DECAY OF SOLUTIONS OF DISPERSIVE EQUATIONS AND POISSON BRACKETS IN ALGEBRAIC GEOMETRY

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Abstract

In the first part of this work we will study the spatial decay of solutions of nonlinear dispersive equations. The starting point will be the *Korteweg-de Vries* (KdV) equation, for which it will be proved that a decay of exponential type is degraded in time, and that the exhibited decay is optimal. More precisely, we will prove the following two theorems:

Theorem I. *For $u_0 \in L^2(\mathbb{R})$ and $T > 0$, let $u \in C\left([0,T]; L^2(\mathbb{R})\right)$ be the solution of the KdV equation with $u(0) = u_0$. Let us suppose that for $a_0 > 0$, $e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R})$. Then*

$$\left\| e^{a(t)\, x_+^{3/2}} u(t) \right\|_{L^2(\mathbb{R})} \leq C \left\| e^{a_0\, x_+^{3/2}} u_0 \right\|_{L^2(\mathbb{R})}, \qquad \text{for every } t \in [0,T],$$

where

$$a(t) = \frac{a_0}{\sqrt{1 + \frac{27}{4} a_0^2 t}}, \qquad t \in [0,T],$$

and $\quad C = C\left(a_0, T, \|u_0\|_{L^2(\mathbb{R})}, \|e^x u_0\|_{L^2(\mathbb{R})}\right).$

Theorem II. *For $T > 0$, $a_0 > 0$ and $0 < \epsilon < \frac{1}{3} a_0$, there exist $u_0 \in \mathcal{S}(\mathbb{R})$ with $e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R})$ and $C > 0$ such that the solution u on $[0,T]$ of the KdV equation with initial datum u_0 satisfies*

$$C\, e^{-g(t)(a_0+\epsilon)\, x^{3/2}} \leq u(t)(x), \qquad \text{for every } t \in [0,T] \text{ and every } x > 0.$$

Here, $g(t)(b) := \frac{b}{\sqrt{1+\frac{27}{4}b^2 t}}$.

In particular, $e^{g(t)(a_0+\epsilon)\, x_+^{3/2}}\, u(t) \notin L^2(\mathbb{R})$, for every $t \in [0,T]$,

In the second part we will make an exposition on Symplectic and Poisson Geometry with connections in Classical Mechanics to motivate a more abstract view of Poisson structures. With these preliminaries we can then give way to a little digression on Integrable Systems, and discuss the notion of complete integratbility in the sense of Liouville.

keywords

KdV equation, evolution dispersive equations, decay properties, Poisson structures, Liouville integrable systems.

Contents

Part I

DECAY OF SOLUTIONS OF DISPERSIVE EQUATIONS

Chapter 1

Introduction

In this thesis we consider the initial value problem (IVP) associated to the *Korteweg-de Vries* (KdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad u = u(x,t), \quad x,t \in \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

and study a decay property of exponential type of its solutions $u(x,t)$ in the positive semiaxis x.

The KdV equation, introduced by Korteweg and de Vries in [KdV], describes the propagation of one-dimensional longwaves of small amplitude in a shallow medium, and has been studied in many aspects, of which we mention the local and global well-posedness, the persistence of solutions that decay exponentially on the right and the unique continuation of solutions. Results on the local and global well-posedness for the IVP (1.1) in the context of Sobolev spaces $H^s(\mathbb{R})$ have been obtained and successively improved in a series of papers of which we cite among others the works of Saut and Temam [ST], Bona and Smith [BS], Bona and Scott [BSc], Kato [K], Kenig, Ponce and Vega [KPV1], [KPV2], Bourgain [B], Colliander, Keel, Staffilani, Takaoka and Tao [CKSTT], Christ, Colliander and Tao [CCT], Guo [G], and Kishimoto [Ki]. In the last two papers, using variants of the methods introduced in [B] and [CKSTT], it was proved that the IVP (1.1) is locally and globally well-posed for initial data in $H^s(\mathbb{R})$ with $s \geq -3/4$. On the other hand, in [KPV3], Kenig, Ponce and Vega showed that for $s < -3/4$ the data solution map is not uniformly continuous, as a map from $H^s(\mathbb{R})$ into the space of continuous functions from $[0,T]$ into $H^s(\mathbb{R})$, $C([0,T];H^s(\mathbb{R}))$.

With regard to exponential decay of solutions, in [K], Kato proved that if $u_0 \in H^2(\mathbb{R})$ is such that $e^{\beta x}u_0 \in L^2(\mathbb{R})$, $\beta > 0$, then, for the global solution u of the IVP (1.1) with initial datum u_0, the following estimates hold:

$$\begin{aligned} &\left\|e^{\beta x}u(t)\right\|_{L^2(\mathbb{R})} \le e^{Kt}\,\left\|e^{\beta x}u_0\right\|_{L^2(\mathbb{R})}, \quad \text{for every } t\ge 0, \quad \text{and}\\ &\int_0^\infty e^{-Kt}\left\|e^{\beta x}\partial_x u(t)\right\|^2_{L^2(\mathbb{R})}\,dt \le \frac{1}{4\beta}\,\left\|e^{\beta x}u_0\right\|^2_{L^2(\mathbb{R})}, \end{aligned} \tag{1.2}$$

where $K = K\left(\beta, \|u_0\|_{L^2(\mathbb{R})}\right)$.

It was also proved in [K] that if u_0 decays polynomially in such a way that $u_0 \in Z_{2n,n} \equiv H^{2n}(\mathbb{R})\cap L^2(|x|^{2n}\,dx)$, $n\in\mathbb{N}$, then, the corresponding solution u on $[0,T]$ of the IVP (1.1) is such that $u\in C\left([0,T];Z_{2n,n}\right)$. From this, it is also concluded in [K] that if, u_0 is in the Schwartz space $\mathcal{S}(\mathbb{R})$, then the solution u of the IVP (1.1) is such that

$$u\in C([0,T];\mathcal{S}(\mathbb{R})). \tag{1.3}$$

This is also true for the linear problem associated to the equation:

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, \quad u = u(x,t), \quad x,t\in\mathbb{R},\\ u(0) = u_0. \end{cases} \tag{1.4}$$

The properties of decay preservation are closely related to the aspect of unique continuation. That is, the determination of local character conditions (on the space of variables x,t) that force the solution of the problem (or the difference of two solutions) to be null.

In [EKPV], Escauriaza, Kenig, Ponce and Vega showed that there exists a constant $a_0>0$ such that if $a>a_0$ and if a solution u of the IVP (1.1) satisfies

$$e^{a\,x_+^{3/2}}u(0)\in L^2(\mathbb{R}) \quad \text{and} \quad e^{a\,x_+^{3/2}}u(1)\in L^2(\mathbb{R}),$$

then $u\equiv 0$. The exponent of order $x^{3/2}$ is related to the decay of the fundamental solution of the IVP (1.1), as we will see further.

The question arises about if, for an initial datum u_0 with $e^{a_0 x_+^{3/2}}u_0\in L^2(\mathbb{R})$, the solution of the IVP (1.1) keeps some decay with exponent of order $x_+^{3/2}$ as time evolves. An affirmative answer to this question was given in [ILP], where it was proved, using weighted energy estimates, that if $e^{a_0 x_+^{3/2}}u_0\in L^2(\mathbb{R})$, then the solution $u(t)$ defined on an interval $[0,T]$ is such that

$$\left\|e^{a(t)\,x_+^{3/2}}u(t)\right\|_{L^2(\mathbb{R})}\le C, \tag{1.5}$$

where $C = C\left(a_0, T, \|u_0\|_{L^2(\mathbb{R})}, \|e^x u_0\|_{L^2(\mathbb{R})}\right)$, and

$$a(t) = \frac{a_0}{\sqrt{1+27\,a_0^2 t}}, \qquad t\in[0,T].$$

Our purpose is to obtain an optimal function $a(t)$, with $a(0) = a_0$ for which (1.5) holds if $e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R})$. In order to intuit what such a function can be, we analyze the behavior of the fundamental solution $S_t(x)$ of the linear problem associated to the IVP (1.1). That is, the solution of the problem

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, \quad x, t \in \mathbb{R}, \\ u(0) = \delta, \end{cases} \tag{1.6}$$

where δ is the Dirac delta function, which is described through the Fourier transform by $\widehat{\delta}(\xi) \equiv 1$. The fundamental solution $(x,t) \mapsto S_t(x)$ is given by the improper integral

$$S_t(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\xi^3}\, e^{ix\xi}\, d\xi. \tag{1.7}$$

By the change of variable $\xi' = \sqrt[3]{3t}\,\xi$ it can be seen that

$$S_t(x) = \frac{1}{\sqrt[3]{3t}}\, A\left(\frac{x}{\sqrt[3]{3t}}\right), \tag{1.8}$$

where A is the Airy function, defined by

$$A(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi^3/3}\, e^{ix\xi}\, d\xi.$$

The Airy function presents the following asymptotic behavior (see [H]):

$$A(x) = \begin{cases} c_0\, x^{-1/4}\, e^{-\frac{2}{3}\, x^{3/2}} + O\left(x^{-7/4}\, e^{-\frac{2}{3}\, x^{3/2}}\right), & \text{if} \quad x > 0 \\ c_1\, r^{-1/4}\, \sin\left(\frac{2}{3} r^{3/2} + \frac{\pi}{4}\right) + O\left(r^{-3/2}\right), & \text{if} \quad r = -x > 0. \end{cases} \tag{1.9}$$

Furthermore, A is a bounded function.

From (1.8) and (1.9), it follows that for $x > 0$,

$$\begin{aligned} S_t(x) &\sim \left(\frac{x}{\sqrt[3]{3t}}\right)^{-1/4} \frac{1}{\sqrt[3]{3t}}\, e^{-\frac{2}{3}\left(\frac{x}{\sqrt[3]{3t}}\right)^{3/2}} \\ &\sim x^{-1/4}\, t^{-1/4}\, e^{-\frac{2}{3\sqrt{3}}\, \frac{x^{3/2}}{\sqrt{t}}}. \end{aligned}$$

Let us notice that for $a_0 > 0$, the exponent in the above expression is $a_0 x^{3/2}$ at the instant $t_0 = \frac{27}{4a_0^2}$. In this way, if we take t_0 as the initial instant, and we measure the time t from that instant on, then the fundamental solution at the instant t will be

$$u(t)(x) = S_{t_0+t}(x) \sim x^{-1/4}\, (t_0 + t)^{-1/4}\, e^{-\frac{a_0}{\sqrt{1+\frac{27}{4}a_0^2 t}}\, x^{3/2}}, \qquad x > 0,\ t > -t_0. \tag{1.10}$$

In this thesis we will prove that the function

$$a(t) = \frac{a_0}{\sqrt{1+\frac{27}{4}\,a_0^2 t}}$$

produces the optimal decay of exponential order 3/2 to the right of the x-axis, as t evolves, when the initial datum satisfies $e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R})$.

In order to formulate in a precise way our theorems we refer to the local existence result obtained in [B] (see also [KPV2]) in the context of the spaces $X_{s,b}$, which, for $s, b \in \mathbb{R}$, are defined by

$$X_{s,b} := \left\{ f \in \mathcal{S}'_F(\mathbb{R}^2) \;\Big|\; \|f\|^2_{X_{s,b}} := \iint (1+|\tau-\xi^3|)^{2b}(1+|\xi|)^{2s}|\widehat{f}(\xi,\tau)|^2\, d\xi\, d\tau < \infty \right\},$$

where $\mathcal{S}'_F(\mathbb{R}^2)$ is the space of tempered distributions in $\mathbb{R}^2$ whose Fourier transform $\widehat{f}$ can be represented through a function of the variables ξ, τ.
For $b > 1/2$, in virtue of the Sobolev embedding theorem, we have that $X_{s,b}$ is continuously embedded in $C(\mathbb{R}; H^s(\mathbb{R}))$ and thus, for $T > 0$, it makes sense to define the space of restrictions $f|_{[0,T]}$ of the elements f in $X_{s,b}$ to the interval $[0, T]$:

$$X_{s,b}([0,T]) := \left\{ f|_{[0,T]} \;\big|\; f \in X_{s,b} \right\},$$

which is provided with the norm

$$\|u\|_{X_{s,b}([0,T])} := \inf \left\{ \|f\|_{X_{s,b}} \;\big|\; f\,|_{[0,T]} = u \right\}.$$

It was proved in [B] that for $s \geq 0$ (later in [KPV2] for $s > -3/4$) and $u_0 \in H^s(\mathbb{R})$, there exist $T_0 = T_0(\|u_0\|_{H^s(\mathbb{R})})$ and $b > 1/2$ such that the IVP (1.1) has a unique solution

$$u \in X_{s,b}([0,T_0]) \hookrightarrow C([0,T_0]; H^s(\mathbb{R})), \tag{1.11}$$

with $u(0) = u_0$. Furthermore, the map $u_0 \mapsto u$ is continuous from $H^s(\mathbb{R})$ into $X_{s,b}([0,T_0])$ and, for the case $s = 0$ the solution can be extended to any interval $[0, T]$. This solution satisfies an integral equation associated to the Duhamel's formula for the KdV equation and, in general, it does not satisfy problem (1.1) in a pointwise way (that is, for each value of $t \in [0, T]$) because the low regularity terms in the equation could be meaningless. However, for s large enough the solution in (1.11) satisfies the differential equation in (1.1) allowing to carry out a priori estimates.

We will also denote $x_+ := \dfrac{1}{2}(|x| + x)$, for $x \in \mathbb{R}$.

We state now our main results.

Theorem I. *For $u_0 \in L^2(\mathbb{R})$ and $T > 0$, let $u \in C\left([0,T]; L^2(\mathbb{R})\right)$ be the solution of the IVP (1.1) with $u(0) = u_0$ described in the previous paragraph. Let us suppose that for $a_0 > 0$,*

$$e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R}).$$

Then

$$\left\| e^{a(t)\, x_+^{3/2}} u(t) \right\|_{L^2(\mathbb{R})} \leq C \left\| e^{a_0\, x_+^{3/2}} u_0 \right\|_{L^2(\mathbb{R})}, \qquad \text{for every } t \in [0,T], \tag{1.12}$$

where

$$a(t) = \frac{a_0}{\sqrt{1 + \frac{27}{4} a_0^2 t}}, \qquad t \in [0,T], \tag{1.13}$$

and $\quad C = C\left(a_0, T, \|u_0\|_{L^2(\mathbb{R})}, \|e^x u_0\|_{L^2(\mathbb{R})}\right)$.

To simplify our notation, for $t \geq 0$ and $b \geq 0$, let us define

$$g(t)(b) := \frac{b}{\sqrt{1 + \frac{27}{4} b^2 t}}. \tag{1.14}$$

Our second result establishes that the function $a(t)$ obtained in Theorem I is optimal. This means that we can not expect a stronger decay than the one given in (1.12). More precisely, we prove the following result:

Theorem II. *For $T > 0$, $a_0 > 0$ and $0 < \epsilon < \frac{1}{3} a_0$, there exist $u_0 \in \mathcal{S}(\mathbb{R})$ with $e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R})$ and $C > 0$ such that the solution u on $[0,T]$ of the IVP (1.1) with initial datum u_0 satisfies*

$$C\, e^{-g(t)(a_0+\epsilon)\, x^{3/2}} \leq u(t)(x), \qquad \text{for every } t \in [0,T] \text{ and every } x > 0.$$

In particular, $e^{g(t)(a_0+\epsilon)\, x_+^{3/2}} u(t) \notin L^2(\mathbb{R})$, for every $t \in [0,T]$.

The proof of Theorem I will be performed in Chapter 2 and is based on an energy estimate which allows to minimize the losses from terms that are discarded from the estimates.

In Chapter 3 we prove Theorem II. For that we will apply Theorem I, the properties of the Airy function and Duhamel's formula.

Chapter 2

Decay of solutions: Proof of Theorem I

We begin by proving an interpolation lemma which will be used to justify integration processes in the proof of Theorem I.

Lemma 2.1. *Let* $f \in H^{\infty}(\mathbb{R})$ *and* $j \in \mathbb{Z}^{+}$. *If* $e^{\beta x} f \in L^2(\mathbb{R})$ *for every* $\beta > 0$, *then* $e^{\beta x}\partial_x^j f \in L^2(\mathbb{R})$ *for every* $\beta > 0$.

Proof.
Let us analyze the case in which $j = 1$:

We want to estimate the $L^2(\mathbb{R})$ norm of $e^{\beta x}\partial_x f$. For this, let us consider a truncation function $\eta \in C^{\infty}(\mathbb{R})$ such that $\eta(x) = 1$ if $x \leq 1$, and $\eta(x) = 0$ if $x \geq 2$. For example, one like that whose graph is shown below:

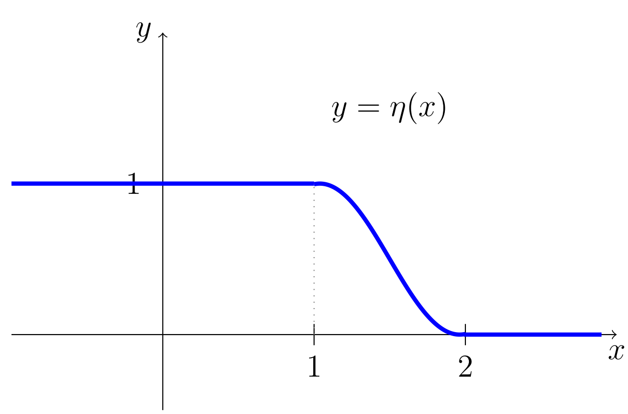

Figure 1

Now, for $n \in \mathbb{Z}^+$ let us define $\eta_n(x) := \eta\left(\frac{x}{n}\right)$ and $\theta_n(x) := \beta \int_0^x \eta_n(s)\, ds,\ \ x \in \mathbb{R}$. It can be seen that, for every $x \in \mathbb{R}$,

$$\theta_n(x) \to \beta\, x \quad \text{and} \quad \theta_n'(x) \to \beta \qquad \text{when } n \to \infty, \tag{2.1}$$

$$\theta_n(x) \leq 2\beta x \quad \text{and} \quad \theta_n(x) \leq 2\beta n, \quad \text{for all } x \in \mathbb{R}, \tag{2.2}$$

$$|\theta_n'(x)| \leq \beta \quad \text{and} \quad |\theta_n''(x)| \leq \frac{C\beta}{n}, \quad \text{for } x \in \mathbb{R}. \tag{2.3}$$

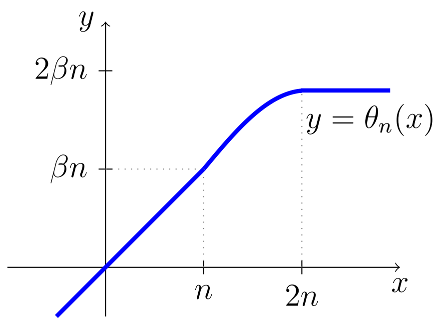

Figure 2

Let us notice that, from (2.2) and from the fact that $f \in H^1(\mathbb{R})$, it follows that $e^{\frac{1}{2}\theta_n}\partial_x f \in L^2(\mathbb{R})$, for every $n \in \mathbb{Z}^+$. In this way, integrating by parts and applying Cauchy-Schwarz inequality we obtain that

$$\begin{aligned}
\int_{\mathbb{R}} e^{\theta_n(x)} \left(\partial_x f\right)^2\, dx &= -\int_{\mathbb{R}} \partial_x\left(e^{\theta_n(x)}\partial_x f\right) f\, dx \\
&= -\frac{1}{2}\int_{\mathbb{R}} \theta_n'(x) e^{\theta_n(x)} \partial_x(f^2)\, dx - \int_{\mathbb{R}} e^{\theta_n(x)} f\, \partial_x^2 f\, dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[\theta_n''(x) + \theta_n'(x)^2\right] e^{\theta_n(x)} f^2\, dx - \int_{\mathbb{R}} e^{\theta_n(x)} f\, \partial_x^2 f\, dx \\
&\leq \frac{1}{2}\int_{\mathbb{R}} \left[\theta_n''(x) + \theta_n'(x)^2\right] e^{\theta_n(x)} f^2\, dx + \left\|\partial_x^2 f\right\|_{L^2(\mathbb{R})} \left[\int_{\mathbb{R}} \left(e^{\theta_n(x)} f\right)^2\, dx\right]^{1/2} \\
&\leq \frac{1}{2}\left(\frac{C\beta}{n} + \beta^2\right)\int_{\mathbb{R}} e^{\beta x} f^2\, dx + \left\|\partial_x^2 f\right\|_{L^2(\mathbb{R})} \left[\int_{\mathbb{R}} e^{2\beta x} f^2\, dx\right]^{1/2}.
\end{aligned} \tag{2.4}$$

From (2.1) and (2.3) we have that for every $x \in \mathbb{R}$,

$$e^{\theta_n(x)} \left(\partial_x f(x)\right)^2 \to e^{\beta\, x} \left(\partial_x f(x)\right)^2 \ \text{ when } n \to \infty.$$

Therefore, applying Fatou's lemma on the left hand side of (2.4) it follows that

$$\int_{\mathbb{R}} e^{\beta\, x} \left(\partial_x f\right)^2\, dx \leq \frac{1}{2}\beta^2 \int_{\mathbb{R}} e^{\beta\, x} f^2\, dx + \left\|\partial_x^2 f\right\|_{L^2(\mathbb{R})} \left[\int_{\mathbb{R}} \left(e^{\beta\, x} f\right)^2\, dx\right]^{1/2}. \tag{2.5}$$

By hypothesis, the integrals involved on the right hand side of (2.5) are finite. This allows us to conclude that $e^{\frac{1}{2}\beta x}\partial_x f \in L^2(\mathbb{R})$.

For the general case, we recursively apply the above reasoning in the following way: since $e^{\frac{1}{2}\beta x}\partial_x f \in L^2(\mathbb{R})$ and $\partial_x^2(\partial_x f) \in L^2(\mathbb{R})$ (because $f \in H^3(\mathbb{R})$) we have that $e^{\frac{1}{4}\beta x}\partial_x^2 f \in L^2(\mathbb{R})$. Continuing in this way and taking into account that $f \in H^\infty(\mathbb{R})$, we can conclude that $e^{\frac{1}{2^j}\beta x}\partial_x^j f \in L^2(\mathbb{R})$.

Since $\beta > 0$ is arbitrary in the hypothesis of the lemma, we could have started with $2^j\beta$ instead of β to obtain that $e^{\beta x}\partial_x^j f \in L^2(\mathbb{R})$, which completes the proof of this lemma.

☑

Next we regularize the initial datum of the IVP (1.1) in a similar way as it was performed in [ILP]. This will allow us to deal with solutions having enough regularity and decay to make a priori estimates on them and, in particular, to apply integration by parts. Later, we will pass to the limit to obtain the result of Theorem I.

We consider a function $\rho \in C_0^\infty(\mathbb{R})$ with $\rho \geq 0$, $supp(\rho) \subset [-1,1]$ and such that

$$\int_{\mathbb{R}} \rho\, dx = 1.$$

For $\epsilon \in (0,1)$, we define

$$\begin{aligned} \rho_\epsilon &:= \frac{1}{\epsilon}\,\rho\left(\frac{\cdot}{\epsilon}\right) \quad \text{and} \\ u_0^\epsilon(x) &:= \rho_\epsilon * u_0(\cdot + \epsilon)(x) \\ &= \int_{\mathbb{R}} \rho_\epsilon(y)\, u_0(x+\epsilon-y)\, dy. \end{aligned}$$

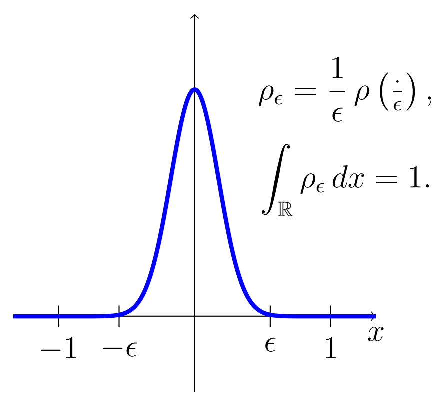

Figure 3

Let us see that

$$u_0^\epsilon \in H^\infty(\mathbb{R}) \quad \text{and} \quad \left\| e^{a_0\, x_+^{3/2}} u_0^\epsilon \right\|_{L^2(\mathbb{R})} \leq \left\| e^{a_0\, x_+^{3/2}} u_0 \right\|_{L^2(\mathbb{R})}. \tag{2.6}$$

Indeed, $u_0^\epsilon \in H^\infty(\mathbb{R})$ because $\rho_\epsilon \in C_0^\infty(\mathbb{R})$. Furthermore, using Minkowski's integral inequality we have that

$$\begin{aligned}
\left\|e^{a_0\,x_+^{3/2}}u_0^\epsilon\right\|_{L^2(\mathbb{R})} &= \left(\int_{\mathbb{R}}\left|e^{a_0\,x_+^{3/2}}u_0^\epsilon(x)\right|^2\,dx\right)^{1/2}\\
&= \left(\int_{\mathbb{R}}e^{2\,a_0\,x_+^{3/2}}\left|\int_{\mathbb{R}}\rho_\epsilon(y)\,u_0(x+\epsilon-y)\,dy\right|^2\,dx\right)^{1/2}\\
&= \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}}e^{a_0\,x_+^{3/2}}\rho_\epsilon(y)\,u_0(x+\epsilon-y)\,dy\right|^2\,dx\right)^{1/2}\\
&\le \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|e^{a_0\,x_+^{3/2}}\rho_\epsilon(y)\,u_0(x+\epsilon-y)\right|^2\,dx\right)^{1/2}\,dy\\
&= \int_{\mathbb{R}}\rho_\epsilon(y)\left(\int_{\mathbb{R}}\left|e^{a_0\,x_+^{3/2}}u_0(x+\epsilon-y)\right|^2\,dx\right)^{1/2}\,dy\\
&= \int_{\mathbb{R}}\rho_\epsilon(y)\left(\int_{\mathbb{R}}\left|e^{a_0\,(x-\epsilon+y)_+^{3/2}}u_0(x)\right|^2\,dx\right)^{1/2}\,dy. \qquad (2.7)
\end{aligned}$$

Now, for $y \in supp(\rho_\epsilon) \subset [-\epsilon,\epsilon]$ we have that $-\epsilon+y \le 0$ and therefore

$$e^{a_0\,(x-\epsilon+y)_+^{3/2}} \le e^{a_0\,x_+^{3/2}}, \quad \text{for every } x \in \mathbb{R}.$$

Then, from (2.7) we obtain that

$$\begin{aligned}
\left\|e^{a_0\,x_+^{3/2}}u_0^\epsilon\right\|_{L^2(\mathbb{R})} &\le \int_{\mathbb{R}}\rho_\epsilon(y)\left(\int_{\mathbb{R}}\left|e^{a_0\,x_+^{3/2}}u_0(x)\right|^2\,dx\right)^{1/2}\,dy\\
&= \left(\int_{\mathbb{R}}\left|e^{a_0\,x_+^{3/2}}u_0(x)\right|^2\,dx\right)^{1/2}\int_{\mathbb{R}}\rho_\epsilon(y)\,dy\\
&= \left\|e^{a_0\,x_+^{3/2}}u_0\right\|_{L^2(\mathbb{R})},
\end{aligned}$$

from which (2.6) follows.

Let us notice that when we apply the above inequality with $a_0 = 0$ we obtain that

$$\|u_0^\epsilon\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}. \qquad (2.8)$$

Proceeding in a similar way as we did to prove (2.6), it can be shown that for every $\beta > 0$,

$$\left\|e^{\beta\,x}u_0^\epsilon\right\|_{L^2(\mathbb{R})} \le \left\|e^{\beta\,x}u_0\right\|_{L^2(\mathbb{R})}. \qquad (2.9)$$

Besides, we have that

$$u_0^\epsilon = \rho_\epsilon * u_0(\cdot+\epsilon) \to u_0 \ \text{ in } L^2(\mathbb{R}), \ \text{ as } \epsilon \searrow 0.$$

For $m \in \mathbb{Z}^+$ we consider the IVP (1.1) with initial datum $u_0^{1/m}$, which, by the global well-posedness theory, has a unique solution

$$u_m \in C^1\left([0,T]; H^\infty(\mathbb{R})\right), \tag{2.10}$$

which satisfies the differential equation in (1.1) in a classical sense in any space $H^s(\mathbb{R})$ with $s \geq 3$. That is, for every $s \geq 3$ and every $t \in [0,T]$,

$$\partial_t u(t) + \partial_x^3 u(t) + u(t)\partial_x u(t) = 0, \quad \text{in } H^{s-3}(\mathbb{R}).$$

Furthermore, from the continuous dependence of solutions of the IVP (1.1) with respect to the initial datum we have that for every $t \in [0,T]$

$$u_m(t) \to u(t) \ \text{ in } L^2(\mathbb{R}), \ \text{ as } m \to \infty. \tag{2.11}$$

Along the proof we will work with u_m and with $u_m(0) = u_0^{1/m}$, which for simplicity in the writing we will continue denoting by u and by $u(0)$, respectively. At the end of the analysis of u_m we will pass to the limit to get the desired result for the solution u with the assumptions of Theorem I.

Proof of Theorem I.
We will find in the proof that $a(t)$ in (1.12) is given by (1.13). For the moment, let us consider that a is a differentiable function on $[0,T]$, with $a(0) = a_0$.

Next we are going to make an a priori estimate of $u \equiv u_m$. For this, let us take an increasing function $\omega \in C^\infty(\mathbb{R})$, so that $\omega(x) = 0$, for $x \in (-\infty, \frac{1}{4}]$ and $\omega(x) = 1$, for $x \in [\frac{1}{2}, \infty)$.

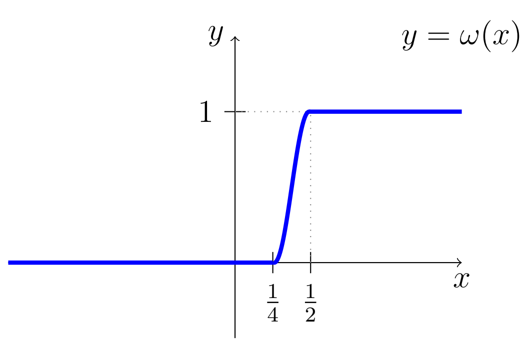

Figure 4

For each positive integer n, we consider a function ψ_n defined in the following fashion:

$$\psi(x,t) \equiv \psi_n(x,t) := \begin{cases} \omega(x)\, a(t)\, x^{3/2}, & \text{if} \quad x \le n, \\ \log\left(P_n(x,t)\right), & \text{if} \quad x > n, \end{cases} \tag{2.12}$$

where, for fixed $t \in [0,T]$, $P_n(x,t)$ is the second degree polynomial in x which coincides with $e^{\omega(x)\, a(t)\, x^{3/2}} = e^{a(t)\, x^{3/2}}$ at $x = n$ together with its two first derivatives. In this way,

$$e^{\psi} \equiv e^{\psi_n} = \begin{cases} e^{\omega(x)\, a(t)\, x_+^{3/2}}, & \text{if} \quad x \le n, \\ P_n(x,t), & \text{if} \quad x > n. \end{cases} \tag{2.13}$$

Remark 2.1. From the definition of $P_n(x,t)$ we see that, for fixed t, e^{ψ} is a C^2 function of x, and, since $\partial_x^3 P_n(x,t) = 0$ for $x > n$, it follows that the third derivative $\partial_x^3 e^{\psi}$ is continuous and bounded for $x < n$, and vanishes for $x > n$, with a saltus at $x = n$. Hence, $\psi = \log(e^{\psi})$ inherits from e^{ψ} the same regularity properties.

For a fixed positive integer m and for $n \in \mathbb{N}$, let us define

$$f \equiv f_{m,n} = u_m\, e^{\psi_n} = u\, e^{\psi}. \tag{2.14}$$

Then, $u = e^{-\psi} f$, and replacing u in the KdV equation we get:

$$\partial_t(e^{-\psi} f) + \partial_x^3(e^{-\psi} f) + (e^{-\psi} f)\, \partial_x(e^{-\psi} f) = 0. \tag{2.15}$$

Proceeding formally, we multiply both sides of (2.15) by e^{ψ} and use the fact that

$$\begin{aligned} &e^{\psi}\, \partial_t(e^{-\psi}\, \cdot\,) = \partial_t - \psi_t, \text{ and} \\ &e^{\psi}\, \partial_x^j(e^{-\psi}\, \cdot\,) = \left(e^{\psi}\, \partial_x e^{-\psi}\right)^j = (\partial_x - \psi_x)^j\,, \text{ for every } j \in \mathbb{N}, \end{aligned} \tag{2.16}$$

to obtain that

$$(\partial_t - \psi_t)\ f + (\partial_x - \psi_x)^3\, f + e^{-\psi} f\ (\partial_x - \psi_x)\ f = 0. \tag{2.17}$$

Let us notice that

$$(\partial_x - \psi_x)^3\, f = \ \partial_x^3 f - 3\, \psi_x\, \partial_x^2 f + \left(3\, \psi_x^2 - 3\, \psi_{xx}\right) \partial_x f + \left(3\, \psi_x \psi_{xx} - \psi_x^3 - \psi_{xxx}\right) f. \tag{2.18}$$

Therefore, equation (2.17) can be written as

$$\begin{aligned} &\partial_t f - \psi_t f + \partial_x^3 f - 3\, \psi_x\, \partial_x^2 f + (3\, \psi_x^2 - 3\, \psi_{xx})\partial_x f + (3\, \psi_x \psi_{xx} - \psi_x^3 - \psi_{xxx}) f \\ &+ e^{-\psi}(\partial_x f) f - e^{-\psi} \psi_x f^2 = 0. \end{aligned} \tag{2.19}$$

In order to justify the formal procedure leading to (2.19) we observe that, for all $\beta > 0$, $e^{\beta\, x} \le C_{\beta,a_0}\, e^{a_0\, x_+^{3/2}}$ which, together with the fact that $e^{a_0\, x_+^{3/2}} u(0) \in L^2(\mathbb{R})$, implies that $e^{\beta\, x} u(0) \in L^2(\mathbb{R})$ for every $\beta > 0$.
According to the result of Kato mentioned in (1.2), for each $t \in [0,T]$, $e^{\beta\, x} u(t) \in L^2(\mathbb{R})$ for every $\beta > 0$. Furthermore, from (2.10), since $u(0) \equiv u_0^{1/m} \in H^\infty(\mathbb{R})$, we have that $u(t) \in H^\infty(\mathbb{R})$ and so, from Lemma 2.1, $e^{\beta\, x}\partial_x^j u(t) \in L^2(\mathbb{R})$ for $j \in \mathbb{Z}^+$. In particular, for $j, k \in \mathbb{N}$,

$$\left(1 + x_+^k\right) \partial_x^j u(t) \in L^2(\mathbb{R}). \tag{2.20}$$

From (2.13), for fixed t, $e^{\psi(x,t)}$ is a second degree polynomial in x if $x > n$ and is bounded if $x \le n$. On the other hand, $\psi_t, \psi_x, \psi_{xx}$ and ψ_{xxx} are bounded functions of the variable x for fixed t (see (2.29) below for the unbounded case $x > n$). Since $f = u\, e^{\psi}$, from the Leibniz's formula for the derivatives of a product, from (2.20), and taking into account Remark 2.1, we have that all terms in (2.19) belong to $L^2(\mathbb{R})$ and equation (2.19) is satisfied in $L^2(\mathbb{R})$ for fixed t.

We now multiply (2.19) by f and integrate by parts with respect to the variable x, on $\mathbb{R}$, letting the variable t fixed. For simplicity in the notation, we write $\int$ instead of $\int_{\mathbb{R}} \cdot\, dx$ and omit the variable t in the expressions. Thus, we obtain that

$$\begin{aligned}&\underbrace{\int (\partial_t f) f}_{A} - \int \psi_t f^2 + \underbrace{\int (\partial_x^3 f) f}_{B} - \underbrace{3\int \psi_x (\partial_x^2 f) f}_{C} + \underbrace{3\int (\psi_x^2 - \psi_{xx})(\partial_x f) f}_{D}\\ &+ \int (3\,\psi_x\psi_{xx} - \psi_x^3 - \psi_{xxx}) f^2 + \underbrace{\int e^{-\psi}(\partial_x f) f^2 - \int e^{-\psi}\psi_x f^3}_{E} = 0.\end{aligned} \tag{2.21}$$

As we have seen in the preceding discussion, the regularity and decay of the terms in C, D, and E allow us to integrate by parts to obtain that

$$B = \int (\partial_x^3 f) f = -\int (\partial_x^2 f)(\partial_x f) = \int (\partial_x f)(\partial_x^2 f) = -\int f(\partial_x^3 f) = -B,$$

thus $B = 0$.

$$\begin{aligned} C = -3\int \psi_x (\partial_x^2 f) f &= 3\int \psi_{xx}(\partial_x f) f + 3\int \psi_x (\partial_x f)^2\\ &= \frac{3}{2}\int \psi_{xx}\partial_x(f^2) + 3\int \psi_x(\partial_x f)^2\\ &= -\frac{3}{2}\int \psi_{xxx} f^2 + 3\int \psi_x (\partial_x f)^2,\end{aligned}$$

$$
\begin{aligned}
D = 3\int(\psi_x^2 - \psi_{xx})(\partial_x f)f &= \frac{3}{2}\int(\psi_x^2 - \psi_{xx})\partial_x(f^2)\\
&= -3\int \psi_x\psi_{xx}\, f^2 + \frac{3}{2}\int \psi_{xxx}\, f^2,
\end{aligned}
$$

$$
\begin{aligned}
E = \int e^{-\psi}(\partial_x f)f^2 - \int e^{-\psi}\psi_x f^3 &= \frac{1}{3}\int e^{-\psi}\partial_x(f^3) - \int e^{-\psi}\psi_x f^3\\
&= \frac{1}{3}\int e^{-\psi}\psi_x f^3 - \int e^{-\psi}\psi_x f^3\\
&= -\frac{2}{3}\int e^{-\psi}\psi_x f^3.
\end{aligned}
$$

Furthermore, $A = \displaystyle\int(\partial_t f)f = \frac{1}{2}\int \partial_t(f^2) = \frac{1}{2}\frac{d}{dt}\int f^2.$

Replacing the equivalent expressions for A, B, C, D y E in (2.21), grouping and simplifying we obtain that

$$
\frac{1}{2}\frac{d}{dt}\int f^2 + 3\int \psi_x(\partial_x f)^2 - \int(\psi_t + \psi_x^3 + \psi_{xxx})f^2 - \frac{2}{3}\int e^{-\psi}\psi_x f^3 = 0. \tag{2.22}
$$

Since $(\partial_x f)^2 \geq 0$ and ψ is an increasing function, we have that $\displaystyle\int \psi_x(\partial_x f)^2 \geq 0$. Hence, from (2.22), it follows that

$$
\frac{1}{2}\frac{d}{dt}\int f^2 \leq \int(\psi_t + \psi_x^3 + \psi_{xxx})f^2 + \frac{2}{3}\int e^{-\psi}\psi_x f^3. \tag{2.23}
$$

Our objective is to apply Gronwall's lemma to estimate $\displaystyle\int f^2$. For this, with respect to the first term on the right hand side of (2.23), we pretend to bound $\psi_t + \psi_x^3 + \psi_{xxx}$ with a constant independent of $t \in [0, T]$ by choosing $a(t)$ in an appropriate way.

We start by studying the terms on the right hand side of (2.23) for $1 \leq x \leq n$, where we know that $\psi = a\, x^{3/2}$ and, in consequence,

$$
\begin{aligned}
&\psi_t = a'\, x^{3/2}, &&\psi_x = \frac{3}{2}\, a\, x^{1/2}, &&\psi_x^3 = \frac{27}{8}\, a^3 x^{3/2},\\
&\psi_{xx} = \frac{3}{4}\, a\, x^{-1/2}, &&\psi_{xxx} = -\frac{3}{8} a\ x^{-3/2}.
\end{aligned}
$$

In this way, the first integrand on the right hand side of (2.23) is

$$
\left(a' x^{3/2} + \frac{27}{8}\, a^3 x^{3/2} - \frac{3}{8}\, a\, x^{-3/2}\right) f^2.
$$

To bound this last expression we take $a(t)$ so that

$$a' x^{3/2} + \frac{27}{8}\, a^3 x^{3/2} = 0.$$

This leads us to state the initial value problem

$$\begin{cases} a'(t) + \dfrac{27}{8}\, a(t)^3 = 0, \\ a(0) = a_0, \end{cases}$$

whose solution is given by

$$a(t) = \frac{a_0}{\sqrt{1 + \frac{27}{4} a_0^2 t}},$$

which is precisely the function $a = a(t)$ given in (1.13), in the statement of Theorem I.

With this choice of a, it turns out that

$$\int_1^n (\psi_t + \psi_x^3 + \psi_{xxx}) f^2 = \int_1^n \left(-\frac{3}{8}\, a\, x^{-3/2} \right) f^2 \leq 0. \tag{2.24}$$

For the second integral on the right hand side of (2.23) we then have that

$$\begin{aligned} \int_1^n e^{-\psi} \psi_x f^3 &= \frac{3}{2}\, a \int_1^n x^{1/2} e^{-\psi} f^3 = \frac{3}{2}\, a \int_1^n x^{1/2} u\, f^2 \\ &\leq \frac{3}{2}\, a_0\, \|x_+^{1/2} u(t)\|_{L^\infty([0,\infty))} \int_1^n f^2 \\ &\leq \frac{3}{2}\, a_0\, \|x_+^{1/2} u(t)\|_{L^\infty([0,\infty))} \int_{\mathbb{R}} f^2. \end{aligned} \tag{2.25}$$

From (2.24) and (2.25) we conclude that the integrals on the right hand side of (2.23) performed on the interval $[1, n]$ are bounded by

$$a_0\, \|x_+^{1/2} u(t)\|_{L^\infty([0,\infty))} \int_{\mathbb{R}} f^2. \tag{2.26}$$

Next, we consider the contribution of the interval (n, ∞) to the integrals of the right hand side of expression (2.23).

Let $\varphi_n(x,t) \equiv \varphi(x,t) = e^{\omega(x) a(t)\, x^{3/2}}$. From the definition of P_n given in (2.12) we have that

$$P(x,t) \equiv P_n(x,t) = \varphi(n,t) + \varphi_x(n,t)(x-n) + \frac{1}{2}\varphi_{xx}(n,t)(x-n)^2.$$

Let us observe that

$$\begin{aligned}
\varphi(n,t) &= e^{a\,n^{3/2}},\\
\varphi_x(n,t) &= \frac{3}{2}\,a\,n^{1/2}\,e^{a\,n^{3/2}} \quad \text{and}\\
\varphi_{xx}(n,t) &= \left(\frac{3}{4}\,a\,n^{-1/2}+\frac{9}{4}\,a^2 n\right) e^{a\,n^{3/2}}.
\end{aligned}$$

Hence

$$P(x,t)=\left[1+\frac{3}{2}\,a\,n^{1/2}(x-n)+\left(\frac{3}{8}\,a\,n^{-1/2}+\frac{9}{8}\,a^2 n\right)(x-n)^2\right]e^{a\,n^{3/2}}. \tag{2.27}$$

From (2.27) we can calculate the derivatives of P:

$$\begin{aligned}
P_t(x,t) &= a'\left[n^{3/2}+\left(\frac{3}{2}\,n^{1/2}+\frac{3}{2}\,a\,n^2\right)(x-n)+\left(\frac{3}{8}\,n^{-1/2}+\frac{21}{8}\,a\,n+\frac{9}{8}\,a^2 n^{5/2}\right)(x-n)^2\right]e^{a\,n^{3/2}},\\
P_x(x,t) &= \left[\frac{3}{2}\,a\,n^{1/2}+\left(\frac{3}{4}\,a\,n^{-1/2}+\frac{9}{4}\,a^2 n\right)(x-n)\right]e^{a\,n^{3/2}},\\
P_{xx}(x,t) &= \left(\frac{3}{4}\,a\,n^{-1/2}+\frac{9}{4}\,a^2 n\right)e^{a\,n^{3/2}}.
\end{aligned} \tag{2.28}$$

Our immediate objective is to estimate $\psi_t+\psi_x^3+\psi_{xxx}$ on the interval (n,∞). Let us observe that on (n,∞), $\psi_n=\log(P_n)\equiv\log(P)$. Therefore

$$\begin{aligned}
\psi_t &= \frac{P_t}{P}, \qquad \psi_x=\frac{P_x}{P}, \qquad \psi_{xx}=\frac{P_{xx}}{P}-\frac{P_x^2}{P^2},\\
\psi_{xxx} &= \frac{P_{xxx}}{P}-3\,\frac{P_xP_{xx}}{P^2}+2\,\frac{P_x^3}{P^3}.
\end{aligned} \tag{2.29}$$

Taking into account that $P_{xxx}=0$ we have that

$$\psi_t+\psi_x^3+\psi_{xxx}=\frac{1}{P^3}\left[P^2P_t+3\,P_x^3-3\,P\,P_xP_{xx}\right]. \tag{2.30}$$

In order to calculate the right hand side of (2.30) we proceed in the following way.

Let $r:=a\,n^{1/2}(x-n)$. Then $r>0$ because $x\in(n,\infty)$. Using the fact that $a'=-\frac{27}{8}\,a^3$ and highlighting the terms of degree 0 in n in the expressions of P and its derivatives in (2.27) and (2.28), we write:

$$
\begin{aligned}
P &= \left[1+\frac{3}{2}\,r+\left(\frac{9}{8}+\epsilon_n^{(1)}\right)r^2\right]e^{a\,n^{3/2}},\\
P_t &= -\frac{27}{8}\,a^3n^{3/2}\left[1+\left(\frac{3}{2}+\epsilon_n^{(2)}\right)r+\left(\frac{9}{8}+\epsilon_n^{(3)}\right)r^2\right]e^{a\,n^{3/2}},\\
P_x &= a\,n^{1/2}\left[\frac{3}{2}+\left(\frac{9}{4}+\epsilon_n^{(4)}\right)r\right]e^{a\,n^{3/2}} \quad \text{and}\\
P_{xx} &= a^2n\left(\frac{9}{4}+\epsilon_n^{(4)}\right)e^{a\,n^{3/2}},
\end{aligned}
\tag{2.31}
$$

where

$$
\begin{aligned}
&\epsilon_n^{(1)}=\epsilon_n^{(1)}(t)=\frac{3}{8\,a\,n^{3/2}},\quad \epsilon_n^{(2)}=\epsilon_n^{(2)}(t)=\frac{3}{2\,a\,n^{3/2}},\\
&\epsilon_n^{(3)}=\epsilon_n^{(3)}(t)=\frac{21}{8\,a\,n^{3/2}}+\frac{3}{8\,a^2n^3}\quad\text{and}\quad \epsilon_n^{(4)}=\epsilon_n^{(4)}(t)=\frac{3}{4\,a\,n^{3/2}}.
\end{aligned}
\tag{2.32}
$$

Next, we calculate (2.30) taking formally $\epsilon_n^{(j)}=0,\;\; j=1,2,3,4,$ for which we have that

$$
\begin{aligned}
P &= \left(1+\frac{3}{2}\,r+\frac{9}{8}r^2\right)e^{a\,n^{3/2}}, & P_t &= -\frac{27}{8}\,a^3n^{3/2}\left(1+\frac{3}{2}\,r+\frac{9}{8}\,r^2\right)e^{a\,n^{3/2}},\\
P_x &= a\,n^{1/2}\left(\frac{3}{2}+\frac{9}{4}\,r\right)e^{a\,n^{3/2}} \quad \text{and} & P_{xx} &= \frac{9}{4}\,a^2n\,e^{a\,n^{3/2}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&P^2P_t+3\,P_x^3-3\,P\,P_xP_{xx}\\
&=\Bigg[\left(1+\frac{3}{2}\,r+\frac{9}{8}\,r^2\right)^2\left(-\frac{27}{8}\,a^3n^{3/2}\right)\left(1+\frac{3}{2}\,r+\frac{9}{8}\,r^2\right)+3\,a^3n^{3/2}\left(\frac{3}{2}+\frac{9}{4}\,r\right)^3\\
&\qquad -3\left(1+\frac{3}{2}\,r+\frac{9}{8}\,r^2\right)\cdot a\,n^{1/2}\left(\frac{3}{2}+\frac{9}{4}\,r\right)\cdot\frac{9}{4}\,a^2n\Bigg]e^{3\,a\,n^{3/2}}\\
&=a^3n^{3/2}e^{3\,a\,n^{3/2}}\left[-\frac{27}{8}\left(1+\frac{3}{2}\,r+\frac{9}{8}\,r^2\right)^3+3\left(\frac{3}{2}+\frac{9}{4}\,r\right)^3-\frac{27}{4}\left(1+\frac{3}{2}\,r+\frac{9}{8}\,r^2\right)\left(\frac{3}{2}+\frac{9}{4}\,r\right)\right].
\end{aligned}
$$

The calculation of the above expression can be performed using a computer software, to obtain that such expression is equal to

$$
a^3n^{3/2}e^{3\,a\,n^{3/2}}\left(-\frac{19683}{4096}\,r^6-\frac{19683}{1024}\,r^5-\frac{19683}{512}\,r^4-\frac{3645}{128}\,r^3-\frac{27}{8}\right)<0. \tag{2.33}
$$

Let us define $\epsilon_n := \left(\epsilon_n^{(1)}, \epsilon_n^{(2)}, \epsilon_n^{(3)}, \epsilon_n^{(4)}\right)$. We see that the coefficients of the polynomials in r appearing in the expressions of P and its derivatives in (2.31) are continuous functions of ϵ_n. Likewise, when calculating $P^2 P_t + 3\,P_x^3 - 3\,P\,P_x P_{xx}$ with general ϵ_n, a similar expression to (2.33) is obtained in which the coefficients of the powers of r are polynomials in ϵ_n, whose values at $\epsilon_n = 0$ are the negative coefficients that appear in (2.33). Furthermore, since $a(t)$ is decreasing, it is clear, from the definition of $\epsilon_n^{(j)}$ in (2.32), that $\epsilon_n^{(j)}(t) \leq \epsilon_n^{(j)}(T)$, for every $t \in [0,T]$ and every $j = 1,2,3,4$, and thus it is observed in (2.32) that $\epsilon_n(t) \to 0$ uniformly in t when $n \to \infty$. Thus, by the continuity of the coefficients of the powers of r as functions of ϵ_n, there is a positive integer N such that if $n > N$, then

$$P_n^2 \partial_t P_n + 3(\partial_x P_n)^3 - 3\,P_n(\partial_x P_n)(\partial_x^2 P_n) < 0, \tag{2.34}$$

for every $x \in (n,\infty)$ and every $t \in [0,T]$.

Since, from (2.31),

$$P = \left[1 + \frac{3}{2}\,r + \left(\frac{9}{8} + \epsilon_n^{(1)}\right) r^2\right] e^{a\,n^{3/2}} \geq e^{a\,n^{3/2}} > 0,$$

we conclude from (2.30) and (2.34) that, for $x \in (n,\infty)$ and $n > N$,

$$(\psi_t + \psi_x^3 + \psi_{xxx}) f^2 \leq 0, \tag{2.35}$$

from which it follows that the first integral on the right hand side of (2.23) on (n,∞) is nonpositive.

In order to estimate the second integral on the right hand side of (2.23) on (n,∞), we recall the expressions for P and P_x given in (2.27) and (2.28) for $x > n$:

$$\begin{aligned} P(x,t) &= \left[1 + \frac{3}{2}\,a\,n^{1/2}(x-n) + \left(\frac{3}{8}\,a\,n^{-1/2} + \frac{9}{8}\,a^2\,n\right)(x-n)^2\right] e^{a\,n^{3/2}} \qquad \text{and} \\ P_x(x,t) &= \left[\frac{3}{2}\,a\,n^{1/2} + \left(\frac{3}{4}\,a\,n^{-1/2} + \frac{9}{4}\,a^2 n\right)(x-n)\right] e^{a\,n^{3/2}}. \end{aligned}$$

From these two expressions and taking into account that $x > n > 1$ and $a(t) \leq a_0$ for all $t \in [0,T]$, we see that

$$
\begin{aligned}
0 \le P_x(x,t) &\le \frac{3}{2}\,a\,n^{1/2}P(x,t) + \frac{3}{4}\,a\,n^{-1/2}(x-n)e^{a\,n^{3/2}} \\
&\le \frac{3}{2}\,a\,n^{1/2}P(x,t) + \frac{3}{4}\,a\,n^{1/2}\,(x-n)e^{a\,n^{3/2}} \\
&\le \frac{3}{2}\,a\,n^{1/2}P(x,t) + \frac{1}{2}\,P(x,t) \\
&\le \left(\frac{3}{2}\,a_0\,x^{1/2} + \frac{1}{2}\right)P(x,t) \\
&\le \frac{3}{2}\,(1+a_0)\,x^{1/2}P(x,t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|\frac{2}{3}\int_n^\infty e^{-\psi}\psi_x\,f^3\right| &= \left|\frac{2}{3}\int_n^\infty \psi_x\,u\,f^2\right| \\
&\le \frac{2}{3}\int_n^\infty |\psi_x|\,|u|\,f^2 = \frac{2}{3}\int_n^\infty \frac{P_x}{P}\,|u|\,f^2 \\
&\le \frac{2}{3}\int_n^\infty \frac{3}{2}\,(1+a_0)\,x^{1/2}|u|\,f^2 \\
&\le (1+a_0)\left\|x_+^{1/2}u(t)\right\|_{L^\infty([0,\infty))}\int_{\mathbb{R}} f^2. \qquad (2.36)
\end{aligned}
$$

From (2.35) and (2.36) we conclude that the integrals on the right hand side of (2.23) performed on the interval (n,∞) are bounded by

$$
(1+a_0)\left\|x_+^{1/2}u(t)\right\|_{L^\infty([0,\infty))}\int_{\mathbb{R}} f^2. \qquad (2.37)
$$

On the other hand, for $x<\frac{1}{4}$ we have that $\psi=0$ and therefore $\psi_t=\psi_x=\psi_{xxx}=0$, and thus

$$
\int_{-\infty}^{1/4}(\psi_t+\psi_x^3+\psi_{xxx})f^2 + \frac{2}{3}\int_{-\infty}^{1/4} e^{-\psi}\psi_x f^3 = 0. \qquad (2.38)
$$

It remains to estimate the right hand side of (2.23) on the interval $[\frac{1}{4},1]$. We remark that we have defined ψ by using the truncation function ω in order to avoid the unboundedness of the third order spatial derivative of $a\,x^{3/2}$ near the origin.

In $[\frac{1}{4}, 1]$ we have that $\psi(x,t) = \omega(x)a(t)x^{3/2}$ and a direct calculation shows that its derivatives are given by

$$\begin{aligned}
\psi_t &= \omega\, a'\, x^{3/2} = -\frac{27}{8}\omega\, a^3 x^{3/2},\\
\psi_x &= a\left(\frac{3}{2}\,\omega\, x^{1/2} + \omega'\, x^{3/2}\right),\\
\psi_{xx} &= a\left(\frac{3}{4}\,\omega\, x^{-1/2} + 3\,\omega'\, x^{1/2} + \omega''\, x^{3/2}\right) \quad \text{and}\\
\psi_{xxx} &= a\left(-\frac{3}{8}\,\omega\, x^{-3/2} + \frac{9}{4}\,\omega'\, x^{-1/2} + \frac{9}{2}\,\omega''\, x^{1/2} + \omega'''\, x^{3/2}\right).
\end{aligned}$$

Using the fact that $0 \le a \le a_0$, taking into account that ω and its derivatives are bounded, that $\omega' \ge 0$, and that $x > \frac{1}{4}$ on the support of such functions, we can conclude that, for $x \in [\frac{1}{4}, 1]$,

$$\begin{aligned}
&\psi_t(x,t) \le 0, && 0 \le \psi_x(x,t) \le C\, a_0,\\
&|\psi_{xx}(x,t)| \le C\, a_0 \quad \text{and} && |\psi_{xxx}(x,t)| \le C\, a_0,
\end{aligned}$$

where C is an absolute constant since it depends only upon the choice of ω.

Therefore,

$$\psi_t + \psi_x^3 + \psi_{xxx} \le C\,(1 + a_0^3). \tag{2.39}$$

Consequently,

$$\begin{aligned}
\int_{1/4}^{1} (\psi_t + \psi_x^3 + \psi_{xxx}) f^2 + \frac{2}{3}\int_{1/4}^{1} e^{-\psi}\psi_x f^3 &\le \int_{1/4}^{1} \left(\psi_t + \psi_x^3 + \psi_{xxx}\right) f^2 + \frac{2}{3}\int_{1/4}^{1} \psi_x\, |u|\, f^2\\
&\le C\,(1 + a_0^3)\int_{1/4}^{1} f^2 + \frac{2}{3}\, C\, a_0 \|u(t)\|_{L^\infty([0,\infty))}\int_{1/4}^{1} f^2\\
&\le C\,(1 + a_0^3)\left(1 + \|u(t)\|_{L^\infty([0,\infty))}\right)\int_{\mathbb{R}} f^2. \qquad (2.40)
\end{aligned}$$

From (2.23), and putting together the results obtained in (2.26), (2.37), (2.38), and (2.40), we conclude that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int f^2 &\le \left((1 + a_0)\,\|x_+^{1/2} u(t)\|_{L^\infty([0,\infty))} + C\,(1 + a_0^3)\left(1 + \|u(t)\|_{L^\infty([0,\infty))}\right)\right)\int f^2\\
&\le C\,(1 + a_0^3)\left(1 + \|(1 + x_+^{1/2})u(t)\|_{L^\infty([0,\infty))}\right)\int f^2. \qquad (2.41)
\end{aligned}$$

Since $1+x_+^{1/2}\le 2e^{x_+}$, we have that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int f^2 &\le C\,(1+a_0^3)\left(1+\|e^{x_+}u(t)\|_{L^\infty([0,\infty))}\right)\int f^2\\
&\le C\,(1+a_0^3)\left(1+\|e^{x}u(t)\|_{L^\infty(\mathbb{R})}\right)\int f^2\\
&\le C\,(1+a_0^3)\left(1+\|e^{x}u(t)\|_{L^2(\mathbb{R})}+\|\partial_x\left(e^x u(t)\right)\|_{L^2(\mathbb{R})}\right)\int f^2\\
&\le C\,(1+a_0^3)\left(1+\|e^{x}u(t)\|_{L^2(\mathbb{R})}+\|e^x\partial_x u(t)\|_{L^2(\mathbb{R})}\right)\int f^2,
\end{aligned}
$$

where C is a universal constant.

Returning to the notation $u=u_m$, with u_m defined in (2.10), and $f=f_{m,n}=u_m e^{\psi_n}$, as in (2.14), we then have that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int e^{2\,\psi_n}u_m(t)^2\,dx &\le C\,(1+a_0^3)\left(1+\|e^{x}u_m(t)\|_{L^2(\mathbb{R})}+\|e^x\partial_x u_m(t)\|_{L^2(\mathbb{R})}\right)\int e^{2\,\psi_n}u_m(t)^2\,dx\\
&\equiv \beta_m(t)\int e^{2\,\psi_n}u_m(t)^2\,dx.
\end{aligned}
\tag{2.42}
$$

In order to apply Gronwall's lemma, let us estimate the integral of β_m on $[0,T]$. Using (1.2) we see that

$$
\begin{aligned}
\int_0^T\beta_m(s)\,ds &= C\,(1+a_0^3)\left[\int_0^T\|e^x u_m(s)\|_{L^2(\mathbb{R})}\,ds+\int_0^T\|e^x\partial_x u_m(s)\|_{L^2(\mathbb{R})}\,ds\right]\\
&\le C\,(1+a_0^3)\left[\int_0^T e^{K_m s}\left\|e^x u_0^{1/m}\right\|_{L^2(\mathbb{R})}ds+\int_0^T e^{\frac{K_m}{2}s}\left(e^{-\frac{K_m}{2}s}\|e^x\partial_x u_m(s)\|_{L^2(\mathbb{R})}\right)ds\right],
\end{aligned}
$$

where $K_m=K\left(\|u_0^{1/m}\|_{L^2(\mathbb{R})}\right)$. In virtue of (2.8) it follows that $K_m\le K\left(\|u_0\|_{L^2(\mathbb{R})}\right)\equiv K$. Therefore, from (1.2) and (2.9),

$$
\begin{aligned}
\int_0^T\beta_m(s)\,ds &\le C\,(1+a_0^3)\left[T\,e^{KT}\|e^x u_0\|_{L^2(\mathbb{R})}+e^{\frac{K}{2}T}\int_0^T e^{-\frac{K_m}{2}s}\|e^x\partial_x u_m(s)\|_{L^2(\mathbb{R})}\,ds\right]\\
&\le C\,(1+a_0^3)\left[T\,e^{KT}\|e^x u_0\|_{L^2(\mathbb{R})}+\sqrt{T}e^{\frac{K}{2}T}\left(\int_0^T e^{-K_m s}\|e^x\partial_x u_m(s)\|^2_{L^2(\mathbb{R})}\right)^{1/2}ds\right]\\
&\le C\,(1+a_0^3)\left[T\,e^{KT}\|e^x u_0\|_{L^2(\mathbb{R})}+\sqrt{T}e^{\frac{K}{2}T}\cdot\frac{1}{2}\left\|e^x u_0^{1/m}\right\|_{L^2(\mathbb{R})}\right]\\
&\le C\,(1+a_0^3)(1+T)\,e^{KT}\,\|e^x u_0\|_{L^2(\mathbb{R})}\,.
\end{aligned}
\tag{2.43}
$$

From (2.42) and (2.43), applying Gronwall's lemma, we conclude that

$$
\begin{aligned}
\int e^{2\,\psi_n(x,t)}u_m(t)^2\,dx &\le \exp\left(\int_0^T\beta_m(s)\,ds\right)\int e^{2\,\psi_n(x,0)}u_m(0)^2\,dx\\
&\le C\int e^{2\,\psi_n(x,0)}u_m(0)^2\,dx,
\end{aligned}
\tag{2.44}
$$

where $C = C\left(a_0, T, \|u_0\|_{L^2(\mathbb{R})}, \|e^x u_0\|_{L^2(\mathbb{R})}\right)$.

Let us observe that from (2.12), $e^{\psi_n(x,t)} \to e^{\omega(x)a(t)x_+^{3/2}}$, for every $x \in \mathbb{R}$ when $n \to \infty$. In order to obtain our result we will make $n \to \infty$ and will apply Fatou's lemma on the left hand side of (2.44), for which we will bound the integrand of the right hand side with the integrable function $e^{2a_0 x_+^{3/2}} u_0^2$ of the variable x. For this, it is enough to prove that for $x \geq n$

$$e^{\psi_n(x,0)} = P_n(x,0) \leq e^{a_0 x^{3/2}}. \tag{2.45}$$

Since $P_n(x,0)$ and $e^{a_0 x^{3/2}}$, and their derivatives up to the second order, coincide at $x = n$, to prove (2.45), it is enough to show that

$$\partial_x^2 P_n(x,0) \leq \frac{d^2}{dx^2}\, e^{a_0 x^{3/2}}, \quad \text{for } \ x \geq n. \tag{2.46}$$

With this purpose we observe from (2.28) that

$$\partial_x^2 P_n(x,0) = \left(\frac{3}{4}\, a_0\, n^{-1/2} + \frac{9}{4}\, a_0^2\, n\right) e^{a_0\, n^{3/2}}.$$

Furthermore,

$$\frac{d^2}{dx^2}\, e^{a_0 x^{3/2}} = \left(\frac{3}{4}\, a_0\, x^{-1/2} + \frac{9}{4}\, a_0^2\, x\right) e^{a_0\, x^{3/2}}. \tag{2.47}$$

Now,

$$\begin{aligned}\frac{d^3}{dx^3}\, e^{a_0 x^{3/2}} &= \left(-\frac{3}{8}\, a_0 x^{-3/2} + \frac{9}{8}\, a_0^2 + \frac{9}{4}\, a_0^2 + \frac{27}{8}\, a_0^3 x^{3/2}\right) e^{a_0 x^{3/2}} \\ &= \frac{27}{8}\, a_0^3 x^{3/2} \left(-\frac{1}{9a_0^2 x^3} + \frac{1}{a_0 x^{3/2}} + 1\right) e^{a_0 x^{3/2}} > 0,\end{aligned}$$

if $x > n$ and $n > (9\, a_0^2)^{-1/3} \equiv N(a_0)$.
Therefore the expression in (2.47) is an increasing function of the variable x for $x \geq n$ if $n \geq N(a_0)$. From this fact, (2.46) is concluded and therefore (2.45) follows. In this way, letting $n \to \infty$ in (2.44), applying Fatou's lemma, and using (2.6), we can conclude that

$$\begin{aligned}\int \left(e^{\omega(x)\, a(t)\, x_+^{3/2}} u_m(t)\right)^2 dx &\leq C \int \left(e^{a_0\, x_+^{3/2}} u_m(0)\right)^2 dx && (2.48)\\ &\leq C \int \left(e^{a_0\, x_+^{3/2}} u_0\right)^2 dx, \quad \text{for every } t \in [0,T]. && (2.49)\end{aligned}$$

Since $\omega \geq 0$ and $0 \leq a \leq a_0$, for $x \in [0,1]$ it follows that

$$e^{a(t)\, x^{3/2}} \leq e^{a(t)} \leq e^{a(t)} e^{\omega(x)\, a(t)\, x^{3/2}} \leq e^{a_0} e^{\omega(x)\, a(t)\, x^{3/2}}.$$

Hence $e^{a(t)\, x_+^{3/2}} \leq C\, e^{\omega(x)\, a(t)\, x_+^{3/2}}$, for every $x \in \mathbb{R}$ and every $t \in [0,T]$. Then, from (2.48), we have that

$$\int_{\mathbb{R}} \left(e^{a(t)\, x_+^{3/2}} u_m(t) \right)^2 \, dx \leq C \int_{\mathbb{R}} \left(e^{a_0\, x_+^{3/2}} u_0 \right)^2 \, dx, \tag{2.50}$$

for every $t \in [0,T]$, and for all $m \in \mathbb{N}$.

Finally, from the continuity of the data-solution map for the IVP (1.1) mentioned in (2.11), we have that for fixed t, $u_m(t) \to u(t)$ in $L^2(\mathbb{R})$ as $m \to \infty$. Then, there is a subsequence $u_{m_j}(t)$ such that $u_{m_j}(t)(x) \to u(t)(x)$, for almost every $x \in \mathbb{R}$, when $j \to \infty$. Thus, applying Fatou's lemma on the left hand side of (2.50) for this subsequence, we obtain that

$$\int_{\mathbb{R}} \left(e^{a(t)\, x_+^{3/2}} u(t) \right)^2 \, dx \leq C \int_{\mathbb{R}} \left(e^{a_0\, x_+^{3/2}} u_0 \right)^2 \, dx, \quad \text{for all } t \in [0,T], \tag{2.51}$$

which completes the proof of Theorem I.

☑

Remark 2.2. For the linear problem (1.4), associated to the IVP (1.1), we can obtain a result similar to that in Theorem I. In this case the proof is simpler due to the absence of the nonlinear term. More specifically, under the same assumptions stated in Theorem I, if u is a solution of the linear problem associated to (1.1) and $e^{a_0\, x_+^{3/2}} u(0) \in L^2(\mathbb{R})$, then

$$\left\| e^{a(t)\, x_+^{3/2}} u(t) \right\|_{L^2(\mathbb{R})} \leq C \left\| e^{a_0\, x_+^{3/2}} u(0) \right\|_{L^2(\mathbb{R})}, \tag{2.52}$$

where $C = C(a_0, T)$.

Proof.
To prove this fact it is enough to notice that for the linear problem the term E coming from the nonlinear part of the KdV equation in equality (2.21), does not appear. Thus (2.23) reduces to

$$\frac{1}{2}\frac{d}{dt} \int f^2 \leq \int (\psi_t + \psi_x^3 + \psi_{xxx}) f^2. \tag{2.53}$$

Therefore, from (2.24), (2.35), (2.38), and (2.39), it follows that

$$\frac{1}{2}\frac{d}{dt} \int f^2 \leq C\, \left(1 + a_0^3\right) \int f^2. \tag{2.54}$$

We then apply Gronwall's lemma, and pass to the limit, as we did in the proof of Theorem I, to estimate the $L^2(\mathbb{R})$ norm of $e^{a(t)\,x_+^{3/2}}u(t)$, in terms of the $L^2(\mathbb{R})$ norm of $e^{a_0\,x_+^{3/2}}u(0)$, this gives (2.52). ☑

Chapter 3

Optimal decay: Proof of Theorem II

In the proof of Theorem II we will construct a function of the Schwartz class which satisfies the conclusions of the theorem. For this function it will be important to study the exponential decay of order 3/2 of its first and second derivatives, for which we now present a result similar to Theorem I for such derivatives. It should be noted that it is possible to show this result with weaker conditions on the initial datum than those we impose here, as it was done for Theorem I. However, for our purposes, it will be enough to take the initial datum in the Schwartz space.

Proposition 3.1. *For $T > 0$ and $u_0 \in \mathcal{S}(\mathbb{R})$ let u be the solution on $[0,T]$ of the IVP (1.1) with initial datum u_0.*

(a) *If $a_0 \geq 0$ and $e^{a_0\, x_+^{3/2}} \partial_x u_0 \in L^2(\mathbb{R})$, then there exist constants C and M, such that*

$$\left\| e^{a(t)\, x_+^{3/2}} \partial_x u(t) \right\|_{L^2(\mathbb{R})} \leq e^{MT} \left\| e^{a_0\, x_+^{3/2}} \partial_x u_0 \right\|_{L^2(\mathbb{R})}, \quad \textit{for every } t \in [0,T],$$

where $M = C(1+a_0^3) \sup\limits_{t\in[0,T]} \left[1 + \|(1+x_+^{1/2})u(t)\|_{L^\infty([0,\infty))} + \|\partial_x u(t)\|_{L^\infty(\mathbb{R})} \right]$, and C is an absolute constant.

(b) *If $a_0 \geq 0$ and $e^{a_0\, x_+^{3/2}} \partial_x^2 u_0 \in L^2(\mathbb{R})$, then*

$$\left\| e^{a(t)\, x_+^{3/2}} \partial_x^2 u(t) \right\|_{L^2(\mathbb{R})} \leq e^{MT} \left\| e^{a_0\, x_+^{3/2}} \partial_x^2 u_0 \right\|_{L^2(\mathbb{R})}, \quad \textit{for every } t \in [0,T],$$

where M is as in (a).

Proof.
Since the KdV equation preserves the Schwartz space, we have that $u(t) \in \mathcal{S}(\mathbb{R})$, for every

$t \in [0,T]$. Let us denote $v := \partial_x u$ and $w := \partial_x^2 u$. Taking derivative with respect to x in the KdV equation we obtain

$$\partial_t \partial_x u + \partial_x^4 u + (\partial_x u)^2 + u \partial_x^2 u = 0.$$

Therefore v satisfies the following equation:

$$\partial_t v + \partial_x^3 v + v^2 + u \partial_x v = 0. \tag{3.1}$$

By differentiating with respect to x once more, we obtain

$$\partial_t \partial_x^2 u + \partial_x^5 u + 3\, \partial_x u\, \partial_x^2 u + u \partial_x^3 u = 0.$$

Therefore w satisfies

$$\partial_t w + \partial_x^3 w + 3\, (\partial_x u)\, w + u \partial_x w = 0. \tag{3.2}$$

For part (a), we imitate the proof of Theorem I. Thus, we define $f \equiv f_n := v\, e^{\psi_n} \equiv v\, e^{\psi}$, multiply (3.1) by e^{ψ}, replace $v = e^{-\psi} f$ in (3.1), apply (2.16), multiply by f, and integrate by parts in the variable x over $\mathbb{R}$, to obtain that

$$\frac{1}{2}\frac{d}{dt}\int f^2 \le \int \left(\psi_t + \psi_x^3 + \psi_{xxx}\right) f^2 + \int u \psi_x f^2 - \frac{1}{2}\int \partial_x u\, f^2. \tag{3.3}$$

For part (b), we proceed with (3.2) as we did with (3.1) by defining $f \equiv f_n := w\, e^{\psi_n} \equiv w\, e^{\psi}$, to obtain that

$$\frac{1}{2}\frac{d}{dt}\int f^2 \le \int \left(\psi_t + \psi_x^3 + \psi_{xxx}\right) f^2 + \int u \psi_x f^2 - \frac{5}{2}\int \partial_x u\, f^2. \tag{3.4}$$

The integration by parts processes performed to get (3.3) and (3.4) are justified since for fixed t, $u(t) \in \mathcal{S}(\mathbb{R})$, $e^{\psi(x,t)}$ has polynomial growth for $x > 0$, and $\psi_t, \psi_x, \psi_{xxx}$ are bounded functions of the variable x.

The first two integrals of the right hand side of (3.3) and (3.4) are estimated as we did in the proof of Theorem I, to obtain estimate (2.41). Therefore (3.3) and (3.4) lead to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int f^2 &\le C(1+a_0^3)\left(1 + \|(1+x_+^{1/2})u(t)\|_{L^\infty([0,\infty))} + \|\partial_x u(t)\|_{L^\infty(\mathbb{R})}\right)\int f^2 \\ &\le M\int f^2,\end{aligned} \tag{3.5}$$

where

$$M = C(1+a_0^3)\sup_{t\in[0,T]}\left[1 + \|(1+x_+^{1/2})u(t)\|_{L^\infty([0,\infty))} + \|\partial_x u(t)\|_{L^\infty(\mathbb{R})}\right].$$

Since $u_0 \in \mathcal{S}(\mathbb{R})$, we have that $u \in C([0,T];\mathcal{S}(\mathbb{R}))$, and thus M is finite.

The rest of the proof is as in the proof of Theorem I, when we apply Gronwall's lemma from (3.5), and take limit when $n \to \infty$ to estimate the $L^2(\mathbb{R})$ norm of $e^{a(t)\, x_+^{3/2}} \partial_x u(t)$ and $e^{a(t)\, x_+^{3/2}} \partial_x^2 u(t)$, in terms of the $L^2(\mathbb{R})$ norms of $e^{a_0\, x_+^{3/2}} \partial_x u_0$ and $e^{a_0\, x_+^{3/2}} \partial_x^2 u_0$, respectively. ☑

For the proof of the Theorem II it will be also convenient to study some monotonicity properties of the function

$$(\tau, b) \mapsto g(\tau)(b) := \frac{b}{\sqrt{1 + \frac{27}{4} b^2 \tau}}, \qquad \text{for } \tau, b \geq 0,$$

defined in (1.14). Let us notice that for fixed $\tau \geq 0$, the function $b \mapsto g(\tau)(b)$ is monotone increasing; and for fixed $b \geq 0$, $\tau \mapsto g(\tau)(b)$ is a monotone decreasing function.

We now highlight two properties of the function g.

Remark 3.1. First, if $b_1 < b_2$, then there exists $\eta = \eta(b_1, b_2) > 0$ such that $g(t)(b_1) \leq g(t)(b_2) - \eta$, for every $t \in [0, T]$. Namely, $\eta = g(T)(b_2) - g(T)(b_1)$:

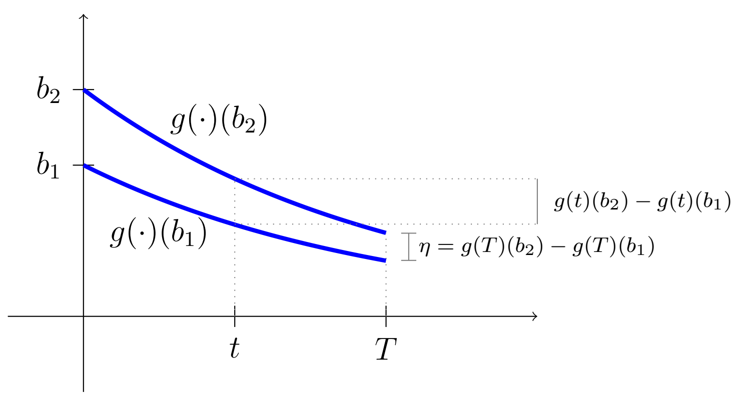

Figure 5

Indeed, to see this, it is enough to notice that the function $G(t) := g(t)(b_2) - g(t)(b_1)$ is a decreasing function on $[0, T]$, for which we use the identity

$$\frac{d}{dt} g(t)(b) = -\frac{27}{8} \left(g(t)(b)\right)^3,$$

which together with the fact that the function $b \mapsto g(t)(b)$ is increasing, implies that

$$G'(t) = \frac{27}{8} \left[\left(g(t)(b_1)\right)^3 - \left(g(t)(b_2)\right)^3 \right] < 0, \quad \text{for every } t \in [0, T].$$

Remark 3.2. Second, if λ, b_1 and b_2 are positive constants such that $\lambda > 1$, $b_1 < b_2$, and $\lambda\, g(T)(b_1) < g(T)(b_2)$, then this last inequality is still valid for every $t \in [0, T]$. That is,

$$\lambda\, g(t)(b_1) < g(t)(b_2), \quad \text{for every } t \in [0, T].$$

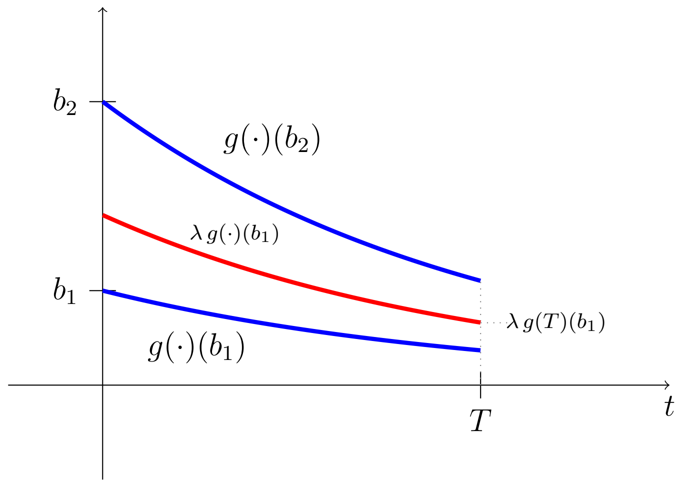

Figure 6

In fact, when we apply the intermediate value theorem to the function $g(T)(\cdot)$, there is $b \in (b_1, b_2)$ such that $g(T)(b) = \lambda\, g(T)(b_1)$. This value of b can be explicitly computed:

$$b = \frac{\lambda\, b_1}{\sqrt{1 + \frac{27}{4} b_1^2 (1-\lambda^2) T}}.$$

Let us fix $t \in [0, T]$ and let us notice that, in virtue of the monotonicity of the function $g(t)(\cdot)$,

$$g(t)(b) < g(t)(b_2). \tag{3.6}$$

Now, we observe that

$$\begin{aligned} g(t)(b) &= \frac{\frac{\lambda\, b_1}{\sqrt{1+\frac{27}{4}b_1^2(1-\lambda^2)T}}}{\sqrt{1 + \frac{27}{4}\left(\frac{\lambda^2\, b_1^2}{1+\frac{27}{4}b_1^2(1-\lambda^2)T}\right) t}} \\ &= \frac{\lambda\, b_1}{\sqrt{1 + \frac{27}{4} b_1^2 (1-\lambda^2) T + \frac{27}{4}\lambda^2 b_1^2 t}} \\ &= \frac{\lambda\, b_1}{\sqrt{1 + \frac{27}{4} b_1^2 t + \frac{27}{4} b_1^2 (1-\lambda^2)(T-t)}} \\ &\geq \frac{\lambda\, b_1}{\sqrt{1 + \frac{27}{4} b_1^2 t}} \\ &= \lambda\, g(t)(b_1), \end{aligned} \tag{3.7}$$

since $\lambda > 1$. From (3.6) and (3.7) it follows that, $\lambda\, g(t)(b_1) \leq g(t)(b) < g(t)(b_2)$.

We now proceed to prove Theorem II which we state again, for convenience.

Theorem II. *For $T > 0$, $a_0 > 0$ and $0 < \epsilon < \frac{1}{3}\, a_0$, there exist $u_0 \in \mathcal{S}(\mathbb{R})$ with $e^{a_0\, x_+^{3/2}} u_0 \in L^2(\mathbb{R})$ and $C > 0$ such that the solution u on $[0,T]$ of the IVP (1.1) with initial datum u_0 satisfies*

$$C\, e^{-g(t)(a_0+\epsilon)\, x^{3/2}} \leq u(t)(x), \qquad \text{for every } t \in [0,T] \text{ and every } x > 0.$$

In particular, $e^{g(t)(a_0+\epsilon)\, x_+^{3/2}}\, u(t) \notin L^2(\mathbb{R})$, for every $t \in [0,T]$.

Proof of Theorem II.
Let us define $a_0^+ := a_0 + \epsilon$ and $a_0^- := a_0 - \epsilon$.

Let us take a function $\varphi \in C_c^\infty(\mathbb{R})$ such that $\varphi \geq 0$, $supp(\varphi) \subset (-1,1)$ and $\int_{\mathbb{R}} \varphi = 1$, and, for $\delta \in (0, 1/2)$ let $\varphi_\delta = \frac{1}{\delta}\varphi\left(\frac{\cdot}{\delta}\right)$.

Let $\{S(t)\}$ be the group associated to the linearized KdV equation, which is defined through the Fourier transform by

$$[S(t)u_0]^{\widehat{}} = e^{i\,t\xi^3}\, \widehat{u_0}.$$

For $\alpha > 0$ small, which will be properly chosen later in the development of the proof, we consider problem (1.1) with initial datum

$$u_{0,\alpha} \equiv u_0 = S(t_0)(\alpha\varphi_\delta),$$

where

$$t_0 = \frac{4}{27(a_0 + \epsilon/3)^2}.$$

Since φ_δ is a function of the Schwartz class, and the group $S(t)$ preserves this class, then $u_0 \in \mathcal{S}(\mathbb{R})$. Besides $u_0 = \alpha S_{t_0} * \varphi_\delta$, where S_t is the fundamental solution (1.7). From the theory of global well-posedness on spaces $H^s(\mathbb{R})$, for $T > 0$, the IVP (1.1) has a unique solution $u_\alpha \equiv u \in C\left([0,T]; \mathcal{S}(\mathbb{R})\right)$, which, given its regularity, satisfies the Duhamel's Formula pointwise; that is,

$$\begin{aligned} u(t) &= S(t)\, u_0 - \int_0^t S(t-\tau)\left(u(\tau)\partial_x u(\tau)\right)\, d\tau \\ &\equiv S(t)u_0 - F(t), \quad \text{for every } t \in [0,T], \end{aligned} \tag{3.8}$$

where the integral in (3.8) is a Bochner integral on any $H^n(\mathbb{R})$ with $n \in \mathbb{N}$, and, for every $x \in \mathbb{R}$ and every $t \in [0,T]$,

$$u(t)(x) = [S(t)\, u_0]\,(x) - \int_0^t [S(t-\tau)\,(u(\tau)\partial_x u(\tau))]\,(x)\, d\tau.$$

Using the properties of the convolution and the asymptotic behavior of the Airy function, given in (1.9), we will prove that for $\delta > 0$ small enough, $x > 1$ and $t \in [0,T]$,

$$\overline{C}\,\alpha\, e^{-g(t)\left(a_0^+\right)x^{3/2}} \le [S(t)u_0]\,(x) \le C\,\alpha\, e^{-g(t)(a_0+\epsilon/4)x^{3/2}}, \tag{3.9}$$

where C and $\overline{C}$ are independent of α and of $t \in [0,T]$.

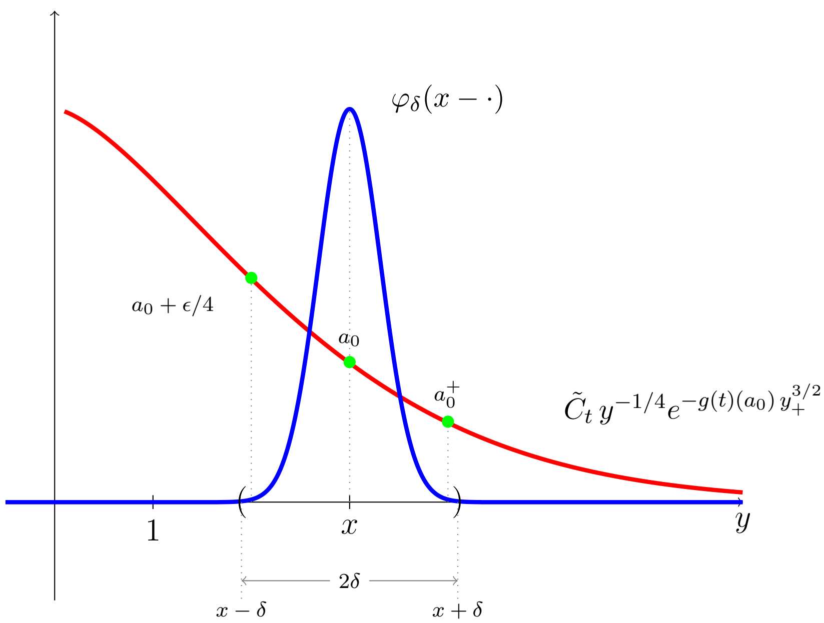

Figure 7

In fact, we have that

$$\begin{aligned} S(t)u_0 &= S(t)\, S(t_0)\,(\alpha\varphi_\delta) = S(t+t_0)(\alpha\varphi_\delta) \\ &= S_{t+t_0} * (\alpha\varphi_\delta) = \alpha\, S_{t_0+t} * \varphi_\delta. \end{aligned}$$

Thus, from the asymptotic behavior of the Airy function in (1.9), and from (1.10), for $x > 1$ and $t \in [0,T]$ it follows that

$$\begin{aligned} [S_{t_0+t} * \varphi_\delta]\,(x) &= \int_{\mathbb{R}} S_{t_0+t}(x-y)\,\varphi_\delta(y)\,dy \\ &\sim C_t \int_{-\delta}^{\delta} (x-y)^{-1/4} e^{-g(t)(a_0+\epsilon/3)\,(x-y)^{3/2}}\varphi_\delta(y)\,dy, \end{aligned} \tag{3.10}$$

where $C_t = C\ (t_0+t)^{-1/4}$.

Now we can estimate $\left[S_{t_0+t} * \varphi_\delta\right](x)$ by taking into account the following remarks:

(i) For $x > 1$, it follows that

$$x - \delta > x - \delta x = x(1-\delta) \ \text{ and } \ x + \delta < x + \delta x = x(1+\delta). \tag{3.11}$$

(ii) For fixed $\tau \geq 0$, $g(\tau)(\cdot)$ is a monotone increasing function; hence,

$$g(\tau)(a_0 + \epsilon/4) \leq g(\tau)(a_0 + \epsilon/3) \leq g(\tau)(a_0 + \epsilon/2) \leq g(\tau)(a_0 + \epsilon).$$

From these properties, using the fact that $x > 1$, $0 < \delta < 1/2$, and $t \geq 0$, we have that

$$\begin{aligned}
\left[S(t)u_0\right](x) &= \alpha\left[S_{t_0+t} * \varphi_\delta\right](x) \\
&\leq C_t\,\alpha \int_{-\delta}^{\delta} (x-y)^{-1/4} e^{-[g(t)(a_0+\epsilon/3)]\,(x-y)^{3/2}} \varphi_\delta(y)\,dy \\
&\leq C_0\,\alpha\,(x-\delta)^{-1/4} e^{-[g(t)(a_0+\epsilon/3)]\,(x-\delta)^{3/2}} \int_{-\delta}^{\delta} \varphi_\delta(y)\,dy \\
&= C\alpha\,(x-\delta)^{-1/4} e^{-[g(t)(a_0+\epsilon/3)]\,(x-\delta)^{3/2}} \cdot 1 \\
&\leq C\alpha \left(\frac{x}{2}\right)^{-1/4} e^{-[g(t)(a_0+\epsilon/3)]\,(1-\delta)^{3/2} x^{3/2}} \\
&\leq C\alpha\, e^{-[g(t)(a_0+\epsilon/3)]\,(1-\delta)^{3/2} x^{3/2}},
\end{aligned} \tag{3.12}$$

where C is independent of $t \in [0, T]$.

We chose $\delta \in (0, 1/2)$ small enough to have that

$$(1-\delta)^{3/2} g(T)(a_0 + \epsilon/3) > g(T)(a_0 + \epsilon/4). \tag{3.13}$$

In view of Remark 3.2 (with $\lambda = (1-\delta)^{-3/2} > 1$) it follows that

$$(1-\delta)^{3/2} g(t)(a_0 + \epsilon/3) > g(t)(a_0 + \epsilon/4), \quad \text{for every } t \in [0, T].$$

Therefore,

$$\begin{aligned}
\left[S(t)u_0\right](x) &\leq C\alpha\, e^{-[g(t)(a_0+\epsilon/3)]\,(1-\delta)^{3/2} x^{3/2}} \\
&\leq C\alpha\, e^{-[g(t)(a_0+\epsilon/4)]\,x^{3/2}}.
\end{aligned} \tag{3.14}$$

On the other hand, from (3.10) and (3.11), it follows that

$$\begin{aligned}
[S(t)u_0](x) &= \alpha\,[S_{t_0+t} * \varphi_\delta](x) \\
&\geq C_T\,\alpha \int_{-\delta}^{\delta} (x-y)^{-1/4} e^{-[g(t)(a_0+\epsilon/3)]\,(x-y)^{3/2}} \varphi_\delta(y)\,dy \\
&\geq C_T\,\alpha\,(x+\delta)^{-1/4} e^{-[g(t)(a_0+\epsilon/3)]\,(x+\delta)^{3/2}} \int_{-\delta}^{\delta} \varphi_\delta(y)\,dy \\
&\geq C_T\,\alpha\,(1+\delta)^{-1/4} x^{-1/4} e^{-[g(t)(a_0+\epsilon/3)]\,(1+\delta)^{3/2} x^{3/2}} \\
&\geq C_T\,\alpha \left(\frac{3}{2}\right)^{-1/4} x^{-1/4} e^{-[g(t)(a_0+\epsilon/2)-\eta]\,(1+\delta)^{3/2} x^{3/2}},
\end{aligned} \tag{3.15}$$

where, according to Remark 3.1, $\eta = \eta(a_0, T, \epsilon) > 0$ is such that

$$g(t)(a_0+\epsilon/3) < g(t)(a_0+\epsilon/2) - \eta, \qquad \text{for every } t \in [0,T].$$

In order to deal with the term $x^{-1/4}$ in (3.15) we use the fact that the function $\theta \mapsto \theta^{1/4} e^{-\theta^{3/2}}$, $\theta \geq 0$, is bounded, and therefore

$$x^{1/4} e^{-\eta\, x^{3/2}} = \eta^{-1/6} (\eta^{2/3} x)^{1/4} e^{-(\eta^{2/3} x)^{3/2}} \leq C\,\eta^{-1/6}.$$

Thus, $x^{-1/4} e^{\eta\,(1+\delta)^{3/2}\,x^{3/2}} \geq x^{-1/4} e^{\eta\, x^{3/2}} \geq \dfrac{\eta^{1/6}}{C}$.
In this way, from (3.15),

$$[S(t)u_0](x) \geq \overline{C}\,\alpha\, e^{-[g(t)(a_0+\epsilon/2)]\,(1+\delta)^{3/2} x^{3/2}},$$

where $\overline{C} = \overline{C}(a_0, T, \epsilon)$.

In addition to the condition for δ stated in (3.13), we chose $\delta \in (0, 1/2)$ small enough in such a way that

$$(1+\delta)^{3/2} g(T)(a_0+\epsilon/2) < g(T)(a_0+\epsilon).$$

In view of Remark 3.2 (with $\lambda = (1+\delta)^{3/2} > 1$) it follows that

$$(1+\delta)^{3/2} g(t)(a_0+\epsilon/2) < g(t)(a_0+\epsilon), \quad \text{for every } t \in [0,T].$$

Therefore,

$$\begin{aligned}
[S(t)u_0](x) &\geq \overline{C}\,\alpha\, e^{-g(t)(a_0+\epsilon)\,x^{3/2}} \\
&= \overline{C}\,\alpha\, e^{-g(t)(a_0^+)\,x^{3/2}},
\end{aligned} \tag{3.16}$$

where $\overline{C}$ does not depend upon $t \in [0,T]$, $x > 1$ and $\alpha > 0$.

Hence (3.9) follows from (3.14) and (3.16).

Let us observe that, applying (3.14) with $t=0$, for u_0 we obtain that

$$|u_0(x)| \leq C\,\alpha\, e^{-(a_0+\epsilon/4)\,x^{3/2}}, \qquad x>1.$$

Then,

$$e^{a_0\,x^{3/2}}\,|u_0(x)| \leq C\,\alpha\, e^{-\frac{\epsilon}{4}\,x^{3/2}}, \quad \text{for } x>1. \tag{3.17}$$

Besides $\|u_0\|_{L^2((-\infty,1])} \leq \|u_0\|_{L^2(\mathbb{R})} = \|\alpha S(t_0)\varphi_\delta\|_{L^2(\mathbb{R})} = \alpha\|\varphi_\delta\|_{L^2(\mathbb{R})}$. From this fact, (3.17), and taking into account that $e^{a_0\,x_+^{3/2}} \leq e^{a_0}$ for $x\leq 1$, we conclude that

$$\|e^{a_0 x_+^{3/2}}u_0\|_{L^2(\mathbb{R})} \leq C\,\alpha, \tag{3.18}$$

where C is independent of α.

Our next step is to show that the integral term $F(t)$ in (3.8) decays as $\alpha^2 e^{-\beta\,x^{3/2}}$, for some $\beta > g(t)\left(a_0^+\right)$, for every $t\in[0,T]$.

Let us fix a_1 and a_2 such that $a_0^- < a_2 < a_1 < a_0$. Taking into account that $\partial_x^2 u_0 = \alpha\,\partial_x^2 S_{t_0} * \varphi_\delta = \alpha\, S_{t_0} * \partial_x^2\varphi_\delta$, and noticing that

$$\int |\partial_x^2\varphi_\delta| = \frac{1}{\delta^2}\int \left|\frac{1}{\delta}\varphi_\delta''\left(\frac{y}{\delta}\right)\right|\,dy \leq \frac{C}{\delta^2},$$

we can imitate for $|\alpha\, S(t_0)\partial_x^2\varphi_\delta|$ the procedure we followed to obtain (3.14) and (3.17) to conclude that $\|e^{a_0 x_+^{3/2}}\partial_x^2 u_0\|_{L^2(\mathbb{R})} \leq C\,\alpha$. Therefore, when we apply part (b) of Proposition 3.1 we obtain that

$$\begin{aligned}\left\|e^{g(t)(a_0)x_+^{3/2}}\,\partial_x^2 u(t)\right\|_{L^2(\mathbb{R})} &\leq e^{M_\alpha T}\,\|e^{a_0 x_+^{3/2}}\partial_x^2 u_0\|_{L^2(\mathbb{R})}\\ &\leq C\,\alpha\, e^{M_\alpha T}, \quad \text{for all}\, t\in[0,T],\end{aligned} \tag{3.19}$$

where $M_\alpha = C(1+a_0^3)\sup_{t\in[0,T]}\|u_\alpha(t)\|_{1,1}$, and $\|\cdot\|_{1,1}$ is the Schwartz semi-norm defined by

$$\|h\|_{1,1} = \|(1+x^2)^{1/2}h\|_{L^\infty(\mathbb{R})} + \|\partial_x h\|_{L^\infty(\mathbb{R})}.$$

Here C is independent of α and $t\in[0,T]$.

It is important to observe now that M_α is bounded by a constant M independent of $\alpha\in[0,1]$. To see this, we compose the function $\alpha\mapsto\alpha\,\varphi_\delta$, which is continuous from $[0,1]$ into $\mathcal{S}(\mathbb{R})$, with the data solution map from $\mathcal{S}(\mathbb{R})$ into $C([0,T];\mathcal{S}(\mathbb{R}))$, which is also continuous. Since the interval $[0,1]$ is compact we have that M_α is bounded by a constant M independent of $\alpha\in[0,1]$. Thus, from (3.19) we have that

$$\left\|e^{g(t)(a_0)x_+^{3/2}}\,\partial_x^2 u(t)\right\|_{L^2(\mathbb{R})} \leq C\,\alpha, \tag{3.20}$$

where C does not depend on $\alpha \in [0,1]$ and $t \in [0,T]$.

Next, we will use the fact that $u(t) \in \mathcal{S}(\mathbb{R})$, the Fundamental Theorem of Calculus, and Cauchy-Schwarz inequality to estimate $\partial_x u(t)(x)$, for $t \in [0,T]$ and $x > 1$:

$$\begin{aligned}
|\partial_x u(t)(x)| &\leq \int_x^\infty \left|\partial_x^2 u(t)(y)\right| \, dy \\
&= \int_x^\infty \left|e^{g(t)(a_0)y^{3/2}} \partial_x^2 u(t)(y)\right| \, e^{-g(t)(a_0)y^{3/2}} \, dy \\
&\leq \left[\int_x^\infty \left|e^{g(t)(a_0)y^{3/2}} \partial_x^2 u(t)(y)\right|^2 \, dy\right]^{1/2} \left[\int_x^\infty e^{-2\, g(t)(a_0)y^{3/2}} \, dy\right]^{1/2}.
\end{aligned}$$

From Remark 3.1, there exists $\eta_1 > 0$ such that $g(t)(a_1) \leq g(t)(a_0) - \eta_1$, for every $t \in [0,T]$. Then, from (3.20)

$$\begin{aligned}
|\partial_x u(t)(x)| &\leq \left\|e^{g(t)(a_0)y_+^{3/2}} \partial_x^2 u(t)\right\|_{L^2(\mathbb{R})} \left(e^{-2\, g(t)(a_1)x^{3/2}} \int_x^\infty e^{-2\eta_1\, y^{3/2}} \, dy\right)^{1/2} \\
&\leq C\, \alpha\, e^{-g(t)(a_1)\, x^{3/2}},
\end{aligned} \tag{3.21}$$

with C independent of α and of $t \in [0,T]$.

Again, by the Fundamental Theorem of Calculus and from (3.21), for $x > 1$ we have that

$$\begin{aligned}
|u(t)(x)| \leq \int_x^\infty |\partial_x u(t)(y)| \, dy &\leq C\, \alpha \int_x^\infty e^{-g(t)(a_1)y^{3/2}} \, dy \\
&\leq C\, \alpha\, e^{-g(t)(a_2)\, x^{3/2}} \int_x^\infty e^{-\eta_2 y^{3/2}} \, dy \\
&\leq C\, \alpha\, e^{-g(t)(a_2)\, x^{3/2}},
\end{aligned} \tag{3.22}$$

where $\eta_2 > 0$ is such that $g(t)(a_2) \leq g(t)(a_1) - \eta_2$, for every $t \in [0,T]$, and C is independent of α and of $t \in [0,T]$.

Let us recall that the integral term $F(t)$ in (3.8) is given by

$$F(t) = \int_0^t S(t-\tau)\left(u(\tau)\partial_x u(\tau)\right) \, d\tau, \qquad t \in [0,T]. \tag{3.23}$$

In order to estimate $F(t)$, we first analyze $\partial_x F(t)$:

$$\partial_x F(t) = \int_0^t S(t-\tau) f(\tau) \, d\tau, \tag{3.24}$$

where $f(\tau) \equiv \partial_x \left(u(\tau)\partial_x u(\tau)\right) = \left(\partial_x u(\tau)\right)^2 + u(\tau)\partial_x^2 u(\tau)$.

First, using (3.21), we notice that for $\tau \in [0,t]$,

$$
\begin{aligned}
\int_1^\infty \left| e^{2\,g(\tau)(a_2)\,y^{3/2}}\,[\partial_x u(\tau)(y)]^2 \right|^2 dy &\le C \int_1^\infty \left| e^{2\,g(\tau)(a_2) y^{3/2}}\,\alpha e^{-2\,g(\tau)(a_1) y^{3/2}} \right|^2 dy \\
&\le C\,\alpha^2 \int_1^\infty \left| e^{2\,g(\tau)(a_2) y^{3/2}}\, e^{-2\,g(\tau)(a_2) y^{3/2}}\, e^{-2\,\eta_2 y^{3/2}} \right|^2 dy \\
&\le C\,\alpha^2 \int_1^\infty e^{-4\,\eta_2\,y^{3/2}}\,dy \le C\,\alpha^2. \qquad (3.25)
\end{aligned}
$$

Besides, from (3.20), (3.22), and part (b) of Proposition 3.1 we obtain that

$$
\begin{aligned}
\int_1^\infty \left| e^{2\,g(\tau)(a_2) y^{3/2}}\,u(\tau)(y)\partial_x^2 u(\tau)(y) \right|^2 dy &= \int_1^\infty \left| e^{\,g(\tau)(a_2) y^{3/2}}\,u(\tau)(y) \right|^2 \left| e^{\,g(\tau)(a_2) y^{3/2}}\,\partial_x^2 u(\tau)(y) \right|^2 dy \\
&\le C\,\alpha^2 \int_1^\infty \left| e^{\,g(\tau)(a_2) y^{3/2}}\,\partial_x^2 u(\tau)(y) \right|^2 dy \\
&\le C\,\alpha^2 \int_1^\infty \left| e^{\,g(\tau)(a_0) y^{3/2}}\,\partial_x^2 u(\tau)(y) \right|^2 dy \le C\,\alpha^2,
\end{aligned}
\tag{3.26}
$$

where the constants in (3.25) y (3.26) are independent of τ and t.

We now include in our estimates the values of $x \in (-\infty, 1]$. For that, we use the fact that the KdV equation preserves the L^2 norm, that is, $\|u(\tau)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$, and apply proposition 3.1 with $a_0 = 0$, and the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ to obtain that

$$
\begin{aligned}
\|f(\tau)\|_{L^2(\mathbb{R})} &= \left\| (\partial_x u(\tau))^2 + u(\tau)\partial_x^2 u(\tau) \right\|_{L^2(\mathbb{R})} \\
&\le \left\| (\partial_x u(\tau))^2 \right\|_{L^2(\mathbb{R})} + \left\| u(\tau)\partial_x^2 u(\tau) \right\|_{L^2(\mathbb{R})} \\
&\le \|\partial_x u(\tau)\|_{L^\infty(\mathbb{R})} \|\partial_x u(\tau)\|_{L^2(\mathbb{R})} + \|u(\tau)\|_{L^\infty(\mathbb{R})} \left\|\partial_x^2 u(\tau)\right\|_{L^2(\mathbb{R})} \\
&\le C \left(\|\partial_x u(\tau)\|_{L^2(\mathbb{R})} + \left\|\partial_x^2 u(\tau)\right\|_{L^2(\mathbb{R})} \right) \|\partial_x u(\tau)\|_{L^2(\mathbb{R})} \\
&\quad + C \left(\|u(\tau)\|_{L^2(\mathbb{R})} + \|\partial_x u(\tau)\|_{L^2(\mathbb{R})} \right) \left\|\partial_x^2 u(\tau)\right\|_{L^2(\mathbb{R})} \\
&\le C \left(\|\partial_x u_0\|_{L^2(\mathbb{R})} + \left\|\partial_x^2 u_0\right\|_{L^2(\mathbb{R})} \right) \|\partial_x u_0\|_{L^2(\mathbb{R})} \\
&\quad + C \left(\|u_0\|_{L^2(\mathbb{R})} + \|\partial_x u_0\|_{L^2(\mathbb{R})} \right) \left\|\partial_x^2 u_0\right\|_{L^2(\mathbb{R})} \\
&\le C\,\alpha^2,
\end{aligned}
\tag{3.27}
$$

where C does not depend upon α and $\tau \in [0,t]$, $t \in [0,T]$.

Hence, from (3.25), (3.26), and (3.27), it follows that

$$
\left\| e^{2\,g(\tau)(a_2)\,x_+^{3/2}}\,f(\tau) \right\|_{L^2(\mathbb{R})} \le C\,\alpha^2,
$$

where C is independent of τ and t, $0 \le \tau \le t \le T$.

When we apply Remark 2.2 from the previous chapter (linear problem) with initial datum $f(\tau)$ and weight $e^{2\,g(\tau)(a_2)\,x_+^{3/2}}$ we obtain that

$$\left\| e^{g(t-\tau)[2\,g(\tau)(a_2)]\,x_+^{3/2}}\, S(t-\tau)f(\tau) \right\|_{L^2(\mathbb{R})} \le C\,\alpha^2, \tag{3.28}$$

where C is independent of α, τ and t, $0 \le \tau \le t \le T$.

Let us observe now that $g(t)(2\,a_2) \le g(t-\tau)\,(2\,g(\tau)(a_2))$, for every τ and t with $0 \le \tau \le t \le T$. In fact,

$$\begin{aligned} g(t-\tau)\,(2\,g(\tau)(a_2)) &= g(t-\tau)\left(\frac{2a_2}{\sqrt{1+\frac{27}{4}a_2^2\tau}}\right) \\ &= \frac{\frac{2a_2}{\sqrt{1+\frac{27}{4}a_2^2\tau}}}{\sqrt{1+\frac{27}{4}\left(\frac{4a_2^2}{1+\frac{27}{4}a_2^2\tau}\right)(t-\tau)}} \\ &= \frac{2a_2}{\sqrt{1+\frac{27}{4}a_2^2\tau+27a_2^2(t-\tau)}} \\ &= \frac{2a_2}{\sqrt{1+27a_2^2t-\frac{81}{4}a_2^2\tau}} \\ &\ge \frac{2a_2}{\sqrt{1+27a_2^2t}} = g(t)(2\,a_2). \end{aligned} \tag{3.29}$$

Consequently, from (3.28) and (3.29) it follows that

$$\left\| e^{g(t)(2a_2)\,x_+^{3/2}}\, S(t-\tau)f(\tau) \right\|_{L^2(\mathbb{R})} \le C\,\alpha^2, \tag{3.30}$$

where C is independent of τ and t, $0 \le \tau \le t \le T$.

In virtue of the Fundamental Theorem of Calculus, Fubini's Theorem, Cauchy-Schwarz inequality, (3.24), and (3.30) we obtain that for $x \ge 0$

$$
\begin{aligned}
|F(t)(x)| &\le \int_x^\infty |\partial_x F(t)(y)|\, dy = \int_x^\infty \left| \left[\int_0^t S(t-\tau)\, f(\tau)\, d\tau \right](y) \right| dy \\
&= \int_x^\infty \left| \int_0^t \left[S(t-\tau)\, f(\tau) \right](y)\, d\tau \right| dy \\
&\le \int_x^\infty \int_0^t \left| \left[S(t-\tau)\, f(\tau) \right](y) \right| d\tau\, dy \\
&= \int_0^t \int_x^\infty \left| \left[S(t-\tau)\, f(\tau) \right](y) \right| dy\, d\tau \\
&= \int_0^t \int_x^\infty \left| e^{g(t)(2a_2)y^{3/2}} \left[S(t-\tau)\, f(\tau) \right](y) \right| e^{-g(t)(2a_2)y^{3/2}}\, dy\, d\tau \\
&\le \int_0^t \left\| e^{g(t)(2a_2)y^{3/2}} S(t-\tau)\, f(\tau) \right\|_{L^2(\mathbb{R})} \left(\int_x^\infty e^{-2\, g(t)(2a_2)\, y^{3/2}}\, dy \right)^{1/2} d\tau \\
&\le C\, \alpha^2\, T \left(\int_x^\infty e^{-2\, g(t)(2a_2)\, y^{3/2}}\, dy \right)^{1/2} \\
&\le C\, \alpha^2\, T\, e^{-g(t)(2a_0^-)\, x^{3/2}} \left(\int_x^\infty e^{-2\, \eta_3\, y^{3/2}}\, dy \right)^{1/2} \\
&\le C\, \alpha^2\, T\, e^{-g(t)(2a_0^-)\, x^{3/2}},
\end{aligned} \tag{3.31}
$$

where $\eta_3 > 0$ is such that, $g(t)(2a_0^-) \le g(t)(2a_2) - \eta_3$, for every $t \in [0,T]$.

Let us notice that if $\epsilon < \frac{1}{3}\, a_0$, then $2a_0^- > a_0^+$. Therefore, from (3.31)

$$
F(t)(x) \le C\, \alpha^2 e^{-g(t)(a_0^+)\, x_+^{3/2}}, \tag{3.32}
$$

with C independent of $x > 1$, $t \in [0,T]$, and of α.

Then, from (3.8), (3.9) and (3.32), it follows that for $x > 0$,

$$
u(t)(x) \ge \overline{C}\, \alpha e^{-g(t)(a_0^+)\, x^{3/2}} - C\, \alpha^2 e^{-g(t)(a_0^+)\, x^{3/2}},
$$

where C and $\overline{C}$ do not depend upon $x > 0$, $t \in [0,T]$, and $\alpha > 0$. Thus, by taking $\alpha = \overline{C}/2C$, we obtain that, for $x > 0$

$$
u(t)(x) \ge \frac{\overline{C}^2}{4C} e^{-g(t)(a_0^+)x^{3/2}},
$$

which concludes the proof of Theorem II.

☑

Part II

POISSON BRACKETS IN ALGEBRAIC GEOMETRY

Chapter 4

Foundations on Geometry and Mechanics

We begin with some preliminary in symplectic tensors and then study some properties of these structures in the context of smooth manifolds. With this tool we can therefore analyze one of its more important applications in the field of theoretical physics: *Hamiltonian formalism* in *classical mechanics*, and end with a central result in this discipline: the *Noether theorem*. A more thorough discussion of the subject can be seen in [Ar], [CaWe], [GuiSte], [Lee] and [LiMa], which have also been references to the following presentation.

4.1 Basic notions on Symplectic Algebra

Let $\mathcal{V}$ be a vector space. A 2-covector ω on $\mathcal{V}$ is said to be *non-degenerate* if for every nonzero vector $v \in \mathcal{V}$, there exists $w \in \mathcal{V}$ such that $\omega(v, w) \neq 0$. It can be proved that this definition is equivalent to:

- ⋄ The linear map $\Phi : \mathcal{V} \to \mathcal{V}^*$, defined by $\Phi(v) = \iota_v \omega = \omega(v, \cdot)$ is a vector space isomorphism.
- ⋄ In terms of some basis for $\mathcal{V}$, the matrix $(\omega_{jk})_{j,k}$ (which represents to ω in this basis) is invertible. If this is the case, this property is in fact independent of the chosen basis for $\mathcal{V}$.

The 2-covector ω is called a *symplectic tensor*, and we say that $\mathcal{V}$ is endowed with a *symplectic structure* or that $(\mathcal{V}, \omega)$ is a *symplectic vector space*.

Example 4.1.1. Let $\mathcal{V}$ be a real vector space of dimension $2n$, and let us fix a basis $\{A_1, B_1, \ldots, A_n, B_n\}$ for $\mathcal{V}$. Let $\{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n\}$ be the corresponding dual basis, for $\mathcal{V}^*$, and $\omega \in \bigwedge^2(\mathcal{V}^*)$ be the 2-covector defined by

$$\omega = \sum_{j=1}^{n} \alpha_j \wedge \beta_j.$$

Let us notice that

$$\begin{aligned} \omega(A_j, A_k) &= \omega(B_j, B_k) = 0, \\ \omega(A_j, B_k) &= -\omega(B_k, A_j) = \delta_{jk}. \end{aligned}$$

If $v = \sum_{j=1}^{n}(a_j A_j + b_j B_j) \in \mathcal{V}$ is such that $\omega(v, w) = 0$ for every $w \in \mathcal{V}$, then

$$a_j = \omega(v, B_j) = 0, \quad \text{and} \quad b_j = -\omega(v, A_j) = 0,$$

for every $j = 1, \ldots, n$. Thus $v = 0$. Hence ω is non-degenerate. We conclude that $(\mathcal{V}, \omega)$ is a symplectic vector space.

Example 4.1.2. Let $\mathcal{V}$ be a vector space of dimension n, and $\mathcal{V}^*$ its dual space. We define a natural symplectic structure on the product space $\mathcal{V} \times \mathcal{V}^*$ by

$$\omega((v_1, \xi_1), (v_2, \xi_2)) := \langle \xi_2, v_1 \rangle - \langle \xi_1, v_2 \rangle,$$

where $(v_1, \xi_1), (v_2, \xi_2) \in \mathcal{V} \times \mathcal{V}^*$ and $\langle \, , \rangle$ denotes the *duality pairing* of $\mathcal{V}$ with $\mathcal{V}^*$.

In what follows, we point out some important facts about symplectic vector spaces. For this purpose we make the following definition.

Definition 4.1.3. Let $(\mathcal{V}, \omega)$ be a symplectic vector space and $\mathcal{S} \subseteq \mathcal{V}$ be a vector subspace. The *symplectic complement* of $\mathcal{S}$ is the vector subspace

$$\mathcal{S}^{\perp} := \{v \in \mathcal{V} \mid \omega(v, w) = 0, \text{ for every } w \in \mathcal{S}\}.$$

The following lemma justifies the label *complement* in the name of $\mathcal{S}^{\perp}$.

Lemma 4.1.4. *With the above notation, $dim(\mathcal{S}) + dim(\mathcal{S}^{\perp}) = dim(\mathcal{V})$.*

Proof.

Let us consider the linear map $\Psi : \mathcal{V} \to \mathcal{S}^*$, defined by $\Psi(v) := \iota_v \omega|_S$, that is, $\Psi(v)(w) = \omega(v, w)$, for every $v \in \mathcal{V}$ and $w \in \mathcal{S}$. Suppose that $f \in \mathcal{S}^*$ and let $\tilde{f} \in \mathcal{V}^*$ be an extension of f to a linear functional on $\mathcal{V}$. Since $\Phi : \mathcal{V} \to \mathcal{V}^*$ is an isomorphism, then there exists

$v \in \mathcal{V}$ in such a way that $\Phi(v) = \tilde{f}$. Then, restricting to $\mathcal{S}^*$, $\Psi(v) = f$. Hence Ψ is a surjective map.
In virtue of the Rank-nullity theorem, and taking into account that $\mathcal{S}^\perp = Ker(\Psi)$, it follows that

$$\begin{aligned} dim(\mathcal{S}^\perp) &= dim(Ker(\Psi)) = dim(\mathcal{V}) - dim(\mathcal{S}^*) \\ &= dim(\mathcal{V}) - dim(\mathcal{S}). \end{aligned}$$

☑

The next proposition can be considered as a symplectic version of the well-known *Gram-Schmidt process.*

Proposition 4.1.5 (Canonical form for a symplectic tensor). *Let ω be a symplectic tensor on a vector space $\mathcal{V}$ over $\mathbb{R}$, of dimension m. Then $\mathcal{V}$ has even dimension $m = 2n$, and there exists a basis $\{A_1, B_1, \ldots, A_n, B_n\}$ for $\mathcal{V}$ such that*

$$\omega = \sum_{j=1}^{n} \alpha_j \wedge \beta_j,$$

where $\{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n\}$ is the corresponding dual basis.
The set $\{A_1, B_1, \ldots, A_n, B_n\}$ with his property is called a symplectic basis for $\mathcal{V}$.

Proof.

We proceed by induction on m, proving that there exists a basis $\{A_1, B_1, \ldots, A_n, B_n\}$ for $\mathcal{V}$, such that

$$\omega(A_j, A_k) = \omega(B_j, B_k) = 0, \quad \text{and} \quad \omega(A_j, B_k) = \delta_{jk}.$$

For $m = 0$, there is nothing to prove. Let us suppose that this proposition holds true for $0 < k < m$.
Let $(\mathcal{V}, \omega)$ be a symplectic vector space of dimension m. Since $m > 0$, we can take $A_1 \in \mathcal{V} - \{0\}$. In addition, since ω is non-degenerate, there exists $B_1 \in \mathcal{V}$ such that $\omega(A_1, B_1) \neq 0$, and this B_1 can be taken in such a way that $\omega(A_1, B_1) = 1$. Notice that the set $\{A_1, B_1\}$ is linearly independent, so $dim(\mathcal{V}) \geq 2$. Let $\mathcal{S}$ be the vector subspace spanned by A_1 and B_1. Hence, from the previous lemma, $dim(\mathcal{S}^\perp) = m - 2$, and it can be easily verified that $(\mathcal{S}^\perp, \omega|_{\mathcal{S}^\perp})$ is a symplectic vector space. Applying the induction hypothesis we get that $dim(\mathcal{S}^\perp) = 2(n-1)$ and there exists a basis $\{A_2, B_2, \ldots, A_n, B_n\}$ for $\mathcal{S}^\perp$, with the required properties. Finally, $\{A_1, B_1, A_2, B_2, \ldots, A_n, B_n\}$ is the basis which satisfies the statement. ☑

Proposition 4.1.6. *Let $\mathcal{V}$ be a vector space of dimension $2n$, and $\omega \in \bigwedge^2(\mathcal{V}^*)$. Then ω is a symplectic tensor if and only if $\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$.*

Proof.

Suppose that ω is a symplectic tensor. Let $\{A_j, B_j\}_{j=1}^n$ be a symplectic basis for $\mathcal{V}$. Then ω can be written as $\omega = \sum_{j=1}^n \alpha_j \wedge \beta_1$, where $\{\alpha_j, \beta_j\}_{j=1}^n$ is the corresponding dual basis. Hence, we verify that ω^n is a volume form:

$$\omega^n = n!(\alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n) \neq 0.$$

Now suppose that ω is degenerate. Then, there exists $v \in \mathcal{V} - \{0\}$ such that $\iota_v\omega = \omega(v, \cdot)$ is the trivial linear map. Since ι is an antiderivation, $\iota_v\omega^n = n(\iota_v\omega) \wedge \omega^{n-1} = 0$. We can extend v to a basis $\{e_j\}_{j=1}^{2n}$, where $e_1 = v$, and such that $\omega^n(e_1, \ldots, e_{2n}) = 0$. Hence $\omega^n = 0$. ☑

4.2 Symplectic Geometry

4.2.1 Smooth Manifolds

Definition 4.2.1. An n-dimensional smooth manifold is a Hausdorff, second countable topological space M together with a collection of open sets $\{U_\alpha\}_{\alpha\in\Lambda}$, called the coordinate charts, such that

- ⋄ The open sets U_α (labeled by a countable set Λ) cover M.
- ⋄ There exist homeomorphisms $\varphi_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha \subseteq \mathbb{R}^n$, such that for any pair of overlapping coordinate charts U_α and U_β the maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

  are smooth (that is, infinitely differentiable) functions in the usual sense of $\mathbb{R}^n$.

Thus, a smooth manifold is a topological space together with an additional structure which makes the differential calculus possible.

As examples of smooth manifolds we find $\mathbb{R}^n$ (*the local model*), which can be covered by one coordinate chart. Also, if M is any smooth manifold and U is a non-empty open subset of M, then U inherits a smooth structure from M in a natural way.

A less trivial example is the n-dimensional unit sphere

$$\mathbb{S}^n := \{r \in \mathbb{R}^{n+1} \mid \|r\| = 1\}.$$

To see that $\mathbb{S}^n$ is indeed a smooth manifold, take the open sets

$$U := \mathbb{S}^n - \{(0, \ldots, 0, 1)\}, \quad \text{and} \quad \tilde{U} := \mathbb{S}^n - \{(0, \ldots, 0, -1)\},$$

which cover $\mathbb{S}^n$, and define local coordinates by *stereographic projections*:

$$\varphi(r_1, \ldots, r_n, r_{n+1}) = \left(\frac{r_1}{1 - r_{n+1}}, \ldots, \frac{r_n}{1 - r_{n+1}}\right) = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$
$$\tilde{\varphi}(r_1, \ldots, r_n, r_{n+1}) = \left(\frac{r_1}{1 + r_{n+1}}, \ldots, \frac{r_n}{1 + r_{n+1}}\right) = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Taking into account that

$$\frac{r_j}{1 + r_{j+1}} = \frac{1 - r_{j+1}}{1 + r_{j+1}} \, \frac{r_j}{1 - r_{j+1}}, \quad \text{for } j = 1, \ldots, n,$$

where $r_{n+1} \neq \pm 1$, it follows that on $U \cap \tilde{U}$, the transition functions

$$\varphi \circ \tilde{\varphi}^{-1}(x_1, \ldots, x_n) = \left(\frac{x_1}{x_1^2 + \cdots + x_n^2}, \ldots, \frac{x_n}{x_1^2 + \cdots + x_n^2}\right),$$

are smooth.

Next, we introduce the notion of the *tangent bundle* and later constructions that can be obtained from this object.

In a more algebraic way, we define a tangent vector v_p at p on M as a real-valued pointwise derivation on the space of germs of smooth functions defined on a neighborhood of p. That is, given real-valued smooth functions f and g, defined on some neighborhood of p, we have:

- ◇ v_p is a linear map over $\mathbb{R}$,
- ◇ $v_p(f) = v_p(g)$, if $f = g$ on some neighborhood of p,
- ◇ v_p satisfies the Leibniz's rule:

$$v_p(f\,g) = f(p)\, v_p(g) + g(p)\, v_p(f).$$

A more geometric way to define tangent vectors is as infinitesimal curves in a space. We will not explore the latter approach in this work.

The set of tangent vectors to M at the point p is denoted by T_pM. It turns out that T_pM is an n-dimensional vector space over $\mathbb{R}$. In local coordinates, if $(U, \varphi = (x_1, \ldots, x_n))$ is a chart for M at p, we have special tangent vectors $\frac{\partial}{\partial x_j}\Big|_p$, defined by

$$\frac{\partial}{\partial x_j}\Big|_p (f) := \frac{\partial}{\partial r_j}(f \circ \varphi^{-1})\Big|_{\varphi(p)} \qquad \text{for } j = 1, \ldots, n,$$

where f is a smooth function defined near p and $r_1, \ldots, r_n$ denote the standard coordinates for $\mathbb{R}^n$. In fact, $\left\{ \frac{\partial}{\partial x_j}\Big|_p \mid j = 1, \ldots, n \right\}$ forms a basis for T_pM.

Let $TM := \sqcup_{p \in M} T_pM = \{(p, v_p) \mid p \in M,\ v_p \in T_pM\}$, and let $\pi : TM \to M$ be the natural projection map, $\pi(p, v_p) = p$. Notice that $\pi^{-1}(\{p\}) = T_pM$. The triple (TM, M, π) is called the *tangent bundle* of M. TM can be endowed with a smooth structure, as follows:

> If (U, φ) is a coordinate chart for M, consider the function
>
> $$\tilde{\varphi} : \pi^{-1}(U) \to U \times \mathbb{R}^n,$$
>
> defined by
>
> $$\tilde{\varphi}\left(p, \sum_{j=1}^n a_j \ \frac{\partial}{\partial x_j}\Big|_p\right) = (\varphi(p), a_1, \ldots, a_n).$$
>
> It can be easily seen that $(\pi^{-1}(U), \tilde{\varphi})$ is a coordinate chart for TM. Thus, TM is a smooth manifold of dimension $2n$. The chart $(\pi^{-1}(U), \tilde{\varphi})$ is called a *trivialization*, and it has the property that for every $q \in U$, the map $\tilde{\varphi}|_{T_qM} : T_qM \to \{q\} \times \mathbb{R}^n$ is a vector space isomorphism.

We also have the *cotangent bundle* T^*M. For $p \in M$, just consider the dual space to the tangent space T_pM, $T_p^*M := \mathrm{Hom}(T_pM; \mathbb{R})$, which is called the *cotangent space* to M at p. Suppose that $(U, \varphi = (x_1, \ldots, x_n))$ is a coordinate chart for M at p. We have seen that $\left\{ \frac{\partial}{\partial x_j}\Big|_p \mid j = 1, \ldots, n \right\}$ forms a basis for T_pM. Then, take the dual basis to the latter, which is usually denoted by $\{dx_j|_p \mid j = 1, \ldots, n\}$. Thus, a generic element $\omega_p \in T_p^*M$ can be written as

$$\omega_p = \sum_{j=1}^n a_j \, dx_j|_p,$$

for some real coefficients $a_1, \ldots, a_n$.

The disjoint union of all the cotangent spaces,

$$T^*M = \sqcup_{p\in M} T_p^*M = \{(p,\omega_p) \mid p \in M,\ \omega_p \in T_p^*M\},$$

with the natural projection map $\pi : T^*M \to M$, $\pi(p,\omega_p) = p$ is called the *cotangent bundle* of M. In a similar way as it was done for TM, T^*M can be endowed with a smooth structure making it into a smooth manifold of dimension $2n$.

A (*smooth*) *vector field* X on M is a (smooth) section of the tangent bundle TM. This means that $X : M \to TM$ is a smooth map such that $\pi \circ X = Id_M$. We denote the space of (smooth) vector fields on M by $\mathfrak{X}^1(M)$.

More generally, for $k \in \mathbb{N}$, we can consider the space of (*smooth*) *multivector fields* of degree k on M to be the space of (smooth) sections of the bundle $\bigwedge^k(TM) = \sqcup_{p\in M} \bigwedge^k(T_pM)$, which is denoted by $\mathfrak{X}^k(M)$. It turns out that the space of multivector fields, $\mathfrak{X}(M) = \bigoplus_{k\in\mathbb{N}} \mathfrak{X}^k(M)$, becomes a *graded Lie algebra*, with the bracket given by the *Schouten-Nijenhuis bracket* (for more details, see [La-GePiVa]).

An *exterior differential k-form* (or *de Rham k-form*) is a smooth section of the bundle $\bigwedge^k(T^*M) = \sqcup_{p\in M} \bigwedge^k(T_p^*M)$. The space of differential k-forms is denoted by $\Omega^k(M)$. In local coordinates, if $p \in M$ and $(U, \varphi = (x_1, \dots, x_n))$ is a coordinate chart for M at p, then $\omega(p) = \omega_p \in \bigwedge^k(T_p^*M)$ looks like

$$\omega_p = \sum_{1\le j_1<\cdots<j_k\le n} a_{(j_1,\dots,j_k)}\, dx_{j_1}|_p \wedge \cdots \wedge dx_{j_k}|_p,$$

for some real coefficients $a_{(j_1,\dots,j_k)}$.

The direct sum $\Omega(M) = \bigoplus_{k\in\mathbb{N}} \Omega^k(M) = \bigoplus_{k=0}^n \Omega^k(M)$ is called the *algebra of exterior differential forms*. Let us notice that this space has the structure of an algebra with respect to the wedge product, $\wedge$.

There is a differential operator of degree 1 on $\Omega(M)$, $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for every k, locally defined by

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\, dx_j, \quad \text{for } f \in \Omega^0(M) = \mathcal{C}^\infty(M;\mathbb{R}),$$

$$d\omega = \sum_J df_J \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k},$$

where $\omega = \sum_J f_J\, dx_{j_1} \wedge \cdots \wedge dx_{j_k} \in \Omega^k(M)$, for $k = 1, \dots, n$, and $J = \{(j_1, \dots, j_k) \mid 1 \le j_1 < \cdots < j_k \le n\}$.

The operator d is called the *exterior derivative* on forms or the *de Rham operator*, and it can be seen that d satisfies:

- ⋄ A graded Leibniz's rule: for every $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

- ⋄ $d \circ d = 0$.

The properties above mentioned tell us that $(\Omega(M), d)$ is a *differential graded commutative algebra*. So, $(\Omega(M), d)$ is a *cochain complex*, called the *de Rham complex*, and its homology is the de Rham cohomology of M:

$$\mathcal{H}^k_{dR}(M) := \frac{\mathcal{Z}^k(M)}{\mathcal{B}^k(M)},$$

where

$$\begin{aligned} \mathcal{Z}^k(M) &:= \ker(d : \Omega^k(M) \to \Omega^{k+1}(M)), \quad \text{and} \\ \mathcal{B}^k(M) &:= \operatorname{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M)) \end{aligned}$$

are the vector spaces of *closed* and *exact* k-forms, respectively.

Let us finish this brief digression on smooth manifolds remarking an important example.

Definition 4.2.2. A *Lie group* G is a group and, at the same time a smooth manifold, such that both structures are compatible in the sense that the group operations

$$\begin{aligned} G \times G \to G, &\quad (g_1, g_2) \mapsto g_1 \cdot g_2, \\ G \to G, &\quad g \mapsto g^{-1} \end{aligned}$$

are smooth maps. Here, $G \times G$ has the usual smooth product structure.
A *Lie algebra* of a Lie group G is the tangent space to G at the identity element, $\mathfrak{g} = T_e G$, with the Lie bracket defined by a commutator of vector fields.

Example 4.2.3. The general linear group $G = GL(n, \mathbb{R})$ is an open set in $\mathbb{R}^{n^2}$, defined by the condition $\det(g) \neq 0$, for $g \in G$. It is therefore a Lie group of dimension n^2. Its Lie algebra is the vector space $\operatorname{End}(\mathbb{R}^n)$ equipped with the usual commutator of linear operators on $\mathbb{R}^n$.

4.2.2 Symplectic Manifolds

Definition 4.2.4. Let M be a smooth manifold. A non-degenerate de Rham 2-form on M is a differential 2-form ω such that ω_p is a non-degenerate 2-covector on T_pM, for

every $p \in M$. As we mentioned before, ω is said to be *closed* if it satisfies the differential equation $d\omega = 0$, where d is the de Rham differential.
A *symplectic form* on M is a non-degenerate closed 2-form. In this case we say that ω is a symplectic structure and that (M, ω) is a *symplectic manifold*.

Notice that if (M, ω) is a symplectic manifold then, M has even dimension (as a consequence of Proposition 4.1.5) and it is an orientable manifold (in virtue of Proposition 4.1.6).

Let us make another remark.

Remark 4.1. From one of the equivalent definitions to ω_p be a non-degenerate 2-covector on T_pM, at every $p \in M$, we can conclude that ω defines an isomorphism

$$\omega^\flat : TM \to T^*M, \tag{4.1}$$

from the tangent bundle onto the cotangent bundle. This morphism is explicitly given by the formula

$$\omega^\flat(v_p) = \omega_p(\cdot, v_p),$$

for every $v_p \in T_pM$ and $p \in M$. The morphism $\omega^\flat$ lifts to a morphism between sections of the tangent and cotangent bundles, that is, between vector fields and differential forms.

Example 4.2.5 (Local model)**.** Let $M = \mathbb{R}^{2n}$ with the standard coordinates $x_1, \ldots, x_n$, $y_1, \ldots, y_n$. The form

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$$

is symplectic. ω is clearly a closed form, and it is non-degenerate because its value at every point of $\mathbb{R}^{2n}$ is the symplectic tensor described in Example 4.1.1. This ω is called the *standard symplectic form* on $\mathbb{R}^{2n}$.
In addition, the set $\left\{ \frac{\partial}{\partial x_j}\Big|_p, \frac{\partial}{\partial y_j}\Big|_p \right\}_{j=1}^{n}$ is a symplectic basis for T_pM, for every point $p \in M = \mathbb{R}^{2n}$.

Example 4.2.6. Let $M = \mathbb{C}^n$, with the standard coordinates $z_1, \ldots, z_n$. The form

$$\omega = \frac{i}{2} \sum_{k=1}^{n} dz_x \wedge d\overline{z}_k$$

is symplectic. In fact, this example is the same that the previous one under the identification $\mathbb{C} \backsimeq \mathbb{R}^{2n}$, $z_k = x_z + i\, y_k$.

Example 4.2.7. Let $M = \mathbb{S}^2 = \{p \in \mathbb{R}^3 \mid \|p\| = 1\}$. Tangent vectors to $\mathbb{S}^2$ at p may be identified with vectors orthogonal to p. On $\mathbb{S}^2$ there is a standard symplectic form, given in terms of the inner and exterior products:

$$\omega_p(u, v) := \langle p, u \times v \rangle, \quad \text{for } u, v \in T_p\mathbb{S}^2.$$

This form is closed because it is of top degree, and it is non-degenerate because

$$\langle p, u \times v \rangle \neq 0 \quad \text{when} \quad u \neq 0.$$

Take for example, $v = u \times p$.

We can construct new symplectic manifolds from old ones, as it is shown in the next proposition.

Proposition 4.2.8. *Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds, and let $M = M_1 \times M_2$ be the product space. Consider the projection maps*

$$\pi_1 : M_1 \times M_2 \to M_1, \quad \textit{and} \quad \pi_2 : M_1 \times M_2 \to M_2.$$

Then, the 2-form ω defined on M by

$$\omega := \pi_1^*\omega_1 - \pi_2^*\omega_2 \tag{4.2}$$

is symplectic.

Proof.

Notice first that

$$\begin{aligned} d\omega &= d(\pi_1^*\omega_1 - \pi_2^*\omega_2) \\ &= \pi_1^* d\omega_1 - \pi_2^* d\omega_2 \\ &= 0, \end{aligned}$$

since ω_1 and ω_2 are closed 2-forms (that is, $d\omega_1 = d\omega_2 = 0$). Hence ω is a closed 2-form on M.

We also observe that the tangent space at a point $p = (p_1, p_2)$ of the product manifold $M = M_1 \times M_2$ may be identified with the direct sum $T_{p_1}M_1 \bigoplus T_{p_2}M_2$. Since $\omega_1|_{p_1}$ and $\omega_2|_{p2}$ are non-degenerate 2-covectors on $T_{p_1}M_1$ and $T_{p_2}M_2$, respectively, we conclude that $\omega|_p$ is a non-degenerate 2-covector on T_pM. ☑

Remark 4.2. In the previous proposition it is also possible to define ω (as it was done in (4.2)) by taking the sum of the pull-backs of ω_1 and of ω_2 instead of their difference.

Next, we are going to point out a special example. It has to do with how to regard the cotangent bundle as a symplectic manifold.

Let Q be any n-dimensional manifold, and $M = T^*Q$ its cotangent bundle. Let us say that the smooth structure on Q is locally described by a coordinate chart $(U, \varphi = (q_1, \ldots, q_n))$. Then, at any point $q \in Q$, the set of differentials $\{dq_1|_q, \ldots, dq_n|_q\}$ forms a basis for T_q^*Q. Consequently, if $p \in T_q^*Q$, then $p = \sum_{j=1}^n p_j \, dq_j|_q$, for some coefficients $p_1, \ldots, p_n$. Hence, we have a local map

$$\begin{aligned} T^*U &\to \mathbb{R}^{2n} \\ (q,p) &\mapsto (q_1, \ldots, q_n, p_1, \ldots, p_n). \end{aligned}$$

It turns out that, $(T^*U, q_1, \ldots, q_n, p_1, \ldots, p_n)$ is a coordinate chart for $M = T^*Q$. Now, we define a 2-form on T^*U by

$$\omega = \sum_{j=1}^n dq_j \wedge dp_j.$$

To see that ω is coordinate-independent, let us consider the 1-form on T^*U given by

$$\tau = \sum_{j=1}^n p_j \, dq_j.$$

Notice that $\omega = -d\tau$. So, let us see that τ is intrinsically defined and then ω so is:

> Let $(T^*U, q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $(T^*U', q'_1, \ldots, q'_n, p'_1, \ldots, p'_n)$ be two coordinate charts. On $T^*U \cap T^*U'$, the two sets of coordinates are related by $p'_j = \sum_{k=1}^n p_k \frac{\partial q_k}{\partial q'_j}$. Then,
>
> $$\begin{aligned} \tau' = \sum_{j=1}^n p'_j \, dq'_j &= \sum_{j=1}^n \left(\sum_{k=1}^n p_k \frac{\partial q_k}{\partial q'_j} \right) \sum_{l=1}^n \left(\frac{\partial q'_j}{\partial q_l} \right) dq_l \\ &= \sum_{j,k,l=1}^n p_k \frac{\partial q_k}{\partial q'_j} \frac{\partial q'_j}{\partial q_l} \, dq_l = \sum_{k,l=1}^n p_k \frac{\partial q_k}{\partial q_l} \, dq_l \\ &= \sum_{k,l=1}^n p_k \, \delta_{kl} \, dq_l = \sum_{k=1}^n p_k \, dq_k = \tau. \end{aligned}$$

The 1-form τ is known as the *tautological form* or *Liouville-Poincaré* 1-*form*, and ω is called the *canonical symplectic form* on the cotangent bundle.

There is an alternative way to construct the tautological 1-form τ, which shows its intrinsic character:

Let $\pi : T^*Q \to Q$ be the projection map. Consider a covector $(p,q) \in T^*Q$, that is, $\pi(p) = q$ and $p \in T_q^*Q$. Differentiating the map π at the point p, we obtain

$$d\pi|_p : T_p(T^*Q) \to T_{\pi(p)}Q = T_qQ.$$

Then, for $v \in T_p(T^*Q)$, $d\pi|_p(v)$ is an element of T_qQ. This allows us to define

$$\tau|_p(v) := p(d\pi|_p(v)).$$

Here is one of the central results in symplectic geometry. The Darboux's theorem tells us that, locally, all symplectic forms look like the standard symplectic structure.

Theorem 4.2.9 (Darboux's theorem). *Let (M,ω) be a symplectic manifold of dimension $2n$. Then, each point $p \in M$ has a coordinate neighborhood U, with local coordinates $x_1, \dots, x_n, y_1, \dots, y_n$, such that*

$$\omega|_U = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

The coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ are called Darboux's coordinates. Thus, in Darboux's coordinates $\omega|_U$ is the standard symplectic form on U.

We will not make a proof of this theorem, since later in the section of Poisson manifolds we will prove the *Weinstein's splitting theorem*, which includes the Darboux's theorem as a special case.

We end this section with a brief digression about the Hamiltonian formalism of classical mechanics.

In a mechanical system the geometric object which models the possible positions of that system is given by a smooth manifold, called the *configuration space*, and the space which models positions and velocities (or momenta) of the given system is the cotangent bundle of the configuration space, known as *phase space*. Notice that the phase space has even dimension. It turns out that this space can be equipped with a symplectic structure (in the same way as it was done with the canonical form on the cotangent bundle).

In this context, a special function plays a decisive role in the evolution of the system. This special function is known as the*Hamiltonian*, and the vector field associated to this function satisfies the property that its integral curves represent the possible paths describing the mechanical system. From these tools and this approach to the study of analytical mechanics, it is discovered that what matters is not the symplectic structure of the phase space, but its nature of Poisson manifold that is obtained by defining an operation on the algebra of smooth functions of this phase space, called the *Poisson bracket*, which makes this algebra a *Poisson algebra* and the evolution of the mechanical system can then be written in terms of this bracket.

Our aim is to become familiar with this language and prove an important result: *Noether's theorem.*

A vector field X on M is said to be *symplectic* if it preserves the symplectic structure ω, that is, $\mathcal{L}_X\omega = 0$. Taking into account that ω is a closed 2-form and using the Cartan's formula, we obtain a characterization of these vector fields, in the following way:

$$\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X(\underbrace{d\omega}_{0}) = d\,\iota_X\omega,$$

so that X is symplectic if and only if $\iota_X\omega$ is a closed 1-form.

A vector field X on M is called *Hamiltonian* if $\iota_X\omega$ is an exact 1-form, that is, if there exists $f \in \Omega^0(M) = \mathcal{C}^\infty(M;\mathbb{R})$, such that $\omega(X,\cdot) = df$. Since $\omega^\flat : TM \to T^*M$ is a vector bundle isomorphism, which lifts to an isomorphism on sections, $\omega^\flat : \mathfrak{X}^1(M) \to \Omega^1(M)$, we can go in the other direction to get this kind of vector fields, as follows:

> Given a smooth function $f \in \mathcal{C}^\infty(M;\mathbb{R})$, we have the 1-form $df \in \Omega^1(M)$. Then we define the *Hamiltonian vector field associated to* f to be the vector field X_f on M that corresponds to df under the map $\omega^\flat$. This means that $X_f := (\omega^\flat)^{-1}(df)$. In other words, X_f is the unique vector field on M which satisfies
>
> $$\omega(X_f,\cdot) = df.$$

Let us fix a smooth function $H \in \mathcal{C}^\infty(M;\mathbb{R})$ (who will play the role of the *Hamiltonian function*). We want to figure out how X_H can be locally written. In Darboux's coordinates $x_1,\ldots,x_n,y_1,\ldots,y_n$ on some open neighborhood U, we write

$$X_H = \sum_{j=1}^{n}\left(a_j\,\frac{\partial}{\partial x_j} + b_j\,\frac{\partial}{\partial y_j}\right),$$

for certain smooth coefficient functions $a_1,\ldots,a_n,b_1,\ldots,b_n$ to be found.

Notice that, on U,

$$\begin{aligned}\omega(X_H,\cdot) &= \sum_{j=1}^{n} dx_j \wedge dy_j \left(\sum_{k=1}^{n}\left(a_k \frac{\partial}{\partial x_k} + b_k \frac{\partial}{\partial y_k}\right), \cdot\right) \\ &= \sum_{j,k=1}^{n} (a_k \delta_{jk}\, dy_j - b_k \delta_{jk}\, dx_j) \\ &= \sum_{j=1}^{n} (-b_j\, dx_j + a_j\, dy_j). \end{aligned} \tag{4.3}$$

On U we also have that

$$dH = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial x_j}\, dx_j + \frac{\partial H}{\partial y_j}\, dy_j\right). \tag{4.4}$$

Since $\{dx_j, dy_j\}_{j=1}^{n}$ gives rise to a basis for T^*U at each point of U, from (4.3) and (4.4) we conclude that the coefficients in the expression of X_H are given by

$$a_j = \frac{\partial H}{\partial y_j}, \quad \text{and} \quad b_j = -\frac{\partial H}{\partial x_j}, \qquad \text{for } j = 1, \ldots, n.$$

Hence, on U,

$$X_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial y_j} \frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial}{\partial y_j}\right). \tag{4.5}$$

Definition 4.2.10. A *(symplectic) Hamiltonian system* is a triplet (M, ω, H), where (M, ω) is a symplectic manifold, called the *phase space* of the system, and $H \in \mathcal{C}^\infty(M;\mathbb{R})$ is called the *Hamiltonian* of the system.

The differential equation on M, associated to the Hamiltonian vector field X_H,

$$\frac{d}{dt}\,\gamma(t) = X_H(\gamma(t)),$$

where t can be interpreted as a time variable, is called the *Hamilton's equation*.

The maximal integral curves $t \mapsto \gamma(t)$ of the vector field X_H are called the *trajectories of motion* of the Hamiltonian system (M, ω, H) and the first integrals of this vector field are also known as the *first integrals* of (M, ω, H).

In Darboux's coordinates, from the expression obtained for X_H in (4.5), the trajectories of motion of the system (M, ω, H), $\gamma(t) = (x_j(t), y_j(t))$, satisfy

$$\dot{x}_j(t) \equiv \frac{d}{dt}\, x_j(t) = \frac{\partial H}{\partial y_j}(\gamma(t)), \quad \text{and} \quad \dot{y}_j(t) \equiv \frac{d}{dt}\, y_j(t) = -\frac{\partial H}{\partial x_j}(\gamma(t)),$$

for $j = 1, \ldots, n$, which are the well-known *Hamilton's equations of motion.*

Proposition 4.2.11. *A smooth function $f \in \mathcal{C}^\infty(M;\mathbb{R})$ is a first integral of (M,ω,H) if and only if $\{f,H\} = 0$, that is, if f and H are in involution (or Poisson commute).*
The space of first integrals of (M,ω,H) is a Lie subalgebra of $\mathcal{C}^\infty(M;\mathbb{R})$, for the Lie subalgebra structure defined by the Poisson bracket.

Proof.

Let us observe that a function $f \in \mathcal{C}^\infty(M;\mathbb{R})$ is a first integral of the vector field X_H if and only if its derivative along this vector field, $X_H(f)$, vanishes. But, $X_H(f) = \{f,H\}$. Thus, f is a first integral of (M,ω,H) if and only if $\{f,H\} = 0$.

For the last statement of the proposition, let us notice that if f and g are first integrals of (M,ω,H) then their Poisson bracket is also a first integral of (M,ω,H) (*Poisson's theorem*) because, by the Jacobi identity,

$$\begin{aligned}\{\{f,g\},H\} &= \{\{f,H\},g\} + \{f,\{g,H\}\} \\ &= \{0,g\} + \{f,0\} \\ &= 0.\end{aligned}$$

☑

A trivial fact, in view of the skew-symmetry of the Poisson bracket, is that H is a first integral of (M,ω,H), because $\{H,H\} = 0$.

Remark 4.3. As a consequence of the preceding proposition and of the skew-symmetry of the Poisson bracket, if $f, g \in \mathcal{C}^\infty(M;\mathbb{R})$ then f is a first integral of (M,ω,g) if and only if (M,ω,f).

From Hamiltonian vector fields we can define an operation on the algebra of smooth functions on a symplectic manifold (M,ω).
We will denote this operation by $\{\,,\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, and it is given by

$$\{f,g\} := \omega(X_f, X_g),$$

for every $f, g \in \mathcal{C}^\infty(M)$. Let us notice that $\{f,g\} = X_g(f)$, so that $\{f,g\}$ is a measure of the rate of change of f along the *Hamiltonian flow* of g.

In Darboux's coordinates,

$$\{f,g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j}\frac{\partial g}{\partial x_j} \right).$$

It turns out that $(\mathcal{C}^\infty(M;\mathbb{R}), \{\,,\})$ becomes a *Poisson algebra*, a notion which will be defined later.

In order to state the Noether's theorem, we give a final definitions.

If (M, ω, H) is a Hamiltonian system, any function $f \in \mathcal{C}^\infty(M;\mathbb{R})$ that is constant on every integral curve of X_H is called a *conserved quantity* of the system. A smooth vector field Y on M is called an *infinitesimal symmetry* of (M, ω, H) if both ω and H are invariant under the flow of Y.

Proposition 4.2.12. *Let (M, ω, H) be a Hamiltonian system.*

(a) *A function $f \in \mathcal{C}^\infty(M;\mathbb{R})$ is a conserved quantity if and only if $\{f, H\} = 0$.*

(b) *The infinitesimal symmetries of (M, ω, H) are precisely the symplectic vector fields Y that satisfy $V(H) = 0$.*

Proof.

(a) It is a consequence of the following observation: if γ_t denotes the flow of the vector field X_H, then

$$\begin{aligned}\frac{d}{dt}(\gamma_t^* f) &= \gamma_t^*(\mathcal{L}_{X_H} f) = \gamma_t^*(\iota_{X_H} df)\\ &= \gamma_t^*(\iota_{X_H}\iota_{X_f}\omega) = \gamma_t^*\omega(X_f, X_H)\\ &= \gamma_t^*\{f, H\}.\end{aligned}$$

(b) Let Y a vector field on M. If σ_t denotes the flow of Y, then

$$\frac{d}{dt}(\sigma_t^*\omega) = \sigma_t^*(\mathcal{L}_Y\omega), \quad \text{and} \quad \frac{d}{dt}(\sigma_t^* H) = \sigma_t^*(\mathcal{L}_Y H) = \sigma_t^* Y(H).$$

From this, part (b) follows.

☑

Conserved quantities turn out to be deeply related to symmetries. The following result of deep consequences in theoretical physics, establishes a bijective correspondence between conserved quantities (modulo additive constants) and infinitesimal symmetries of a Hamiltonian system.

Theorem 4.2.13 (Noether's theorem). *Let (M, ω, H) be a Hamiltonian system. If f is any conserved quantity, then its Hamiltonian vector field X_f is an infinitesimal symmetry. Conversely, if $\mathcal{H}^1_{dR}(M) = 0$, then each infinitesimal symmetry is the Hamiltonian vector field of a conserved quantity, which is unique up to addition of a function that is constant on each component of M.*

Proof.

Suppose that f is a conserved quantity, then from the preceding proposition, 4.2.12, it follows that $\{f, H\} = 0$. Hence $X_f H = \{H, f\} = -\{f, H\} = 0$, so H is constant along the flow of X_f. Since ω is invariant along the flow of any Hamiltonian vector field we conclude that X_f is an infinitesimal symmetry.

Now suppose that $\mathcal{H}^1_{dR}(M)$ is trivial, and let X be an infinitesimal symmetry of (M, ω, H). Then X is symplectic by definition, that is, $\iota_X \omega$ is a closed 1-form. Since $\mathcal{H}^1_{dR}(M) = 0$, *Poincaré's lemma* allows us to write $\iota_X \omega = df$, for some $f \in \mathcal{C}^\infty(M; \mathbb{R})$, so X is a Hamiltonian vector field. In fact, $X = X_f$. By part (b) of the previous proposition we have that

$$\{f, H\} = -\{H, f\} = -X_f(H) = -X(f) = 0,$$

thus f is a conserved quantity. Finally, let us notice that if $g \in \mathcal{C}^\infty(M; \mathbb{R})$ is any other function with the property $X_g = X = X_f$, then

$$d(g - f) = \omega(X_g - X_f, \cdot) = 0,$$

so $g - f$ must be constant on each component of M. ☑

Chapter 5

Poisson Structures

In this chapter, we present some basic notions of Poisson structures, making an abstraction of the essential ideas developed in the last section of the previous chapter, on symplectic geometry. Then we see how these tools come into play in contexts of Poisson varieties and of Poisson manifolds. A more thorough discussion of the subject can be seen in [AdvMVa], [BhVi], [DuZu], and [La-GePiVa], which have also been references to the following presentation.

5.1 Preliminary notions

In geometric terms, a Poisson structure on a smooth manifold M associates to every smooth function H on M, a vector field X_H on M. When it comes to classical mechanics, this vector field leads to the equations of motion, taking as Hamiltonian function H. The essential ingredient here is the Poisson bracket, defined on smooth functions on M, requiring that this is a Lie bracket in order to make valid the Poisson's theorem, which states that the Poisson bracket of two constants of motion is itself a constant of motion.

In algebraic terms, a (generally infinite-dimensional) vector space A is considered, endowed with two different algebraic structures to identify:

- ◇ A commutative and associative multiplication.
- ◇ A Lie bracket.

With the first one, a commutative associative algebra is obtained. With the last one, a Lie algebra is obtained. In addition, both structures are compatible. Roughly speaking, this compatibility condition, which will be mentioned later, is what allows us to get derivations on A from elements of A. Here, derivations play the same role of vector fields.

In what follows, we fix a ground field $\mathbb{F}$ of characteristic zero, keeping in mind, as usual examples, the fields $\mathbb{R}$ or $\mathbb{C}$.

Definition 5.1.1. A *Poisson algebra* is an $\mathbb{F}$-vector space A equipped with two binary operations: $\cdot\,, \{\,,\}\,: A \times A \to A$, such that

- $\diamond$ $(A, \cdot)$ is a commutative associative algebra over $\mathbb{F}$, with 1.
- $\diamond$ $(A, \{\,,\})$ is a Lie algebra over $\mathbb{F}$.
- $\diamond$ Both structures are compatible in the sense that

$$\{f \cdot g, h\} = f \cdot \{g, h\} + g \cdot \{f, h\}, \quad \text{for every } f, g, h \in A. \tag{5.1}$$

In this case the Lie bracket $\{\,,\}$ is called a *Poisson bracket*.

Typical examples of Poisson algebras are the algebra of regular functions on an (affine algebraic) variety, and the algebra of smooth (or holomorphic) functions on a smooth (or complex) manifold.

Let us notice that a skew-symmetric bilinear map $\{\,,\} : A \times A \to A$ which satisfies (5.1), will be a Poisson bracket on A whenever Jacobi identity is hold:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \quad \text{for every } f, g, h \in A. \tag{5.2}$$

It is in this context that, in modern language, Poisson's theorem comes alive. Let us fix an element $H \in A$. We will say that $f \in A$ is a *constant of motion relative to* H if $\{f, H\} = 0$.

Theorem 5.1.2 (Poisson). *If f and g are constants of motion relative to H, then so it is $\{f, g\}$.*

Proof.

By hypothesis we have that $\{f, H\} = \{g, H\} = 0$. Thus, in view of Jacobi identity we get that

$$\begin{aligned}\{\{f, g\}, H\} &= \{\{f, H\}, g\} - \{\{g, H\}, f\}\\ &= \{0, g\} - \{0, f\}\\ &= 0,\end{aligned}$$

which means that $\{f, g\}$ is also a constant of motion relative to H.
Here we have used that $\{0, a\} = 0$, for every $a \in A$, as an easy consequence of property (5.1). ☑

In the algebraic language, property (5.1) means that for every $H \in A$ the linear map $f \mapsto \{f, H\}$ is a derivation on A.
We recall that a linear map $\mathcal{D} : A \to A$ is called a derivation on A (with values in A) if

$$\mathcal{D}(f \cdot g) = \mathcal{D}(f) \cdot g + f \cdot \mathcal{D}(g), \quad \text{for every } f, g \in A, \tag{5.3}$$

that is, when $\mathcal{D}$ satisfies Leibniz's rule.

In analogy with vector fields on a smooth manifold, we denote by $\mathfrak{X}^1(A)$ the Lie algebra of derivations of A, where the Lie bracket $[\,,]$ on $\mathfrak{X}^1(A)$ is given by the usual commutator, that is,

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1, \quad \text{for } D_1, D_2 \in \mathfrak{X}^1(A).$$

In fact, Poisson bracket defined on 5.1.1 leads to a biderivation on A, in virtue of skew-symmetry and property (5.1). We recall that a bilinear map $\mathcal{B} : A \times A \to A$ is called a biderivation on A (with values in A) if for every $f \in A$, $\mathcal{B}(\cdot, f)$ and $\mathcal{B}(f, \cdot)$ are derivations on A. The $\mathbb{F}$-vector space of biderivations on A is denoted by $\mathfrak{X}^2(A)$.

As it is usual when dealing with algebraic structures, it is convenient to have a way to compare them. This leads to the notion of morphisms between Poisson algebras.

Definition 5.1.3. Let $(A_1, \cdot_1, \{\,,\}_1)$ and $(A_2, \cdot_2, \{\,,\}_2)$ be two Poisson algebras over $\mathbb{F}$. A linear map $\varphi : A_1 \to A_2$ is called a *morphism of Poisson algebras* if for every $f, g \in A_1$ it holds that

- ◇ $\varphi(f \cdot_1 g) = \varphi(f) \cdot_2 \varphi(g)$,
- ◇ $\varphi(\{f, g\}_1) = \{\varphi(f), \varphi(g)\}_2$.

From the previous definition it is clear that a Poisson morphism respects both algebraic structures, being a morphism of commutative associative algebras, and a morphism of Lie algebras. In addition, if $\varphi : A_1 \to A_2$ is a morphism of Poisson algebras and φ is bijective, then $\varphi^{-1} : A_2 \to A_1$ is also a morphism of Poisson algebras. In this case φ is called an *isomorphism of Poisson algebras.*

Next, given a Poisson algebra $(A, \cdot, \{\,,\})$, we distinguish two algebraic substructures: *subalgebra* and *ideal* refer to the associative multiplication and, *Lie subalgebra* and *Lie ideal* refer to the Lie bracket.

Definition 5.1.4. Let $(A, \cdot, \{\,,\})$ be a Poisson algebra and let $B \subseteq A$ be a vector subspace. Then

- B is a *Poisson subalgebra* of A if it is a subalgebra and a Lie subalgebra of A. That is,
$$B \cdot B \subseteq B \quad \text{and} \quad \{B, B\} \subseteq B.$$
- B is a *Poisson ideal* of A if it is an ideal and a Lie ideal of A. That is,
$$B \cdot A \subseteq B \quad \text{and} \quad \{B, A\} \subseteq B.$$

If B is a Poisson subalgebra of A then B becomes itself a Poisson algebra. It turns out that the inclusion map $\iota : B \hookrightarrow A$ is a morphism of Poisson algebras if and only if B is a Poisson subalgebra of A.
If B is a Poisson ideal of A then the quotient A/B inherits a Poisson bracket from A. In a similar way, the projection map $\pi : A \twoheadrightarrow A/B$ is a morphism of Poisson algebras if and only if B is a Poisson ideal of A.

So far, for a fixed field $\mathbb{F}$, we have defined a category whose objects are the Poisson algebras over $\mathbb{F}$ and whose morphisms are the morphisms of Poisson algebras.

Next, we point out important objects and facts when dealing with Poisson algebras.

Definition 5.1.5. Let $(A, \cdot, \{\,,\})$ be a Poisson algebra and let $H \in A$. The derivation $X_H := \{\cdot, H\}$ of A is called a *Hamiltonian derivation* and we call H a *Hamiltonian* associated to X_H. We define

$$Ham(A) := \{X_H \mid H \in A\},$$

the $\mathbb{F}$-vector space of Hamiltonian derivations of A, so that we have an $\mathbb{F}$-linear surjective map

$$\begin{aligned} X \colon A &\to Ham(A) \\ H &\mapsto X_H. \end{aligned}$$

An element in the kernel of the last map is called a *Casimir*. In other words, $H \in A$ is a Casimir if $X_H(f) = \{f, H\} = 0$, for every $f \in A$. We denote the set of Casimir elements by

$$Cas(A) := \{H \in A \mid \{f, H\} = 0, \text{ for every } f \in A\}.$$

In virtue of bilinearity of $\{\,,\}$, $Cas(A)$ is an $\mathbb{F}$-vector space. Actually, it is the center of the Lie algebra $(A, \{\,,\})$.

We summarize some basic important facts in the next proposition.

Proposition 5.1.6. *Let $(A, \cdot, \{\ ,\ \})$ be a Poisson algebra.*

(1) *$Cas(A)$ is a subalgebra of $(A, \cdot)$, which contains the image of $\mathbb{F}$ in A, under the natural inclusion $a \mapsto a \cdot 1$.*

(2) *If A has no zero divisors, then $Cas(A)$ is integrally closed in A.*

(3) *$Ham(A)$ is not an A-module (in general). Instead,*

$$X_{f \cdot g} = f\, X_g + g\, X_f, \quad \textit{for every } f, g \in A.$$

(4) *$Ham(A)$ is a $Cas(A)$-module.*

(5) *The map $A \to \mathfrak{X}^1(A)$, defined by $H \mapsto -X_H$ is a morphism of Lie algebras. As a consequence, $Ham(A)$ is a Lie subalgebra of of $\mathfrak{X}^1(A)$.*

(6) *The Lie algebra sequence*

$$0 \longrightarrow Cas(A) \longrightarrow A \xrightarrow{-X} Ham(A) \longrightarrow 0$$

is a short exact sequence.

Proof.

(1) Let us see first that $Cas(A)$ is a subalgebra of $(A, \cdot)$.

Let f and g be two Casimir elements of A. Notice that $f \cdot g$ is also a Casimir because property (5.1):

$$\begin{aligned}\{h, f \cdot g\} &= -\{f \cdot g, h\} = -f \cdot \{g, h\} - g \cdot \{f, h\} \\ &= f \cdot \{h, g\} + g \cdot \{h, f\} \\ &= f \cdot 0 + g \cdot 0 = 0, \quad \text{for every } h \in A.\end{aligned}$$

This shows that $Cas(A)$ is a subalgebra of $(A, \cdot)$. Now, if $a \in \mathbb{F}$ and $f \in A$ then

$$\begin{aligned}\{f, a \cdot 1\} &= -\{a \cdot 1, f\} = -a \cdot \{1, f\} = -a \cdot \{1 \cdot 1, f\} \\ &= -a \cdot 1 \cdot \{1, f\} - a \cdot 1 \cdot \{1, f\} \\ &= -2a \cdot \{1, f\} = 2\{f, a \cdot 1\},\end{aligned}$$

and hence $\{f, a \cdot 1\} = 0$. So $a \cdot 1 \in Cas(A)$, for every $a \in \mathbb{F}$.

(2) Let $f \in A$ be integral over $Cas(A)$. This means that there exists a monic polynomial $p(T) \in Cas(A)[T]$, such that $p(f) = 0$, which we can assume to be the one of smallest degree. We are going to show that $f \in Cas(A)$.

If $deg(p) = 1$, it is clear that $f \in Cas(A)$.

Let us suppose therefore that $d := deg(p) > 1$. From property (5.1) we have that $0 = \{p(f), g\} = p'(f)\{f, g\}$, for every $g \in A$, where p' denotes the formal derivative of p. But $p'(f) \neq 0$, as $d^{-1}p'$ would otherwise be a monic polynomial of degree $d-1$, such that $d^{-1}p'(f) = 0$, contradicting the minimality of d. Since A has no zero divisors, it follows that $\{f, g\} = 0$, for every $g \in A$, which shows that $f \in Cas(A)$.

(3) Set $f, g, h \in A$. Then

$$\begin{aligned} X_{f\cdot g}(h) &= \{h, f \cdot g\} = -\{f \cdot g, h\} \\ &= -f \cdot \{g, h\} - g \cdot \{f, h\} \\ &= f \cdot \{h, g\} + g \cdot \{h, f\} \\ &= f\, X_g(h) + g\, X_f(h). \end{aligned}$$

Thus, $X_{f\cdot g} = f\, X_g + g\, X_f$.

(4) In order to prove that $Ham(A)$ is a $Cas(A)$-module, it is enough to show that if $H \in A$ and $f \in Cas(A)$ then $f\, X_H \in Ham(A)$. In fact, if $g \in A$ then, by property (5.1),

$$\begin{aligned} f\, X_H(g) = f \cdot \{g, H\} &= \{g, f \cdot H\} + H \cdot \{f, g\} \\ &= \{g, f \cdot H\} + H \cdot 0 = X_{f\cdot H}(g), \end{aligned}$$

because $f \in Cas(A)$, which implies that $\{f, g\} = 0$.

Thus, $f\, X_H(g) = X_{f\cdot H}(g)$, for every $g \in A$. Hence $f\, X_H = X_{f\cdot H} \in Ham(A)$.

(5) We have mentioned that X is a an $\mathbb{F}$-linear map, then so it is $-X : A \to \mathfrak{X}^1(A)$. Let us see that this map preserves the Lie bracket; that is, set $f, g \in A$ and let us show that $[-X_f, -X_g] = -X_{\{f,g\}}$. In fact, if $h \in A$ then, by Jacobi identity,

$$\begin{aligned} -X_{\{f,g\}}(h) &= -\{h, \{f, g\}\} = \{f, \{g, h\}\} + \{g, \{h, f\}\} \\ &= -\{\{g, h\}, f\} - \{\{h, f\}, g\} \\ &= -X_f(\{g, h\}) - X_g(\{h, f\}) \\ &= X_f(\{h, g\}) - X_g(\{h, f\}) \\ &= X_f(X_g(h)) - X_g(X_f(h)) \\ &= [X_f, X_g](h) = [-X_f, -X_g](h). \end{aligned}$$

We conclude that $-X : A \to \mathfrak{X}^1(A)$ is a morphism of Lie algebras. In particular, the image of this map is a Lie subalgebra of $\mathfrak{X}^1(A)$, but $-X(A) = -\,Ham(A)$, then so it is $Ham(A)$.

(6) In the sequence $0 \longrightarrow Cas(A) \hookrightarrow A \xrightarrow{-X} Ham(A) \longrightarrow 0$ we have that $Cas(A)$ and $Ham(A)$ are Lie subalgebras of A (because $Cas(A)$ is the center of $(A, \{\,,\})$) and $(\mathfrak{X}^1(A), [\,,])$, respectively. It is clear that $\iota : Cas(A) \hookrightarrow A$ is injective and $-X : A \twoheadrightarrow Ham(A)$ is surjective, so it is enough to show that $Cas(A) = Ker(-X)$.

First, if $f \in A$ is such that $-X_f = 0$, then $X_f = 0$, thus $f \in Cas(A)$.

On the other hand, if $f \in Cas(A)$ and $g \in A$ then $-X_f(g) = -\{g, f\} = \{f, g\} = 0$, thus $-X_f = 0$ and therefore $f \in Ker(-X)$. This shows that the short exact sequence above is exact.

☑

In the case where A is an algebra of functions on some variety or manifold, the properties of a Poisson bracket on A acquire a geometrical meaning.
Next we will make a brief introduction to Poisson varieties and Poisson manifolds, discovering this way beautiful applications of Poisson brackets in algebraic and differential geometry.

5.2 Poisson Varieties

Let us recall that an *affine variety* is an irreducible algebraic subset M of an affine space $\mathbb{F}^d$. Algebraic in the sense that M is the zero locus of a family of polynomials in d variables. In this context we can consider the prime ideal $\mathcal{I}$ of $\mathbb{F}[x_1, \ldots, x_d]$, which consists of all polynomial functions vanishing on M. It turns out that $\mathbb{F}[x_1, \ldots, x_d]/\mathcal{I}$ becomes a finitely generated, commutative associative algebra, which can be regarded as an algebra of functions on M, since the evaluation of elements of $\mathbb{F}[x_1, \ldots, x_d]/\mathcal{I}$ at points of M is a well-defined function. We will denote this algebra by $\mathcal{F}(M) := \mathbb{F}[x_1, \ldots, x_d]/\mathcal{I}$, which is called the *affine coordinate ring* of M. Let us notice that $\mathcal{F}(M)$ has no zero divisors, since M is irreducible.

Definition 5.2.1. Let M be an affine variety and suppose that $\mathcal{F}(M)$ is equipped with a Lie bracket $\{\,,\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$, which makes $(\mathcal{F}(M), \cdot, \{\,,\})$ into a Poisson algebra. Then $(M, \{\,,\})$ is said to be an *affine Poisson variety*.

Let us see now how to compute the Poisson bracket of two functions, in the case of an affine Poisson variety.

Proposition 5.2.2. *Let $\{\,,\}$ be a Poisson bracket on $A = \mathbb{F}[x_1, \ldots, x_d]$. Then the Poisson bracket of f and g in A is given by*

$$\{f, g\} = \sum_{j,k=1}^{d} \{x_j, x_k\} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}. \tag{5.4}$$

Proof.

Since both sides of (5.4) are bilinear in f and g, it is enough to show (5.4) in the case when f and g are monomials in $x_1, \ldots, x_d$. Let us reasoning according to the degree of f and g.

If f or g is a monomial of total degree 0, then the right hand side of (5.4) is zero but this is also the case for the left hand side, since constant functions are Casimirs.

Equality in (5.4) is also evident when f and g are monomials of degree 1. Take into account that $\partial f/\partial x_j$ and $\partial g/\partial x_k$ are deltas of Kronecker in this situation; thus, the double sum reduces to the only bracket term which consists of f and g.

Let us suppose that (5.4) holds when $deg(f) + deg(g) \leq n$, for some $n \geq 2$, and let us show that it holds for all f and g such that $deg(f) + deg(g) = n + 1$. Let f and g be non-constant monomials such that $deg(f) + deg(g) = n + 1$; by skew-symmetry we can assume that $deg(f) > 1$. Then, there exist monomials $f_1, f_2 \in A$ with lower degrees than $deg(f)$, such that $f = f_1 f_2$. Now we use the recursion hypothesis and the fact that $\{\,,\}$ is a biderivation to get

$$\begin{aligned}
\{f, g\} &= \{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\} \\
&= f_1 \sum_{j,k=1}^{d} \{x_j, x_k\} \frac{\partial f_2}{\partial x_j} \frac{\partial g}{\partial x_k} + f_2 \sum_{j,k=1}^{d} \{x_j, x_k\} \frac{\partial f_1}{\partial x_j} \frac{\partial g}{\partial x_k} \\
&= \sum_{j,k=1}^{d} \{x_j, x_k\} \left(f_1 \frac{\partial f_2}{\partial x_j} + f_2 \frac{\partial f_1}{\partial x_j} \right) \frac{\partial g}{\partial x_k} \\
&= \sum_{j,k=1}^{d} \{x_j, x_k\} \frac{\partial (f_1 f_2)}{\partial x_j} \frac{\partial g}{\partial x_k} \\
&= \sum_{j,k=1}^{d} \{x_j, x_k\} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}.
\end{aligned}$$

☑

Next we present the notion of a morphism of (affine) Poisson varieties.

Definition 5.2.3. Let $(M_1, \{\ ,\ \}_1)$ and $(M_2, \{\ ,\ \}_2)$ be two Poisson varieties. A morphism of varieties $\varphi : M_1 \to M_2$ is called a *Poisson morphism* if the dual morphism $\varphi^* : \mathcal{F}(M_2) \to \mathcal{F}(M_1)$ is a morphism of Poisson algebras.

Here, the dual morphism φ^* is defined as $\varphi^*(f) = f \circ \varphi$, for every $f \in \mathcal{F}(M_2)$. In this case, the condition that $\varphi^* : (\mathcal{F}(M_2), \{\ ,\ \}_2) \to (\mathcal{F}(M_1), \{\ ,\ \}_1)$ is a morphism of Lie algebras can be written as

$$\{f, g\}_2 \circ \varphi = \{f \circ \varphi, g \circ \varphi\}_1, \quad \text{for every } f, g \in \mathcal{F}(M_2).$$

In conclusion, for a fixed field $\mathbb{F}$, we have defined a category whose objects are (affine) Poisson varieties and whose morphisms are Poisson morphisms between Poisson varieties, defined as above.

We finish this section with the notion of the *rank* of a Poisson structure.

Lemma 5.2.4. *Let $(M, \{\ ,\ \})$ be an affine Poisson variety and let $p \in M$. The rank of the Poisson matrix $\mathcal{X} = (\{\overline{x_j}, \overline{x_k}\})_{j,k}$ evaluated at p is independent of the chosen generators $\overline{x_1}, \ldots, \overline{x_d}$ of $\mathcal{F}(M)$. ($\overline{x_j}$ denotes the class of x_j in the quotient $\mathcal{F}(M) = \mathbb{F}[x_1, \ldots, x_d]/\mathcal{I}$.)*

Proof.

It is enough to prove that if $\overline{x_1}, \ldots, \overline{x_d}$ are generators of $\mathcal{F}(M)$ and $\overline{x_0}$ is an arbitrary element of $\mathcal{F}(M)$, then the matrices $\mathcal{X}_p = (\{\overline{x_j}, \overline{x_k}\}(p))_{1 \leq j,k \leq d}$ and $\tilde{\mathcal{X}}_p = (\{\overline{x_j}, \overline{x_k}\}(p))_{0 \leq j,k \leq d}$ have the same rank. Let us say that $\overline{x_0} = f(\overline{x_0}, \ldots, \overline{x_d})$, written as a polynomial in $\overline{x_1}, \ldots, \overline{x_d}$. According to the proposition 5.2.2,

$$\{\overline{x_j}, \overline{x_0}\}(p) = \sum_{k=1}^{d} \{\overline{x_j}, \overline{x_k}\}(p)\, \frac{\partial f}{\partial x_k}(p),$$

then the zeroth column of $\tilde{\mathcal{X}}_p$ is a linear combination of the other columns of $\tilde{\mathcal{X}}_p$, that is, the columns of $\mathcal{X}_p$. Hence, $\tilde{\mathcal{X}}_p$ and $\mathcal{X}_p$ have the same rank. ☑

In view of the last lemma, the following definition makes sense.

Definition 5.2.5. For a Poisson variety $(M, \{\ ,\ \})$ and a point $p \in M$, the rank of the Poisson matrix of $\{\ ,\ \}$ with respect to an arbitrary system of generators of $\mathcal{F}(M)$, evaluated at p, is called the *rank* of $\{\ ,\ \}$ at p, denoted by $Rk_p\{\ ,\ \}$. The *rank* of $\{\ ,\ \}$, denoted by $Rk\{\ ,\ \}$ is the maximum $\max_{p \in M} Rk_p\{\ ,\ \}$.

We highlight some important facts about the rank of a Poisson structure in the context of Poisson varieties.

Proposition 5.2.6. *Let $(M, \{\,,\})$ be an affine Poisson variety.*

(*i*) *For every $p \in M$, $Rk_p\{\,,\}$ is an even number.*

(*ii*) *For each $s \in \mathbb{N}$, let us define*

$$M_{(s)} := \{p \in M \mid Rk_p\{\,,\} \geq 2s\} \subseteq M.$$

Then $M_{(s)}$ is open. In particular, the set $\mathcal{U} := \{p \in M \mid Rk_p\{\,,\} = Rk\{\,,\}\}$ is open and dense in M.

(*ii*) *$Rk\{\,,\}$ is at most equal to the dimension of M.*

Proof.

(*i*) It is a consequence that, for a system of generators $\overline{x_1}, \ldots, \overline{x_d}$ of $\mathcal{F}(M)$, the Poisson matrix of $\{\,,\}$ at $p \in M$ is the skew-symmetric matrix $\mathcal{X}_p = (\{\overline{x_j}, \overline{x_k}\}(p))_{j,k}$, whose rank is even.

(*ii*) Let us consider the open subset $R_s \subseteq \mathfrak{gl}_d$ of all $d \times d$ matrices of rank greater than or equal to $2s$. Since $M_{(s)}$ is the inverse image of R_s by the continuous map $\mathcal{X} : M \to \mathfrak{gl}_d$, defined by $p \mapsto \mathcal{X}_p$, then $M_{(s)} \subseteq M$ is open. Now, since the considered topology here is the Zarisky topology, these open subsets are dense as soon as they are non-empty.

(*iii*) Let us consider $p \in M$ and let us say that $Rk_p\{\,,\} = 2r$. Now, let us take $p' \in M_{(s)}$, which is a smooth point of M (i.e. $dim(T_pM)$ attains its minimal value, precisely the dimension of M, $dim(M)$). Such a point exists because $M_{(r)}$ and the set of smooth points of M are both dense in M.

Then, $Rk_p\{\,,\} \leq Rk_{p'}\{\,,\} \leq dim(I_{p'}/I_{p'}^2) = dim(M)$.

Since p is an arbitrary point of M in the last formula, we conclude that $Rk\{\,,\} \leq dim(M)$.

☑

5.3 Poisson Manifolds

In this section we will consider both real and complex manifolds. So, when we say that M is a manifold and $\varphi : M \to N$ is a map, we will be meaning one of the two possible contexts: M is a smooth manifold and $\varphi : M \to N$ is a smooth map, or, M is a complex manifold and $\varphi : M \to N$ is an holomorphic map. In a similar way, $\mathcal{F}(M)$ will be the algebra of real-valued smooth functions or the algebra of complex-valued holomorphic functions on M.

For every $p \in M$, the *tangent space to M at p* is T_pM, which consists of *pointwise derivations* at p. Specifically, $\delta_p \in T_pM$ is a linear form on the vector space of all *function germs* at p, satisfying the Leibniz's rule:

$$\delta_p(f\, g) = f(p)\, \delta_p(g) + g(p)\, \delta_p(f),$$

where f and g are function germs at p.

The dual space to T_pM is the *cotangent space to M at p*, denoted by T_p^*M. We denote by $\mathfrak{X}^1(M)$ the $\mathcal{F}(M)$-module of *vector fields* on M. It is usually convenient to to think a vector field $X \in \mathfrak{X}^1(M)$ through its action on local functions, in the following way: if U is a non-empty open subset of M and $f \in \mathcal{F}(U)$, then $X(f) : U \to \mathcal{F}$ is given by $X(f)(p) = X_p(f)$, for every $p \in U$. Here, $X_p \in T_pM$.

As a consequence of Hadamard lemma (see [La-GePiVa]), the basic *pointwise skew-symmetric biderivations* $\frac{\partial}{\partial x_j}\Big|_p \wedge \frac{\partial}{\partial x_k}\Big|_p$ span the vector space of all pointwise skew-symmetric biderivations at p, which is denoted by $\bigwedge^2(T_pM)$.

A *bivector field* on M is a map $P : M \to \bigwedge^2(TM)$, such that for every $p \in M$, $P_p \in \bigwedge^2(T_pM)$, and for every open subset $U \subseteq M$, and $f, g \in \mathcal{F}(U)$, $P(f,g) \in \mathcal{F}(U)$, where $P(f,g)$ is the function on U defined by $P(f,g)(q) = P_q(f,g)$, for every $q \in U$.
In local coordinates, (U,x), P can be written as

$$P = \sum_{1 \le j < k \le d} P(x_j, x_k)\, \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}. \tag{5.5}$$

Definition 5.3.1. Let Π be a bivector field on a manifold M. We say that Π is a Poisson structure on M if for every open subset $U \subseteq M$, the restriction of Π to U makes $\mathcal{F}(U)$ into a Poisson algebra. In this case, (M, Π) is called a *Poisson manifold.*

In bracket notation, $\{f, g\} = \Pi(f, g)$, where $f, g \in \mathcal{F}(U)$. $\{\,,\}$ is a *Poisson bracket.* So, according to (5.5), Π can also be written as

$$\Pi = \sum_{1 \le j < k \le d} \{x_j, x_k\}\, \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}. \tag{5.6}$$

Since the Poisson bracket is a biderivation, it vanishes whenever one of its arguments is constant. In this way, to every Poisson bracket on M we can associate an $\mathcal{F}(M)$-linear map

$$\begin{aligned}\Pi\colon \Omega^1(M)\times\Omega^1(M) &\to \mathcal{F}(M)\\ (df,dg) &\mapsto \{f,g\}.\end{aligned}$$

This Π is called the *Poisson tensor* associated to $\{\,,\}$, and the latter bracket can be reconstructed from Π:

$$\{f,g\}=\Pi(df,dg)\equiv\Pi(df\wedge dg).$$

From Π we obtain a map

$$\tilde{\Pi}:\Omega^1(M)\to\mathfrak{X}^1(M),$$

given by $\tilde{\Pi}(df)(g):=\Pi(dg\wedge df)=\{g,f\}$.
The preceding reasoning shows that $\tilde{\Pi}(df)=\{\cdot,f\}=X_f$, for every $f\in\mathcal{F}(M)$, where X_f is the Hamiltonian vector field associated to the function f.

We denote the bundle map $T^*M\to TM$ that corresponds to $\tilde{\Pi}$ with the same letter. Notice that, in the case where M is a symplectic manifold, this $\tilde{\Pi}$ is just the inverse map of the isomorphism $\omega^\flat$, given in (4.1).

There is an important characterization for a given bivector field to be a Poisson structure. We make a little discussion about this fact.

Given a bivector field Π on M, a necessary and sufficient condition for Π to define a Poisson structure is that $[\Pi,\Pi]_S=0\in\mathfrak{X}^3(M)$, where $[\,,]_S$ denotes the Schouten-Nijenhuis bracket. Let us see how this condition is achieved.
Let Π and Π' be two bivector fields on M. For the next computation we will use the bracket notation, let us say that $\Pi=\{\,,\}$ and $\Pi'=\{\,,\}'$. Then, the Schouten-Nijenhuis bracket of Π and Π' is the trivector field given by

$$\begin{aligned}[\Pi,\Pi']_S(f,g,h):=&\{\{f,g\},h\}'+\{\{g,h\},f\}'+\{\{h,f\},g\}'\\ &+\{\{f,g\}',h\}+\{\{g,h\}',f\}+\{\{h,f\}',g\}.\end{aligned}$$

So that

$$[\Pi,\Pi]_S(f,g,h)=2(\,\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}\,),$$

for every $f,g,h\in\mathcal{F}(M)$. The above computation shows that Π is a Poisson structure if and only if $\{\,,\}$ satisfies the Jacobi identity, if and only if $[\Pi,\Pi]_S=0$.

The following proposition tells us a way to compare two given Poisson manifolds.

Proposition 5.3.2. *Let $\Phi : M \to N$ be a map between two Poisson manifolds (M, Π) and (N, Π'). Then Φ is a Poisson map if and only if $\bigwedge^2(T\Phi)\Pi = \Pi'$, that is, for every $p \in M$, $\bigwedge^2(T_p\Phi)\Pi_p = \Pi'_p$.*

Proof.

Let p be a point of M, (U, x) be a coordinate chart for M around p, and f, g be two functions defined on a neighborhood of $\Phi(p)$. We have that

$$\Pi = \sum_{1\le j<k\le d} \{x_j, x_k\} \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}.$$

Then,

$$\begin{aligned}
\wedge^2(T_p\Phi)\Pi_p(f,g) &= \sum_{1\le j<k\le d} \wedge^2(T_p\Phi)\left(\{x_j, x_k\}(p)\left.\frac{\partial}{\partial x_j}\right|_p \wedge \left.\frac{\partial}{\partial x_k}\right|_p\right)(f,g) \\
&= \sum_{1\le j<k\le d} \{x_j, x_k\}(p)\, T_p\Phi\left(\left.\frac{\partial}{\partial x_j}\right|_p\right) \wedge T_p\Phi\left(\left.\frac{\partial}{\partial x_k}\right|_p\right)(f,g) \\
&= \sum_{1\le j<k\le d} \{x_j, x_k\}(p)\left.\frac{\partial}{\partial x_j}\right|_p \wedge \left.\frac{\partial}{\partial x_k}\right|_p (f\circ\Phi, g\circ\Phi) \\
&= \{f\circ\Phi, g\circ\Phi\}(p).
\end{aligned}$$

Setting $\{\,,\}' = \Pi'$, we have that $\Pi'_{\Phi(p)}(f,g) = \{f,g\}'(\Phi(p))$, so that $\bigwedge^2(T_p\Phi)\Pi_p = \Pi'_p$ if and only if $\{f\circ\Phi, g\circ\Phi\}(p) = \{f,g\}'(\Phi(p))$, that is, if and only if Φ is a Poisson map. ☑

A standard construction for a new Poisson manifold from old ones is the product of Poisson manifolds. For the next proposition, let us indicate the Poisson structure by the bracket notation in order to get no confusion with the projection maps.

Proposition 5.3.3. *Let $(M_1, \{\,,\}_1)$ and $(M_2, \{\,,\})_2$ be two Poisson manifolds. The product $M_1 \times M_2$ has a natural Poisson structure such that the projection maps*

$$\pi_1 : M_1 \times M_2 \to M_1, \quad \textit{and} \quad \pi_2 : M_1 \times M_2 \to M_2$$

are Poisson morphisms.

Proof.

In order for π_1 and π_2 to be Poisson morphisms it is necessary and sufficient to define

$$\{\pi_1^* f_1, \pi_1^* g_1\} := \pi_1^*\{f_1, g_1\}_1, \quad \text{and} \quad \{\pi_2^* f_2, \pi_2^* g_2\} := \pi_2^*\{f_2, g_2\}_2,$$

for every $f_1, g_1 \in \mathcal{F}(M_1)$ and $f_2, g_2 \in \mathcal{F}(M_2)$. In addition, we set $\{\pi_1^* f_1, \pi_2^* f_2\} := 0$ for every $f_1 \in \mathcal{F}(M_1)$ and $f_2 \in \mathcal{F}(M_2)$. These definitions extend uniquely to a skew-symmetric biderivation $\{\,,\}$ on $\mathcal{F}(M_1 \times M_2)$. Here is important to point out that the Poisson matrix of $\{\,,\}$ with respect to the system of local coordinates coming from local coordinates on M_1 and on M_2 has a block form, where each block is the pull-back under π_1^* or π_2^* (according to the case) of the Poisson matrix with respect to those local coordinates on M_1 and on M_2. Hence, the Jacobi identity is satisfied. ☑

In the previous chapter we made a discussion about a *Hamiltonian vector field* X_H, associated to a *Hamiltonian function* $H \in \mathcal{F}(M)$. More generally, a vector field X is said to be a *locally Hamiltonian vector field* if there exists $H \in \mathcal{F}(U)$ such that $X = X_H$ on U. In this case, H is called a *local Hamiltonian* of X. According to (5.6), if (U, x) is a coordinate chart and $H \in \mathcal{F}(U)$, then X_H can be written as

$$X_H = \sum_{j,k=1}^{d} \{x_j, x_k\} \frac{\partial H}{\partial x_j} \frac{\partial}{\partial x_k}. \tag{5.7}$$

Hamiltonian vector fields have a special behavior together with the bivector field Π, provided that the latter is a Poisson structure, as it is stated in the following proposition.

Proposition 5.3.4. *Let (M, Π) be a Poisson manifold. The Lie derivative of Π with respect to every (locally) Hamiltonian vector field is zero. As a consequence, the flow of each (locally) Hamiltonian vector field preserves the Poisson structure.*

Proof.

Let U be an open subset of M and let us consider $f, g \in \mathcal{F}(U)$. We want to show that if $H \in \mathcal{F}(U)$ then $\mathcal{L}_{X_H}\Pi(f, g) = 0$. For this, we will use the classical formula for the Lie derivative of a tensor (in this case of a bivector field), that is,

$$\mathcal{L}_X P(f, g) = X(P(f, g)) - P(X(f), g) - P(f, X(g)),$$

where $P \in \mathfrak{X}^2(U)$ and $X \in \mathfrak{X}^1(U)$.
As it is usual, let us denote $\Pi = \{\,,\}$. Then,

$$\begin{aligned}
\mathcal{L}_{X_H}\Pi(f, g) &= X_H(\{f, g\}) - \{X_H(f), g\} - \{f, X_H(g)\} \\
&= \{\{f, g\}, H\} - \{\{f, H\}, g\} - \{f, \{g, H\}\} \\
&= \{\{f, g\}, H\} + \{\{H, f\}, g\} + \{\{g, H\}, f\} \\
&= 0,
\end{aligned}$$

because Π is a Poisson structure, so $\{\,,\}$ satisfies the Jacobi identity. ☑

The previous fact motivates the following definition.

Definition 5.3.5. A vector field X is called a *Poisson vector field* if the Lie derivative of Π with respect to X vanishes, that is, $\mathcal{L}_X \Pi = 0$.

Thus, proposition 5.3.4 states that every Hamiltonian vector field is a Poisson vector field.

To finish this section, next we present a deep classical result on Poisson geometry, which states that, in the neighborhood of a point where the rank of the Poisson structure is $2r$, the Poisson manifold is a product of a symplectic manifold of dimension $2r$, and a Poisson manifold which has rank zero at the origin. The proof is taken from [La-GePiVa].

Theorem 5.3.6 (Weinstein's splitting theorem). *Let (M, Π) be a Poisson manifold. Let $x \in M$ be an arbitrary point and denote the rank of Π at x by $2r$. There exists a coordinate neighborhood U of x with coordinates $q_1, \ldots, q_r, p_1, \ldots, p_r, z_1, \ldots, z_s$, centered at x, such that, on U,*

$$\Pi = \sum_{j=1}^{r} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial p_j} + \sum_{1 \le k,l \le s} \varphi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l}, \tag{5.8}$$

where the functions φ_{kl} are (smooth or holomorphic) functions which depend on $z = (z_1, \ldots, z_s)$ only, and which vanish when $z = 0$.
Such local coordinates $q_1, \ldots, q_r, p_1, \ldots, p_r, z_1, \ldots, z_s$ are called splitting coordinates, centered at x.

Proof.

We proceed by induction on r.
For the case $r = 0$, it is clear that for every Poisson manifold (M, Π) and for every point x such that the rank of Π at x is zero, an arbitrary system of local coordinates $(z_1, \ldots, z_d)$, centered at x, works.
Let $r \in \mathbb{Z}^+$ and assume that the theorem is valid for every Poisson manifold, at every point where the rank is $2(r-1)$.
Let (M, Π) be a Poisson manifold and let $x \in M$ be a point for which $Rk_x\Pi = 2r$. We will show that the theorem holds for (M, Π) at x.
Since $Rk_x\Pi > 0$, there exists a function p_1 on a neighborhood of x, which can be taken in such a way that $p_1(x) = 0$, whose Hamiltonian vector field X_{p_1} does not vanish at x. Since $X_{p_1}(x) \neq 0$, there exists by *straightening theorem* (see [La-GePiVa]) a system of

coordinates $(q_1, y'_2, \dots, y'_d)$ on a neighborhood U' of x, centered at x, such that $X_{p_1} = \frac{\partial}{\partial q_1}$. Writing $\{\,,\} = \Pi$, it follows that

$$
\begin{aligned}
&\{q_1, p_1\} = X_{p_1}(q_1) = \frac{\partial q_1}{\partial q_1} = 1,\\
&[X_{q_1}, X_{p_1}] = X_{\{p_1, q_1\}} = -X_1 = 0 \quad \text{and},\\
&X_{p_1}(y'_j) = 0, \quad \text{for } j = 2, \dots, d, \quad \text{on } U.
\end{aligned}
$$

Writing X_{q_1} in terms of these coordinates,

$$
X_{q_1} = \xi_1 \frac{\partial}{\partial q_1} + \sum_{j=2}^{d} \xi_j \frac{\partial}{\partial y'_j},
$$

where $\xi_1 = X_{q_1}(q_1) = \{q_1, q_1\} = 0$, and all the coefficients $\xi_2, \dots, \xi_d$ are independent of q_1, since X_{q_1} and X_{p_1} commute. In addition, let us notice that

$$
\sum_{j=2}^{d} \xi_j(x) \frac{\partial p_1}{\partial y'_j} = X_{q_1}(p_1)(x) = \{p_1, q_1\}(x) = -1,
$$

so that the vector field X_{q_1} is independent of q_1 and does not vanish at x. Applying the *straightening theorem* once more, we may introduce a system of coordinates $(q_1, p'_1, y_3, \dots, y_d)$ on a neighborhood of x, centered at x, where $p'_1, y_3, \dots, y_d$ depend on $y'_2, \dots, y'_d$ only, with

$$
\frac{\partial}{\partial p'_1} = -\sum_{j=2}^{d} \xi_j \frac{\partial}{\partial y'_j} = -X_{q_1}.
$$

Substituting p'_1 by p_1 we consider $q_1, p_1, y_3, \dots, y_d$, which is also a system of coordinates on a neighborhood U of x, since

$$
\frac{\partial p_1}{\partial p'_1} = -X_{q_1}(p_1) = \{q_1, p_1\} = 1,
$$

in a neighborhood of x. Since $p'_1, y_3, \dots, y_d$ depend on $y'_2, \dots, y'_d$ only, $\frac{\partial}{\partial q_1}$ has the same meaning in both coordinate systems, so that the Poisson brackets take in the new coordinates the following form:

$$
\begin{aligned}
&\{q_1, p_1\} = 1,\\
&\{q_1, y_j\} = -X_{q_1}(y_j) = \frac{\partial y_j}{\partial p'_1} = 0,\\
&\{p_1, y_j\} = -X_{p_1}(y_j) = \frac{\partial y_j}{\partial q_1} = 0,
\end{aligned}
$$

for $j = 3, \dots, d$, and we conclude that, in terms of the coordinates $q_1, p_1, y_3, \dots, y_d$, Π is given by

$$\Pi = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \sum_{3 \le k,l \le d} \{y_k, y_l\} \frac{\partial}{\partial y_k} \wedge \frac{\partial}{\partial y_l}. \tag{5.9}$$

As an easy consequence of the Jacobi identity for $\Pi = \{\,,\}$, we get that $\{\{y_k, y_l\}, p_1\} = \{\{y_k, y_l\}, q_1\} = 0$, therefore $\{y_k, y_l\}$ is independent of q_1 and p_1, for all k, l. The Jacobi identity also yields that the second term in (5.9) defines a Poisson structure Π' on a neighborhood V of the origin of $\mathbb{F}^{d-2}$. It turns out that Π' has rank $2(r-1)$ at 0, and by the induction hypothesis, there exist local coordinates $q_2, \dots, q_r, p_2, \dots, p_r, z_1, \dots, z_s$, centered at 0, such that Π' takes on V the following form:

$$\Pi' = \sum_{j=2}^{r} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial p_j} + \sum_{1 \le k,l \le s} \varphi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l}.$$

In terms of the system of coordinates $q_1, q_2, \dots, q_r, p_1, p_2, \dots, p_r, z_1, \dots, z_s$, which is centered at x, Π takes the required form (5.8). ☑

Example 5.3.7. A prime example of a Poisson manifold is that of a symplectic manifold (M, ω), that is, ω is a non-degenerate closed de Rham 2-form. Such a manifold carries a Poisson structure, which is defined for smooth functions $f, g \in \mathcal{C}^\infty(M; \mathbb{R})$ by

$$\{f, g\} := \omega(X_f, X_g),$$

where for $H \in \mathcal{C}^\infty(M; \mathbb{R})$, the Hamiltonian vector field X_H is defined by

$$\omega(X_H, \cdot) = dH.$$

This notation is coherent with the one we had before in order to write $X_H = \{\cdot, H\}$.
The previous theorem shows that every Poisson manifold which has constant rank is a symplectic manifold and that for any Poisson manifold the Hamiltonian vector fields define a generalized distribution whose leaves inherit a natural symplectic structure. (For more details, see [AdvMVa])

Example 5.3.8. It is possible to describe all Poisson structures on $\mathbb{C}^2$ because in this situation the Jacobi identity is trivially held for any skew-symmetric biderivation on $\mathcal{F}(\mathbb{C}^2)$. Let us denote the standard coordinates on $\mathbb{C}^2$ by x and y. Then, every Poisson bracket on $\mathbb{C}^2$ is of the form

$$\{f, g\} = \varphi \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \right), \tag{5.10}$$

for some $\varphi \in \mathcal{F}(\mathbb{C}^2)$, where $f, g \in \mathcal{F}(\mathbb{C}^2)$. In fact, it is easy to see that $\varphi = \{x, y\}$, and for any $\varphi \in \mathcal{F}(M)$ the formula (5.10) defines a Poisson structure on $\mathbb{C}^2$.

Example 5.3.9. Any constant skew-symmetric $d \times d$ matrix is the matrix of a Poisson structure on $\mathbb{C}^d$. This Poisson structure is known as the *constant Poisson structure*. Using the classification theorem for skew-symmetric bilinear forms there exists a linear system of coordinates $x_1, \dots, x_d$ of $\mathbb{C}^d$ with respect to which the Poisson matrix takes the form

$$X = \begin{pmatrix} 0 & Id_r & 0 \\ -Id_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, the rank of the Poisson structure is $2r$

Example 5.3.10. Given a smooth manifold M, we can equip this with the trivial Poisson structure: $\{f, g\} = 0$, for all $f, g \in \mathcal{F}(M) = \mathcal{C}^\infty(M)$. The rank of M is zero everywhere, and the symplectic leaves are precisely the points of M.

Example 5.3.11. Let M be a connected symplectic manifold and N be an arbitrary smooth manifold, equipped with the trivial Poisson structure. Then $M \times N$ is a Poisson manifold with symplectic leaves $\{M \times \{q\} \mid q \in N\}$.

Example 5.3.12. The coadjoint orbits of a Lie group G can be realised as the symplectic leaves of the Poisson manifold $\mathfrak{g}^*$ (see [AdvMVa]). Let us mention that the Poisson bracket on $\mathfrak{g}^*$ is given by

$$\{f, h\}(\varphi) = \langle \varphi, [df|_\varphi, dh|_\varphi] \rangle, \tag{5.11}$$

for $f, h \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and $\varphi \in \mathfrak{g}^*$. Here, $df|_\varphi$ and $dh|_\varphi$ are interpreted as elements of $\mathfrak{g}$ when computing the bracket. Rewriting (5.11) for $h = H$ (in order to recognize a Hamiltonian function) as

$$X_H|_\varphi(f) = \langle \varphi, -ad_{dH|_\varphi} df|_\varphi \rangle = \langle ad^*_{dH|_\varphi}\varphi, df|_\varphi \rangle,$$

we find that the Hamiltonian vector field X_H is given, at $\varphi \in \mathfrak{g}^*$ by

$$X_H|_\varphi = ad^*_{dH|_\varphi}\varphi,$$

where $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ indicates the *adjoint representation* of the Lie algebra $\mathfrak{g}$.

For example, if we consider the Lie group $G = SU(2)$, then $\mathfrak{g}^* = \mathfrak{su}(2) \simeq \mathbb{R}^3$, and the coadjoint action consists of rotations about the origin. The corresponding orbits are concentric spheres about the origin, $\{\partial B(0; r) \mid r \geq 0\}$, each of which is symplectic.

Corollary 5.3.12.1. *The only local invariant of a regular (smooth or complex) Poisson manifold* (M, Π) *is its rank,* $Rk\,\Pi$.

Chapter 6

A brief discussion on Integrable Systems

In this chapter, we give some basic notions, results and examples of a Integrable Systems in the context of Poisson varieties and of Poisson manifolds. A more thorough discussion of the subject can be seen in [AdvMVa], [Fo] [Lee] and [Va], which has also been a reference to the following presentation.

6.1 Geometric precedents

The aim of this section is to present the notion of a *generalized distribution*, which has been already mentioned in previous sections, and a central result in this subject: the *Frobenius theorem.*

Let M be a (smooth or complex) manifold of dimension n. Instead of having a tangent vector at each point of M, as is the case of a vector field on M, one may have a k-dimensional subspace of the tangent space T_pM. In this way we arrive to the notion of a *distribution.*

A *(generalized) k-dimensional distribution* E on M is a datum of a k-dimensional subspace E_p of T_pM, for every $p \in M$. We say that E is smooth (or holomorphic) if for every $p \in M$ there exist smooth (or holomorphic) vector fields $V_1, \ldots, V_k$ on a neighborhood U of p, such that for every $q \in U$, E_q is the subspace spanned by the tangent vectors $V_1|_q, \ldots, V_k|_q$.

There is also a similar version of *integral curves* in the case of distributions. An *integral manifold* is a k-dimensional connected immersed submanifold N of M such that $T_qN = E_q$, for every $q \in N$.

Example 6.1.1. In $\mathbb{F}^n$, the vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ span a smooth (or complex if $\mathbb{F} = \mathbb{C}$) distribution of rank k. The k-dimensional affine subspaces parallel to $\mathbb{F}^n$ are integral manifolds.

Unlike integral curves, integral manifolds need not exist in general, even locally. We point out an important obstruction: if X and Y are two vector fields which are tangent to some submanifold N (this means that $X|_q, Y|_q \in T_qN$, for every $q \in N$) then their Lie bracket $[X, Y]$ is also tangent to N.

We say that E is an *involutive distribution* if for every vector fields X and Y on M, such that $X|_p, Y|_p \in E_p$, for every $p \in M$, their commutator $[X, Y]$ also holds the last property: $[X, Y]|_p \in E_p$, for every $p \in M$.

E is said to be *completely integrable* if for every point $p \in M$ there exists an integral manifold of E everywhere of maximal dimension which contains p.

Given a k-dimensional distribution $E \subseteq TM$, we say that a coordinate chart $(U, \varphi = (x_1, \dots, x_n))$ is *flat* for E if $\varphi(U)$ is a cube in $\mathbb{F}^n$, and at points of U, E is spanned by the first k coordinate vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$.
A *foliation* of dimension k on M is a collection $\mathcal{F}$ of disjoint, connected, non-empty, immersed k-dimensional submanifolds of M (called the *leaves* of the foliation), whose union is M, and such that in a neighborhood of each point $p \in M$ there exists a flat chart for $\mathcal{F}$.

Example 6.1.2. The collection of all k-dimensional affine subspaces of $\mathbb{F}^n$ parallel to $\mathbb{F}^k \times \{0\}$ is a k-dimensional foliation of $\mathbb{F}^n$.

Example 6.1.3. The collection of open rays of the form $\{\lambda x \mid \lambda > 0\}$ as x ranges over $\mathbb{F}^n - \{0\}$ is a 1-dimensional foliation of $\mathbb{F}^n - \{0\}$.

Example 6.1.4. If M and N are connected manifolds, the collection $\{M \times \{q\} \mid q \in N\}$ forms a foliation of $M \times N$, each of whose leaves is isomorphic to M.

The *Frobenius theorem* states that if E is a (smooth or holomorphic) k-dimensional distribution, then the following conditions are equivalent:

- ◇ E is involutive.

- ◇ E is completely integrable.
- ◇ E arises from a k-dimensional foliation on M.

6.2 Algebraic integrability in Hamiltonian systems

Recall that an *affine Poisson variety* $(M, \{\,,\})$ is an affine variety M (defined over $\mathbb{C}$) with a Poisson algebra structure $\{\,,\}$ on its algebra of regular functions $\mathcal{F}(M)$.

In the algebraic geometric context a vector field is a section of the *tangent sheaf*, which is well-defined, even at singular points. For instance, to every $f \in \mathcal{F}(M)$ there is an associated vector field X_f, defined by $X_f = \{\cdot, f\}$, which is called the *Hamiltonian vector field associated to f*.

Recall also that a regular function whose associated Hamiltonian vector field is zero is called a *Casimir*. The Casimirs form a subalgebra of $\mathcal{F}(M)$, denoted by $Cas(M)$.

Let $\mathcal{A}$ be a subalgebra of $\mathcal{F}(M)$. To every point $p \in M$ we may associate an algebra homomorphism $\chi_p : \mathcal{A} \to \mathbb{C}$, given by $\chi_p(f) := f(p)$, for every $f \in \mathcal{A}$. To this point p we can also associate the ideal $\{f - \chi_p(f) \mid f \in \mathcal{A}\}$, which is a point in $Spec(\mathcal{A})$, the *spectrum* of $\mathcal{A}$. Thus, we have a natural map

$$\begin{aligned} \pi_{\mathcal{A}} \colon M &\to Spec(\mathcal{A}) \\ p &\mapsto \{f - \chi_p(f) \mid f \in \mathcal{A}\}. \end{aligned}$$

We denote the *Krull dimension* of $\mathcal{A}$ by $dim(\mathcal{A})$. It turns out that, if $\mathcal{A}$ is finitely generated then $dim(\mathcal{A}) = dim(Spec(\mathcal{A}))$.

Definition 6.2.1. Let $(M, \{\,,\})$ be an affine Poisson variety and let $\mathcal{A}$ be a subalgebra of $\mathcal{F}(M)$.

- ◇ $\mathcal{A}$ is called *involutive* if $\{\mathcal{A}, \mathcal{A}\} = 0$.
- ◇ We say that $\mathcal{A}$ is *complete* if for any $f \in \mathcal{F}(M)$ one has $\{f, \mathcal{A}\} = 0$ if and only if $f \in \mathcal{A}$.

The triplet $(M, \{\,,\}, \mathcal{A})$, where $\mathcal{A}$ has the above two properties is called a *complete involutive Hamiltonian system*.

Lemma 6.2.2. *Let* $(M, \{\,,\})$ *be an affine Poisson variety. Then*

$$dim(Cas(M)) \leq CoRk\,\{\,,\} := dim(M) - Rk\,\{\,,\}.$$

Proof.

Let us consider a general fiber F of the map $M \to Spec(\mathcal{A})$, which is also induced by the inclusion map $\mathcal{A} \hookrightarrow \mathcal{F}(M)$. We are going to use the fact that

$$dim(M) - dim(F) = dim(Cas(M)).$$

We also use the fact that $Rk\,\{\,,\}_F = Rk\,\{\,,\}$, because F is a general fiber.
Since $dim(F)$ is equal to the number of independent derivations of $\mathcal{F}(M)$, and $Rk\,\{\,,\}_F$ is equal to the number of independent Hamiltonian derivations of $\mathcal{F}(M)$, at a general point of F, then

$$dim(Cas(M)) = dim(M) - dim(F) \leq dim(M) - Rk\,\{\,,\} \equiv CoRk\,\{\,,\}.$$

☑

Proposition 6.2.3. *Let* $(M, \{\,,\}, \mathcal{A})$ *be a complete involutive Hamiltonian system. Then*

$$dim(\mathcal{A}) \leq dim(M) - \frac{1}{2} Rk\,\{\,,\}.$$

Proof.

Let F be a general fiber of the map $A \to Spec(\mathcal{A})$, and let $\{\,,\}_F$ be the induced Poisson structure on F. We use the fact that

$$dim(F) = dim(M) - dim(\mathcal{A}). \tag{6.1}$$

Next, we claim that involutivity of $\mathcal{A}$ implies that independent derivations can be constructed using elements of $\mathcal{A}$.
For this, recall that the ideal F is generated by the functions $f - \chi_p(f)$, where $p \in M$ is arbitrary but fixed and f ranges over $\mathcal{A}$. For any $g \in \mathcal{A}$ we have

$$X_g(f - \chi_p(f)) = \{f, g\} = 0,$$

hence X_g is tangent to the locus defined by the ideal F, which is the same F.

Now, we are going to show that the elements of A lead to $dim(A) - dim(Cas(M))$ independent derivations. For this, let us consider a nested sequence of subalgebras

$$Cas(M) = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_r = \mathcal{F}(M),$$

where $dim(\mathcal{A}_{j+1}) = dim(\mathcal{A}_j) + 1$, for every $j = 0, \ldots, r-1$. In particular, $r = Rk\,\{\,,\}$. For any $j = 0, \ldots, r$, let us denote by n_j the number of independent vector fields coming from $\mathcal{A}_j$. Then $n_j \leq n_{j+1} \leq n_j + 1$, $n_0 = 0$ and $n_r = r$, so that $n_j = j$, for all $j = 0, \ldots, r$. From this and taking into account that $dim(F)$ is equal to the number of independent derivations of $\mathcal{F}(M)$ at an arbitrary point of F, we obtain

$$dim(F) \geq dim(A) - dim(Cas(\mathcal{A})). \tag{6.2}$$

From Lemma 6.2.2, (6.1), and (6.2) we find

$$dim(\mathcal{A}) \leq \frac{1}{2}\,(dim(M) + dim(Cas(M))) \leq dim(M) - \frac{1}{2}\,Rk\,\{\,,\}.$$

☑

The previous proposition motivates to make the following definition.

Definition 6.2.4. If $(M, \{\,,\})$ is an affine Poisson variety whose algebra of Casimirs is maximal and $\mathcal{A}$ is a complete involutive subalgebra of $\mathcal{F}(M)$ then $\mathcal{A}$ is called *integrable* if

$$dim(\mathcal{A}) = dim(M) - \frac{1}{2}\,Rk\,\{\,,\}.$$

The triplet $(M, \{\,,\}, \mathcal{A})$ is then called and *integrable Hamiltonian system* and each non-zero vector field in $Ham(\mathcal{A}) = \{X_f \mid f \in \mathcal{A}\}$ is called an *integrable vector field.*
The dimension of $\mathcal{A}$ is called the *dimension* or the *degrees of freedom* of the integrable Hamiltonian system. M is called its *phase space* and $\mathcal{A}$ its *base space.*

6.3 Integrable systems on Poisson manifolds

The aim of this section is to give a notion of integrability in the sense of Liouville for Poisson manifolds and present the *Liouville's theorem.*

The next definition is the analog version of Definition 6.2.1, given in the case of affine Poisson varieties.

Definition 6.3.1. Let $(M, \{\,,\})$ be a Poisson manifold and let $f, g \in \mathcal{F}(M)$. We say that f and g are in *involution* (or *Poisson commute*) if $\{f, g\} = 0$. For a subset $\mathcal{A}$ of $\mathcal{F}(M)$ we say that $\mathcal{A}$ is *involutive* if any two elements of $\mathcal{A}$ are in involution.

In order to give an interesting example, let us make a brief digression on *bi-Hamiltonian manifolds.*

Some Poisson manifolds carry another Poisson structure. This fact is relevant in the study of Hamiltonian vector fields in relation to integrability.

Definition 6.3.2. Let M be a manifold and let $\{\,,\}_1, \dots, \{\,,\}_s$ be s Poisson structures such that any linear combination of them is also a Poisson structure. Then these Poisson brackets are called *compatible Poisson structures*, and M equipped with these Poisson structures is called a *multi-Hamiltonian manifold.*
In the cases $s = 2, 3$, it is usual to say that M is *bi-Hamiltonian*, respectively *tri-Hamiltonian* manifold.

If P_1 and P_2 denote the Poisson bivector fields that correspond to two Poisson brackets $\{\,,\}_1$ and $\{\,,\}_2$ on M, then, for $\lambda_1, \lambda_2 \in \mathbb{F}$, $\lambda_1 \{\,,\}_1 + \lambda_2 \{\,,\}_2$ is a Poisson bracket if and only if

$$[\,\lambda_1 P_1 + \lambda_2 P_2\,,\, \lambda_1 P_1 + \lambda_2 P_2\,]_S = 0,$$

which is equivalent to $[P_1, P_2]_S = 0$ (in the non trivial cases $\lambda_1, \lambda_2 \neq 0$), since $[P_1, P_1]_S = 0$ and $[P_2, P_2]_S = 0$. From this observation, it can be concluded that s Poisson structures $\{\,,\}_1, \dots, \{\,,\}_s$ are compatible if and only if these s Poisson structures are pairwise compatible.

Consider a bi-Hamiltonian manifold $(M, \{\,,\}_1, \{\,,\}_2)$, where $\{\,,\}_2$ is not a scalar multiple of $\{\,,\}_1$. A vector field X is called *bi-Hamiltonian vector field* if it is Hamiltonian with respect to both Poisson structures.

We are now ready to illustrate the following example, which keeps the same spirit that Remark 4.3, given at the end of the first chapter, in the context of symplectic manifolds.

Example 6.3.3. Let X be a bi-hamiltonian vector field on a bi-Hamiltonian manifold $(M, \{\,,\}_1, \{\,,\}_2)$. Then, there exist functions $f, g \in \mathcal{F}(M)$, such that $X = \{\cdot, f\}_1$ and $X = \{\cdot, g\}_2$. The special fact to note is that f and g are in involution with respect to both brackets, because

$$\begin{aligned}
\{f, g\}_2 &= X(f) = \{f, f\}_1 = 0, \quad \text{and} \\
\{g, f\}_1 &= X(g) = \{g, g\}_2 = 0.
\end{aligned}$$

More generally, suppose that there is a sequence of functions $\mathcal{A} = \{f_j \mid j \in \mathbb{Z}\} \subseteq \mathcal{F}(M)$, such that $\{\cdot, f_j\}_2 = \{\cdot, f_{j+1}\}_1$. These data are known as a *bi-hamiltonian hierarchy.* In this case, for any $j < k$ in $\mathbb{Z}$,

$$\{f_j, f_k\}_1 = \{f_j, f_{k-1}\}_2 = \{f_{j+1}, f_{k-1}\}_1 = \cdots = \{f_k, f_j\}_1,$$

so that $\{f_j, f_k\}_1 = 0$, by skew.symmetry of $\{\,,\}_1$. It follows that $\mathcal{A}$ is involutive with respect to $\{\,,\}_1$. Using the same argument, $\mathcal{A}$ is also involutive with respect to $\{\,,\}_2$.

Given s functions $f_1, \ldots, f_s \in \mathcal{F}(M)$, we denote by $F = (f_1, \ldots, f_s)$ the s-tuple of this data. Since $\{f, g\} = X_g(f)$ and since $[X_f, X_g] = -X_{\{f,g\}}$ (as it was proved on Proposition 5.1.6), for any $f, g \in \mathcal{F}(M)$, the following proposition follows immediately, without any difficulty.

Proposition 6.3.4. *Let $(M, \{\,,\})$ be a Poisson manifold and let us suppose that $F = (f_1, \ldots, f_s)$ is involutive. Then,*

(*a*) *The Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_s}$ commute.*

(*b*) *The subalgebra of $\mathcal{F}(M)$, generated by the functions $f_1, \ldots, f_s$ is also involutive.*

A version of Poisson theorem also appears in the context of Poisson manifolds, as it might be expected.

Proposition 6.3.5 (Poisson). *Let $(M, \{\,\})$ be a Poisson manifold and let $f, g, h \in \mathcal{F}(M)$. If $\{f, h\} = 0$ and $\{g, h\} = 0$ then $\{\{f, g\}, h\} = 0$.*

Proof.

Is is a consequence of the Jacobi identity, in the same way as it was done in Theorem 5.1.2. ☑

Let us fix a function $H \in \mathcal{F}(M)$, which can be thought of as the Hamiltonian of a mechanical system. Let us suppose that $F = (f_1, \ldots, f_s)$ is an s-tuple of elements in $\mathcal{F}(M)$, not necessarily in involution, but each of them in involution with H. In this case, X_H is tangent to each of the hypersurfaces $f_j = \textit{constant}$, thus, all functions $f_1, \ldots, f_s$ are constant on the trajectories of X_H. This is the reason why functions in involution with H are classically called *constants of motion* or *conserved quantities*.

Finding enough *independent* constants of motion is very useful for the explicit integration Hamilton's equations. Next, we make precise the definition of the word *independent* in the last assertion, and point out some important facts in the general context we are working with.

Definition 6.3.6. Let $(M, \{\,,\})$ be a Poisson manifold and suppose that $F = (f_1, \ldots, f_s)$ is an s-tuple of elements in $\mathcal{F}(M)$. We say that F is *independent* when the open subset on which the differentials $df_1, \ldots, df_s$ are independent is dense in M.

This means that $F = (f_1, \dots, f_s)$ is independent if and only if the set

$$\mathcal{U}_F := \{p \in M \mid \ df_1|_p \wedge \cdots \wedge df_s|_p \neq 0\}$$

is a dense open subset of M. We can express locally this condition in other way: for a point $p \in M$, let us take local coordinates $x_1, \dots, x_n$ in a neighborhood of p. Then, $p \in \mathcal{U}_F$ if and only if the matrix

$$\left(\frac{\partial f_j}{\partial x_k} \right)_{1 \leq j \leq s,\ 1 \leq k \leq n}$$

has rank s. It is evident that $s \leq dim(M)$ when $F = (f_1, \dots, f_s)$ is independent.

Proposition 6.3.7. *let $(M, \{\ ,\ \})$ be a Poisson manifold of rank $2r$ and suppose that $(f_1, \dots, f_s)$ is independent.*

(a) If $f_1, \dots, f_s$ are Casimirs then $s \leq dim(M) - 2r$.

(b) If $(f_1, \dots, f_2)$ is involutive then $s \leq dim(M) - r$.

(c) If $F = (f_1, \dots, f_s)$ is involutive with $s = dim(M) - r$, then

$$dim(span\{X_{f_1}|_p, \dots, X_{f_s}|_p\}) \leq r,$$

for any $p \in \mathcal{U}_F$, with equality if $p \in \mathcal{U}_F \cap M_{(r)}$.

Proof.

We only show part (a). The rest of the proof can be found in [AdvMVa].

For $p \in M$ let us consider the map $\tilde{\Pi}_p : T_p^*M \to T_pM$ given by

$$\tilde{\Pi}_p(df|_p) := X_f|_p = \{\cdot, f\}(p),$$

for every $f \in \mathcal{F}(M)$. Recall that $Rk_p\{\ ,\ \}$ denotes the rank of $\tilde{\Pi}_p$, and that for every f in $Cas(M)$ the covector $df|_p$ belongs to $Ker(\tilde{\Pi}_p)$, whose dimension is $dim(M) - Rk_p\{\ ,\ \}$. Let $F = (f_1, \dots, f_s)$ be independent and let p be any point in the non-empty open set $\mathcal{U}_F \cap M_{(r)}$. Then $df_1|_p, \dots, df_s|_p$ are independent. If $f_1, \dots, f_s$ are Casimirs then we have that

$$s \leq dim(Ker(\tilde{\Pi}_p)) = dim(M) - 2r.$$

☑

Now we are going to present the analog version of Definition 6.2.4, in the context of Poisson manifolds

Definition 6.3.8. Let $(M, \{\,,\})$ be a Poisson manifold of rank $2r$ and set a s-tuple $F = (f_1, \ldots, f_s)$ of elements in $\mathcal{F}(M)$. We say that F is *completely integrable*, in the sense of Liouville, if it is involutive, independent and $s = dim(M) - r$. In this case, $(M, \{\,,\}, F)$ is said to be a *completely integrable system.*
The vector fields X_{f_j} are then called *integrable vector fields*, and F, regarded as a map with values in $\mathbb{F}$, is called the *momentum map.*
We say that the integer r is the number of *degrees of freedom* of the integrable system and we call $2r$ its rank.

Let us notice that on $\mathcal{U}_F \cap M_{(r)}$ the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_s}$ define an *integrable* distribution D of rank r. The integral manifolds of D are the leaves of a foliation $\mathcal{F}$. Let us define F'_p, the *invariant manifold* of $F = (f_1, \ldots, f_s)$ that passes through $p \in \mathcal{U}_F \cap M_{(r)}$, as the leaf of $\mathcal{F}$ passing through p. It turns out that F'_p is an (embedded) submanifold of M.

We finish this section with the Liouville Theorem for real integrable systems. The proof of this theorem is taken from [La-GePiVa].

Theorem 6.3.9. *Let $(M, \{\,,\}, F)$ be a real integrable system of rank $2r$, where $F = (f_1, \ldots, f_s)$.*

- (a) *If F'_p is compact then there exists a diffeomorphism from F'_p to the torus $\mathbb{T}^r = (\mathbb{R}/\mathbb{Z})^r$, under which the vector fields $X_{f_1}, \ldots, X_{f_s}$ are mapped to linear vector fields.*
- (b) *If F'_p is not compact, but the flow of each of the vector fields X_{f_j} is complete on F'_p then there exists a diffeomorphism from F'_p to a cylinder $\mathbb{R}^{r-l} \times \mathbb{T}^l$ $(0 \leq l < r)$, under which the vector fields X_{f_j} are mapped to linear vector fields.*

Proof.

For $j = 1, \ldots, r$, let us denote the flow of the integrable vector field X_{f_j} by $\Phi^{(j)}$. We are going to make the assumption that each $\Phi^{(j)}$ is complete on F'_p, that is, $\Phi^{(j)}$ is defined for all $t \in \mathbb{R}$. By ordering the functions f_j, if it is necessary, we may suppose that the first r vector fields $X_{f_1}, \ldots, X_{f_r}$ are independent at p. These vector fields are then independent at every point of F'_p. In fact, since these Hamiltonian vector fields pairwise commute,

$$\mathcal{L}_{X_{f_j}}(X_{f_1} \wedge \ldots \wedge X_{f_s}) = \sum_{i=1}^{r} X_{f_1} \wedge \ldots \wedge [X_{f_j}, X_{f_i}] \wedge \ldots \wedge X_{f_r} = 0,$$

for $j = 1, \ldots, s$. This means that $X_{f_1} \wedge \ldots \wedge X_{f_s}$ is conserved by the flow of each one of the vector fields $X_{f_1}, \ldots, X_{f_s}$. In particular, since this $r-$vector field is non-vanishing at p, it

is non-vanishing on F_p'. Therefore F_p' is a leaf of the distribution defined by $X_{f_1}, \ldots, X_{f_r}$ in a neighborhood of F_p'.
By the completeness and commuativity of the vector fields $X_{f_1}, \ldots, X_{f_s}$ on F_p' we can define an action of $\mathbb{R}^r$ on F_p' by

$$\mathbb{R}^r \times F_p' \to F_p', \qquad ((t_1, \ldots, t_r), q) \mapsto \Phi_{t_1}^{(1)} \circ \cdots \circ \Phi_{t_r}^{(r)}(q).$$

Since F_p' is the integral manifold through p of the distribution defined by the first r integrable vector fields, this action is transitive on F_p' and F_p' becomes a homogeneous space. This action is also locally free, because the vector fields X_{f_j} are independent at every point of F_p'. Therefore the stabilizer is a discrete subgroup of $\mathbb{R}^r$, let us say S_p, and F_p' is diffeomorphic to $\mathbb{R}^r/S_p$. If F_p' is compact, then S_p must be a lattice, so $\mathbb{R}^r/S_p$ is a torus, smoothly embedded into M. Otherwise S_p is a discrete subgroup whose rank l is at most $r-1$ and $\mathbb{R}^r/S_p$ is diffeomorphic to $\mathbb{R}^{r-l} \times \mathbb{T}^l$. By construction, the vector fields X_{f_j} are mapped to translation-invariant vector fields in both cases.

☑

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