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CERTAIN CLASSES OF UNIVALENT ANALITIC FUNCTIONS

by

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§1. Introduction. In the present paper we shall study the subclasses of starlike, convex, meromorphically starlike, and meromorphically convex functions. Our results extend, generalize and unify the existing results. We base the development of our paper on classical methods. In some instances our results are completely new.

§2. Some classes of univalent functions. 2.1. Let m and M be arbitrary fixed real numbers which satisfy the relation $(m,M) \in E$ where

$$(2.1) \qquad E = \{(m,M) : m > \frac{1}{2},\ |m - 1| < M \leq m\}.$$

Let us denote by $S(m,M)$ and $K(m,M)$ the class of functions of the form

$$(2.2) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n ,$$

regular in the unit disc $D = \{z \mid |z| < 1\}$ and satisfying there the conditions

$$(2.3) \qquad \left| z \frac{f'(z)}{f(z)} - m \right| < M$$

and

$$(2.4) \qquad \left| 1 + z \frac{f''(z)}{f'(z)} - m \right| < M ,$$

respectively, for $(m,M) \in E$.*

Further let us denote by $\Gamma(m,M)$ and $\Sigma(m,M)$ the classes of functions of the form

$$(2.5) \qquad g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n ,$$

regular in the disc $D_0 = \{z \mid 0 < |z| < 1\}$, having a simple pole at the origin, and satisfying the conditions

$$(2.6) \qquad \left| z \frac{g'(z)}{g(z)} + m \right| < M$$

and

$$(2.7) \qquad \left| 1 + z \frac{g''(z)}{g'(z)} + m \right| < M ,$$

respectively, for $(m,M) \in E$. If we take

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* For further references and other subsequent information we refer to [7].

(2.8) $$a = \frac{M^2 - m^2 + m}{M}$$

and

(2.9) $$b = \frac{m - 1}{M}$$

then the conditions (2.3), (2.4), (2.6) and (2.7) are equivalent to

(2.10) $$z \frac{f'(z)}{f(z)} = \frac{1 + a\, w_1(z)}{1 - b\, w_1(z)} ,$$

(2.11) $$1 + z \frac{f''(z)}{f'(z)} = \frac{1 + a\, w_2(z)}{1 - b\, w_2(z)} ,$$

(2.12) $$z \frac{g'(z)}{g(z)} = - \frac{1 + a\, w_3(z)}{1 - b\, w_3(z)} ,$$

and

(2.13) $$1 + \frac{z\, g''(z)}{g'(z)} = - \frac{1 + a\, w_4(z)}{1 - b\, w_4(z)} ,$$

respectively, for some $w_j(z)$, $j=1,2,3,4$, regular and satisfyingthe conditions $|w_j(0)| = 0$, $|w_j(z)| < 1$ in D. In particular, if we choose

$$a = \frac{\alpha - 2N\alpha + N}{N} , \quad b = \frac{N - 1}{N}$$

and make $N \to \infty$ then (2.10), (2.11), (2.12) and (2.13), respectively, imply that

(2.14) $$\operatorname{Re} \left\{\frac{zf'(z)}{f(z)}\right\} > \alpha ,$$

(2.15) $$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha ,$$

(2.16) $$\text{Re}\left\{\frac{zg'(z)}{g(z)}\right\} < -\alpha ,$$

and

(2.17) $$\text{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\} < -\alpha$$

where $0 \leqslant \alpha < 1$.

But functions satisfying (2.14), (2.15), (2.16) and (2.17), respectively, are called starlike univalent functions, convex univalent functions, meromorphic starlike functions, and meromorphic convex functions of order α, and their classes are denoted by $S^*(\alpha)$, $K(\alpha)$, $\Gamma^*(\alpha)$ and $\Sigma(\alpha)$, respectively. Thus $S^*(\alpha) = S(m,M)$, $K(\alpha) = K(m,M)$, $\Gamma^*(\alpha) = \Gamma(m,M)$ and $\Sigma^*(\alpha) = \Sigma(m,M)$, for $0 \leqslant \alpha < 1$ and $M \to \infty$.

In 1964, M.S.Robertson [13] proposed the problem of proving that if $f(z) \in S^*(\alpha)$ (or $K(\alpha)$) then

$$\left(\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt\right) \in S^*(\alpha) \quad (\text{or } K(\alpha))$$

for $c = 1$ and $\alpha = 0$.

Subsequently, the problem was solved by Libera [10], and generalized by Bernardi [4] and by Bajpai and Srivastava [2a]. The first author found that the above result is true for all $c > -1$ and $0 \leqslant \alpha < 1$. In [1], analogous results for the meromorphic classes $\Sigma^*(\alpha)$ and $\Gamma^*(\alpha)$ are also obtained. In the present paper, we shall extend the results to $S(m,M)$, $K(m,M)$, $\Sigma(m,M)$ and $\Gamma(m,M)$.

2.2. To prove our theorems we need the following lemma due to I.S. Jack [8].

LEMMA 2.2.1. *Suppose that* $w(z)$ *is analytic for* $|z| \leqslant r < 1$, $w(0) = 0$ *and* $|w(z_1)| = \max_{|z|=r} |w(z)|$ *then* $z_1 w'(z_1) = k\, w(z_1)$ *where* $k \geqslant 1$.

2.3. In this section we shall prove

THEOREM 2.3.1. *If* $f \in S(m,M)$ *and* $F(z)$ *is defined by*

$$(2.18)\qquad F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt, \qquad c > -\frac{1-a}{1+b}$$

where a,b *are defined by the formulas* (2.8) *and* (2,9), *and* $(m,M) \in E$ *then* $F \in S(m,M)$.

Proof. Let us choose a function $w(z)$ regular in D such that

$$w(0) = 0 \quad \text{and} \quad z\frac{F'(z)}{F(z)} = \frac{1+a\;w(z)}{1-b\;w(z)} .$$

From (2.18) we get

$$(2.19)\qquad z\,\frac{f'(z)}{f(z)} - m = \frac{(1-m) + (a+bm)\,w(z)}{1 - b\,w(z)} + \frac{(a+b)\,z\,w'(z)}{\{1-bw(z)\}\{(1+c)+(a-bc)w(z)\}} .$$

Now suppose that it were possible to have $M(r,w) = \max_{|z|=r} |w(z)| = 1$ for some $r < 1$. At the point z_o where this occurs we would have $|w(z_o)| = 1$

(but clearly $|w(z)| \neq 1$). Then, by lemma 2.2.1, there is a point z_o such that

$$(2.20) \qquad z_o\, w'(z_o) = k\, w(z_o), \qquad k \geq 1.$$

From (2.19) and (2.20) we have

$$(2.21) \qquad z_o \frac{f'(z_o)}{f(z_o)} - m \equiv \frac{N(z_o)}{D(z_o)}$$

where

$$(2.22) \quad N(z_o) = (1-m)(1+c) + [(1+c)(a+bm)+(a-bc) -(a-bc)m+k(a+b)]w(z_o)+(a-bc)(a+bm) \cdot w^2(z_o)$$

and

$$(2.23) \quad D(z_o) = (1+c)+(a-2bc-b)w(z_o)-b(a-bc)w^2(z_o).$$

If we take

$h = (1-m)(1+c)$, $d = (1-m)(a-bc)+(1+c)(a+bm)+k(a+b)$,

$e = (a-bc)(a+bm)$, $j = (a-bc)-b(1+c)$ and $z = b(a-bc)$

then

$$(2.24) \qquad N(z_o) = h+d\, w(z_o) + e\, w^2(z_o)$$

and

$$(2.25) \qquad D(z_o) = (1+c)+j\, w(z_o)-z\, w^2(z_o).$$

Now, using $|w(z_o)| = 1$, we have

$$(2.26) \quad |N(z_o)|^2 = (h^2+d^2+e^2)+2(e+h)d\,\mathrm{Re}\{w(z_o)\} + 2\, eh\, \mathrm{Re}\{w^2(z_o)\}$$

and

$$(2.27) \quad |D(z_o)|^2 = (1+c)^2+j^2+z^2+2(1+c-z)jRe\{w(z_o)\} + 2\, z(1+c)\, Re\{w^2(z_o)\}.$$

Also

$$(2.28) \quad |N(z_o)|^2-M^2|D(z_o)|^2 = A+2B\, Re\{w(z_o)\} + 2CRe\{w^2(z_o)\}$$

where

$$A = (h^2+d^2+e^2)-M^2\{(1+c)^2+j^2+z^2\}$$

$$= k(a+b)\left[k(a+b)+2M(1+c)-2Mb(a-bc)\right]$$

$$B = (e+h)d-M^2j(1+c-z) = Mk(a+b)\{(a-bc)-b(1+c)\}$$

and

$$C = eh + M^2z(1+c)$$

$$= (1-m)(1+c)(a-bc)(a+bm)+M^2b(a-bc)(1+c) = 0.$$

Since $C = 0$, from (2.28) it is clear that

$$(2.29) \quad |N(z_o)|^2 - M^2|D(z_o)|^2 \geqslant 0 ,$$

provided $A + 2B \geqslant 0$.

Now

$$A + 2B = k(a+b)\left[k(a+b)+2M(1+c) - 2Mb(a-bc)+2M(a-bc)-2Mb(1+c)\right] \geqslant 0.$$

$$A - 2B = k(a+b)\left[k(a+b)+2M(1+c) - 2Mb(a-bc)-2M(a-bc)+2Mb(1+c)\right] \geqslant 0.$$

Thus we have proved (2.29) which along with (2.21) gives $\left| z_o \dfrac{f'(z_o)}{f(z_o)} - m \right| \geqslant M.$

But this is a contradiction to the fact that $f \in S(m,M)$. So we can not have $M(r,w) = 1$. Since this is true for every $r < 1$ and since $M(0,w)=0$ it is clear that we must have $M(r,w) < 1$ and so $|w(z)| < 1$ for $|z| < 1$. Therefore, $F \in S(m,M)$. ∎

COROLLARY 2.3.1. *If* $f \in K(m,M)$ *and* F *is defined by* (2.18) *then* $F \in K(m,M)$, *provided* $c \geqslant (1-a)/(1+b)$.

Proof: We can write (2.18) in the form

$$z\,F'(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}\cdot t\,f'(t)dt.$$

Since $f \in K(m,M)$ it is easy to see that $zf'(z) \in S(m,M)$. Therefore, by Theorem 2.3.1 we get $zF'(z) \in S(m,M)$, which implies $F(z) \in K(m,M)$. ∎

Remark 1. In theorem 2.3.1, if we put $m = M$ and $m \to \infty$ then the results of Bernardi in [4] follow. If

$$m = \frac{\alpha-2\alpha N+N}{N}, \quad M = \frac{N-1}{N} \quad \text{and} \quad N \to \infty$$

then the results of Bajpai in [2] follow. Finally, if $m = M$, $c = 1$ and $m \to \infty$ then the results of Libera in [10] follow.

THEOREM 2.3.2. *If* $f \in S^*(\alpha)$, $g \in S(m,M)$ *and* $F(z)$ *is defined by*

$$(2.22)\quad F(z) = \frac{c+2}{z^{c+1}}\int_0^z t^{c-1}\,f(t)\,g(t)\,dt, \quad c \geqslant 0$$

then $F \in S^*(\alpha)$ *if* $0 \leqslant \alpha < 1$ *and* m *and* M *satisfy*

$$m \geqslant \frac{4c+3+5\alpha}{4(c+1+\alpha)}, \; |m-1| < M$$

$$|m-1| < M \leqslant m-1 + \frac{1-\alpha}{2(c+1+\alpha)} \}.$$

Remark 2. Let us take $G(z) = \frac{f(z)\, g(z)}{z}$. Then (2.22) reduces to

$$F(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c\, G(t)dt, \qquad c \geqslant 0$$

Bernardi [4] proved that $F(z) \in S^*(0)$ if $G(z) \in S^*(0)$. If we take $f(z)$ and $g(z)$ such that

$$z\,\frac{f'(z)}{f(z)} = \frac{1-z}{1+z} \quad \text{and} \quad z\,\frac{g'(z)}{g(z)} = 1 - \frac{z}{2(c+1)}$$

then $f \in S^*(0)$, $g(z) \in S(1, 1/2(c+1))$

and

$$z\,\frac{G'(z)}{G(z)} = \frac{2(c+1) - (2c+3)\,z - z^2}{2(c+1)(1+z)}.$$

If we take z real between $(\sqrt{4c^2+20c+17} - (2c+3))/2$ and 1 then it is easily seen that $\mathrm{Re}\{zG'(z)/G(z)\} < 0$ and so $G(z) \notin S^*(0)$. But by theorem 2.3.2 we have $F(z) \in S^*(\alpha)$.

Following the lines of the proof of theorem 2.3.1 we also have:

THEOREM 2.3.3. *Let* $f \in \Gamma(m, M)$ *and* $F(z)$ *be defined by*

$$(2.23) \qquad F(z) = \frac{c}{z^{c+1}} \int_0^z t^c\, f(t)\, dt.$$

Then $F \in \Gamma(m,M)$ *if* $c \geqslant \max\{\frac{a+b}{1-b}, 1\}$.

COROLLARY 2.3.2. *If* $f \in \Sigma(m,M)$ *and* $F(z)$ *is defined by* (2.23) *then* $F \in \Sigma(m,M)$, *provided* $c \geqslant \max\{\frac{a+n}{1-b}, 1\}$.

Proof. We can write (2.23) as

$$z\,F'(z) = \frac{c}{Z^c+1} \int_o^z t^c \cdot t\, f'(t)\, dt.$$

Since $f \in \Sigma(m,M)$ we have $zf'(z) \in \Sigma(m,M)$ and hence from theorem 2.3.3 we get $zF'(z) \in \Sigma(m,\underline{M})$. So $F(z) \in (m,M)$. ■

Remark 3. If we take $m = M$ and $m \to \infty$ then the results of Bajpai in [2] follow from theorem 2.3.3.

An analogue of Theorem 2.3.2 for meromorphic functions is the following.

THEOREM 2.3.4. *Let* $f \in \Gamma^*(\alpha)$, $g \in \Gamma(m,M)$ *and let* $F(z)$ *be defined by*

$$(2.24) \quad F(z) = \frac{c}{z^{c+1}} \int_o^z t^{c+1} f(t)\, g(t)\, dt, \qquad c \geqslant 1,$$

Then $F \in \Gamma^*(\alpha)$, *provided* m *and* M *satisfy*

$$m \geqslant \frac{4c+3(1-\alpha)}{4(c+1-\alpha)}, \quad \|m-1\| < M \leqslant (m-1) + \frac{1-\alpha}{2(c+1-\alpha)}$$

Remark 4. Let us take $G(z) = z\,f(z)\,g(z)$. Then (2.24) reduces to

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c G(t)dt, \qquad c \geqslant 1.$$

Bajpai [1] has proved that $F \in \Gamma^*(0)$ if $G(z) \in \Gamma'^*(0)$. If we take f(z) and g(z) such that

$$z\frac{f'(z)}{f(z)} = -\frac{1-z}{1+z} \quad \text{and} \quad z\frac{g'(z)}{g(z)} = -1 + \frac{z}{2(c+1)}$$

then $f(z) \in \Gamma^*(0)$ and $g(z) \in \Gamma(m,M)$ for $m = 1$ and $M = \frac{1}{2(c+1)}$. But $z\frac{G'(z)}{G(z)} = -\frac{2(c+1)-(2c+3)z-z^3}{2(c+1)z}$.

If we take z real and between $(\sqrt{4c^2+20c+17} - (2c+3))/2$ and 1 then it is easily seen that $\operatorname{Re}\{z\frac{G'(z)}{G(z)}\} > 0$ and so $G(z) \notin \Gamma^*(0)$. But by theorem 2.3.4 we have $F \in \Gamma^*(\alpha)$.

We have omitted the proofs of theorems 2.32, 2.33, and 2.34 since they all follow the same lines of the proof of Theorem 2.3.1.

§3. A subordination to a certain class of analytic functions. 3.1. It is well known that convex functions are starlike with respect to the origin. In 1933 A. Marx [12] and E. Strohacker [14] proved that if $f(z) \in K(0)$ then $f(z) \in S^*(\beta)$ where $\beta \geqslant \frac{1}{2}$. This result is sharp as can be seen from the function $z/(1-z)$. In 1971, I.S. Jack [8] generalized this result and proved the following.

THEOREM A. (Jack) *If* $f(z) \in K(\alpha)$ *then* $f(z) \in S^*(\beta(\alpha))$ *where*

(3.1) $$\beta(\alpha) \geqslant \frac{(2\alpha-1)-\sqrt{9-4\alpha+4\alpha^2}}{4}$$

But this bound for $\beta(\alpha)$ is not sharp. Jack [8] conjectured that

$$(3.2) \qquad \beta(\alpha) = \begin{cases} \dfrac{(1-2\alpha)}{4^{1-\alpha}(1-2^{2\alpha-1})} & \text{if } \alpha \neq \tfrac{1}{2} \\[2ex] \dfrac{1}{\log 4} & \text{if } \alpha = \tfrac{1}{2} \end{cases}$$

Recently T.H. MacGregor [11] has settled this conjecture. MacGregor's proof is very nice and independent of any classical result. His result is as follows.

THEOREM B (MacGregor) *If is convex of order* α, i.e. $f \in K(\alpha)$, *then* $z\,\dfrac{f'(z)}{f(z)} \ll J(z)$, i.e. $z\,\dfrac{f'(z)}{f(z)}$ *is subordinate to* $J(z)$, *where*

$$(3.3) \qquad J(z) = \begin{cases} \dfrac{(2\alpha-1)z}{(1-z)^{2(1-\alpha)}\{1-(1-z)^{2\alpha-1}\}} & \textit{if } \alpha = \tfrac{1}{2} \\[2ex] -\dfrac{z}{(1-z)\log(1-z)} & \textit{if } \alpha = \tfrac{1}{2} \end{cases}$$

In this section we have proved a similar result for the classes $S(m,M)$ and $K(m,M)$ defined in section 2. In proving our results we follow procedures developed by MacGregor.

3.2. We need the following lemmas for the proof of our theorem.

LEMMA 3.2.1 *Suppose that the function* T *and*

S *are analytic in* D, T(0) = 0 = S(0), *and* S *maps* D *onto a (possibly many sheeted) region which is starlike with respect to the origin. If*

(3.4) $$\mathrm{Re}\left\{\frac{T'(z)}{S'(z)}\right\} > \delta \quad \text{for } |z| < 1$$

then

(3.5) $$\mathrm{Re}\left\{\frac{T(z)}{S(z)}\right\} > \delta \quad \text{for } |z| < 1\,,$$

and if

(3.6) $$\mathrm{Re}\left\{\frac{T'(z)}{S'(z)}\right\} < \delta \quad \text{for } |z| < 1$$

then

(3.7) $$\mathrm{Re}\left\{\frac{T(z)}{S(z)}\right\} < \delta \quad \text{for } |z| < 1\,.$$

The first half of the lemma can be found in [10], and for $\delta = 0$ in [4]. It appears complete in [11].

LEMMA 3.2.2(**) *If we define* G(z) *in* D *as follows, for* $a \neq b$

(3.8) $$G(z) = \frac{az(1-bz)^{-(a+b/b)}}{(1-bz)^{-a/b}-1}\,, \quad G(0) = 1$$

then G(z) *is univalent in* D.

Proof: If we write

(3.9) $$F(z) = \frac{(1-bz)^{-a/b}}{a}$$

(**) G(z) is defined by its limiting values a a = 0 and b = 0.

then, by logarithmic differentiation we get

$$(3.10) \qquad z\,\frac{F'(z)}{F(z)} = \frac{az(1-bz)^{-(a+b)/b}}{(1-bz)^{-a/b} - 1}$$

From (3.8) we have

$$(3.11) \qquad G(z) = z\,\frac{F'(z)}{F(z)}$$

Let us rewrite G(z) as

$$(3.12) \qquad G(z) = \frac{-a/b}{1+G_1(z)} \quad ,$$

where

$$(3.13) \qquad G_1(z) = \frac{(1-bz)^{(a+b)/b} - 1}{b\ z} .$$

Rewrite $G_1(z)$ in terms of $G_2(z)$ so that

$$(3.14) \qquad G_2(z) = G_1(z) + \frac{a+b}{b} = \frac{a(a+b)}{bz}\int_0^z G_3(t)dt$$

where

$$(3.15) \qquad G_3(z) = \frac{1-(1-bz)^{((a+b)/b)-1}}{a} .$$

Differentiating $G_3(z)$ and then differentiating logarithmically and taking the real parts in both sides, we have

$$(3.16) \qquad \mathrm{Re}\ \{1+z\,\frac{G''_3(z)}{G'_3(z)}\} = \frac{(1-a|z|)(1-b|z|)}{|1-bz|^2} > 0.$$

Since $G_3(0) = 0$ and $G'_3(0) = 1$, $G_3(z)$ is univalent and convex. Using Corollary 2.31 we observe that $G_2(z)$ is convex univalent (of course not normalized). This is turn implies that G(z) is univalent. As a remark we point out here that the univalence of $G_2(z)$ and hence of $G(z)$ can also be established in the following way. As in (3.16), we find that

$$(3.17)\qquad \left|1+z\frac{G''_3(z)}{G'_3(z)} - \frac{1-ab}{1-b^2}\right| < \begin{cases} \dfrac{a-b}{1-b^2} & \text{if } a > b \\ \dfrac{b-a}{1-b^2} & \text{if } a < b\,. \end{cases}$$

This implies that

$$(3.18)\qquad G_3(z) \in K\left(\frac{1-ab}{1-b^2}, \frac{|a-b|}{1-b^2}\right)$$

Then by theorem (2.3.1) it follows that

$$(3.19)\qquad G_4(z) \in K\left(\frac{1-ab}{1-b^2}, \frac{|a-b|}{1-b^2}\right)$$

where

$$(3.20)\qquad G_4(z) = \frac{2}{z}\int_0^z G_3(t)\,dt.$$

Since $G_2(z) = \dfrac{(a+b)G_4(z)}{2b}$, $G_2(z)$ is univalent and so $G(z)$ is univalent. But (3.19) is a stronger conclusion than (3.16). ∎

LEMMA 3.2.3: *If* $F(z)$ *is defined by* (3.9) *and* $a < b$ *then*

$$(3.21)\qquad \frac{1-(1+br)^{-a/b}}{a\,r} \leq \operatorname{Re}\left\{\frac{F(z)}{z}\right\} \leq \frac{(1-br)^{-a/b}-1}{a\,r}$$

provided $b^2 < a$ *when* $a+b > 1$.

<u>Proof</u>: Let $z = r\,e^{i\theta}$ and $1\text{-}bz = Re^{i\phi}$. Then

$$(3.22)\quad \operatorname{Re}\left\{\frac{F(z)}{z}\right\} = \frac{R^{-a/b} r\cos(\frac{a}{b}\phi+\theta) - r\cos\theta}{a\,r^2}$$

$$\equiv P(r,\theta)$$

where $R = (1 - 2br\cos\theta + b^2r^2)^{\frac{1}{2}}$ and $\tan\theta = -\frac{br\sin\theta}{1-br\cos\theta}$. Therefore,

$$(3.23)\qquad \frac{dR}{d\theta} = \frac{br\sin\theta}{R}$$

and

$$(3.24)\qquad \frac{d\phi}{d\theta} = \frac{br(br-\cos\theta)}{R^2}$$

Since $(1\text{-}bz) = Re^{i\theta}$ and $\operatorname{Re}(1\text{-}bz) > 0$, then $\phi = 0$ if $\theta = 0$ or $\theta = \pi$. From (3.22), we have

$$(3.25)\quad \frac{\partial P(r,\theta)}{\partial\theta} = \{-ar\,R^{-(a+2b)/b}\cos(\tfrac{a}{b}\phi+\theta) - R^{-a/b}\sin(\tfrac{a}{b}\phi+\theta)\times(\frac{ar(br-\cos\theta)}{R^2} + 1) + \sin\theta\}\,/ar^2.$$

Define $G_1(r)$ by

$$(3.26)\qquad \left.\frac{\partial^2 P(r,\theta)}{\partial\theta^2}\right|_{\theta=0} \equiv \frac{G_1(r)}{a\,r}.$$

Differentiating $G_1(r)$ we get

$$(3.27)\quad G'_1(r) = -a(a+b)r(1-br)^{-(a+3b)/2}\{1+(a+b)r\}.$$

If $a > 0$ then $G'_1(r) < 0$. Hence $G_1(r)$ is a decreasing function of r so $G_1(r) \leqslant G_1(0) = 0$.

In this case $\left.\frac{\partial^2 P(r,\theta)}{\partial\theta^2}\right|_{\theta=0} \leqslant 0.$

Similarly, if $a < 0$ then $G_1(r)$ is an increasing function of r, so $G_1(r) \geqslant G_1(0) = 0$. In this case we obtain the same inequality. This implies that $\mathrm{Re}\{\frac{F(z)}{z}\}$ is maximum if $\theta = 0$. Define $G_2(r)$ by

$$(3.28)\qquad \left.\frac{\partial^2 P(r,\theta)}{\partial\theta^2}\right|_{\theta=\pi} \equiv -\frac{G_2(r)}{ar}\ .$$

Differentiating $G_2(r)$ with respect to r we have

$$(3.29)\qquad G'_2(r) = -a(a+b)r(1+br)^{-(a+3b)/b}\{1-(a+b)r\}.$$

Now two cases arise.

<u>Case 1</u>. $a+b < 1$. In this case $\{1-(a+b)r\} > 0$. So $G'_2(r) < 0$ if $a > 0$, and $G'_2(r) > 0$ if $a < 0$. In the first case $G_2(r)$ is a decreasing function or r so $G_2(r) < G_2(0) = 0$ and so

$$\left.\frac{\partial^2 P(r,\theta)}{\partial\theta^2}\right|_{\theta=\pi} > 0.$$

If $a < 0$, $G_2(r)$ is an increasing function of r so $G_2(r) \geqslant G_2(0) = 0$ and the same inequality is obtanined. This implies that $\mathrm{Re}\{\frac{F(z)}{z}\}$ is minimum for $\theta = \pi$.

<u>Case 2</u>. $a+b > 1$. In this case $0 < 1/(a+b) < 1$ and a and b are positive. So we have $G_2(r) \leqslant \max\{G_2(0),\ G_2(1)\}$. Now

$$G_2(1) \equiv Q(a,b) = 1-(1+b)^{-((a+2b)/b)}((1+a+b)^2-a)$$

$$\leqslant G_2(0) = 0$$

if

$$(1+b)^2(1+b)^{a/b} \leqslant (1+a+b)^2 - a .$$

But the above inequality is satisfied if

$$(1+b)^2(1+b)^{a/b} \leqslant (1+b)^2(1+a) \leqslant (1+a+b)^2 - a$$

or if $b^2 \leqslant a$ since $a > 0$.

But $b^2 < a$ is equivalent to the inequality

$$M^3 - m^2 M + mM - m^2 + 2m - 1 > 0.$$

The above inequality is always satisfied if $1/2 \leqslant b \leqslant ((M^2+4M)^{1/2}-M)/2$. Hence, we find that $G_2(r) \leqslant 0$ and so $\partial^2 P(r,\theta)/\partial\theta^2 \big|_{\theta=\pi} > 0$, implying that the minimun of $\mathrm{Re}(F(z)/z)$ is attained at $\theta = \pi$. This completes the proof of the lemma. ∎

LEMMA 3.2.4. *If* $H(z) = \dfrac{1+az}{1-bz}$, $a \leqslant b$, *and* $G(z)$ *is defined by* (3.8) *then*

(3.30) $$H_k(z) = k\, H(z) + (1-k)\, G(z)$$

is univalent in *for* $k \geqslant 1$, *provided* $b^2 < a$ *whenever* $a+b > 1$.

<u>Proof:</u> We begin by showing

(3.31) $$\mathrm{Re}\left\{\frac{G'(z)}{H'(z)}\right\} < 1 \quad \text{for} \quad z \in D.$$

We see that

(3.32) $$G'(z) = \frac{a(1-bz)^{-(a+2b)/b}\{(1-bz)^{-a/b}-(1+az)\}}{\{(1-bz)^{-a/b} - 1\}^2}$$

and

$$(3.33) \qquad H'(z) = \frac{a+b}{(1-bz)^2} .$$

From (3.32) and (3.33) we get

$$(3.34) \qquad \frac{G'(z)}{H'(z)} = - \frac{1}{a+b} \left(\frac{T(z)}{S(z)} \right)$$

where

$$(3.35) \qquad T(z) = a(1-bz)^{-a/b}\{1+az-(1-bz)^{-a/b}\}$$

and

$$(3.36) \qquad S(z) = S_1^2(z)$$

with

$$(3.37) \qquad S_1(z) = (1-bz)^{-a/b}-1 .$$

It is easy to see that $S_1(z)/a \in K(m,M)$ and hence belongs to $S(m,M)$, so $S(z)$ is bivalent and satisfies the condition

$$(3.38) \qquad \left| z \frac{S'(z)}{S(z)} - 2m \right| < 2M.$$

We have

$$(3.39) \qquad \frac{T'(z)}{S'(z)} = \frac{(a-b)z}{2F(z)} - a$$

where $F(z)$ is given by (3.9).

Now by using the result (3.21) in Lemma 3.2.3, we get in D

$$(3.40) \qquad \text{Re}\left\{\frac{T'(z)}{S'(z)}\right\} \geqslant \frac{a-b}{2} . \frac{a}{1-(1+b)^{-a/b}} - a, \quad \text{if} \quad a < b.$$

From Lemma 3.2.1 and (3.40) we get

$$(3.41) \qquad \mathrm{Re}\left\{\frac{T(z)}{S(z)}\right\} \geqslant \frac{a-b}{2} \frac{a}{1-(1+b)^{-a/b}} - a, \text{ if } \ a < b.$$

From (3.34) and (3.41) we have

$$(3.42) \qquad \mathrm{Re}\left\{\frac{G'(z)}{H'(z)}\right\} \leqslant - \frac{1}{a+b} \left\{\frac{a-b}{2} . \frac{a}{1-(1+b)^{-a/b}} - a\right\}$$

$$= \frac{a}{a+b} - \frac{a(a-b)}{2(a+b)\{1-(1+b)^{-a/b}\}}, \text{ if } a < b.$$

To prove (3.31) it is sufficient to prove that

$$(3.43) \qquad - \frac{a(a-b)}{2\{1-(1+b)^{-a/b}\}} \leqslant b \ .$$

Since we are considering the case $b > a$ and we know that $(a+b) \geqslant 0$, b is always positive but a may be either positive or negative. If $a \geqslant 0$, $\{1-(1+b)^{-a/b}\} \geqslant 0$. If $a < 0$, then $\{1-(1+b)^{-a/b}\} < 0$; therefore, (3.43) is equivalent to

$$(3.44) \qquad a^2-ab+2b-2b(1+b)^{-a/b} > 0 \ , \quad \text{if } 0 < a < b$$

and

$$(3.45) \qquad a^2-ab+2b-2b(1+b)^{-a/b} < 0, \quad \text{if } a < b \text{ and } a < 0.$$

Let us write

$$(3.46) \qquad A(a,b) = a^2-ab+2b-2b(1+b)^{-a/b}.$$

Differentiating $A(a,b)$ with respect to a, we have,

$$(3.47) \qquad \frac{\partial A(a,b)}{\partial a} = 2a-b+2(1+b)^{-a/b}\log(1+b)$$

and

$$(3.48) \qquad \frac{\partial^2 A(a,b)}{a^2} = \frac{2}{b} \{b-(1+b)^{-a/b}\log^2(1+b)\}$$

$$\geqslant \frac{2}{b}\{b-\log(1+b)\} \equiv \frac{2}{b}U(b), \quad \text{if} \quad a > 0.$$

Also we have

$$(3.49) \qquad U'(b) = 1 - \frac{2\log(1+b)}{1+b} \equiv \frac{V(b)}{1+b}$$

and

$$(3.50) \qquad V'(b) = 1 - \frac{2}{1+b} = -\frac{1-b}{1+b} < 0.$$

Thus it follows that $V(b)$ is a decreasing function of b and

$$V(b) \geqslant V(1) = 2(1-\log 2) > 0.$$

Thus $U'(b) > 0$. Hence $U(b)$ is an increasin function of a for all fixed b. But

$$(3.51) \qquad \frac{\partial A(a,b)}{\partial a} \geqslant \left|\frac{\partial A(a,b)}{\partial a}\right|_{a=o} = -b+2\log(1+b) \equiv T(b).$$

Clearly $T'(b) > 0$ and so $T(b) \geqslant T(0) = 0$. Thus $\frac{\partial A(a,b)}{\partial a} > 0$. Hence $A(a,b)$ is an increasing function of a and $A(a,b) \geqslant A(0,b) = 0$. Thus (3.44) is proved.

The situation in case $a < 0$ and $a < b$ is slightly different. Since neither $\frac{\partial A(a,b)}{\partial a}$ nor $\frac{\partial A(a,b)}{\partial b}$ is a purely increasingor decreasing function, we shall determine the sign of the second derivative. From (3.48) we have

$$(3.52) \qquad \frac{\partial^2 A(a,b)}{\partial^2 a} \geqslant \frac{2}{b}\{b-(1+b)\log^2(1+b)\} \equiv B(b).$$

Now,

$$B'(b) = -\frac{2}{b}\log(1+b)\{2b-\log(1+b)\}$$

$$= -\frac{2}{b^2}\log(1+b)\ U_1(b).$$

This implies that $U_1(b)$ is an increasing function of b and $U_1(b) \geqslant U_1(0) = 0$. Hence $B'(b) < 0$. Thus $B(b) \geqslant B(1) = 2(1-2)\log^2 2) \geqslant 0$. Therefore, the second derivative of A(a,b) is positive. Now $a+b \geqslant 0$ and we are considering $a < b$, $a < 0$, so $0 \geqslant a \geqslant -b$ and $A(0,b) = 0 = A(-b,b)$. Hence, by Roll's theorem, it follows that A(a,b) is positive in $-b < a < 0$. This completes the proof of the fact that

$$\mathrm{Re}\ \{\frac{G'(z)}{H'(z)}\} < 1 \quad \text{in } D.$$

Now we show that H_k is univalent. Clearly the facts that $H(z) = 1+(a+b)N(z)$ and N(z) is convex, imply that H is convex in D. Since H is convex and (3.31) is satisfied in the case $a < b$, it follows from the argument of Pommerenke that

$$(3.53) \qquad \mathrm{Re}\ \{\frac{G(z_2)-G(z_1)}{H(z_2)-H(z_1)}\} < 1 \quad \text{for} \quad z_1, z_2 \in D.$$

Let us assume $H_k(z)$ is not univalent. Then, we must have $H_k(z_1) = H_k(z_2)$ for some distinct z_1 and z_2 in D. This implies that

$$(3.54) \qquad \{\frac{G(z_2) - G(z_2)}{H(z_2) - H(z_1)}\} = \frac{k}{k-1} > 1.$$

But (3.54) contradicts (3.53). Hence, $H_k(z)$ must be univalent. This completes the proof of the lemma. ∎

LEMMA 3.2.5. $H(z) \ll H_k(z)$ *in* D, *under the condition of lemma* 3.2.4.

Proof. Since H_k is univalent by lemma 3.2.4 and $H(0) = H_k(0)$, the subordination follows if $H(D) \subset H_k(D)$. Clearly, H maps D onto the circle $|w-m| < M$. Also if $z = e^{i\theta}$, we obtain

$$(3.55) \qquad |w-m| = \left| \frac{1+a\ e^{i\theta}}{1-b\ e^{i\theta}} - m \right| = M.$$

Hence, H maps the boundary of D onto the boundary of the circle $|w-m| < M$. Thus, the lemma will be proved if we show that points in the boundary of H_k. satisfy

$$(3.56) \qquad |w-m| \geqslant M.$$

Suppose $|z_1| = 1$, $w_1 = \lim_{z \to z_1} H(z)$, $w_2 = \lim_{z \to z_2} G(z)$.

Now we want to prove that

$$(3.57) \qquad |k\ w_1 + (1-k)w_2 - m| \geqslant M$$

which will be satisfied if

$$(3.58) \qquad |k||w_1-m| - |1-k||w_2-m| \geqslant M.$$

Using (3.55) in (3.58), we see that (3.57) is satisfied if

$$kM - |1-k||w_2-m| \geqslant M.$$

Thus the inequality follows if $|w_2-m| \leqslant M$. However, this is obviously true from Corollary 2.3.1. Hence, the lemma is proved. ∎

3.3 In this section we shall prove the following theorem.

THEOREM 3.3.1. *If* $f \in K(m,M)$ *and* G *is defined by* (3.8) *then* $z \frac{f'(z)}{f(z)} \ll G(z)$ *in* D *for* $b > a$, *provided* $b^2 < a$ *if* $a+b > 1$.

Proof. We shall follow the lines of the proof developed by T.H.MacGregor [11]. If we write $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$, where F(z) is define by (3.9) and $f \in K(m,M)$, then $A_2 = a+b$, and $|a_2| \leqslant A_2$. Further, $|a_2| = A_2$ if and only if

(3.59) $\quad f(z) = \frac{1}{a} e^{i\eta}\{(1-b\, e^{i\eta}z)^{-a/b}-1\}$, η real.

This result is due to Z.J.Jakubowski [7]. Now if $g(z) = z \frac{f'(z)}{f(z)}$ and G(z) is defined by (3.9), and further, if we write

(3.60) $$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

and

(3.61) $$G(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

then $|b_1| \leqslant B_1$ is equivalent to $|a_2| \leqslant A_2$. Since $b_1 = a_2$ and $B_1 = A_2$, it follows that $|b_1| < B_1$ is equivalent to $|a_2| < A_2$. Also $|b_1| = B_1$ only if $g(z) = G(e^{i\eta}z)$, where η is real. As

(3.62) $$G(e^{i\eta}z) \ll G(z)$$

for η real, we continue the argument by assuming $|b_1| < B_1$. If we set $\Delta_r = \{z: |z| < r\}$ (obviously $\Delta_1 = D$) then $|b_1| < B_1$ implies that $g(\Delta_r) \subset G(\Delta_r)$ for sufficiently small values of r.

This subordination implies that there exists

(3.63) $w(z) = G^{-1}(g(z))$,

analytic for $|z| < r$, satisfying $w(0) = 0$ and

(3.64) $|w(z)| < r$

for sufficiently small values of r.

Let $\rho = \mathrm{Sup}\{r: 0 < r < 1,$ w is analytic for $|z| < r,$ and satisfies (3.64) for $|z| < r\}$. We need only to show that $\rho = 1$. On contrary let us assume $0 < \rho < 1$. Then $w(z)$ is analytic for $|z| < \rho$ and $|w(z)| < \rho$ for $|z| < \rho$. We first show that $w(z)$ is analytic for $|z| \leqslant \rho$. We know that $g \ll G$ in $\Delta\rho$ and thus

(3.65) $g(\overline{\Delta}_\rho) \subset G(\overline{\Delta}_\rho)$.

Since $G(\overline{\Delta}\rho) \subset G(\Delta_1)$ it follows that $g(\Delta_{\rho+\varepsilon}) \subset G(\Delta_1)$ for all sufficiently small values of $\varepsilon > 0$. Therefore, because G is univalent in Δ_1, equation (3.63) defines w as an analytic function on $\Delta_{\rho+\varepsilon}$. Since w is analytic for $|z| \leqslant \rho$, the definition of ρ implies that there is a number z_1 such that $|z_1| = \rho$ and $|w(z_1)| = \rho$. Then by Jack's lemma 2.2.1 there exists a real number k such that

(3.66) $z_1 w'(z_1) = k\, w(z_1)$ for some z_1 and $k \geqslant 1$.

Since $h \ll H$, where H is defined by $H(z) = \dfrac{1+az}{1-bz}$ and

(3.67) $h(z) = 1 + z\,\dfrac{f''(z)}{f'(z)}$

in D, we may write $h(z) = H(\phi(z))$ where $\phi(z)$ is analytic in D, $|\phi(z)| < 1$ and $\phi(0) = 0$. Writing in terms of g, we may express $h(z) = H(\phi(z))$ in the form

$$(3.68) \qquad z\,\frac{g'(z)}{g(z)} + g(z) = H(\phi(z))$$

Equation (3.63) implies that $g(z) = G(w(z))$ and $g'(z) = G'(w(z))\,w'(z)$. If we use these relations at $z = z_1$, then, we have

$$(3.68) \qquad \frac{k w(z_1)\,G'(w(z_1))}{G(w(z_1))} + G(w(z_1)) = H(\phi(z_1)).$$

Since $H(z) = z\,\frac{G'(z)}{G(z)} + G(z)$, equation (3.69) is the same as

$$(3.70) \qquad H_k(w(z_1)) = H(\phi(z_1)),$$

where $H_k(z)$ is defined in (3.34). Because of lemma 3.2.4, $\psi = H_k^{-1}(H(\phi))$ is analytic in D, $|\psi(z)| < 1$ and $\psi(0) = 0$. Equation (3.70) implies that

$$(3.71) \qquad H_k(w(z_1)) = H_k(\psi(z_1)).$$

Since H_k is univalent in D, $w(z_1)$ and $\phi(z_1)$ are equal. In particular it follows that $|\psi(z_1)| = |w(z_1)| = \rho = |z_1|$. Equality in Schwarz's lemma is possible only if $\psi(z) = z\,e^{i\delta}$, δ real. Thus we have $H_k(\psi) = k\,H(\psi) + (1-k)\,G(\psi) = k\,H(z\,e^{i\delta}) + (1-k)\,G(z\,e^{i\delta})$.

Also $$H(e^{i\delta}z) = \frac{1 + a\,z\,e^{i\delta}}{1 - b\,z\,e^{i\delta}} = 1 + (a+b)z\,e^{i\delta} + \cdots$$

and

$$G(e^{i\delta}\,z) = 1 + (b-1)z\,e^{i\delta} + \cdots$$

Hence,

$$H_k(\psi) = 1+\{k(a+b)+(1-k)\ (b-1)\}z\ e^{i\delta}+\dots\ .$$

Now, if $\phi(z) = c_1 z + c_2 z^2+\dots$ then by comparing coefficients in $\psi = H_k^{-1}(H(\phi))$ we obtain

$$(a+b)c_1 = [(b-1) + (1-a)k]\ e^{i\delta}.$$

This equation gives $k = \dfrac{(a+b)c_1\ e^{-i\delta}+(1-b)}{1+a}$.

But $k \geqslant 1$, therefore, we must have $c_1 e^{-i\delta} \geqslant 1$. For the bounded function $\phi(z)$ we know $|c_1| \leqslant 1$. Hence $|c_1| = 1$ or $c = e^{i\delta}$, $h(z) = H(e^{i\delta}z)$. This yields for all real δ, $|b| = B_1$. This is a contradiction. Hence, we must have $\rho = 1$, which proves the theorem. ■

If $f(z)$ is in $K(m,M)$ then from theorem 3.3.1 we have $f(D) \subset G(D)$. Hence we get the following results as corollaries.

COROLLARY 3.3.1 *If* $f(z)$ *belongs to* $K(m,M)$ *and* $b \geqslant a$ *then*

$$\frac{a}{(1+br)\{(1+br)^{a/b}-1\}} \leqslant \left|\frac{f'(z)}{f(z)}\right| \leqslant \frac{a}{(1-br)\{1-(1-br)^{a/b}\}}$$

provided $b^2 < a$ *when* $a+b > 1$.

COROLLARY 3.3.2 *If* $f(z)$ *belongs to* $K(m,M)$ *and* $b > a$ *then* $f(z)$ *belongs to* $S(m',M')$ *where*

$$m' = \frac{1}{2}\left[\frac{a}{(1-b)\{1-(1-b)^{a/b}\}} + \frac{a}{(1+b)\{(1+b)^{a/b}-1\}}\right]$$

and

$$M' = \frac{1}{2}\left[\frac{a}{(1-b)\{1-(1-b)^{a/b}\}} - \frac{a}{(1+b)\{(1+b)^{a/b}-1\}}\right],$$

provided $b^2 < a$ *if* $a+b > 1$.

§4. On radius of starlikeness of some classes of functions. 4.1. We need the following lemma.

LEMMA 4.1.1. *If* $g(z) \in K(\alpha)$ *then*

$$(4.1) \qquad \left|\frac{zg'(z)}{g(z)}\right| \leqslant B(\alpha, r), \quad \text{where}$$

$$B(\alpha,r) = \begin{cases} \dfrac{2(\alpha-1)r}{(1-r)^{2(1-\alpha)}\{1-(1-|z|)^{2\alpha-1}\}}, & \alpha \neq \frac{1}{2} \\[2ex] \dfrac{r}{(1-r)\log(1-r)}, & \alpha = \frac{1}{2} \end{cases}$$

Proof. We have stated a result of T.H. MacGregor in Section 3 as theorem D, which gives $\frac{zg'(z)}{g(z)} << J(z)$ where $J(z)$ is given by (3.3). Hence

$$(4.2) \qquad \left|\frac{zg'(z)}{g(z)}\right| \leqslant |J(z)| \leqslant B(\alpha,r). \quad ■$$

4.2. In this section we prove the following theorem.

THEOREM 4.2.1 *Let* $F \in S(m,M)$ *and* $F(z)$ *be defined by* (2.18) *and* $r(a,b)$ *be the unique positive root of the equation*

(4.3) $(a+2b+d)-2(ad+bd+b+d)r-\{2(b^2-d^2)+(a+d)$

$$+2b(1-d^2)-d(ad+b^2)\}r^2-2d\{(a+b)+b(b+d)r^3\}$$

$$-d(ad+2bd+b^2)\ r^4 = 0.$$

Then, f(z) *is starlike of order* β *for* $|z| < r_o$, *where* r_o *is the smallest positive root of the equation*

$$(4.4)\qquad (1-\beta)-\{\beta(b+d)+a+b+2d\}r + d(a+b\beta)r^2 = 0$$

if $r_o \leqslant r(a-b)$, *otherwise* r_o *is the smallest positive root of the equation*

$$(4.5)\qquad (E-1+bd)-(1+bd)x + \sqrt{(1-d)\{(1-d)+(1+dx\}} \cdot \sqrt{\{(1+2a+4b-b^2)+(1+b^2)x\}} = 0$$

where

$$(4.6)\qquad x = \frac{1+r^2}{1-r^2},\qquad E = -\beta(b+d)+2d-(a+b),$$

and

$$d = \frac{a-bc}{c+1}\ .$$

This result is sharp.

Proof. Since $F \in S(m,M)$ there exists a regular function w(z) with $w(0) = 0$, $|w(z)| < 1$ and

$$(4.7)\qquad \frac{zF'(z)}{F(z)} = \frac{1 + a\ w(z)}{1 - b\ w(z)}$$

From (4.7) and (2.18) we get

$$(4.8)\qquad \frac{f(z)}{F(z)} = \frac{1 + \frac{a-bc}{c+1}\ w(z)}{1 - bw(z)} = \frac{1 + dw(z)}{1 - bw(z)}$$

Differentiating (4.8) logarithmically with respect to z and using (4.7), we get,

$$(4.9) \qquad \mathrm{Re}\{\frac{zf'(z)}{f(z)} - \beta\} \geqslant -\beta + \mathrm{Re}\{\frac{1+aw(z)}{1-bw(z)}\}$$

$$+ (b+d)\ \mathrm{Re}\ \{\frac{w(z)}{(1-bw(z))(1+dw(z))}\}$$

$$- \frac{(b+d)\ (r^2-|w(z)|^2)}{(1-r^2)|1-bw(z)|\ |1+dw(z)|}.$$

Here we have used the well know inequality

$$|zw'(z) - w(z)| < \frac{r^2-|w(z)|^2}{1-r^2}.$$

If we take

$$(4.10) \qquad p(z) = \frac{1+dw(z)}{1-bw(z)}$$

it is easy to see that

$$(4.11) \qquad |p(z) - A| \leqslant B$$

where

$$(4.12) \qquad A = \frac{1+dbr^2}{1-b^2r^2}$$

and

$$(4.13) \qquad B = \frac{(b+d)r}{1-b^2r^2}.$$

Substituting value of $w(z)$ from (4.10) in (4.9) we get

$$(4.14) \quad \mathrm{Re}\ \{\frac{zf'(z)}{f(z)} - \beta\} \geqslant \frac{1}{b+d}\ [E-d\ \mathrm{Re}\ \{\frac{1}{p(z)}\}$$

$$+ (a+2b)\ \mathrm{Re}\ \{p(z)\}$$

$$- \frac{r^2|bp(z)+d|^2 - |p(z) - 1|^2}{(1-r^2)|p(z)|}] .$$

If we take $p(z) = A + u + iv$, $|p(z)| = R$, and use (4.12) and (4.13) in (4.14), we get

$$(4.15) \quad \text{Re}\{\frac{zf'(z)}{f(z)} - \beta\}$$

$$\geq \frac{1}{b+d}\Big[E - \frac{d(A+u)}{R^2} + (a+2b)(A+u) - \frac{B^2-u^2-v^2}{R} \times$$

$$\frac{1-b^2r^2}{1-r^2}\Big] \equiv \frac{1}{b+d} P(u,v) .$$

Differentiating $P(u,v)$ partially with respect to v we get

$$(4.16) \quad \frac{\partial P(u,v)}{\partial v} = \frac{v}{R}\Big[\frac{d2(A+u)}{R^3} + \{2 + \frac{B^2-u^2-v^2}{R}\}(\frac{1-b^2r^2}{1-r^2})\Big] .$$

If $d \geq 0$, the quantity in the square bracket is positive. If $d < 0$ we see that

$$\frac{1-b^2r^2}{1-r^2} + \frac{d(A+u)}{R^3} \geq 1 + \frac{d(1+br)^2}{(1-dr)^2} \geq 0$$

and therefore the quantity in the square bracket in (4.16) is positive. So $\frac{\partial P(u,v)}{\partial v} \geq 0$ if $v \geq 0$, and $\frac{\partial P(u,v)}{\partial v} < 0$ if $v < 0$. Therefore,

$$(4.17) \quad \min_v P(u,v) = P(u,0) = E - \frac{d}{R} + (a+2b)R -$$

$$- \frac{B^2-(R-A)^2}{R}\left(\frac{1-b^2r^2}{1-r^2}\right) \equiv P(R)$$

where $R = A + u$.

$P'(R)$ is an increasing function of R and

$P'(R_o) = 0$ where

$$(4.18) \qquad R_o = \left| \frac{(1-d)(1+dr^2)}{(a+2b+1)-(a+2b+b^2)r^2} \right|^{\frac{1}{2}}.$$

Again we see that $P'(A+B) \geqslant 0$, therefore $R_o \leqslant A+B$. Since $P'(R)$ is an increasing function of R and $A-B \leqslant R \leqslant A+B$ we have

$$(4.19) \qquad \min_R P(R) = \begin{cases} P(A-B) & \text{if } 0 \leqslant R_o \leqslant A-B \\ P(R_o) & \text{if } A-B \leqslant R_o \leqslant A+B \end{cases}$$

$$= \begin{cases} \dfrac{(b+d)\left|(1-\beta)-\{\beta(b-d)+a+b+2d\}r+d(a+b\beta)r^2\right|}{(1-dr)(1+br)} & \text{if } R_o \leqslant A-B \\ (E-1+bd)-(1+bd)x \\ \quad + \sqrt{(1-d)\{(1-d)+(1+d)x\}\{(1+2a+4b+b^2)+(1-b^2)x\}} & \text{if } R_o \geqslant A-B \end{cases}$$

where $x = (1+r^2)/(1-r^2)$.

Let us take

$$(4.20) \qquad Q(r) = (A-B)^2 - R_o^2 = \left(\frac{1-dr}{1+br}\right)^2 - \frac{(1-d)\,(1+dr^2)}{(a+2b+1)-(a+2b+b^2)r^2}.$$

Then $Q(r)$ is a decreasing function of r and $Q(0) = \frac{(a+b)+(b+d)}{(a+b)+(1+b)} \geqslant 0$, $Q(1) = -\frac{2(1-d)(b+d)}{(1+b)(1-b^2)} \leqslant 0$.

Therefore $Q(r)$ has unique root in $(0,1)$, call it

$r(a,b)$. Hence if $r \leqslant r(a,b)$, $Q(r) \geqslant 0$, i.e. $A-B \geqslant R_o$, and if $r \geqslant r(a,b)$, $Q(r) \leqslant 0$, i.e $A-B \leqslant R_o$. So from (4.19) and (4.20) the result follows. ■

The equatily in (4.4) is attained for the function $F(z) = z(1-bz)^{-(a+b)/b}$, and that in (4.5) for the function $F(z) = z(1-2k\,bz+b^2\,z^2)^{-(a+b)/2b}$ where k is given by

$$\frac{1+k(a-b)r-br^2}{1-2kbr+b^2r^2} = \left\{\frac{(1-d)\ (1+dr^2)}{(a+2b+1)-(a+2b+b^2)r^2}\right\}^{\frac{1}{2}} .$$

Similarly, by using the method of Theorem 4.2.1, the following theorems follow.

THEOREM 4.2.2. *If* $f(z)$ *is regular in* D *and satisfies* (2.22) *where* $F \in S^*(\beta)$ *and* $g \in S(m,M)$ *then* $f(z)$ *is univalent and starlike of order* β *in* $|z| < r_o$, *where* r_o *is the smallest positive root of the equation.*

$$\begin{aligned}(1-\beta)(c+2)&-\{(c+2)(a+2b-b\beta)+2(1-\beta)(2-\beta)\}r\\ &+\{2b(1-\beta)(2-\beta)-(1-\beta)(c+2\beta)\\ &-2(c+1+\beta)(a+b)\}\,r^2-(c+2\beta)(a+b\beta)r^3\\ &=0.\end{aligned}$$

This result is sharp.

THEOREM 4.2.3. *If* $f(z)$ *is regular in* D *and satisfies* (2.22) *where* $F \in S^*(\beta)$ *and* $g \in K(\alpha)$, *then* f(z) *is starlike of order* β *for* $|z| < r_o$ *where* r_o *is the smallest positive root the equation*

$$(c+2)(2-\beta)+2\{(c+\beta+1)-(1-\beta)(2-\beta)\}r+\beta(c+2\beta)r^2$$
$$-(1+r)\{(c+2)+(c+2\beta)r\}B(\alpha,r) = 0$$

where

$$B(\alpha,r) = \begin{cases} \dfrac{(2\alpha-1)r}{(1-r)^{2(1-\alpha)}\{1-(1-r)^{2\alpha-1}\}}, & \alpha \neq \frac{1}{2} \\[2ex] \dfrac{r}{(1-r)\log(1-r)}, & \alpha = \frac{1}{2} \end{cases}$$

This result is sharp.

THEOREM 4.2.4. *If* f(z) *is regular in* D *and satisfies* (2.22) *where* $F \in S^*(\beta)$ *and* $g(z)/z \in P(\alpha)$ *then* f(z) *is univalent and starlike of order* β *in* $|z| < r_o$, *where* r_o *is the smallest positive root of the equation*

$$(c+2)(1-\beta)-2\{(c+2)(1-\alpha\beta)+(1-\beta)(2-\beta)\}r$$
$$-2\{c(3-4\alpha-\beta+\alpha\beta)+(3+2\beta-8\alpha+6\alpha\beta-\beta^2-2\alpha\beta^2)\}r^2$$
$$+2\{(c+2\beta)(2\alpha-\alpha\beta-1)-(2\alpha-1)\ (1-\beta)(2-\beta)\}r^3$$
$$-\ (2\alpha-1)(1-\beta)(c+2\beta)r^4 = 0.$$

The result is sharp.

THEOREM 4.2.5. *Let* $F \in \Gamma(m,M)$ *and* f(z) *be defined by* (2.23) *and* r(a,b) *be the unique positive root of the equation*

$$(a+d)+2\{d(a+b)-(d-b)\}r+\{2(b^2-d^2)-(a+d) + d(ad+b^2)\}r^2-2d\{(a+b)+b(d-b)\}r^3 - d(ad+b^2)r^4 = 0$$

and $d \leqslant 0$. *Then* $f(z)$ *is meromorphic starlike of order* β *for* $|z| < r_o$, *where* r_o *is the smallest positive root of the equation*

$$(1-\beta)+\{(a+b+2d)-(b+d)\beta\}r+(ab+bd+d^2-bd\beta)r^2 = 0$$

if $0 < r_o \leqslant r(a,b)$, *and that of the equation*

$$(E-1+bd)-(1+bd)x + \sqrt{(1+d)\{(1+d)+(1-d)x\}\{(1-2a+b^2)+(1-b^2)x\}} = 0$$

if $r(a,b) \leqslant r_o$ *where* $x = (1+r^2)/(1-r^2)$, $E = (a-b)-(d-b)\beta$ *and* $d = (a+b+c)/c$.

Equality is attained for the functions

$$F(z) = \frac{(1+bz)^{(a+b)/b}}{z}$$

$$F(z) = \frac{\left[(1-bz)^{1+k}(1+bz)^{1-k}\right]^{(a+b)/2b}}{z}$$

where k is determined from

$$\frac{1-k(a+b)z+abz^2}{1-b^2z^2} = \frac{(1+d)\ (1-dr^2)}{(1-a)+(a-b^2)r^2}\ ^{\frac{1}{2}}.$$

The above sequence of theorems contains the results of [2(b)], [3], [5], [9] and makes use of a result given in [6].

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