AN INTERPOLATING CURVE SUBDIVISION SCHEME BASED ON DISCRETE FIRST DERIVATIVE

UN ESQUEMA DE SUBDIVISIÓN INTERPOLANTE BASADO EN LA PRIMERA DERIVADA DISCRETA

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ABSTRACT: This paper develops a new scheme of four points for interpolating curve subdivision based on the discrete first derivative (DFDS), which reduces the apparition of undesirable oscillations that can be formed on the limit curve when the control points do not follow a uniform parameterization. We used a set of 3000 curves whose control points were randomly generated. Smooth curves were obtained after seven steps of subdivision using five schemes DFDS, Four-Point (4P), New four-point (N4P), Tight four-point (T4P) and the geometrically controlled scheme (GC4P). The tortuosity property was evaluated on every smooth curve. An analysis for the frequency distributions of this property using the Kruskal-Wallis test reveals that DFDS scheme has the lowest values in a close range.

KEYWORDS: Curve subdivision, curve interpolation, four-point subdivision scheme.

RESUMEN: En este artículo se desarrolla un nuevo esquema de cuatro puntos para la subdivisión interpolante de curvas basado en la primera derivada discreta (DFDS), el cual, reduce la formación de oscilaciones indeseables que pueden surgir en la curva límite cuando los puntos de control no obedecen a una parametrización uniforme. Se empleó un conjunto de 3000 curvas cuyos puntos de control fueron generados aleatoriamente. Curvas suaves fueron obtenidas tras siete pasos de subdivisión empleando los esquemas DFDS, Cuatro-puntos (4P), Nuevo de cuatro-puntos (N4P), Cuatro-puntos ajustado (T4P) y el Esquema interpolante geométricamente controlado (GC4P). Sobre cada curva suave se evaluó la propiedad de tortuosidad. Un análisis de las distribuciones de frecuencia obtenidas para esta propiedad, empleando la prueba de Kruskal-Wallis, revela que el esquema DFDS posee los menores valores de tortuosidad en un rango más estrecho.

PALABRAS CLAVE: Subdivisión de curvas, interpolación de curvas, esquema de subdivisión de cuatro puntos.

1. INTRODUCTION

Curves are a powerful tool used in engineering to interpolate or approximate a set of points, with the aim to provide pleasant shapes [1]. Splines, NURBS and Subdivision are the most common techniques.

Subdivision has become a popular technique in computer graphics to create smooth curves and surfaces. Currently, it is the industry standard for character animation. It dates back to the work of Chaikin [2], where the corners of a control polygon are iteratively trimmed until a smooth curve is obtained in the limit. This an approximating scheme with C^1 continuity.

Four point schemes for interpolating curve subdivision have two fundamental challenges: first, obtain C^2 continuity on the whole curve, and the second is to prevent the apparition of undesirable oscillations. Several schemes have been proposed to date from Dubuc's work [3], which are characterized by having at most C^1 continuity interpolating the given points. Recent techniques have achieved curves with C^2

continuity, but do not interpolate the original data properly.

This paper focuses on the undesirable oscillations which may appear in the limit curve after an interpolating subdivision process. This problem has received little attention. Recently Hernandez et al. [4] proposed a new interpolatory subdivision scheme based on incenter subdivision which avoids undesirable oscillations.

2. PREVIOUS WORKS

For interpolating subdivisions it is common to use geometrical local schemes, such as four-point or six-point schemes. Earlier works such as Dubuc's [3] and Deslauriers and Dubuc's [5] propose four-point interpolating schemes, which insert a new point by fitting a cubic polynomial to neighboring points over uniformly spaced parameter values. The idea behind this scheme is that if the original control points fall on a defined polynomial, then the next level's control points must also lie on the same polynomial. This four-point scheme is exact for cubic polynomials. Equations 1 and 2 describe the new control points as a function of the old control points.

$$Q_{2i}^{k+1} = Q_i^k \tag{1}$$

$$Q_{2i+1}^{k+1} = -\frac{1}{16}Q_{i-1}^k + \frac{9}{16}Q_i^k + \frac{9}{16}Q_{i+1}^k - \frac{1}{16}Q_{i+2}^k \tag{2}$$

One generalization is presented by Dyn et al. in [6], who uses a four-point based scheme with a tension parameter w ; the limit curve presents C^1 continuity when $0 < w < \frac{1}{8}$. Hechler et al. [7] showed that the w tension parameter generates C^1 limit curves, if and only if, $0 < w < w$, where $w* \approx 0.19273$. One improvement is presented by Nira Dyn et al. in [8], where one new four-point scheme is developed with the tension parameter w , where the limit curve presents C^2 continuity, and the resulting curve is near to be interpolating.

The characteristic of previous works has been that the tension parameter is always constant. Marinov et al. [9] presents a new scheme with a variable tension parameter, which is locally adapted according to the geometry of the control polygon. Beccari et al. [10] proposes a subdivision scheme which provides C^1 continuity with a single tension parameter that can be either arbitrarily increased or appropriately chosen.

In the same way Floater [11] derives an algorithm for expressing the *n - th* order divided differences of the scheme at level $j+1$ as an affine combination of *n - th* order divided differences at level *j*. This algorithm could be a useful tool for analyzing the smoothness of the scheme, especially when the grid points are irregularly spaced.

In Dubuc's work [3], uniform parameter values were assumed. Nira Dyn et al. [12] replaced those values by chordal and centripetal values since parameterization is updated at each refinement level. Related to this, Floater [13] derived an approximation property of four-point interpolating curve subdivision, based on local cubic polynomial fitting.

A step of subdivision can be considered as a sequence of simple, highly local stages. Dodgson et al. [14] proposes to create families of schemes by manipulating the stages of a subdivision step. Applications in the field of finite elements or construction of functions are provided by Zhijie [15] and Floater [16].

There are some works on four-point subdivision for binary and ternary schemes. This is shown in Siddiqui [17-18] and Hassan et al., Hed, Beccari et al. and Ko et al. [19-22].

3. CURVE SUBDIVISION

A Subdivision is an iterative process to get a smooth curve or surface from a coarse mesh. A simple definition is given by Augsdorfer: "*Given a sequence of vertices, subdivision is a process by which, in each refinement step, new vertices are inserted as linear combinations of old vertices. Repeating the process leads eventually to a smooth limit curve*" [14].

A Subdivision can be interpolating or approximating, depending on whether the limit curve passes through the control points. The spirit of subdivision was presented by Chaikin in [2].

3.1. Four-point interpolatory subdivision

It is an interpolating scheme presented by Nira Dyn et al. in [6] which inserts a new vertex $(k+1)-th$ in every subdivision step, given a tension parameter w without deleting old vertexes. The topological rules are simple; for each edge on a control polygon, a new vertex preserving old vertexes is inserted, as shown in Figure 1.

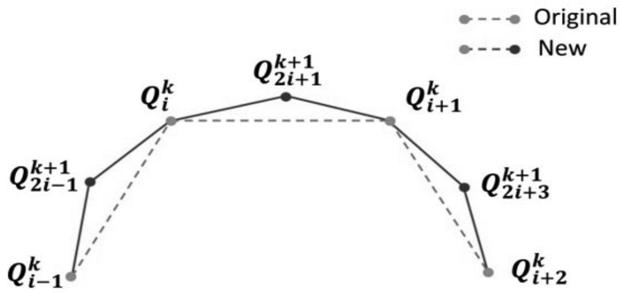


Figure 1. Topological rules for a Four-point subdivision scheme.

Let the control points be specified by $Q_{i=-2}^{n+2}$ where $Q \in \mathbb{R}^d$. The geometrical rules are specified by Equations 3 and 4.

$$Q_{2i}^{k+1} = Q_i^k;\ -1 \le i \le 2^k n + 1 \tag{3}$$

$$Q_{2i+1}^{k+1} = \left(\frac{1}{2} + w\right)\left(Q_i^k + Q_{i+1}^k\right) + w\left(Q_{i-1}^k + Q_{i+2}^k\right); \tag{4}$$

This scheme produces a C^1 continuity curve.

3.2. New four-point subdivision

In search for a subdivision with C^2 continuity, Nira Dyn et al. presents in [8] several schemes close to be interpolating. In this paper, only those subdivision schemes with four-points will be referenced. One of these is the new four-point subdivision scheme, which is based on data interpolation by a cubic polynomial (Eq. 5) evaluated on $t = \frac{1}{4}$ and $t = \frac{3}{4}$.

$$Q(t) = \sum_{j=-1}^{2} L_j(t) Q_j \tag{5}$$

Where $L_j(t) = \prod_{k=-1, k\neq j}^{2} \frac{t-k}{j-k}$

Topological rules express that for every edge, two new vertexes are created, and the old ones do not exist anymore in the next subdivision step. This is shown in Figure 2.

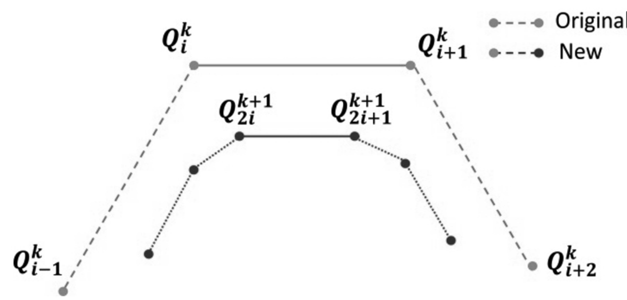


Figure 2. Topological rules for new Four-point subdivision scheme.

The Geometrical rules are expressed by:

$$Q_{2i}^{k+1}\left(\frac{1}{4}\right) = -\frac{7}{128} Q_{-1}^k + \frac{105}{128} Q_0^k + \frac{35}{128} Q_1^k - \frac{5}{128} Q_2^k \tag{6}$$

$$Q_{2i+1}^{k+1}\left(\frac{3}{4}\right) = -\frac{5}{128} Q_{-1}^k + \frac{35}{128} Q_0^k + \frac{105}{128} Q_1^k - \frac{7}{128} Q_2^k \tag{7}$$

3.3 Tight four-point subdivision

Nira Dyn et al. [8] considering the new four-point subdivision as a perturbation of Chaikin's scheme introduces a tension parameter w , producing an extended scheme. For $w \approx 0.013723$ this new scheme is named tight four-point subdivision, which is C^2 continuous. The topological rules are the same of the New Four-Point Subdivision. The Geometrical rules are expressed by Equation 8 and 9.

$$Q_{2i}^{k+1} = -7wQ_{-1}^k + \left(\frac{3}{4} + 9w\right) Q_0^k + \left(\frac{1}{4} + 3w\right) Q_1^k - 5wQ_2^k \tag{8}$$

$$Q_{2i}^{k+1} = -5wQ_{-1}^k + \left(\frac{1}{4} + 3w\right) Q_0^k + \left(\frac{3}{4} + 9w\right) Q_1^k - 7wQ_2^k \tag{9}$$

3.4 Geometrically controlled four point interpolatory scheme

Marinov et al. [9] proposes several four-point subdivision schemes based on the classical four-point scheme using Equation 10 and 11, with variable tension parameters w_i^k defined on Eq. 12, which is adjusted according to the geometry of the control polygon.

$$Q_{2i}^{k+1} = Q_i^k \tag{10}$$

$$Q_{2i+1}^{k+1} = \left(Q_i^k + Q_{i+1}^k\right)\left(w_i^k - \frac{1}{2}\right) - w_i^k\left(Q_{i-1}^k + Q_{i+2}^k\right) \tag{11}$$

Where

$$w_i^k = f\left(g(i, k)\right) \tag{12}$$

Experimentally, Marinov found that the four-point scheme produces visually pleasing curves when the insertion rule includes equidistant edges. The proposed functions are:

$$g(i,k) = \frac{3\left|e_i^k\right|}{\left|e_{i-1}^k\right| + \left|e_i^k\right| + \left|e_{i+1}^k\right|} \tag{13}$$

$$f(x) = \begin{cases} W_x & 0 \le x \le 1 \\ \dfrac{w(3-x)}{2} & 0 < x \le 3 \end{cases} \tag{14}$$

Another variant is:

$$g(i,k) = \frac{3\left|e_i^k\right|}{\left|e_{i-1}^k\right| + \left|e_i^k\right| + \left|e_{i+1}^k\right|} \tag{15}$$

$$f(x) = \begin{cases} W_x & 0 \le x \le 1 \\ W & 0 < x \le 3 \end{cases} \tag{16}$$

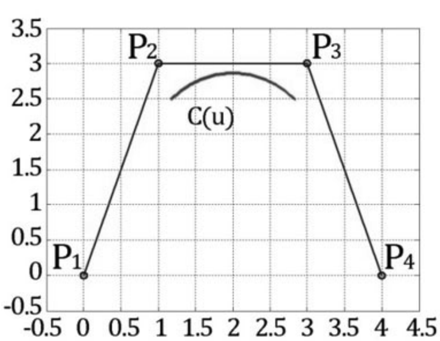
Where $g(i,k) = 0$ if $\left|e_{i-1}^k\right| + \left|e_i^k\right| + \left|e_{i+1}^k\right| = 0$

4. DFDS SCHEME

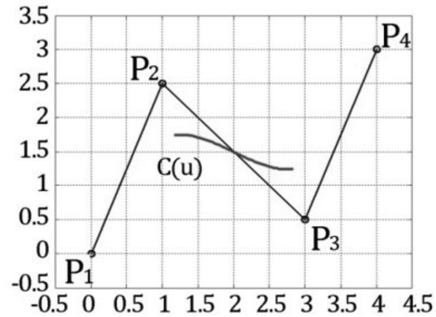
A cubic Spline $C(u)$ is a parametric curve from a control point set $\{P_i\}$ with C^2 continuity, described by Eq. 17. A good description of how this formula is obtained is shown in [23]. Figure 3 shows a typical Spline.

$$c(u) = \left(\frac{1}{6}\right)\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \tag{17}$$

for $u \in [0,1]$



a. First example for a cubic spline



b. Second example for a cubic spline

Figure 3. Cubic Spline $C(u)$ for control points.

This Spline definition is approximate. For an interpolating definition it is necessary to redefine the Eq. 17, in terms of a point set $\{Q_i\}$ where $Q_i \in C(u)$. One selection for u_i is shown in Figure 4.

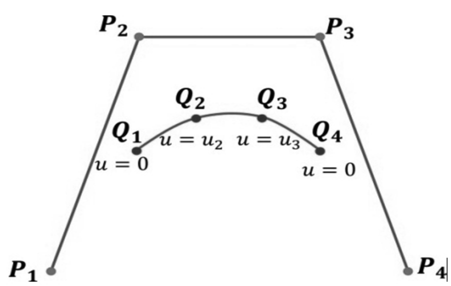


Figure 4. Set of Q_i points over curve $C(u)$.

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \left(\frac{1}{6}\right) \begin{bmatrix} 0 & 0 & 0 & 1 \\ u_2^3 & u_2^2 & u_2 & 1 \\ u_3^3 & u_3^2 & u_3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \tag{18}$$

Equation 17 can be expressed in a simple way in terms of Eq. 18 as:

$$C(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} N \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} \tag{19}$$

Where N is:

$$\begin{bmatrix} \dfrac{1}{u_2 u_3} & \dfrac{1}{u_2(u_2-u_3)(u_2-1)} & \dfrac{1}{u_3(u_2-u_3)(u_3-1)} & \dfrac{1}{(u_2-1)(u_3-1)} \\ \dfrac{u_2+u_3+1}{u_2 u_3} & \dfrac{u_3+1}{u_2(u_2-u_3)(u_2-1)} & \dfrac{u_2+1}{u_3(u_2-u_3)(u_3-1)} & \dfrac{u_2+u_3}{(u_2-1)(u_3-1)} \\ \dfrac{1}{u_2}-\dfrac{1}{u_3}-1 & \dfrac{u_3}{u_2(u_2-u_3)(u_2-1)} & \dfrac{u_2}{u_3(u_2-u_3)(u_3-1)} & \dfrac{u_2 u_3}{(u_2-1)(u_3-1)} \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

It has infinite solutions for $0 < u_2 < u_3 < 1$.

In order to developing our method, a smooth interpolating curve $C(u)$ is assumed so that:

$$\left.\frac{\|\Delta Q\|}{\Delta u}\right|_{Q_2}^{-} \approx \left.\frac{\|\Delta Q\|}{\Delta u}\right|_{Q_2}^{+} \rightarrow \frac{\|Q_{12}\|}{u_2 - 0} = \frac{\|Q_{23}\|}{u_3 - u_2} \tag{20}$$

$$\left.\frac{\|\Delta Q\|}{\Delta u}\right|_{Q_3}^{-} \approx \left.\frac{\|\Delta Q\|}{\Delta u}\right|_{Q_3}^{+} \rightarrow \frac{\|Q_{23}\|}{u_3 - u_2} = \frac{\|Q_{34}\|}{1 - u_3} \tag{21}$$

Solving Equations 20 and 21 simultaneously:

$$u_2 = \frac{\|Q_{12}\|}{\|Q_{12}\| + \|Q_{23}\| + \|Q_{34}\|} \tag{22}$$

$$u_3 = \frac{\|Q_{12}\| + \|Q_{23}\|}{\|Q_{12}\| + \|Q_{23}\| + \|Q_{34}\|} \tag{23}$$

The and values are obtained. The new point is shown in Figure 5.

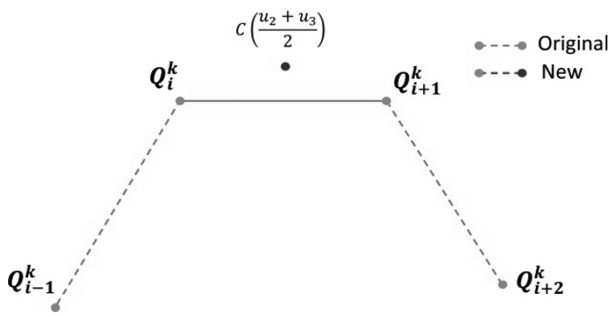


Figure 5. Topological rules for proposed scheme.

5. TORTUOSITY

Tortuosity is an intrinsic curve property of having twists and turns.

A formal definition of tortuosity was not found in the literature reviewed, but several criteria have been applied to measure it, such as presented by Kalitzeos et al. [24], which are resumed in Table 1.

The main applications of this property are found in porous media and ophthalmology, as can be viewed in Bullit et al. and Dougherty et al. [24-26]. In the attempt to measure this property, several metrics have been developed. The Table 1 describes the most important.

The most recent tortuosity indexes are based on the curvature, which has shown to be more robust in the calculation of this property. This work presents a discrete form considering the existence of a distance Δt uniformly spaced between data points of the polygon, which is consistent with the concept of subdivision. Equation 24 presents the continuous version of the curvature definition.

Table 1. Definitions of tortuosity index.

Index	Definition
DF: Distance factor	$\frac{\text{Arc length}}{\text{Chord length}}$
ICM: Inflection count points	$(\text{Number of inflection points} + 1) * \text{DF}$
SOAM: Sum of angles metric	$\frac{\sum\sqrt{\text{Tangencial angle}^2 + \text{Torsional}^2}}{\text{Chord length}}$
τ_1	$\frac{\text{Arc length}}{\text{Chord length}} - 1$
τ_2	Total curvature
τ_3	Total square curvature
τ_4	$\frac{\text{Total curvature}}{\text{Arc length}}$
τ_5	$\frac{\text{Total square curvature}}{\text{Arc length}}$
τ_6	$\frac{\text{Total curvature}}{\text{Chord length}}$
τ_7	$\frac{\text{Total square curvature}}{\text{Chord length}}$

$$\mathcal{X}(t) = \frac{\|\mathrm{r}'(t) \times \mathrm{r}''(t)\|}{\|\mathrm{r}'(t)\|^3} \tag{24}$$

For the discrete case of curvature, Figure 6 shows a neighboring scheme of the point to be evaluated.

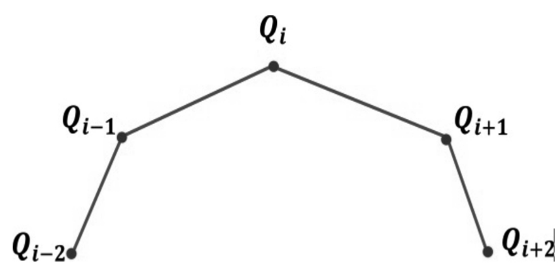


Figure 6. Five-point polygonal curve.

Equations 25, 26 and 27 present an approximation to the first derivative at points Q_i , Q_{i-1} and Q_{i+1} respectively.

$$Q_i' = \frac{\frac{Q_i - Q_{i-1}}{\Delta t} + \frac{Q_{i+1} - Q_i}{\Delta t}}{2} = \frac{Q_{i+1} - Q_{i-1}}{2\Delta t} \tag{25}$$

$$Q_i' = \frac{Q_i - Q_{i-2}}{2\Delta t} \tag{26}$$

$$Q_i' = \frac{Q_{i+2} - Q_i}{2\Delta t} \tag{27}$$

Using these results and the same concept, it is possible to calculate the value of Q_i'' , which is expressed in Eq. 28.

$$Q_i'' = \frac{Q_{i-2} - 2Q_i + Q_{i+2}}{4\Delta t} \tag{28}$$

Taking these results into Equation 24 and simplifying leads to the expression presented in Equation 30.

$$\mathcal{X}(t) = \frac{\left\| \frac{(Q_{i+1} - Q_{i-1})}{2\Delta t} \times \frac{(Q_{i-2} - 2Q_i + Q_{i+2})}{4\Delta t} \right\|}{\left\| \frac{Q_{i+1} - Q_{i-1}}{2\Delta t} \right\|^3} \tag{29}$$

$$\mathcal{X}(t) = \frac{\|(Q_{i+1} - Q_{i-1}) \times (Q_{i-2} - 2Q_i + Q_{i+2})\|}{\|Q_{i+1} - Q_{i-1}\|^3} \tag{30}$$

6. RESULTS

In this section, we compare the proposed subdivision scheme (DFDS) with the classical Four-point subdivision scheme (4P), New four-point subdivision scheme (N4P), the Tight four-point subdivision scheme (T4P) and the Geometrically controlled subdivision scheme (GC4P), which were described in section 2. Figure 7a, shows the control points for the first example. Figures 7-b to 7.f show the resulting curves after seven subdivision steps.

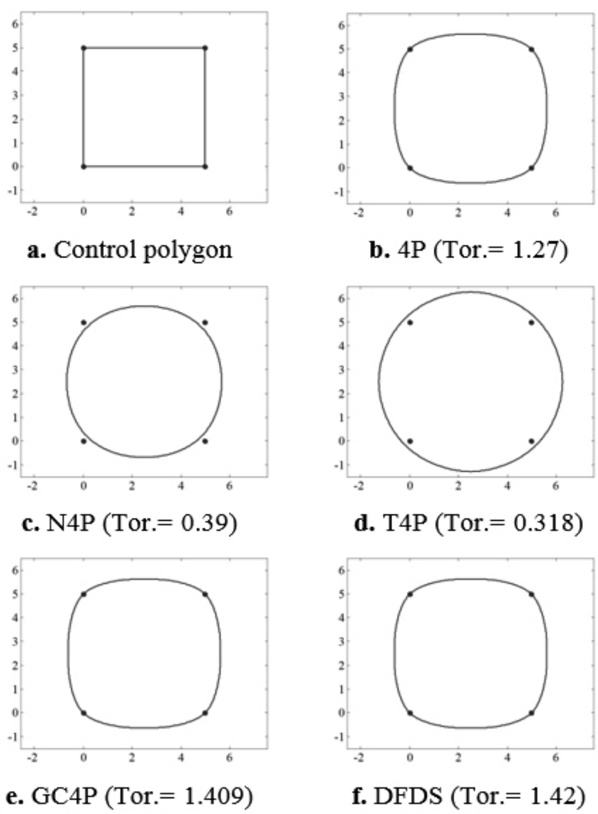


Figure 7. Subdivision applied to uniformly spaced control points.

Figure 8a shows the control points for the second example. Figures 8-b to 8.f show the resulting curves after seven subdivision steps.

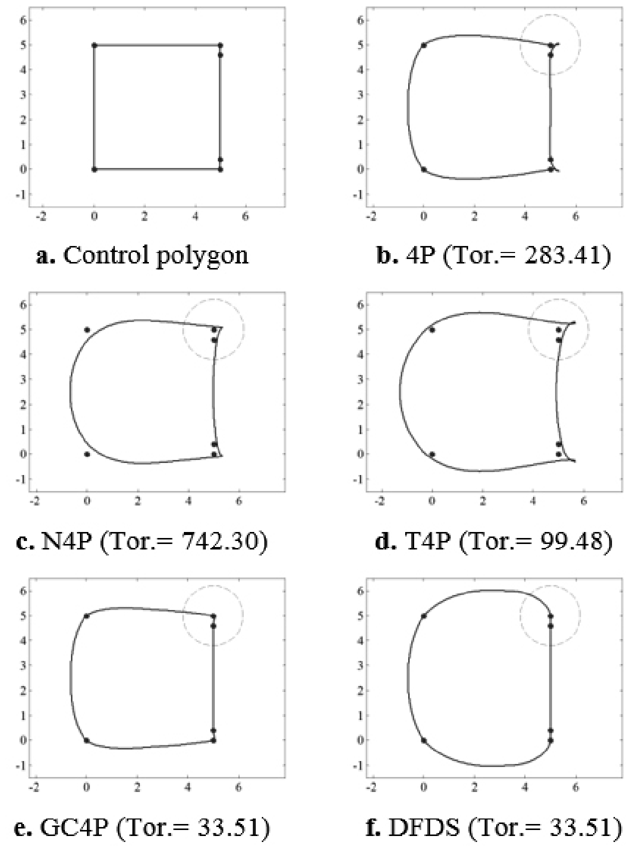


Figure 8. Subdivision applied to non-uniformly spaced control points.

It can be seen that in the presence of non-homogeneously spaced control points unwanted oscillations can be present, as shown by Figures 8b, 8c and 8d.

A set of 3000 curves was randomly generated, each one had nine control points. The curves were interpolated one by one with each one of the interpolation methods. In all cases there were seven subdivision steps. The discrete curvature was evaluated at all points of each curve, taking the measure of tortuosity as its maximum value.

Figures 9 to 13 show a histogram for each evaluated subdivision method, the median is plotted over each histogram. A normality analysis reveals that none of the distributions are normal, the analysis was performed using a non-parametrical test, Kruskal-Wallis.

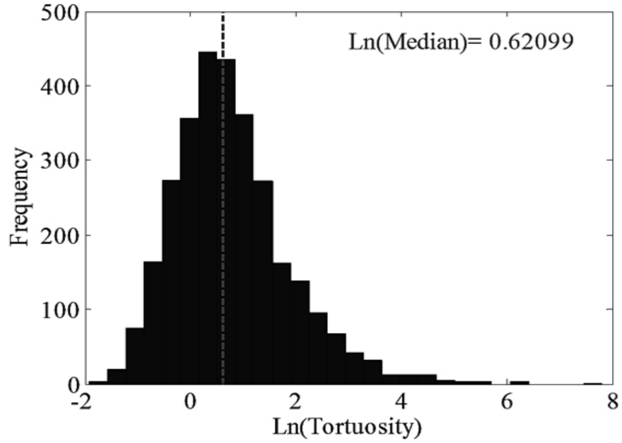


Figure 9. Histogram for the maximum curvature of the four-point scheme.

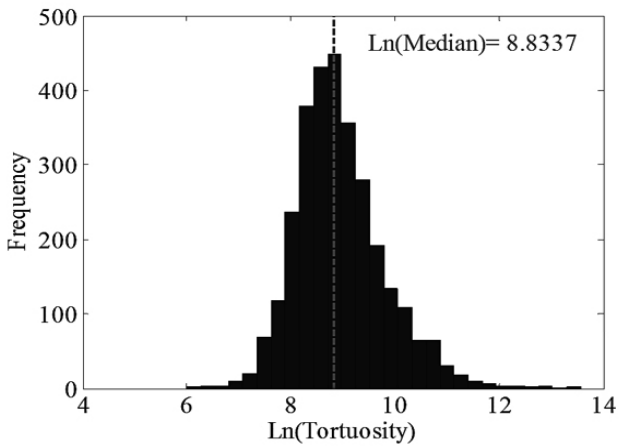


Figure 10. Histogram for maximum curvature of the new scheme of four points.

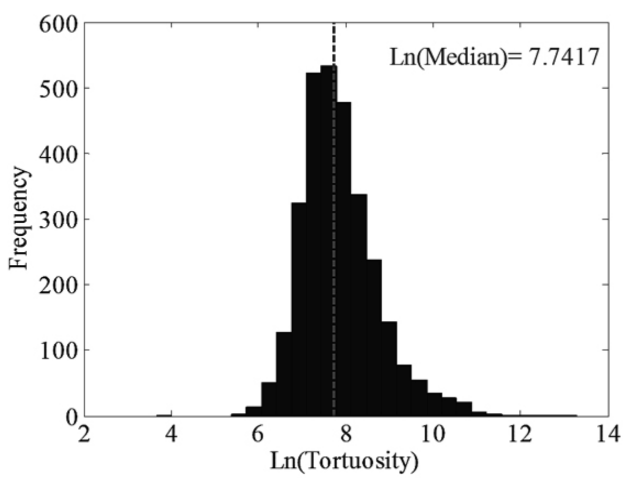


Figure 11. Histogram for the maximum curvature for the tight four point scheme

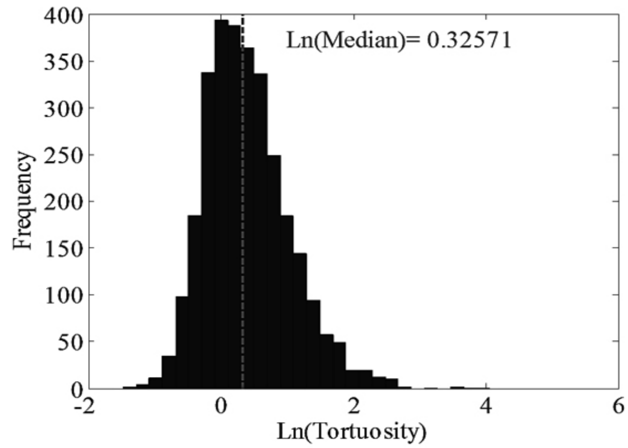


Figure 12. Histogram for the maximum curvature of GC4P.

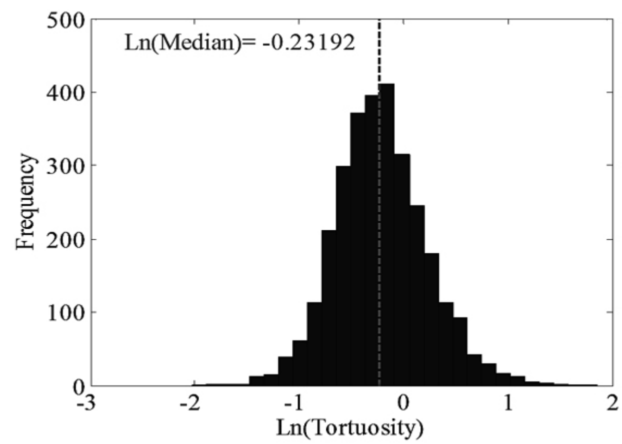


Figure 13. Histogram for the maximum curvature of DFDS.

A comparative analysis using a box and whisker plot between these distributions is shown in Figure 14, which reveals that DFDS has a lower tortuosity median that the other subdivision methods. Values of medians are shown in Table 2, where DFDS has a lower value.

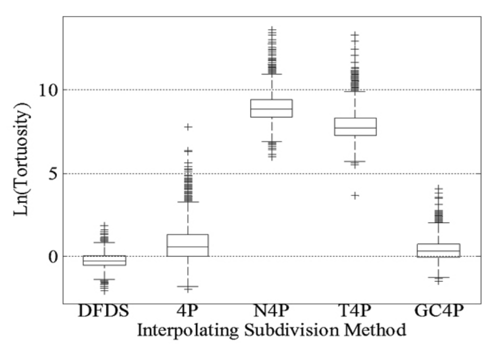


Figure 14. Box and whisker plot for subdivision schemes.

A Kruskal-Wallis analysis shows that the tortuosity data using DFDS, 4-P, New 4-P, Tight 4-P and GC Subdivision come from populations with different medians. Since the P value of the F test is less than 0.05, we can conclude that the new scheme has a lower tortuosity median than the others methods, as shown in Table 3.

Table 2. Maximum curvature median

Subdivision Method	Max. Curv. Median
Four Point	1.8608
New Four Point	6.8617e+03
Tight Four point	2.3025e+03
Marinov	1.3850
DFDS	0.7930

Table 3. Kruskal-Wallis ANOVA Table for maximum curvature over each interpolation method

Source	SS 1e+11	df	MS	χ^2 1e+4	Prob $> \chi^2$
Groups	2.230	4	5.58e+10	1.19	0
Error	0.576	14995	3.84e+6		
Total	2.810	14999			

6. CONCLUSION AND FUTURE WORK

This work presents DFDS, a new four-point based interpolating curve subdivision scheme. The most important advantage is the low tendency to produce artifacts, as evidenced by the ANOVA analysis, where it is compared with the other methods. Another advantage is the absence of weights or tension parameters for interpolation. The shape of the limit curve is only a function of the neighboring geometry. It is expected to have C^1 continuity because it is based on the first derivative.

Future work is directed to extend the new scheme to surfaces, reverse curve subdivision and reverse surface subdivision.

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