Dynamic Model for the Multiproduct Inventory Optimization with Multivariate demand.
Modelación dinámica para la optimización de inventarios multiproducto con demanda multivariada.

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Abstract

Inventory Policies for the industry are a need to its profitability and to provide adequate service to the clients. The final stage in the internal logistic of the supply chain, the storage and the final disposal, must have a correct planning, but this requires a correct management of the final products demand. New models based on Bayesian techniques are proposed in this thesis in order to do forecasts with few data and for short periods. Also, new Inventory models, and the respective optimization for a type of industry that has a storage center and final supply of its products.

The work is divided in two phases. The first phase will provide a better form to forecast, and the second, the form to do optimization, to give to the industry adequate quantities to order, save, and transport to fulfill service to the clients.

Keywords: Forecasts, Bayesian Techniques, Dynamic Linear Models, Inventory Models.

Resumen

Las políticas de inventario para la industria son una necesidad para su rentabilidad y para proveer un servicio adecuado a los clientes. Es necesario buscar herramientas que faciliten a la industria una buena planeación en la etapa final de la logística interna de la cadena de suministro, tanto en el almacenamiento, como en la disposición final, pero ésto requiere un manejo adecuado de la demanda de los productos finales. En ésta Tesis Doctoral se proponen nuevos modelos basados en técnicas bayesianas, con el fin de hacer pronósticos con pocos datos y a corto plazo. También, se proponen Modelos de inventario y su respectiva optimización, para un tipo de industria que tenga un centro de almacenamiento y distribución final de sus productos. El trabajo se divide en dos fases. La primera fase proporcionará una mejor forma de predecir, y la segunda, la forma de hacer la optimización de modelos de inventario multiproducto, para dar a la industria de las cantidades adecuadas para ordenar, guardar y transporte para cumplir con el servicio a los clientes.

Palabras clave: Pronósticos, Técnicas Bayesianas, Modelos Lineales Dinámicos, Modelos de Inventarios.

Chapter 1

Introduction

The future of an organization can be affected by problems related to a bad inventory management, or to bad demand forecasting. Such problems could reduce profits, or the management service, and they could increase costs, among other aspects. Some of the causes of these problems can be tied to wrong practices like over-ordering or shortfalls of existences and wrong forecasts, conducting the organization to be unprepared to face unexpected changes in demand or prices, in relation to other aspects (Gutiérrez and Vidal, 2008; Sarimveis et al., 2008; Sethi et al., 2003).

In this chapter it is presented the problem, the hypothesis, the expected contributions, and a summary of the content of this Doctoral Thesis.

1.1 Problem Definition

It is increasingly necessary to make a proper planning of inventories in the industry, since several effects can occur and impact, decisively, the future of the organizations. These effects may affect profitability, good service and costs, among other things. Some of these conflicting practices are over-ordering or shortage of stocks, or bad estimation of supply times, or demand forecasting, leaving the organizations to be unprepared to respond to their markets.

Industry needs accurate and fast methods to be efficient (Silver, 1981; Nenes et al., 2010; Flynn and Garstka, 1990; Chou et al., 2013; Vargas, 2009; Jeyanthi and Radhakrishnan, 2010). Planning, evaluation and control of inventories are activities of vital importance in order to fulfill the objectives of the organizations, specially, in the manufacturing fields (Valencia et al., 2015). *"The strategy used to determine how to manage inventory is known as a firm's inventory policy"* (Simchi-Levi et al., 2008). And this Inventory Policy for an organization, depends on many characteristics of the supply chain of the inherent economical sector. For an industry, this leads to search adequate Inventory Policies that provide the costs and profits needed to achieve the best possible results in order to be sustainable. This leads to propose Inventory Models that can make a good representation of the variables inherent to the industry management.

In general, the main characteristics to consider in an inventory model are: the demand behavior, the number of periods of the planning horizon, the cost of products and inventory holding cost, the service level, lead time, among others that will be detailed later (Simchi-Levi et al., 2008). The demand behavior could be stochastic or deterministic, but, nowadays, the stochastic demand has been a very investigated topic because of the impact it has over the inventory policy (Wang et al., 2011; Sarimveis et al., 2008; Gutiérrez and Vidal, 2008; Taleizadeh et al., 2011). Besides this, it is important to take into account, the kind of objective to optimize; and this considers the static or dynamic variables.

Organizations currently face many dynamics by showing static models as inadequate to cope with them (Gutiérrez and Vidal, 2008; Valencia et al., 2015). Unexpected drastic changes in some variables of inventory models could generate bad consequences to the organization, leading to significant effects. Some causes of this could be casual events in the market; for example, modifications in representative money taxes, vulnerability of systems, uncertainty in the demands of terminated products which affect the orders, uncertainty in lead times as well as the costs and profits (Simchi-Levi et al., 2008; Ventura et al., 2013). In special, the demand of final products has a dynamic which makes the Inventory planning very uncertain (Gutiérrez and Vidal, 2008; Jimenez Sanchez, 2005; Sarimveis et al., 2008).

The demand of final product is one of the most important variables in order to propose an appropriate Inventory Policy (Simchi-Levi et al., 2008), because of the treatment that it receives (Bes and Sethi, 1988; Feng et al., 2006; Gutiérrez and Vidal, 2008; Kumar et al., 2012; Samaratunga et al., 1997; Schwartz and Rivera, 2006; Sethi et al., 2003). The stochastic formulation of the demands has been one of the most studied cases in the literature of these problems. In order to produce forecasts of sales of final products, one of the most used models has been the ARIMA (Autorregressive Moving Average) and also, the Exponential Smoothing, but, how well do these models predict the demand? The answer to this question depends, not only on the way to consider the components of the time series, as the seasonality, randomness, trend, cycles, but also, on the theoretical structure used to do the estimation based on assumptions over the errors or random perturbations ϵ_i. These are tested with the estimated residuals, which are the result of the difference between the adjusted response values (with model) and the observed values. In summary, the most important assumptions of ARIMA, Classical Regression models for time series are: normality distribution, constant variance, and independence for residuals. For Exponential Smoothing models, it is necessary to test the non-correlation of residuals, but not normality or homogeneity, because these models are not estimated based on normal distribution, but based on an optimization method.

Sometimes, the assumptions about the residuals do not receive so much attention when a prediction is done, and, if they are not actually fulfilled in a particular problem, possible bias can affect parameters estimations. Therefore, it can generate wrong predicted values (Valencia, 2010; Armstrong et al., 2014). In Regression Analysis, besides these assumptions, there is another problem called multicolinearity which sometimes could happen if there is not an appropriate exploratory analysis, because it could introduce a variance inflation factor, and this is due to the non-independence among co-variables. There is also another problem about the identification of seasonal behavior in Regression and ARIMA models, because, as Armstrong et al. (2014) affirm, there is less accuracy detecting seasonal component, when there are not enough historical data.

The compliance with the assumptions of most of the classical forecasting models can be explored with graphical methods, or verified with analytical tests. These will be explained in detail later. For example, to check the constant variance it is useful to study the behavior of residuals vs. fitted values. In order to determine the adjustment of residuals to a normal distribution, it is recommended to use a plot like Quantile-Quantile normal (QQ -norm) (Montgomery et al., 2006; Valencia et al., 2014b). Among the analytical methods, there are: Shapiro Wilks or Jarque bera tests for normality, Bartlett or Levene tests for homogeneity of variance, and Ljung Box for non-correlated residuals(Caridad y Ocerin, 1998).

Sometimes ARIMA models do not show better accuracy than other models like Exponential Smoothing (Makridakis et al., 1979) or empirical distribution, as Cohen and Dunford (1986) use them. In fact, they also affirm that normality of SARIMA models is not achieved many times. Makridakis et al. (1979) compare 111 time series data with randomness, and which are not, necessarily, stable, estimating different kind of models. They estimate the indicator Mean Absolute Percentage Error (MAPE) for fitting and forecast data, and they conclude that simple models perform well in comparison with ARIMA models.

Bayesian techniques are based on theoretical structures which are different from classical models. These can be useful to forecast demands as many authors have shown (West and Harrison, 1997; Lee et al., 2003; Pedroza, 2006; Bermúdez et al., 2009; Petris et al., 2009; Yelland and Lee, 2003; Yelland, 2010; Fúquene et al., 2015), and, also, these techniques can be helpful in optimization methods (Bolstad, 1986). The question: why should a Bayesian approach be considered to the analysis of time series? This can be answered as follows: first, because of the mistakes that are committed when a classical model is estimated. These kinds of models fails in the theoretical assumptions of its residuals, as it was previously mentioned. Second, many times there are few historical data (sales of a product), or total absence of them. Third, priori distributions can be used for Bayesian Methods in the first step of this estimation; and, fourth, there is evidence of many works already presented about Non-Bayesian methods (Broemeling and Shaarawy, 1988; Gutiérrez and Vidal, 2008; Valencia et al., 2014b, 2015). One hypothesis to verify in this Doctoral Thesis is if the Bayesian models are a good alternative to produce better forecasts than other classical models, in cases of drastic changes of the time series. But, can a Bayesian univariate model perform better than a multivariate Bayesian model when there are multiple products? It is another hypothesis to verify.

It is necessary to define criteria to evaluate the forecasts of the demands in a multiproduct scenario, when the models do not follow the same theoretical structures.

Optimization of an Inventory Model in a time horizon T, needs accurate demands (Chen and Lee, 2004). Gutiérrez and Vidal (2008); Valencia et al. (2015) present reviews of statistical models for demands and different aspects of Inventory Policies, and in Valencia et al. (2015) a conclusion about some important requirements of these inventory models is that the dynamics must be considered in its formulation and optimization, because of the evident uncertainty of variables, in special, in multiproduct demands. Other conclusion is, that, not many authors have used Bayesian Techniques for these type of optimization in relation to this type of models.

One of the purposes of the Industry is, with no doubt, to minimize the total production and inventory costs along the time horizon of planning, and this has been considered in different works (Bes and Sethi, 1988; Buffett and Scott, 2004; Dunbar and Desa, 2005; Feng et al., 2006; Hausman and Peterson, 1972; Jeyanthi and Radhakrishnan, 2010; Samaratunga et al., 1997; Sani and Kingsman, 1997; Sethi et al., 2003; Taleizadeh et al., 2011; Urrea and Torres, 2006; Wang et al., 2011). Costs can be considered fixed or variable, according to the company process or external factors. But not always this cost minimization is an objective. Also, the level of inventory quantity saved can be an objective (Arrow et al., 1958; Arslan et al., 2007; Sarimveis et al., 2008; Wang et al., 2005). There are other problems that consider a maximization of an objective, like profits or inventory level. In Song (1998), the allowed level of inventory is maximized, in Dawande et al. (2006) the objective is to achieve the maximum orders fulfillment, and others consider the maximization of the expected profit (Choi et al., 2003; Chou et al., 2013; Gao and Ting, 2009).

The solutions of such Inventory policies can be achieved by using different kinds of techniques. Sometimes statistical and simulation methods are useful in order to find possible solutions. Linear Programming, Linear Integer-Mixed programming, and also, an organized form to search solutions, known as heuristics, can also be programmed. However, this last technique does not always provide the optimal solution of a proposed model. Many researchers have developed a Metaheuristic method to get better solutions. "Metaheuristic is an iterative master process that guides and modifies the operations of subordinate heuristics to produce efficiently high-quality solutions" (Silver, 2004). In this way, Tabu Search algorithm can be used to minimize the inventory cost of the internal logistic of an organization, according to Valencia et al. (2014a), where ARIMA and Time Series Regression Models were used to forecast demand. This optimization technique was applied to the same products with the same time horizon of the real situation of the company, and a saving of 20% was found in inventory costs, with a 100% service level. Also, Genetic Algorithms can be used for efficient supply chain management (Jeyanthi and Radhakrishnan, 2010). Among these techniques, can a Bayesian technique be appropriate to help in the optimization of an Inventory Policy?

The Bayesian Statistics has shown strength to be used in the mentioned forecasts, to help with the optimization, when there are few historical data, in a complete absence of these, or in the presence of the before mentioned drastic changes. They can also use probability distributions based on stochastic dependence facilitating an update of important variables as predicting demand in a multivariate level. These Bayesian techniques can be used to forecast demands concerning univariate or multivariate level, which are used to plan production and storage, in an Optimal Inventory Policy for multiple products, as it is proposed in this Doctoral Thesis.

This Doctoral Thesis focuses in two problems for an industry: The first, related to final demand forecasting, because this is one of the principal factors causing uncertainty, and the second, related to inventory planning, due to the increasing necessity for the industry to be prepared to order and store properly to maintain an adequate service for clients, trying to face the several effects that can occur and impact decisively the future of the organization. It is necessary to propose an appropriate inventory policy in a general framework of an adequate storage, which is inherent to all the chain; but in this Doctoral Thesis, the proposal is in the final stage of a supply chain. It means, the storage of final multiple products, to provide an appropriate service.

A fundamental question of this research is to determine if the multivariate estimation of the demands of terminated products, with Bayesian techniques, contributes to the optimization of a Multiproduct Dynamic Inventory Model.

In summary, these are important hypothesis to solve through this Doctoral Thesis:

- Why should a Bayesian approach be considered to the analysis of time series?
- Are the Bayesian models a good alternative to produce better forecasts than other classical models, in cases of drastic changes of the time series?
- Can a Bayesian univariate model perform better than a multivariate Bayesian model when there are multiple products?
- Can a Bayesian technique be appropriate to help in the optimization of an Inventory Policy?

1.2 Objectives

1.2.1 General Objectives

To develop a methodology for optimizing dynamic multiproduct inventory modeling with multivariate demand.

1.2.2 Specific Objectives

- To select the main variables involved with the final stage of the supply chain, for terminated products in a company manufacturing.
- To conduct a review of state-space models that have been proposed for upgrading demand using Bayesian statistics.
- To define criteria for evaluating the multiproducts demand prediction performed with Bayesian techniques.
- To propose a methodology for the analysis of the temporal evolution of the inventory system, allowing to be represented with a state space model.
- To develop an optimization model of multiproduct inventory, according to multivariate demand, with the conditions, variables, and parameters established.
- To simulate the optimization of the dynamic model proposed, integrating Bayesian techniques properly.
- To select a study of case.
- To apply the model(s) developed to the selected case, using a statistical software.
- To validate the optimization of a dynamic model for the inventory management proposed.

1.3 Contributions of this Doctoral Thesis

In summary, the contributions are:

- To develop a novel model to forecast in short term with few data, based on Bayesian techniques, designed in R program.
- To develop a novel Multi-product Inventory Model, which optimization is based on a dynamic algorithm. The optimal solution for the inventory model provides the quantities to order, store and transport in an appropriate time horizon for the industry, by maximizing profits.
- To provide a methodological proposal applied to an industry, with an experimental design, to find the best form to make a correct Inventory policy, based on the results of the designed Algorithm in R program, according to the initial conditions of the industry.

The methodological proposal for the dynamic Inventory Model, includes two forms to do optimization, the first one considers previous demand forecasts, derived from a comparison, from which the best form was chosen. The second one, considers a simulation of the demands, and a calculation of an Expected Value for the objective function proposed.

The forecast models are designed in an application with free access to the industry, that compares final demand forecasts between classical and Bayesian models, for time series with the volatility, seasonality and non-stationary characteristics, using software R. The application considers univariate and multivariate time series models, and it provides the best possible model, using MAPE criteria for fitting or forecasts.

Next, the Inventory Policies will be proposed, with the algorithm, to find optimal solutions, using software R, that will be validated with real data. In this Doctoral Thesis, it will be shown the structure, model representation, and solutions of an Inventory system for the final stage of a supply chain of a company that needs to do forecasts and then, to store inventories and distribute them, based on adequate modeling techniques and computer technologies that generate competitiveness and profitability.

The dynamic inventory model can use a dynamic Bayesian prediction, in order to make a control that allows, in turn, optimize the desired inventory levels.

1.4 Summary of the content

The chapter two contains a State of the Art review, extending the principal topics of the Thesis related with Inventory structures, works associated with Statistical applications to the demand prediction, or to the optimization process, among other aspects. The problem was divided in two phases, in order to argue the importance to choose the proposed models. In chapter three, the phase of Statistical models to study and estimate the demands, will be developed, by choosing the best models to forecast in the related case. Chapter four describes the phase two, with the Multiproduct Inventory Model proposals, and the respective processes to do the Optimization, for establishing the best way to program the inventory management for a company with the specifications related. Finally, the conclusions chapter summarizes the results and the hypothesis that could be supported by the research.

Chapter 2

State of the art review

The logistics of a supply chain in an organization could be internal or external. This Doctoral thesis will focus in the internal logistics, specifically, in the final stage of inventory management.

Planning, programming, evaluation and control of manufacture inventories in the industry are important activities in order to fulfill the objectives previously defined by the company. The Inventory Management could be done for raw materials, products in process, or final products. The problem studied in this Doctoral Thesis will be focused on final products. Further, this work could be linked with inventories of products in process and with raw materials.

Inventory Optimization is not always easy to achieve due to the uncertainty present in the demand fluctuations (Valencia et al., 2015). In many situations, the problem is so hard to solve, that finding good solutions about how much to order, to store, and to send, requires more accurate and faster mathematical and statistical techniques.

Few works have been proposed related to Bayesian applications in Inventory problems. Hansson (1998) shows how to implement a solver of Bayesian problems (Bayesian Problem Solver - BPS) to programing problems. Part of the motivation is that a chosen action depends on the form to take decisions under uncertainty. Mockus (2002) uses Bayesian Heuristic Approach-(BHA) to facilitate the choice of a heuristic among three, minimizing the completion time of all the tasks.

In this chapter, the proposals that have appeared in the literature of inventory models will be reviewed. First, the state of the art will be presented about the dynamics in the inventory management, characteristics of inventory models, and how to work with them, according to the literature. In a second stage, this Doctoral Thesis will present a review of the state of art of some classical techniques and also, some Bayesian models for forecasting. An emphasis will be done as well in the principal aspects of the time series components, and the residuals assumptions of some of these models to do the estimation that could cause problems in classical models, as it was previously mentioned.

After this, in order to define the kind of models to be proposed, it is necessary to characterize the conditions, parameters and variables of inventory systems of an industry, and also, the conditions of the proposed models to do forecast in this Doctoral Thesis.

2.1 Dynamics of the Inventory Management

Uncertainty has been a common factor in some systems of the industry that affects many decisions (Rocquigny, 2012; Sarimveis et al., 2008). Sarimveis et al. (2008) present a review of 187 references about dynamic models related to supply chain, external, and also, internal to the organization. The authors show different factors included in the optimization of production and also, inventory models, indicating that static models are not enough to face problems like unexpected fluctuations in the demand. They affirm that inventory or production systems can be seen as dynamic programs, because they have *an uncertain and changing nature* (Sarimveis et al. (2008), p. 3542).

Drastic changes in some variables of inventory models could generate bad consequences to the organization, leading to significant effects. These changes could be caused by casual events in the market, for example, jumps in the representative money taxes, vulnerability of systems, high uncertainty in the demands, which affect orders, lead times and consequently, costs and profits (Simchi-Levi et al., 2008; Ventura et al., 2013).

Some strategies for the fulfillment of its processes can be done by the organization, and all of them have posterior effects in the internal systems. Other way to say this is: if we denote with d the actions applied to a system, with x an input variable, and with z an output variable of a system; then, the actions “d”, taken in a state (x, z), of a system, may impact the system on later times. This is the reason why it is necessary to quantify the uncertainty via a measure of imperfect information on such state on some possible output variable of interest (Rocquigny, 2012).

The state of a system often involves a dynamic variation on time, but at the same time, characteristics of this could involve a time series on inputs, outputs and events of that system (Rocquigny, 2012).

The word *dynamics* on inventory models means unexpected changes on the variability of the system, causing uncertainty in variables like demand, lead times, prices or costs, among others, which are associated to inventory management.

Inventory models that contemplate all fixed factors in time are not robust, because they do not incorporate random fluctuations on internal logistics, for example, the bullwhip effect (Simchi-Levi et al., 2008), this represents strong variations over demand, a violation in the assumption of independence, or inventory policies with constant requests when the demand, really, has a random and dependent variation (Bes and Sethi, 1988; Gutiérrez and Vidal, 2008; Sarimveis et al., 2008; Shoesmith and Pinder, 2001). This suggests that the management of such inventories with fixed demand for any period, or simply a constant expected value, is not appropriate.

It is necessary to define here the principal characteristics to plan Inventory Management in the internal logistics of an organization, in addition to the mentioned variables, in order to give concepts about the appropriate policy to reach the objectives searched.

In general, the main characteristics related to an inventory model, and which were previously mentioned, will be explained in this review of the state of art, by emphasizing, more, on the behavior of the stochastic demand, and on some dynamic inventory models. Besides this, it will be also defined, the number of periods of the planning horizon, the cost of the product, the inventory holding cost, the cost of ordering, the service level, and the lead time, among others, as follows:

- *Demand* is one of the principal determinants for the storage on manufacturing companies, finding it in deterministic or stochastic forms, and modeled in different ways, or, with different schemes. Moreover, it is very important to know or to forecast the demand of final products, which, would guarantee an optimal planning of inventory models of the final stage of a supply chain.

  Those Organizations that contemplate inventory storage, often face dynamics where, static inventory models are not enough to do an adequate representation of their supply system. Many authors review or present models to forecast the demand under uncertainty, or randomness, and also, inventory models involving this randomness (Simchi-Levi et al., 2008; Gutiérrez and Vidal, 2008; Sarimveis et al., 2008).

The demand forecasts have some problems such as changes in the distribution function, producing a lack of stability in the time series, so, *"...a time series is unstable if there are frequent and significant changes in the distribution"* (Hillier and Hillier (2007),p., 397). This problem is cited by different authors (Braun et al., 2003; Sarimveis et al., 2008; Valencia et al., 2015). However, in some situations, the models of interest cannot meet some theoretical assumptions, like normality in residuals or constant variance. In other situations, the researcher does not have the enough required data to carry out the estimation of the model.

- *Lead time* is considered constant to simplify the inventory modeling in many works (Baker and Urban, 1988; Chou et al., 2013; Silver, 1981). Silver (1981) affirms: *"This is the most used case in literature"* and in few cases this is assumed with a random component (Song, 1998) that can be characterized with a probabilistic distribution like a Normal one.

- *Price* is determinant in demand variation for some inventory policies, since, sometimes, the sales depend on the price or the price variations of the product. Urban and Baker (1997) cite the elasticity model of the demand rate: $D = \alpha p^{-e}$, but it is a deterministic model; in these cases, prices can be random (Chen and Lee, 2004; Sarimveis et al., 2008; Ventura et al., 2013; Yang and Fung, 2013). In other cases, prices are incorporated on the Objective Function (Gallego and Ryzin, 2013) and they are optimized, but often, these prices are assumed fixed (Bassok et al., 2011; Félix and Nunes, 2003).

- *Objective function* is the equation to be optimized. One of the principal objectives in Inventory Systems, is to minimize the costs, as many authors have reviewed and worked on this; (Sarimveis et al., 2008; Valencia et al., 2014a; Bes and Sethi, 1988; Buffett and Scott, 2004; Dunbar and Desa, 2005; Feng et al., 2006; Hausman and Peterson, 1972; Jeyanthi and Radhakrishnan, 2010; Sani and Kingsman, 1997; Sethi et al., 2003; Taleizadeh et al., 2011;

Urrea and Torres, 2006; Wang et al., 2011). But, there are not many researches about the maximization of profits (Choi et al., 2003; Chou et al., 2013).

In some cases, a minimization of random costs, is considered (Rocquigny, 2012; Wang, 2006; Ventura et al., 2013). Ventura et al. (2013) develop a *mixed integer nonlinear programming model, to determine an optimal inventory policy that coordinates the transfer of materials between consecutive stages of the supply chain from period to period while properly placing purchasing orders to selected suppliers and satisfying customer demand on time. The proposed model minimizes the total variable cost, including purchasing, production, inventory, and transportation costs. The model can be linearized for certain types of cost structures*.

Profits are the remainder of incomes minus costs. Some authors have provided works related to profit optimization or to uncertainty in profits (Choi et al., 2003; Chou et al., 2013; Gao and Ting, 2009).

Vidal and Goetschalckx (1997) describe a review where different models are proposed to do production optimization. They use a profit maximization as one of the objective functions of this kind of models.

Inventory level at the t-*th* time, measured as the simple balance inventory equation: $I_t = I_{t-1} + x_t - D_t$, where x_t is the quantity to order, and D_t is the demand rate at t-*th* period.

These restrictions of inventory balance represent the way in which the product enters and leaves from the storage center, where the inventories are stored from one period to another.

In some works, I_t is an output of a model, or, an objective function (Salinas et al., 2013; Samaratunga et al., 1997). But there is a work where it is considered to do a minimization of deviations in relation to the required level (Wang et al., 2005). In this case, it is shown a representation of a Dynamic Linear Model, and where a Kalman Filter is a tool in order to proceed with the optimization.

2.2 General Inventory Models

Inventories handling has been considered as one of the most important aspects of the production and logistics processes of a company. Many inventory-decisions are difficult because they can have important effects on customer service and costs (Ventura et al., 2013; Simchi-Levi et al., 2008). So, it is fundamental to define that *"the strategy used to determine how to manage inventory is known as a firm's inventory policy"* (Simchi-Levi et al., 2008). And this inventory policy depends on many characteristics of the supply chain, related, for example, to the internal logistics of the company.

Inventory can appear on these type of forms in the logistics of the industry: Raw material, Work-in-process (WIP), or finished products inventory. *"Every one of these needs their own inventory control mechanism or control"* (Simchi-Levi et al., 2008). The necessity of holding these kind of inventory can be caused because:

- The demand can be modeled, but there is uncertainty almost all the time when forecasts must be produced. Some causes of these uncertainty could be: a) Short life cycle of products; so there are few historical data. b) The presence of other competing products in the market, for example, due to free commercial trades. But, it is important to consider that a high uncertainty can also be caused by unexpected changes in prices, taxes, or a wrong definition of the forecasts. In order to find an effective inventory policy, a stochastic demand can be modeled with different techniques (Makridakis et al., 1979; Bes and Sethi, 1988; Gutiérrez and Vidal, 2008; Sarimveis et al., 2008; Shoesmith and Pinder, 2001; Valencia et al., 2015), depending on the kind of production, the type of products, components of the time series, among other aspects.
- Other uncertainties are due to changes in supplier costs, and lead times.
- Costs involved in transportation, because there are some companies that encourage large-size of shipments (Simchi-Levi et al., 2008).

Some of the recommended characteristics to take into account, in order to find an effective inventory policy, by Simchi-Levi et al. (2008) are:

1. *Demand*. If the demand is unknown, it must be predicted with the appropriate tools.

2. *Replenishment lead time*. It must be known at the time the order is placed, or it could be uncertain.

3. *Number of products*. The inventory of one of them could affect the other(s), because of space and budget requirements.

4. *Length of the planning horizon*.

5. *Costs*. These are: a. Order cost, that contemplates product and transportation cost. b. Inventory holding cost, which must consider maintenance, and obsolescence, among others (Simchi-Levi et al., 2008).

6. *Service level*. It is necessary to specify an acceptable level, despite it is impossible to meet 100%.

It is increasingly important to develop Inventory Models based on dynamic variables like demand, but also, lead times, and not many works in Colombia present this (Valencia et al., 2015; Gutiérrez and Vidal, 2008).

We could state then an important question here: Should the order be equal to, higher or smaller than the demand? There are some additional conditions proposed on inventory modeling of real situations of the industry that could help to answer these aspects.

In order to answer the question, first, some generalities about inventory models of the literature, are presented in the next sub-section. Then, in chapter four, the Inventory Policy will be proposed and validated with real data.

2.2.1 Inventory Policies

In order to find an effective management of inventory, the most common policy is the Economic Order Quantity (EOQ), which has, among other problems, a risk to lead the firm to have overstock or decreasing profits (Simchi-Levi et al., 2008). Other two common kinds of policies are the Continuous Review Policy, and the Periodic Review Policy.

- Continuous Review Policy: the inventory is continuously reviewed; an order is placed when the inventory reaches a specific level or a reorder point. Here, when the inventory reaches a level R, an order of size Q is placed, so this is called the (Q,R) policy.

- Periodic Review Policy: the inventory is reviewed in regular periods, and an order is placed after each review.

Despite they are too similar policies, the first one is the most recommended when the company has proper computerized tools to do the review. This also assumes a random demand, a fixed cost to place orders *k*, an inventory holding cost per unit and per unit time, a replenishment lead time from the supplier to the distributor, and a service level to reach (Hillier and Hillier, 2007; Simchi-Levi et al., 2008).

Taha (2004) presents these policies, but he refers to them as models, and he also shows the *Dynamic Model of Economic Order Quantity* (p. 443), that has a periodic review considering a dynamic but deterministic demand. There are two models included in this section presented by the author: Model without/with preparing cost, where he explains a general dynamic algorithm, which establishes the optimal policy by reviewing period by period these aspects: the dynamic and deterministic demand, the balance inventory constraints and the objective function costs for different schemes of orders. The orders are set in a very similar way stated by Wagner and Whitin (1958), that is the sum of demands at the beginning of the time horizon, and it decreases as time increases.

Other type of policies are related to Multistage, Multiple Installations, Multi-echelon, Multiple products (Valencia et al., 2015), and some of them appear with combinations of the previous kinds of models or with other variables, factors and mathematical functions, representing the models which are optimized to find an effective policy.

These bases are used by many authors who propose mathematical models composed by an objective function, and constraints, in order to fulfill the objectives of the particular firm or a group of companies having the same interest.

2.2.2 Inventory Models with dynamics Structures

As it was previously mentioned, many kinds of inventory models have the demand as the main variable that represents changes every time (Valencia et al., 2015). This aspect is also pointed out by Sarimveis et al. (2008), who describe different schemes of Inventory Models with dynamics.

It is necessary to clear that dynamic programming is a different conception to the case when a model has just some factors, or parameters, with dynamic behavior. But, Scarf (2002) affirms that Richard Bellman, known author of Dynamic programming (Bellman, 1957), was convinced that "all the optimization problems with a dynamic structure could be formulated fruitfully, and solved as dynamic programs". Sarimveis et al. (2008) point out that for some of the references reviewed, *"...dynamic programming serves as a tool for proving the existence of optimal feedback control laws and characterize their general form. However, it is not employed as a computational tool due to the course of dimensionality which is prevalent even for simplified, medium-scale supply networks. In order to solve these complex stochastic control problems, some kind of simplifying assumptions, decompositions or approximations need to be considered".* (Sarimveis et al. (2008), p., 3544). This leads to permit different forms of techniques to provide solutions, not only dynamic programming, but also, heuristics, for example.

Some of the representative models related to these topics that involve dynamics, will be presented here.

Scarf (2002) describes different Inventory Models with dynamics, and cites one paper entitled "Bayes Solutions to the Statistical Inventory Problem"; the paper studies a dynamic inventory problem, in which the purchase cost is strictly proportional to the quantity purchased. The innovation in his paper is to allow the density function of the demand to depend on an unknown parameter, upon which, he assumes a prior distribution.

Crowston et al. (1973) consider a problem of production planning of seasonal product, in a multistage form, it is, with a dynamic variation. The problem is to determine the production quantities of the various components and assemblies at each period to minimize expected costs, and they use two periods of Bayesian Forecast Revision over the demands in the period.

Taha (2004), (p. 448), explains an algorithm, called the Inventory Dynamic Model with preparing cost. He presents an exact and an heuristic dynamic algorithm. The explanations are presented as follows:

- Z_i: Order quantity.
- D_i: Demand for period i.
- x_i: Inventory at the beginning of the period i.
- K_i: Preparing cost in period i.
- h_i: Unit inventory holding cost from period i to i+1.
- c_i: Unit production cost for unities.

The function corresponding to the cost production for period i is:

$$C_i(z_i) = \begin{cases} 0 & if \quad z_i = 0 \\ K_i + c_i(z_i) & if \quad z_i > 0 \end{cases} \tag{2.1}$$

(2.2)

Where $c_i(z_i)$ is the production marginal cost for z_i. It is possible that the cost changes according to the quantities that will be sold every period.

A general dynamic programming algorithm is described by Taha (2004) as follows.

Without considering shortfalls, the inventory model proposal is based on the minimization of the sum of production costs and storage for all the n periods. For simplicity, this model assumes that the storage cost is based on the final inventory of the period i: $x_{i+1} = x_i + z_i - D_i$. But the value z_i is the order to fulfill the demand in period $i + 1$. Besides this:

$$0 \leq x_{i+1} \leq D_{i+1} + \ldots + D_n \tag{2.3}$$

In this equation it can be recognized than the resting inventory, after period i, can satisfy the demand for the rest of periods $(i + 1, \ldots, n)$. The objective function, which is a minimization of costs is a recursive equation estimated for every time (Taha (2004), p. 449)

$$f_1(x_2) = min_{0 \leq z_1 \leq D_1 + x_2} \{C_1(z_1) + h_1 x_2\} \quad (2.4)$$

$$f_i(x_{i+1}) = min_{0 \leq z_i \leq D_i + x_{i+1}} \{C_i(z_i) + h_i x_{i+1} + f_{i-1}(x_{i+1} + D_i - z_i)\} \quad (2.5)$$

Sethi et al. (2003) establish an inventory model related to a dynamic structure. They seek to minimize the expected total cost, and the demand is a random variable for two kinds of orders emissions. A general description of the variables and parameters of this model are shown in table 2.1.

Sethi et al. (2003) affirm that at the beginning of each period, the on-hand inventory and the demand information are updated. At the same time, decisions on how much to order using fast and slow delivery modes are made. Fast and slow orders are delivered at the end of the current period and at the end of the next period, respectively. A forecast-update-dependent (s, S)-type policy is shown to be optimal. The authors also explain a dynamic programming function, and a theorem to prove the optimality of the type of policy for the fast and the slow orders.

Yokoyama (2002) presents a minimization of holding inventories and transportation cost, and considers that the demand at each period and for each client is a mutually independent random variable that follows a stationary normal distribution. He uses and Order Up to Level policy, using a periodic review of the inventory level. Some of the assumptions of his model are that each one of these centers can distribute to multiple clients, and that each consumer could be supplied by multiple distribution centers, every one of these have an Order-up-to-R policy. The author also assumes two kinds of decision variables: transportation quantity and the inventory target. In order to solve the problem, he applies and compares two processes: a random local search, and a genetic algorithm, and he finds a better solution in the second one.

Bailey (1973), uses a dynamic programming technique as an approach to do optimization of the model they present about a maximization of profits. The appendix of Bailey's paper shows an explanation about the Dynamic Programming Model, and he affirms that this technique *"provides a systematic procedure for determining the sequence of interrelated decisions that optimizes overall effectiveness if the problem at hand conforms to certain characteristics"* (Bailey (1973), p. 572).

The objective is the maximization of profits in Bailey (1973). This problem has constraints re-

Variable	Description
N	The total periods of time
F_k	The fast-order quantity in period k, $k = 1, \dots N.$, a decision variable
S_k	The slow-order quantity in period k, $k = 1, \dots N - 1.$, a decision variable
c_k^f, c_k^s	The unit cost for fast and slow-orders, respectively, in period k.
K_k^f, K_k^s	The fixed cost for fast and slow-orders, respectively, in period k.
R_k^1	First determinant (random variable) of demand in period k, observed in period $k - 1$.
$Q_k(.)$	Distribution function of R_k^1.
R_k^2	Second determinant (random variable) of the demand in period k,
at the end of period k.	
v_k	The third determinant (a constant) of demand in period k.
D_k	The demand in period k, modeled as a function $g_k(R_k^1, R_k^2, v_k)$.
$G_k(.)$	The distribution function of the demand D_k.
X_k	The Inventory level at the beginning of the period k.
X_{N+1}	The inventory level at the end of the last period N.
$H_k(x)$	The inventory holding/backlog cost when $X = x$.
$H_{N+1}(x)$	The inventory holding cost when $X_{N+1} = x \geq 0$.
$E_{(R_k^2, R_{k+1}^1)}[.]$:	The conditional expectation with respect to R_k^2 and R_{k+1}^1.
$Y_k = X_k + S_{k-1}$	The inventory position at the beginning of period k.
$\delta_k = 0$	when $x \leq 0$;1 when $x > 0$.

Table 2.1: Variables and parameters definitions. Source: Sethi et al., 2003

lated to a inventory balance equation $I_i = I_{i-1} + x_i - y_i$, a limit for the level of variability in production, using an interval without exceeding the capacity, limits for the inventories, among others.

Other form to analyze dynamics are based on Control Theory, because these techniques can be used to do optimization in a time horizon. Wang et al. (2005) present a Model Predictive Control optimization problem, according to the minimization of the next equation.

$$J = \sum_{l=1}^{P} Q_e(l)((\hat{y}(k+l|k) - r(k+l))^2 + \sum_{l=1}^{M} Q_{\Delta\mu}(l)(\Delta\mu(k+l-1|k))^2 + \sum_{l=1}^{M} Q_{\mu}(l)(\mu(k+l-1|k) - \mu_{target}(k+l-1|k))^2 \quad (2.6)$$

Subject to:

$$\mu_{min} < \mu(k+l-1|k) < \mu_{max} \quad (2.7)$$

$$\Delta\mu_{min} < \Delta\mu(k+l-1|k) < \Delta\mu_{max} \quad (2.8)$$

These authors develop a dynamic model based on the next balance inventory equation:

$$I_i(k+1) = I_i(k) + Y_j C_j(k-\theta_j) - C_{j+1}(k) \quad (2.9)$$

Other kinds of theories could be proposed in order to formulate solutions to optimize the inventory problem of an industry.

2.2.3 Wagner & Whitin Inventory Models

Some of the classical inventory models have been formulated based on the Wagner & Whitin model (Wagner and Whitin, 1958) proposal about cost minimization. This model and theorems proposed have been reviewed and cited by different authors about inventory models (Vargas, 2009; Fleischhacker and Zhao, 2011; Ventura et al., 2013; Baker and Urban, 1988; Sarimveis et al., 2008). Ventura et al. (2013) affirm *that a relevant problem in supply chain logistics is to determine the appropriate levels of inventory at the various stages involved in a supply chain.* The basic problem is to determine the production quantities for each period, so that all demands are satisfied on time at minimal production and inventory cost. Wagner and Whitin (1958) use dynamic programming to find an exact solution for the uncapacitated version of the problem.

Some generalities about the Wagner and Whitin (1958) model are presented in this Doctoral Thesis, and the theorem 2, that is important in order to develop this research. The parameters and assumptions of this model are presented in table 2.2.

Parameter	Assumption
d_t	Demand for period t, $dt > 0$
C	Per unit order cost
K	Fixed order cost, when an order is placed
$\delta(y)$	1 if an order is placed: $y > 0$; 0 otherwise
h	Holding cost per unit per period
I_0	Initial inventory (which is zero)
L_d	Lead times, which are zero
	Decision Variables
y_t	Order, placed at the start of the period t.
I_t	Inventory, is charged at the end of the period

Table 2.2: Parameters and variables of the Wagner & Whitin inventory models. Source: Simchi-Levi et al. (2005).

The model formulated by Wagner & Whitin, cited in (Simchi-Levi et al., 2005), assumes that we have a sequence of orders over a T period planning horizon. The assumptions of the model are shown in table 2.2.

$$\min Z = \sum_{i=1}^{T} [K\delta(y_t) + hI_t] \tag{2.10}$$

Subject to:

$$I_t = I_{t-1} + y_t - d_t \tag{2.11}$$

The last expression is called the inventory balance equation which accounts the demand that must be provided every period.

And,

$$I_t = 0 \tag{2.12}$$

Where $y_t, I_t > 0, t = 1, \ldots T$.

An order could be zero or the sum of some demands (Wagner and Whitin (1958), *Theorem 2* p. 91).

Theorem 2 affirms: *"there exists an optimal program such that for all t"*:

$x_t = 0$ or $x_t = \sum_{j=t}^{k} d_j$, for some k, $t < k < N$

Proof: Since all demands must be met, any other value for x_t *implies there exists a period* $t* > t$ *such that* $I_{x_t*} > 0$*; but Theorem 1 assures that it is sufficient to consider programs in which such a condition does not arise. The implication of Theorem* 2 *is that we can limit the values of I in the objective function about a cost minimization (p. 90), for period t to zero and the cumulative sums of demand for periods t up to N. If initial inventory is zero, then only* $N(N + 1)/2$ *different values of I in to over the entire N periods need be examined.*

That will permit for this Doctoral thesis, to formulate schemes to order, by selecting between doing or not doing orders in every period *t*, but programming the sums with dynamic variations, as it was proposed by Taha (2004), which was mentioned previously.

Given the possible impact of transportation costs in both supplier selection and inventory replenishment at each stage of the supply chain in today enterprises, Ventura et al. (2013), consider in their model proposed, *purchasing, production, inventory, and transportation costs over a planning horizon with time varying demand considering quality constraints for the suppliers, capacity constraints for suppliers and the manufacturer, and inventory capacity constraints at all the stages*.

The scenario described above can be viewed as a generalization of one of the most studied problems in production and inventory planning for a single facility, called the dynamic inventory lot-size problem, related to the problem explained in this research, in the form to connect the system described about inventory holding and varying demands.

Ventura et al. (2013) address both supplier selection and inventory management decisions in a supply chain, by studying the production and distribution of a single product in a serial supply chain structure. An example of this situation is a manufacturer that purchases raw parts from various preferred suppliers. These raw parts are stored at the manufacturing facility or processed into final products. These products are either stored at the manufacturer level or transported to a warehouse. At the warehouse stage, either products are stored there or transported to a distribution center (DC). In general, the DC may serve products to an entire market area or a set of retailers.

Ventura et al. (2013) present a multi-period inventory lot-sizing model, for a single product in a supply chain, designed to the inventory management. This model considers raw-materials, inventories in process and finished products. The total variable cost is minimized, including purchasing, production, inventory and transportation costs.

Assuming that there is a set K_D of intermediate warehouse/distribution stages, where the last stage n_k is the last supplier stage, that ships products to customers, they show an important theorem ((Ventura et al., 2013) , p. 261), that will be presented next.

Theorem 2 (Ventura et al., 2013). The supply chain inventory problem with supply selection can only be feasible if for every demand node (n_k, t), $1 + \sum_{k'=m_{n_k}}^{n_k-1} l_{k'} \leq t < 1 + \sum_{k'=0}^{n_k-1} l_{k'}$ the following condition holds:

$$\sum_{k'=k_t}^{n_k} \left(y_{k'-1}^0 + i_{k'}^0 \right) - \sum_{t'=1}^{t-1} d^{t'} \geq d^t \tag{2.13}$$

Where,

$$k_t = min \left\{ n_k, k \in K : \sum_{k'=k}^{n_k-1} l_{k'} < t \right\} \tag{2.14}$$

Where $y_{k'-1}^0$ represents the replenishment order (units of finished products) for period k', and $i_k'^0$ is the inventory initial level (units) held at stage k. The total lead time from 0 to stage n_k is $\sum_{k'=0}^{n_k-1} l_{k'}$. The first term on the left $\sum_{k'=k_t}^{n_k} \left(y_{k'-1}^0 + i_{k'}^0 \right)$, is the accumulated initial inventory and pending orders accessible to node (n_K, t) (stage n_k, period t). $\sum_{t'=1}^{t-1} d^{t'}$ is the accumulated demand until period $t-1$. Therefore, the total quantity on the left is the maximum possible available inventory minus the accumulated demand until period $t-1$, before attempting to satisfy the demand in stage n_k at period t. This term must be higher than the demand to be fulfilled in period t. Otherwise, the problem is not feasible.

This theorem is important to show that a Multiperiod-inventory model should consider the sums of orders and inventories, discounting the demands until period $(t - 1)$, in order to provide an appropriate solution to the dynamic formulation, as it will be shown in this Doctoral thesis. This is a similar conception to the sums previously formulated by Wagner and Whitin (1958), that will be used here to formulate the Multiproduct Inventory Model and its correspondent proposed optimization process.

2.2.4 Techniques to solve optimization of Inventory problems

In order to find optimal solutions for Inventory Models, classical Analytical techniques are very common, as Linear programming, when the objective function and all the constraints are linear. The linear Integer Mixed Programming is useful when continuous and integer variables are included in the model. Other not very common techniques are the bounded variables algorithms, the Benders and the Branch and Bound algorithms, the Decomposition technique (Sarimveis et al., 2008), the Stochastic Dynamic Dual Programming, Dynamic Programming(Scarf, 2002), and Control Theory techniques (Braun et al., 2003; Sarimveis et al., 2008). There are also heuristic and metaheuristic techniques, that are combined among them, and it is possible that they use simulation (Zanakis and Evans, 1981; Silver, 2004), in order to find the best possible solution not always the global optimum.

Heuristics and Meta-heuristics. The heuristics are simple procedures, that can present approximate solutions (not always the optimal) to hard problems in an easy and fast way (Zanakis and Evans, 1981). These techniques will probably need a lower quantity of constraints, and they *"permit the use of models that are more representative of the real world"*(Silver (2004), p. 937).

Many authors have explored these techniques to find a very good approximation to the optimal solution of an inventory policy (Arslan et al., 2007; Fouskakis and Draper, 2002; Hausman and Peterson, 1972; Jeyanthi and Radhakrishnan, 2010; Taleizadeh et al., 2011; Urrea and Torres, 2006; Zanakis and Evans, 1981). Jeyanthi and Radhakrishnan (2010) study genetic algorithms to Optimize Multiproduct Inventory. Ant Colony algorithms can also be used, as in Wu et al. (2012) to do an optimization of a production system. Sometimes, statistical and simulation techniques are used in order to find possible solutions to inventory problems which are not linear, or that have so many variables or constraints that these classical structures become very difficult to solve (Silver, 2004; Valencia et al., 2014a). But, these methods do not always provide the optimal solutions.

Fouskakis and Draper (2002) affirm: *one alternative to solve objective functions, is to do stochastic search, where optimum value search involves randomness in a constructive form* (p. 315). The authors review three optimization techniques: Simulated Annealing, SA, Genetic Algorithms, GA, and Tabu Search, TS, and they apply those techniques to an objective function based on an expected profit maximization.

There is a difference between heuristics and meta-heuristics procedures: *"a meta-heuristic is an iterative master process that guides and modifies the operations of subordinate heuristics to produce efficiently high-quality solutions"* Silver (2004).

Tabu Search algorithm has been used to minimize the inventory cost of the internal logistics of an organization, according to (Valencia et al., 2014a), and showed a better result when it was compared with a policy where there was only a previous demand forecast process, using ARIMA and Time Series Regression Models. This optimization technique (Valencia et al., 2014a), was applied to the same products with the same time horizon of the real situation of a company, and a saving of 20% was found in inventory costs, and a 100% service level. In Urrea and Torres (2006), it is used a Tabu Search algorithm in order to find the optimum level of orders.

Genetic algorithms can be used for efficient supply chain management (Jeyanthi and Radhakrishnan, 2010), in a multi-product scenario, but it is not frequent to analyze scenarios like these, in inventory models. A recent work in Colombia, Palacio and Adarme (2014), proposed a model of multiproduct inventory among companies, to minimize costs of the logistics in an operational context of urban distribution.

In all the mentioned cases about inventory optimization, the demand has been predicted previously.

Other works like Wang et al. (2005) and Braun et al. (2003) have used a technique based on *Control Theory* in order to manage inventories, *the Model Predictive Control (MPC)*. MPC has become an important tool to help the organization in planning of policies, for example, in semiconductor demand networks (Wang et al., 2005).

2.3 Statistical techniques to Forecast Demand

A time series is a realization of a stochastic process; the data are obtained and registered in fixed intervals of time, of a particular variable. One of the objectives of a time series model is to identify the behavior of historical data, by doing an adequate representation of such behavior, and with the past of these data, try to predict the future, or to forecast it. The time series can have repetitive patterns, that are identified as components (Bowerman and Oconnell, 2007), which are:

- Trend, as the component that represents the growth (or decreasing) over a time series.
- Seasonality, which are the oscillations produced and repeated in short periods of time, associated to dynamic factors, for example: customs, climate, vacations, among others (Bowerman and Oconnell, 2007).
- Cycle, that is a fluctuation behavior lasting more than a year o more than a seasonal period, and sometimes it is associated to the changing economical conditions.
- Irregular fluctuations, which are random variations, representing a remind in a time series, after explaining the trend, cycles, and seasonal patterns (Bowerman and Oconnell, 2007).

Two of the most common and studied components, of a time series, are the trend, and seasonality. The problems of detecting a signal and then estimating or extracting the wave form of the seasonal patterns are of great interest in many areas of the engineering (Shumway and Stoffer, 2006). In some cases these components are difficult to find, or are not easily identifiable. There are many kinds of variations in the search of models to forecast variables; for example, in the co-variables introduced in the estimation process, among others.

The demand forecasts are important for the industry, because of the mentioned need to have adequate predictions to be considered in the planning of good inventory policy, and all these uses have been reviewed by different authors, like Gutiérrez and Vidal (2008), and Valencia et al. (2015).

Besides the use of probability distributions to forecast demands of final products, there are also classical statistical models like ARIMA, Exponential Smoothing, Regression Linear Model (Makridakis et al., 1979; Chen, 2011; Valencia et al., 2014b). These have been used to find future values of demands of terminated products at a factory, and after this, these forecasts are used to look for solutions of optimal inventory problems (Bes and Sethi, 1988; Cohen and Dunford, 1986; Sarimveis et al., 2008).

Some of the general concepts of these classical models are presented next.

The correlation of two variables refers to that: the values of the responses of one variable depend on the values of the other variable. In a time series, the correlation measure is called autocorrelation, and it indicates if the values of the time series depend on any of the past values of the same time series.

One of the principal aspects of a time series description, is to identify the form of the autocorrelation of a series, providing the state of dependence in the data, when a comparison is done among different periods. The autocorrelation permits to know if the dependent variable of a period is linearly related with the values of the dependent variable of another period. This function is also important to describe the behavior of the grade of dependence, seasonality, and this permits to identify the covariables that could be included in a model as Regression, or $ARIMA(p, d, q)$, o explain appropriately the response.

2.3.1 Classical statistical models

ARIMA Model

The classical $ARIMA(p, d, q)$ models, developed over the $70's$, by George Box and Gwilym Jenkins (Bowerman and Oconnell, 2007), have been widely studied (Cohen and Dunford, 1986; Bowerman and Oconnell, 2007; Diebold, 1999; Makridakis et al., 1979; Chen, 2011). They incorporate characteristics of the same time series according to the autocorrelation, providing predictions based on its past values. A time series Regression Model is also a technique based on maximum likelihood estimation to find the fixed parameters in order to write the equation to do inferences, and, of course, forecasts with new data. The general form of an ARIMA model with order (p,d,q), could be represented as: $\phi(B)\nabla^d y_t = \theta(B)\epsilon_t$. Where Y_t is the time series data, ϵ_t is the error term, B is the back shift operator that is defined as $By_t = y_{t-1}$, and $\nabla = 1 - B$ (Chen, 2011). For the estimation of these models, four steps must be considered, as it is cited by Chen (2011):

- Identification of $ARIMA(p, d, q)$ structure, based on Autocorrelation and Partial autocorrelation definitions.
- Estimation of the unknown parameters.
- Goodness of fit tests of the estimated residuals.
- Forecast future unknown values.

In R program (R Core Team, 2014), the ARIMA model can be estimated with the function auto.arima, which provides the best possible fit according to parameters like the number of possible autorregressive, seasonal, differencing components that can be identifiable according to the Autocorrelation and Partial Autocorrelation functions.

The selection of the type of model depends on specific autocorrelation and partial autocorrelation functions, as they are presented in the table 2.3:

Model	Autocorrelation Function (ACF)	Partial autocorrelation Function (PACF)
MA(q)	Fast decrease to zero after q lag (order q)	Exponential or slow decreasing, sinusoidal.
AR(p)	Exponential or slow decreasing, sinusoidal	Fast decrease to zero after lag p (order p).
ARMA(p, q)	Exponential or slow decreasing, sinusoidal	Exponential or slow decreasing, sinusoidal.

Table 2.3: Identification of an ARIMA model. Source: Bowerman et al., 2007

Exponential Smoothing (ES) Model

Exponential Smoothing (ES) is another very used statistical technique (Bermúdez et al., 2009; Wang, 2006; Chen, 2011). It is based on the moving average technique, with a weight of the values of past periods of the same series. The forecast is based on: $F_{t+1} = \alpha Y_t + (1 - \alpha)F_t$. Here, F_{t+1} is the forecast value of the response, in period $t + 1$, the observed variable is Y_t, and α is the smoothed constant Chen (2011). The fitting is adjusted in a time horizon using an optimization of the Sum of Squared Errors (SSE) value, searching for the value of α that minimizes such sum. It could incorporate a level such as a trend and a seasonality components of the data in variations of these, like Holt Winter's method.

Classical Regression Model

A classical Multiple Regression Model can incorporate multiple co-variables in order to establish a relation among these and a response variable Y; and provides an interpretation of the effect that each one of these variables, also called explanatory, causes over the response. A general form of these models is $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \epsilon$. Where ϵ is a random variable with the assumptions based on mean zero, constant variance, with a Normal distribution behavior, and uncorrelation among temporal values (Bowerman and Oconnell, 2007; Valencia et al., 2014b).

As it was previously mentioned in Chapter 1, the most important assumptions for ARIMA and also classical Regression models, for the residuals after the estimation of these for a time series are: normality distribution, constant variance, and independence for residuals. For Exponential Smoothing models, it is important to check the independence in errors, not the normality, because of the optimization estimation method.

The principal assumptions to be tested over the residuals of an ARIMA or a Regression Model Tabares et al. (2014); Valencia et al. (2014b), and the form to verify them, are:

- Normal Distribution, that is explored with graphs like QQ norm, where theoretical residuals are estimated using Normal Standard Distribution, and are plotted against estimated standardized residuals. The plotted point should be very linear, and must be inside the interval $(-2, 2)$. If there are isolated points that are outside these bands, there is an apparent non normality. But it is precise to do analytical tests like Shapiro or Jarque Bera tests (Jarque and Bera, 1987; Valencia et al., 2014b).

- Constant Variance, or homogeneous variance, explored with the graph between residuals vs fitted values, that will show a constant form if there is uniformity around zero residual, and non constant variance if there is a conical behavior along the residuals, or inappropriate trends. Some of the analytical tests to prove homogeneity are Levene and Bartlett (Shoemaker, 2003; Tabares et al., 2014).

- Uncorrelated residuals, explored with a graph of Autocorrelation of residuals, and, if there are lags with values outside the bands, there is evidence of autocorrelation, but it can also be tested with Ljung-Box (Bowerman and Oconnell, 2007).

If some of the tests are not actually fulfilled in a particular problem, possible bias can affect the parameters estimations. Therefore, the model will produce wrong predicted values (Valencia, 2010; Armstrong et al., 2014). There is also another problem about the form to identify seasonal behavior in Regression and ARIMA models, because, as Armstrong et al. (2014) says, "there is less accuracy when there are not enough historical data", and, in general, for classical models, it is better to have more historical data in order to identify the correct behavior of the series.

If the tests indicate problems, some times transformations can be used. However, in some cases, these transformations do not work. This leads to possible bias to the results. (Makridakis et al., 1979) affirms that *"the ultimate test of any forecast is whether or not it is capable of predicting any future events accurately"*. In this sense, it is better to measure the predictive capacity of the estimated models, and there are error indicators, like the Root of the Mean Square Error (RMSE), the Mean Absolute Deviation (MAD), and the Mean of absolute Percentage error (MAPE).

2.3.2 Bayesian alternatives

Finding better methodologies for forecasting in difficult situations is the first aim of this research, because it is the first step in order to define a good inventory policy.

For the industry, it may be necessary to find models that do not require too much historical data to make predictions of demand. Sometimes, the industry must do a correct planning about new products, or short term ones; and they have no idea about how much to order or program on inventories, and there are no data available. These techniques to forecast can also help when special conditions of the time series are found, like: changes in the seasonal component, level drastic changes. Besides this, as it was previously mentioned, when theoretical assumptions of many classical models cannot be fulfilled, the researcher could have problems with lack of accuracy and trust-ability. These gaps could be filled using Bayesian techniques, which are explored to develop forecasts in numerous investigations (Alba and Mendoza, 2007; Fei et al., 2011; Gill, 2007; Harrison and Stevens, 1976; West and Harrison, 1997; Tabares et al., 2014; Lee et al., 2003; Pedroza, 2006; Bermúdez et al., 2009; Petris et al., 2009; Yelland and Lee, 2003; Yelland, 2010; Fúquene et al., 2015).

There are special Bayesian models for doing forecasts, like the Bayesian Regression, the Bayesian Dynamic Linear Models (Zellner, 1996; West and Harrison, 1997; Petris et al., 2009). The first Bayesian treatment of the Dynamic Linear Models (DLM), in the statistical literature was done by Harrison and Stevens (1976), and since then, much work has been undertaken to extend the theoretical basis and applications (Petris et al., 2009; Fúquene et al., 2015; Pole et al., 1988).

Crowston et al. (1973) proposes a Bayesian approximation to forecast the demand and shows, with different types of heuristics, that the production costs can be optimized, assuming a high capacity, and considering N periods of demands. Few works consider Bayesian models related to operations research, in especial, inventories (Choi et al., 2003; Nechval et al., 2011).

In this line, State Space models have been formulated for many reasons in engineering or population dynamics. Harrison and Stevens (1976) formulated these models in the framework of Bayesian forecasts over a period of equally-spaced discrete time, and here, they call them as: Dynamic Linear Models (DLM). State-Space models consider a time series as outputs of dynamic systems perturbed by random disturbances (Petris et al., 2009), and, in this way, they can be used to generate forecasts. Some generalities of these models and the Bayesian framework are summarized in the next sections.

2.3.3 Fitting and Forecasts indicators

The most used indicators, to compare forecast, are the MAPE, Mean Absolute Percentage Error or the MSE, Mean Square Error (Makridakis et al., 1979; Bowerman and Oconnell, 2007), not only used in classical, but also, in Bayesian works, were it is precise to compare forecasts of different models (Petris et al., 2009; Rojo and Sanz, 2010; Armstrong et al., 2014). Petris et al. (2009) affirms that MAPE (Mean of Absolute Percentage Error) indicator, also used by Makridakis et al. (1979), is one the most used statistics to compare models, besides MAD (Mean of Absolute Deviations) and RMSE (Root of Mean Square Error), and the measures of demands are always positive values, so, a percentage as MAPE can provide a good information about the accuracy of the studied models. Additionally, it expresses, in a simple form, an amount of error associated to the real measure of the response, providing, to anyone who does not even understand what is a big or very small quantity, a proportion of the possible bias committed. These are the reasons to use the MAPE measures in order to do comparisons among statistical models, for making decisions based on one response behavior.

2.4 Bayesian Processes

The Bayesian Analysis uses the Bayes' Theorem, and it does not need a big sample data to do inferences. These techniques have assumptions that are different with respect to classical models. For example, some parameters of probability distributions are random variables. This includes prior information quantified in a probability distribution (Gelman et al., 2004; Gill, 2007) known as $\xi(\theta)$, also, data information represented by: $y_1, \ldots, y_n$, which is included in the likelihood function $L(y_1, \ldots, y_n|\theta)$.

The form to estimate the posterior distribution can be explained as follows. Using the Bayes Theorem, the prior distribution $\xi(\theta)$ times likelihood, takes to the posterior distribution: $\xi(\theta|y_1, \ldots, y_n)$ (Gill, 2007; Congdon, 2002).

Then, in order to find a distribution for forecasting, it is possible to estimate the predictive distribution, as an integral of the distribution of the variable to be predicted, times the posterior (Gill, 2007; Congdon, 2002).

Bayesian Inference requires the use of prior information for the parameter(s), then, prior probability distributions are selected for the parameters, but often, there is not so much information, and that is the reason to require sometimes, knowledge from experts, because the data never speak, completely, by themselves. Besides this, the data information is also involved in this process to build Bayesian estimations.

A practical form to do bayesian estimation is to consider *conjugate priors distributions*. If the density of the prior distribution belongs to the same family of the posterior, then, it is conjugate (Gill, 2007).

Bayesian approach, according to Broemeling and Shaarawy (1988), unify and simplify the analysis of time series.

In summary, in a Bayesian process to do forecast, these definitions can be considered:

- Prior Distribution: Distribution of the parameters. This can be informative or non informative.

- Posterior Distribution: The product between the prior distribution and the likelihood function of the data.

- Predictive Distribution: The integral of the distribution of the data to be predicted, times the posterior distribution.

Broemeling and Shaarawy (1988) present a review of Bayesian Analysis applied to Autorregressive Moving Average processes (ARMA) as a formulation, but it does not contemplate the Dynamic Linear Models, or State-Space Models.

It is necessary to begin defining the usual Bayesian models to predict the demand of final products under study, by doing a review of some other authors defined models, and also, the novel designed models. In the next chapter, it will be done a comparison among different models, and a selection process of the best.

2.4.1 Bayesian Regression Model

The general Bayesian Regression Model has many kinds of variations, but it can be presented in a general form to understand the general mathematical process. In this section it is shown some of the results of the model presented in Zellner (1996). The author builds this model from the assumptions: A prior non informative distribution: $\frac{1}{\sigma}$, for model parameters β, σ, a Normal Distribution for data, thereby likelihood of data sample; and the product between this prior distribution and likelihood generates posterior parameter distribution. Followed to this, the integral of the distribution of the data to be predicted, times the posterior distribution, leads to the predictive Bayesian distribution, which is the finally distribution used to do forecasts of the response variable.

Suppose Y is a vector that contains all the information of the serie: $Y = (y_1, y_2, \ldots, y_T)$ and $Y_{-1} = (y_0, y_1 \ldots, y_{T-1})$, a vector of lag 1, and we use the matrix form of the general model: $Y = X\beta$. The likelihood will be as it follows.

$$L(y|y_0, \beta) \propto \frac{1}{\sigma^T} exp^{-\frac{1}{2\sigma^2}(Y-X\beta)'(Y-X\beta)} \tag{2.15}$$

Posterior and Predictive Distributions for the Bayesian Regression Model

Assuming a non informative prior distribution for the parameters β, σ:

$$\xi(\beta, \sigma) \propto \frac{1}{\sigma} \tag{2.16}$$

With the product between these two (the likelihood, eq. 2.15, and the prior, eq. 2.16), the posterior distribution is 2.17:

$$\xi(\beta, \sigma | y_0, Y) \propto \frac{1}{\sigma^{T+1}} exp^{-\frac{1}{2\sigma^2}(Y-X\beta)'(Y-X\beta)} \tag{2.17}$$

Re-expressing the term of the exponent, it will be:

$$(Y - X\beta)'(Y - X\beta) = Y'Y - 2X'\beta Y + \beta' X' X\beta \tag{2.18}$$

$$X'X\left(\beta'\beta - 2\beta(X'X)^{-1}X'Y\right) + Y'Y = (\beta - \hat{\beta})'X'X(\beta - \hat{\beta}) + Y'Y \tag{2.19}$$

Where $\hat{\beta} = (X'X)^{-1}X'Y$. So posterior distribution for β will be:

$$\xi(\beta, \sigma | y_0, Y) \propto \frac{1}{\sigma^{T+1}} exp^{-\frac{1}{2\sigma^2}\left[(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})+Y'Y\right]} \tag{2.20}$$

The expression 2.20 will be used to calculate the Predictive distribution.

Predictive distribution for the Bayesian Regression Model

The predictive distribution is the integral of the product between the distribution for the new data, and the posterior. Using the posterior 2.20, for the parameters β, σ, then:

$$f(Y_{T+1} | y_0, y_T) = \int\int f(Y_{T+1} | \beta, \sigma, Y)\xi(\beta, \sigma | y_0, Y) d\sigma d\beta \tag{2.21}$$

And assuming $S_{+1} = Y_{+1} - X_{+1}\beta$ and $S = Y - X\beta$, it is:

$$f(Y_{T+1} | y_0, y_T) \propto \int\int \frac{1}{\sigma^{T+2}} exp^{\frac{1}{2\sigma^2}(S'_{+1}S_{+1}+S'S)} d\sigma d\beta \tag{2.22}$$

According to the properties of the Inverse Gamma distribution (Zellner, 1996; Casella and Berger, 2002), and after algebra work it leads to 2.23:

$$f(Y_{T+1}|y_0, y_T) \propto \int K \frac{1}{\left[\left(\frac{T+1}{2}\right)(S'_{+1}S_{+1} + S'S)\right]^{(T+1)/2}} d\beta \tag{2.23}$$

Which is a Student-t. Where $K = \frac{\Gamma[(T+1)/2]}{\sqrt{2\pi}}$

The term of the denominator of the equation 2.23, can be re-expressed in terms of β parameters. Doing the term of the denominator $C = S'_{+1}S_{+1} + S'S$:

$$\begin{aligned} C &= (Y_{+1} - X_{+1}\beta)'(Y_{+1} - X_{+1}\beta) + (Y - X\beta)'(Y - X\beta) \\ C &= Y'_{+1}Y_{+1} - Y'_{+1}X\beta - \beta' X'_{+1}Y_{+1} + X'_{+1}X_{+1}\beta'\beta + Y'Y - Y'X\beta - X'Y\beta + X'X\beta'\beta \\ C &= \beta'\beta(X'_{+1}X_{+1} + X'X) - 2\beta'(X'_{+1}Y_{+1} + X'Y) + Y'_{+1}Y_{+1} + Y'Y \end{aligned} \tag{2.24}$$

Completing the square, and doing $M^{-1} = (X'_{+1}X_{+1} + X'X)^{-1}$:

$$C = \left[\beta'\beta - 2\beta' M^{-1}(X'_{+1}Y_{+1} + X'Y)\right] M + Y'_{+1}Y_{+1} + Y'Y \tag{2.25}$$

Doing $\tilde{\beta} = M^{-1}(X'_{+1}Y_{+1} + X'Y)$

$$\begin{aligned} C &= \left[(\beta - \tilde{\beta})' M(\beta - \tilde{\beta}) - \tilde{\beta}M\tilde{\beta} + Y'_{+1}Y_{+1} + Y'Y\right] \\ C &= \left[(\beta - \tilde{\beta})' M(\beta - \tilde{\beta}) + K\right] \\ C &= K\left[(\beta - \tilde{\beta})' MK^{-1}(\beta - \tilde{\beta}) + 1\right] \end{aligned} \tag{2.26}$$

Where $K = Y'_{+1}Y_{+1} + Y'Y - \tilde{\beta}M\tilde{\beta}$

$$f(y_{t+1}|y_t) \propto \int \frac{1}{\left(\frac{T+1}{2}\right)\left[K\left((\beta - \tilde{\beta})' MK^{-1}(\beta - \tilde{\beta}) + 1\right)\right]^{(T+1)/2}} d\beta \tag{2.27}$$

$$f(y_{t+1}|y_t) \propto \frac{1}{\left(Y'_{+1}Y_{+1} + Y'Y - \tilde{\beta}M\tilde{\beta}\right)^{(\nu+q)/2}} (2.28)$$

After an algebraic process, shown in Zellner (1996), pg. 73 it is:

$$f(y_{t+1}|y_t) \propto \frac{1}{\left[\nu + (Y_{+1} - X_{+1}\hat{\beta})' H (Y_{+1} - X_{+1}\hat{\beta})\right]^{(\nu+q)/2}} \tag{2.28}$$

Where the parameter estimation $\hat{\beta} = (X'X)^{-1}X'Y$, and $H = \frac{1}{s^2}(I - X_{+1}MX_{+1})$. Besides, the degrees of freedom: $\nu = T - p$; p: Number of parameters; q: Number of forecasting position. The power of the denominator can be changed, depending on the number of the forecasting position (q); for y_{T+1}, $q = 1$, the exponent will be: $(\nu + 1)/2 = (T - 2 + 1)/2 = (T - 1)/2$, and for y_{T+2}, $(\nu + 2)/2 = (T - 2 + 2)/2 = T/2$. This is the Predictive Distribution to do forecasts for the Bayesian Regression Model (BRM) presented by Zellner (1996), with diffuse Prior distribution for the parameters.

2.4.2 New Bayesian Regression Model with Normal Prior Distribution

For this Doctoral Thesis, a novel model will be proposed to do forecasts for the demands. This is a Bayesian Regression Model with the Normal, as the Prior distribution for the parameters β, introducing modifications to the original proposed by Zellner (1996). As it is discussed by Gelman et al. (2004) or Zellner (1996), the Normal Distribution can be used instead of a Non informative. In this thesis, it will be changed the original non informative prior distribution, to the informative distribution, "Normal", for the model parameters β. It is a joined distribution with σ_1, with $\sigma_1 = \sigma_0\sigma$. Moreover, in this research, it will be introduced a dynamic modification to the prior parameters, called the vector β_0. And the analytical process to obtain the predictive distribution will be presented.

- Likelihood function

  Assuming that the value of y_0 is known, and if all the observations of responses, are collected in a vector: $Y = (y_0, y_1, y_2, \ldots, y_T)$. The model can be expressed as: $Y = X\beta = \lambda Y_{-1} + \alpha X_t$, the likelihood function is:

$$L(y|y_0, \beta) \propto \sigma^{-T} exp^{\left[-\frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta)\right]} \tag{2.29}$$

Assuming the Normal distribution as the prior with the parameters β and σ_1, $N(\beta, \sigma_1)$, with $\sigma_1 = \sigma_0\sigma$.

$$
\begin{aligned}
\xi(\beta,\sigma) &\propto \frac{1}{\sigma_1} exp^{\left[-\frac{1}{2\sigma_1^2}(\beta-\beta_0)'(\beta-\beta_0)\right]} \\
\xi(\beta,\sigma) &\propto \frac{1}{\sigma_0\sigma} exp^{\left[-\frac{1}{2(\sigma_0\sigma)^2}(\beta-\beta_0)'(\beta-\beta_0)\right]}
\end{aligned}
\tag{2.30}
$$

- Posterior Distribution.

  With the product between the likelihood and the prior (eq. 2.29 and eq. 2.30), and changing: $\tau = \frac{1}{\sigma^2}$, $\tau_0 = \frac{1}{\sigma_0^2}$, then, the equation of the posterior distribution changes to the expression 2.32.

$$
\xi(\beta,\tau|\tau_0,\beta_0,y_0,y_t) \propto \tau^{(T+1)/2}\tau_0^{1/2} exp^{\left[-\frac{\tau}{2}(Y-X\beta)'(Y-X\beta)-\frac{\tau\tau_0}{2}(\beta-\beta_0)'(\beta-\beta_0)\right]} \tag{2.31}
$$

$$
\xi(\beta,\tau|\tau_0,\beta_0,y_0,y_t) \propto \tau^{\frac{T+1}{2}}\tau_0^{\frac{1}{2}} exp^{-\frac{\tau}{2}\left[(Y-X\beta)'(Y-X\beta)+\tau_0(\beta-\beta_0)'(\beta-\beta_0)\right]} \tag{2.32}
$$

- Predictive Distribution

  Let Y_+, be the h dimensional vector to be predicted, and X_+ is the design matrix with hxp dimension. Here it is also necessary to use the Normal distribution, $f(Y_+|\beta,\tau) = N_h(X_+\beta, \tau I)$, of the original data to calculate the integral that leads to the predictive distribution.

  If $S = Y - X\beta$ and $S_+ = Y_+ - X_+\beta$.

$$
\begin{aligned}
f(Y_+|y_0,Y) &= \int_{\Re}\int_0^{\infty} f(Y_+|\beta,\tau)\xi(\beta,\tau|Y_0,Y)d\tau d\beta \\
f(Y_+|y_0,Y) &\propto \tau_0^{\frac{1}{2}}\int\int \tau^{\frac{T+2}{2}} exp^{-\frac{\tau}{2}\left[(S'S)+(S'_+S_+)+\tau_0(\beta-\beta_0)'(\beta-\beta_0)\right]} d\tau d\beta \\
f(Y_+|Y_0,Y) &\propto \tau_0^{\frac{1}{2}}\int\int \frac{\Gamma\left[\frac{T+4}{2}\right]}{D^{\frac{T+4}{2}}} * \frac{D^{\frac{T+4}{2}}}{\Gamma\left[\frac{T+4}{2}\right]}\tau^{\frac{T+2}{2}} exp^{-\frac{\tau}{2}[D]} d\tau d\beta \\
f(Y_+|Y_0,Y) &\propto \tau_0^{\frac{1}{2}}\int \left[(S'S)+(S'_+S_+)+\tau_0(\beta-\beta_0)'(\beta-\beta_0)\right]^{-\frac{T+4}{2}} d\beta
\end{aligned}
\tag{2.33}
$$

Expanding the expression of the brackets in the integral of (2.33):

$$
\begin{aligned}
D &= Y'Y - 2\beta X'Y + \beta' X'X\beta + Y_+'Y_+ - 2\beta X_+'Y_+ \\
&+ \beta' X_+'X_+\beta + \beta'\tau_0\beta - 2\beta'\beta_0\tau_0 + \beta_0'\tau_0\beta_0 \\
D &= \beta'(X_+'X_+ + X'X + \tau_0)\beta - 2\beta'(X'Y + X_+'Y_+ + \tau_0\beta_0) + Y'Y + Y_+'Y_+ + \beta_0'\tau_0\beta_0 \\
D &= \left[\beta'\beta - 2\beta' M^{-1}(X'Y + X_+'Y_+ + \tau_0\beta_0) + M^{-1}(Y'Y + Y_+'Y_+ + \beta_0'\tau_0\beta_0)\right] M \qquad (2.34)
\end{aligned}
$$

Where $M^{-1} = (X_+'X_+ + X'X + \tau_0)^{-1}$. Let $\beta_n = M^{-1}(X'Y + X_+'Y_+ + \beta_0\tau_0)$. Then,

$$
D = (\beta - \beta_n)'M(\beta - \beta_n) - \beta_n'M\beta_n + Y'Y + Y_+'Y_+ + \beta_0'\tau_0\beta_0 \qquad (2.35)
$$

$$
f(Y_+|Y_0, Y) \propto \tau_0^{\frac{1}{2}} \int [(\beta - \beta_n)'M(\beta - \beta_n) + Y'Y + Y_+'Y_+ + \beta_0'\tau_0\beta_0 - \beta_n'M\beta_n]^{-\frac{T+4}{2}} d\beta \qquad (2.36)
$$

Doing $A = Y'Y + Y_+'Y_+ + \beta_0'\tau_0\beta_0 - \beta_n'M\beta_n$.

After solving:

$$
f(Y_+|Y_0, Y) \propto \tau_0^{\frac{1}{2}} \int [(\beta - \beta_n)'M(\beta - \beta_n) + A]^{-\frac{T+4}{2}} d\beta \qquad (2.37)
$$

The integral in 2.37 is a Student-t distribution.

$$
\begin{aligned}
f(Y_+|y_0, Y) &\propto \tau_0^{\frac{1}{2}} \int \left[A\left((\beta - \beta_n)'A^{-1}M(\beta - \beta_n) + 1\right)\right]^{-\frac{T+4}{2}} d\beta \\
f(Y_+|y_0, Y) &\propto \tau_0^{\frac{1}{2}} [Y'Y + Y_+'Y_+ + \beta_0'\tau_0\beta_0 - \beta_n'M\beta_n]^{-\frac{T+4}{2}} \qquad (2.38)
\end{aligned}
$$

Expression 2.38 is a Student-t distribution, but we can change the internal terms, to get a function of Y_+.

Expanding the expression in the brackets of (2.38):

$$I = Y_+'Y_+ - \beta_n' M\beta_n + Y'Y + \beta_0'\tau_0\beta_0 \tag{2.39}$$

Doing $M^{-1} = (X_+'X_+ + X'X + \tau_0)^{-1}$ and $\beta_n = M^{-1}(X'Y + X_+'Y_+ + \beta_0\tau_0)$

$$\begin{aligned}
I &= Y_+'Y_+ - M^{-1}(X'Y + X_+'Y_+ + \beta_0\tau_0)MM^{-1}(X'Y + X_+'Y_+ \\
&+ \beta_0\tau_0) + Y'Y + \beta_0'\tau_0\beta_0 \\
I &= Y_+'Y_+ + Y'Y + \beta_0'\tau_0\beta_0 - M^{-1}((X'Y)'(X'Y) + (X_+'Y_+)'(X_+'Y_+) \\
&+ (\beta_0\tau_0)'\beta_0\tau_0 + 2(X'Y)'(X_+'Y_+) + 2(X'Y)'(\beta_0\tau_0) + 2(X_+'Y_+)'(\beta_0\tau_0)) \\
I &= Y_+'(I - X_+M^{-1}X_+)Y_+ - 2Y_+X_+M^{-1}(X'Y + \beta_0\tau_0) \\
&+ \beta_0'\tau_0\beta_0 - M^{-1}(YX'X'Y + \tau_0\beta_0\beta_0\tau_0 + 2X'Y\beta_0\tau_0) + Y'Y \\
I &= \left[Y_+'Y_+ - 2Y_+X_+(I - M^{-1}X_+'X_+')^{-1}M^{-1}(X'Y + \beta_0\tau_0)\right](1 - M^{-1}X_+'X_+') \\
&+ (\beta_0'\tau_0\beta_0 - M^{-1}\tau_0\beta_0\beta_0\tau_0) + (Y'(I - M^{-1}X'X)Y - 2M^{-1}X'Y\beta_0\tau_0) \\
I &= [(Y_+ - Y_n)'(Y_+ - Y_n) - Y_n'Y_n](1 - M^{-1}X_+'X_+') \\
&+ (\beta_0'\tau_0\beta_0 - M^{-1}\tau_0\beta_0\beta_0\tau_0) \\
&+ (Y'Y - 2M^{-1}(I - M^{-1}X'X)^{-1}X'Y\beta_0\tau_0)(I - M^{-1}X'X) \\
I &= [(Y_+ - Y_n)'(Y_+ - Y_n) - Y_n'Y_n](1 - M^{-1}X_+'X_+') \\
&+ (\beta_0'\tau_0\beta_0 - M^{-1}\tau_0\beta_0\beta_0\tau_0) + [(Y - Y_m)'(Y - Y_m) - Y_m'Y_m](I - M^{-1}X'X)
\end{aligned} \tag{2.40}$$

Where: $Y_n = (I - X_+M^{-1}X_+)^{-1}X_+M^{-1}(X'Y + \beta_0\tau_0)$, using a definition of inverted difference in matrices (Zellner, 1996).

$$\begin{aligned}
Y_n &= (I + X_+(X'X)^{-1}X_+)M^{-1}X_+(X'Y + \beta_0\tau_0) \\
Y_n &= (I + (X'X)^{-1}X_+'X_+)(X'X + \tau + X_+'X_+)^{-1}X_+(X'Y + \beta_0\tau_0) \\
Y_n &= (I + (X'X)^{-1}X_+'X_+)(I + (X'X + \tau_0)^{-1}X_+'X_+)^{-1}X_+(X'X + \tau_0)^{-1}(X'Y + \beta_0\tau_0)
\end{aligned} \tag{2.41}$$

τ_0 is a very small quantity, so, $Y_n = (X'X + \tau_0)^{-1}X_+(X'Y + \beta_0\tau_0) = X_+\tilde{\beta}$. Besides, some terms disappear due to the proportionality. By doing $A = (Y - Y_m)'(Y - Y_m)$, and $Y_m = M^{-1}(I - M^{-1}X'X)^{-1}X'\beta_0\tau_0$, and simplifying the final predictive distribution, it leads to:

$$\begin{aligned} I &= \left[(Y_+ - Y_n)'(Y_+ - Y_n) - Y_n'Y_n\right](1 - M^{-1}X_+'X_+') \\ &+ \left[(Y - Y_m)'(Y - Y_m) - Y_m'Y_m\right](I - M^{-1}X'X) \\ I &= \left[(Y_+ - Y_n)'(Y_+ - Y_n) + (Y - Y_m)'(Y - Y_m)\right] \end{aligned} \tag{2.42}$$

$$\begin{aligned} f(Y_+|Y_0, Y) &\propto \left[[(Y_+ - Y_n)'(Y_+ - Y_n) + A]\right]^{-\frac{T+4}{2}} \\ f(Y_+|Y_0, Y) &\propto \left[((Y_+ - Y_n)'A^{-1}(Y_+ - Y_n) + 1)A\right]^{-\frac{T+4}{2}} \end{aligned} \tag{2.43}$$

The expression (2.43) is a Student-t distribution, with mean: $Y_n = X_+\tilde{\beta}$, with ν degrees of freedom, and Variance, $\frac{\nu}{\nu-2}A = \frac{\nu}{\nu-2}(Y - Y_m)'(Y - Y_m)$, which mean is different to the result of Zellner (1996), with non informative prior distribution.

2.4.3 Predictive Model innovation in the Prior distribution

Assuming the final model in 2.43, which is a Student-t distribution, with mean: $Y_n = X_+\tilde{\beta}$. If we assume, besides, an expert knowledge that will change the β_0 parameter, for every time t, like, β_{0t} it will permit that a user can do variations to the prior distribution, internally, posterior and, finally, the predictive distribution will also change, according the time goes on.

That leads to the next expression of the mean, to do predictions for every $t + 1$ period:

$$Y_n = (X'X + \tau_0)^{-1}X_+(X'Y + \beta_{0,t+1}\tau_0) \tag{2.44}$$

2.5 Bayesian Dynamic Linear Model (BDLM)

A Dynamic Linear System is a general mechanism for representing the State Space Model of univariate and multivariate systems. The State Space models are a powerful technique for the analysis and forecasting of time series data, applied to areas like econometrics, signal processing, genetics and population dynamics (Petris, 2010).

The state estimation of a system is an important problem for engineering, and the Kalman Filter is a technique to do recursive estimation of the parameters of this system. But one of the most known applications of this filter is related to the Dynamic Linear Models in the framework of Bayesian forecasting, developed by Harrison and Stevens (1976). The estimation of Dynamic Linear Models, implies four principal steps for estimation and prediction: Establishing initial conditions, estimation of the next stage, prediction of new observation, and updating the parameters (Peña and Guttman, 1988).

Specifically, the standard form to represent the dynamics in statistical areas, involves one step at a time of the linear dependence about the parameter of the system, say θ. The general process for this analysis is explained by different authors (Meinhold and Singpurwalla, 1983; Harrison and Stevens, 1976; West and Harrison, 1997). There are also new developments in these models, for example, in creating robust algorithms to establish robust procedures to outliers (Peña and Guttman, 1988; Pericchi and Pérez, 2010; Fúquene et al., 2015), and tests to demonstrate stability (Tsurumi, 1988).

General Dynamic Linear Model

The following is a general representation of a *Dynamic Linear Model*(DLM), whose estimation is built with a Bayesian Process, and a recursive Kalman Filter (Meinhold and Singpurwalla, 1983).

Let $Y_t, Y_{t-1}, \ldots, Y_1$ be the data, denoting the observed values of a variable of interest at times $t, t-1, \ldots, 1$. Y_t depends on the parameter unobserved θ_t, known as the *State of nature*. A DLM is a representation of a Normal Prior Distribution for this state vector θ_t and a pair of equations, that is expressed with the linear relation between Y_t and θ_t, and specified with the equations:

$$\begin{array}{lll} \text{Observation equation:} & Y_t = F_t\theta_t + \nu_t, & \nu_t \sim N(0, V_t) \\ \text{System equation:} & \theta_t = G_t\theta_{t-1} + w_t, & w_t \sim N(0, W_t) \end{array} \tag{2.45}$$

where F_t and G_t are known quantities, and may or not change with time. G_t is the evolution matrix which describes a relation between times $t-1$ and t for parameter θ_t; and F_t describes the information of the system or response, for example, it can be the $px1$ vector of explanatory variables. ν_t and w_t are independent random Gaussian vectors with mean zero and variance matrices V_t and W_t, and independent with each other.

At time $t-1$: $(\theta_{t-1}|Y_{t-1}) \sim N(\hat{\theta}_{t-1}, \Sigma_{t-1})$, where $\hat{\theta}_{t-1}$ and Σ_{t-1} are initial mean and variance.

An easy example of this problem, is a State Space Regression model cited in Tsurumi (1988), who presents a test created by LaMotte and McWhorter Jr (1978) (p. 90):

$$\begin{array}{ll} Y_t = x_t\beta_t + \epsilon_t, & \epsilon_t \sim N(0, \sigma_t^2) \\ \beta_t = \beta_{t-1} + \mu_t, & \mu_t \sim N(0, \sigma_\mu^2 D) \end{array} \tag{2.46}$$

Y_t is the response variable, and x_t is a known co-variable. D is a known matrix. β_t is represented as a random walk with independent random variable μ_t.

Other example is the *linear growth model*, also called, local linear trend (Petris et al., 2009).

$$\begin{array}{ll} Y_t = \mu_t + \nu_t, & \nu_t \sim N(0, \mathbf{V}_t) \\ \mu_t = \mu_{t-1} + \beta_{t-1} + \mu_{1,t}, & \mu_{1,t} \overset{iid}{\sim} N(0, \sigma_\mu^2) \\ \beta_t = \beta_{t-1} + \mu_{2,t}, & \mu_{2,t} \overset{iid}{\sim} N(0, \sigma_\mu^2) \end{array} \tag{2.47}$$

$$G_t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; F_t = \begin{bmatrix} 1 & 0 \end{bmatrix}; W_t = \begin{bmatrix} \sigma_\mu^2 & 0 \\ 0 & \sigma_\beta^2 \end{bmatrix} \tag{2.48}$$

The matrices G_t, F_t, and the covariance matrices V_t and W_t are constant. In this case, the model is said to be time invariant.

These kind of processes are applied to special cases of changes in time series, as it is exposed by Pole et al. (1988), (p. 167), who cite: *"The first comprehensive Bayesian treatment of these models in the statistical literature was Harrison and Stevens (1976), and since then, motivated both by academic interest and the perceived requirements of practitioners, much work has been undertaken to extend the theoretical basis and range of application of these models"*.

Dynamic State Space models provide a basis for constructing forecasting models for reasons as simplicity, structuring, and insights into model construction (Pole et al., 1988). *"The choice of the state-space form of the DLM as an appropriate framework within which to model many forecasting system rests in the twin foundations of simplicity and interpretbility. Its simplicity derives from having just two constituents: an observation equation relating the values actually recorded, and a system equation that defines the evolutionary dynamics of the individual components describing these conditions"* (p. 168).

2.5.1 Bayesian Dynamic Linear Model with Kalman Filter

Inferences in DLM involves two operations: evolution to build up the prior and posterior distributions, and updating to incorporate new observations. In order to understand how these two components work, there are some techniques, like a Kalman filter, that also produces forecasts. There are different forms to build this filter, and one of the simplest forms to understand it, is the recursive Kalman Filter explained by (Meinhold and Singpurwalla, 1983), who represent this model as:

$$\begin{array}{llll} \text{Observation equation:} & Y_t & = & F_t\theta_t + v_t, \quad v_t \sim N(0, V_t) \\ \text{System equation:} & \theta_t & = & G_t\theta_{t-1} + w_t, \quad w_t \sim N(0, W_t) \end{array} \tag{2.49}$$

Where the first line is the Observation equation, and the second one is the system equation.

The recursive Kalman Filter does an estimation of the state of the parameter θ_t, and also, the prediction of the response.

In order to explain the process, first, we must be focused on time $t-1$. and the observed data until this period, it is, Y_{t-1}. Here, let Y and θ are the vectors of responses and parameters of the system, respectively (Meinhold and Singpurwalla, 1983).

Process

The recursive process can be described as follows.

- Starts from initial values of the parameters: θ_0, and a variance matrix $\sum$, at time 0.
- Generate θ_t with a Normal Prior Distribution with mean: $\mu = G_t\hat{\theta}_{t-1}$ and variance: $R_t = G_t\sum_t G'_t + W_t$
- Update the Normal posterior distribution: $(\theta_t|Y_{t-1}) \sim N\left(\hat{\theta}_{t-1}, \hat{\sum}_{t-1}\right)$.
- Estimate the mean and the variance of the posterior distribution, as follows:

  Mean:

$$\hat{\theta}_t = G_t\hat{\theta}_{t-1} + R_t[F'_t(V_t + F_tR_tF'_t)]^{-1}e_t \tag{2.50}$$

  Variance:

$$\sum_t = R_t - R_tF'_t(V_t + F_tR_tF'_t)^{-1}F_tR_t \tag{2.51}$$

- Update the posterior distribution $(\theta_{t-1}|Y_{t-1}) \sim N(\hat{\theta}_{t-1}, \sum_{t-1})$, where the mean and the variance are shown in the previous equations.
- Estimate the system equation: $\theta_t = G_t\theta_{t-1} + w_t$, using the estimated value $\hat{\theta}_{t-1}$.
- Estimate the observation equation: $\hat{Y}_t = F_tG_t\hat{\theta}_t + v_t$
- Predict $\hat{Y}_p$ using the Normal predictive distribution, with mean $\hat{Y}_t$ and variance R_t
- Estimate the error $e_t = Y_0 - Y_p$ and cycle re-starts.

Derivation of the Posterior Distribution

Suppose $a_t = \hat{\theta}_{t-1}$ mean of θ_t at initial condition, and m_t the posterior mean.

- Parameter Prior distribution.

$$\xi(\theta_t) \propto exp\left\{-\frac{1}{2}(\theta_t - a_t)^{'} R_t^{-1}(\theta_t - a_t)\right\} \tag{2.52}$$

- Data distribution

$$L(Y_t|\theta_t) \propto exp\left\{-\frac{1}{2}(Y_t - F_t'\theta_{t-1})^{'} V_t^{-1}(Y_t - F_t'\theta_{t-1})\right\} \tag{2.53}$$

  (West and Harrison, 1997)

- Posterior distribution

  For parameter θ_t, the posterior distribution is:

$$\xi(\theta_t|D_t) \propto L(Yt|\theta_t)\xi(\theta_t|D_{t-1}) \tag{2.54}$$

  Taking natural log and multiply by -2:

$$-2ln(\xi(\theta_t|D_t)) \propto (Y_t - F_t'\theta_{t-1})^{'} V_t^{-1}(Y_t - F_t'\theta_{t-1}) + (\theta_t - a_t)^{'} R_t^{-1}(\theta_t - a_t) \tag{2.55}$$

  Expanding the expression, it will be:

$$\theta_t'(R_t^{-1} + F_t'V_t^{-1}F_t)\theta_t - 2\theta_t(R_t^{-1}a_t + F_t'V_t^{-1}Y_t) + constant \tag{2.56}$$

  According to the results of proved theorem in West and Harrison (1997):

$$C_t^{-1} = R_t^{-1} + F_t'V_t^{-1}F_t$$

$$C_t^{-1}m_t = R_t^{-1}m_t + F_t'V_t^{-1}F_tm_t = R_t^{-1}m_t + F_t'V_t^{-1}Y_t \tag{2.57}$$

Which conduces to:

$$\theta_t'C_t^{-1}\theta_t' - 2\theta_tC_t^{-1}m_t + constant \tag{2.58}$$

$$(\theta_t - m_t)'C_t^{-1}(\theta_t - m_t) + constant$$

$$\xi(\theta_t|D_t) \propto exp\left\{-\frac{1}{2}(\theta_t - m_t)'C_t^{-1}(\theta_t - m_t)\right\} \tag{2.59}$$

The Normal Posterior Distribution presented, applies to univariate and multivariate cases (West and Harrison, 1997).

– Forecast according to the final steps explained at the past subsection, using the observation equation and the Normal Predictive Distribution.

2.5.2 Dynamic Linear Regression

Different kinds of components, such as trends, seasonality, or stochastic evolution, could also be included in these models. One general form to present these components can be in a Dynamic Linear Regression, as it will be presented in this section.

(Petris et al., 2009) describes a Dynamic Linear Regression Model in a univariate form as:

$$\begin{aligned} Y_t &= x_t'\theta_t + \nu_t & \nu_t &\sim N(0, \sigma_t^2) \\ \theta_t &= G_t\theta_{t-1} + w_t & w_t &\sim N(0, W_t) \end{aligned} \tag{2.60}$$

Where $x_t = [x_{1,t}, \ldots, x_{p,t}]$ are the values of the p explanatory variables at time t, which are fixed. The matrix G_t has an order pxp, and F_t has order mxp for every time t (m: observation equation for every time, which is 1 for univariate cases).

It is possible to define the evolution matrix G_t as an identity and $\mathbf{W}_t$ is a diagonal matrix, this corresponds to model the parameters as independent random walks (Petris et al., 2009).

For example,

$$\begin{aligned} Y_t &= \theta_{t1} + \theta_{t2} x_t + \nu_t \quad \nu_t \sim N(0, \sigma_t^2) \\ \theta_t &= G_t \theta_{t-1} + w_t \qquad\quad w_t \sim N(0, W_t) \end{aligned} \tag{2.61}$$

Where $\theta_{t1} = \theta_{t1-1} + w_{t1}$ and $\theta_{t2} = \theta_{t2-1} + w_{t2}$, and $V_t = \sigma_t^2$. The model can be described with the matrices:

$$G_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; F_t = \begin{bmatrix} 1 & x_t \end{bmatrix}; W_t = \begin{bmatrix} \sigma_{t1}^2 & 0 \\ 0 & \sigma_{t2}^2 \end{bmatrix} \tag{2.62}$$

(Petris, 2010) developed a package in R in order to proceed with the estimation and generate Bayesian forecasts of these kinds of DLM. He explains generalities about models that were exposed more detailed in the book of 2009 (Petris et al., 2009).

Other example can be related to the addition of terms of lagged variables in the observation equation, as it is expressed in equation (2.63).

$$\begin{aligned} Y_t &= \lambda_t Y_{t-1} + \alpha_t x_t + \nu_t \quad \nu_t \sim N(0, \sigma_t^2) \\ \lambda_t &= \lambda_{t-1} + w_{t\lambda} \\ \alpha_t &= \alpha_{t-1} + w_{t\alpha} \end{aligned} \tag{2.63}$$

The parameters are random walks, with $\mathbf{w}_t = (w_{t\lambda}, w_{t\alpha})$, so $\mathbf{w}_t \sim N(0, W_t)$, and $V_t = \sigma_t^2$ (Zellner, 1996).

$$G_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; F_t = \begin{bmatrix} Y_{t-1} & x_t \end{bmatrix}; W_t = \begin{bmatrix} \sigma_{t\lambda}^2 & 0 \\ 0 & \sigma_{t\alpha}^2 \end{bmatrix} \tag{2.64}$$

2.5.3 DLM for Multivariate Time Series

Assuming that the quantity y is observed for m statistical units over time, for example, products, companies, etc., so, we have a multivariate time series $Y_t = (Y_{1,t}, \ldots, Y_{m,t})$. These models introduce dependence along the $m - variate$ process. Here, the Seemingly Unrelated Time Series equations (SUTSE) is a multivariate model that assumes that all the m series can be modeled by the same DLM structure, the same observation Matrix F, and the same system matrix G. P-dimensional state vector have the same interpretation. These DLM's, can assume that the m processes follow the same time-invariant matrices F and G, and could be represented by the expression (2.65) (Petris et al., 2009):

$$\begin{array}{ll} \text{Observation equation:} & Y_{i,t} = F\theta_t^{(i)} + v_{i,t} \\ \text{System equation:} & \theta_t^{(i)} = G\theta_{t-1}^{(i)} + w_{i,t} \end{array} \tag{2.65}$$

Where $v_{i,t} \sim N(0, V_i)$, and $w_{i,t} \sim N(0, W_i)$.

Here, it is assumed that all the m series follow the same type of dynamics, but different values for $Y_{i,t}$. Additional theory related about these processes can be found in Petris et al. (2009) and in West and Harrison (1997).

General framework

Assuming that the information set available at time t is $D_t = Y_t, D_{t-1}$, and that initial prior distribution at t=0 is multivariate normal: $(\theta_0|D_0) \sim N[m_0, \mathbf{C}_0]$, for some mean m_0 and variance $\mathbf{C}_0$ known (West and Harrison, 1997).

As the theorem 15.1 proposed by West and Harrison (1997), who cite: "One-step forecast and posterior distribution in the model just defined are given, for each t, as follows":

- Posterior Distribution at $t - 1$, for some mean m_{t-1} and variance $\mathbf{C}_{t-1}$ known.

$$(\theta_{t-1}|D_{t-1}) \sim N[m_{t-1}, \mathbf{C}_{t-1}] \tag{2.66}$$

- Prior Distribution for parameters at time t.

$$(\theta_t|D_{t-1}) \sim N[a_t, \mathbf{R}_t] \tag{2.67}$$

Where: $\mathbf{a}_t = \mathbf{G}_t m_{t-1}$ and $\mathbf{R}_t = \mathbf{G}_t \mathbf{C}_{t-1} \mathbf{G'}_t + \mathbf{W}_t$

- One-Step forecast:

$$(\mathbf{Y}_t|D_{t-1}) \sim N[\mathbf{f}_t, \mathbf{Q}_t] \tag{2.68}$$

Where: $\mathbf{f}_t = \mathbf{F'}_t a_{t-1}$ and $\mathbf{Q}_t = \mathbf{F'}_t \mathbf{R}_{t-1} \mathbf{F}_t + \mathbf{V}_t$

- Posterior Distribution for θ_t at t

$$(\theta_t|D_t) \sim N[\mathbf{m}_t, \mathbf{C}_t] \tag{2.69}$$

Where: $\mathbf{a}_t = a_t + \mathbf{A}_t \mathbf{e}_t$ and $\mathbf{C}_t = \mathbf{R}_t - \mathbf{A}_t \mathbf{Q}_t \mathbf{A'}_t$

$\mathbf{A}_t = \mathbf{R}_t \mathbf{F}_t \mathbf{Q}_t^{-1}$ and $e_t = \mathbf{Y}_t - \mathbf{f}_t$

An Example of a Multivariate Model

It is possible to use the level and the trend components but also, the seasonal representation, for example factor and Fourier models, in the model structure, in order to give a correct explanation of the response.

An example of a representation of these models with level and trend, where there are *m* time series $Y_{m,t}$ is the expression (2.70):

$$\begin{aligned} y_{i,t} &= \alpha_{i,t} + \beta_{i,t} x_t + \nu_{i,t}, & \nu_{i,t} &\overset{iid}{\sim} N(0, \sigma_i^2) \\ \alpha_{i,t} &= \alpha_{i,t-1} + w_{t\alpha,i}, & w_{t\alpha,i} &\overset{iid}{\sim} N(0, \sigma^2_{w_\alpha,i}) \\ \beta_{i,t} &= \beta_{i,t-1} + w_{t\beta,i}, & w_{t\beta,i} &\overset{iid}{\sim} N(0, \sigma^2_{w_\beta,i}) \end{aligned} \tag{2.70}$$

Where

$$F_t = \begin{bmatrix} 1 & x_t \end{bmatrix}; G_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

State vector: $\theta_t^{(i)} = (\alpha_{i,t}, \beta_{i,t})$, for $i = 1, \ldots, m$, and a vector of parameters: $\varphi = (\sigma_i^2, \sigma_{w_1,i}^2, \sigma_{w_2,i}^2)$. Where $\sigma_{w_1,i}^2$, $\sigma_{w_2,i}^2$ are the variances of the parameters.

To introduce a dependent level and slope, can be governed by correlated inputs (Petris et al., 2009), across the m series $Y_{m,t}$.

A matricial representation of this model is:

$$\begin{aligned} y_t &= (F_t \otimes I_m)x_t + v_t & v_t &\overset{iid}{\sim} N(0, V) \\ \theta_t &= (G_t \otimes I_m)\theta_{t-1} + w_t & w_t &\overset{iid}{\sim} N(0, W) \end{aligned} \tag{2.71}$$

Where

$$y_t = \begin{bmatrix} y_{1,t} \\ \vdots \\ y_{m,t} \end{bmatrix}; \theta_t = \begin{bmatrix} \alpha_{1,t} \\ \vdots \\ \alpha_{m,t} \\ \beta_{1,t} \\ \vdots \\ \beta_{m,t} \end{bmatrix}; v_t = \begin{bmatrix} v_{1,t} \\ \vdots \\ v_{m,t} \end{bmatrix}; w_{pm,t} = \begin{bmatrix} w_{11,t} \\ \vdots \\ w_{pm,t} \end{bmatrix}$$

The matrix **W** is the variance covariance structure of the parameters for the m DLM's. A matrix of p blocks (Petris et al., 2009) (p.128), every one with order mxm, in total **W** has the order *pmxpm*.

For example, if $m = 2$, then we will have four parameters: $\theta_t = (\alpha_{1,t}, \alpha_{2,t}, \beta_{1,t}, \beta_{2,t})$, and 2x2 error components $\mathbf{w}_t = (w_{t\alpha,1}, w_{t\alpha,2}, w_{t\beta,1}, w_{t\beta,2})$. Then, the matrix **W** will have an order 4x4, with variances in the diagonal: $(\sigma_{w_\alpha,1}^2, \sigma_{w_\alpha,2}^2, \sigma_{w_\beta,1}^2, \sigma_{w_\beta,2}^2)$, and covariance in order to consider dependence.

With the R software it is possible to build the evolution for every parameter, and also, to do forecasts of the response, using the dlm package created by Petris Petris (2010), in univariate or multivariate structures.

Estimation

Petris et al. (2009) (p.150), in the *proposition 4.1*, present a common case of conjugate Bayesian Inference that can be followed, in order to build a DLM, by showing that the predictive density is a Student-t. In multivariate extensions, when there are m time series models, it can be assumed a diagonal matrix $\mathbf{W}$ for state parameters.

When the samples are large, the posterior distribution has an asymptotic behavior that can be used to do estimations of the parameters by using Maximum Likelihood Estimation, MLE. For a sequence of independent and identically distributed (*i.i.d*), random vectors Y_t, conditioned to θ (that has a prior distribution $\pi(\theta)$), if n is large, it can be proved that the posterior distribution $\pi(\theta|y_1, \ldots, y_n)$ can be approximated by means of a normal density (Petris et al., 2009), centered at the MLE estimator $\hat{\theta}_n$, this implies that in these cases, frequentist and bayesian estimators do not differ so much.

For other kind of distributions, which can not be analytically deduced, the estimation of parameters could be approximated with Monte Carlo Methods using simulation, like Gibbs sampling (Petris, 2010). Gibbs draws a sample from the conditional posterior distribution of interest, in two possible forms: the parameters, given the data and the unobserved states, or the states given the data and the parameters.

Petris (2010) package, dlm, in the software R (R Core Team, 2014), has been developed in order to do filter, Maximum Likelihood Estimation, Gibbs sampling, and also forecasting of these DLM models.

2.6 Linear Mixed Models (LMM)

In contrast to multiple Linear Regression Models, that only have a fixed component $X\beta$, in the Linear Mixed Models, there is an additional component which expresses a random component, Zb, where Z is a matrix with known covariables, introducing a random variation for the repeated observations, which corresponds, for example, to different products that have chronological values of sales. b is a random vector of effects, which are predicted; and the response values are contained in the vector y (Valencia, 2010). This model considers these two components, fixed and random. The following is the expression according to the general form of the model Lange and Ryan (1989), cited and used by Valencia (2010):

$$\underset{T\times 1}{y} = \underset{T\times p}{\mathbf{X}}\ \underset{p\times 1}{\beta} + \underset{T\times Nr}{\mathbf{Z}}\ \underset{Nr\times 1}{b} + \underset{T\times 1}{\varepsilon} \tag{2.72}$$

- y is the vector that contains the components of response, y_{ij}, with i individuals: $i = 1, ...N$, $j = 1, \ldots, n_i$.
- n_i is the quantity of measures by individual.
- N: total of individuals.
- $\mathbf{X}$ fixed covariables design matrix, relating parameters β with y.
- $\mathbf{Z}$ design matrix of random effects relating vector b with y.

$$\mathbf{Z} = diag(Z_1, Z_2, \ldots, Z_N) \quad y \quad b = (b_1', b_2', \ldots, b_N')', \quad \dim(b_i) = r \times 1$$

$$\begin{aligned} \varepsilon &\sim N(0, \mathbf{R}) \\ b &\sim N(0, \mathbf{B}) \end{aligned}$$

- $T = \sum_{i=1}^{N} n_i$. Where p is the number of covariables of the design matrix $\mathbf{X}$ plus 1. r is the number of co-variables of the Design matrix $\mathbf{Z}$ plus 1 (the random intercept).
- b_i are independent among them, and with ε_i and identically distributed.
- $\mathbf{R} = \sigma_e^2 I$ Variance Matrix of Residuals.
- $\mathbf{B}$ Variance matrix of random effects.

It is an interest, to estimate the parameters $\beta' s$ and to predict the $b_i' s$, and variance components.

This LMM presented in this Doctoral Thesis is based in the Maximum Likelihood Estimation with the frequentist techniques, not the Bayesian theory, as the Bayesian DLM.

2.7 Synthesis of the problems related to Inventory Planning

Inventory planning at the industry is a necessity, because, the previously mentioned problems, impact the future of an organization. Some of the frequent problems are:

- Inadequate levels of inventories.
- Inadequate service levels.
- Sometimes the supply times are not fulfilled.
- Mistakes in demand forecasts.
- Industry must be prepared for drastic changes, associated to its functioning; changes dealing with factors such as demand, seasonality and irregular fluctuations, among others.
- The demand of final products has a dynamic which makes the inventory planning uncertain (Gutiérrez and Vidal, 2008; Jimenez Sanchez, 2005; Sarimveis et al., 2008).
- Industry needs accurate and fast methods to be efficient (Silver, 1981; Nenes et al., 2010; Flynn and Garstka, 1990; Chou et al., 2013; Vargas, 2009; Jeyanthi and Radhakrishnan, 2010).
- Bayesian techniques can be used to forecast demands, but also, to do optimization, but there are not many works in this sense (Bolstad, 1986; West and Harrison, 1997). There are two important questions about forecast models that will be answered in this research. Can the estimation be very good with univariate models? or can a multivariate estimation model be better?
- It is necessary to define criteria to evaluate the forecasts of the demands in a multiproduct scenario.
- Optimization of an Inventory Model in a time horizon T, needs accurate demands (Chen and Lee, 2004)

And the principal contributions to the State of the art can be summarized in:

- It is necessary to consider the dynamics in the industry systems, because fixed factors are not robust to face the problems that it produces; this conduces to search techniques with more advantages in comparison to others, to find optimal inventory policies.
- Theoretical assumptions about the residuals of classical models estimations, do not receive so much attention when a prediction is done, and, if they are not actually fulfilled in a particular problem, possible bias can affect parameters estimations.
- The question: why should a Bayesian approach be considered to the analysis of time series? This can be answered as follows: 1. Some mistakes are committed when a classical models are estimated, and if these are omitted, a bias can be done in forecasts. 2. Many times the industry has few historical data, which are not enough to do estimations of classical models. 3. In Bayesian Analysis, different priori distributions can be used, also, it is possible to an update process every time, for parameters or probability distributions. 4. Many works already presented about Non-Bayesian methods.
- Bayesian approximations to forecast the demand are not so frequently studied for inventory problems, in special, for multiproduct problems. These techniques have advantages, as: use of few data, facilitate changes in probability prior distributions, change and updating of the parameters, and predicted data, but also, they could be applied to do the optimization of inventory models, (Choi et al., 2003; Nechval et al., 2011; Valencia et al., 2014b).
- State Space models have been formulated for reasons like dynamics in engineering, and it is possible to formulate them, considering a time series as outputs of dynamic systems, and, in this way, they can be used to generate forecasts for inventory management, but also, to represent inventory levels as outputs of these kind of systems.

2.7.1 Problems to Solve

After the review process, it is possible to identify two problems that do not have so much research, when an industry must provide a good planning policy of the final inventories.

1. Forecast demands of final products for few historical data, in short terms.
2. Multiproduct-Multiperiod Inventory Optimization, when there are few historical data, with Bayesian techniques.

2.7.2 Proposal

This Doctoral Thesis will present the problem in two principal phases of work.

1. Forecast models of final products for few historical data, in short terms, establishing a comparison between some classical and some Bayesian univariate and multivariate models, and proposing the best models.
2. Multiproduct-Multiperiod Inventory Optimization proposal, when there are few historical data, using Bayesian techniques.

Chapter 3

Phase 1: Forecasting the Demand

3.1 Forecast Models

In this chapter we will simulate short time series under control conditions. For each simulated series, an estimation process will be executed using classical and Bayesian techniques. After that, a comparison will be done. The objective is to answer that if the time series behaves with seasonal and drastic changes in variability, which of the techniques works better to estimate the model to forecast the demand for different number of periods of prediction and length of the time series, in such a way, that the demand predicted values are useful for the inventory Policy to be proposed in the chapter 4.
In order to make decisions, about what is or which are the best models, first, it is going to be shown the example of the estimation of every model with the real data of fuel sales, and after, these models will be used in the simulation, in order to compare results of forecast values.

The estimations are organized according to a Factorial Design of Experiments, a technique that permits the control of the factors: Distribution of time series data, periods to do forecasts, model, and also, equation of the regression models. The response variable of the designs is the MAPE of forecast, obtained after the estimation of time series simulated, for all the univariate and multivariate models used. The decisions to be held will be related with the best possible models and equations to use after changing the size of data, and also, the mean and probability distribution of the time series simulated.

The models to be compared for the time series simulation are presented bellow:

Univariate Models

- Classical Models: ARIMA, and Exponential Smoothing (ES).
- Linear Models (LM). Classical Multiple Regression. The same formulas of the Bayesian Regression will be used in order to do the comparison.
- Bayesian Regression Models (BRM): different co-variables, and a variation of percentiles.
- Bayesian Dynamic linear Models (BDLM). Two kinds of these models will be used (BDLM1, BDLM2).

Multivariate Models

- Bayesian Dynamic Linear Models of two kinds: linear growth or polynomial, and seasonal plus polynomial.
- Linear Mixed Models (LMM). These models have a multivariate structure, according to multiple product demands (Lange and Ryan, 1989). Seasonal co-variables will be added in order to find the best result.

The MAPE indicator, as it was used in Makridakis et al. (1979), can be estimated in two forms: for the fitting and for the forecast values. As it was previously mentioned, MAPE is a measure that is used for different authors in order to examine the quality of adjustment and forecasts, in classical and Bayesian models (Makridakis et al., 1979; Bowerman and Oconnell, 2007; Petris et al., 2009; Rojo and Sanz, 2010). In this research, after simulation of time series with size n, data will be divided in a group of length $n - k$ and another with length k values. The estimation of all the models will be done for $n-k$ values, and the forecasts derived from those models, to the k data, in order to compare forecasts with real values with the Mean Absolute Percentage Error (MAPE). This MAPE for forecasts will give the criteria to choose the best model in this type of comparisons, because the principal purpose when forecasting is held, is to achieve a very good future horizon (Makridakis et al., 1979; Valencia et al., 2014b).

The first three models, ARIMA, ES and LM, will be estimated in first place, and a summary of the process is going to be shown. After this, the other bayesian models will be estimated and detailed in order to understand the results and error indicators that every one of the models provide.

3.1.1 Summary of the Bayesian Regression Model with Normal Prior distribution (BRM).

This is the summary of the process shown in the past chapter, where it was presented the structure in which, this model uses the Prior Normal distribution for the parameters β, with dynamic variation presented as a novel model for this Doctoral Thesis.

- General Regression model:

$$y_t = \beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k + \epsilon_t \tag{3.1}$$

  Here, $\epsilon_t \sim N(0, \sigma^2)$; which are independent, and identically distributed. σ could have a non informative distribution, but here, as it will be shown, the marginal distribution is Gamma.

Prior, Posterior and Predictive distributions

- Prior Distribution for the parameters β and σ_1.

  Supposing a Normal distribution as the prior for β and σ_1 parameters, $N(\beta_0, \sigma_1)$, with $\sigma_1 = \sigma_0 \sigma$, this will lead to the next equation.

$$\xi(\beta,\sigma) \propto \frac{1}{\sigma_0\sigma} exp^{\left[-\frac{1}{2(\sigma_0\sigma)^2}(\beta-\beta_0)'(\beta-\beta_0)\right]} \tag{3.2}$$

- Posterior Distribution of β

  By doing: $\tau = \frac{1}{\sigma^2}$, $\tau_0 = \frac{1}{\sigma_0^2}$, the posterior distribution is re-expressed, in the next form. By doing a product between the Prior Distribution and the likelihood, then:

$$\xi(\beta,\tau|\tau_0,\beta_0,Y_0,Y) \propto \tau^{\frac{T+1}{2}} exp^{-\frac{\tau}{2}A\left[(\beta-\tilde{\beta})'A^{-1}(X'X+\tau_0)(\beta-\tilde{\beta})+1\right]} \tag{3.3}$$

  Which is a Normal-Gamma: $N(\tilde{\beta},(X'X+\tau_0)^{-1})Ga(\frac{T+3}{2},D)$

- Predictive distribution

  The final Predictive Distribution to do forecasts with this model is finally:

$$f(Y_+|Y_0,Y) \propto \left[((Y_+ - Y_n)'A^{-1}(Y_+ - Y_n)+1)A\right]^{-\frac{T+4}{2}} \tag{3.4}$$

  Which is a Student-t distribution, with mean: $Y_n = X_+\tilde{\beta}$, with ν degrees of freedom, and Variance, $\frac{\nu}{\nu-2}$.

$$A = \frac{\nu}{\nu-2}(Y-Y_m)'(Y-Y_m) \tag{3.5}$$

3.1.2 Example 1- Application to real data of combustible demands for a Colombian gas station.

It is important to do, in first place, a descriptive analysis of the time series, in order to establish the adequate components and possible patterns of the behavior. This will permit to include possible covariables in models like the ARIMA or Regression. Also, if a seasonal behavior is detected, it can be possible to establish this kind of components in Exponential Smoothing.

In second place, all the forecast models of this work, will be estimated to the real case. In this Doctoral Thesis, this applications will permit to help to understand the process of the statistical models to be presented. In a third place, after the case study, a simulation study will be presented, and with these simulations, different Designs of Experiments will be shown in order to answer to all the formulated hypothesis. All of these results are developed in R program, with functions and algorithms, used and designed in this Doctoral Thesis.

The real demands of products, are the sales of three kinds of fuel: Corriente, Extra and Diesel, from a gas station in Colombia. The data cannot be shown due to confidentiality of the company, but they consist on 92 daily sales of these fuels, in a period between November 2014 and January 2015. Here, the estimation of all the models for the three time series will be done in the software R, and the Multivariate Dynamic Bayesian Models will be done by using the packages designed and explained by Petris et al. (2009).

As it can be seen in the values of the covariance between pairs of time series data, in the matrix shown below, named *Covar*, some of these values are high. For example, between sales of corriente and extra fuel, the covariance value is 2422.15, between corriente and diesel, is 4031.42, and between extra and diesel, −145.29.

$$Covar = \begin{bmatrix} 31885.56 & 2422.15 & 4031.42 \\ 2422.15 & 1460.34 & -145.29 \\ 4031.42 & -145.29 & 11635.17 \end{bmatrix}$$

Besides this, the covariances which are different from zero, reflect presence of correlation between the respective time series. This matrix is used in this work as a base to do the estimation of the multivariate Bayesian models.

On the other side, the figure 3.1, that presents the autocorrelation function (ACF) and Partial autocorrelation (PACF) shows that in the three series, there are significantly high lags to be considered for the model estimations. In the first line of the graph, there is a lag of order seven, in the second, there is a lag of order one and seven, which are out of the bands (of zero values), and in the third one, the seventh lag is high enough to provide conclusions about seasonal behavior with this order. This fact implies that these can be the lag terms added to the ARIMA, and also, they can be covariables to the Regression Models. Besides, this seasonal component leads to consider additional kinds of explanatory variables for the classical as well as for the Bayesian Regression Model.

Using the Ljung Box test (table 3.1), it is possible to see that the two first series (Corriente and Extra sales) do not have stationary behavior, because the p-value is lower than the significance level of 5%.

Box-Ljung test			
	X-squared	Df	P-value
Corriente	18.292	7	0.01072
Extra	75.963	7	9.137e-14
Diesel	8.0757	7	0.326

Table 3.1: Tests for non-correlation for Fuel predictions

The first two series do not show normality (table 3.2), according to the Jarque bera test, which is more recommended for time series data.

Jarque Bera Test			
	X-squared	Df	P-value
Corriente	18.905	7	7.851e-05
Extra	13.582	7	0.001124
Diesel	3.3219	7	0.19

Table 3.2: Jarque Bera Tests for Fuel predictions- Normality tests

The last time series (Diesel) fits to a Normal Distribution, according to the Jarque Bera test (table 3.2).

In summary, the possible covariables that can help to explain adequately the response of fuel sales are: lags of order one, and seven; also, trigonometric components, as $cos\left(\frac{2\pi t}{L}\right)$, $sin\left(\frac{2\pi t}{L}\right)$; and indicators for every period of the seasonal component, for example, on Tuesday (1 to this day, 0 other case), on Wednesday (1 to this day, 0 other case), etc.

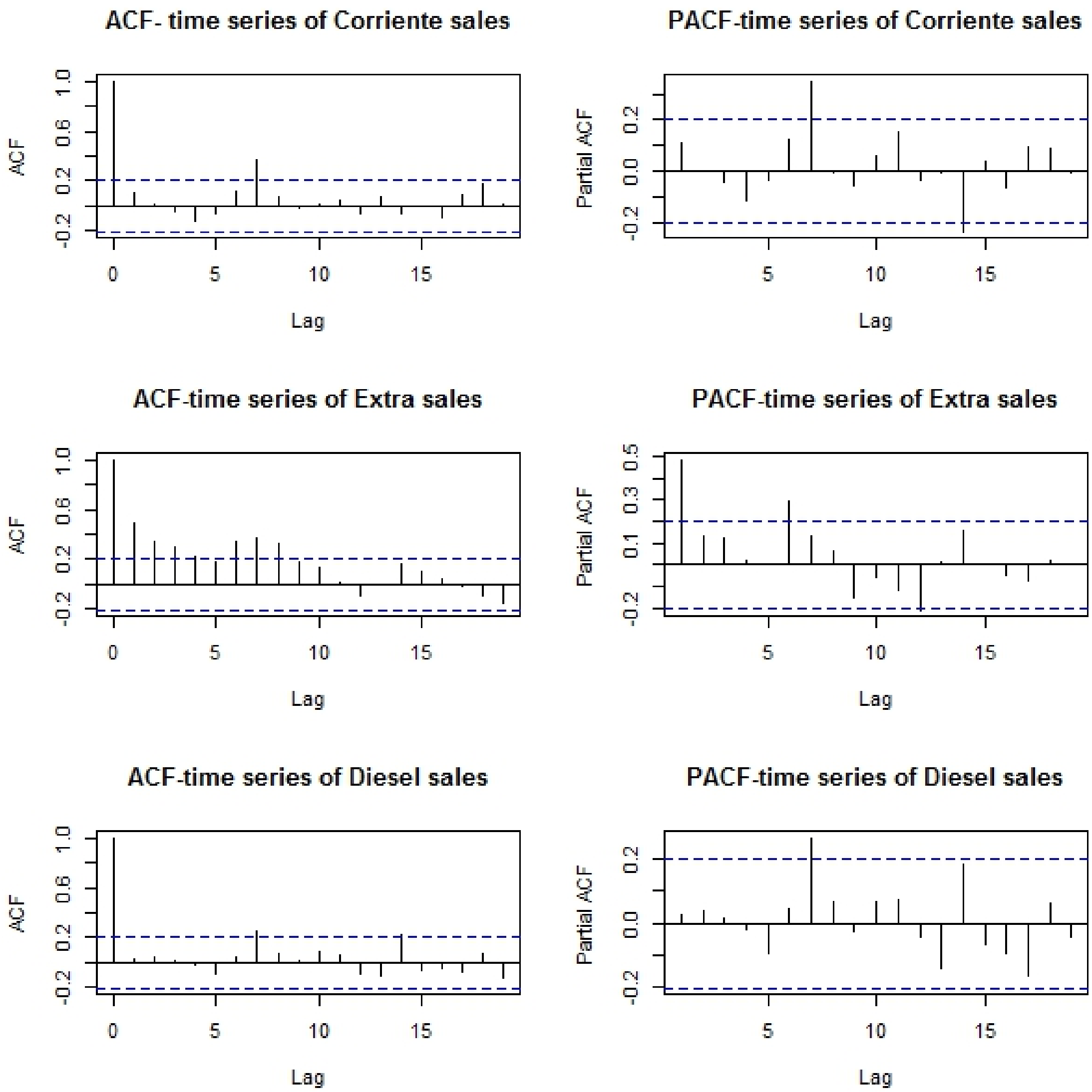

Figure 3.1: Autocorrelation and Partial Autocorrelation for fuel sales.

Classical Linear Model (LM) applied to real data of combustible demands.

The estimation of these three models will be done with the R program. For the Regression Classical Model (LM), the lm function must read covariables according to specifications of the researcher, in this case, according to the descriptive analysis done before.

Regression Model (LM)

The general equation that will be shown here for the three time series of fuel, is the expression (3.6).

$$Y_t = \alpha_0 + \beta_1 y_{t-1} + \gamma_2 Indicator_2 + \ldots + \gamma_7 Indicator_7 \tag{3.6}$$

The Anova tables for all the models of the time series are shown in the table 3.3. This Anova table can indicate the significance of the co-variables included in order to identify which one provides a better explanation for the response. It can be appreciated that the co-variable indicator introduced in the model, is significant at 5%, but the lag covariable is only significant for the linear regression of the Extra fuel. Besides this, the validation tests of the assumptions are not fulfilled for the three time series.

Model for fuel Corriente				
Covariable	Sum Sq	Df	F value	P-value-$Pr(>F)$
(Intercept)	1468710.02	1	53.46	0.00
y_{t-1}	64617.54	1	2.35	0.13
indicator	389756.95	6	2.36	0.04
Residuals	2088136.93	76		
Model for fuel Extra				
Covariable	Sum Sq	Df	F value	P-value-$Pr(>F)$
(Intercept)	1004.26	1	1.02	0.32
y_{t-1}	35712.57	1	36.12	0.00
indicator	15034.91	6	2.53	0.027
Residuals	75135.06	76		
Model for fuel Diesel				
Covariable	Sum Sq	Df	F value	P-value-$Pr(>F)$
(Intercept)	801344.44	1	94.25	0.00
y_{t-1}	291.29	1	0.03	0.85
indicator	323135.88	6	6.33	0.00
Residuals	646177.09	76		

Table 3.3: Anova Tables for three fuel time series

Normality test			
Fuel	Statistic	parameter	p.value
Corriente	52.25	2	0.00
Extra	5.424	2	0.0664
Diesel	40.6	2	0.00
Non-correlation test			
Corriente	11.63	7.00	0.08
Extra	12.59	7.00	0.08
Diesel	4.74	7.00	0.69
Constant Variance test			
Corriente	1.22	6	0.31
Extra	0.585	6	0.74
Diesel	1.34	6	0.24

Table 3.4: Assumptions of the Linear Model estimation

The normality assumption for residuals (in the table 3.4) is only fulfilled by extra fuel, because it is the only that has a p-value higher than 5%. The non-correlation test is accepted for the three tests for the residuals of the three estimated models, and also the constant variance, but MAPE values estimated do not show so much accuracy, to trust in the predictions.

Fuel	Fitting MAPE	Forecast MAPE
Corriente	11.1	6.3
Extra	48.44	38.7
Diesel	24.7	17.3

Table 3.5: MAPES (%) for the fuel sales for the Linear Model estimation

The estimations of the LM models for every fuel time series, produces the MAPE values seen in the table 3.5, here, it can be appreciated that the lower MAPE values are the calculated for the fuel Corriente, but this model does not fulfill normality assumption. When a transformation over the response value is done, this assumption is not satisfied either.

ARIMA and ES

For the ARIMA model, the function auto.arima from the R program, is an automatic process helping to find the best possible combination of lag terms of the model. For ES, the function ets, provides also an automatic selection.

Summarizing the MAPE results of these two classical models, the table 3.6 shows that the minimum results are for the fuel Corriente.

Bayesian Regression Model (BRM) to real data of combustible demands.

The past Predictive distribution must have covariables, and also, the prior parameter values of τ_0 and seasonal components. In the designed program presented in this Doctoral Thesis, the inputs to be typed in the R program, in order to run the model and do the estimations, are:

ARIMA		
Fuel	Fitting MAPE	Forecast MAPE
Corriente	12	10
Extra	51.2	95.2
Diesel	35.9	69.88
ES		
Corriente	13	7.3
Extra	52.8	95.5
Diesel	36.2	65.5

Table 3.6: MAPES (%) for the fuel sales for ARIMA and ES estimation

- τ_0=1 if it is not specified a different value by the user.
- The time series data (series)
- The formula with the covariables (formulation)
- Initial Percentile (Stad)
- Value of n-k (cuts)
- Indicator variables (indicator)
- Number of seasonal pattern (L)
- Criteria to chose the best model, between: Fitting and Forecast MAPE.
- Number of simulation (sim) for the internal optimization of the vector β_0, this will permit to begin the forecast with the best fitting.
- Value of the first period to forecast, it is, $cuts + 1$ if the forecast values are part (at the end) of the data.
- Periods to do forecasts, k.

The results of the designed algorithm are the fitted values, and its respective MAPE value, the forecast values, and, its respective forecast MAPE, the percentile that was chosen as the best

(minimum), according to the criteria (Fitting or Forecast), and the equation of the model used to do the estimation. In this example the equation is the same as the used for the regression model, with a lag variable, and the indicator,
$Y_t = \alpha_0 + \beta_1 y_{t-1} + \gamma_2 Indicator_2 + \ldots + \gamma_7 Indicator_7$. It is used the criteria of minimum forecast MAPE. The results of the MAPE values and the best percentile, are shown in the table 3.7.

Fuel	Fitting MAPE	Forecast MAPE	Best percentile
Corriente	11	4.55	45.79
Extra	68.5	22.2	95
Diesel	33.96	9.64	91.2

Table 3.7: MAPES (%) and percentile, results from the New BRM.

The MAPE values to do forecast, are lower than the previous classical models, when it is considered the same periods to forecast, seven days.

Figure 3.2 shows the fitting and forecast values provided by the designed algorithm, for the sales of fuel regular. It is possible to see that the dashed line is very close to the real values, in especial, in the forecast periods, after the vertical line.

Figures 3.3 and 3.4 show the fitting and forecast for Extra and Diesel fuels. It can be seen that there are some periods where the fitted line is close to the real values, in especial, when the values goes down, and at the end, but in all the movement of the horizon, the two lines are not totally close.

3.1.3 New Bayesian Dynamic Linear Model (BDLM1)

In the review of the state of the art, it was presented the general theory about Dynamic Linear Models (DLM) and a procedure to estimate the Kalman Filter process, and also, a general recursive Kalman Filter applied to the estimation of a DLM, that was used, as well, in Valencia and Correa (2013). Here, the Bayesian Dynamic Linear Model (BDLM1) explained in the

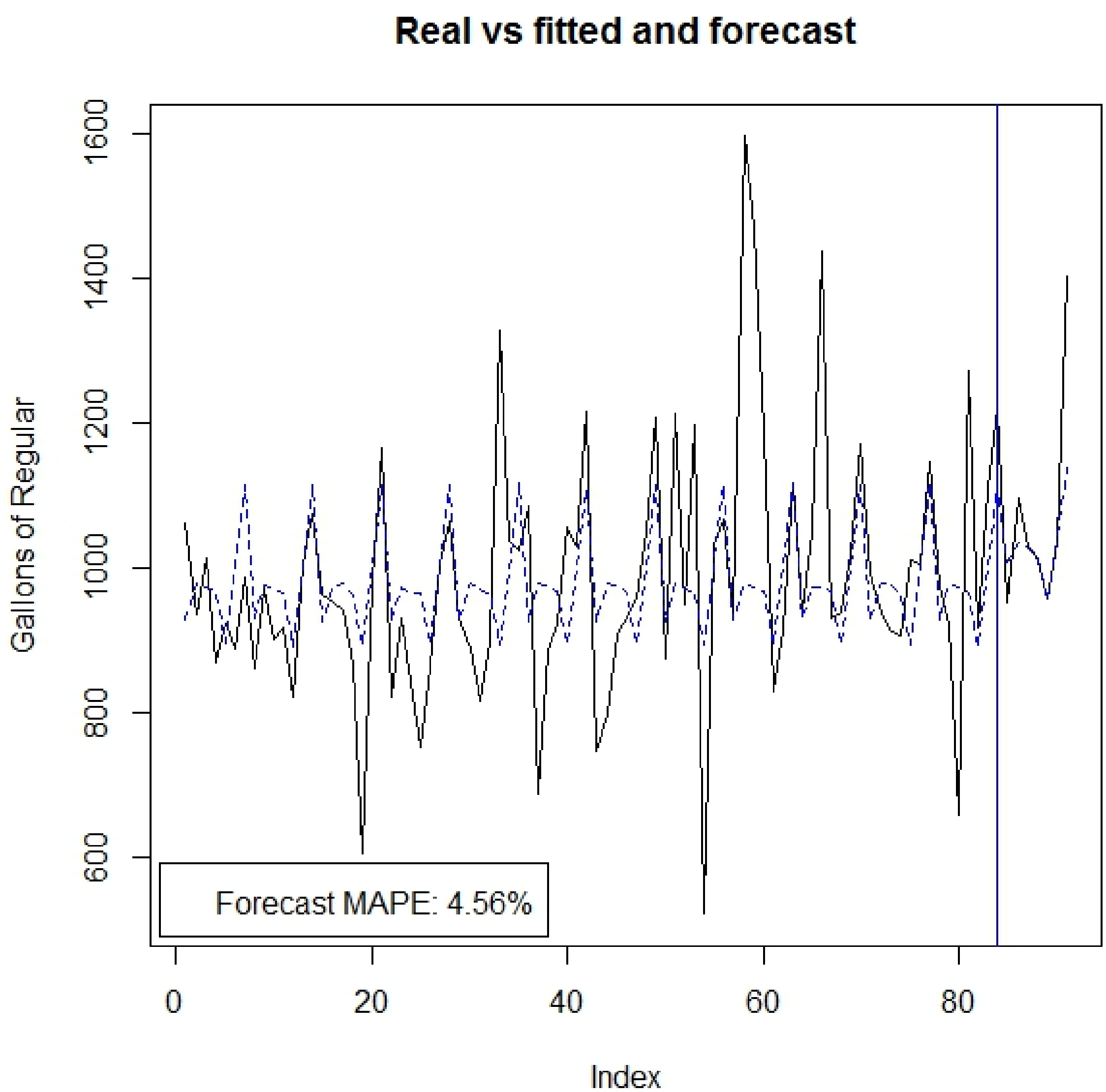

Figure 3.2: Bayesian Regression Model for Regular fuel.

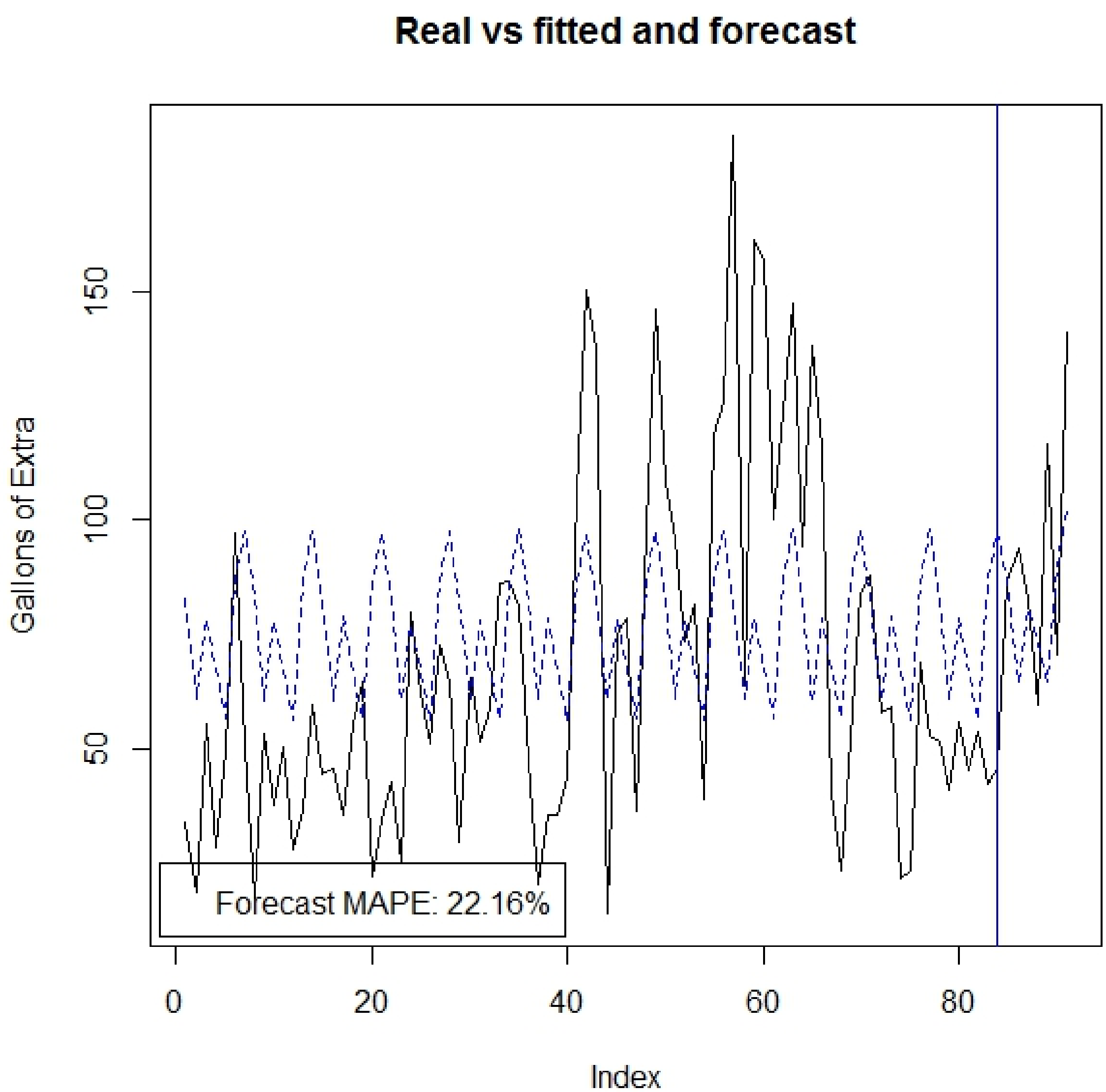

Figure 3.3: Bayesian Regression Model for Extra fuel.

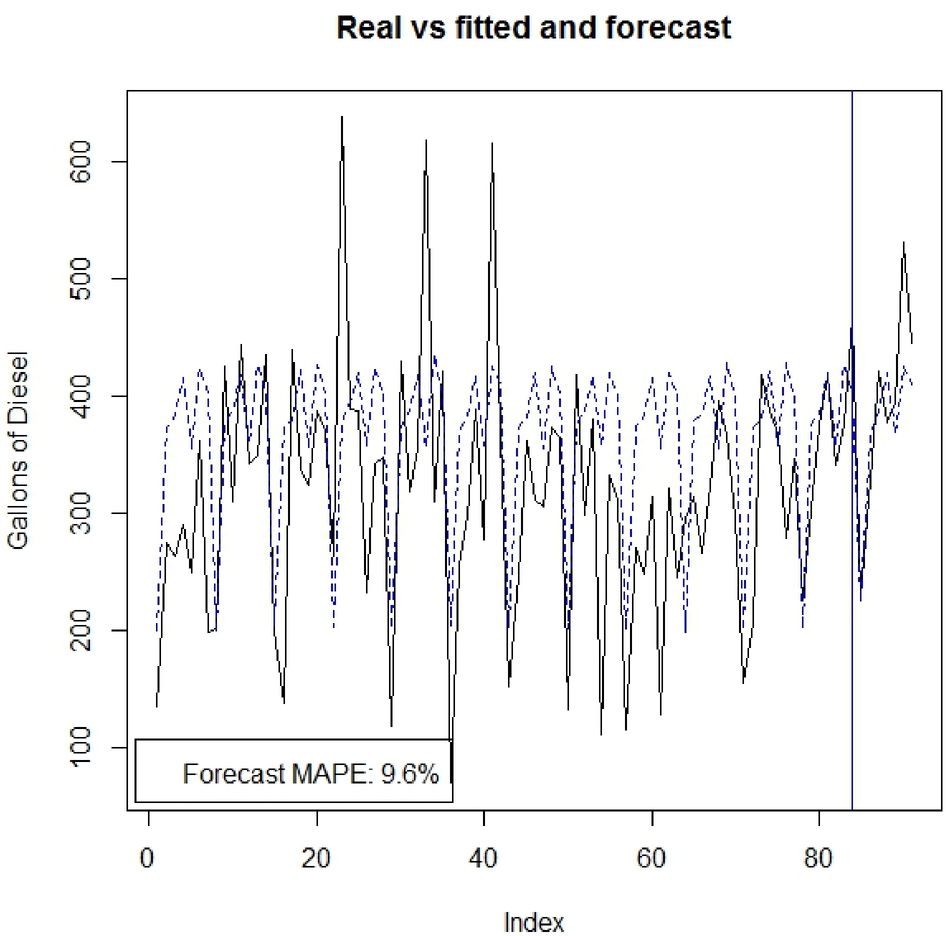

Figure 3.4: Bayesian Regression Model for Diesel fuel.

past chapter, as an approach of this Doctoral Thesis, has a modification, and will be used in the comparison of models estimated to the time series simulated data.

Here, it will be proposed a model that uses percentages of changes as the information of the system, and also, as its evolution, according to the previously exposed theory about the recursive Kalman Filter explained by (Meinhold and Singpurwalla, 1983):

$$\begin{array}{llcll} \text{Observation equation:} & Y_t & = & F_t\theta_t + \nu_t, & \nu_t \sim N(0, V_t) \\ \text{System equation:} & \theta_t & = & G_t\theta_{t-1} + w_t, & w_t \sim N(0, W_t) \end{array} \tag{3.7}$$

3.1.4 Bayesian Dynamic Linear Model with linear growth (BDLM2)

The linear growth DLM, also called, local linear trend developed by Petris et al. (2009), will be used here. It can be expressed as in the equation (3.8)):

$$\begin{array}{ll} Y_t = \mu_t + \nu_t, & \nu_t \sim N(0, \mathbf{V}_t) \\ \mu_t = \mu_{t-1} + \beta_{t-1} + \mu_{1,t}, & \mu_{1,t} \overset{iid}{\sim} N(0, \sigma^2_\mu) \\ \beta_t = \beta_{t-1} + \mu_{2,t}, & \mu_{2,t} \overset{iid}{\sim} N(0, \sigma^2_\mu) \end{array} \tag{3.8}$$

$$G_t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; F_t = \begin{bmatrix} 1 & 0 \end{bmatrix}; W_t = \begin{bmatrix} \sigma^2_\mu & 0 \\ 0 & \sigma^2_\beta \end{bmatrix} \tag{3.9}$$

The complete matrix F, for all the model, will have an order of $18x18$, and the covariance matrices V_t and W_t will be supported in the covariance matrix of the data.

3.1.5 Multivariate Bayesian Dynamic Linear Model (MBDLM1)

Assume three time series processes, with an observation equation: $y_{i,t} = \mu_{i,t} + \nu_{i,t}$. The general form of a SUTSE model (Seemingly unrelated time series equations) of a linear growth model, is:

$$
\begin{aligned}
y_{1,t} &= \mu_{1,t} + \nu_{1,t}, \\
y_{2,t} &= \mu_{2,t} + \nu_{2,t}, \qquad (\nu_{1,t}, \nu_{2,t}, \nu_{3,t})' \sim N(0, \mathbf{V}) \\
y_{3,t} &= \mu_{3,t} + \nu_{3,t}, \\
\mu_{1,t} &= \mu_{1,t-1} + \beta_{1,t-1} + w_{1,t} \\
\mu_{2,t} &= \mu_{2,t-1} + \beta_{2,t-1} + w_{2,t} \\
\mu_{3,t} &= \mu_{3,t-1} + \beta_{3,t-1} + w_{3,t} \\
\beta_{1,t} &= \beta_{1,t-1} + w_{4,t} \qquad (w_{1,t}, w_{2,t}, w_{3,t}, w_{4,t}, w_{5,t}, w_{6,t})' \sim N(0, \mathbf{W}) \\
\beta_{2,t} &= \beta_{2,t-1} + w_{5,t} \\
\beta_{3,t} &= \beta_{3,t-1} + w_{6,t}
\end{aligned} \tag{3.10}
$$

Where

$$
F_t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \dots \end{bmatrix}; G_t = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

besides,

$$
\mathbf{V} = \begin{bmatrix} \sigma_1^2 & \sigma_{21} & \sigma_{31} \\ \sigma_{21} & \sigma_2^2 & \sigma_{32} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{bmatrix}; \mathbf{W} = \begin{bmatrix} \mathbf{W}_{\mu(3x3)} & \mathbf{0}_{(3x3)} \\ \mathbf{0}_{(3x3)} & \mathbf{W}_{\beta(3x3)} \end{bmatrix}
$$

$\mathbf{V}$ is a variance covariance matrix of the vector of errors, $(\nu_{1,t}, \nu_{2,t}, \nu_{3,t})$, of the observation equation, with order $3x3$. $\mathbf{W}$ is a block diagonal matrix, with order $6x6$, with variances $(\sigma^2_{w1}, \sigma^2_{w2}, \sigma^2_{w3})$ for the W_μ matrix and $(\sigma^2_{w4}, \sigma^2_{w5}, \sigma^2_{w6})$, for W_β matrix. There are also covariances in order to consider dependence. But it is possible to fix one of these (W_μ, W_β) with zeros, that will lead to have constant parameters in all time horizons.

3.1.6 Example 2- Application of MDLM1 to the real data of combustible demands for a Colombian gas station.

The performance of the multivariate models will be shown by applying the models explained to a real time series data, the fuel sales mentioned before, and using the R program and the package dlm and codes designed by Petris (2010). The first model to show is the MDLM1, then, the MDLM2; showing the movements of the parameters from one time to another, and estimating forecasts, with the respective MAPE indicators.

Figure 3.5 shows the adjustment of the Multivariate Bayesian Dynamic Linear Model (MB-DLM1).

It can be seen that the fitting of the MDLM1 is relatively good, but the performance of the forecasts are not good, because the predicted points go to different directions, from the ones of the real values, as it is shown in figure 3.5.

Besides the fitting and forecast, it is possible to see the dynamic parameters, because this model permits it. The Figure 3.6 shows the intercept parameters, and the polynomial component for every fuel sales, respectively, Corriente, Extra, Diesel.

The estimation of forecast MAPE, for a period of 7 days provides the results, Corriente 16.63%, Extra 47.84%, and Diesel 28.51%. The R-code to estimate the model was adapted from the code help provided by (Petris, 2009), that is in the appendix A.

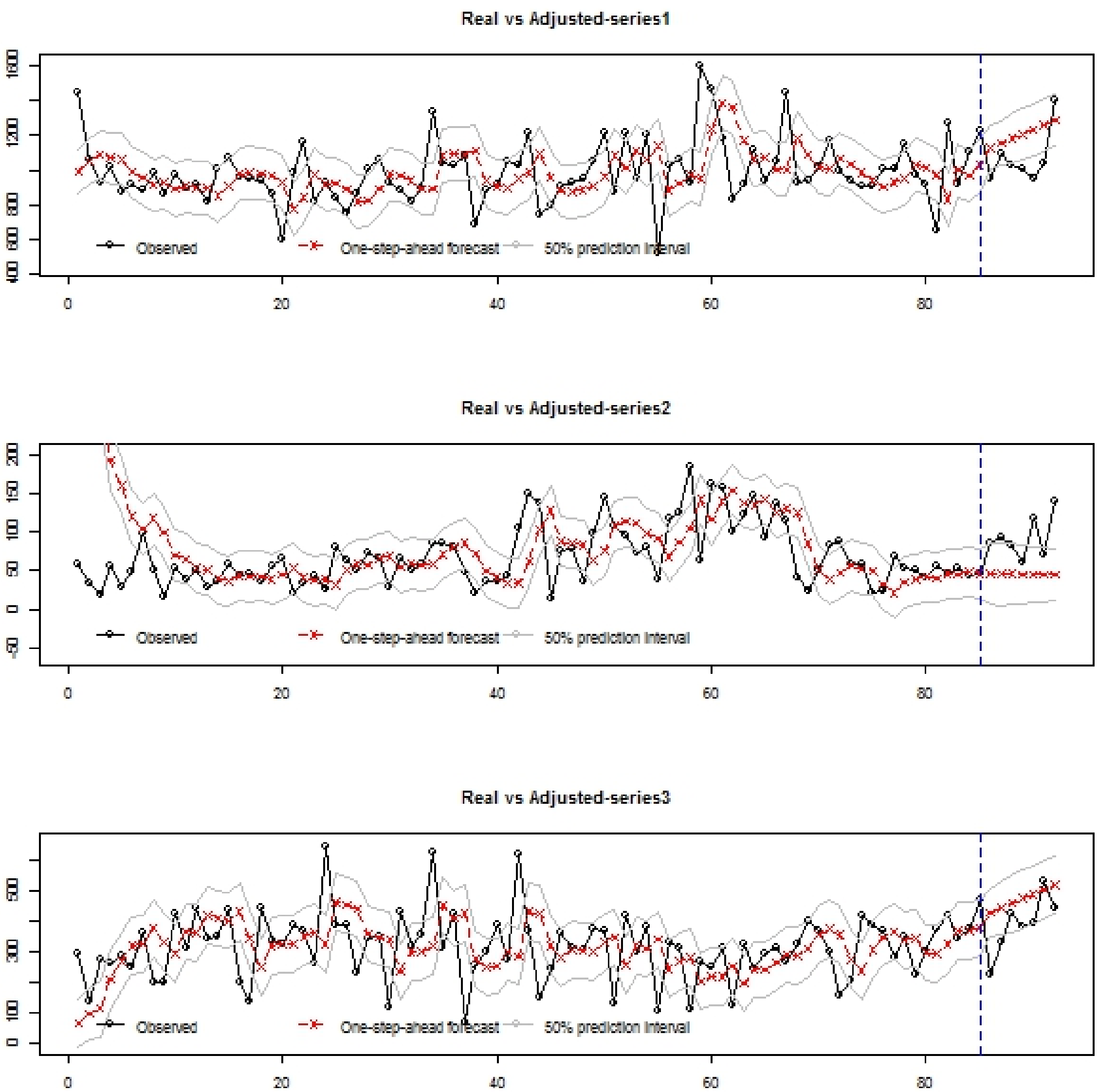

Figure 3.5: Multivariate Bayesian Dynamic Linear Model-1 for fuel sales

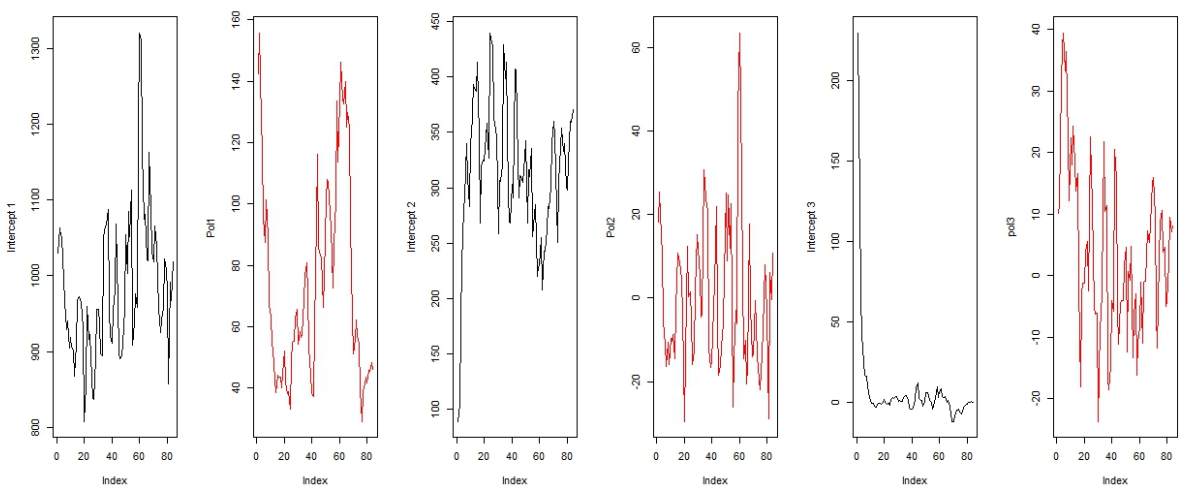

Figure 3.6: Dynamic parameters of the Multivariate Bayesian Dynamic Linear Model-1 (MDLM1)

3.1.7 Example 3- Application of MDLM2 to the real data of combustible demands for a Colombian gas station.

The MDLM2 is applied for the three real time series data of the same initial example. Fixing the same periods to forecast as the estimation exposed above, (7 days), it is possible to see that the MDLM2 can make a good representation of the seasonal behavior, along the horizon period of fitting, and also, of forecasts (figure 3.3).

As the figure 3.7 shows, the predictions are, in general, close to the real values, for fitted as well as for forecasts, concerning this MBDLM2. This accuracy can also be appreciated in the forecast MAPE values for the 7 days: for Corriente 9.897%, Extra 24.149%, and for Diesel: 18.223%. Values that are lower than those of the MDLM1, despite these are the same real time series data.

The R-code to do the estimation of this model is presented in the appendix B.

The Figure 3.8 shows the dynamic intercept parameters, and one indicator of the six seasonal patterns that the model contemplates. This shows that the parameters are not fixed in all the periods, and moreover, there are some periods where they are higher, which can lead to possible interpretation of a change in the level of the series.

3.1.8 Example 4- Application of LMM to the real data of combustible demands for a Colombian gas station.

The same data of the fuel sales is used here to do the estimation of the Linear Mixed Model presented in the chapter 2. As it was mentioned, this is a model estimated with frequentist techniques, not Bayesian. The estimation considers the sales of the multiple products, in this case, they are three, and it is possible to do forecasts simultaneously of the three products. The estimations of the model that can be analyzed in order to do inferences are the Anova table, the standard deviation of the random component, and the validation tests of the residuals about

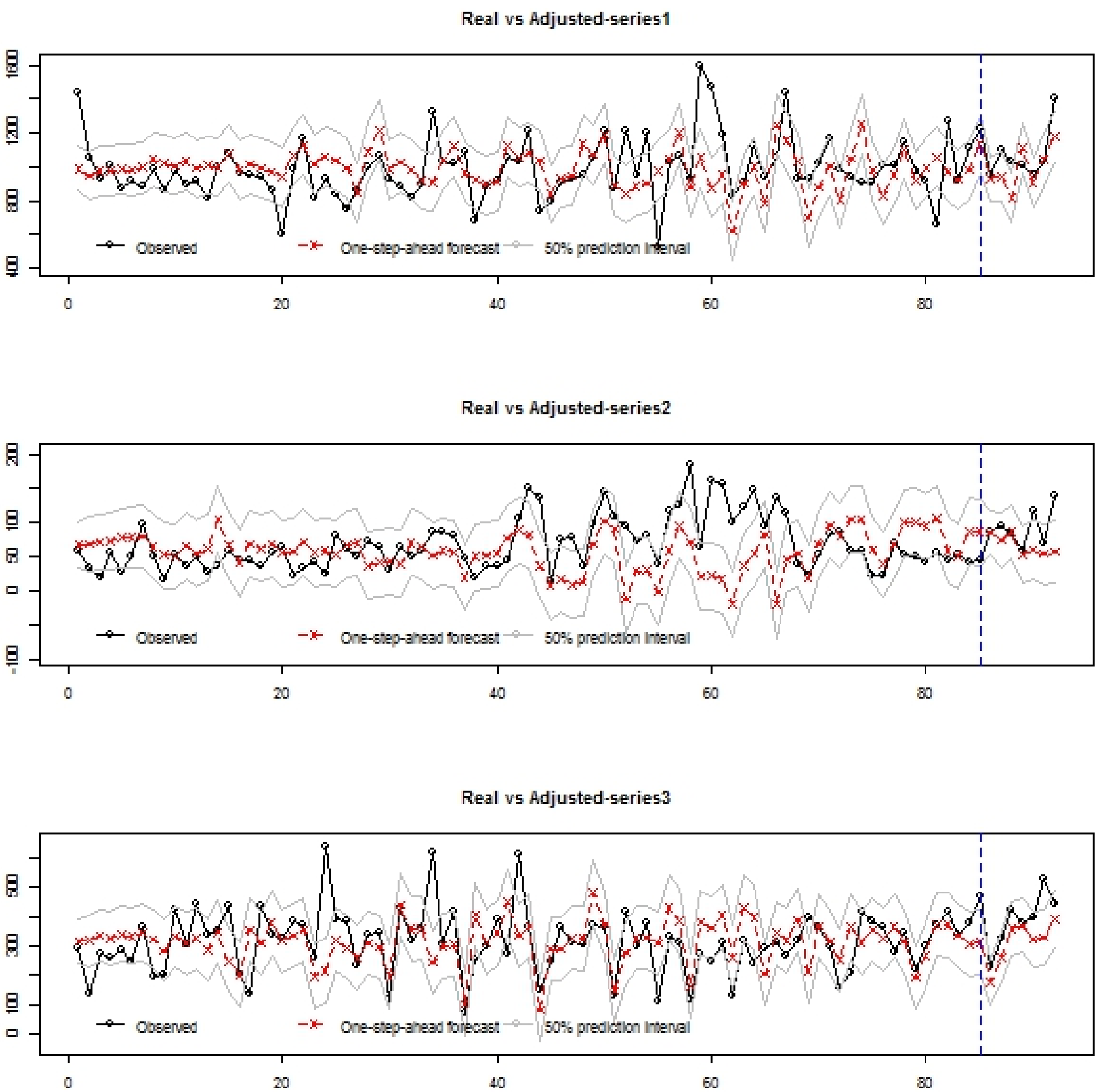

Figure 3.7: Multivariate Bayesian Dynamic Linear Model-2 for fuel sales

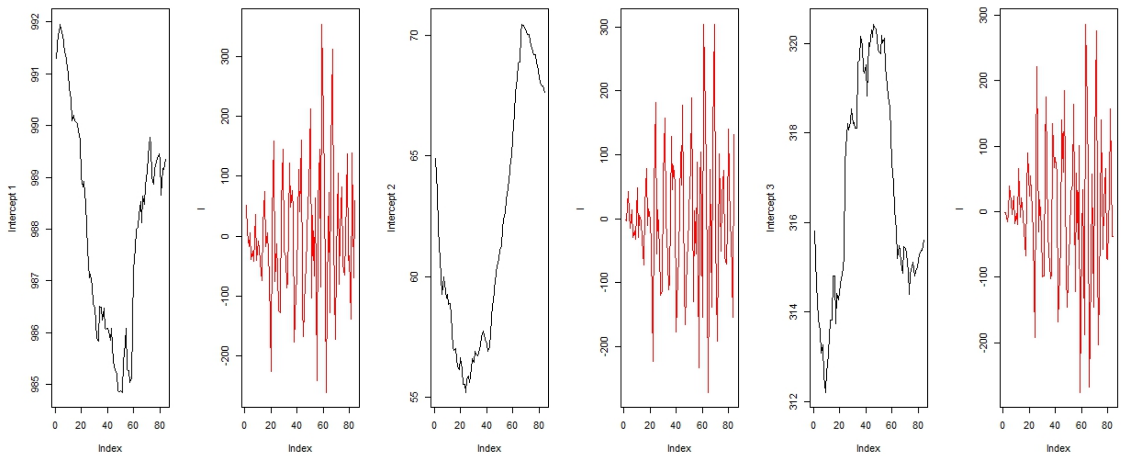

Figure 3.8: Dynamic parameters of the Multivariate Bayesian Dynamic Linear Model-2 (MDLM2)

normality and constant variance, and this is done with 85 days of sales for every product, but the forecast is done with the 7 periods left.

The model estimated for this fuel sales example has the next general equation.

$$y = \beta_0 + b_0 + \beta_2 I_2 + \ldots + \beta_7 I_7 + \varepsilon \tag{3.11}$$

Where the responses, in the vector y, are the sales of the three fuel time series data expose before, together.

b_0 is the random intercept, which changes according to each one of the products. I_i is the period indicator, which represents seasonality. The vector of fixed coefficients represents first, the intercept β_0, and after, the effects of every indicator $\beta_2, \ldots, \beta_7$, that in this case, are the days of the week minus one that is not in the model, and it is represented by the general intercept. But, this model provides the form to add the random effects to the fixed, according to the quantity of random components specified by the researcher. This model has only one random component, the vector b_0, which differ for every product, so, there will be three different values of intercept, respectively, for every fuel, Corriente, Extra and Diesel.

Intercept	Residual
473.6	131.26

Table 3.8: Standard Deviation of Random effect

It is possible to find a standard deviation of 473.6 for the random intercept (table 3.8), which is not negligible, in fact, is a quantity that justifies the LMM estimation.

	Chisq	Df	Pr(>Chisq)
Intercept	3.0669	1	0.0799
Indicator-day	29.7927	6	$4.304e-05$

Table 3.9: ANOVA table of linear mixed model applied to fuel sales

It is possible to see that the fixed intercept, and the parameters of the factor indicator-day, are significant at level 10%, because p-value (0.0799 and $4.304e-05$) are lower, as it is seen in table 3.9. But the residuals of this model do not fit to a normal distribution, because P-value of the test of Jarque Bera ($2.2e-16$) is lower than 5%. When a transformation is done, the result also leads to the non-normality for residuals. This leads to a possible bias in the inferences.

The sum of the random and fixed coefficient leads to final coefficients of the model, in order to be used to do predictions. This sums are valid for the intercept for the model estimated in this example (table 3.10), because it has a fixed and a random component for every product.

Product	Intercept
Corriente	1007.41
Extra	99.03
Diesel	344.66

Table 3.10: Intercepts of the LMM for fuel sales

Vector of the coefficients of the seasonal indicators are the same for all the products in all

periods:
$(\beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7) = (-23.85, 40.984, -91.5, -27.28, -65.6, -19.92)$.

The forecast MAPE calculated for 7 periods, in order to do a comparison with the Multivariate Bayesians, are: 10.6%, 93.08%, 69.74%.

The multivariate Dynamic Linear models with Bayesian theory, presented above, provides a better performance of the accuracy of the forecast MAPE, than this mixed model estimation. At the end, the differences in the results of the LMM vs MBDLM's are the techniques employed, because in the Bayesians (MDLM1 and MDLM2), the parameters are dynamic, they vary from one period to another, and in the LMM, the parameters do not change for every period involved in the estimations.

The advantages are in the analysis of the behavior for every period, in order to provide tools in the identification of periodical patterns of the demands, as in this case shown.

3.2 Time Series Simulation and Estimation of Models

Some of the answers to be achieved with the simulation process must be directed to find the best models to supply the need of a good representation for a demand behavior that presents: unexpected changes, seasonal component, non stationary, lack of stability in time series, lack of historical data, non normality distributions, or high variance.

It is important to find good models because these problems can affect the decisions making about production and storage in business, as it was mentioned before (Correa and Gómez, 2009; Diebold, 1999; Ventura et al., 2013). And also, to know if it can be possible that Bayesian techniques are good alternatives to produce more robust estimations.

The real time series shown in the before examples, about the demand of the three types of fuels, represent the kind of behavior of the time series that is desired to be simulated in this section, because of the seasonality, autocorrelation, covariance, and non normality in the response.

In this research, it is proposed first, to simulate scenarios of time series under normality and non normality behaviors, and with these bases, to estimate different models, and to compare them, to determine the best form to represent adjustment and forecast of the simulated data, using a Design of experiments, analyzed with the R program.

The figure 3.9 presents the process employed in the simulation, followed by a Design of experiments for the comparison among models.

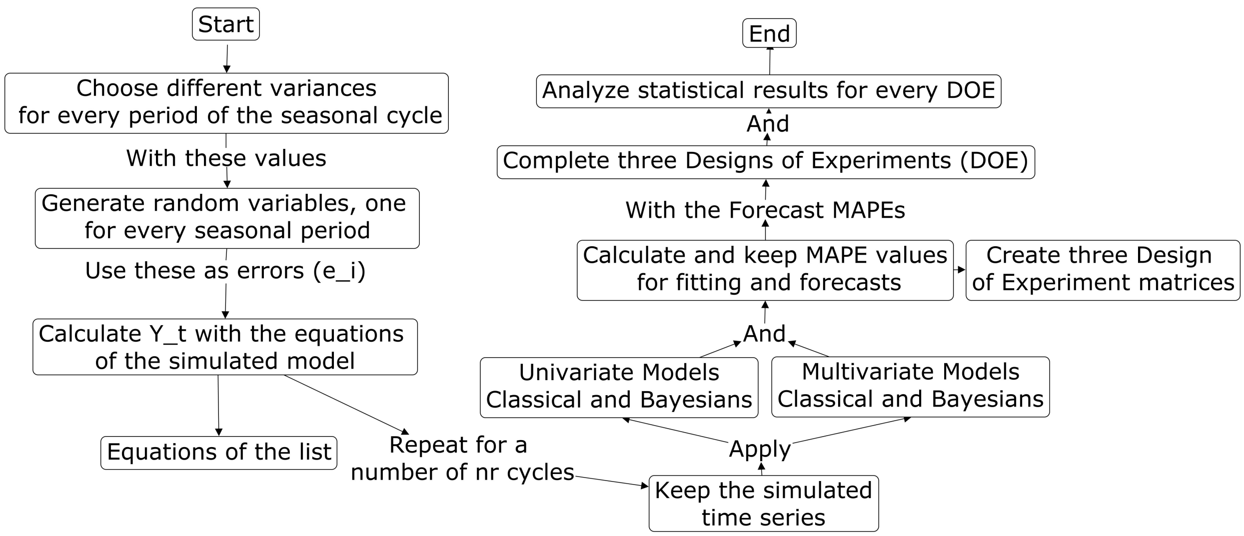

Figure 3.9: Process employed in the time series simulation

This process consists on doing the simulation of N values: $y_1, y_2, \ldots y_N$. In order to do it, different variances are previously fixed for every period of a seasonal pattern, these are taken by every probability distribution to obtain a random variable, and include it into an equation to calculate every value of y_i. Here, it is possible to use different probability distributions changing variances or skew parameter terms for every periods, as it corresponds to Skew Probability Distributions. The schemes employed for the univariate comparison, were:

– Variable with Normal distribution, and dynamic variance.

- Equation of regression, as $Y_t = \beta_0 + \beta_1 * Y_{t-1} + e_t$. Error variable ($e_t$) with Skew Normal (SN) distribution, changing the variance and fixing one skew parameter for every period.

  The next items have the same equation of regression.
- Error variable with Skew t (ST) distribution, changing the variance and fixing one skew parameter for every period.
- Error variable with Gamma distribution, changing the variance for every period.
- Error variable with Poisson distribution, changing the variance for every period.

And for the multivariate comparison, the only difference is that the probability distributions were: Multivariate Skew Normal distribution, and Multivariate Skew-t distribution, and a matrix of variance-covariance where used.

The simulation study is done in order to select the best model to forecast; among the ARIMA, an Exponential Smoothing (ES), a classical linear regression (LM), a Bayesian Regression (BRM), two Bayesian Dynamic Linear Models (BDLM): percentage changes and linear growth (BDLM1, BDLM2). Also, multivariate Bayesian models (MBDLM1, MBDLM2), and the Linear Mixed Model, as it was presented before. In summary, the models to be estimated for every time series simulated, are:

Univariate Models

- Classical Models: ARIMA, and Exponential Smoothing (ES).
- Linear Models (LM). Classical Multiple Regression. The same formulas of the Bayesian Regression will be used in order to do the comparison.
- Bayesian Regression Models (BRM): different co-variables, and a variation of percentiles.
- Bayesian Dynamic linear Models (BDLM). Two kinds of this model will be used (BDLM1, BDLM2).

Multivariate Models

- Bayesian Dynamic Linear Models of two kinds: linear growth or polynomial, and seasonal plus polynomial.
- Linear Mixed Models (LMM). These models have a multivariate structure, according to multiple product demands (Lange and Ryan, 1989). Seasonal co-variables will be added in order to find the best result.

The estimated models are compared using the MAPE for adjustment and forecasts, after making a partition of the data. Forecast MAPE will be the final decision criteria to choose the best models for every case, because the models do not have the same validation tests, neither, they have the same kind of inferences. Different authors make comparisons between different Bayesian models and others, with MAD, MAPE, and RMSE (Petris et al., 2009; Rojo and Sanz, 2010).

A second class of multivariate simulation of vector auto-regressive time series, will be done and two Multivariate Dynamic linear Models will be estimated to find the best possible forecast model. The simulation will recreate cases when there is not a Normal Distribution behavior for the series of interest.

$$MAPE = \frac{1}{k}\frac{|\hat{z}_{t+1} - z_{t+1}|}{z_{t+1}} \tag{3.12}$$

Where $\hat{z}_{t+1}$ is the predicted value for the demand in the period $t+1$; Z_{t+1} is the real value for the demand in the period $t+1$, which is $(N-k)+1$, in a one step forecast.

This process is repeated 100 times, getting a final percentage that explains the frequency of times in which every model is chosen as the best to do adjustment and forecast of every simulated series.

The quantity of simulated data is fixed in N=63. For every case, a partition will be done: $N-k$ values, to do the adjustment, and k will be the data left to forecast.

3.2.1 Description of the Statistical Model

Figures 3.10 shows one time series simulated, and the figure 3.11, shows its ACF and PACF behavior. This demonstrates the behavior of the seasonality, dynamic variance and non normal distribution used to simulate the data.

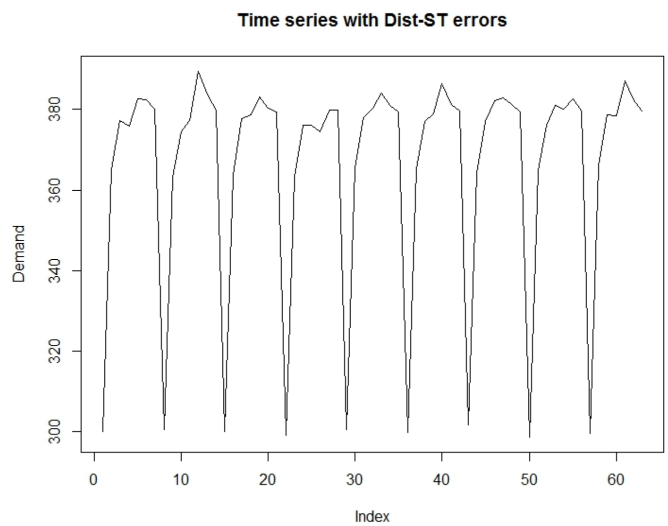

Figure 3.10: Behavior of the time series simulated. Source: By the authors.

The simulated series present seasonality, because of the repetitive patterns, and the non constant variance that it demonstrates (see figures 3.10, 3.11). In especial, the figure 3.11 shows the ACF and PACF behavior about the simulation done. The ACF shows autocorrelations out from the significant bands, in seasonal periods, which demonstrate the dependence in seasonality.

3.2.2 Comparison of the models estimations

First, in order to understand how the comparison will be done, the six models will be estimated to one simulated time series data, and they will be compared. In order to estimate the Classical

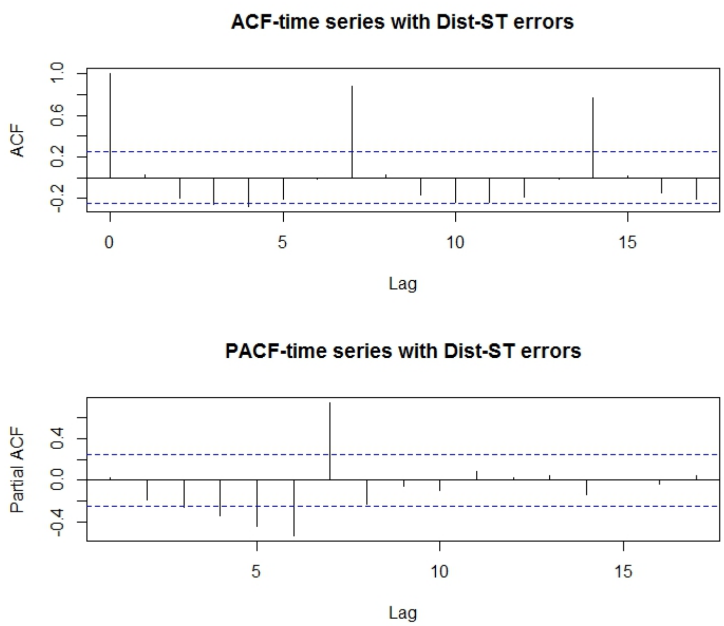

Figure 3.11: ACF and PACF of Time series simulated

regression model and the Bayesian Regression, i is necessary to define an equation, and it will be the same in order to establish a better comparison of the two models.

The equation of the two regressions, for this section, is: $Model : Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 t + \beta_3 Sin + \sum_i^{L-1} \gamma_i I_i$, it is, the same covariables in the two cases. After, a comparison with other equations will be done.

Then the models will be applied to the different scenarios of the simulated time series data, and a forecast MAPE will be shown to do comparisons. For all the cases of simulations, the time series simulated with Skew Normal, Skew t, Gamma and Poisson distributions for the errors, with dynamic variance, the comparison among models provides the results shown in tables 3.11 and 3.12.

Distribution		**MAPE OF FORECASTS** (%)					
		ARIMA	**ES**	**LM**	**BRM**	**BDLM1**	**BDLM2**
Normal	Adjusted	0.54	0.54	0.53	0.83	7.89	3.41
	Forecasted	0.26	0.26	0.33	0.33	8.34	0.47
Skew Normal	Adjusted	5.33	5.33	0.58	0.80	10.81	9.61
($\lambda = 50$)	Forecasted	22.99	23.00	17.83	0.95	5.87	6.87
Skew T	Adjusted	5.06	5.55	0.40	0.57	13.14	9.65
($\lambda = 50$)	Forecasted	23.12	23.41	19.08	1.20	5.74	6.89
Gamma	Adjusted	17.06	31.65	3.93	4.88	64.52	38.67
	Forecasted	8.93	1.79	24.72	7.73	65.54	47.45
Poisson	Adjusted	18.43	20.77	4.35	4.70	30.19	24.81
	Forecasted	16.54	17.78	21.30	6.33	30.54	23.11

Table 3.11: MAPE values for the comparison of models, mean=200, periods to forecast=7, N=63.

In the first case of table 3.11, for the data simulated with normal distribution, the ARIMA or ES model have the lowest MAPE (0.26%), but the others have also low MAPE values, except for BDLM1. In the next two cases (Skew Normal and Skew T), the λ represents the skew parameter used in these distributions. The best MAPE of forecasted values is the Bayesian

Regression Model (BRM) (0.95%, 1.2%), followed by the BDLM1 and BDLM2, which can be good alternatives.

In the fourth case, for the data simulated with Gamma distribution, the Exponential Smoothing is chosen as the best, followed by the BRM. In the Poisson simulation case, the BRM is chosen.

After changing the number of data simulated, the models chosen are not always the same as in the past case.

Distribution		**MAPE OF FORECASTS** (%)					
		ARIMA	ES	LM	BRM	BDLM1	BDLM2
Normal	Adjusted	0.51	0.51	0.45	0.68	6.19	6.11
	Forecasted	0.33	0.33	0.47	0.44	12.77	0.55
Skew Normal	Adjusted	4.85	4.85	0.41	0.66	10.07	11.58
(λ = 50)	Forecasted	22.80	22.80	15.39	1.96	8.17	7.37
Skew T	Adjusted	5.00	5.00	0.41	0.69	12.56	11.70
(λ = 50)	Forecasted	23.41	23.41	18.95	1.96	4.95	7.95
Gamma	Adjusted	23.67	31.71	3.64	4.77	30.43	40.70
	Forecasted	5.19	2.03	25.06	12.05	71.58	46.06
Poisson	Adjusted	20.14	21.51	4.27	6.97	45.33	29.82
	Forecasted	16.81	15.89	23.89	11.35	34.15	17.10

Table 3.12: MAPE values for the comparison of models, mean=200, periods to forecast=7, N=35.

In table 3.12, ARIMA is chosen as the best model for simulated data with Normal distribution. In the next two cases, also the BRM has the minimum MAPE for Skew normal and Skew t distributions. For simulated data with Gamma distribution, the ES is the best model, followed by ARIMA. And, in the last case, for the simulated data under Poisson distribution, the option is BRM. As we can see, the BDLM1 and BDLM2 have MAPE values which are good alternatives in some simulated data, like the Normal for BDLM2, and SN or ST for BDLM1.

But, what is the best model, if we change some factors like: mean, number of the data (size), kind of distribution, or equation of the regressions?

After a variation in different scenarios, it can be useful to estimate a Design of experiments (DOE), a Factorial Design, where the response to evaluate is the MAPE of the forecast values. This analysis can allow to find the most influential factors in order to choose a particular model for forecasting.

3.2.3 Design of Experiments for simulation

In this section, two Designs of Experiments will be created, factorial designs. The first will have these factors: the probability distribution to obtain time series data, the models employed, the mean of the data that will be: 30 and 200; and the size of data, 35 and 63. The second will have the factors: the probability distribution to obtain time series data, the models with specific equations, the mean of the data; and the size of data. The sizes are not higher, because one of the hypothesis of this Doctoral thesis is that Bayesian models perform well when there are few historical data.

The levels of the factors are shown in tables 3.13, 3.14, 3.15.

ARIMA
Exponential Smoothing
Linear Model
Bayesian Regression
Bayesian Dynamic Linear Model 1-percentage change
Bayesian Dynamic Linear Model 2-linear growth, polynomial 2

Table 3.13: Levels of the factor Model for univariate comparison.

In the table 3.14, for the indicator variable, $k = 1, \dots, (L-1)$. Where L is the seasonal period The factor model will be separated of the equation of the regression models, because it is

Eq 1	$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 t + \beta_3 sin + \sum_k^{L-1} \gamma_k indicator_k$
Eq 2	$y_t = \beta_0 + \beta_1 y_{t-1}$
Eq 3	$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 t$
Eq 4	$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_3 sin$
Eq 5	$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_3 sin + \sum_k^{L-1} \gamma_k indicator_k$
Eq 6	$y_t = \beta_0 + \beta_1 y_{t-1} + \sum_k^{L-1} \gamma_k indicator_k$
Eq 7	$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 t + \beta_3 sin$

Table 3.14: Equations of the regression models.

necessary to identify in general, the best kind (s) of models to forecast. The equation will be explored as a block, to see the impact, and also, as a factor if it deserves.

Looking for low variability, some other factors will be fixed. The skew parameter of the distributions Skew normal and Skew t, will be fixed in 50, and the forecast validation data will be 7 in all cases, to calculate MAPE of forecast.

Skew Normal
Skew T
Gamma
Poisson

Table 3.15: Levels of the factor distributions for simulated data

Exploratory plots

After the exploration of preliminary figures of Box Plot (Figs. 3.12, 3.13, 3.14), it is possible to see that the distributions of MAPE of forecasts change depending on the model. In especial, the Bayesian Regression Model (BRM), in figure 3.12, shows a lower distribution, when we compare it with the classical Linear Model (LM), and maybe, the ARIMA. It is also possible to find how Bayesian Dynamic Linear Models 1 and 2 have a variability that can be due to the distributions used, or that can affect the inferences.

In figure 3.13, it is possible to see that the Gamma distribution has a high variance, despite it does not show differences in mean or medians from one box to another.

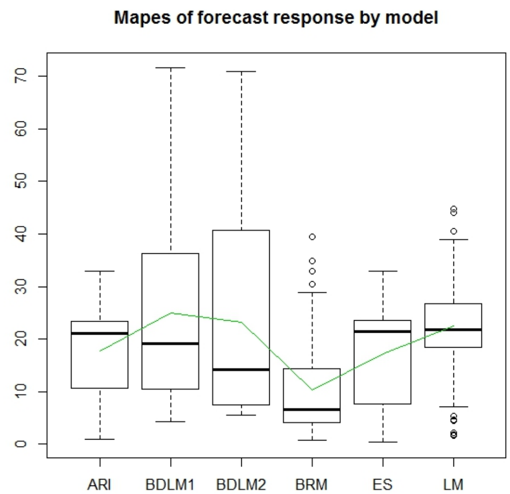

Figure 3.12: Box plot for MAPE of forecasts vs Model

The Box plot for the MAPE vs size in figure 3.14, does not show differences in mean or medians, and neither MAPE vs distribution.

These results provide ideas about the possible significance or not, of the factors and interactions. But, according to the exploration, there is not evidence that the factor size of data can affect the response, and, neither, that this response can be affected by the interaction among models and the mean and size of data.

Estimation of the ANOVA table

The previous descriptive results can be verified using the ANOVA tables. Observing the p values higher than 5%, of the interactions: Model-size and Model-mean of data and of the factor size, in the preliminary Anova Table (3.16), it leads to conclude the non significance

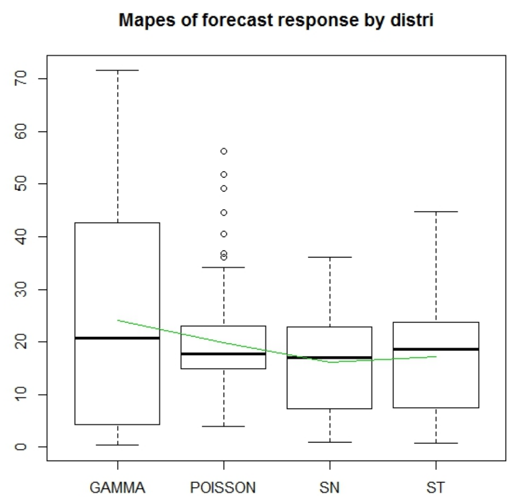

Figure 3.13: Box plot for MAPE of forecasts vs distribution of simulated data

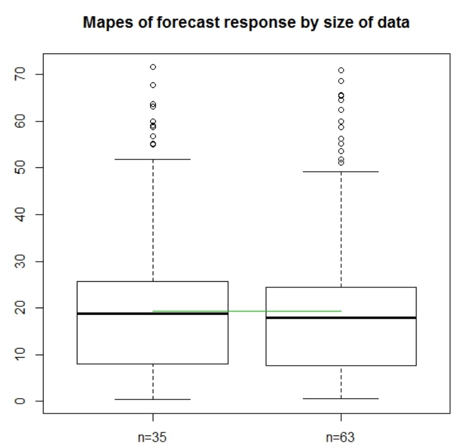

Figure 3.14: Box plot for MAPE of forecasts vs size of simulated data

of these two interactions, besides of the factor size of data, on the response: Forecast MAPE (%).

	Sum Sq	Df	F value	Pr(>F)
Distri	429.54	4	602.54	0.0000
Model	186.30	5	209.07	0.0000
Size	0.01	1	0.05	0.8164
Meandata	1.36	1	7.62	0.0059
Distri:model	407.70	15	152.51	0.0000
Model:size	0.21	5	0.24	0.9452
Model:meandata	1.35	5	1.51	0.1833
Residuals	113.35	636		

Table 3.16: Preliminary Anova Table for model comparison with simulated time series.

The non significant factors were eliminated one by one, so, at the end no interactions remained. And the residuals did not have the normality expected, because p-values of the residuals were lower than 5%. So, a transformation is necessary in order to obtain good performance in the results.

The model is re-estimated without the non meaningful factors, but, despite the significance, the residuals do not fulfill the assumptions of normality and constant variance needed, so the model is not trustable. Another model is re-estimated with no interactions (table 3.17), and using a transformation in the response value, a power of 0.6, so, the new response is taken as: $y^* = y^{0.6}$, because this is an optimal power to find the normality in the residuals. It is also considered a blocking factor that represents the kind of equation used for the regression, and it is efficient because it explains a Mean Square Sum (MSB) higher than the MSE of the model, so, it adds value to the design.

In the final ANOVA table (3.17), it is possible to see that all factors considered are significant, at level 5%, but the block Equation contributes to explain the variance estimation, because its MSB, 5.96, is higher than the MSE, 5.29.

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Distribution	3	110.74	36.91	6.98	0.0001
Mean of data	1	36.38	36.38	6.88	0.0089
Model	5	588.42	117.68	22.24	0.0000
BL.Equation	6	35.77	5.96	1.13	0.3450
Residuals	656	3470.96	5.29		

Table 3.17: Final Anova Table for model comparison with simulated time series.

Besides, the normality and variance assumptions of this model with transformed response, are fulfilled. The Jarque Bera Test has a p-value = 0.1425, and the Levene's Test for Homogeneity of Variance, when the factor mean of data is used to group, has a P-value of 0.6019, higher than 5% of significance level, so, normality and constant variance are accepted.

In the table of coefficients (table 3.18), it is possible to see that in presence of SN and ST distributions, the forecast MAPE can be more easily reduced than with other distributions used for data simulation. In this table (3.18), it is possible to see that the BRM has the lowest value of the coefficient estimated.

	Estimate	Std. Error	t value	$Pr(>\|t\|)$
(Intercept)	5.9832	0.3549	16.86	0.0000
Dist-POISSON	-0.1194	0.2510	-0.48	0.6345
Dist-SN	-0.9467	0.2510	-3.77	0.0002
Dist-ST	-0.7677	0.2510	-3.06	0.0023
Mean-data200	-0.4654	0.1775	-2.62	0.0089
Model-BDLM1	1.1470	0.3074	3.73	0.0002
Model-BDLM2	0.7114	0.3074	2.31	0.0210
Model-BRM	-1.6254	0.3074	-5.29	0.0000
Model-ES	-0.1540	0.3074	-0.50	0.6165
Model-LM	0.9756	0.3074	3.17	0.0016
BL.EQE2	0.0484	0.3320	0.15	0.8842
BL.EQE3	0.1717	0.3320	0.52	0.6053
BL.EQE4	0.4373	0.3320	1.32	0.1883
BL.EQE5	-0.0195	0.3320	-0.06	0.9531
BL.EQE6	-0.3775	0.3320	-1.14	0.2559
BL.EQE7	0.1881	0.3320	0.57	0.5712

Table 3.18: Effects of linear Model for the first Design of Experiments.

For the Tukey mean differences tests, in table 3.19, the hypothesis to be tested in every case is $H_0 : \mu_1 - \mu_2 = 0$ vs $H_1 : \mu_1 - \mu_2 \neq 0$. In this table, the names of the levels in the left of the first column, produce higher values of means than in the right. When the difference is significant, the p-value is lower than 5%, and it also can be seen in the interval, if the two limit values are positive. For example, in the first line, the interval (-0.47, 0.83) indicates that the mean of forecast MAPES are equal for the distributions SN and ST; and in the second line (0.18, 1.47) indicates that the mean of MAPE is higher for Poisson distribution, than for SN. Then, it can be seen that Gamma distribution has a higher mean than the other distribution if they are not equal. The factor means of data have also significant differences, indicating that the higher mean (200) produces lower MAPE of forecast.

It can also be seen that the BRM is in the second place of the differences for all comparisons involved. It means that BRM produces the lower MAPES of forecasts, but other good model is the exponential smoothing. The other two Bayesian models: BDLM1 and BDLM2 have equal effects, but when they are compared to the BRM, this last is better. When BDLM's are compared to the classical linear model, the dynamic Bayesian produces, in general, better results.

In general, from these analysis, it can be concluded that BRM produces significantly lower MAPE of forecast than the other proposed models.

Differences of means of the factor distribution					
	Difference	lower	upper	p Value adj	Decision
ST-SN	0.18	-0.47	0.83	0.89	EQUAL
POISSON-SN	0.83	0.18	1.47	0.01	DIFFERENT
GAMMA-SN	0.95	0.30	1.59	0.00	DIFFERENT
POISSON-ST	0.65	0.00	1.29	0.05	DIFFERENT
GAMMA-ST	0.77	0.12	1.41	0.01	DIFFERENT
GAMMA-POISSON	0.12	-0.53	0.77	0.96	EQUAL
Differences of means in factor mean of the data					
30-200	0.47	0.12	0.81	0.01	DIFFERENT
Differences of means of the factor Model					
ES-BRM	1.47	0.59	2.35	0.00	DIFFERENT
ARI-BRM	1.63	0.75	2.50	0.00	DIFFERENT
BDLM2-BRM	2.34	1.46	3.22	0.00	DIFFERENT
LM-BRM	2.60	1.72	3.48	0.00	DIFFERENT
BDLM1-BRM	2.77	1.89	3.65	0.00	DIFFERENT
ARI-ES	0.15	-0.72	1.03	1.00	EQUAL
BDLM2-ES	0.87	-0.01	1.74	0.06	EQUAL
LM-ES	1.13	0.25	2.01	0.00	DIFFERENT
BDLM1-ES	1.30	0.42	2.18	0.00	DIFFERENT
BDLM2-ARI	0.71	-0.17	1.59	0.19	EQUAL
LM-ARI	0.98	0.10	1.85	0.02	DIFFERENT
BDLM1-ARI	1.15	0.27	2.03	0.00	DIFFERENT
LM-BDLM2	0.26	-0.61	1.14	0.96	EQUAL
BDLM1-BDLM2	0.44	-0.44	1.31	0.72	EQUAL
BDLM1-LM	0.17	-0.71	1.05	0.99	EQUAL

Table 3.19: Difference in means for the first Design of Experiments

Second Analysis-Second Design of experiments

Another model which produces a lower MSE than before (5.13), is to consider only two factors: distribution of simulated data, and the models specifying the equation used, but also, with the transformed response. ANOVA table (table 3.20) shows two significant factors at the level 5%.

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Distri	3	110.74	36.91	7.19	0.0001
Equation	17	790.31	46.49	9.06	0.0000
Residuals	651	3341.23	5.13		

Table 3.20: Anova Table, factors: Equation of model and distribution

Here, both factors are significant at 5%. Besides, in the coefficients table 3.21, it can be seen that in BRM, equation 6 that has two covariables: Y_{t-1} and the seasonal indicator, has the lowest effect of this model, followed by Eq-1, and after, Eq-5. Besides, the model also accepts the hypothesis of normality and constant variance at 5%.

The differences of means were tested also with the Tukey technique, whose results are shown in tables 3.22 to 3.24.

In table 3.22 it is possible to see the differences between means of forecast MAPE for the factor distribution of data.

For the comparison between pairs of means of the factor equation of the model, of the forecast MAPE (table 3.23), in the first 17 lines are the pairs compared to the BRM-equation 6. It can be seen that the first 7 lines decide equality, and they are only compared among BRM equations, but, in the rest of the lines, the BRM equations are compared with different models.

	Estimate	Std. Error	t value	$Pr(>\|t\|)$
(Intercept)	5.8146	0.2622	22.18	0.0000
Dist-POISSON	-0.1194	0.2472	-0.48	0.6293
Dist-SN	-0.9467	0.2472	-3.83	0.0001
Dist-ST	-0.7677	0.2472	-3.11	0.0020
Eq-BDLM1	1.1470	0.3027	3.79	0.0002
Eq-BDLM2	0.7114	0.3027	2.35	0.0191
Eq-BRME1	-2.6338	0.6055	-4.35	0.0000
Eq-BRME2	-0.8264	0.6055	-1.36	0.1728
Eq-BRME3	-0.3695	0.6055	-0.61	0.5419
Eq-BRME4	-0.7787	0.6055	-1.29	0.1989
Eq-BRME5	-2.5659	0.6055	-4.24	0.0000
Eq-BRME6	-2.9661	0.6055	-4.90	0.0000
Eq-BRME7	-1.2373	0.6055	-2.04	0.0414
Eq-ES	-0.1540	0.3027	-0.51	0.6111
Eq-LME1	1.0087	0.6055	1.67	0.0962
Eq-LME2	-0.2946	0.6055	-0.49	0.6268
Eq-LME3	-0.1067	0.6055	-0.18	0.8602
Eq-LME4	2.2007	0.6055	3.63	0.0003
Eq-LME5	1.1820	0.6055	1.95	0.0513
Eq-LME6	0.6736	0.6055	1.11	0.2663
Eq-LME7	2.1653	0.6055	3.58	0.0004

Table 3.21: Effects of linear model for the second Design of Experiments

Differences of means of the factor distribution					
	Difference	lower	upper	p Value adj	Decision
ST-SN	0.18	-0.46	0.82	0.89	EQUAL
POISSON-SN	0.83	0.19	1.46	0	DIFFERENT
GAMMA-SN	0.95	0.31	1.58	0	DIFFERENT
POISSON-ST	0.65	0.01	1.29	0.04	DIFFERENT
GAMMA-ST	0.77	0.13	1.4	0.01	DIFFERENT
GAMMA-POISSON	0.12	-0.52	0.76	0.96	EQUAL

Table 3.22: Fist table-Difference in means for the second Design of Experiments

Differences of means of the factor equation of model					
	Difference	Lower	Upper	P Value adj	Decision
BRME1-BRME6	0.33	-2.47	3.14	1	EQUAL
BRME5-BRME6	0.4	-2.41	3.21	1	EQUAL
BRME7-BRME6	1.73	-1.08	4.53	0.78	EQUAL
BRME2-BRME6	2.14	-0.67	4.95	0.4	EQUAL
BRME4-BRME6	2.19	-0.62	4.99	0.36	EQUAL
BRME3-BRME6	2.6	-0.21	5.4	0.11	EQUAL
LME2-BRME6	2.67	-0.13	5.48	0.08	EQUAL
ES-BRME6	2.81	0.69	4.93	0	DIFFERENT
LME3-BRME6	2.86	0.05	5.67	0.04	DIFFERENT
ARI-BRME6	2.97	0.85	5.09	0	DIFFERENT
LME6-BRME6	3.64	0.83	6.45	0	DIFFERENT
BDLM2-BRME6	3.68	1.56	5.8	0	DIFFERENT
LME1-BRME6	3.97	1.17	6.78	0	DIFFERENT
BDLM1-BRME6	4.11	1.99	6.23	0	DIFFERENT
LME5-BRME6	4.15	1.34	6.95	0	DIFFERENT
LME7-BRME6	5.13	2.33	7.94	0	DIFFERENT
LME4-BRME6	5.17	2.36	7.97	0	DIFFERENT
BRME5-BRME1	0.07	-2.74	2.87	1	EQUAL
BRME7-BRME1	1.4	-1.41	4.2	0.96	EQUAL
BRME2-BRME1	1.81	-1	4.61	0.71	EQUAL
		……			

Table 3.23: Second table-Difference in means for the second Design of Experiments

Differences of means of the factor equation of model					
	Difference	lower	upper	p Value adj	Decision
BRME7-BRME5	1.33	-1.48	4.13	0.97	EQUAL
BRME2-BRME5	1.74	-1.07	4.55	0.77	EQUAL
BRME4-BRME5	1.79	-1.02	4.59	0.73	EQUAL
BRME3-BRME5	2.2	-0.61	5	0.35	EQUAL
LME2-BRME5	2.27	-0.53	5.08	0.29	EQUAL
ES-BRME5	2.41	0.29	4.53	0.01	DIFFERENT
LME3-BRME5	2.46	-0.35	5.26	0.17	EQUAL
ARI-BRME5	2.57	0.45	4.69	0	DIFFERENT
LME6-BRME5	3.24	0.43	6.05	0.01	DIFFERENT
BDLM2-BRME5	3.28	1.16	5.4	0	DIFFERENT
LME1-BRME5	3.57	0.77	6.38	0	DIFFERENT
BDLM1-BRME5	3.71	1.59	5.83	0	DIFFERENT
LME5-BRME5	3.75	0.94	6.55	0	DIFFERENT
LME7-BRME5	4.73	1.93	7.54	0	DIFFERENT
LME4-BRME5	4.77	1.96	7.57	0	DIFFERENT
		......			

Table 3.24: Third table-Difference in means for the second Design of Experiments

All the results of differences in means are not shown here because they are so extensive.

In table 3.23, the BRM-Equation 6, has the lower MAPE value of forecasts, compared to almost all the models: BDLM1, BDLM2, ARI, ES, and LM (equations 1, 3, 4, 5, 6, 7, not 2). In the table 3.24, BRM-Equation 5 is equal to other BRM equations, but lower than many other models. Among the same BRM, there are not many significant differences. The model BDLM1 has equal mean than models like ES and some equations of LM, and BDLM2. This result confirms that the proposed BDLM1 model designed in this Doctoral Thesis can be also a good alternative to do forecast.

Until here, it can be confirmed that BRM produces better forecasts when the univariate data have non normality, but, they do have seasonality as well as abrupt changes. Then, the hypothesis verified is that Bayesian models are a good alternative to produce good forecasts in cases of drastic changes.

3.2.4 Repeating the most important cases

After repeating every particular simulation a thousand times, it is possible to find the frequency of choice of the best adjusted model. The estimation of the six models will be shown for the Normal distribution simulation, followed by the Skew Normal and T, by changing the skew parameter in four cases. Using a fixed scenario: Skew T distribution for the demand and errors, with the skew dynamic parameter taking the values of: 1, 10, 50 and 100, and the mean values of data: 30, 200, and 500. Finally, the simulation with Gamma and Poisson distributions will be used.

The performed simulations for the fixed scenarios with mean of data: 30 and 200, Skew Normal distribution for the data, and equation with covariables: Y_{t-1} and the *indicator*, which results are shown in table 3.25, confirms that this model BRM proposed in this Doctoral Thesis is a very good alternative for forecasting, because it is the most selected almost all times. In the cases with mean 30, the BRM is chosen 83% (of 1000 simulations), and with mean 200, this percentage in 100%.

Mean: 30				
Skew-parameter	1	10	50	100
ARIMA	0.00	0.00	0.00	0.00
ES	0.00	0.00	0.00	0.00
LM	0.00	0.00	0.00	0.00
BRM	83.33	83.33	83.33	83.33
BDLM1	0.00	16.67	0.00	0.00
BDLM2	16.67	0.00	16.67	16.67
Mean: 200				
Skew-parameter	1	10	50	100
ARIMA	0	0	0	0
ES	0	0	0	0
LM	0	0	0	0
BRM	100	100	100	100
BDLM1	0	0	0	0
BDLM2	0	0	0	0

Table 3.25: Selection of models for simulation with Skew Normal distribution.

In the scenario of the Skew t distribution (table 3.26) used to simulate data, it can be seen that the model with the most frequent selection is also the Bayesian Regression. With mean 30, and skew parameters 1 and 10, the BRM is followed by the BDLM2 in 10%, and for mean 200, it is followed by the BDLM1 in 16.67%, for skew parameter 50.

Skew-parameter	1	10	50	100
Mean: 30				
ARIMA	0.00	0.00	0.00	0.00
ES	0.00	0.00	0.00	0.00
LM	0.00	0.00	0.00	0.00
BRN	90.00	90.00	100.00	100.00
BDLM1	0.00	0.00	0.00	0.00
BDLM2	10.00	10.00	0.00	0.00
Mean: 200				
ARIMA	0.00	0.00	0.00	0.00
ES	0.00	0.00	0.00	0.00
LM	0.00	0.00	0.00	0.00
BRN	100.00	100.00	83.33	100.00
BDLM1	0.00	0.00	16.67	0.00
BDLM2	0.00	0.00	0.00	0.00

Table 3.26: Selection of models for simulation of Skew t distribution

3.2.5 Multivariate Comparison

The models estimated until here have univariate theoretical structures, and are used to univariate time series. What happens if different time series present multivariate behavior, with a covariance among them? are there different results? is there a different model with better behavior?

This questions will be answered with two process. First, by applying all the models, univariate and multivariate, to a real case, of sales of fuel in a Colombian gas station, and second, by simulating a multivariate time series with three variables using Skew Normal and Skew T probability distributions, with a covariance matrix, and a seasonal behavior, and after, applying the models to the time series simulated, in order to estimate the response MAPE, and compare the results obtained of all estimated models.

It will be fixed the size of the data in 63, because it was found no significant effect of this factor, and a factorial design will be used to answer what is the best form for forecasting: with univariate or with multivariate models.

Factors: Model (table 3.27), multivariate distribution for data simulation (Skew Normal and Skew t Multivariate distributions), and periods to forecast (7 and 14).

Univariate Models
ARIMA
Exponential Smoothing
Linear Model
Bayesian Regression
Bayesian Dynamic Linear Model 1-percentage change
Bayesian Dynamic Linear Model 2-linear growth, polynomial 2
Multivariate Models
Multivariate Bayesian Dynamic Linear Model 1-linear growth
Multivariate Bayesian Dynamic Linear Model 2-linear growth plus seasonal
Linear Mixed Model- seasonal co-variables

Table 3.27: Univariate and Multivariate models estimated.

Example 5- Comparisons of models estimated to the real data of combustible demands for a Colombian gas station.

The application among all the different models will be first shown for the same real data of combustible demand, exposed with the previous examples.

- Data: Daily sales of three kinds of fuel of a gas station: Extra, Corriente, Diesel.
- Period: November 1 2014 to January 31, 2015.
- Horizon time of forecasts: 14 days of January 2015.

The table 3.28 shows the forecast MAPES for the estimated models to the time series of the three kinds of fuel.

	MAPE OF FORECASTS (%)-Fuel sales								
	ARIMA	ES	LM	BRM	BDLM1	BDLM2	MDLM1	MDLM2	MIXED
Corriente	12.38	12.79	11.50	10.17	16.23	16.93	14.47	12.37	88.00
Extra	94.00	94.58	39.82	30.99	37.97	26.58	58.50	45.22	88.14
Diesel	68.94	69.48	68.14	9.14	79.82	19.60	17.50	15.25	88.58

Table 3.28: MAPES Comparison for the three Fuel sales. 14 forecast periods.

As it can be seen in the table 3.28, the best model to do forecasts is the BRM (MAPES: 10.17%, 30.99%, 9.14%), followed by LM in Corriente and Extra fuels, and the model MDLM2, that produces lower results than ARIMA or ES for the three series.

3.2.6 Multivariate Simulation

The results of the simulation of multivariate time series, using the Skew Normal and Skew T multivariate distributions, and posterior estimation of models for data, are shown in table 3.29. The BRM-eq 6 produces better results than all the other models (MAPE of forecasts 4.47%, 3.45%, 4.92%). For the first series, this model is followed by the Mixed model, and

MDLM2. Multivariate Dynamic linear models (1 and 2), have a good result compared to the classical models, and univariate DLM's, but apparently not better than the BRM-eq 6.

Besides this, apparently, the mixed model used is not better than the rest of models in all series.

	ARIMA	ES	LM	BRM	BDLM1	BDLM2	MDLM1	MDLM2	MIXED
Skew t distribution									
Series 1	7.84	8.10	4.63	4.47	12.28	7.06	7.33	5.11	4.72
Series 2	19.18	19.18	13.03	3.45	11.38	5.85	12.99	5.94	21.01
Series 3	48.23	48.22	42.91	4.92	13.27	19.19	18.27	5.88	48.29
Skew Normal distribution									
Series 1	7.07	7.07	2.41	1.90	6.96	6.20	7.00	3.74	2.64
Series 2	16.07	16.07	14.83	2.46	9.44	15.59	12.09	3.53	16.27
Series 3	46.33	46.14	38.16	9.95	16.94	13.63	12.40	12.05	46.76

Table 3.29: Comparison of models after multivariate simulation. 7 forecast periods.

For the Skew Normal distribution for the data (table 3.29), the best model chosen is also BRM-eq 6, in all the three series generated with multivariate simulation. Figure 3.15 shows a comparison, where models BRM, and MDLM2 have an apparent closer estimation, in comparison with the others.

After obtained the simulated time series data, a factorial design of experiments was estimated. The results of the experimental design are shown in next section.

3.2.7 Results of the experimental design for multivariate arrangements

In the exploratory box plots, it is possible to see that the factor distribution of the data has no apparent differences for the forecast MAPE response, but, the factor model has differences, and it is possible to identify that the BRM, and the MDLM2 have apparent lower values than others.

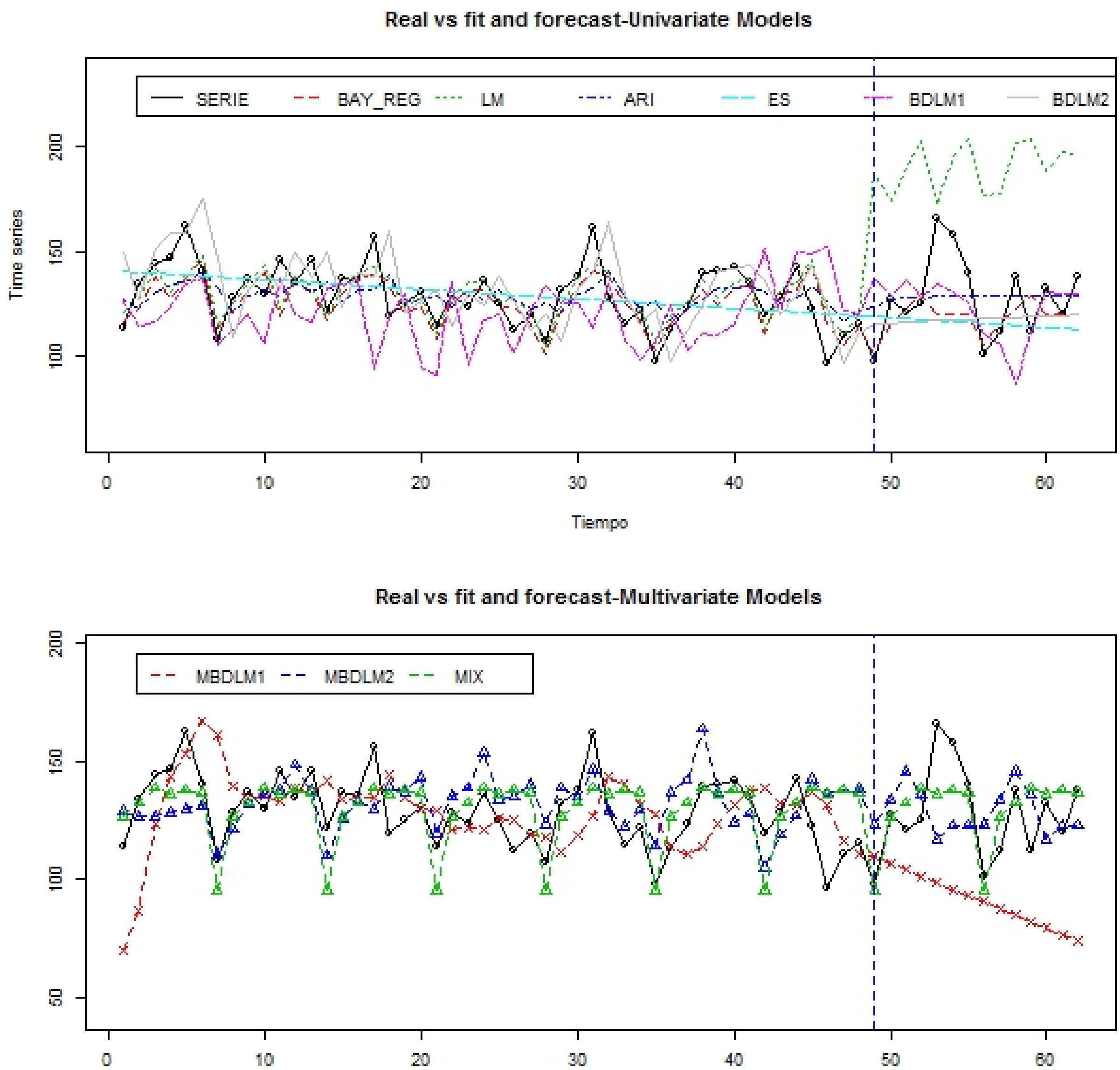

Figure 3.15: Comparison of Univariate and Multivariate models

In the first ANOVA table 3.30, it is possible to identify that there was no significant interactions, neither some factors. The non significant interactions and factors were eliminated one by one, so, at the end no interactions remained. The residuals do not have the normality expected. So a transformation is necessary in order to obtain good performance in the results ($y^* = y^{0.6}$). A final significant model is identified, and the results are shown in the final Anova table 3.31.

	Sum Sq	Df	F value	Pr(>F)
(Intercept)	9.40	1	19.18	0.0000
Model	3.42	8	0.87	0.5416
Distribution	0.00	1	0.00	0.9760
Periods	0.02	1	0.04	0.8411
Model:distribution	0.25	8	0.06	0.9999
Model:periods	0.43	8	0.11	0.9989
Distribution:periods	0.00	1	0.00	0.9663
Model:distribution:periods	0.32	8	0.08	0.9996
Residuals	88.21	180		

Table 3.30: First Anova Table for model comparison, for the multivariate time series simulation.

The final ANOVA table 3.31, reflects two significant factors: model and periods. Besides, after verifying the residuals, the p-value of the Jarque Bera test was 0.5734, and the p-value of the Levene test was 0.8732, leading to the acceptance of the hypothesis of normality and constant variance of the residuals.

The table of coefficients (3.32), shows the BRM with the lowest value, which leads to lower MAPE values with this model. The next model with low value is the MDLM2.

In the first table of difference of means 3.33, it is possible to verify that the BRM produces the lowest mean, because the intervals of the differences are positive, indicating that the mean in first place is higher than the mean in the second place, but it is significantly equal to the MDLM2, because the interval of the difference goes from a negative value to a positive:

	Sum Sq	Df	F value	Pr(>F)
Intercept	172.01	1	394.31	0.0000
Model	40.82	8	11.70	0.0000
Periods	1.91	1	4.38	0.0376
Residuals	89.86	206		

Table 3.31: Final Anova Table for the multivariate time series simulation

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
Intercept	2.8219	0.1421	19.86	0.0000
BDLM1	-0.3377	0.1907	-1.77	0.0780
BDLM2	-0.5545	0.1907	-2.91	0.0040
BRM	-1.3148	0.1907	-6.90	0.0000
ES	0.0002	0.1907	0.00	0.9990
LM	-0.3235	0.1907	-1.70	0.0913
MDLM1	-0.3581	0.1907	-1.88	0.0618
MDLM2	-1.0851	0.1907	-5.69	0.0000
MIXED	-0.1386	0.1907	-0.73	0.4680
Periods_{14}	0.1881	0.0899	2.09	0.0376

Table 3.32: Effects of linear model for the Design of experiments of the multivariate simulation

$(-0.37, 0.83)$, in the first line.

In this table 3.33, the MDLM2 produces lower means than the models: ARIMA, ES, LM, MIXED, BDLM1, and MDLM1, but it is significantly equal to the BDLM2.

In table 3.34, it is easy to appreciate that the difference in the means of the factor periods, show that if periods are lower, the forecast is better, in general, with p-value of 4%. It is also appreciable that the Mixed model, which is the other classical multivariate model that can consider co-variables like seasonal terms, is not significantly better than any other model, it just produces equal mean results to ARIMA, ES, and LM. Besides this, residuals of the mixed model do not reach normal distribution, neither constant variance.

It is demonstrated until here, that BRM and BDLM2 produced better forecasts than other models, when multivariate data are analyzed with non normality, seasonality, and with abrupt changes, so, this hypothesis is verified: Multivariate bayesian models are a good alternative to produce good forecasts in cases of drastic changes for multivariate demand.

Multivariate demand can be predicted using the Bayesian Regression model proposed and designed in R, as a novel proposal for this Doctoral Thesis. Besides this, a very good alternative is the Multivariate Dynamic Linear Model with seasonality and linear growth (MDLM2), which is found in the dlm package designed by (Petris, 2010).

3.3 Elicitation process for the studied case of combustible demands.

Expert Judgment in the prediction of variables like demand could improve results. Österholm (2009) presented a weighted linear combination, also referred as linear opinion pool, boarding several conflicting scenarios in order to predict different variables. This technique can be used by applying low and high weighted values, like those that come from different sources of forecasting models.

Differences of means of the factor model					
	Difference	lower	upper	p Value adj	Decision
MDLM2-BRM	0.23	-0.37	0.83	0.95	EQUAL
BDLM2-BRM	0.76	0.16	1.36	0	DIFFERENT
MDLM1-BRM	0.96	0.36	1.55	0	DIFFERENT
BDLM1-BRM	0.98	0.38	1.57	0	DIFFERENT
LM-BRM	0.99	0.39	1.59	0	DIFFERENT
MIXED-BRM	1.18	0.58	1.77	0	DIFFERENT
ARIMA-BRM	1.31	0.72	1.91	0	DIFFERENT
ES-BRM	1.31	0.72	1.91	0	DIFFERENT
BDLM2-MDLM2	0.53	-0.07	1.13	0.13	EQUAL
MDLM1-MDLM2	0.73	0.13	1.32	0.01	DIFFERENT
BDLM1-MDLM2	0.75	0.15	1.35	0	DIFFERENT
LM-MDLM2	0.76	0.16	1.36	0	DIFFERENT
MIXED-MDLM2	0.95	0.35	1.54	0	DIFFERENT
ARIMA-MDLM2	1.09	0.49	1.68	0	DIFFERENT
ES-MDLM2	1.09	0.49	1.68	0	DIFFERENT
MDLM1-BDLM2	0.2	-0.4	0.79	0.98	EQUAL
BDLM1-BDLM2	0.22	-0.38	0.81	0.97	EQUAL
LM-BDLM2	0.23	-0.37	0.83	0.95	EQUAL
MIXED-BDLM2	0.42	-0.18	1.01	0.42	EQUAL
ARIMA-BDLM2	0.55	-0.04	1.15	0.09	EQUAL
ES-BDLM2	0.55	-0.04	1.15	0.09	EQUAL

Table 3.33: First table-Difference in means of the final factorial design-Multivariate time series simulation

Differences of means of the factor model					
	Difference	lower	upper	p Value adj	Decision
BDLM1-MDLM1	0.02	-0.58	0.62	1	EQUAL
LM-MDLM1	0.03	-0.56	0.63	1	EQUAL
MIXED-MDLM1	0.22	-0.38	0.82	0.97	EQUAL
ARIMA-MDLM1	0.36	-0.24	0.96	0.63	EQUAL
ES-MDLM1	0.36	-0.24	0.96	0.63	EQUAL
LM-BDLM1	0.01	-0.58	0.61	1	EQUAL
MIXED-BDLM1	0.2	-0.4	0.8	0.98	EQUAL
ARIMA-BDLM1	0.34	-0.26	0.94	0.7	EQUAL
ES-BDLM1	0.34	-0.26	0.94	0.7	EQUAL
MIXED-LM	0.18	-0.41	0.78	0.99	EQUAL
ARIMA-LM	0.32	-0.27	0.92	0.75	EQUAL
ES-LM	0.32	-0.27	0.92	0.75	EQUAL
ARIMA-MIXED	0.14	-0.46	0.74	1	EQUAL
ES-MIXED	0.14	-0.46	0.74	1	EQUAL
ES-ARIMA	0	-0.6	0.6	1	EQUAL
Differences of means of the factor periods					
14-7	0.19	0.01	0.37	0.04	DIFFERENT

Table 3.34: Second table-Difference in means of the final factorial design-Multivariate time series simulation

It is necessary to choose the best form to do forecasts. In this case, two criteria are used for every case: adjusted and forecast MAPE.

Bases of the questionnaire for the studied case.

First, it is necessary to investigate for the existence of the seasonality. Time depends on the problem, for example, which days (periods) of the week (month, year), do you think that the demand is high or low. If the seasonality is found, next questions will have to be done for every periods of the seasonal period.

Percentiles.

In order to obtain the values for every period of sales, and through these, to build an empirical distribution of the predicted values, it is necessary to formulate the next questions day by day.

- What do you consider is the real minimum of all values of sales of product . . .
- Which value could be low, but not the minimum.
- Which value is very common in the regular period (from: . . . to: . . .)
- Which value is so high that is not common but there could be higher than this . . .
- Which value is so high that is almost impossible to have a higher value in this regular period . . .
- The same questions will be done if there is another period which shows seasonality for predicted data.

There were some difficulties in getting the information. The expert was in other city and did not have much time to solve all the questions for the different days of predictions proposed, so she finally told me some generalities of the behavior of the sales for the three fuels.

Other difficulty is that this process is specifically for a particular case, because it requires the expertise knowledge. An specific process will be used here for the case of the combustible demands provided. The expert is the person who does the planning of the orders defined the next aspects for every product and period, because it is a forecasting process. But the methodology can be generalized for different kind of products.

It is important to know that the company is interested in observing the results of 15 or 31 days of the month of January, in order to do a real comparison, and the need they have about managing the periods of countability close.

After finishing the interview, the gathering of the data starts. Empirical distribution will be analyzed, in order to understand the behavior of the data.

Corriente fuel.

- The day Saturday, the Corriente fuel increases the actual level, minimum in 300 gallons, maximum in 450
- For day with no motorcycle, which is a policy of that city, for the last Tuesday of the month, the level of this gas reduces between 200 and 300 gallons.

Extra fuel.

- In the weekends, this fuel increases the proper mean in about 30 to 45 gallons, but if this weekend has festive, there is a higher increase, between 35 and 50 gallons at maximum.

Diesel fuel.

- In the weekends, this fuel decreases, because it is used principally by public transportation. The decreasing level is between 20 to 30% of the real level of the mean.

The distributions used to estimate the empirical elicited distribution consists in doing a sample of three data, given by the expert, because it is not possible to get more data to do the process. Then, a minimum, a mean and a maximum, are used with a discrete probability to occur, then a sampling is done, generating a sample of the mean with a posterior distribution, and generating a predicted value with an uniform distribution, which mean was updated. The distributions are explained in the next chapter, in the section of bayesian optimization.

In the past chapter, the equation 6 of the BRM was the best, but it is statistically equal to the other equations for the BRM models, the minimum difference of the results is with the equation 1:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 t + \beta_3 sin + \sum_{k}^{L-1} \gamma_k indicator_k$$

for $k = 1, \ldots, (L - 1)$, L, minus 1, where L is the seasonal period. In table 3.35 it is possible to appreciate differences for the real fuel demands, compared to the elicited forecast values.

Forecast MAPE (%)-Fuel sales									
	ARIMA	ES	LM	BRM	BDLM1	BDLM2	MDLM1	MDLM2	ELICIT
7 days forecasts									
Corriente	12.46	12.58	13.66	9.19	27.75	14.10	17.26	12.07	9.87
Extra	93.64	94.46	52.45	32.00	38.80	34.97	80.66	50.73	60.46
Diesel	69.24	70.07	67.33	9.20	42.44	27.38	16.69	15.00	20.7
	ARIMA	ES	LM	BRM	BDLM1	BDLM2	MDLM1	MDLM2	ELICIT
15 days forecasts									
Corriente	15.43	14.88	17.74	9.86	16.33	16.60	105.91	21.58	11.963
Extra	86.62	88.51	74.83	38.35	66.29	98.70	90.88	87.00	51.177
Diesel	68.44	86.13	67.84	24.21	38.83	25.15	32.42	24.83	22.417
	ARIMA	ES	LM	BRM	BDLM1	BDLM2	MDLM1	MDLM2	ELICIT
31 days forecasts									
Corriente	9.51	7.31	12.17	5.57	14.19	19.26	16.63	9.90	9.256
Extra	95.25	95.45	44.71	24.06	26.62	47.80	47.84	24.15	27.958
Diesel	69.89	65.52	68.86	8.96	22.18	32.45	28.51	18.22	21.392

Table 3.35: Comparison for three fuel sales, including elicitation process, for 7-15-31 forecast days.

As we can realize (table 3.35), the expert elicitation forecast do not show better results than the BRM model for 7, 15, nor 31 forecast days, but this elicitation process is an additional alternative for example, when a new station is going to be opened, having an idea about mean initial levels. Until here, the proposed BRM produces a good performance to do forecasts.

3.4 Proposal to do forecast with validation through simulation.

As a novel approach, in order to generate trust-able forecasts, it is proposed a method, consisting in selecting a model by validation using simulation. Three distributions will generate multiproduct time series (three time series for the case of fuel): Skew Normal (SN), Skew T (ST) multivariate distributions, and the simulation using elicitation process directed by the expert.

Table 3.36 shows error indicators, comparing the simulated series data with the real values. For every case (SN, ST and Elicitation), the indicators MAD, RMSE and MAPE will be provided, estimated between a last part of the real series according to periods, in the month January, and the simulated series, in order to know which one of the simulated cases has the lower values. The minimum indicators will provide results about the best possible representation of the reality.

The table 3.36 shows a better performance of elicitation simulation data is appreciable. So, these can be the better kinds of reference to predict data on February month.

The decision is to use the Elicitation to simulate new data, because it has lower values of forecast MAPES, compared to the cases of SN and ST. Then, the 16 days of February are simulated, and these data are joined to the rest time series. Then the models are fitted for all the available data, and the equation used for regression is Eq-6, doing a partition: 92 know data for fitting (November 1, to January 31), and 16 of February (simulated), for forecasting.

Table 3.37 shows the result of every scenario (SN, ST and Elicitation). Where the best are the Elicitation process, but, as it can be seen, the forecasts of fuel shows most accuracy with LMM, but this model was not trustable, according to the Normality problems shown before. The other possible were, ES, ARIMA, and BRM. For Extra, the BDLM1, MDLM2 and BRM have good results, and for Diesel, the BRM, BDLM1 and MDLM2.

	IRMSE	IMAD	IMAPE
SN multivariate distribution for extended data			
CORRIENTE	1984.9	19038.4	19.2
EXTRA	487.8	4679.2	73.3
DIESEL	1038.5	9961.4	28.6
ST multivariate distribution for extended data			
CORRIENTE	2901.3	27828.3	28.4
EXTRA	415.1	3981.0	67.0
DIESEL	1399.5	13423.8	40.2
Elicitation forecasts			
CORRIENTE	1389.1	13324.1	12.7
EXTRA	314.3	3015.0	53.1
DIESEL	952.4	9134.7	22.1

Table 3.36: Indicators for validation with elicitation, and extended data

Fitting Mape (%)									
Serie	ARI	ES	LM	BRM	BDLM	BDLM2	MBDLM1	MBDLM2	MIX
CORRIENTE	12.3	12.5	10.9	10.6	28.4	20.2	15.6	14.4	11.7
EXTRA 2	49.7	51.1	49.3	42.3	$31.1x10^7$	60.0	95.4	63.5	77.7
DIESEL	35.5	34.8	24.6	22.8	113.9	51.2	41.0	29.4	29.6
Forecast Mape (%)									
SN multivariate distribution for extended data									
CORRIENTE	11.5	11.4	11.1	12.7	11.9	33.8	33.4	13.8	12.6
EXTRA	91.2	90.8	41.0	46.8	336	203.6	261.5	86.2	92.9
DIESEL	69.1	59.2	64.5	28.7	35.8	35.3	70.4	27.8	69.1
ST multivariate distribution for extended data									
CORRIENTE	14.3	15.5	16.0	22.1	21.3	19.1	18.3	19.4	19.0
EXTRA	92.4	92.1	42.4	205.4	$10.1x10^{20}$	464.5	427.0	190.3	93.8
DIESEL	73.4	64.9	68.6	48.2	104.4	69.0	113.6	61.6	73.2
Elicitation forecasts									
CORRIENTE	14.1	13.8	14.2	16.2	20.8	29.2	31.1	18.1	13.0
EXTRA	91.5	91.1	43.0	39.7	26.0	76.9	109.4	35.4	93.1
DIESEL	70.2	60.7	65.1	14.7	15.2	63.0	107.8	22.2	70.1

Table 3.37: MAPE for fitting and forecast-Validation process. 16 days of forecast.

Table 3.38 shows the particular results for the 16 days of February, in the algorithm that chooses the best model according to one criteria: Fitting or Forecast MAPE.

SERIE	MAPE	PERCENTILE	BEST MODEL
Criteria minimum fitting MAPE (%)			
CORRIENTE	16.1	31.9	BRM
EXTRA	15.3	NA	LM
DIESEL	14.8	10.46	BRM
Criteria minimum Forecast MAPE (%)			
CORRIENTE	14.4	NA	ES
EXTRA	22.9	80.4	BRM
DIESEL	14.4	5.46	BRM

Table 3.38: Forecasts using elicitation extended series.

In most cases, BRM is still the best model chosen (table 3.38) to do forecasts, and the MAPE to forecasts seems to be close to elicitation, but is also possible to have more approximated solutions using other equation.

3.5 Synthesis of the chapter

- The Bayesian Regression Model, with equation 6, which has the variables first lag of the series (Y_{t-1}), and the season indicator (7 categories minus 1), showed the best performance for the Forecast MAPE response, for univariate and multivariate responses.
- The Bayesian Multivariate Dynamic Linear Model with linear growth (polynomial 2-Petris et al. (2009)), plus seasonal factors, has a good performance, statistically equal to the BRM-eq 6 model.
- The application of the model comparisons, using the designed algorithm in R, to real data about fuel sales generates the same results found in simulation cases.

- It was shown that not always it is necessary to consider the covariance values between time series data in order to provide good performance of the forecasts.
- Expert elicitation of empirical distributions for daily forecast are an alternative when there are no data. But, in general, for this comparison among models, results are not better for elicitation process, than for BRM.
- The validation proposal to do forecasts can be useful in order to use expert knowledge as a guide to the decision making.

Chapter 4

Phase 2: Inventory Models Proposal

As it was pointed in the State of Art chapter, the finished products inventory needs its own inventory control mechanism or just, to have any control (Simchi-Levi et al., 2008). Also, another important aspect mentioned is to do an adequate planning of the demands. In this chapter, it will be presented the proposal used to determine how to manage the final inventory, for a general type of industries, and the respective algorithm was designed in R to find optimal solutions to manage it, as a novel approach.

In the chapter three, it was demonstrated that the multivariate demands can be predicted with the Novel Bayesian Regression Model proposed and designed here. And, as it was previously mentioned, this model had a better performance than the ones shown in that chapter, and, even some models with multivariate structures. Also, a Bayesian Multivariate DLM, by Petris et al. (2009) can be a second alternative in relation to the demand forecasting. This is a preliminary phase that should be carried out in order to continue designing an adequate Inventory Policy.

A general framework to represent the inventory policies that are proposed in this Doctoral Thesis, have the next elements:

– Dynamic demand for sales periods

– Storage centers to dispose the final products for clients, or to supply.

- Capacity of storage
- Transportation of material to the storage centers.

And the cost that these aspects involve:

- Cost of the product.
- Cost of storage from one period to another.
- Cost of transportation.
- If it is the case, sometimes it implies a cost of order requirement.

A motivational context to do this work is the inventory problem of a gas station of Colombia that expends three kinds of fuel: Corriente, Extra and Diesel. The company provided information of: daily demands for each one of the fuel sold by one service station in the time interval from November 1, 2014, to January 31, 2015. They also provided the quantities of fuel that entered to the service station in the respective periods, and, the quantities left in inventory at the end of every day. In the same way, they provided a cost of the products, storage, transportation, and the price in order to estimate total profits at the end of every day in the time interval, and to do the validation with the partition of the data, the group of all the January values. The distribution of the fuels, has been detailed in a thesis of the Msc. in System Engineering of *Universidad Nacional de Colombia* (Calle, 2015), who explored an Inventory and Routing problem of the distribution of this fuel in a network of gas stations.

The demand behavior in this problem, fits to the generalities described in this Doctoral thesis proposal. As it was shown in the past chapter, the demands present autocorrelation in different lag periods, seasonality, non stationary and non normality. Besides, all the models presented, were estimated to these demand information.

The company has planned the inventories for this gas station, with an anticipated month, considering daily characteristics of the demands, according to the aspects pointed in the Elicitation section of the past chapter. That is why they prefer 30 days of planning.

4.1 Heuristic-Optimization Multiproduct Inventory Model (HOMINV)

The purpose of the Multiproduct Inventory Model proposed here is to provide an adequate inventory policy that maximizes profits, and provide orders, inventory values, costs, but also, quantities to transport and all the costs involved.

Different types of policies are going to be proposed, looking for the best alternative. In this case, the problem can require many periods, and for every one, variables as demands, final and initial inventories and orders can be required, and therefore, many variables can be required. Also, some constraints are not linear. For these reasons, we propose a heuristic to find the best possible solution for the inventory model proposal.

In this section, first, it will be presented a general mathematical model of Inventory management, to be optimized. Then, it will be proposed the schemes that contemplate different forms to do orders. After this, it will be done a modification of the original model, then, combinations of models and types of orders, to create inventory policies, and it will be designed a final heuristic that incorporates different policies into one which provides the best approximated solution. In order to choose a good form to do this plan, an experimental design will help to make a better decision related to the way of programming inventories. These policies will assume the theorems exposed in the State of Art review, from the authors Wagner and Whitin (1958) and Taha (2004), who establish optimal policies by reviewing period by period, the balance inventory constraints, for different forms to do the orders in a general dynamic algorithm, with some differences.

In this research, the objective function is related to the profits, and therefore, to the costs, because the prices and costs are considered fixed in the analyzed periods. Besides this, a transportation cost is considered here, and it will be presented more forms to do orders according to the maximum capacities of the transportation. These schemes are incorporated into a function that permits to calculate all the forms to order for a defined interval of periods, $t = 1, \ldots, T$, for every product that is considered for the Inventory policy to be provided. An algorithm for every policy is designed to provide results, in the R program.

The forecasts of the daily demand values will be inputs of this model, according to the final results of the previous chapter.

The policies, the performance of the algorithms, and a Design of Experiments, will be applied to the real case of the combustible demands of a gas station in Colombia, to provide a solution for the same periods of the last month that the company provided, in order to define if the designed policies contribute or not, to the currently company policy, or if these will give advices in order to find better profits.

4.1.1 Variables and parameters of the Proposed Inventory Models

The table 4.1 shows the parameters, decision variables or outputs related, that are part of the general model.

Variables and Parameters	
T	Periods of the horizon time
c_i	Cost of the product i ($i = 1, 2 \ldots, K$)
h_i	Cost of holding inventory for product i
Y_{it}	Demand of product i, period t (t=1, ..., T)
Ctr	Transportation cost
I_{io}	Initial Inventory for the i-th product
Cap_i	Capacity of storage for the i-th product
$Capc$	Capacity of the compartments of the cars
$Cmis_{it}$	Cost of missing values
nc	Number of compartments in every car
s	Random variable, Stock
Output Variables	
I_{it}	Final Inventory at period t
$Cars_t$	Number of cars to be sent at period t
Decision Variable	
X_{it}	Ordered quantity to be supplied at the beginning of period.

Table 4.1: Variables definitions of the proposed Inventory Model

Where $i = 1, \ldots, k$ products, and $t = 1, \ldots, T$ periods.

4.1.2 Proposed Inventory Model 1

This is the general formulation of the inventory model proposed, based on the variables definitions given in 4.1. This model will be called Model 1 (M1), for purposes of the algorithm explanations.

$$\begin{aligned} \text{Maximize z} &= \textstyle\sum_{i=1}^{k} \sum_{t=1}^{T} \{p_i * min(y_{it}, x_{it} + I_{i(t-1)})\} - \left(\sum_{i=1}^{k} \sum_{t=1}^{T} c_i x_{it} \right. \\ &+ \left. \textstyle\sum_{i=1}^{k} \sum_{t=1}^{T} h_i(I_{it}) + \sum_{t=1}^{T} Ctr * Cars_t \right) \end{aligned} \tag{4.1}$$

Subject to

An Inventory Balance Constraint, for the i-*th* product, the t-*th* period.

$$I_{it} = max\{0, I_{i(t-1)t} + x_{it} - y_{it}\} \tag{4.2}$$

A capacity constraint,

$$x_{it} \leq Cap_i \tag{4.3}$$

A number of cars,

$$Cars_t = \textstyle\sum_{i=1}^{k} \left\lceil \left| \frac{1}{nc} \left\lceil \left| \frac{x_{it}}{Capc} \right| \right\rceil \right| \right\rceil \tag{4.4}$$

All variables are greater or equal to 0.

The transportation quantities, and the respective cost, will depend on the number of cars to be used, $Cars_t$ and these cars depend on their compartments, *nc*, and these in their capacities, *Capc*, assuming equal values. When the quantity to order x_{it} is divided into *Capc*, a number of compartments will be obtained, the integer part of these compartments are divided into the number *nc*, so, the integer part of the final fraction will produce the total of cars required to take the order for every time t as it is represented in the expression (4.4). This expression is replaced in eq. (4.1).

4.1.3 Proposed Inventory Model 2

A different objective function is proposed, considering missing quantities. Also, it is aggregated a restriction that limits the inventory amount, when it is so much high.

$$Miss_{it} = max(0, y_{it} - sold_{it}) \tag{4.5}$$

For $i = 1, \dots k$, $t = 1, \dots T$.

Where $sold_{it}$ is the amount available to be sold in period t for product i.

The profits must be reduced in the cost:

$$\sum_{i=1}^{K} \sum_{t=1}^{T} (Miss_{it} Cmis_{it}) \tag{4.6}$$

- Model 2 (M2):

So, the Objective function will be the expression (4.7):

$$\begin{aligned} \text{Maximize z} &= \textstyle\sum_{i=1}^{k} \sum_{t=1}^{T} \{p_i * min(y_{it}, x_{it} + I_{i(t-1)})\} - \left(\sum_{i=1}^{k} \sum_{t=1}^{T} c_i x_{it} + \sum_{i=1}^{k} \sum_{t=1}^{T} h_i(I_{it})\right. \\ &+ \left.\textstyle\sum_{t=1}^{T} Ctr * Cars_t + \sum_{i=1}^{K} \sum_{t=1}^{T} (Miss_{it} Cmis_{it})\right) \end{aligned} \tag{4.7}$$

Subject to the same constraints that Model 1, which objective function is (4.1), but with the addition of the Limit of inventories in the expression (4.8):

If in any period, the inventory of (t-1) period is higher than 1.5 times the demand of period t, so, the orders must be reduced for the t-period. It is,

$$\text{If} \qquad I_{(i,t-1)} > 1.5 * y_{it}, \tag{4.8}$$

Then in period t, an order cannot be sent in this case.

4.1.4 Inventory Policies.

Five policies are designed and presented in this proposal, in order to find the best form to make decisions. They consist of the combination of a form to do orders, a set of constraints and an objective function, that provides the optimal solution.

In the next sections it will be introduced the Schemes to order 1, 2 and 3. One combination is, for example, Model 1 with Schemes 1, to create the policy 11 (P11). The process of this policy is explained in the figure 4.1. The Scheme 1 (S1) corresponds to a matrix with a deterministic order schemes explained in the table 4.2. The Model 1 (M1), is the same inventory model explained in eqs. (4.1, ..., 4.4). The matrix S1 calculated with the previous forecasts of the demands, is then read by the function that calculates constraints according to a Policy that provides a solution.

Figure 4.1 considers the general process of the first policy 11 (P11).

The process of the Policy 12, that combines the Model 1 with Schemes 2, is explained in the figure 4.2.

In total, the policies will be: Policy 11 (P11), Policy 12 (P12), Policy 21 (P21), Policy 22 (P22), Policy 13 (P13).

4.1.5 Algorithms to find the optimal solution

The optimal form to order will be a vector x_{it} that will be chosen with an algorithm based on 4.9, that will depend on the r-*th* position of orders.

Doing x_{itr}, the r-*th* vector of orders, then, this can be represented in general by 4.9, but, the vectors are obtained with the table 4.2, explained below.

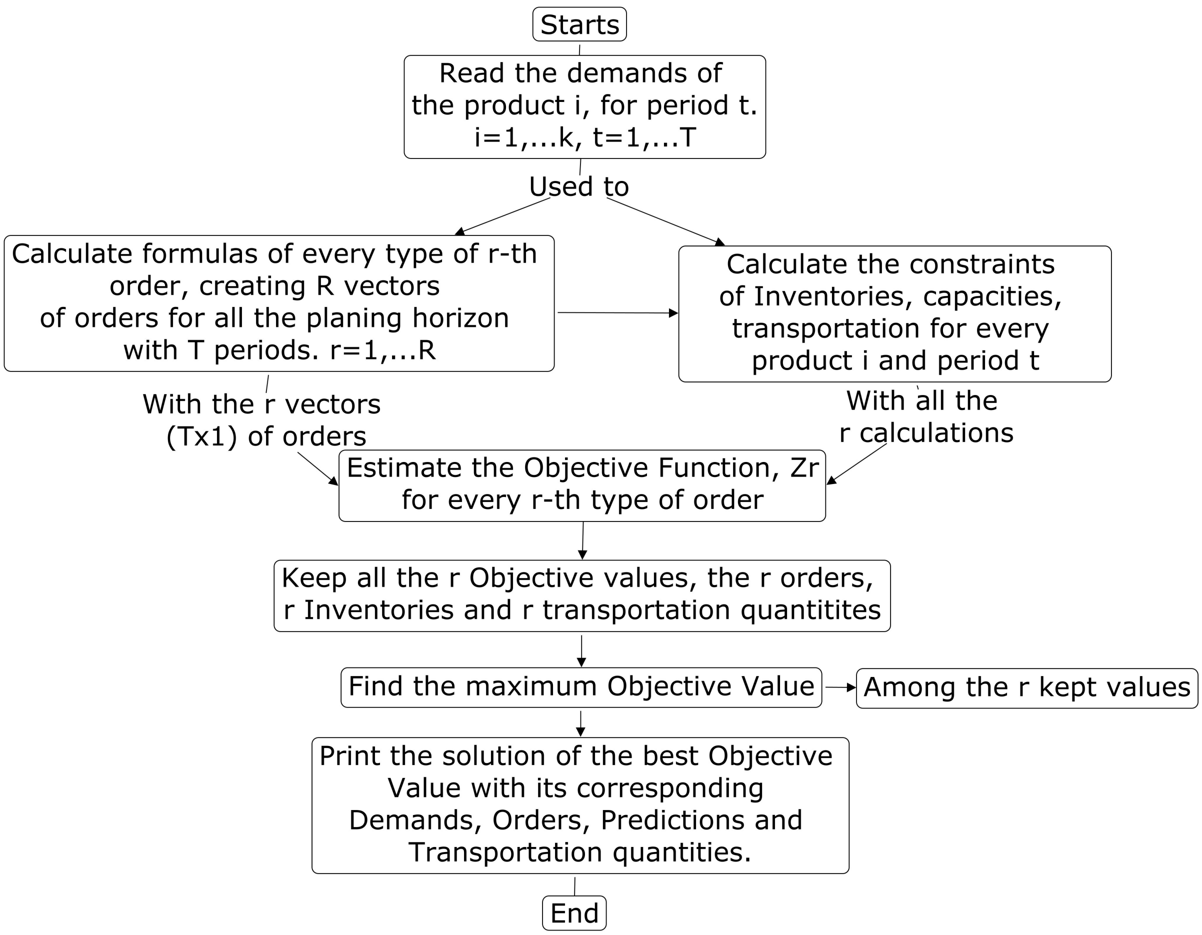

Figure 4.1: Process of Policy 11 (P11)

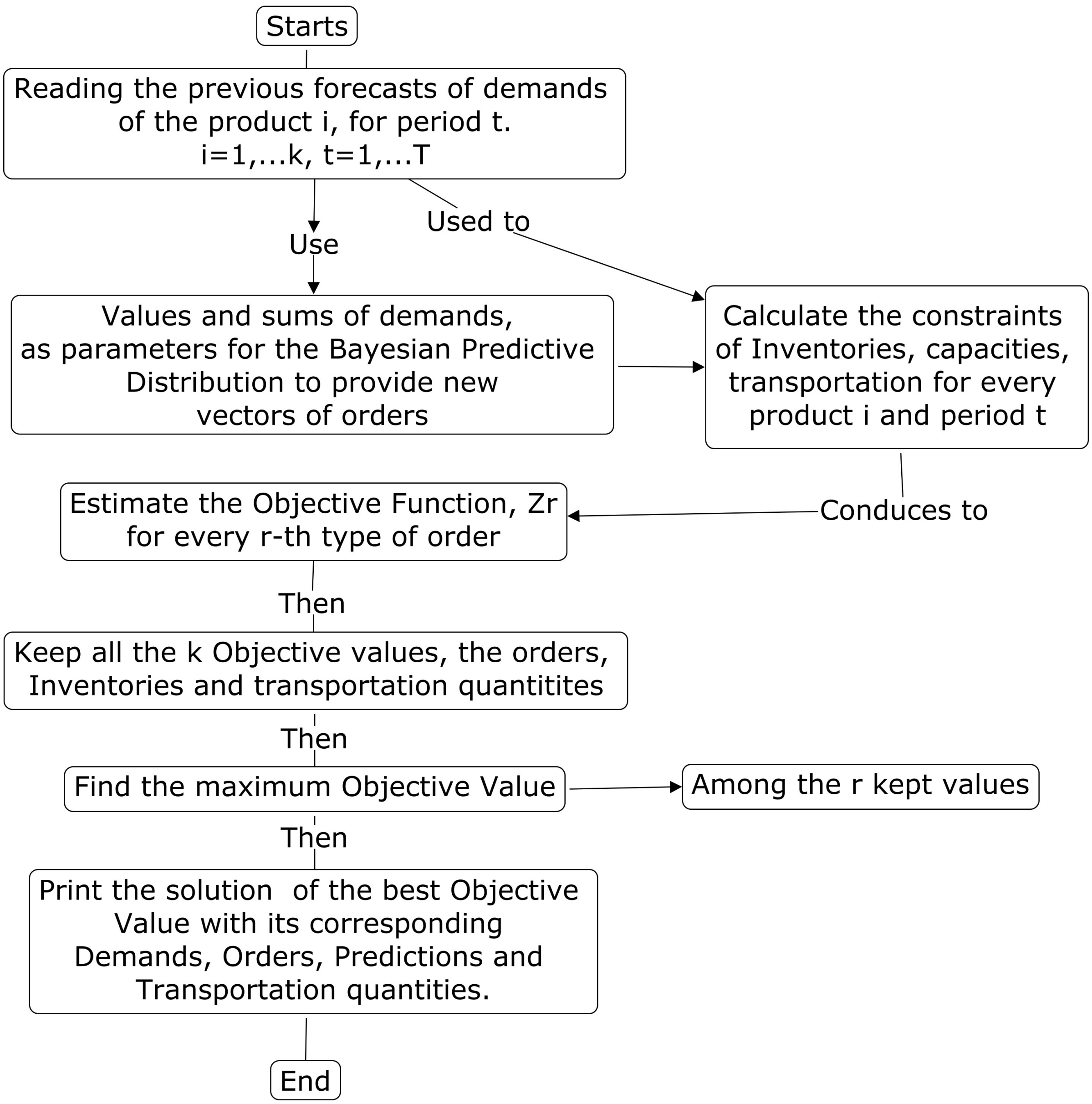

Figure 4.2: Process of Policy 12 (P12)

$$x_{itr} = \begin{cases} 0, \\ \sum_{t=p}^{T} Y_{it} \\ Y_{it} + s \end{cases} \tag{4.9}$$

- Here, r is the number of the type of order, that is a vector with T elements, calculated in a deterministic form (in the table 4.2), $r = 1, \ldots, R$. The vector of orders will be replaced in the formulas of the constraints of the Inventory Model, in order to detect which one of these produces the best results, in a r-*th* position. So, the maximum of Z, is the max of all the R values calculated. Then, the optimal of the objective function will be $Z = max\{Z_1, \ldots, Z_R\}$, and it must give one optimal solution, the values of x_{it} that maximizes Z, from the x_{itr} generated, and also one I_{it}, and one $Cost$. The figure 4.1 can help to understand the process.
- Doing $sold_{itr} = min(y_{it}, x_{itr} + I_{i(t-1)r})$, the quantity of effective sales of product i at the period t, for the r-*th* type. The optimal solution of this quantities will be part of the final optimal solution when a maximum of Z, from the R values, are chosen.

Schemes to order 1

This first scheme of orders is called Schemes 1 (S1) in this proposal. It was formulated with a base on the theorem 2 of (Wagner and Whitin (1958), p. 91), as in the State of the art was shown, that affirms: *"there exists an optimal program such that for all t"*:

$x_t = 0$ or $x_t = \sum_{j=t}^{k} d_j$, for some k, $t < k < N$

The Proof was presented in the State of Art chapter. So, this theorem is the base to know that an optimal policy will be achieved if the orders formulation are based on the values: zero, or the sum of demands in a such a way that the order contemplates all the demands left of the time interval until the last, or, that this sum discounts one by one, the demands, until it is obtained the demand value of the t-*th* period. This is also explained by Taha (2004) (p. 448), called the Inventory Dynamic Model with preparing cost. It also uses the initial formulation of the inventory model proposed by Wagner and Whitin (1958).

Table 4.2 consigns, which in this Doctoral Thesis proposal is called, S1, or Schemes 1, to be used in the first policy of the inventory model proposed.

Order type	1	2	3	4	5	...	t	...	T
1	$\sum_{t=1}^{T} Y_t$	0	0	0	0	...	0	...	0
2	Y_1	$\sum_{t=2}^{T} Y_t$	0	0	0	...	0	...	0
3	Y_1	Y_2	$\sum_{t=3}^{T} Y_t$	0	0	...	0	...	0
				...					
	$\sum_{t=1}^{2} Y_t$	0	$\sum_{t=3}^{4} Y_t$	0	$\sum_{t=5}^{6} Y_t$	...	...	...	0
⋮	Y_1	Y_2	Y_3	Y_4	Y_5	...	...	$\sum_{t=t-2}^{T-1} Y_t$	0
R	Y_1	$\sum_{t=2}^{3} Y_t$	0	Y_4	$\sum_{t=5}^{6} Y_t$	0	...	...	0

Table 4.2: Schemes 1. Types to order in Inventory Model

Explanations of table 4.2.

1. To order in the first period all the demands estimated for the planning horizon (first line of table 4.2).
2. To order in the first period only the demand for period 1 and in the second period, the sum of the remaining demands.
3. To order in the first two periods the respective demand for the day, and in the third one, the sum of remaining demands.
4. ...Continue successively, until ordering all the periods of the time interval. So, until here, these quantities depend on how many periods are fixed in the problem.
5. To order every two periods in this way: in the first, order the demand, in the second, to order for two periods (second and third), and without ordering on the third, then to order the demand in the fourth, and in the fifth, to order for two periods, ..., until $T - 1$, and not to order at the final period.

6. To order every three periods, with the third, being the sum of two demands.
7. To order every four periods.
8. To order every five periods.

As an additional aspect, studying the solutions of the algorithm, the Economic Order Quantity (EOQ) was never chosen as an optimal solution for this problem, then, it was discarded from the types of orders.

Schemes to order 2

The second scheme, S2, uses the demands as means of the predictive Bayesian distribution, and the sums of the schemes 1 as limits, to allow the generation of orders, with a Bayesian process explained in next subsection. These demands are incorporated into the formulas of S1, given in the table 4.2.

Bayesian process to obtain the predictive distribution

This is the procedure Schemes 2 (S2). The assumptions of this process will be:

– The data distribution is uniform ($Data \sim unif(a_1, b_1)$). Let "μ", the mean of this distribution, with a prior, Truncated Normal Distribution with parameters μ_0: mean; σ_0: standard deviation, that will be assumed to be constant; a: inferior limit; and b: superior limit.

$$\xi(\mu) = \frac{f(\mu|data)}{(\phi(\mu,\sigma,b,x) - \phi(\mu,\sigma,a,x))} = \frac{1}{\sqrt{2\pi}\sigma_0}\frac{e^{-\frac{(\mu-\mu_0)^2}{(2\sigma_0^2)}}}{(F(b)-F(a))} \propto e^{-\frac{(\mu-\mu_0)^2}{(2\sigma_0^2)}} \tag{4.10}$$

It results to be proportional to the constants (eq. 4.10), which do not depend on the parameter "μ". As it was mentioned in the State of Art chapter, the product between the prior distribution and the likelihood function will lead to a posterior distribution. In this section, this posterior distribution will be for the mean of the uniform distribution.

- Final equation of the posterior distribution is eq. (4.11). It does not depend on the parameters, and σ_0 is assumed to be constant.

$$\xi(\mu, \sigma_0|data) = \xi(\mu, \sigma_0) * \frac{n}{(b_1 - a_1)}$$

$$\xi(\mu, \sigma_0|data) \propto \frac{n}{(b_1 - a_1)} * \frac{f(\mu, \sigma_0|datos)}{(F(b) - F(a))} \propto \frac{1}{\sigma_0} e^{-\frac{(\mu-\mu_0)^2}{(2\sigma_0^2)}} \tag{4.11}$$

- The predictive Bayesian distribution (in eq. (4.12)), is used to calculate the orders, after the mean is updated by using the posterior (4.11) for every time t. Previously, the demands are read, and with these, their sums are calculated, according to the explanations of schemes 1.

$$P(X_{t+1}|x) \propto \int_{-\infty}^{\infty} \frac{1}{(b_1 - a_1)} * \frac{1}{\sigma_0} e^{\left(\frac{-(\mu-\mu_0)^2}{2\sigma_0^2}\right)} d\mu = \frac{\sqrt{2\pi}}{(b_1 - a_1)} \propto \frac{1}{(b_1 - a_1)} \tag{4.12}$$

Example of Inventory Policy 12 applied to the combustibles inventories of the Colombian gas station.

The plot between the profits and costs applied to the products of Corriente, extra and Diesel, that are produced by the Policy 12 (P12), with the Model 1 and the Schemes 2, is presented in the figure 4.3.

The policy will be shown here for 15 days of forecasts, as an example.

The information of the orders of every product appears in the final solution of the algorithm. The solutions of the fuel Corriente are shown in the table 4.3. In the complete solution, it is

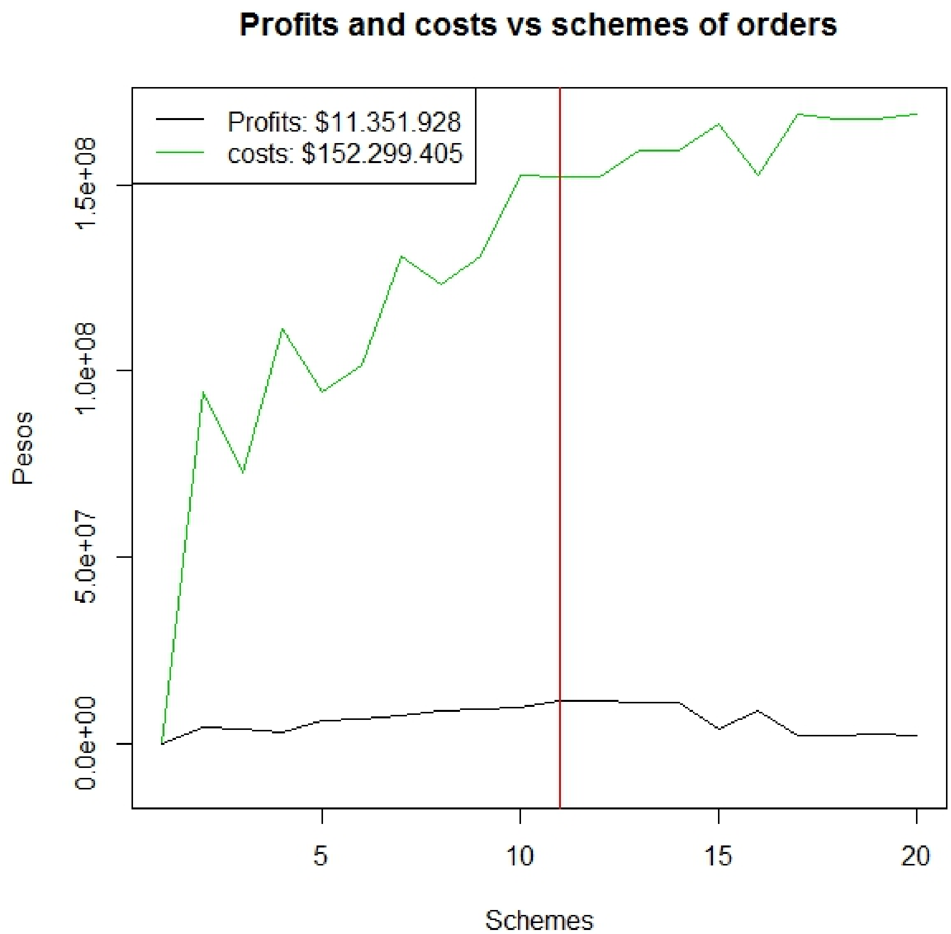

Figure 4.3: Profits vs costs. 15 periods. With no Initial Inventories, and P12.

Day	Demand	Order	Cost	Inventory	Income	Profit (not minus transport)
1	1096.86	0.00	63635.24	903.14	8729908.74	8666273.50
2	1120.79	2295.11	16317709.72	2077.46	8920343.73	-7397365.99
3	1049.42	0.00	72435.64	1028.04	8352349.70	8279914.06
4	845.77	2393.34	17044941.71	2575.61	6731457.17	-10313484.55
5	886.88	0.00	118987.72	1688.73	7058708.96	6939721.24
6	896.58	0.00	55814.63	792.15	7135886.59	7080071.95
7	931.68	2376.46	16902179.84	2236.93	7415235.55	-9486944.29
8	945.68	0.00	90981.72	1291.25	7526648.02	7435666.30
9	1027.03	2373.27	16907909.76	2637.49	8174170.77	-8733738.99
10	1074.24	0.00	110146.31	1563.25	8549910.38	8439764.07
11	909.12	0.00	46089.77	654.13	7235679.71	7189589.94
12	948.89	2386.12	16959936.40	2091.35	7552213.92	-9407722.49
13	938.90	0.00	81201.72	1152.45	7472725.79	7391524.08
14	940.50	2449.78	17448705.30	2661.73	7485448.25	-9963257.04
15	938.01	0.00	121453.44	1723.72	7465619.20	7344165.76
16	1018.88	0.00	49662.98	704.84	8109285.82	8059622.84
17	1087.64	2377.57	16892898.82	1994.77	8656529.94	-8236368.88
18	948.50	0.00	73720.14	1046.27	7549096.38	7475376.24
19	982.90	2396.98	17062501.74	2460.35	7822891.55	-9239610.19
20	967.18	0.00	105208.93	1493.17	7697798.35	7592589.43
21	956.22	0.00	37833.34	536.95	7610591.59	7572758.25
22	946.34	2457.76	17461718.26	2048.37	7531904.14	-9929814.12
23	1022.48	0.00	72284.51	1025.89	8137897.63	8065613.12
24	1098.53	2491.06	17722376.20	2418.42	8743185.94	-8979190.25
25	979.35	0.00	101397.07	1439.07	7794633.12	7693236.05
26	1008.56	2446.80	17442919.00	2877.31	8027167.24	-9415751.76
27	990.97	0.00	132911.43	1886.34	7887158.88	7754247.45
28	974.13	0.00	64274.17	912.21	7753107.83	7688833.66
29	960.41	2389.66	17002542.31	2341.46	7643926.27	-9358616.04
30	1030.99	0.00	92335.37	1310.47	8205670.10	8113334.73
31	1109.21	0.00	14180.33	201.25	8828213.53	8814033.20

Table 4.3: Application of Policy 22, 31 periods. Initial Inventory 212 fuel Corriente

possible to see the quantities: demands, final inventories, costs, Profits, number of cars, and transportation costs, for all the kinds of fuel. The total profits are \$59′030.896.

It can be seen that the orders are not sent in all the periods, and the costs and profits can be obtained, according to the previously demand forecasts, read by the algorithm. The complete table of the appendix C shows the results for the three fuels.

4.1.6 Design of experiments for Inventory Policies

The experimental design will help in the decision making in order to get solutions in a better form. It is necessary to define the factors as: initial inventories, and the policies to be contrasted. The experimental design will be factorial, because all the levels will be considered, and there are not a big quantity of factors to be controlled, in order to do a fractional design.

The levels of the factors and the response variable to be considered in the factorial design, will be:

- Factor Initial inventories (Io): a. No initial inventories (Noinv); b. Low initial: 1000, 80, 300 gallons (Io183) for Corriente, Extra and Diesel fuels, respectively; c. High inventories: 2000, 1000, 2000 (Io212), for Corriente, Extra and Diesel fuels, respectively.
- Factor Policy: P11, P12, P21, P22, P13. And the fixed values will be the costs, prices, and periods as it will be seen in the next three designs of experiments.
- The response variable of the DOE is: Profits of the sales of all the three products in all the time horizon, in Colombian pesos.

The figure 4.4 represents the general process that helps to find the best possible Inventory Policy, and the form to do the design of experiments, for these kinds of industries, at the end of its final internal logistic chain.

For the Schemes 3 in the Policy 13 (P13), the order is replaced by the total capacities in integer quantity of the corresponding compartment. For example, if an order is 150, then the order is replaced by 1000.

The differences in initial inventories and the type of policy can lead to different profits amounts. The periods obviously will give different quantities, because of the costs and incomes.

Descriptive Statistics of profits

It is possible to see that each one, P12 and P22, produces apparently, high levels of Profits, compared with the others, but the variability must be due to the initial inventory (Io). It will

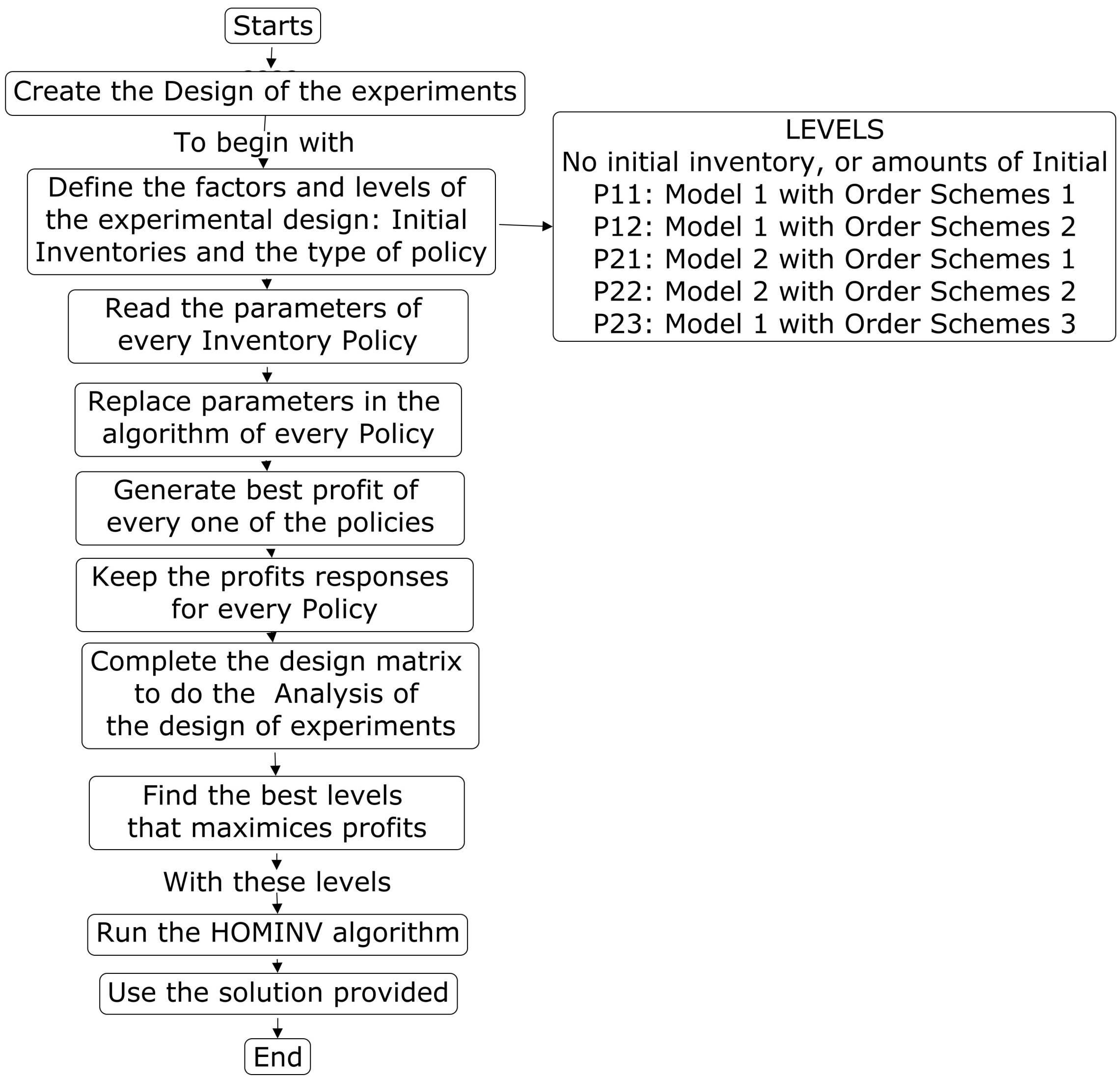

Figure 4.4: Process of the Design of Experiments of Inventory Policies.

be verified in the posterior analysis.

	7	15	31
p11	14.100.320,26	20.797.878,00	33.208.490,68
p12	17.120.194,24	28.143.593,33	39.139.063,30
p13	12.121.710,90	12.839.849,33	15.610.035,60
p21	14.842.692,76	13.975.275,50	20.151.014,23
p22	16.447.066,23	27.151.451,33	37.615.205,90

Table 4.4: Means of profits by periods and by policies

The profits are very different according to every time horizon (see table 4.4), this is the reason to do different Design of experiments, according to every interval of periods to plan (7, 15 or 31).

Design of experiments - Inventory policy for 7 days of planning

For 7 days of planning, the figure 4.5 shows that the initial inventory in zero level produces lower profits, as it is expected, because it is necessary to buy products and to pay transport at the beginning of the planning horizon. This implies that the industry can analyze different forms to plan inventories depending on the initial quantities, the budget to buy products, the costs of these, and their holding capacities.

This behavior for the levels of policies is similar, but they have different scales of values, for every interval of periods analyzed, as it will be shown in next section.

The ANOVA Table 4.5 estimates the significance of the factors; Initial Inventories (Io), and Policy, on the response, profits. With a significance level of 10%, the factor policy does not have significance,because the p-value is higher than this level.

Due to the variability behavior of these data, it was necessary to do a transformation over the response $y* = y^{(1.5)}$. In the Anova table (4.5), with the transformed response, it is possible

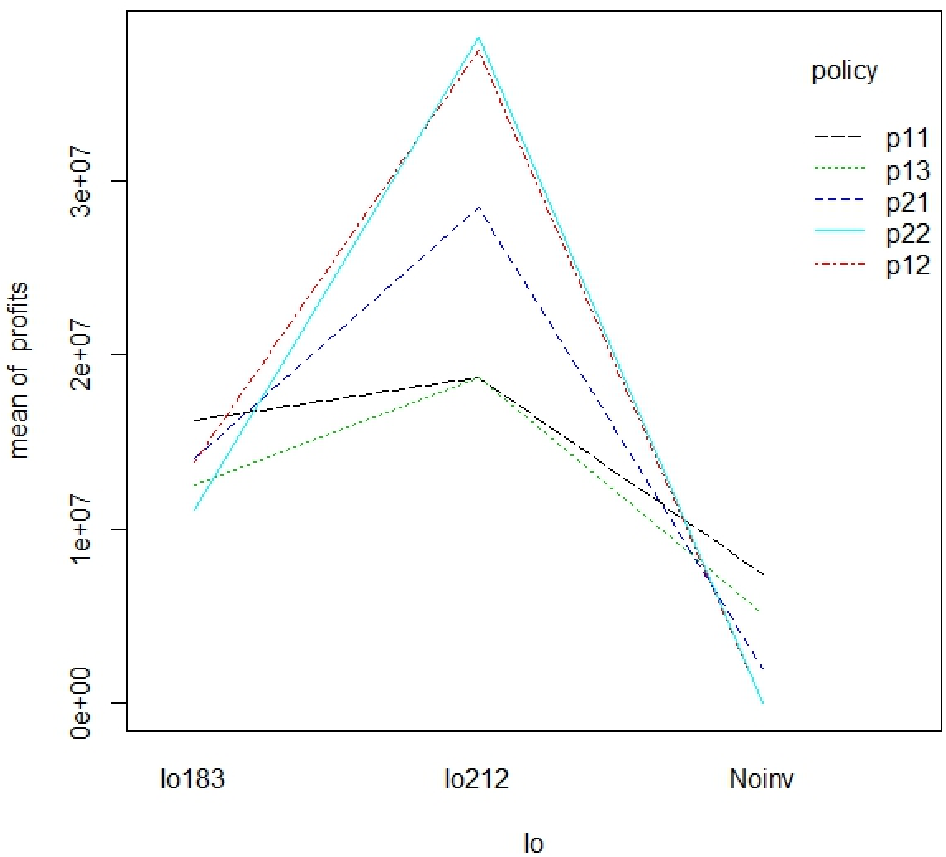

Figure 4.5: Interaction of Policy and Initial Inventory. Response Profits. 7 periods.

	Sum Sq	Df	F value	Pr(>F)
(Intercept)	4.033×10^{21}	1	2.51	0.1278
Io	9.385×10^{22}	2	29.25	0.0000
policy	14.0285×10^{21}	4	2.19	0.1056
Residuals	33.687×10^{22}	21		

Table 4.5: Anova Table for response profits of Inventory Models. 7 days

to identify the significant impact of the initial inventories, with a p value lower than the level 5%. The policy is efficient enough as to include it in the analysis.

The table for effects (4.6) shows that, to have amounts in the initial inventories, produces more profits. The policies P12, and P22, contribute significantly, to the increasing of the profits.

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
(Intercept)	30948371551.2	19518754361.67	1.59	0.1278
Io212	105406920317.8	17911637424.4841	5.88	0.0000
IoNoinv	-32039307343.8	19518754361.67	-1.64	0.1156
Policyp12	56889602229.22	26491669012.3	2.15	0.0436
Policyp13	-9790116221.89	23123824482.78	-0.42	0.6763
Policyp21	14114653492.2	23123824482.78	0.61	0.5482
Policyp22	35615610261.2	23123824482.78	1.54	0.1384

Table 4.6: Effects of the linear model for response profits of Inventory Models. 7 days

This design, with the transformed response, achieves adequately the normality and constant variance assumptions, so no transformation is necessary in order to find a better model.

The policy was considered as a blocking factor, and it does not show statistical differences, but, as the table 4.7 shows, the initial level of inventories produces significant differences, with 95% of confidence; for example, for the level Io212, this mean of profits is higher than the other two levels of initial inventories. In the figure 4.5 it was also possible to appreciate a benefit for policy P12 in the level of initial inventories 2000, 1000 and 2000 gallons, but if

	diff	lwr	upr	p adj	Decision
Io183-Noinv	41420220413.08	295304511.23	82545136314.93	0.1	DIFFERENT
Io212-Noinv	146827140730.88	105702224829.03	187952056632.73	0	DIFFERENT
Io212-Io183	105406920317.8	66633977771.24	144179862864.37	0	DIFFERENT

Table 4.7: Difference in means for response profits of Inventory Models. 7 days

there is zero inventory, the policy P11 seems to be better.

Design of experiments - Inventory policy for 15 days of planning

When a design of experiments for 15 periods is run, the two factors are also significant, with a level of significance of 5% (table 4.8).

	Sum Sq	Df	F value	Pr(>F)
(Intercept)	1553407117499104.50	1	31.36	0.0000
Io	2806927933792558.50	2	28.33	0.0000
policy	1223830156034678.00	4	6.18	0.0016
Residuals	1139438180080087.50	23		

Table 4.8: Anova Table for response profits of Inventory Models. 15 days

The policy and the Initial Inventory (Io) are significant at a level 5%.

According to the table 4.9, the policies that produce more profits are P12 and P22. And the initial inventory specifies that it is better to have initial inventories in order to have higher profits, which is a logical result, because if there are no initial amounts, the company must buy the product in the same periods planned, and this, will consequently lead to lower profits in this time interval.

For the test of differences in means (table 4.10), it can be appreciated that the Policies P11, P12, P21, P22 have higher means of profits than the policy 13 (with 95% of confidence), so

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
(Intercept)	19038425.9000	3399929.0829	5.60	0.0000
IoIo212	14387567.1000	3147722.6828	4.57	0.0001
IoNoinv	-9109210.8000	3147722.6828	-2.89	0.0082
policyp12	7345715.3333	4063692.5096	1.81	0.0838
policyp13	-7958028.6667	4063692.5096	-1.96	0.0624
policyp21	-6822602.5000	4063692.5096	-1.68	0.1067
policyp22	6353573.3333	4063692.5096	1.56	0.1316

Table 4.9: Effects of linear model for response profits of Inventory Models. 15 days

this policy P13, is not recommended. Besides this,each one of these policies, P12 and P22, also produces higher profits than P21. This can also be seen in figure 4.6.

The residuals of this design also fulfill the requirements of normality and constant variance, at 5% of significance.

Design of experiments - Inventory policy for 31 days of planning

The table 4.11, shows that the two factors, policy and Initial Inventory (Io) are significant to explain the profits.

The table for effects (4.12) shows, in a similar form than the previous design, that policies 12 and 22 produce the higher increases over the profits for the 31 days of planning period.

The results are also similar for the initial inventory (Io) factor. The initial inventory coefficients specify that is better to have more initial inventories in order to have higher profits (table 4.12).

According to the difference in means of the table 4.13, the best policies are P12 and P22, because they generate higher mean of profits, with a confidence level of 95%. But, it was found a significant equality to the policy P11, and this to P21. But, P13 does not show in any

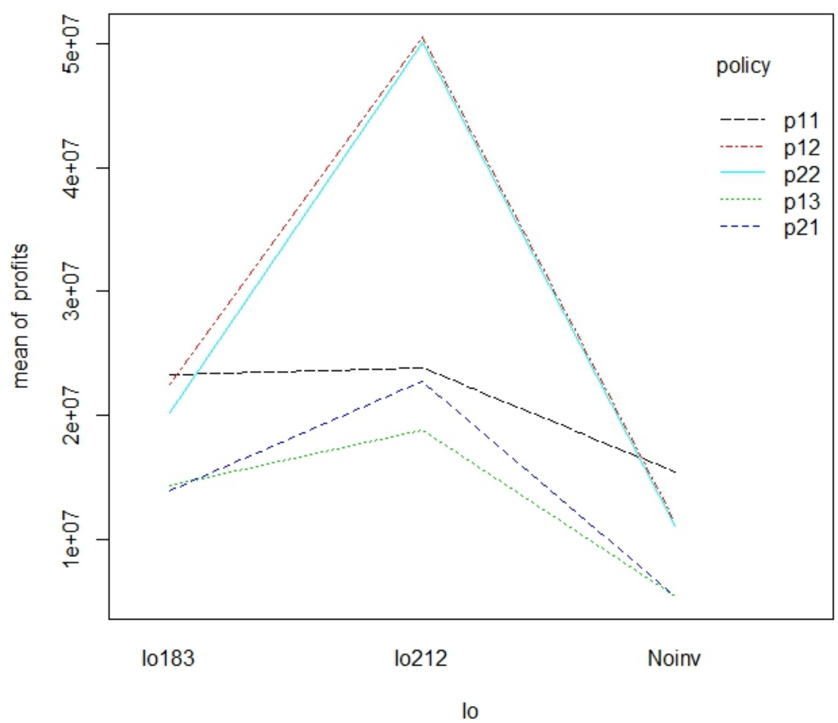

Figure 4.6: Interaction of Policy and Io. Response Profits. 15 periods.

	diff	lwr	upr	p adj	Decision
p21-p13	1135426.17	-9479972.69	11750825.03	1	EQUAL
p11-p13	7958028.67	-2657370.19	18573427.53	0.32	EQUAL
p22-p13	14311602	3696203.14	24927000.86	0.01	DIFFERENT
p12-p13	15303744	4688345.14	25919142.86	0.01	DIFFERENT
p11-p21	6822602.5	-3792796.36	17438001.36	0.47	EQUAL
p22-p21	13176175.83	2560776.97	23791574.69	0.03	DIFFERENT
p12-p21	14168317.83	3552918.97	24783716.69	0.02	DIFFERENT
p22-p11	6353573.33	-4261825.53	16968972.19	0.53	EQUAL
p12-p11	7345715.33	-3269683.53	17961114.19	0.39	EQUAL
p12-p22	992142	-9623256.86	11607540.86	1	EQUAL
Io183-Noinv	9109210.8	2328736.71	15889684.89	0.02	DIFFERENT
Io212-Noinv	23496777.9	16716303.81	30277251.99	0	DIFFERENT
Io212-Io183	14387567.1	7607093.01	21168041.19	0	DIFFERENT

Table 4.10: Difference in means for response profits of Inventory Models. 15 days

	Sum Sq	Df	F value	Pr(>F)
(Intercept)	3830608265899293.00	1	54.66	0.0000
Io	4714006932663745.00	2	33.63	0.0000
policy	2713348120579523.00	4	9.68	0.0001
Residuals	1611783122648776.00	23		

Table 4.11: Anova Table for response profits of Inventory Models. 31 days

	Estimate	Std. Error	t value	Pr(> |t|)
(Intercept)	29896631.8400	4043689.2788	7.39	0.0000
Io212	20050030.8000	3743728.8115	5.36	0.0000
IoNoinv	-10114454.2700	3743728.8115	-2.70	0.0127
Policyp12	5930572.6167	4833133.1133	1.23	0.2322
Policyp22	4406715.2167	4833133.1133	0.91	0.3713
Policyp13	-17598455.0833	4833133.1133	-3.64	0.0014
Policyp21	-13057476.4500	4833133.1133	-2.70	0.0127

Table 4.12: Effects of linear model for response profits of Inventory Models. 31 days

	diff	lwr	upr	p adj	Decision
p21-p13	4540978.63	-9745881.58	18827838.84	0.88	EQUAL
p11-p13	17598455.08	3311594.87	31885315.29	0.01	DIFFERENT
p22-p13	22005170.3	7718310.09	36292030.51	0	DIFFERENT
p12-p13	23529027.7	9242167.49	37815887.91	0	DIFFERENT
p11-p21	13057476.45	-1229383.76	27344336.66	0.08	EQUAL
p22-p21	17464191.67	3177331.46	31751051.88	0.01	DIFFERENT
p12-p21	18988049.07	4701188.86	33274909.28	0.01	DIFFERENT
p22-p11	4406715.22	-9880144.99	18693575.43	0.89	EQUAL
p12-p11	5930572.62	-8356287.59	20217432.83	0.74	EQUAL
p12-p22	1523857.4	-12763002.81	15810717.61	1	EQUAL
Io183-Noinv	10114454.27	738898.57	19490009.97	0.03	DIFFERENT
Io212-Noinv	30164485.07	20788929.37	39540040.77	0	DIFFERENT
Io212-Io183	20050030.8	10674475.1	29425586.5	0	DIFFERENT

Table 4.13: Difference in means for response profits of Inventory Models. 31 days

case a significantly higher mean to the others. So, the four policies: P11, P12, P21, P22, will be considered in the rest of this chapter, in order to plan a correct policy.

As in the other periods, the high level of initial inventories (2000, 1000 and 2000), produces the higher values of profits. It is important to consider that there will be some periods where there are no inventory, so, it is necessary to select the best possible combination of policies. Figure 4.7, represents the possible interaction between the factors: policy, and Io. It is possible to appreciate that for 31 days, the policy P11 or also, P12, could provide a good performance for the profits, followed by policy P22.

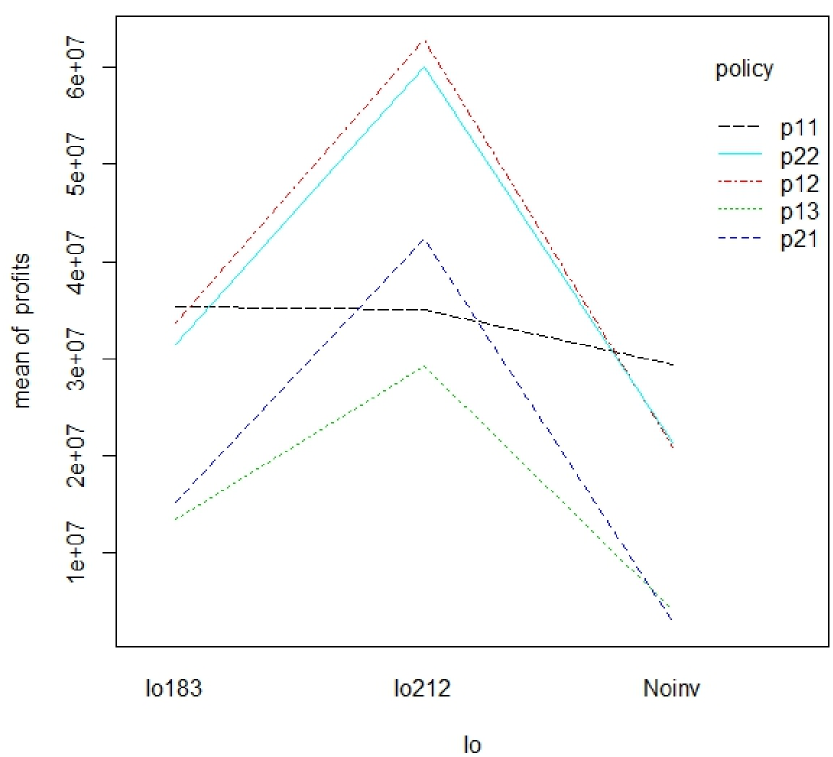

Figure 4.7: Interaction of Policy and Io. Response Profits. 31 periods.

Policy P13 produces a reduction in profits in general, as it can be seen in the figure 4.7, so, this policy should not be considered. This implies that it is not good to take cars with complete capacity in the first period.

4.1.7 Algorithm of the HOMINV

The policies corresponding to P11, P12, P22 and P21, are included in a final algorithm explained in the figure 4.8. The designed Algorithm, combines the schemes S1, S2, and models, M1 and M2, that generates the four policies explained before, P11, P12, P21, and P22. The optimal solution will give the orders, inventory holding quantities from one day to the other, transportation quantities, costs and the final profits. In order to organize the heuristic, it is necessary to define initial inventories,and read the respective demands. Every policy will produce a list of solutions, and the best of every one, will permit a comparison to find the best possible solution. The heuristic is designed in the program R, and it will provide an alternative to find an optimal solution, after defining initial inventories, and after knowing the respective demands.

4.1.8 Example - HOMINV applied to the real case of combustible inventory management.

The Model HOMINV will be applied to the real case of combustible inventory management.

The company provided, besides the demands, their inventory quantities management for the period of January, in order to do comparisons between the proposal and the current management of the company. In summary:

- Data: Daily sales of three kind of fuel of one service station: Corriente, Extra and Diesel.
- Period: November 1, 2014 to January 31, 2015.
- Horizon time of planning: The 31 days of January 2015. The month of January will be used to compare results between the proposal and the real.

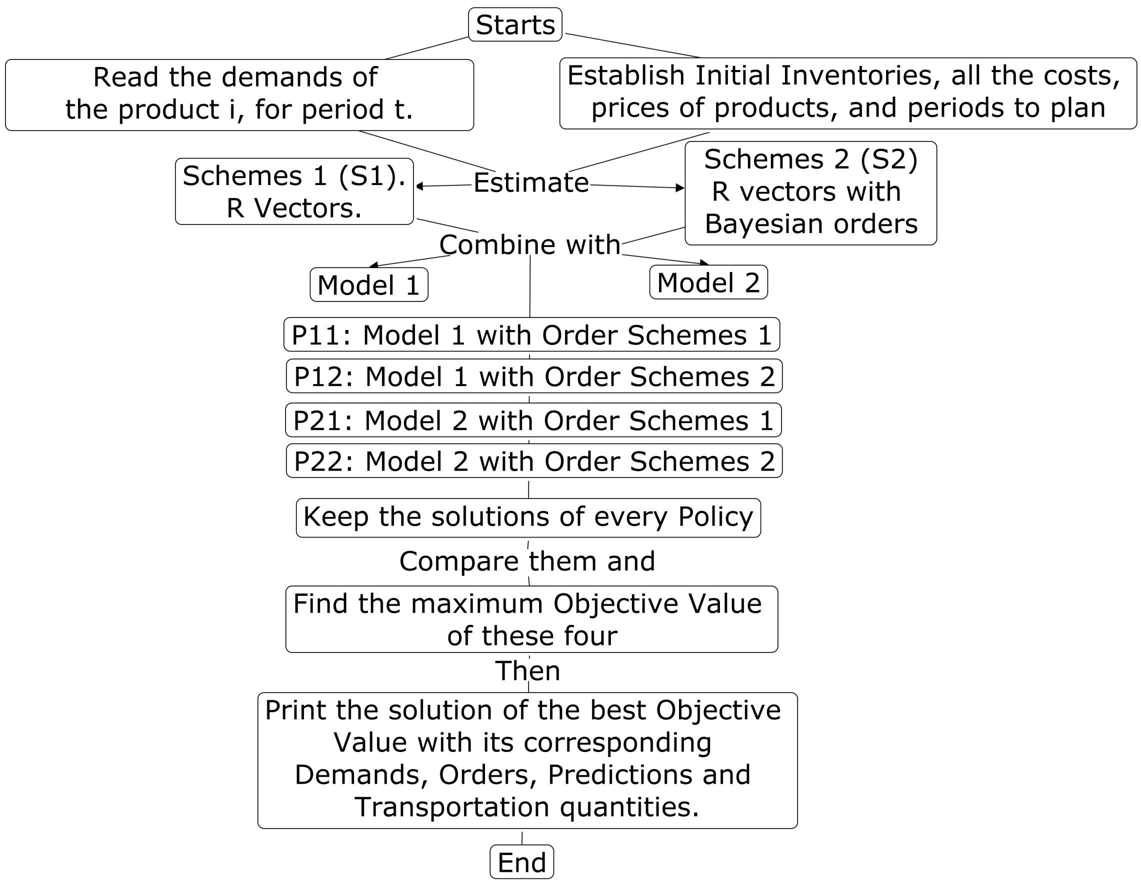

Figure 4.8: Model HOMINV.

– Results: orders, inventories, transportation quantities, profits, and costs.

Alternative

As it was shown, it is important to have an initial inventories amount, in order to have a most optimum level. But if the industry must buy the product in the horizon time of planning, then what can be done?

The algorithm HOMINV was applied for the 31 days of the January month of 2015, in order to plan the inventory management for the three combustible of this gas station. In first place, it will be applied for zero 0 inventories. After reading the demands, the proposed algorithm provides the profits about $\$29'341.414$, but it is not a correct policy, because the orders must be sent daily, which is not factible. In this case, it will be recommended that the company should do first a planning of 15 days and do a periodic review, as real values of demand be kept. The first 15 days will be considered with no inventory, and in the other half of the month (16 days), the initial inventory is the final of the first. The requirements of final inventory at the first 15 days reduces considerably the profits.

When the algorithm of HOMINV is used, the maximum possible profits is $\$ - 792.168.6$ for the 15 days, but for the 16 days, when Inventory has the initial quantities: 1000, 330, 220, for fuels Corriente, Extra and Diesel, respectively, and finishing with no inventories, the profits are $\$24'261.771, 8$ so, the total profit would be $\$23'469.603, 2$ approximately, for the month of January.

But, in order to do a validation of the algorithm, a comparison with real values is going to be estimated.

Comparison to the current situation of the combustible data.

Assume the planning of the scenario: 31 days, and the actual conditions of Initial Inventories of January for the company, which are to have: Io=(4300, 1000, 2150), (Io4,1,2), for Corriente, Extra and Diesel fuel respectively.

- With the initial inventories as the company had in the first day of january (Io4,1,2), which they must buy before, the best profits found with the algorithm HOMINV, has an amount simulated of $\$80'246.612$.
- The real values for the company in the same month of January, are $\$39'682.382,19$. When we replace the orders provided for the algorithm HOMINV, as if they were ordered in this month, we calculate the formulas of effective sold quantities, the inventory balance for every day, the costs, by adding the orders to the holding quantities and transportation, the profits, according to the found effective sales, and, the profits that the company could get, were $\$74'219.641,50$. That would be saving around $\$34.537.259,31$ (87% of the real). A part of the solution is shown in table 4.14, the rest in the appendix D.

Day	Demand	Ord	Cost	Invent	Inco	Profit
1	1096.9	0.0	225693.2	3203.1	8729908.7	8504215.5
2	1120.8	0.0	146722.6	2082.4	8920343.7	8773621.1
3	1049.4	0.0	72780.3	1032.9	8352349.7	8279569.4
4	845.8	0.0	13187.6	187.2	6731457.2	6718269.6
5	886.9	1783.5	12642653.0	1083.7	7058709.0	-5583944.0
6	896.6	0.0	13187.6	187.2	7135886.6	7122699.0
7	931.7	1877.4	13307676.8	1132.8	7415235.5	-5892441.2
	...					

Table 4.14: Best policy 31 days (Io4,1,2)

The policy provides orders for all the three products (Appendix D), but at the end of the planning horizon it returns zero inventories in the last day.

In this solution (Appendix D) it can be seen that the orders are not sent in all the periods. For Corriente fuel, the orders are sent in 13 days, for Extra Fuel, only one day, the day 21 of January. Also, for Diesel, orders are sent only 7 days in the month. The solution shows the cost, and profits for every day, and at the end, the number of cars.

Restriction of final inventories in HOMINV

A second scenario is run, with a restriction of final inventories in order to begin the next period. When the simulated orders and demands are used, the simulated profits were $\$59'343.789$, but, when the orders are used in real situation, and the effective sales are recalculated, the profits are $\$68.034.045, 77$, the amount saved is $\$28.351.663, 58$ (71% of real amount, $\$39'682.382, 19$). The complete plan is shown in the appendix E.

If we take into account the exact final inventories that the company had at the end of January, the profits are $\$51.254.540, 05$ with the same orders except in the last day, these are saving nearly 12 million of pesos, (29% of real amount, $\$39'682.382, 19$). But leaving the same amounts in the final inventory that the company had, it is not optimal, because these are quantities that are above 200% of the daily demands for every product, so it is not recommended, it only shows that the algorithm has a logical result, but the best form to do the inventory management is saving less inventories than what they had.

This results provides elements to say that the algorithm HOMINV has so much advantages, and that could be a good alternative to use, for these, and many other industries with the same frame of systems.

4.2 Multiproduct Inventory Model 2: Stochastic-Optimization Multiproduct Inventory Model (SOMINV)

Here, the inventory policy is based on the same variables and parameters than the previous model HOMINV. The purpose is also searching for maximizing profits, according to the same model formulated in previous section (4.1, 4.2, 4.3, 4.4). But as a novel approach, I propose to use the predictive Student-t distribution in order to simulate random variable associated to the demand, according to the same Bayesian Regression Model (BRM) exposed in chapter 2.

These are the general characteristics of the model SOMINV:

- The predictive distribution of the demand can be used to simulate, and after, the demand values can be replaced in the constraints and the objective function of the model, estimating these equations in a frequency (nsim) of times.
- The objective function can be calculated as an expected value, because it is a linear combination of the demands, which are random variables. The expected value of a sum of random variables, is the sum of the expected values of all the random variables, in this case, the demands.

4.2.1 Inventory Model

The general model has the same mathematical formulation as before (4.1, 4.2, 4.3, 4.4). The process to estimate it changes, by using simulation.

$$\begin{aligned} \text{Maximize E[z]} &= \sum_{i=1}^{k}\sum_{t=1}^{T}\{p_i * min(y_{it}, x_{it} + I_{i(t-1)})\} - \Big(\sum_{i=1}^{k}\sum_{t=1}^{T} c_i x_{it} \\ &+ \sum_{i=1}^{k}\sum_{t=1}^{T} h_i(I_{it}) + \sum_{t=1}^{T} Ctr * Cars_t\Big) \end{aligned}$$

Subject to

An Inventory Balance Constraint, for the i-*th* product, the t-*th* period.

$$I_{it} = max\{0, I_{i(t-1)} + x_{it} - y_{it}\}$$

A capacity constraint,

$$x_{it} \leq Cap_i$$

A number of cars,

$$Cars_t = \sum_{i=1}^{k}\left\lceil\left|\frac{1}{nc}\left\lceil\left|\frac{x_{it}}{Capc}\right|\right\rceil\right|\right\rceil$$

And the random variable is the demand, y_{it}, which will be derived from the Student-t predictive distribution of BRM.

4.2.2 Steps for Stochastic Inventory model-SOMINV

In this model, the predictive distribution of the demand is used for simulation, but also, the percentiles of the distribution are used. The next procedure was followed:

- Simulate the demand N times, getting N vectors with T dimension, of predicted y_{ij}.
- Estimate the previous policies, every N time simulated, obtaining the objective function.
- Repeat this process for different percentiles of the predictive distribution for the demand, until finding the best structure. The percentiles are used to estimate the demands with the Student-t distribution, that the Bayesian Regression Model uses, so, this will permit to have different vectors as means for the simulation with same distribution. The process will be, 1. Change percentile, 2. Forecast the Demands with BRM 3. Use previos values as means to simulate N demands every period, 4. Obtain the mean of z for all periods, T.
- Make a comparison with real values of sales and inventories left in the company. This validation will be done for the percentiles used for the simulation.

4.2.3 Results of SOMINV

The results in this section will be based on the case of a fuel Station (the same case exposed in the past sections).

It is important to remind that, as it was explained in the section of BRM, there was found a percentile of the Student-t distribution, when the optimal value for every fuel was obtained, with the criteria of minimum forecast MAPE. The table 4.15 shows the results when every level of the BRM percentiles are changed, taking a low level in 20%, the optimum is the best level found by the program of the model prediction of every product, for Corriente, 58%, for Extra, 48.7%, and for Diesel 26.5%; and the high is 75%.

The initial Inventories are fixed in three levels: No inventories, I183 (1000, 80 and 300, respectively, for Corriente, Extra, and Diesel), and I212 (2000, 1000 and 2000).

Scenarios (Percentile, %)	IO	I183	I212
Low (20)	25110552	32315088	59363402
Optimum	29655282	37807851	63113560
High (75)	36207203	42795394	67420873

Table 4.15: Profits after changing percentile in three scenarios, for 31 days, with no final inventories.

Scenarios	Low	Optimum	High
Mean of Profit	25448838.9	29781872.2	36061433.3
Sd of profit	337638.0	364928.1	571657.4

Table 4.16: Mean of simulated profits. Three percentile Simulation -Initial Inventory 0

The simulation means are very similar to the particular results, because the standard deviation is not so high, it is just around 1.4% of the mean, 1.3%, 1.2%, 1.6% respectively.

The variability for these scenarios is higher.

Scenarios	low	Medium	high
Mean of Profit	76355560.3	63404533.2	70153345.5
Sd of profit	4567661.7	17906348.2	17382167.7

Table 4.17: Three percentile simulation-Initial Inventories 4300, 1000, 2150

In a high level of inventories, when a validation with real data is estimated, the profits are lower than the optimum level scenario, 23.237.977, 05$, leading to a worse scenario than before with the most adequate prediction.

4.2.4 Synthesis of the chapter

– An Inventory dynamic model was proposed, in order to represent a system that is changing in time, but with a vectorial form to see it, and using some of the theorems and formulations to program an optimization process, described by authors as Wagner and Whitin (1958), and Taha (2004).

- The algorithm HOMINV to solve the problem, is efficient when the demand is read from another type of processes, and for different forms to fix the initial inventories. It was shown a better performance for a real case studied, providing better profits than the real.
- Solutions of the algorithm SOMINV lasts more than the HOMINV ones, and in order to have practical solutions, it can be useful to use only the percentile, because it leads to a good alternative solution. Besides this, a higher percentil scenario does not behave better than the optimum solution found with the BRM.
- The Bayesian Regression Model is the best alternative of forecasting models, for both, univariate and multivariate data analyzed, but, another alternative, corresponds to a Multivariate Bayesian Dynamic Linear Model with seasonal and polynomial components, designed by Petris in program R (Petris et al., 2009). The best prediction found with the comparative algorithm designed in this Doctoral Thesis, can be read by the HOMINV, to do the optimization process of the novel inventory model, because this inventory model permits the input of a previous predicted demand, independent of the model.

Chapter 5

Conclusions

This Doctoral Thesis proposes a novel model to do forecasts, using few data, and for short term: a Bayesian Regression Model with Normal prior distribution and dynamic innovation in prior parameters (BRM), designing the algorithm in the program R. It is compared to other models with a simulation process, and it was applied to a real case of combustible demands of a Colombian gas station, obtaining very good performance of this model.

A novel Inventory dynamic model (HOMINV) was also proposed, with efficient performance, which optimization is based on a dynamic algorithm. The optimal solution for the inventory model provides the quantities to order, store and transport in an appropriate time horizon for the industry, by maximizing profits.

It is also developed a methodological proposal applied to an industry, with an experimental design, to find the best form to make a correct Inventory policy, based on the results of the designed Algorithm in R program, according to the initial conditions of the industry.

Another novel stochastic model was proposed (SOMINV), but the computational time is not as fast as model HOMINV to provide solutions, and HOMINV provided good results for the real case where it was applied. The proposed models, HOMINV and SOMINV, represent a dynamic system that also use principles to program a dynamic optimization process, previously proposed by Wagner and Whitin (1958) or by Taha (2004), and it was designed in the program R.

The algorithm to propose the Inventory Management, was explained with detailed steps that can be reproduced with different programs. It was shown to be efficient when it was applied to different scenarios of initial inventories. It also, can read demands from different possible forecasts previously done. The performance demonstrated better profits compared to the current situation of the combustible demand of the gas station in Colombia.

In the process of this research, a State of The art was built, where different variables, models, advance techniques were studied, providing a base for the high advantages of the Bayesian Models techniques to help with prediction. These techniques also can help with optimization process, in especial, for the Dynamic Model of the Inventory Optimization proposed, helping to the organization to face to the nowadays drastic variations.

Then, many of the variables involved with end of the supply chain, like demand, prices, storage levels, among others, were studied.

A review of state-space models that have been analyzed and described in the thesis, mentioning Dynamic Linear Models, and also, the approach of some authors of the management of Dynamics in Inventory problems (Wagner and Whitin, 1958; Taha, 2004; Ventura et al., 2013), and also, theory Control. It was necessary to present theory about State Space representations in the frame of Dynamic Bayesian Linear Models, proposed by West and Harrison (1997), models that permit the updating system response, and prediction one step at a time.

Different authors have proposed indicators to do comparisons among these models and others like bayesians, like MAD, RMSE, used in many historical works when a model is fitted to time series data. Forecast MAPE values has not been so much explored, but bayesian authors like Petris et al. (2009), use it as a criteria to choose one model or other to forecast.

It was necessary to compare among univariate and multivariate models in order to measure the forecast capacity they had, with an adequate form to simulate time series data, with algorithms that were designed in R. And, as it was proved, the univariate performance of the Bayesian Regression Model, was one of the best for almost all the simulated cases, or real data. This output becomes input for the inventory model proposed, designed here for a multiproduct system.

Different papers were written around this research, or with an associated analysis, the tittles can be seen in the Appendix F.

5.1 Future lines

- The solutions of the Inventory Models like the proposed, can also be compared with techniques as metaheuristics as Tabu, ACO, Genetic Algorithms, and also, a theory Control technique can be proposed. In the Theory Control Line, it is possible to do an extension of the proposed optimization shown in (Wang et al., 2005), where it is considered a minimization of deviations in relation to a required level, and a Kalman Filter is a tool in order to proceed with the optimization of a Dynamic Linear Model with the output I_t.
- Models like Neural Networks are not very related to this work, because they require a lot of data, but these relatively new models: Generalized Additive Models (GAMs) could be compared with the proposed here.
- Changes in the prior distribution of the parameters every period can be considered. Also, a change in distribution of data could be proposed.
- Possible process to improve the optimization: Fix percentiles for the predictive Bayesian distribution, simulate vectors of demands for every percentile, and find a maximum level of the profits in the sample, using the limits of a confidence interval. The maximum limit or bound of this interval could be the new Objective. This could also be possible for simulated orders, instead of demands.
- This work is related with the new tendency of Operational Research, called Analytics, because this has a connection established among: Statistical description, prediction, simulation and optimization, involving many other lines of research.

5.2 Appendix

Appendix A

Code of MDLM1 in program R

The program code to do the estimation of the MDLM1 is shown below.

Rcode-Linear Growth Model-Multivariate Bayesian Dynamic Linear Model-MDLM1

```
## R code, Multivariate DLM-Linear Growth with filtering####
mod <- dlmModPoly(2)
mod$FF <- mod$FF %x% diag(3)
mod$GG <- mod$GG %x% diag(3)
W2 <- mcovar #Covariance Matrix of data, as a seed.
W1=W2
mod$W <- bdiag(W1, W2)*0.01
V <- mcovar #Covariance Matrix of data
mod$V <- V
mod$m0 <- c(mean(serie3co[,1]),0,mean(serie3co[,2]),0,mean(serie3co[,3]),0)
mod$C0 <- diag(6) * 1e3
mod <- as.dlm(mod)
simuFilt <- dlmFilter(serie3co, mod)
forecasted=dlmForecast(simuFilt,periods)
```

```
fore3=forecasted$f

sdev <- residuals(simuFilt)$sd
sdev2=sdev[1:periods,]
lwr <- rbind(simuFilt$f,fore3) + qnorm(0.25) * rbind(sdev,sdev2)  #qnorm es negativo
upr <- rbind(simuFilt$f,fore3) - qnorm(0.25) * rbind(sdev,sdev2)
```

Appendix B

Code of MDLM2 in program R

The program code to do the estimation of the MDLM2 is shown below.

Rcode-Linear Growth Model plus seasonal components-Multivariate Bayesian Dynamic Linear Model-MDLM2

```
#####Seasonal and Linear Growth####
mod2=dlmModPoly(1)+dlmModSeas(7)
mod2$FF <- mod2$FF %x% diag(3)
mod2$GG <- mod2$GG %x% diag(3)
W1=diag(0,3)
W2 <- diag(rep(c(diag(mcovar)[1],mcovar[2],mcovar[3],diag(mcovar)[2],mcovar[6],diag(mcovar)[
W2[1, 2] <- W2[2, 1] <- mcovar[2]
W2[4, 5] <- W2[5, 4] <- mcovar[3]
W2[5, 6] <- W2[6, 5] <- mcovar[6]
W2[10, 11] <- W2[11, 10] <- mcovar[2]
W2[11, 12] <- W2[12, 11] <- mcovar[3]
W2[16, 17] <- W2[17, 16] <- mcovar[6]
```

```
W2[17, 18] <- W2[18, 17] <- mcovar[2]
mod2$W <- bdiag(W1, W2)*0.1
V <- mcovar #Covariance Matrix of data
mod2$V <- V
mod2$m0 <- c(colMeans(serie3co),rep(0,18))
mod2$C0 <- diag(21) * 1e2
mod2 <- as.dlm(mod2)
##
simuFilt2 <- dlmFilter(serie3co, mod2)
forecasted=dlmForecast(simuFilt2,periods)
fore3=forecasted$f

sdev <- residuals(simuFilt2)$sd
sdev2=sdev[1:periods,]
lwr <- rbind(simuFilt2$f,fore3) + qnorm(0.25) * rbind(sdev,sdev2)  #qnorm es negativo
upr <- rbind(simuFilt2$f,fore3) - qnorm(0.25) * rbind(sdev,sdev2)
#una
par(mfrow=c(3,1),cex=0.5)
plot(series3[,1], type='o', ylim=c(min(series3[,1])-80, max(series3[,1])+20),
main='Real vs Adjusted-series1')
lines(window(c(simuFilt2$f[,1],fore3[,1]), type='o', lty=2, pch=4),col=2,pch=4,lty=2,type='o
lines(lwr[,1],col='grey')
lines(upr[,1],col='grey')
legend("bottomleft", inset = 0.05,
       legend=c("Observed", "One-step-ahead forecast", "50\% prediction interval"),
       pch=c(1,4,1), lty=c(1,2,1), col=c(1,2,'grey'), bty='n')
abline(v=n-periods,col='blue',lty=2, pch=4)
#dos
plot(series3[,2], type='o', ylim=c(min(series3[,2])-80, max(series3[,2])+20), xlab="", ylab="
lines(window(c(simuFilt2$f[,2],fore3[,2]), type='o', lty=2, pch=4),col=2,pch=4,lty=2,type='o
lines(lwr[,2],col='grey')
```

```
lines(upr[,2],col='grey')
legend("bottomleft", inset = 0.05,
       legend=c("Observed", "One-step-ahead forecast", "50% prediction interval"),
       pch=c(1,4,1), lty=c(1,2,1), col=c(1,2,'grey'), bty='n')
abline(v=n-periods,col='blue',lty=2, pch=4)
#tres
plot(series3[,3], type='o', ylim=c(min(series3[,3])-80, max(series3[,3])+20), xlab="", ylab="
lines(window(c(simuFilt2$f[,3],fore3[,3]), type='o', lty=2, pch=4),col=2,pch=4,lty=2,type='o
lines(lwr[,3],col='grey')
lines(upr[,3],col='grey')
legend("bottomleft", inset = 0.05,
       legend=c("Observed", "One-step-ahead forecast", "50\% prediction interval"),
       pch=c(1,4,1), lty=c(1,2,1), col=c(1,2,'grey'), bty='n')
abline(v=n-periods,col='blue',lty=2, pch=4)
resid <- residuals(simuFilt2, type = "raw", sd = FALSE)
M1_aj=mean(abs(resid[,1])/series3[1:corte,1])
M2_aj=mean(abs(resid[,2])/series3[1:corte,2])
M3_aj=mean(abs(resid[,3])/series3[1:corte,3])
map_adjus=c(M1_aj,M2_aj,M3_aj)
M1_fo=mean(abs(fore3[,1]-series3[(b:n),1])/series3[(b:n),1])
M2_fo=mean(abs(fore3[,2]-series3[(b:n),2])/series3[(b:n),2])
M3_fo=mean(abs(fore3[,3]-series3[(b:n),3])/series3[(b:n),3])$
```

Appendix C

Policy P22, (Io212), all products.

Day	Demand-cor	ord	Cost	Invent	Inc	Pro-notr	Demand-ext	ord	Cost	Invent
1	1096.86	0.00	63635.24	903.14	8729908.74	8666273.50	94.71	0.00	84391.43	905.29
2	1120.79	2295.11	16317709.72	2077.46	8920343.73	-7397365.99	95.50	0.00	75488.89	809.79
3	1049.42	0.00	72435.64	1028.04	8352349.70	8279914.06	61.23	0.00	69781.37	748.57
4	845.77	2393.34	17044941.71	2575.61	6731457.17	-10313484.55	26.82	0.00	67281.16	721.75
5	886.88	0.00	118987.72	1688.73	7058708.96	6939721.24	1.27	0.00	67162.35	720.47
6	896.58	0.00	55814.63	792.15	7135886.59	7080071.95	26.15	0.00	64725.11	694.33
7	931.68	2376.46	16902179.84	2236.93	7415235.55	-9486944.29	38.68	0.00	61119.25	655.65
8	945.68	0.00	90981.72	1291.25	7526648.02	7435666.30	42.87	0.00	57123.19	612.78
9	1027.03	2373.27	16907909.76	2637.49	8174170.77	-8733738.99	64.06	0.00	51151.58	548.72
10	1074.24	0.00	110146.31	1563.25	8549910.38	8439764.07	66.53	0.00	44950.11	482.19
11	909.12	0.00	46089.77	654.13	7235679.71	7189589.94	49.98	0.00	40290.63	432.21
12	948.89	2386.12	16959936.40	2091.35	7552213.92	-9407722.49	26.46	0.00	37823.79	405.75
13	938.90	0.00	81201.72	1152.45	7472725.79	7391524.08	41.58	0.00	33947.34	364.16
14	940.50	2449.78	17448705.30	2661.73	7485448.25	-9963257.04	43.51	0.00	29890.95	320.65
15	938.01	0.00	121453.44	1723.72	7465619.20	7344165.76	42.52	0.00	25927.02	278.13
16	1018.88	0.00	49662.98	704.84	8109285.82	8059622.84	63.25	0.00	20031.06	214.88
17	1087.64	2377.57	16892898.82	1994.77	8656529.94	-8236368.88	72.10	0.00	13309.94	142.78
18	948.50	0.00	73720.14	1046.27	7549096.38	7475376.24	62.34	0.00	7498.37	80.44
19	982.90	2396.98	17062501.74	2460.35	7822891.55	-9239610.19	41.26	0.00	3651.75	39.17
20	967.18	0.00	105208.93	1493.17	7697798.35	7592589.43	52.38	0.00	0.13	0.00
21	956.22	0.00	37833.34	536.95	7610591.59	7572758.25	50.85	1000.00	9410480.14	949.15
22	946.34	2457.76	17461718.26	2048.37	7531904.14	-9929814.12	47.78	0.00	84025.69	901.37
23	1022.48	0.00	72284.51	1025.89	8137897.63	8065613.12	67.11	0.00	77769.35	834.26
24	1098.53	2491.06	17722376.20	2418.42	8743185.94	-8979190.25	78.26	0.00	70474.16	756.00
25	979.35	0.00	101397.07	1439.07	7794633.12	7693236.05	72.23	0.00	63740.97	683.77
26	1008.56	2446.80	17442919.00	2877.31	8027167.24	-9415751.76	53.29	0.00	58772.81	630.47
27	990.97	0.00	132911.43	1886.34	7887158.88	7754247.45	62.01	0.00	52992.69	568.47
28	974.13	0.00	64274.17	912.21	7753107.83	7688833.66	59.00	0.00	47493.09	509.47
29	960.41	2389.66	17002542.31	2341.46	7643926.27	-9358616.04	54.95	0.00	42370.61	454.52
30	1030.99	0.00	92335.37	1310.47	8205670.10	8113334.73	72.75	0.00	35588.50	381.77
31	1109.21	0.00	14180.33	201.25	8828213.53	8814033.20	84.77	0.00	27686.33	297.00

Table C.1: Inventory Policy 22-31 days. Initial Inventory (Io212), three fuels

Day	Inco	Pro-notr	Demand-Dies	ord	Cost	Invent	Inco	Gnotrans	Cars	Compar	CostoTran
1	986749.83	902358.39	266.69	0.00	118506.69	1733.31	2069215.90	1950709.21	0.00	0.00	0.00
2	995018.81	919529.93	321.57	0.00	96520.88	1411.74	2495069.39	2398548.51	1.00	3.00	300000.00
3	637917.19	568135.83	351.02	0.00	72521.82	1060.73	2723544.01	2651022.19	0.00	0.00	0.00
4	279443.09	212161.93	265.50	0.00	54369.85	795.23	2059983.46	2005613.61	1.00	3.00	300000.00
5	13278.86	-53883.49	374.63	0.00	28756.68	420.60	2906722.36	2877965.68	0.00	0.00	0.00
6	272405.20	207680.09	334.49	1000.00	6911257.72	1086.12	2595294.72	-4315963.00	1.00	1.00	300000.00
7	403018.68	341899.43	313.45	0.00	52827.22	772.67	2432049.24	2379222.02	1.00	3.00	300000.00
8	446631.08	389507.88	262.46	0.00	34883.13	510.21	2036393.78	2001510.65	0.00	0.00	0.00
9	667434.10	616282.51	319.39	0.00	13046.75	190.83	2478111.32	2465064.57	1.00	3.00	300000.00
10	693125.37	648175.26	342.48	0.00	1.52	0.00	1480615.83	1480614.31	0.00	0.00	0.00
11	520780.37	480489.74	241.50	1000.00	6888858.54	758.50	1873810.14	-5015048.40	1.00	1.00	300000.00
12	275712.87	237889.08	353.64	0.00	27680.18	404.86	2743891.98	2716211.80	1.00	3.00	300000.00
13	433262.46	399315.11	330.14	1000.00	6910478.38	1074.72	2561571.00	-4348907.38	1.00	1.00	300000.00
14	453373.99	423483.04	314.07	0.00	52005.48	760.65	2436861.37	2384855.89	1.00	3.00	300000.00
15	443040.79	417113.77	264.87	0.00	33896.42	495.78	2055114.69	2021218.27	0.00	0.00	0.00
16	658979.12	638948.06	314.29	0.00	12408.51	181.49	2438565.25	2426156.74	0.00	0.00	0.00
17	751204.70	737894.76	336.21	1000.00	6894792.04	845.28	2608630.11	-4286161.93	2.00	4.00	600000.00
18	649546.63	642048.26	237.13	0.00	41579.71	608.16	1839862.96	1798283.25	0.00	0.00	0.00
19	429928.97	426277.22	338.71	0.00	18422.18	269.45	2628043.13	2609620.95	1.00	3.00	300000.00
20	408147.88	408147.75	327.12	0.00	0.58	0.00	2090649.36	2090648.78	0.00	0.00	0.00
21	529763.47	-8880716.67	313.73	1000.00	6883920.44	686.27	2434213.22	-4449707.21	1.00	2.00	300000.00
22	497865.30	413839.61	267.39	0.00	28639.19	418.89	2074655.73	2046016.55	1.00	3.00	300000.00
23	699256.97	621487.62	309.51	1000.00	6912847.65	1109.37	2401526.88	-4511320.76	1.00	1.00	300000.00
24	815368.11	744893.95	331.31	0.00	53195.72	778.06	2570663.77	2517468.05	1.00	3.00	300000.00
25	752554.81	688813.84	237.50	0.00	36958.13	540.56	1842730.69	1805772.56	0.00	0.00	0.00
26	555279.84	496507.03	327.16	0.00	14590.27	213.40	2538425.91	2523835.63	1.00	3.00	300000.00
27	646031.85	593039.16	324.39	1000.00	6897782.00	889.02	2516911.75	-4380870.25	1.00	1.00	300000.00
28	614678.03	567184.94	312.99	0.00	39382.82	576.02	2428494.84	2389112.02	0.00	0.00	0.00
29	572529.36	530158.76	269.43	0.00	20961.78	306.59	2090520.56	2069558.78	1.00	3.00	300000.00
30	758021.44	722432.93	305.15	0.00	98.69	1.44	2367657.30	2367558.61	0.00	0.00	0.00
31	883209.04	855522.72	326.96	0.00	3.26	0.00	11199.34	11196.09	0.00	0.00	0.00

Table C.2: Continuation-Inventory Policy 22-31 days. Initial Inventory (Io212), three fuels

Appendix D

Best Policy with no final inventory restrictions

Policy to be compared with actual situation of the company.

Day	Demand-cor	ord	Cost	Invent	Inco	Pro-notr	Demand-ext	ord	Cost	Invent
1	1096.86	0.00	225693.24	3203.14	8729908.74	8504215.50	94.71	0.00	84391.43	905.29
2	1120.79	0.00	146722.59	2082.35	8920343.73	8773621.14	95.50	0.00	75488.89	809.79
3	1049.42	0.00	72780.32	1032.93	8352349.70	8279569.38	61.23	0.00	69781.37	748.57
4	845.77	0.00	13187.60	187.16	6731457.17	6718269.57	26.82	0.00	67281.16	721.75
5	886.88	1783.46	12642652.96	1083.75	7058708.96	-5583944.00	1.27	0.00	67162.35	720.47
6	896.58	0.00	13187.60	187.16	7135886.59	7122698.99	26.15	0.00	64725.11	694.33
7	931.68	1877.36	13307676.76	1132.84	7415235.55	-5892441.21	38.68	0.00	61119.25	655.65
8	945.68	0.00	13187.60	187.16	7526648.02	7513460.42	42.87	0.00	57123.19	612.78
9	1027.03	2000.00	14173742.72	1160.13	8174170.77	-5999571.95	64.06	0.00	51151.58	548.72
10	1074.24	0.00	6051.46	85.89	8549910.38	8543858.92	66.53	0.00	44950.11	482.19
11	909.12	1858.01	13164441.65	1034.77	7235679.71	-5928761.94	49.98	0.00	40290.63	432.21
12	948.89	0.00	6051.46	85.89	7552213.92	7546162.45	26.46	0.00	37823.79	405.75
13	938.90	1879.40	13314597.64	1026.39	7472725.79	-5841871.85	41.58	0.00	33947.34	364.16
14	940.50	0.00	6051.46	85.89	7485448.25	7479396.79	43.51	0.00	29890.95	320.65
15	938.01	1956.89	13866104.37	1104.77	7465619.20	-6400485.16	42.52	0.00	25927.02	278.13
16	1018.88	0.00	6051.46	85.89	8109285.82	8103234.35	63.25	0.00	20031.06	214.88
17	1087.64	2000.00	14162336.32	998.24	8656529.94	-5505806.38	72.10	0.00	13309.94	142.78
18	948.50	0.00	3505.15	49.75	7549096.38	7545591.23	62.34	0.00	7498.37	80.44
19	982.90	1950.08	13811919.26	1016.93	7822891.55	-5989027.71	41.26	0.00	3651.75	39.17
20	967.18	0.00	3505.15	49.75	7697798.35	7694293.21	52.38	0.00	0.13	0.00
21	956.22	1902.56	13475640.20	996.08	7610591.59	-5865048.61	50.85	1000.00	9410480.14	949.15
22	946.34	0.00	3505.15	49.75	7531904.14	7528399.00	47.78	0.00	84025.69	901.37
23	1022.48	2000.00	14164381.39	1027.27	8137897.63	-6026483.76	67.11	0.00	77769.35	834.26
24	1098.53	0.00	0.71	0.00	8176035.56	8176034.85	78.26	0.00	70474.16	756.00
25	979.35	1987.91	14077899.18	1008.56	7794633.12	-6283266.06	72.23	0.00	63740.97	683.77
26	1008.56	0.00	0.00	0.00	8027167.24	8027167.24	53.29	0.00	58772.81	630.47
27	990.97	1965.10	13914763.57	974.13	7887158.88	-6027604.69	62.01	0.00	52992.69	568.47
28	974.13	0.00	0.00	0.00	7753107.83	7753107.83	59.00	0.00	47493.09	509.47
29	960.41	1991.41	14104086.89	1030.99	7643926.27	-6460160.62	54.95	0.00	42370.61	454.52
30	1030.99	0.00	0.00	0.00	8205670.10	8205670.10	72.75	0.00	35588.50	381.77
31	1109.21	0.00	11.09	0.00	0.00	-11.09	84.77	0.00	27686.33	297.00

Table D.1: Best policy 31 days (Io4,1,2), with no final inventory restrictions

Day	Inco	pro-notr	Demand-Dies	ord	Cost	Invent	Inco	Pr-notr	Cars	Compartm	CostoTran
1	986749.83	902358.39	266.69	0.00	128762.19	1883.31	2069215.90	1940453.71	0.00	0.00	0.00
2	995018.81	919529.93	321.57	0.00	106776.38	1561.74	2495069.39	2388293.01	0.00	0.00	0.00
3	637917.19	568135.83	351.02	0.00	82777.32	1210.73	2723544.01	2640766.69	0.00	0.00	0.00
4	279443.09	212161.93	265.50	0.00	64625.35	945.23	2059983.46	1995358.11	0.00	0.00	0.00
5	13278.86	-53883.49	374.63	0.00	39012.18	570.60	2906722.36	2867710.18	1.00	2.00	300000.00
6	272405.20	207680.09	334.49	0.00	16143.22	236.12	2595294.72	2579151.50	0.00	0.00	0.00
7	403018.68	341899.43	313.45	1000.00	6900082.72	922.67	2432049.24	-4468033.48	1.00	3.00	300000.00
8	446631.08	389507.88	262.46	0.00	45138.63	660.21	2036393.78	1991255.15	0.00	0.00	0.00
9	667434.10	616282.51	319.39	0.00	23302.25	340.83	2478111.32	2454809.07	1.00	2.00	300000.00
10	693125.37	648175.26	342.48	0.00	0.02	0.00	2644465.83	2644465.81	0.00	0.00	0.00
11	520780.37	480489.74	241.50	1000.00	6888858.54	758.50	1873810.14	-5015048.40	1.00	3.00	300000.00
12	275712.87	237889.08	353.64	0.00	27680.18	404.86	2743891.98	2716211.80	0.00	0.00	0.00
13	433262.46	399315.11	330.14	1000.00	6910478.38	1074.72	2561571.00	-4348907.38	1.00	3.00	300000.00
14	453373.99	423483.04	314.07	0.00	52005.48	760.65	2436861.37	2384855.89	0.00	0.00	0.00
15	443040.79	417113.77	264.87	0.00	33896.42	495.78	2055114.69	2021218.27	1.00	2.00	300000.00
16	658979.12	638948.06	314.29	0.00	12408.51	181.49	2438565.25	2426156.74	0.00	0.00	0.00
17	751204.70	737894.76	336.21	1000.00	6894792.04	845.28	2608630.11	-4286161.93	1.00	3.00	300000.00
18	649546.63	642048.26	237.13	0.00	41579.71	608.16	1839862.96	1798283.25	0.00	0.00	0.00
19	429928.97	426277.22	338.71	0.00	18422.18	269.45	2628043.13	2609620.95	1.00	2.00	300000.00
20	408147.88	408147.75	327.12	0.00	0.58	0.00	2090649.36	2090648.78	0.00	0.00	0.00
21	529763.47	-8880716.67	313.73	1000.00	6883920.44	686.27	2434213.22	-4449707.21	2.00	4.00	600000.00
22	497865.30	413839.61	267.39	0.00	28639.19	418.89	2074655.73	2046016.55	0.00	0.00	0.00
23	699256.97	621487.62	309.51	1000.00	6912847.65	1109.37	2401526.88	-4511320.76	1.00	3.00	300000.00
24	815368.11	744893.95	331.31	0.00	53195.72	778.06	2570663.77	2517468.05	0.00	0.00	0.00
25	752554.81	688813.84	237.50	0.00	36958.13	540.56	1842730.69	1805772.56	1.00	2.00	300000.00
26	555279.84	496507.03	327.16	0.00	14590.27	213.40	2538425.91	2523835.63	0.00	0.00	0.00
27	646031.85	593039.16	324.39	1000.00	6897782.00	889.02	2516911.75	-4380870.25	1.00	3.00	300000.00
28	614678.03	567184.94	312.99	0.00	39382.82	576.02	2428494.84	2389112.02	0.00	0.00	0.00
29	572529.36	530158.76	269.43	0.00	20961.78	306.59	2090520.56	2069558.78	1.00	2.00	300000.00
30	758021.44	722432.93	305.15	0.00	98.69	1.44	2367657.30	2367558.61	0.00	0.00	0.00
31	883209.04	855522.72	326.96	0.00	3.26	0.00	11199.34	11196.09	0.00	0.00	0.00

Table D.2: Continuation-Best policy 31 days (Io4,1,2), with no final inventory restrictions

Appendix E

Best policy with final Inventory restrictions.

Policy to be compared with actual situation of the company. A final inventory restriction is added.

Day	Demand-cor	ord	Cost	Invent	Inco	Pro-notr	Demand-ext	ord	Cost	Invent
1	1096.86	0.00	225693.24	3203.14	8729908.74	8504215.50	94.71	0.00	84391.43	905.29
2	1120.79	0.00	146722.59	2082.35	8920343.73	8773621.14	95.50	0.00	75488.89	809.79
3	1049.42	0.00	72780.32	1032.93	8352349.70	8279569.38	61.23	0.00	69781.37	748.57
4	845.77	0.00	13187.60	187.16	6731457.17	6718269.57	26.82	0.00	67281.16	721.75
5	886.88	1783.46	12642652.96	1083.75	7058708.96	-5583944.00	1.27	0.00	67162.35	720.47
6	896.58	0.00	13187.60	187.16	7135886.59	7122698.99	26.15	0.00	64725.11	694.33
7	931.68	1877.36	13307676.76	1132.84	7415235.55	-5892441.21	38.68	0.00	61119.25	655.65
8	945.68	0.00	13187.60	187.16	7526648.02	7513460.42	42.87	0.00	57123.19	612.78
9	1027.03	2000.00	14173742.72	1160.13	8174170.77	-5999571.95	64.06	0.00	51151.58	548.72
10	1074.24	0.00	6051.46	85.89	8549910.38	8543858.92	66.53	0.00	44950.11	482.19
11	909.12	1858.01	13164441.65	1034.77	7235679.71	-5928761.94	49.98	0.00	40290.63	432.21
12	948.89	0.00	6051.46	85.89	7552213.92	7546162.45	26.46	0.00	37823.79	405.75
13	938.90	1879.40	13314597.64	1026.39	7472725.79	-5841871.85	41.58	0.00	33947.34	364.16
14	940.50	0.00	6051.46	85.89	7485448.25	7479396.79	43.51	0.00	29890.95	320.65
15	938.01	1956.89	13866104.37	1104.77	7465619.20	-6400485.16	42.52	0.00	25927.02	278.13
16	1018.88	0.00	6051.46	85.89	8109285.82	8103234.35	63.25	0.00	20031.06	214.88
17	1087.64	2000.00	14162336.32	998.24	8656529.94	-5505806.38	72.10	0.00	13309.94	142.78
18	948.50	0.00	3505.15	49.75	7549096.38	7545591.23	62.34	0.00	7498.37	80.44
19	982.90	1950.08	13811919.26	1016.93	7822891.55	-5989027.71	41.26	0.00	3651.75	39.17
20	967.18	0.00	3505.15	49.75	7697798.35	7694293.21	52.38	0.00	0.13	0.00
21	956.22	1902.56	13475640.20	996.08	7610591.59	-5865048.61	50.85	1000.00	9410480.14	949.15
22	946.34	0.00	3505.15	49.75	7531904.14	7528399.00	47.78	0.00	84025.69	901.37
23	1022.48	2000.00	14164381.39	1027.27	8137897.63	-6026483.76	67.11	0.00	77769.35	834.26
24	1098.53	0.00	0.71	0.00	8176035.56	8176034.85	78.26	0.00	70474.16	756.00
25	979.35	1987.91	14077899.18	1008.56	7794633.12	-6283266.06	72.23	0.00	63740.97	683.77
26	1008.56	0.00	0.00	0.00	8027167.24	8027167.24	53.29	0.00	58772.81	630.47
27	990.97	1965.10	13914763.57	974.13	7887158.88	-6027604.69	62.01	0.00	52992.69	568.47
28	974.13	0.00	0.00	0.00	7753107.83	7753107.83	59.00	0.00	47493.09	509.47
29	960.41	1991.41	14104086.89	1030.99	7643926.27	-6460160.62	54.95	0.00	42370.61	454.52
30	1030.99	0.00	0.00	0.00	8205670.10	8205670.10	72.75	0.00	35588.50	381.77
31	1109.21	2218.42	15787328.21	2218.42	0.00	-15787328.21	84.77	0.00	27686.33	297.00

Table E.1: Best policy, for 31 days of planning (Io4,1,2). Final Inventory restriction.

Day	Inco	Pro-notr	Demand-Dies	ord	Cost	Invent	Inco	Pro-notr	Cars	Compartm	CostoTransp
1	986749.83	902358.39	266.69	0.00	128762.19	1883.31	2069215.90	1940453.71	0.00	0.00	0.00
2	995018.81	919529.93	321.57	0.00	106776.38	1561.74	2495069.39	2388293.01	0.00	0.00	0.00
3	637917.19	568135.83	351.02	0.00	82777.32	1210.73	2723544.01	2640766.69	0.00	0.00	0.00
4	279443.09	212161.93	265.50	0.00	64625.35	945.23	2059983.46	1995358.11	0.00	0.00	0.00
5	13278.86	-53883.49	374.63	0.00	39012.18	570.60	2906722.36	2867710.18	1.00	2.00	300000.00
6	272405.20	207680.09	334.49	0.00	16143.22	236.12	2595294.72	2579151.50	0.00	0.00	0.00
7	403018.68	341899.43	313.45	1000.00	6900082.72	922.67	2432049.24	-4468033.48	1.00	3.00	300000.00
8	446631.08	389507.88	262.46	0.00	45138.63	660.21	2036393.78	1991255.15	0.00	0.00	0.00
9	667434.10	616282.51	319.39	0.00	23302.25	340.83	2478111.32	2454809.07	1.00	2.00	300000.00
10	693125.37	648175.26	342.48	0.00	0.02	0.00	2644465.83	2644465.81	0.00	0.00	0.00
11	520780.37	480489.74	241.50	1000.00	6888858.54	758.50	1873810.14	-5015048.40	1.00	3.00	300000.00
12	275712.87	237889.08	353.64	0.00	27680.18	404.86	2743891.98	2716211.80	0.00	0.00	0.00
13	433262.46	399315.11	330.14	1000.00	6910478.38	1074.72	2561571.00	-4348907.38	1.00	3.00	300000.00
14	453373.99	423483.04	314.07	0.00	52005.48	760.65	2436861.37	2384855.89	0.00	0.00	0.00
15	443040.79	417113.77	264.87	0.00	33896.42	495.78	2055114.69	2021218.27	1.00	2.00	300000.00
16	658979.12	638948.06	314.29	0.00	12408.51	181.49	2438565.25	2426156.74	0.00	0.00	0.00
17	751204.70	737894.76	336.21	1000.00	6894792.04	845.28	2608630.11	-4286161.93	1.00	3.00	300000.00
18	649546.63	642048.26	237.13	0.00	41579.71	608.16	1839862.96	1798283.25	0.00	0.00	0.00
19	429928.97	426277.22	338.71	0.00	18422.18	269.45	2628043.13	2609620.95	1.00	2.00	300000.00
20	408147.88	408147.75	327.12	0.00	0.58	0.00	2090649.36	2090648.78	0.00	0.00	0.00
21	529763.47	-8880716.67	313.73	1000.00	6883920.44	686.27	2434213.22	-4449707.21	2.00	4.00	600000.00
22	497865.30	413839.61	267.39	0.00	28639.19	418.89	2074655.73	2046016.55	0.00	0.00	0.00
23	699256.97	621487.62	309.51	1000.00	6912847.65	1109.37	2401526.88	-4511320.76	1.00	3.00	300000.00
24	815368.11	744893.95	331.31	0.00	53195.72	778.06	2570663.77	2517468.05	0.00	0.00	0.00
25	752554.81	688813.84	237.50	0.00	36958.13	540.56	1842730.69	1805772.56	1.00	2.00	300000.00
26	555279.84	496507.03	327.16	0.00	14590.27	213.40	2538425.91	2523835.63	0.00	0.00	0.00
27	646031.85	593039.16	324.39	1000.00	6897782.00	889.02	2516911.75	-4380870.25	1.00	3.00	300000.00
28	614678.03	567184.94	312.99	0.00	39382.82	576.02	2428494.84	2389112.02	0.00	0.00	0.00
29	572529.36	530158.76	269.43	0.00	20961.78	306.59	2090520.56	2069558.78	1.00	2.00	300000.00
30	758021.44	722432.93	305.15	0.00	98.69	1.44	2367657.30	2367558.61	0.00	0.00	0.00
31	883209.04	855522.72	326.96	653.91	4515508.94	653.91	11199.34	-4504309.60	2.00	4.00	600000.00

Table E.2: Continuation-Best policy, for 31 days of planning (Io4,1,2). Final Inventory restriction.

Appendix F

Articles

- "Un modelo dinámico para el pronóstico de energía diaria", in Revista Ingenieria Industrial, Universidad del Bio Bio de Chile,
  *http://revistas.ubiobio.cl/index.php/RI/article/view/*9/9

- "Inventory planning with dynamic demand. A state of art review", in Revista DYNA,
  *http://www.revistas.unal.edu.co/index.php/dyna/article/view/*42828

- "Métodos estadísticos clásicos y Bayesianos para el pronóstico de demanda. Un análisis comparativo", in Revista de la Facultad de Ciencias,
  *http://revistas.unal.edu.co/index.php/rfc/article/view/*49775

- "Inventory model using bayesian dynamic linear model for demand forecasting", in Revista Investigación y Desarrollo,
  *http://revistas.uptc.edu.co/revistas/index.php/ingenieria_sogamoso/article/view/*3937

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