $x \in K_i$ and $K = K_1 \cap \ldots \cap K_n$. We can even say that the K_i's are in $\mathbb{C}_\cap$, otherwise we decompose K_i into elements of $\mathbb{C}_\cap$.

Now let $y \in \cap C(\mathbb{C}_\cap, x) \setminus \cap C(\mathbb{C}, x)$, then

- $\forall K \in \mathbb{C}_\cap\colon\ x \in K \Rightarrow y \in K$ and
- $\exists K \in \mathbb{C}\colon\ x \in K \wedge y \notin K$.

Hence, there exists a $K \in C(\mathbb{C}, x) \setminus C(\mathbb{C}_\cap, x)$ with $y \notin K$. We can decompose K into elements of $\mathbb{C}_\cap$ as we saw before: $K = K_1 \cap \ldots \cap K_n$. This means there exists a K_i with $K_i \in \mathbb{C}_\cap$, $x \in K_i$ and $y \notin K_i$ which is a contradiction. Hence, $\cap C(\mathbb{C}_\cap, x) = \cap C(\mathbb{C}, x)$.

□

Proposition 53. $N_1^{\mathbb{C}_3} = N_2^{\mathbb{C}_3}$

Proof. $\mathbb{C}_3$ is unary. □

The other three other operators are not equivalent to the five we saw before.

Example 19. *Let $\mathbb{C} = \{3, 12, 13, 123\}$, then $\mathbb{C}_3 = \{1, 12, 3\}$. We have that $N_1^{\mathbb{C}}(1) = \{1\} = N_3^{\mathbb{C}_1}(1)$ and $N_3^{\mathbb{C}_3}(1) = \{12\} = N_4^{\mathbb{C}_3}(1)$.*

Example 20. *Let $\mathbb{C} = \{12, 123\}$, then $\mathbb{C}_1 = \{12, 123\}$. We have that $N_1^{\mathbb{C}_1}(1) = \{12\}$ and $N_3^{\mathbb{C}_1}(1) = \{123\}$.*

Example 21. *Let* $\mathbb{C} = \{1, 12, 13\}$*, then* $\mathbb{C}_3 = \{1, 12, 13\}$*. We have that* $N_3^{\mathbb{C}_3}(1) = \{1\}$ *and* $N_4^{\mathbb{C}_3}(1) = \{123\}$.

It is possible to conclude that the first group exists of four groups of equivalent operators:

- $N_1^{\mathbb{C}}, N_1^{\mathbb{C}_1}, N_1^{\mathbb{C}_3}, N_1^{\mathbb{C}_\cap}, N_2^{\mathbb{C}_3}$
- $N_3^{\mathbb{C}_1}$
- $N_3^{\mathbb{C}_3}$
- $N_4^{\mathbb{C}_3}$

4.3.2 Group 2

Proposition 54. $N_2^{\mathbb{C}} = N_2^{\mathbb{C}_1}$

Proof. For all $x \in U$ it holds that $\mathrm{md}(\mathbb{C}, x) = \mathrm{md}(\mathbb{C}_1, x)$. □

4.3.3 Group 3

Proposition 55. $N_1^{\mathbb{C}_2} = N_3^{\mathbb{C}}$

Proof. First, note that $\mathbb{C}_2 \subseteq \mathbb{C}$. By definition, we have that

- $N_1^{\mathbb{C}_2}(x) = \cap\{K \in \mathbb{C} \mid x \in K \wedge (\exists y \in U)(K \in \mathrm{MD}(\mathbb{C}, y)\}$ and
- $N_3^{\mathbb{C}}(x) = \cap\{K \in \mathbb{C} \mid K \in \mathrm{MD}(\mathbb{C}, x)\}$.

If $K \in \mathrm{MD}(\mathbb{C}, x)$, then $x \in K$ and $(\exists y \in U)(K \in \mathrm{MD}(\mathbb{C}, y)$. On the other hand, if $x \in K$ and $(\exists y \in U)(K \in \mathrm{MD}(\mathbb{C}, y)$, then there exists a $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$ and since $y \in K$, $y \in K'$. Now, $K \in \mathrm{MD}(\mathbb{C}, y)$, so $K = K'$ and $K \in \mathrm{MD}(\mathbb{C}, x)$.

We conclude that both operators are equivalent. □

Proposition 56. $N_3^{\mathbb{C}} = N_3^{\mathbb{C}_2} = N_3^{\mathbb{C}_\cap}$

Proof. It is trivial that $\mathrm{MD}(\mathbb{C}, x) = \mathrm{MD}(\mathbb{C}_2, x)$.

We prove that $\mathrm{MD}(\mathbb{C}, x) = \mathrm{MD}(\mathbb{C}_\cap, x)$. First, take $A \in \mathrm{MD}(\mathbb{C}, x)$ and $K \in \mathbb{C}_\cap$ with $x \in K$ and $A \subseteq K$. Since $K \in \mathbb{C}$, $A = K$ and thus $A \in \mathbb{C}_\cap$ and $A \in \mathrm{MD}(\mathbb{C}_\cap, x)$. Second, take $A \in \mathrm{MD}(\mathbb{C}_\cap, x)$ and $K \in \mathbb{C}$ with $x \in K$ and $A \subseteq K$. If $K \in \mathbb{C}_\cap$, then $A = K$. If $K \notin \mathbb{C}_\cap$, then there exists $K_1, \ldots, K_n \in \mathbb{C}_\cap$ with $K = K_1 \cap \ldots \cap K_n$. Then for all i, $A \subseteq K_i$ and thus, $A = K_i$ for all i. Again we can conclude that $A = K$. Hence, $A \in \mathrm{MD}(\mathbb{C}, x)$. □

The fifth operator of this group, $N_1^{\mathbb{C}_4}$, is not equivalent to the other four.

Example 22. *Let* $\mathbb{C} = \{12, 23, 13\}$*, then* $\mathbb{C}_2 = \{12, 23, 13\}$ *and* $\mathbb{C}_4 = \{123\}$*. We have that* $N_1^{\mathbb{C}_2}(1) = \{1\}$ *and* $N_1^{\mathbb{C}_4}(1) = \{123\}$.

We can conclude that the third group exists of two groups of equivalent operators:

- $N_1^{\mathbb{C}_2}, N_3^{\mathbb{C}}, N_3^{\mathbb{C}_2}, N_3^{\mathbb{C}_\cap}$
- $N_1^{\mathbb{C}_4}$

4.3.4 Group 4

Proposition 57. $N_2^{\mathbb{C}_2} = N_4^{\mathbb{C}_2}$

Proof. For all $x \in U$, $\cup\mathrm{md}(\mathbb{C}_2, x) \subseteq \cup\mathrm{MD}(\mathbb{C}_2, x)$, hence, $N_2^{\mathbb{C}_2}(x) \subseteq N_4^{\mathbb{C}_2}(x)$. Take $K \in \mathrm{MD}(\mathbb{C}_2, x)$, then $K \in \mathbb{C}_2$ and $x \in K$. Let $S \in \mathbb{C}_2$ with $x \in S$ and $S \subseteq K$. Since $S \in \mathbb{C}_2$, there exists a $y \in U$ such that $S \in \mathrm{MD}(\mathbb{C}, y)$. Since $S \subseteq K$, $y \in K$, $K \in \mathbb{C}_2 \subseteq \mathbb{C}$ and since S maximal for y, we have that $S = K$. Hence, $K \in \mathrm{md}(\mathbb{C}_2, x)$ and $N_2^{\mathbb{C}_2}(x) \supseteq N_4^{\mathbb{C}_2}(x)$. □

Proposition 58. $N_4^{\mathbb{C}} = N_4^{\mathbb{C}_2} = N_4^{\mathbb{C}_\cap}$

Proof. For all $x \in U$ we have that $\mathrm{MD}(\mathbb{C}, x) = \mathrm{MD}(\mathbb{C}_2, x) = \mathrm{MD}(\mathbb{C}_\cap, x)$. □

The fifth operator is not equivalent with the other.

Example 23. *Let* $\mathbb{C} = \{12, 23, 34, 14\}$*, then* $\mathbb{C}_2 = \{12, 23, 34, 14\}$ *and* $\mathbb{C}_4 = \{124, 123, 234, 134\}$. *We have that* $N_2^{\mathbb{C}_2}(1) = \{124\}$ *and* $N_2^{\mathbb{C}_4}(1) = \{1234\}$.

The fourth group decomposes in two groups of equivalent operators:

- $N_2^{\mathbb{C}_2}, N_4^{\mathbb{C}}, N_4^{\mathbb{C}_2}, N_4^{\mathbb{C}_\cap}$
- $N_2^{\mathbb{C}_4}$

4.3.5 Group 5

The two operators are not equivalent.

Example 24. *Let* $\mathbb{C} = \{12, 23, 13\}$*, then* $\mathbb{C}_4 = \{12, 23, 13\}$. *We have that* $N_3^{\mathbb{C}_4}(1) = \{1\}$ *and* $N_4^{\mathbb{C}_4}(1) = \{123\}$.

4.3.6 Group 6

The two operators are not equivalent.

Example 25. *Let* $\mathbb{C} = \{1, 23, 14, 123\}$*, then* $\mathbb{C}_\cap = \{14, 23, 123\}$ *and* $\mathbb{C}_1 = \{1, 14, 23\}$. *We have that* $N_2^{\mathbb{C}_\cap}(1) = \{1234\}$ *and* $N_4^{\mathbb{C}_1}(1) = \{14\}$.

It is possible to conclude that there are 13 groups of equivalent neighborhood operators, shown in table 4.2.

The concepts of minimal and maximal description are related with a covering. For different coverings, different minimal and maximal descriptions are obtained. Some preliminary results about the operators defined in Table (4.1), are:

Table 4.2: Groups of neighborhood operators.

A	$N_1^{\mathbb{C}}, N_1^{\mathbb{C}_1}, N_1^{\mathbb{C}_3}, N_1^{\mathbb{C}_\cap}, N_2^{\mathbb{C}_3}$	H	$N_2^{\mathbb{C}_\cap}$
B	$N_2^{\mathbb{C}}, N_2^{\mathbb{C}_1}$	I	$N_1^{\mathbb{C}_4}$
C	$N_3^{\mathbb{C}_1}$	J	$N_2^{\mathbb{C}_2}, N_4^{\mathbb{C}}, N_4^{\mathbb{C}_2}, N_4^{\mathbb{C}_\cap}$
D	$N_3^{\mathbb{C}_3}$	K	$N_3^{\mathbb{C}_4}$
E	$N_4^{\mathbb{C}_3}$	L	$N_2^{\mathbb{C}_4}$
F	$N_4^{\mathbb{C}_1}$	M	$N_4^{\mathbb{C}_4}$
G	$N_1^{\mathbb{C}_2}, N_3^{\mathbb{C}}, N_3^{\mathbb{C}_2}, N_3^{\mathbb{C}_\cap}$		

Proposition 59. *Let $\mathbb{C}$ be a covering of U, then $\mathbb{C}_3$ is an unary covering and $\underline{apr}_{N_1}^{\mathbb{C}_3} = \underline{apr}_{N_2}^{\mathbb{C}_3}$.*

Proof. The elements in $\mathbb{C}_3$ are the neighborhoods $N_1^{\mathbb{C}}(x)$, so $\mathbb{C}_3 = \{N_1^{\mathbb{C}}(x)\}$ and it is unary, because the minimal description of each $x \in U$ has only an element, $|md(\mathbb{C}, x)| = 1$. In this case the neighborhood $N_1^{\mathbb{C}_3}$ and $N_2^{\mathbb{C}_3}$ are the same and therefore the approximation operators $\underline{apr}_{N_1}^{\mathbb{C}_3}$ and $\underline{apr}_{N_2}^{\mathbb{C}_3}$ are equal. □

Proposition 60. *For a covering approximation space and the coverings defined above, we have:*

a. $md(\mathbb{C}, x) = md(\mathbb{C}_1, x)$.

b. $MD(\mathbb{C}, x) = MD(\mathbb{C}_2, x) = MD(\mathbb{C}_\cap, x)$.

Proof. (a.). Let us recall that $\mathbb{C}_1$ is the covering with the minimal sets of $\mathbb{C}$, so the minimal sets of $\mathbb{C}_1$ are equals to $\mathbb{C}_1$. (b). Similarly for maximal description. □

Corollary 4.1. *The relations among neighborhood system are:*

a. $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_1}$ *and* $N_2^{\mathbb{C}} = N_2^{\mathbb{C}_1}$.

b. $N_3^{\mathbb{C}} = N_3^{\mathbb{C}_2} = N_3^{\mathbb{C}_\cap}$ *and* $N_4^{\mathbb{C}} = N_4^{\mathbb{C}_2} = N_4^{\mathbb{C}_\cap}$.

Proof. Let us recall that N_1 and N_2 are defined from $md(\mathbb{C}, x)$ and N_3 and N_4 are defined from $MD(\mathbb{C}, x)$. □

Corollary 4.2. *The relations among approximation operators are:*

a. $\overline{apr}_{N_1}^{\mathbb{C}} = \overline{apr}_{N_1}^{\mathbb{C}_1}$ *and* $\overline{apr}_{N_2}^{\mathbb{C}} = \overline{apr}_{N_2}^{\mathbb{C}_1}$.

b. $\overline{apr}_{N_3}^{\mathbb{C}} = \overline{apr}_{N_3}^{\mathbb{C}_2} = \overline{apr}_{N_3}^{\mathbb{C}_\cap}$ *and* $\overline{apr}_{N_4}^{\mathbb{C}} = \overline{apr}_{N_4}^{\mathbb{C}_2} = \overline{apr}_{N_4}^{\mathbb{C}_\cap}$.

□

The example 26 is used to show differences between $\underline{\mathrm{apr}}_{M_i}$ and $\underline{\mathrm{apr}}_{N_i}$. We will concentrate only in operators $\underline{\mathrm{apr}}_{N_i}$, because operators $\underline{\mathrm{apr}}_{M_i}$, satisfy the same properties.

Example 26. *For the covering* $\mathbb{C} = \{1, 12, 13, 24, 123, 234\}$ *of* $U = 1234$ *we have the lower approximations for the operators* $\underline{\text{apr}}_N$ *in Table 4.3:*

Table 4.3: Lower approximations for subsets of $A = 1234$.

A	$\underline{\text{apr}}_{M_1}$	$\underline{\text{apr}}_{M_2}$	$\underline{\text{apr}}_{M_3}$	$\underline{\text{apr}}_{M_4}$	$\underline{\text{apr}}_{N_1}$	$\underline{\text{apr}}_{N_2}$	$\underline{\text{apr}}_{N_3}$	$\underline{\text{apr}}_{N_4}$
1	1	∅	1	∅	1	1	∅	∅
2	∅	∅	∅	∅	2	∅	∅	∅
3	3	3	∅	∅	3	∅	∅	∅
4	4	∅	4	∅	∅	∅	∅	∅
12	1	∅	1	∅	12	1	∅	∅
13	13	3	1	∅	13	1	∅	∅
14	14	∅	14	∅	1	1	∅	∅
23	3	3	∅	∅	23	∅	23	∅
24	24	24	∅	∅	24	4	∅	∅
34	34	3	4	∅	3	∅	∅	∅
123	13	13	1	1	123	123	1	1
124	124	∅	14	∅	124	124	∅	∅
134	134	3	14	∅	13	1	∅	∅
234	234	234	4	4	234	4	234	4
1234	1234	1234	1234	1234	1234	1234	1234	1234

4.4 Order Relation for Neighborhood Operators

In the previous section, we have established which neighborhood operators based on coverings are equivalent. Furthermore, we have proved that two different neighborhood operators define two different approximation operators. To this end, it suffices to study the order relations between different sets of equivalent neighborhood operators to obtain the order relations for approximation operators based on neighborhoods.

In Proposition 32, it is proven that for a fixed covering $\mathbb{C}$, the following order relations for neighborhood operators hold:

a. $N_1^{\mathbb{C}} \leq N_2^{\mathbb{C}} \leq N_4^{\mathbb{C}}$,

b. $N_1^{\mathbb{C}} \leq N_3^{\mathbb{C}} \leq N_4^{\mathbb{C}}$.

Now, it is possible to establish order relations between neighborhood operators based on different coverings. The first step is related with the neighborhood operator N_1.

Proposition 61. *Let* $\mathbb{C}$ *be a covering, then*

a. $N_1^{\mathbb{C}} \leq N_1^{\mathbb{C}_2}$,

b. $N_1^{\mathbb{C}_2} \leq N_1^{\mathbb{C}_4}$.

Proof. For (i), we show that $N_1^{\mathbb{C}}(x) \subseteq N_1^{\mathbb{C}_2}(x)$ for all $x \in U$. By definition, it holds that $\mathbb{C}_2 \subseteq \mathbb{C}$. Furthermore, $C(\mathbb{C}_2, x) \subseteq C(\mathbb{C}, x)$. This implies that $\cap C(\mathbb{C}, x) \subseteq \cap C(\mathbb{C}_2, x)$, so $N_1^{\mathbb{C}}(x) \subseteq N_1^{\mathbb{C}_2}(x)$.

For (ii), take $x \in U$ and $y \in N_1^{\mathbb{C}_2}(x)$. Then for all $K \in \mathbb{C}_2$ with $x \in K$ it holds that $y \in K$. Take $K' \in C(\mathbb{C}_4, x)$, then there exist $K_1, \ldots, K_n \in \mathbb{C}_2$ such that $K' = K_1 \cup \ldots \cup K_n$. Since $x \in K'$, there exists a $K_i \in \{K_1, \ldots, K_n\}$ such that $x \in K_i$. Hence, $y \in K_i$ and thus $y \in K'$. We conclude that $y \in N_1^{\mathbb{C}_4}(x)$. □

The order relation $N_1^{\mathbb{C}_2} \leq N_1^{\mathbb{C}}$ cannot hold.

Example 27. *Let* $U = \{1,2,3,4\}$ *and* $\mathbb{C} = \{\{1\},\{1,2\},\{1,3\},\{2,4\},\{3,4\},\{1,2,3\},\{2,3,4\}\}$*, then* $N_1^{\mathbb{C}}(1) = \{1\}$ *and* $N_1^{\mathbb{C}_2}(1) = \{1,2,3\}$.

The order relation $N_1^{\mathbb{C}_4} \leq N_1^{\mathbb{C}_2}$ cannot hold.

Example 28. *Let* $U = \{1,2,3\}$ *and* $\mathbb{C} = \{\{1,2\},\{1,3\},\{2,3\}\}$*, then* $N_1^{\mathbb{C}_2}(1) = \{1\}$ *and* $N_1^{\mathbb{C}_4}(1) = \{1,2,3\}$.

We continue with neighborhood operator N_2.

Proposition 62. *Let* $\mathbb{C}$ *be a covering, then*

a. $N_2^{\mathbb{C}_3} \leq N_2^{\mathbb{C}}$,

b. $N_2^{\mathbb{C}} \leq N_2^{\mathbb{C}_\cap}$,

c. $N_2^{\mathbb{C}_\cap} \leq N_2^{\mathbb{C}_2}$,

d. $N_2^{\mathbb{C}_2} \leq N_2^{\mathbb{C}_4}$.

Proof. For $(a.)$, take $x \in U$ and $y \in N_2^{\mathbb{C}_3}(x)$, then there exists a $K \in \mathrm{md}(\mathbb{C}_3, x)$ with $y \in K$. Since $\mathbb{C}_3$ is unary, K is the unique set in $\mathrm{md}(\mathbb{C}_3, x)$. By definition of $\mathbb{C}_3$, $K = \cap\mathrm{md}(\mathbb{C}, x)$. Hence, for all $K' \in \mathrm{md}(\mathbb{C}, x)$ it holds that $y \in K'$. We conclude that $y \in N_2^{\mathbb{C}}(x)$.

For $(b.)$, take $x \in U$ and $y \in N_2^{\mathbb{C}}(x)$, then there exists a $K \in \mathrm{md}(\mathbb{C}, x)$ with $y \in K$. We need to prove that there exists a $K' \in \mathrm{md}(\mathbb{C}_\cap, x)$ such that $y \in K'$. If $K \in \mathbb{C}_\cap$, take $K' = K$. Then $K' \in \mathrm{md}(\mathbb{C}_\cap, x)$ and $y \in K'$, hence $y \in N_2^{\mathbb{C}_\cap}(x)$. If $K \notin \mathbb{C}_\cap$, then there exist $K_1, \ldots, K_n \in \mathbb{C}_\cap$ with $K = K_1 \cap \ldots \cap K_n$. Take $K' = K_1$, then $y \in K'$. Let $L \in \mathbb{C}_\cap$ with $x \in L$ and $L \subseteq K'$. Then $L \in \mathbb{C}$ with $x \in L$ and $L \subseteq K$. Since $K \in \mathrm{md}(\mathbb{C}, x)$, it holds that $L = K$, hence $L = K'$. We conclude that $K' \in \mathrm{md}(\mathbb{C}_\cap, x)$ and $y \in N_2^{\mathbb{C}_\cap}(x)$.

For $(c.)$, take $x \in U$ and $y \in N_2^{\mathbb{C}_\cap}(x)$, then there exists a $K \in \mathrm{md}(\mathbb{C}_\cap, x)$ with $y \in K$. Since $K \in \mathbb{C}_\cap \subseteq \mathbb{C}$, there exists a $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$. This means that $y \in K'$ and $K' \in \mathbb{C}_2$. We prove that $K' \in \mathrm{md}(\mathbb{C}_2, x)$. Take $L \in \mathbb{C}_2$ with $x \in L$ and $L \subsetneq K'$. Since $K' \in \mathrm{MD}(\mathbb{C}, x)$, L cannot be in $\mathbb{C}_2$. Hence, $K' \in \mathrm{md}(\mathbb{C}_2, x)$ and $y \in N_2^{\mathbb{C}_2}(x)$.

For $(d.)$, take $x \in U$ and $y \in N_2^{\mathbb{C}_2}(x)$, then there exists a $K \in \mathrm{md}(\mathbb{C}_2, x)$ with $y \in K$. Since $K \in \mathbb{C}_2$, there exists a $K' \in \mathbb{C}_4$ with $K \subseteq K'$ and there exists a $K'' \in \mathrm{MD}(\mathbb{C}_4, x)$ with $K' \subseteq K''$. Hence, $y \in K''$ and $y \in N_2^{\mathbb{C}_4}(x)$. □

The order relations $N_2^{\mathbb{C}} \leq N_2^{\mathbb{C}_3}$, $N_2^{\mathbb{C}_\cap} \leq N_2^{\mathbb{C}}$ and $N_2^{\mathbb{C}_2} \leq N_2^{\mathbb{C}_\cap}$ cannot hold.

Example 29. *Let* $U = \{1,2,3,4\}$ *and* $\mathbb{C} = \{\{1\},\{1,2\},\{1,3\},\{2,4\},\{3,4\},\{1,2,3\},\{2,3,4\}\}$*, then*

(i) $N_2^{\mathbb{C}_3}(2) = \{2\}$ *and* $N_2^{\mathbb{C}}(2) = \{1,2,4\}$,

(ii) $N_2^{\mathbb{C}}(1) = \{1\}$ *and* $N_2^{\mathbb{C}_\cap}(1) = \{1,2,3\}$, *(iii)* $N_2^{\mathbb{C}_\cap}(2) = \{1,2,4\}$ *and* $N_2^{\mathbb{C}_2}(2) = \{1,2,3,4\}$.

The order relation $N_2^{\mathbb{C}_4} \leq N_2^{\mathbb{C}_2}$ cannot hold.

Example 30. *Let* $U = \{1,2,3,4\}$ *and* $\mathbb{C} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$*, then* $N_2^{\mathbb{C}_2}(1) = \{1,2,4\}$ *and* $N_2^{\mathbb{C}_4}(1) = \{1,2,3,4\}$.

Next, we discuss neighborhood operator N_3.

Proposition 63. *Let* $\mathbb{C}$ *be a covering, then*

a. $N_3^{\mathbb{C}_1} \leq N_3^{\mathbb{C}}$,

b. $N_3^{\mathbb{C}} \leq N_3^{\mathbb{C}_4}$.

Proof. For (*a*.), take $x \in U$ and $y \in N_3^{\mathbb{C}_1}(x)$, then we need to prove that $y \in \cap\{K \mid K \in \mathrm{MD}(\mathbb{C},x)\}$. Take $K \in \mathrm{MD}(\mathbb{C},x)$, then there exists a $K' \in \mathrm{MD}(\mathbb{C}_1,x)$ with $K' \subseteq K$. Since $y \in N_3^{\mathbb{C}_1}(x)$, it holds that $y \in K'$ and thus $y \in K$. Hence, $y \in N_3^{\mathbb{C}}(x)$.

For (*b*.), we know by [54] that $N_3^{\mathbb{C}} = N_1^{\mathbb{C}_2}$. By Propositions 61 and 62, we know that $N_1^{\mathbb{C}_2} \leq N_1^{\mathbb{C}_4} \leq N_3^{\mathbb{C}_4}$. We conclude that $N_3^{\mathbb{C}} \leq N_3^{\mathbb{C}_4}$. □

The neighborhood operator $N_3^{\mathbb{C}_3}$ is not comparable with the other neighborhood operators N_3.

Example 31. *Let* $U = \{1,2,3,4\}$ *and* $\mathbb{C} = \{\{1\},\{2\},\{1,4\},\{2,4\},\{3,4\},\{1,3,4\}\}$*, then* $N_3^{\mathbb{C}}(4) = N_3^{\mathbb{C}_1}(4) = \{4\}$*,* $N_3^{\mathbb{C}_4}(4) = \{1,2,3,4\}$ *and* $N_3^{\mathbb{C}_3}(4) = \{3,4\}$.

Example 32. *Let* $U = \{1,2,3,4\}$ *and* $\mathbb{C} = \{\{1\},\{4\},\{1,2,4\},\{1,3,4\},\{1,2,3\}\}$*, then* $N_3^{\mathbb{C}}(4) = N_3^{\mathbb{C}_1}(4) = \{1,4\}$*,* $N_3^{\mathbb{C}_4}(4) = \{1,2,3,4\}$ *and* $N_3^{\mathbb{C}_3}(4) = \{4\}$.

Example 32 also shows that $N_3^{\mathbb{C}_4} \leq N_3^{\mathbb{C}}$ cannot hold. We illustrate that the order relation $N_3^{\mathbb{C}} \leq N_3^{\mathbb{C}_1}$ cannot hold as well.

Example 33. *Let* $U = \{1,2,3,4\}$ *and* $\mathbb{C} = \{\{1\},\{1,2\},\{1,3\},\{2,4\},\{3,4\},\{1,2,3\},\{2,3,4\}\}$*, then* $N_3^{\mathbb{C}}(1) = \{1,2,3\}$ *and* $N_3^{\mathbb{C}_1}(1) = \{1\}$.

Finally, we discuss the order relation for neighborhood operators based on N_4.

Proposition 64. *Let* $\mathbb{C}$ *be a covering, then*

a. $N_4^{\mathbb{C}_3} \leq N_4^{\mathbb{C}_1}$,

b. $N_4^{\mathbb{C}_1} \leq N_4^{\mathbb{C}}$,

c. $N_4^{\mathbb{C}} \leq N_4^{\mathbb{C}_4}$.

Proof. For (*a*.), take $x \in U$ and $y \in N_4^{\mathbb{C}_3}(x)$, then there exists a $K \in \mathrm{MD}(\mathbb{C}_3,x)$ with $y \in K$. Then there exists a $K' \in \mathbb{C}_1$ with $K \subseteq K'$ and there exists a $K'' \in \mathrm{MD}(\mathbb{C}_1,x)$ with $K' \subseteq K''$. Thus, $y \in K'' \subseteq N_4^{\mathbb{C}_1}(x)$.

For (*b*.), take $x \in U$ and $y \in N_4^{\mathbb{C}_1}(x)$, then there exists a $K \in \mathrm{MD}(\mathbb{C}_1,x)$ with $y \in K$. Thus, $K \in \mathbb{C}_1 \subseteq \mathbb{C}$, so there exists a $K' \in \mathrm{MD}(\mathbb{C},x)$ with $K \subseteq K'$. Hence, $y \in K'$ and $y \in N_4^{\mathbb{C}}(x)$.

For $(c.)$, take $x \in U$ and $y \in N_4^{\mathbb{C}}(x)$, then there exists a $K \in \mathrm{MD}(\mathbb{C}, x)$ with $y \in K$. Take $K' = \cup \mathrm{MD}(\mathbb{C}, x) \in \mathbb{C}_4$, then $y \in K'$ and there exists a $K'' \in \mathrm{MD}(\mathbb{C}_4, x)$ with $K' \subseteq K''$. Hence, $y \in K'' \subseteq N_4^{\mathbb{C}_4}(x)$. □

The order relations $N_4^{\mathbb{C}_1} \leq N_4^{\mathbb{C}_3}$, $N_4^{\mathbb{C}} \leq N_4^{\mathbb{C}_1}$ and $N_4^{\mathbb{C}_4} \leq N_4^{\mathbb{C}}$ cannot hold.

Example 34. *Let* $U = \{1,2,3,4\}$ *and* $\mathbb{C} = \{\{1\},\{1,2\},\{1,3\},\{2,4\},\{3,4\},\{1,2,3\},\{2,3,4\}\}$*, then*

(i) $N_4^{\mathbb{C}_3}(1) = \{1\}$ *and* $N_4^{\mathbb{C}_1}(1) = \{1,2,3\}$,

(ii) $N_4^{\mathbb{C}_1}(2) = \{1,2,4\}$ *and* $N_4^{\mathbb{C}}(2) = \{1,2,3,4\}$,

(iii) $N_4^{\mathbb{C}}(1) = \{1,2,3\}$ *and* $N_4^{\mathbb{C}_4}(1) = \{1,2,3,4\}$.

Therefore there are 13 groups of neighborhood operators based on coverings. A diagram with the order relations studied is constructed.

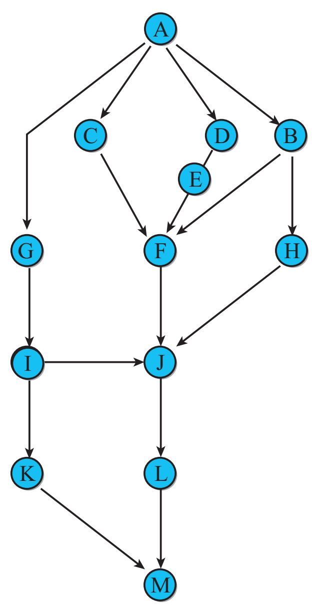

Figure 4.2: Partial order for neighborhood operators.

Integrating the diagrams in Figure (3.4) and Figure (4.2) we have the diagram of Figure (4.3). In this case, we put together the equivalent operators. The operators A to M correspond to operators in diagram of Figure (4.2). The operators Y and Z can be expressed as a relation based operators, using other relations different from $N_i^{\mathbb{C}}$ as was established in [91].

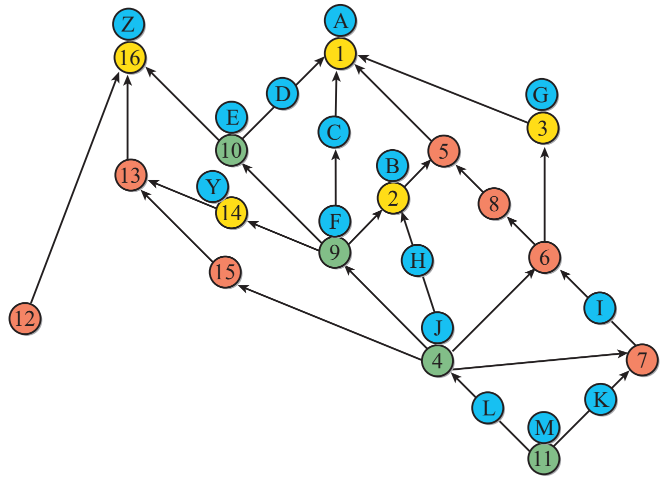

Figure 4.3: Partial order for neighborhood and covering based operators.

4.5 Summary

This chapter studies relationships between relation based and covering based rough sets, In particular it has been shown that each element based definition approximation $\underline{apr}_{N_i}$ and $\overline{apr}_{N_i}$ and each granule based definition approximation $\underline{apr}''_{\mathbb{C}}$ and $\overline{apr}''_{\mathbb{C}}$ can be treated as a relation based approximation operator. We also defined new approximation operators combining coverings and neighborhood operators. Finally a connection between the framework of covering based approximation operators and the framework of relations based approximation operators has been established.

CHAPTER 5

Conclusions

Two different frameworks of approximation operators in covering based rough sets were unified in a general framework of dual pairs. The most important contribution of this work is related to the systematic study of approximation operators in covering based rough sets. The first element in this study is the characterization of approximation operators, using notions of duality, conjugacy and adjointness. The second element is the construction of partial order relation between the operators. A partial order relation for lower and upper approximation operators for covering based rough sets and relation based have been established.

The characterization of approximation operators can be used to deduce properties of approximation. For example, if a similarity relation R is used in datasets with continuous attributes, the pair of approximation operators $\underline{apr}_R$ and $\overline{apr}_R$ is a Galois Connection. The order relation between approximation operators can be used to improve some discretization and attribute reduction algorithms. For example, the Structured Genetic Algorithm used for selecting approximation operators, selects generally maximal elements in the Hasse diagram presented in section 3.3.5.

The characterization of approximation operators was published by International Journal of Approximate Reasoning 55 (2014) and the partial order relation was published in Information Sciences 284 (2014). One Journal's reviewer said: "this work will not have a major impact as it is a very specialized work in the research on rough sets". Two years later, I realize that this work had aroused interest among some researchers and now is an important reference for them. The theoretical aspects of a subject, usually have a bigger impact than a specific application.

A second contribution of this work is the establishment of some relationships between the most important generalization of rough set theory: Covering based and relation based rough sets. A first relationship was introduced in Chapter 4, where new approximation operators were obtained. In particular each element based definition approximation $\underline{apr}_{N_i}$ and $\overline{apr}_{N_i}$ and each granule based definition approximation $\underline{apr}''_{\mathbb{C}}$ and $\overline{apr}''_{\mathbb{C}}$ can be treated as a relation based approximation operator.

The relationship between covering and relation based rough sets was published in Lecture Notes on Artificial Intelligence 8537 (2014) and the generalization was submitted to Information Sciences Journal (2015).

This work has also helped to develop new lines of work, such as the generalization of these approximation operator's properties in fuzzy context, led by Dr. Chris Cornelis, University of Ghent in Belgium, some work proposals of Dr. Piero Plagiani of Research Group on Knowledge and Communication Models in Italy and other work proposal from Dr. Rafael Bello, Universidad Central de las Villas Cuba for the study of coverings obtained from similarity relation in datasets.

6

Future work

A very important continuation of this work involves studying order relationships that hold between the various approximation operators, such as $\overline{apr}''_{\mathbb{C}}(A) \subseteq \overline{apr}'_{\mathbb{C}}(A)$. Such order relations have already been studied partially in [93], where they are induced by means of entropy and co-entropy measures, generalized for covering based rough sets, and [39], which studies order relations between various types of neighborhood-related in covering based rough sets. Since approximation operators are used frequently in data analysis applications of rough sets such as feature selection and classification (see e.g. [10] in the case of covering-based rough sets), order relationships can be meaningfully used to guide the selection of appropriate pairs of approximation operators.

Formal concept analysis (FCA) and rough set theory have the formal context as a common framework [69, 84]. A formal concept (U, A, R) is defined by two finite sets U, A, and a binary relation from U to A. A regular formal context defines a covering $\mathbb{C}_A$ as the set of object sets of attributes $a \in A$. A dual pair of approximation operators is defined from this covering, using the operator $\underline{apr}'_{\mathbb{C}}$. In this point there are at least two interesting ways of work. The first one is the attribute reduction problem, as is showed in [69]. On the other hand, and according to the approximation frameworks presented, we have other approximation operators associated with a particular covering $\mathbb{C}$. Therefore it makes sense to study the properties and the order relation among these approximation operators.

I would like to obtain further characterizations of covering based approximation operators. In particular, an interesting question is whether there exist dual pairs of approximation operators that are both idempotent and adjoint. Also, I plan to study different order relations among pairs of approximation operators, for example the extension of the Entropy based order relation defined by Zhu and Wen in [94].

I want to explore relationships between approximation operators $\underline{apr}'_{\mathbb{C}}$ and $\overline{apr}'_{\mathbb{C}}$ with relation based rough sets and to establish other relationships between the new approximation operators defined in Section 4.2.2.

Grothendieck topologies and the relationship with rough sets theory was presented in [52]. It is an interesting topic that deserve study.

Finally, in 2013 some ideas about matroids and the relationship with rough sets, were studied in the Information Networks Theory seminary, led by professor Humberto Sarria at department of mathematics, Universidad Nacional de Colombia. Here, there are many ideas to develop.

APPENDIX A

Evolutionary approach

This appendix presents a technique for attribute selection and data discretization from the hybridization of rough set theory and structured genetic algorithms. It is not a systematic study of algorithm efficiency but an illustration of how we can implement some ideas. From a theoretical point of view the algorithms maintain the quality of classification as an optimum value in any classification problem.

Quality of classification measures the number of instances that can be classified without ambiguity and it can be obtained from the lower approximations. Clearly it depends on the approximation operator used. In this way, the attribute selection and data discretization problems can be seen as an optimization problem with a constrain given by the quality of classification.

A.1 Introduction

The attribute selection and data discretization are considered important tasks in the data mining and machine learning areas. In one hand, the attribute selection can be presented as finding a subset of *m* attributes from a total of *n* attributes defining a data table, with no significant loss of performance when is used to classify the records in the data table, according to a dependent attribute. On the other hand, attribute discretization is based on searching for cuts that determine intervals for real or integer valued attributes (values that lie within each interval are then treated as indiscernible).

There are several research works that apply rough set theory to the attribute selection and discretization problems, and combine it with evolutionary algorithms [13, 29, 35, 38, 41, 42, 43]. Evolutionary algorithms (EA) are efficient search methods based on the principles of natural selection in which some genetic operators, applied to encoded versions (chromosomes) of candidate solutions, are employed to generate better solutions. There are several different EA techniques, in this work we used the classical genetic algorithm and the evolutionary algorithm that adapts the operator rates while it is solving the optimization problem proposed in [19, 18]. The Hybrid Adaptive Evolutionary Algorithm is a mixture of ideas borrowed from Evolutionary Strategies, decentralized control adaptation, and central control adaptation.

Some attribute selection (reduction) algorithms, based on genetic algorithms, were presented in [2, 13, 35] and it is focused on improving the fitness function to obtain a reduct (considering the rough sets associated to the attributes). The fitness function used for Xu, et al. in [75] consider the number of elements in positive region and the number of selected attributes. In our proposal attribute reduction problem in rough sets can be seen like an optimization problem with a constrain given for the classification quality. However, such technique does not guarantee to get a reduct. Partalain and Shen present, in [48], a new approach on tolerance rough sets model, for feature (attribute) selection and use the positive region and boundary for defining a suitable metric for continuous attribute.

We present a technique for attribute selection and data discretization from the rough set theory and genetic algorithm points of view. Section A.2 describes the basic evolutionary/rough approach to attribute selection. Section A.3 presents the evolutionary/rough approach to attribute discretization. Section A.4 introduces the structured evolutionary algorithm that combines the previous two approaches. Section A.5 extends the structured evolutionary algorithm in order to include the approximation operators to be used. Finally, Section A.6 draws some conclusions.

A.2 Attribute Selection

We extend the approach proposed by Xu, et al. in [75] of considering the attribute selection as an optimization problem. Basically, we try to minimize the number of attributes S with the constrain $\gamma_P(U) = 1$, using an evolutionary algorithm. Here, $\gamma_P(U)$ is the quality of classification value defined in rough set theory.

$$\gamma_P(U) = \frac{\sum |L(Y_i)|}{|U|} \tag{A.1}$$

Using a penalization method, the fitness function can be written as follows:

$$f = r_m - \beta|1 - \gamma_P(U)| - S \tag{A.2}$$

where r_m is the minimum expected number of attributes and it is a simple translation of fitness function, S is the number of selected attributes in an iteration and β is a parameter that needs to be tuned. The negative sign is used because minimization. In the case where $\gamma_P(U) = 1$, the global optimum value must be zero.

We encoded, a candidate solution, as a binary string, one bit per each attribute, where a zero ('0') in position i means the attribute i must be ignored. The classical binary crossover and bit mutation were used by the evolutionary algorithm.

A.2.1 Experiments

We run a classical Genetic Algorithm during 100 iterations, with a population of 100 individuals, a crossover rate of 0.8, a mutation rate of 0.001 and a Elitism selection mechanism, on two different data sets (zoo and colombian coffee growers). The Elitism selection operator is defined in terms of two parameters the elite (10%) and cull (10%) percentages. The zoo data set is taken from UCI repository [1]. It has 101 examples with 16 condition attributes and a decision attribute with seven different classes. All attributes are categorical and do not have missing data. The Colombian coffee growers, which information was collected from a survey applied to growers in different Colombian municipalities. It contains information about 338 coffee growers with 55 condition

[1] http://archive.ics.uci.edu/ml/

attributes and one decision attribute. The decision attribute is an evaluation made by Colombian Coffee Federation and it defines the price they can get for their production. The values of decision attribute are: Preferential, Strategic and Evaluated. All the 55 attributes are categorical and are related with conditions of houses, farming and technology used in the diary working.

A.2.2 Results

Our approach was able to select in general, five (5) out of sixteen (16) attributes for the zoo data set and seven (7) out of fifty five (55) in the coffee data set, maintaining the quality of the classification equals to one. In some cases it found sets of a smaller number of attributes, but with a quality slightly below from one. These results depend on determining an appropriated value of the β parameter. Table A.1 shows the tuning of parameter β for the coffee dataset.

Table A.1: Tuning parameter β for Coffe dataset.

Values	Attributes	Quality	Fitness
$\beta = 1$	0	0	6
$\beta = 5$	4	0.83	2.1691
$\beta = 10$	5	0.91	1.1253
$\beta = 20$	5	0.95	1.0087
$\beta = 50$	6	0.99	0.7084
$\beta = 80$	7	1.0	0
$\beta = 100$	7	1.0	0
$\beta = 150$	7	1.0	0
$\beta = 1000$	8	1.0	-1

When the constrain, given by quality, is satisfied the factor $\beta|1-\gamma_P(U)|$ is equals to zero and the fitness is equals to the difference between the expected number of attributes and the number of attribute found by the algorithm. As we can see lower values of β influences in a low number of attributes but does not care about quality. By the other hand, higher values of β ensures quality but not the minimal number of attributes. We found appropriate values of β for the goal.

A.3 Attribute Discretization

In order to discretize numerical attributes, we use the idea behind the LEM2 algorithm proposed in [21]. The LEM2 algorithm is based on an attribute-value pair blocks. For an attribute-value pair (a, v), ($a \in \mathcal{A}$ and $v \in V_a$) a block is a set of all cases from U such that for attribute a have value v. For numerical attributes the algorithm computes blocks in a different way than for symbolic attributes. First, it sorts all values of a numerical attribute. Then it computes cut-points as averages for any two consecutive values of the sorted list. For each cut point x creates two blocks, the first block contains all cases for which values of the numerical attribute are smaller than x, the second block contains remaining cases, i.e., all cases for which values of the numerical attribute are larger than x as we can see in Figure A.1.

All the block form a covering $\mathbb{C}$ of U. The induced covering is the set of all blocks computed in this way, together with blocks defined by symbolic attributes. Sets in the induced covering can be ordered and indexed in an unique form. Take the data base shown in Table A.2 as example of the LEM2 algorithm.

Table A.2: Decision system with mixed data.

Objects	**Attributes**			
Patient	A_1	A_2	A_3	Class
1	good	1.0	180	normal
2	good	1.6	240	sick
3	good	1.4	280	normal
4	bad	1.4	240	normal
5	bad	2.0	280	sick
6	bad	1.0	320	sick

The set of candidate cuts for each one of the numerical attributes are shown in Figure A.1. So, the induced covering is (see Equation A.3).

$$\mathbb{C} = \{123, 456, 16, 2345, 1346, 25, 12346, 5, 1, 23456, 124, 356, 12345, 6\} \tag{A.3}$$

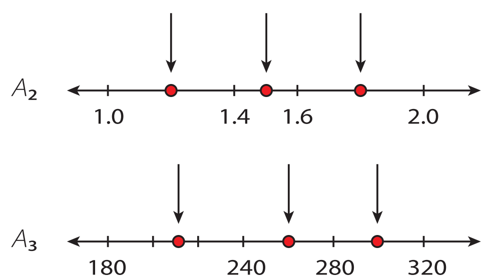

Figure A.1: Possible cuts from dataset in Table 2.

Each set in the induced covering is represented with a bit in a binary string. Here, a bit '1' means the corresponding set is present in the covering. Following our previous example, the 14-bit array represents the covering $\mathbb{C} = \{123, 16, 2345, 1346, 1, 23456, 6\}$

1	0	1	1	1	0	0	0	1	1	0	0	0	1

As can be noticed, some chromosomes cannot represent a covering, so we introduce a constrain in the model: "The solution must represent a covering".

The function must maximize the quality of classification and minimize the number of sets in the covering, therefore it is a multi objective problem. In order to make it simpler, we use the weighting method for converting the multi objective problem in a single objective problem. In this way, the fitness function is defined as a linear combination of three parameters:

$$f(c, s, q) = \alpha c - \beta q - \gamma s + \epsilon$$

where, c is the number of elements of U covered by $\mathbb{C}$, q is the quality of classification, s is the number of sets in $\mathbb{C}$, α, β, γ and ϵ are positive constants. For our example, with the fitness function defined as:

$$f(c,s,q) = 4c - 1.5s - 2q - 17,$$

the optimal value is $f(6,4,0) = 1$ for the array 00001100000011. This bit-array represents the covering $\mathbb{C} = \{1346, 25, 12345, 6\}$ and represents the cuts showed in Figure A.2:

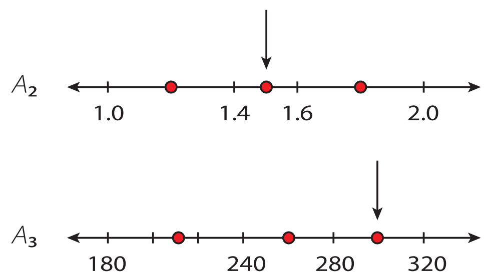

Figure A.2: Cuts selected by the genetic algorithm.

A.4 Attribute Discretization and Selection

A Structured Genetic Algorithm (S-GA) is a variation of the Genetic Algorithm, which categorizes genes into levels, as is shown in Figure A.3. Genes in higher levels are able to control the activation state of genes in lower levels. In this section we use S-GA to solve attribute reduction problem [14]. A chromosome is represented by a set of independent substrings at different levels, with higher level genes controlling neutrality of lower genes. Only those genes currently active in the chromosome contribute to the fitness and genetic operations.

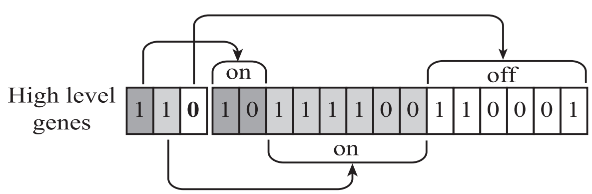

Figure A.3: The structured Genetic algorithm.

Following our example, for the covering in Equation A.3, (representing the possible cuts in Table A.2), we need a 14-bit array. The structured GA, requires three additional bits for the attributes, so each chromosome needs a 17-bit array.

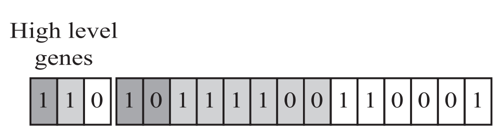

Figure A.4: Codification of a structured Genetic algorithm.

According to solution shown in Figure A.4 it does not include the three last sets in the covering. In this case the covering represented is $\mathbb{C} = \{123, 16, 2345, 1346, 25\}$.

The fitness function is defined as a linear combination of the objectives:

$$f(c,q,s,att) = \alpha c - \beta q - \gamma s - \delta att - \epsilon$$

where, *att* is the number of attributes, c is the number of elements covered by $\mathbb{C}$, q is the quality of classification, s is the number of sets in $\mathbb{C}$, and α, β, γ, δ and ϵ are positive constants. The quality of classification is given by the equation

$$\gamma_P(U) = \frac{\sum_i |\underline{apr}(Y_i)|}{|U|}, \tag{A.4}$$

where $\underline{apr}$ is a lower approximation operator and P is a subset of attributes.

With the fitness function defined in Equation A.5 and the quality of classification, the optimal value was $f(6,2,4,0) = 2$.

$$f(c,s,q,att) = 3c - s - 3q - 2att \pm \epsilon \tag{A.5}$$

Different solution are shown in Figure A.5.

High level genes

0	1	1	0	1	0	0	1	1	0	0	0	0	0	0	1	1
0	1	1	1	0	0	0	1	1	0	0	0	0	0	0	1	1
0	1	1	0	0	0	0	1	1	0	0	0	0	0	0	1	1
0	1	1	1	1	0	0	1	1	0	0	0	0	0	0	1	1

Figure A.5: Solutions for structured Genetic algorithm.

A.5 Selecting approximation operators

Clearly the quality of classification and other metrics used in rough sets, depend on the approximation operator used in classification. Coding a group of pairs of approximation operators in chromosome is possible to use the Structured GA for selecting the most appropriate approximation operator for a specific task. In this case we want to minimize the training error.

Following with our example, for the covering in Equation 3,(representing the possible cuts in Table A.2), needs a 14-bit array, the structured GA, requires three bits for the attributes and three additional bits for a group of eight pairs of approximation operators selected form the framework. So each chromosome needs a 20-bit array.

The list of pairs of approximation operators used in this case is:

1. $(L_2^{\mathbb{C}}, H_5^{\mathbb{C}})$, $(001)_2$
2. $(L_2^{\mathbb{C}}, H_6^{\mathbb{C}})$, $(010)_2$
3. $(L_1^{\mathbb{C}}, H_2^{\mathbb{C}})$, $(011)_2$
4. $(L_1^{\mathbb{C}}, H_3^{\mathbb{C}})$, $(100)_2$
5. $(\underline{apr}_{N_2}^{\mathbb{C}}, \overline{apr}_{N_2}^{\mathbb{C}})$, $(101)_2$
6. $(\underline{apr}_{N_3}^{\mathbb{C}}, \overline{apr}_{N_3}^{\mathbb{C}})$, $(110)_2$

7. $(\underline{apr}^{\mathbb{C}}_{N_4}, \overline{apr}^{\mathbb{C}}_{N_4})$, $(111)_2$

8. $(L_2^{\mathbb{C}}, H_7^{\mathbb{C}})$ (Default case), $(000)_2$

In this case, the fitness function is the same function defined before. A switch controls the approximation operator selected, the default case is $(L_2^{\mathbb{C}}, H_7^{\mathbb{C}})$. The fitness function for the Example dataset was defined as:

$$f(c,s,q,att) = -2att - t - 3q + 3c - 8 \pm \epsilon \tag{A.6}$$

the optimal value can be reached with the approximation operators 0, 1 and 2. Figure A.6 shows some solution obtained by the Structured Evolutionary Algorithm.

Attributes			A1		A2						A3						Appr. Op.		
1	1	0	0	1	0	1	1	1	0	0	1	0	1	0	0	1	0	0	1
1	1	0	0	1	0	1	1	1	0	0	1	0	0	0	1	1	0	1	0
1	1	0	0	1	0	1	1	1	0	0	1	0	1	0	1	0	0	0	0

Figure A.6: Solutions of S-GA with selection.

In this case the pairs of approximation operators selected by S-GA are: $(L_2^{\mathbb{C}}, H_7^{\mathbb{C}})$, $(L_2^{\mathbb{C}}, H_6^{\mathbb{C}})$ and $(L_2^{\mathbb{C}}, H_7^{\mathbb{C}})$. The percentages of selection of approximation operators are the following.

- $(L_2^{\mathbb{C}}, H_7^{\mathbb{C}})$, 33% ($H_7^{\mathbb{C}}$ is self-conjugate).
- $(L_2^{\mathbb{C}}, H_6^{\mathbb{C}})$, 18% (A dual pair).
- $(L_2^{\mathbb{C}}, H_5^{\mathbb{C}})$, 45% (A Galois Connection).

We apply our approach to the Iris data set. The possible cuts for Iris data set determine a covering of 238 sets. Considering the four attributes and the same eight approximation operators the chromosome for the sGA needs a 254-bit array.

The fitness function was defined as:

$$f(c,att,t,q) = 3c - 2att - 4t - 5q.$$

for c the number of elements covered, att, number of attributes selected, t is the number of sets in $\mathbb{C}$ and q number of elements in the boundary. The optimal value was $f(150,2,13,3) = 387$, for $c = 150$, $att = 2$, $t = 13$ and $q = 3$.

The sGA found a reduct of two attributes (petal length and petal width), a covering with 13 sets. The operator selected was $(L_2^{\mathbb{C}}, H_6^{\mathbb{C}})$ and the number of elements in lower approximation for the classes: Setosa (X), Versicolor (Y) and Virginica (Z) were: $|\underline{apr}(X)| = 50$, $|\underline{apr}(Y)| = 49$ and $|\underline{apr}(Z)| = 48$, i.e. the number of elements in the boundary was 3.

A.6 Summary

This appendix presents the implementation of a structured genetic algorithm for attribute reduction in rough set theory like an optimization problem with a constrain given by quality. The algorithm can find a set of cuts for discretizing data and reducts. The last structured GA was built to select the appropriate approximation operator.

The selected attributes importance and the minimal number of attributes in a reduct were confirmed using ROSE2, an implementation of rough sets theory which is appropriate for small data sets. The Structured Genetic Algorithm not only helps to reduce attribute, it also selects approximation operators for minimizing the training error.

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