Sistemas de coordenadas

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Contenido

Prólogo

El objetivo del presente escrito, es el de facilitar al estudiante de las carreras de ingeniería, la asimilación clara de los conceptos matemáticos tratados.

Desde luego que los escritos que se presentan no son originales, ni pretenden serlo, toda vez que es una recopilación organizada y analizada de difernte textos y de mi experiencia personal.

Este escrito constituye un material de consulta obligada de los estudiantes, el cual les genera un diálogo directo con el profesor.

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Capítulo 1

Sistemas de Coordenadas

1.1 Sistemas cartesiano, cilíndrico y esférico

Se estudiarán en esta sección los sistemas cartesiano, cilíndrico y esférico. Un punto P(x,y,z) puede representarse por sus coordenadas (x,y,z). Los intervalos de las variables x,y,z son

$$-\infty < x < \infty, \quad -\infty < y < \infty, \quad -\infty < z < \infty$$

Un vector F en coordenadas cartesianas, puede escribirse como

$$F(x,y,z) = A_x a_x + A_y a_y + A_z a_z = A_x i + A_y j + A_z k = (A_x, A_y, A_z)$$

donde a_x, a_y, a_z son los vectores unitarios en las direcciones x, y, z.

Un punto P en coordenadas cilíndricas se representa por (ρ, φ, z) y r es el radio del cilindro que pasa por el punto P, φ es el ángulo en el plano $z = 0$ y se mide desde el eje x positivo, como se mide θ en coordenadas polares y z igual que el sistema cartesiano. Los intervalos de las variables son

$$0 \leq \rho < \infty, \quad 0 \leq \varphi \leq 2\pi, \quad -\infty < z < \infty$$

Un vector F en coordenadas cilíndricas puede escribirse como

$$F(\rho, \varphi, z) = A_\rho a_\rho + A_\varphi a_\varphi + A_z a_z = (A_\rho, A_\varphi, A_z)$$

donde a_ρ, a_φ, a_z son vectores unitarios en la direccion ρ,φ, z y son mutuamente perpendiculares, a_ρ apunta en la dirección de aumento de r, a_φ apunta en la dirección de aumento de φ, y a_z en la dirección positiva de z, en consecuencia

$$a_r \bullet a_\rho = a_\varphi \bullet a_\varphi = a_z \bullet a_z = 1$$

$$a_\rho \bullet a_\varphi = a_\varphi \bullet a_z = a_z \bullet a_\rho = 0$$

$$a_\rho \times a_\varphi = a_z \qquad a_\varphi \times a_z = a_\rho \quad a_z \times a_\rho = a_\varphi$$

Las relaciones entre las variables (x,y,z) del sistema de coordenadas cartesianas y las sistema de coordenadas cilíndrícas (ρ, φ, z) vestán dadas por figura

$$\rho = \sqrt{x^2 + y^2}, \qquad \tan\varphi = \frac{y}{x}, \qquad z = z \quad ó$$

$$x = \rho\cos\varphi, \; y = \rho\sin\varphi \quad z = z$$

Las ecuaciones $\rho = \sqrt{x^2 + y^2}$, $\tan\varphi = \dfrac{y}{x}$, $z = z$ sirven para transformar de coordenadas cartesianas (x,y,z) a coordenadas cilíndricas (ρ, φ, z) y las ecuaciones $x = \rho\cos\varphi$, $y = \rho\sin\varphi \quad z = z$ de coordenadas cilíndricas a coordenadas cartesianas.

Las relaciones entre los vectores unitarios (a_x, a_y, a_z) y (a_ρ, a_φ, a_z) están dados por

$$a_x = \cos\varphi \; a_\rho - \sin\varphi \; a_\varphi$$

$$a_y = \sin\varphi \; a_\rho + \cos\varphi \; a_\varphi$$

$$a_z = a_z$$

ya que como

$$a_x = \alpha_\rho a_\rho + \alpha_\varphi a_\varphi \quad \text{entonces}$$

$$a_x \bullet a_\rho \; = (\alpha_\rho a_\rho + \alpha_\varphi a_\varphi) \bullet a_\rho \; = \alpha_\rho = \cos\varphi \quad \text{y}$$

$$a_x \bullet a_\varphi \; = (\alpha_\rho a_\rho + \alpha_\varphi a_\varphi) \bullet a_\varphi \; = \alpha_\varphi = \cos(90 + \varphi) = -\sin\varphi$$

por lo tanto

$$a_x = \alpha_\rho a_\rho + \alpha_\varphi a_\varphi = a_x = \cos\varphi a_\rho - \sin\varphi a_\varphi$$

y

$$a_y = \alpha_\rho a_\rho + \alpha_\varphi a_\varphi \quad \text{entonces}$$

$$a_y \bullet a_\rho \; = (\alpha_\rho a_\rho + \alpha_\varphi a_\varphi) \bullet a_\rho \; = \alpha_\rho = \cos(90 - \varphi) = \sin\varphi \; \text{y}$$

$$a_y \bullet a_\varphi \; = (\alpha_\rho a_\rho + \alpha_\varphi a_\varphi) \bullet a_\varphi \; = \alpha_\varphi = \cos\varphi$$

por lo tanto

$$a_y = \alpha_\rho a_\rho + \alpha_\varphi a_\varphi = \sin\varphi \; a_\rho + \cos\varphi \; a_\varphi$$

$$a_z = a_z$$

que en forma matricial se puede escribir como

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_\rho \\ a_\varphi \\ a_z \end{pmatrix}$$

y como

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

por lo tanto la matriz

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

es ortogonal, es decir $A.A^T = A^T.A = I$ entonces $A^{-1} = A^T$ entonces

$$\begin{pmatrix} a_\rho \\ a_\varphi \\ a_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

Ahora observemos que

$$F(x,y,z) = A_x a_x + A_y a_y + A_z a_z =$$

$$= A_x \left(\cos\varphi\ a_\rho - \sin\varphi\ a_\varphi\right) + A_y \left(\sin\varphi\ a_\rho + \cos\varphi\ a_\varphi\right) + A_z a_z =$$

$$= \left(A_x \cos\varphi + A_y \sin\varphi\right) a_\rho + \left(-A_x \sin\varphi + A_y \cos\varphi\right) a_\varphi + A_z a_z =$$

$$= A_\rho a_\rho + A_\varphi a_\varphi + A_z a_z$$

por lo tanto

$$A_\rho = A_x \cos\varphi + A_y \sin\varphi$$

$$A_\varphi = -A_x \sin\varphi + A_y \cos\varphi$$

$$A_z = A_z$$

que se puede escribir en forma de matriz así

$$\begin{pmatrix} A_\rho \\ A_\varphi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

y

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} A_\rho \\ A_\varphi \\ A_z \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\varphi \\ A_z \end{pmatrix}$$

$$A_x = \cos\varphi\ A_\rho - \sin\varphi\ A_\varphi$$

$$A_y = \sin\varphi\ A_\rho\ + \cos\varphi\ A_\varphi$$

$$A_z = A_z$$

Ahora observemos que

$$F(x,y,z) = A_x a_x + A_y a_y + A_z a_z = A_\rho a_\rho + A_\varphi a_\varphi + A_z a_z \text{ entonces}$$

$$F \bullet a_x = A_x = A_\rho(a_\rho \bullet a_x) + A_\varphi(a_\varphi \bullet a_x) + A_z(a_z \bullet a_x)$$

$$F \bullet a_y = A_y = A_\rho(a_\rho \bullet a_y) + A_\varphi(a_\varphi \bullet a_y) + A_z(a_z \bullet a_y)$$

$$F \bullet a_z = A_z = A_\rho(a_\rho \bullet a_z) + A_\varphi(a_\varphi \bullet a_z) + A_z(a_z \bullet a_z)$$

Que en forma matricial se puede escribir como

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} a_\rho \bullet a_x & a_\varphi \bullet a_x & a_z \bullet a_x \\ a_\rho \bullet a_y & a_\varphi \bullet a_y & a_z \bullet a_y \\ a_\rho \bullet a_z & a_\varphi \bullet a_z & a_z \bullet a_z \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\varphi \\ A_z \end{pmatrix}$$

El sistema de coordenadas esféricas es mas apropiado para solucionar problemas con simetría esférica. Un punto P puede representarse como (r, θ, φ), con r la distancia del origen al punto P, θ el angulo comprendido entre el eje z positivo y el valor de posición de P, φ el mismo de las cilíndricas, es decir, $0 \leq r < \infty,\ \ 0 \leq \theta \leq \pi,\ \ 0 \leq \varphi \leq 2\pi$.

Un vector F puede escribirse en coordenadas esféricas como

$$F(r,\theta,\varphi) = A_r a_r + A_\theta a_\theta + A_\varphi a_\varphi$$

con

$$a_r \bullet a_r = a_\theta \bullet a_\theta = a_\varphi \bullet a_\varphi = 1$$

$$a_r \bullet a_\theta = a_\theta \bullet a_\varphi = a_\varphi \bullet a_r = 0$$

$$a_r \times a_\theta = a_\varphi \qquad a_\theta \times a_\varphi = a_r \qquad a_\varphi \times a_r = a_\theta$$

Las variables (x, y, z) pueden relacionarsen con las variables (r, θ, φ) por medio de las expresiones

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}, \quad \tan\varphi = \frac{y}{x}$$

ó por

$$x = r\sin\theta\cos\varphi, \; y = r\sin\theta\sin\varphi, \; z = r\cos\theta$$

Los vectores a_x, a_y, a_z y a_r, a_θ, a_φ estan relacionados por

$$a_x = \sin\theta\cos\varphi \; a_r + \cos\theta\cos\varphi \; a_\theta - \sin\varphi \; a_\varphi$$

$$a_y = \sin\theta\sin\varphi \; a_r + \cos\theta\sin\varphi \; a_\theta + \cos\varphi \; a_\varphi$$

$$a_z = \cos\theta \; a_r - \sin\theta \; a_\theta$$

es decir

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} a_r \\ a_\theta \\ a_\varphi \end{pmatrix}$$

y

$$\begin{pmatrix} a_r \\ a_\theta \\ a_\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

Ahora

$$F(x, y, z) = A_x a_x + A_y a_y + A_z a_z =$$

$$A_x \left(\sin\theta\cos\varphi \; a_r + \cos\theta\cos\varphi \; a_\theta - \sin\varphi \; a_\varphi\right) +$$

$$A_y \left(\sin\theta\sin\varphi \; a_r + \cos\theta\sin\varphi \; a_\theta + \cos\varphi \; a_\varphi\right) +$$

$$A_z \left(\cos\theta \; a_r - \sin\theta \; a_\theta\right) =$$

$$\left(A_x \sin\theta\cos\varphi + A_y \sin\theta\sin\varphi + A_z \cos\theta\ \right) a_r +$$

$$\left(A_x \cos\theta\cos\varphi + A_y \cos\theta\sin\varphi - A_z \sin\theta\right) a_\theta +$$

$$\left(-A_x \sin\varphi + A_y \cos\varphi\right) a_\varphi = A_r a_r + A_\theta a_\theta + A_\varphi a_\varphi$$

por lo tanto

$$A_r = A_x \sin\theta\cos\varphi + A_y \sin\theta\sin\varphi + A_z \cos\theta$$

$$A_\theta = A_x \cos\theta\cos\varphi + A_y \cos\theta\sin\varphi - A_z \sin\theta$$

$$A_\varphi = -A_x \sin\varphi + A_y \cos\varphi$$

luego

$$\begin{pmatrix} A_r \\ A_\varphi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

y así

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\varphi & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\varphi \end{pmatrix}$$

$$= \begin{pmatrix} a_r \bullet a_x & a_\theta \bullet a_x & a_\varphi \bullet a_x \\ a_r \bullet a_y & a_\theta \bullet a_y & a_\varphi \bullet a_y \\ a_r \bullet a_z & a_\theta \bullet a_z & a_\varphi \bullet a_z \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\varphi \end{pmatrix}$$

por lo tanto

$$A_x = \sin\theta\cos\varphi\ A_r + \cos\theta\cos\varphi\ A_\theta\ - \sin\varphi\ A_\varphi$$

$$A_y = \sin\theta\sin\varphi\ A_r + \cos\theta\sin\varphi\ A_\theta + \cos\varphi\ A_\varphi$$

$$A_z = \cos\theta\ A_r - \sin\varphi\ A_\theta$$

1.2 El operador ∇ en coordenadas cilíndricas.

Recordemos que

$$\rho = \sqrt{x^2 + y^2}, \ \ \tan\varphi = \frac{y}{x}$$

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$$

Ahora

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x} =$$

$$= \frac{\partial}{\partial r}\frac{x}{\sqrt{x^2 + y^2}} - \frac{\partial}{\partial \varphi}\frac{y}{x^2 + y^2} + \frac{\partial}{\partial z} 0 =$$

$$= \frac{\partial}{\partial r}\frac{\rho\cos\varphi}{\rho} - \frac{\partial}{\partial \varphi}\frac{\rho\sin\varphi}{\rho^2} = \frac{\partial}{\partial \rho}\cos\varphi - \frac{\partial}{\partial \varphi}\frac{\sin\varphi}{\rho}$$

por tanto

$$\frac{\partial}{\partial x} = \cos\varphi \frac{\partial}{\partial \rho} - \frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi}$$

Ahora

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial z}{\partial y} =$$

$$= \frac{\partial}{\partial r}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial}{\partial \varphi}\frac{x}{x^2+y^2} + \frac{\partial}{\partial z}0 =$$

$$= \frac{\partial}{\partial \rho}\frac{\rho \sin\varphi}{\rho} - \frac{\partial}{\partial \varphi}\frac{\rho\cos\varphi}{\rho^2} = \sin\varphi\frac{\partial}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}$$

por tanto

$$\frac{\partial}{\partial y} = \sin\varphi\frac{\partial}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}$$

y

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial z} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial z} + \frac{\partial}{\partial z}\frac{\partial z}{\partial z} = \frac{\partial}{\partial z}$$

Así que

$$\nabla = \frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z =$$

$$= \left(\cos\varphi\frac{\partial}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\right)a_x + \left(\sin\varphi\frac{\partial}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\right)a_y + \frac{\partial}{\partial z}a_z =$$

$$= \left(\cos\varphi\frac{\partial}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\right)(\cos\varphi\ a_\rho - \sin\varphi\ a_\varphi) +$$

$$\left(\sin\varphi\frac{\partial}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\right)(\sin\varphi\ a_\rho + \cos\varphi\ a_\varphi) + \frac{\partial}{\partial z}a_z =$$

$$\cos^2\varphi\ \frac{\partial}{\partial \rho}a_\rho - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\cos\varphi\ a_\rho - \cos\varphi\sin\varphi\ \frac{\partial}{\partial \rho}a_\varphi + \frac{\sin^2\varphi}{\rho}\frac{\partial}{\partial \varphi}a_\varphi +$$

$$\sin^2\varphi\ \frac{\partial}{\partial \rho}a_\rho + \cos\varphi\sin\varphi\ \frac{\partial}{\partial \rho}a_\varphi + \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\cos\varphi\ a_\rho + \frac{\cos^2\varphi}{\rho}\frac{\partial}{\partial \varphi}a_\varphi + \frac{\partial}{\partial z}a_z =$$

$$= \frac{\partial}{\partial \rho}a_\rho + \frac{1}{\rho}\frac{\partial}{\partial \varphi}a_\varphi + \frac{\partial}{\partial z}a_z$$

por tanto

$$\nabla = \frac{\partial}{\partial \rho}a_\rho + \frac{1}{\rho}\frac{\partial}{\partial \varphi}a_\varphi + \frac{\partial}{\partial z}a_z$$

1.3 El operador ∇ en coordenadas esféricas.

Recordemos que

$$r = \sqrt{x^2+y^2+z^2}, \ \ \tan\theta = \frac{\sqrt{x^2+y^2}}{z}, \ \ \tan\varphi = \frac{y}{x} \text{ y que}$$

$$\nabla = \frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z$$

Ahora

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x} =$$

$$= \frac{\partial}{\partial r}\frac{x}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial \theta}\frac{zx}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} + \frac{\partial}{\partial \varphi}\left(\frac{-y}{x^2+y^2}\right) =$$

$$= \frac{\partial}{\partial r}\frac{r\cos\varphi\ \sin\theta}{r} + \frac{\partial}{\partial \theta}\frac{r\cos\theta\ r\cos\varphi\ \sin\theta}{r^2 r\sin\theta} + \frac{\partial}{\partial \varphi}\left(\frac{-r\sin\varphi\sin\theta}{r^2\sin^2\theta}\right) =$$

$$= \frac{\partial}{\partial r}\cos\varphi\ \sin\theta + \frac{\partial}{\partial \theta}\frac{\cos\theta\ \cos\varphi}{r} + \frac{\partial}{\partial \varphi}\left(\frac{-\sin\varphi}{r\sin\theta}\right)$$

luego

$$\frac{\partial}{\partial x} = \cos\varphi\ \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta\ \cos\varphi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

Ahora

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial y} =$$

$$\frac{\partial}{\partial r}\frac{y}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial \theta}\frac{zy}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} + \frac{\partial}{\partial \varphi}\left(\frac{x}{x^2+y^2}\right) =$$

$$= \frac{\partial}{\partial r}\frac{r\sin\varphi\ \sin\theta}{r} + \frac{\partial}{\partial \theta}\frac{r\cos\theta\ r\sin\varphi\ \sin\theta}{r^2 r\sin\theta} + \frac{\partial}{\partial \varphi}\left(\frac{r\cos\varphi\sin\theta}{r^2\sin^2\theta}\right) =$$

$$= \sin\varphi\ \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta\ \sin\varphi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

por tanto

$$\frac{\partial}{\partial y} = \sin\varphi\ \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta\ \sin\varphi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

Ahora

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial z} =$$

$$= \frac{\partial}{\partial r}\frac{z}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial \theta}\frac{-\sqrt{x^2+y^2}}{(x^2+y^2+z^2)} + \frac{\partial}{\partial \varphi}(0) =$$

$$= \frac{\partial}{\partial r}\frac{r\cos\theta}{r} + \frac{\partial}{\partial \theta}\frac{-r\sin\theta}{r^2} + \frac{\partial}{\partial \varphi}(0) = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

por tanto

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

entonces

$$\nabla = \frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z =$$

$$\left(\cos\varphi\ \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta\ \cos\varphi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)(\sin\theta\cos\varphi\ a_r + \cos\theta\cos\varphi\ a_\theta - \sin\varphi\ a_\varphi) +$$

$$\left(\sin\varphi\ \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta\ \sin\varphi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)(\sin\theta\sin\varphi\ a_r + \cos\theta\sin\varphi\ a_\theta + \cos\varphi\ a_\varphi) +$$

$$\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)(\cos\theta\ a_r - \sin\theta\ a_\theta) = \frac{\partial}{\partial r}a_r + \frac{1}{r}\frac{\partial}{\partial \theta}a_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}a_\varphi$$

haciendo las operaciones algebraicas, por tanto

$$\nabla = \frac{\partial}{\partial r}a_r + \frac{1}{r}\frac{\partial}{\partial \theta}a_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}a_\varphi$$

1.4 La divergencia en coordenadas cilindricas

$$divF(x,y,z) = \nabla \bullet F = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \left(\cos\varphi\frac{\partial}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\right)(\cos\varphi\ A_\rho - \sin\varphi\ A_\varphi) +$$

$$+\left(\sin\varphi\frac{\partial}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\right)(\sin\varphi\ A_\rho + \cos\varphi\ A_\varphi) + \frac{\partial A_z}{\partial z} =$$

$$= \left(\cos\varphi\frac{\partial}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\right)(\cos\varphi\ A_\rho) - \left(\cos\varphi\frac{\partial}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\right)(\sin\varphi\ A_\varphi) +$$

$$\left(\sin\varphi\frac{\partial}{\partial\rho}+\frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)(\sin\varphi\ A_\rho)+\left(\sin\varphi\frac{\partial}{\partial\rho}+\frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)(\cos\varphi\ A_\varphi)+\frac{\partial A_z}{\partial z}=$$

$$\cos\varphi\frac{\partial}{\partial\rho}(\cos\varphi\ A_\rho)-\frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}(\cos\varphi\ A_\rho)-\cos\varphi\frac{\partial}{\partial\rho}(\sin\varphi\ A_\varphi)+$$

$$\frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}(\sin\varphi\ A_\varphi)+\sin\varphi\frac{\partial}{\partial\rho}(\sin\varphi\ A_\rho)+\frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}(\sin\varphi\ A_\rho)+$$

$$\sin\varphi\frac{\partial}{\partial\rho}(\cos\varphi\ A_\varphi)+\frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}(\cos\varphi\ A_\varphi)+\frac{\partial A_z}{\partial z}=$$

$$\cos^2\varphi\frac{\partial A_\rho}{\partial\rho}+\frac{\sin^2\varphi}{\rho}A_\rho-\frac{\sin\varphi\cos\varphi}{\rho}\frac{\partial A_\rho}{\partial\varphi}-\cos\varphi\sin\varphi\frac{\partial A_\varphi}{\partial\rho}+\frac{\sin\varphi\cos\varphi}{\rho}A_\varphi\ +$$

$$\frac{\sin^2\varphi}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\sin^2\varphi\frac{\partial A_\rho}{\partial\rho}+\frac{\cos^2\varphi}{\rho}A_\rho+\frac{\cos\varphi\sin\varphi}{\rho}\frac{\partial A_\rho}{\partial\varphi}+\sin\varphi\cos\varphi\frac{\partial A_\varphi}{\partial\rho}+$$

$$-\frac{\cos\varphi\sin\varphi}{\rho}A_\varphi+\frac{\cos^2\varphi}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}=$$

$$=\frac{\partial A_\rho}{\partial\rho}+\frac{1}{\rho}A_\rho+\frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)+\frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}$$

por tanto

$$divF(x,y,z)=\nabla\bullet F=\frac{1}{\rho}\frac{\partial}{\partial r}(\rho A_\rho)+\frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}$$

1.5 La divergencia en coordenadas esféricas

$$divF(x,y,z)=\nabla\bullet F=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=$$

$$\left(\cos\varphi\ \sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta\ \cos\varphi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)(\sin\theta\cos\varphi\ A_r+\cos\theta\cos\varphi\ A_\theta-\sin\varphi\ A_\varphi)+$$

$$\left(\sin\varphi\ \sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta\ \sin\varphi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)(\sin\theta\sin\varphi\ A_r+\cos\theta\sin\varphi\ A_\theta+\cos\varphi\ A_\varphi)+$$

$$\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(\cos\theta\ A_r-\sin\theta\ A_\theta)=$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

por lo tanto

$$divF(x,y,z) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

1.6 El Rotacional en coordenadas cilindricas

$$RotF = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) a_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) a_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) a_z =$$

$$= \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\varphi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right) a_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) a_\varphi + \frac{1}{\rho}\left(\frac{\partial\left(\rho A_\varphi\right)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right) a_z$$

pues llamemos

$$A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) a_x =$$

$$\left(\left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)(A_z) - \frac{\partial}{\partial z}\left(\sin\varphi\ A_\rho + \cos\varphi\ A_\varphi\right)\right)\left(\cos\varphi\ a_\rho - \sin\varphi\ a_\varphi\right) =$$

$$\left(\left(\sin\varphi\frac{\partial A_z}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial A_z}{\partial\varphi}\right) - \left(\sin\varphi\ \frac{\partial A_\rho}{\partial z} + \cos\varphi\ \frac{\partial A_\varphi}{\partial z}\right)\right)\left(\cos\varphi\ a_\rho - \sin\varphi\ a_\varphi\right) =$$

$$\left(\sin\varphi\frac{\partial A_z}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial A_z}{\partial\varphi} - \sin\varphi\ \frac{\partial A_\rho}{\partial z} - \cos\varphi\ \frac{\partial A_\varphi}{\partial z}\right)\left(\cos\varphi\ a_\rho - \sin\varphi\ a_\varphi\right) =$$

$$\left(\sin\varphi\cos\varphi\frac{\partial A_z}{\partial\rho} + \frac{\cos^2\varphi}{\rho}\frac{\partial A_z}{\partial\varphi} - \sin\varphi\ \cos\varphi\frac{\partial A_\rho}{\partial z} - \cos^2\varphi\ \frac{\partial A_\varphi}{\partial z}\right) a_\rho +$$

$$\left(-\sin^2\varphi\frac{\partial A_z}{\partial\rho} - \frac{\cos\varphi\sin\varphi}{\rho}\frac{\partial A_z}{\partial\varphi} + \sin^2\varphi\ \frac{\partial A_\rho}{\partial z} + \cos\varphi\sin\varphi\ \frac{\partial A_\varphi}{\partial z}\right) a_\varphi$$

$$B = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) a_y =$$

$$\left(\cos\varphi\ \frac{\partial A_\rho}{\partial z} - \sin\varphi\ \frac{\partial A_\varphi}{\partial z} - \cos\varphi\ \frac{\partial A_z}{\partial\rho} + \frac{\sin\varphi}{\rho}\frac{\partial A_z}{\partial\varphi}\right)\left(\sin\varphi\ a_\rho + \cos\varphi\ a_\varphi\right) =$$

$$\left(\cos\varphi\sin\varphi\ \frac{\partial A_\rho}{\partial z} - \sin^2\varphi\ \frac{\partial A_\varphi}{\partial z} - \cos\varphi\ \sin\varphi\frac{\partial A_z}{\partial\rho} + \frac{\sin^2\varphi}{\rho}\frac{\partial A_z}{\partial\varphi}\right) a_\rho +$$

$$\left(\cos^2\varphi\ \frac{\partial A_\rho}{\partial z} - \sin\varphi\cos\varphi\ \frac{\partial A_\varphi}{\partial z} - \cos^2\varphi\ \frac{\partial A_z}{\partial \rho} + \frac{\sin\varphi\cos\varphi}{\rho}\frac{\partial A_z}{\partial\varphi}\right) a_\varphi$$

$$C = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) a_z =$$

$$\left[\left(\cos\varphi\frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)(\sin\varphi A_\rho + \cos\varphi A_\varphi) - \left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)(\cos\varphi A_\rho - \sin\varphi A_\varphi)\right] a_z =$$

$$= (\cos\varphi\sin\varphi\frac{\partial A_\rho}{\partial r} - \frac{\sin\varphi\cos\varphi}{\rho}A_\rho - \frac{\sin^2\varphi}{\rho}\frac{\partial A_\rho}{\partial\varphi} + \cos^2\varphi\frac{\partial A_\varphi}{\partial\rho} + \frac{\sin^2\varphi}{\rho}A_\varphi - \frac{\sin\varphi\cos\varphi}{\rho}\frac{\partial A_\varphi}{\partial\varphi}$$

$$-\cos\varphi\sin\varphi\frac{\partial A_\rho}{\partial\rho} + \sin^2\varphi\frac{\partial A_\varphi}{\partial\rho} + \frac{\sin\varphi\cos\varphi}{\rho}A_\rho - \frac{\cos^2\varphi}{\rho}\frac{\partial A_\rho}{\partial\varphi} + \frac{\cos^2\varphi}{\rho}A_\varphi + \frac{\sin\varphi\cos\varphi}{\rho}\frac{\partial A_\varphi}{\partial\varphi})a_z$$

Ahora

$$A + B + C = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right) a_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) a_\varphi + \left(\frac{A_\varphi}{\rho} - \frac{\partial A_\rho}{\rho\partial\varphi} + \frac{\partial A_\varphi}{\partial\rho}\right) =$$

$$= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right) a_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) a_\varphi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi}\right) a_z$$

1.7 El rotacional de F en coordenadas esféricas.

En forma análoga se deduce que el rotacional de F en coordenadas esféricas viene dado por

$$RotF = \nabla\times F = \frac{1}{r^2\sin\theta}\begin{vmatrix} a_r & ra_\theta & r\sin\theta a_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix} =$$

$$= \frac{1}{r\sin\theta}\left(\frac{\partial(A_\varphi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right) a_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial(rA_\varphi)}{\partial r}\right) a_\theta + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right) a_\varphi$$

El laplaciano de

$$V = \nabla\bullet\nabla V = \nabla^2 V = \left(\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z\right)\bullet\left(\frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z\right) =$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial^2 z}$$

1.8 El laplaciano coordenadas cilíndricas

$$\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z =$$

$$\left(\frac{\partial}{\partial \rho}\cos\varphi - \frac{\partial}{\partial \varphi}\frac{\sin\varphi}{r}\right)(\cos\varphi\ a_\rho - \sin\varphi\ a_\varphi)+\left(\frac{\partial}{\partial \rho}\sin\varphi + \frac{\partial}{\partial \varphi}\frac{\cos\varphi}{\rho}\right)(\sin\varphi\ a_\rho + \cos\varphi\ a_\varphi)+\frac{\partial a_z}{\partial z}$$

$$\frac{\partial a_\rho}{\partial \rho} + \frac{1}{\rho}a_\rho + \frac{1}{\rho}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z}$$

y como

$$\frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z = \frac{\partial V}{\partial \rho}a_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \varphi}a_\varphi + \frac{\partial V}{\partial z}a_z$$

entonces

$$V = \nabla \bullet \nabla V = \nabla^2 V = \left(\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z\right) \bullet \left(\frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z\right) =$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial^2 z} =$$

$$= \left(\frac{\partial a_\rho}{\partial \rho} + \frac{1}{\rho}a_\rho + \frac{1}{\rho}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z}\right) \bullet \left(\frac{\partial V}{\partial \rho}a_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \varphi}a_\varphi + \frac{\partial V}{\partial z}a_z\right) =$$

$$= \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}$$

1.9 El Laplaciano en coordenadas esféricas

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \varphi^2}\right)$$