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The complex use of tonal consonance in music: microscopic and macroscopic rules in harmony and melody

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I dedicate this thesis to my parents, who always have supported me to reach my dreams.

"We then see how precisely the irrationality of the periods of revolution of Jupiter and Saturn in respect to each other prevents their mutual perturbations from repeating themselves at one spot, whereby they would become dangerous. The result of this irrationality is that, appearing rarely and always at a different place, such perturbations must again balance each other; this is comparable to the dissonances in music which resolve themselves once more into harmony."

The World as Will and Representation, Arthur Schopenhauer

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Abstract

This thesis contributes to the understanding of the connection between the tonal consonance phenomenon and the rules associated to the construction of musical pieces. It begins from the tonal consonance properties of sounds and ends with the emergence of macroscopic phenomena. The systems studied are: the *marimba de chonta* music, melodic lines of Western music, and *secco recitatives* of operas. At the microscopic level, we show how tonal consonance connects the timbre of *marimba de chonta* with traditional tunings and musical practices. At this level, we also extended the traditional concept of interval size, and we used chords as the unit of analysis for the harmony of *secco recitatives*. On the macroscopic level, we found that the new representation of intervals, together with an entropy extremalization principle, is suitable for approximately reproducing the selection of intervals in melodic lines. Besides, a conserved macroscopic quantity emerges, which is empirically related to the mean dissonance of a melodic line, hence connecting psychoacoustics with complexity.

Keywords: Consonance; *marimba de chonta*; melody; musical interval; entropy; *secco recitative*; tuning.

Resumen

Esta tesis contribuye al entendimiento de la conección entre el fenómeno de la consonancia tonal y las reglas asociadas a la construcción de piezas musicales. Ésta comienza desde las propiedades de consonancia de los sonidos y termina con la emergencia de fenómenos macroscópicos. Los sistemas estudiados son: música de marimba de chonta, líneas melódicas de música occidental, y recitativos *seccos* de óperas. En el nivel microscópico, mostramos cómo la consonancia tonal conecta el timbre de la marimba de chonta con las afinaciones y prácticas musicales tradicionales. En éste nivel, también extendemos el concepto tradicional de tamaño interválico, y usamos acordes como unidad de análisis para la armonía de recitativos *seccos*. En el nivel macroscópico, encontramos que la nueva representación de intervalos, junto con un principio de extremalización de la entropía, es apropiada para reproducir aproximadamente la selección de intervalos en líneas melódicas. Además, una cantidad macroscópica conservada emerge, la cual está empíricamente relacionada con la disonancia promedio de una línea melódica, por tanto conectando psicoacústica con complejidad.

Palabras clave: Afinación, consonancia; entropía; intervalo musical; marimba de chonta; melodía; recitativo *secco*.

Contents

List of Abbreviations and Symbols

This thesis uses the following abbreviations and symbols:

Abbreviation	**Meaning**
12-TET	Twelve-tone equal-tempered
CCD	Complementary cumulative distribution
CD	Cumulative distribution
FFT	Fast Fourier transform
PD	Probability distribution
R.E.	Relative error
SPL	Sound pressure level
T.U.	Total uncertainty

Symbol	**Unit**	**Meaning**
α	Dimensionless	Fundamental frequency ratio
δ	Dimensionless	Dissonance level between two pure tones
ϵ	J/m^3	Average energy density
ε	Hz^2	Difference in the squares of the frequencies measure using bins in histograms
λ	Hz^{-2}	Lagrange multiplier
ρ	kg/m^3 (SI)	Density
σ	Depends on the context	Standard deviation
σ^2	Depends on the context	Variance
φ	Dimensionless	Phase of a sinusoidal wave

$\mathcal{A}$	Dimensionless	Number of different pitches inside the *ambitus* of a melodic line
a	In the consonance models is dimensionless	Amplitude
a_{norm}	Dimensionless	Normalized amplitude
Avg.	Depends on the context	Average
ΔAvg.	Depends on the context	Uncertainty in the measure of the average
$\mathcal{B}$	m (SI)	Length of the bar
D_{KL}	$Nats$	Kullback-Leibler divergence (relative entropy)
D	Dimensionless	Dissonance level between two complex tones
$\langle D \rangle$	Dimensionless	Mean dissonance level of a melodic line associated to their melodic intervals
$D_F(\alpha)$	Dimensionless	Dissonance level between two complex tones with a fundamental frequency ratio α
E	Pa (SI)	Young's modulus
f	Hz (SI)	Frequency
$\mathcal{K}$	m (SI)	Radius of gyration
$\mathcal{I}$	$Watt/m^2$ (SI)	Intensity
$\mathcal{I}_0$	$Watt/m^2$ (SI)	Hearing threshold associated to the intensity ($10^{-12} Watt/m^2$)
IL	dB	Intensity measured in a logarithmic scale
i	Without units	Tonic chord in a minor scale
I	Without units	Tonic chord in a major scale
iv	Without units	Chord of the fourth degree in a minor scale (subdominant function)
IV	Without units	Chord of the fourth degree in a major scale (subdominant function)
l	*phons*, *sones*	Loudness
l_{norm}	Dimensionless	Normalized loudness
L	*semitones*	Interval size

$\mathbb{L}$	$semitones \times Hz$	Effective interval size containing information about the contribution of the average location in the register
$\mathfrak{L}$	$semitones \times Hz^2$	Effective interval size that takes into account the contribution of the average location of intervals in the register as well as their dispersion
p	Without units	Probability
Δp	Pa (SI)	Variations of pressure
Δp_{ref}	Pa (SI)	Hearing threshold associated to the pressure variations ($2 \times 10^{-5} Pa$)
$\tilde{p}_a$	Without units	Probability of ascending intervals in a melodic line
$\tilde{p}_d$	Without units	Probability of descending intervals in a melodic line
$\tilde{p}_u$	Without units	Probability of unisons in a melodic line
r	Dimensionless	Average frequency ratios
s	*Steps*	Relative distance between marimba bars
S	*Bits*	Shannon entropy.
S_f	*Bits*	Final Shannon entropy
S_{max}	*Bits*	Maximum Shannon entropy
V	Without units	Chord of the fifth degree in a major scale, or in a harmonic minor scale (dominant function)
$\mathcal{W}$	m (SI)	Width of the bar

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Introduction

Music is one of the most generalized, startling and puzzling creations of human beings. The understanding of cognitive processes behind the conveying of musical information has been studied since the time of the ancient Greeks, when Pythagoras found that two sounds produced by vibrating strings of equal tension and density produce a pleasant sensation when the ratio between their lengths goes as the ratio between two small natural numbers [1, 2, 3]. Aaron Copland proposed four essential elements in music: rhythm, melody, harmony and tone color. From his perspective, rhythm is connected with a physical motion, melody with a mental emotion, harmony with an intellectual conception, and tone color with timbre [4]. Each of these four characteristics is relevant in conveying musical information and can be studied independently.

Musical information involves sensations, emotions, intellectual constructions and meaning, and many authors associate the use of consonance with the communication process involved [4, 5, 6, 7, 8, 9].From the perspective of the nature of tonality, consonance and dissonance give rise to emotions through tension and relaxation due to the passage from satisfaction to dissatisfaction, and again back to satisfaction [6]. From this perspective, conveying emotions in music requires a broad exploration of consonance and dissonance.

At the most basic level, music can be described as being made of sounds that can be combined either simultaneously or successively. From the perspective of complexity, a musical piece is made of microscopic constituent elements assembled following sets of rules with musical relevance, and as a whole it shows emergent phenomena that do not correspond to the individual properties of the constituents. Hence, the issue of the definition, selection and organization of the constituent elements in music, subject to formal rules and to the creativity of the composers, is of utmost relevance.

Rules are present in musical pieces at several levels, from microscopic to macroscopic, and order emerges at several hierarchical levels. This structure leads to the possibility of considering different types of constituent elements.

From the physics point of view, Fourier analysis shows that a sound wave produced by a musical instrument can be described as the superposition of several harmonic waves, each one called a pure tone that is characterized by a single frequency, an amplitude, and a phase. Hence, from this perspective, pure tones can be considered as the most basic microscopic constituent elements of music.

From the psychoacoustics point of view, which connects the acoustical stimuli with the auditory sensation [2], human beings cannot perceive the individual pure tones that compose a single sound produced by a musical instrument. Humans perceive the entire superposition as a unit called a complex tone, which is characterized by a pitch[1] that usually is strongly related to the fundamental frequency[2], a loudness, and a timbre. Loudness and timbre are strongly associated with the amplitudes and frequencies of the pure tones in the superposition; it is the specific pattern of the spectrum. Furthermore, complex tones can be considered as the most basic microscopic constituent elements of music in terms of psychoacoustics.

From the music point of view, the most important microscopic element is the musical note. The musical note is commonly characterized by the combination of pitch, in the case of definite pitch instruments, and rhythm figure (time duration).

Up to now, three different possible constituent elements emerge: pure tones, complex tones, and musical notes. Combinations of two or more of these elements give rise to new microscopic structures with higher levels of complexity and new properties. For example, the combination of two or more complex tones constitute important microscopic structures in music, such as musical intervals and chords. The combinations of complex tones exhibit new properties, such as the perceived distance between pitches (musical interval size) and the consonance level. At a higher level of complexity, motifs, musical phrases, and sections can also be defined as constituent elements.

Many studies in psychoacoustics have studied perceptual features associated with the consonance properties of complex tones, and how these properties can be used in order to produce musical scales for a particular type of musical instrument. In this process, timbre becomes one of the most relevant features to take into account.

The perception of timbre is strongly connected to the spectrum of the complex tones [1], and the phenomenon of consonance also depends on the features of the spectrum. Therefore, the consonance approach connects the physical properties of a musical instrument with the selection of a set of elements to be used in the composition of a musical piece.

This approach, based exclusively on the physical properties of sound, is independent of the cultural context of the listeners, and is formally known as tonal or sensory consonance [10].

Regarding the rules used for composing musical pieces, at the most basic level we have those involved in the selection of the collection of pitches that constitute a specific tuning. As we mention above, while these rules are strongly related with the consonance phenomenon, they are also related with musical practices, as with transposition. At a higher level of complexity, we find composition rules, such as the suppression of the tritone and the parallel fifths in most Western music of the Baroque period [11], and the organization of sections in the sonata form of the Classical period [12, pp.787-788].

The work of connecting different levels of complexity in music has been tackled using several strategies. One of the most important approaches involves the use of statistical methods

[1]In the case of definite pitch musical instruments such as strings, pipes, and marimbas, as opposed to indefinite pitch instruments as drums and cymbals.

[2]Lowest frequency pure tone in the superposition.

to study how musical elements are selected and organized in a musical piece.

This thesis contributes to the understanding of the connection between different levels of complexity in music, specifically to the relation between the tonal consonance properties of musical intervals, and their use in music. This connection can be considered as natural in most Western music, in the sense that many rules found in harmony and melody are related with the size and location in the register of musical intervals; properties that are strongly connected with the tonal consonance phenomena.

Chapter 1 presents previous works about the tonal consonance phenomenon, the relation of consonance with the tuning of a musical instrument through the concept of the musical interval, and the macroscopic phenomena associated with the organization of musical intervals in a musical score.

Chapter 2 defines the relation between the musical practices found in the *marimba de chonta* music and the tonal consonance phenomenon associated with the timbre of this instrument. The *marimba de chonta* music, traditional chants and dances from the south pacific region of Colombia and the Esmeraldas province of Ecuador were inscribed by UNESCO in the "Intangible Cultural Heritage of Humanity" list [13, 14, 15, 16]. Despite this, many aspects of the tunings and the musical practices associated with the marimba chonta remain poorly understood [13, 17]. The lack of information concerning this traditional music has led to its endangerment of disappearance, because of the influence of Western music and the introduction of the diatonic 12-tone equal-tempered marimbas in the territories. In order to collect musical samples, this research included an expedition to the Pacific coast of Colombia, with the participation of the author of this thesis, musicians and researchers from the Conservatory of Music, the Department of Physics, and the School of Cinema and Media Arts of the Universidad Nacional de Colombia. This thesis contributes to the understanding of the tunings and musical practices of the *marimba de chonta* music. The research shows that these two aspects are strongly connected, and that this relation is fundamental for the correct conservation of this music, which is absolutely different from Western marimba music.

Chapter 3 presents the relation between the consonance phenomenon in music and the emergence of macroscopic properties in musical pieces, which are connected to the selection made by composers of musical intervals. The first part of this chapter develops a representation that uses the fundamental frequencies of pitches in order to extend the concept of musical interval size. The traditional concept of interval size captures relevant information about musical features, but misses information concerning the locations of intervals in the register (an important feature to measure the level of tonal consonance [2, 18]) and about musical processes such as transposition [12, p. 860]. The second part of this chapter contains an application to a set of melodic lines, finding that the representation developed is suitable for reproducing approximately the final selection of musical intervals, as well as for describing musical features such as the asymmetry in the use of ascending and descending intervals, and transposition processes.

Chapter 4 contains results obtained in an internship of one month carried out by the author at the Bariloche Atomic Centre and the Balseiro Institute in Argentina, under the advice of Professor Damian Zanette. This chapter defines chords as constitutive elements and explores organizational properties of *secco recitatives* from eight operas. Most of the information about the use of harmony in *secco recitatives* has been lost [19], and currently any information about this matter is highly relevant. One of the main goals of the internship was the construction of the database, which was produced manually by extracting and analyzing each chord from the musical scores. The results obtained in this chapter show that the methodology developed leads to new information related to the harmony of the *secco recitatives*, that should be relevant for future studies. As an important remark, the results of this chapter constitute an exploration beyond the main objectives of this thesis.

1. Tonal consonance, scales, and musical intervals

1.1. Tonal consonance

This chapter presents basic concepts about tonal consonance, the models developed by Sethares and Vassilakis to quantify the dissonance level, the relation between the tonal consonance theory and musical intervals, and finally the statistical analysis of the use of intervals in musical pieces.

1.1.1. Some basic concepts

In certain range of fundamental frequencies, 20 Hz to 20 KHz, and with changes of pressure in the air $\Delta P/P_{atm}$ between 2×10^{-10} and 2×10^{-4}, sound waves can excite the human brain-ear system [3, p. 190].

There are many musical instruments in which the sound produced, a *complex tone*, can be characterized by the superposition of a set of sinusoidal waves, each one with a particular frequency of oscillation, amplitude, and phase. Each single sinusoidal component of the set is called a *partial* [1, p. 56] or a *pure tone* [1, p. 147], the lowest frequency of the set is called the *fundamental frequency*, and the remaining frequencies are called *overtones* [1, p. 56]. In many cases the overtones turn out to be very nearly integer multiples of the fundamental frequency, for this case the fundamental frequency and the overtones are called *harmonics*, in such a way that the first harmonic corresponds to the fundamental frequency, the second harmonic to the first overtone, and so on [1, p. 56]. If the frequencies of the overtones do not correspond to integer multiples of the fundamental frequency, this phenomenon is called *inharmonicity*.

1.1.2. Pythagoras' postulate about consonance

Pythagoras (5th century BC) found that two sounds emitted by strings of equal tension and density, with the ratio between lengths L_i and L_j related by two natural numbers n and m produce a pleasant sensation when these numbers are small [3]. Then, in the 17th century, Galileo Galilei found that the Pythagoras postulate can expressed in terms of an small natural numbers ratio between the fundamental frequencies of the two sounds f_i and f_j. The Pythagoras' postulate can expressed using the fundamental frequencies as [1, 2, 20]:

$$\frac{L_i}{L_j} = \frac{f_j}{f_i} = \frac{n}{m} \tag{1-1}$$

This consonance phenomenon is present in melody, harmony, timbre, and musical tuning [2, 20, 21], and many authors have related consonance with conveying sophisticated musical information as emotions and meaning [4, 5, 6, 7]. From the perspective of the nature of tonality, consonance and dissonance give rise to emotions through the succession of tension and relaxation [5] in passages from satisfaction to dissatisfaction and back again to satisfaction [6, 8].

1.1.3. Roughness and tonal consonance

In 19th century Helmholtz found that the amount of partials shared by two sounds plays an important role in their total consonance level [3, 22]. He found that the degree of consonance of pairs of pure tones is related to the beats or shocks produced by fluctuations in the peak intensity, occurring at the frequency difference of the pure tunes [22]; specifically, the perception of high levels of dissonance is related with the perception of roughness due to rapid beats [20].

The superposition of two pure tones with different frequencies $f_i = \frac{\omega_i}{2\pi}$ and $f_j = \frac{\omega_j}{2\pi}$, amplitudes a_i and a_j, and phases φ_i and φ_j is [23]

$$\begin{gathered} a_i cos(\omega_i t + \varphi_i) + a_j cos(\omega_j t + \varphi_j) = \\ (a_i + a_j) cos(\omega_+ t + \varphi_+) cos(\omega_- t + \varphi_-) + (a_j - a_i) sin(\omega_+ t + \varphi_+) sin(\omega_- t + \varphi_-) \\ \omega_+ = \frac{\omega_i + \omega_j}{2}; \quad \omega_- = \frac{\omega_i - \omega_j}{2}; \quad \varphi_+ = \frac{\varphi_i + \varphi_j}{2}; \quad \varphi_- = \frac{\varphi_i - \varphi_j}{2}. \end{gathered} \tag{1-2}$$

The resulting wave has a rapid frequency $\frac{f_i + f_j}{2}$ modulated by a slow frequency $|f_i - f_j|$. Figure **1-1** shows the superposition of two sinusoidal waves of equal amplitudes, and with frequencies $f_1 = 220\,Hz$ (blue) and $f_2 = 240\,Hz$ (red). The two waves begin with the same phase at the time 0 seconds. The rapid frequency is $230\,Hz$, and the slow one is $20\,Hz$.

If the frequencies f_i and f_j are close ($f_j = f_i \pm \Delta f$) and the conditions: $|\omega_i - \omega_j| \ll \omega_i + \omega_j$ and $|f_i - f_j| \lessapprox 15 Hz$ are satisfied, our ear-brain system perceived an average frequency $(f_i + f_j)/2$ with an amplitude oscillating at a frequency $|f_i - f_j|$ [24]. This phenomenon is known as *real beats*. An example is the superposition of two pure tones with frequencies $440\,Hz$ and $= 442\,Hz$.

If we have two frequencies that are not close between them, the ear-brain system can distinguish the two sounds independently. However, in some cases (for example the superposition of two pure tones with frequencies $440\,Hz$ and $= 882\,Hz$) our ear-brain system can not

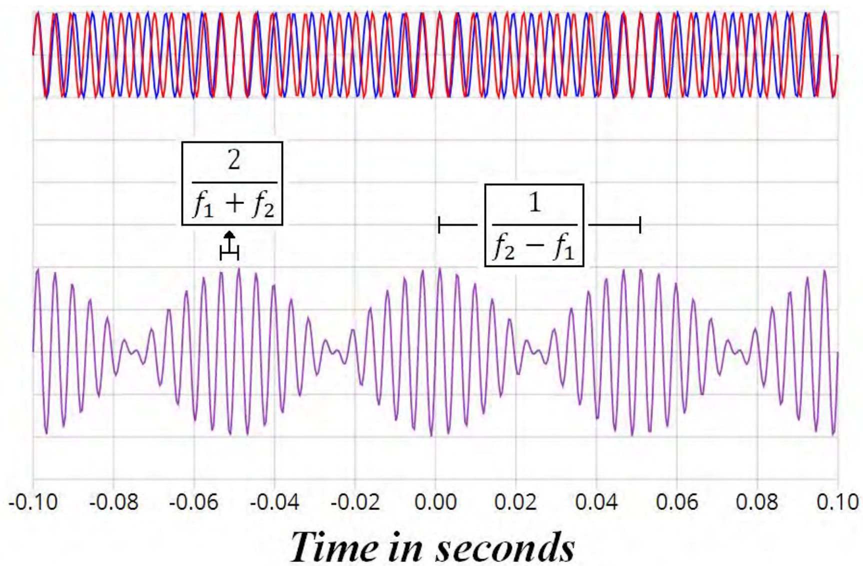

Figure 1-1.: Superposition of two sine waves $f_1 = 220\ Hz$ (blue) and $f_2 = 240\ Hz$ (red)

perceive the beats related with the frequency difference, even so a beat is perceive (in the example this beat has a frequency of $2Hz$). This phenomenon is called *virtual beats* or *second order beats* [2, 24], they result from the neural processing [2], and they are related with the non-tuning of the corresponding musical interval (in the example the musical interval is the octave in which $f_2/f_1 = 2$) [24].

Beats can also be heard when the pure tones are presented separately to our ears, these phenomena are called *binaural beats* [1, 24].

How far the frequencies must be to produce beats, and how they are related with the level of consonance depends, not only on the frequency difference, but also on the actual values of such frequencies, commonly measured using the center frequency $(f_1 + f_2)/2$ [2] which takes into account the mean location in the register.

In the study of Reinier Plomp and Willem Levelt about the level of consonance of pure tones [18], they found that, fixing a pure tone (with a frequency called *base frequency*) and moving the another one in the register, there is a transition range between consonance and dissonance related with a *critical bandwidth* that depends on the frequency difference of the corresponding pure tones. If the frequency difference is greater than a critical band, they sound *consonant* but if the frequency difference is smaller than a critical band, they sound *dissonant* [1].

This approach to consonance using the roughness produced by the superposition of sounds is known as tonal or sensory, because it is based on the physical properties of the stimulus independently of cultural conventions [10].

1.1.4. Quantifying the level of dissonance: the Sethares and Vassilakis models

William Sethares proposed a mathematical function to describe the empirical results for pure tones obtained by Plomp and Levelt, using a function of the form

$$\delta = exp\left[-\mathfrak{A}(f_2 - f_1)\right] - exp\left[-\mathfrak{B}(f_2 - f_1)\right], \tag{1-3}$$

with δ the dissonance level, and $\mathfrak{A} = 3.5$ and $\mathfrak{B} = 5.75$ determining the rates at which the function rises and falls [21]. In equation (1-3), the same frequency difference can be obtained using different combinations of f_1 and f_2, however the dissonance level is also dependent of the location in the register of such frequencies. To take into account this effect, Sethares proposed a function of the form [21, 25]

$$\delta \propto exp\left[-\mathfrak{A}\mathfrak{s}(f_2 - f_1)\right] - exp\left[-\mathfrak{B}\mathfrak{s}(f_2 - f_1)\right], \tag{1-4}$$

with $\mathfrak{s} = 0.24/(0.0207f_1 + 18.96)$. Figure **1-2** shows the dissonance level (normalized to 1 in all cases) as a function of the the frequency difference $f_2 - f_1$ for different base frequencies f_1.

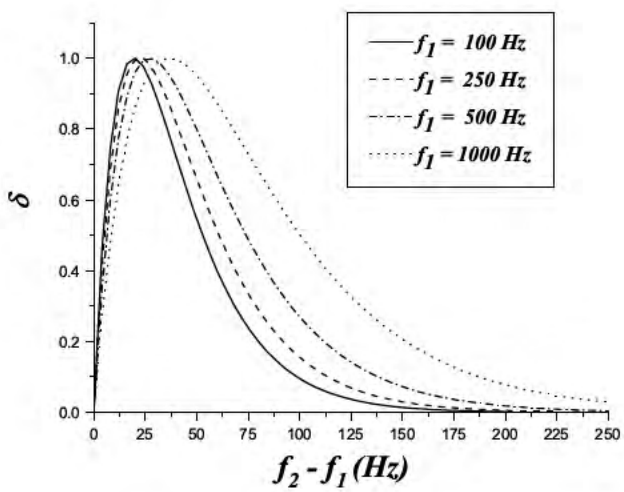

Figure 1-2.: Normalized Dissonance level δ as a function of the frequency difference $f_2 - f_1$ ($f_2 \geq f_1$) for different base frequencies f_1

Figure **1-2** shows that if the two pure tones have the same frequency, they are perceived as consonant, moving one of the frequencies a little with respect to the another one, slow beats are produce increasing the level of dissonance. After approximately $15Hz$, the frequency of the beats produces a sensation of roughness that is perceived as highly dissonant, after this maximum of dissonance the brain begin to distinguish the two sounds independently, but the roughness remains [25]. Finally the roughness sensation disappear, the two sounds are perceived independently, and the dissonance level decreases. Notice that the peak of maximum dissonance tends to move to higher frequency differences when the base frequency increases.

It is important to mention that the tonal consonance approach (associated to the roughness) corresponds to a particular way to understand the consonance, which not necessary corresponds to the meaning of this concept given in music through history [26, 27].

1.1.5. Effect of the amplitude

Since human ear is insensitive to phase for most combinations of harmonic vibrations [28], then the tonal consonance can be considered as phase independent, however the amplitude of sound waves contribute to the perception of loudness associated to the sonorous stimulus. Sethares proposed a seminal model that includes the amplitude contribution in the measure of the dissonance level. [21, 25, 29]. For the superposition of two pure tones of frequencies f_i and f_j, with amplitudes a_i and a_j, respectively, the dissonance level δ is

$$\delta = a_{max}a_{min}\left[e^{-b_1\mathfrak{s}(f_{max}-f_{min})} - e^{-b_2\mathfrak{s}(f_{max}-f_{min})}\right], \tag{1-5}$$

with $f_{max} = max(f_i, f_j)$, $f_{min} = min(f_i, f_j)$, $a_{max} = max(a_i, a_j)$, $a_{min} = min(a_i, a_j)$, $b_1 = 3.5$, $b_2 = 5.75$, and $\mathfrak{s} = 0.24/(0.0207f_{min} + 18.96)$.

This model (first Sethares model) produces good results when the amplitudes of the two pure tones have similar values, however the dissonance contribution is to small when the amplitudes differ considerable.

Vassilakis modified this model by including the dependence of roughness on intensity, the amplitude fluctuation degree, and amplitude fluctuation rate [30, pp. 197-198],[31].

$$\delta = (0.5)[(a_{max})(a_{min})]^{0.1}\left[\frac{2a_{min}}{a_{max}+a_{min}}\right]^{3.11}\left[e^{-b_1\mathfrak{s}(f_{max}-f_{min})} - e^{-b_2\mathfrak{s}(f_{max}-f_{min})}\right], \tag{1-6}$$

This model improves the predicted dissonance when the amplitudes of the two pure tones differ significantly.

Sethares proposed another model (second Sethares model), which takes into account that the amplitude of the beating is given by the minimum of the two amplitudes, and that the loudness of the roughness is proportional to the loudness of the beating [32].

$$\delta = l_{min}\left[e^{-b_1\mathfrak{s}(f_{max}-f_{min})} - e^{-b_2\mathfrak{s}(f_{max}-f_{min})}\right], \tag{1-7}$$

where l_{min} is the minimum loudness between the loudnesses produced by the two pure tones.

As the dissonance level is dimensionless, the amplitudes and the loudness must also be dimensionless.

1.1.6. Measurement of loudness

The loudness l is a subjective attribute of sound that depends strongly on pressure [1], however the frequency of the stimulus also contribute.

Sound pressure level

The sound intensity $\mathcal{I}$ corresponds to the sound power per unit of area, with units in the *SI* system of $[Watt/m^2]$. The lower limit for human perception of intensity is $10^{-12}[Watt/m^2]$ and the upper limit (pain limit) is $1[Watt/m^2]$ [2]. Commonly, the intensity is measured using a logarithmic scale. The sound intensity level IL can be express using a reference sound intensity $\mathcal{I}_0$ as:

$$IL = 10 \times log\frac{\mathcal{I}}{\mathcal{I}_0}, \tag{1-8}$$

and the units of IL are the decibels $[dB]$ and normally the reference sound intensity I_0 is taken as the hearing threshold $10^{-12}[Watt/m^2]$ (in this case $IL = 0dB$).

In order to study stationary waves which energy flux is zero and the intensity can not be defined using equation (1-8), the variations of pressure Δp have been used as the relevant quantity, and defined as [2]:

$$SPL = 20 \times log\left(\frac{\Delta p}{\Delta p_{ref}}\right). \tag{1-9}$$

Numerically, the last equation expresses the same as equation (1-8), however their meaning differ. The sound pressure level (SPL) has units of $[dBspl]$, and the hearing threshold Δp_{ref} in the case of air is approximately $2 \times 10^{-5}[N/m^2]$ [2].

Equal loudness curves

The psychological perception of loudness does not correspond directly to measurement of physical intensity. Figure **1-3** shows equal loudness curves, it is possible to appreciate that, in order to produce the same sensation of loudness in the low frequencies of the human perception range, we need more intensity than in the middle range. Notice that there is a minima between 3000 Hz and 4000 Hz.

The region of greatest sensitivity for human hearing (3000 Hz - 4000 Hz) can be explained by resonance properties of the auditory canal[33]. Modeling the auditory canal as a closed cylindrical tube, the pressure variation (indicated as Δp in Figure **1-4**) is minimum at the open end, and it reaches a maximum at the closed end. Figure **1-4** shows the first two modes of vibration of the auditory canal modeled as a closed tube. The auditory canal has a length of approximately 2.4 cm. Calculating the corresponding fundamental frequency associated to a length of $L' = 2.4$ cm, and using the velocity of sound in the air as $v_s = 344\ m/s$, a fundamental frequency $f = v_s/(4L') \approx 3.7\ KHz$ is obtained, which corresponds approximately to the location of the minima in the equal loudness curves (see Figure **1-3**).

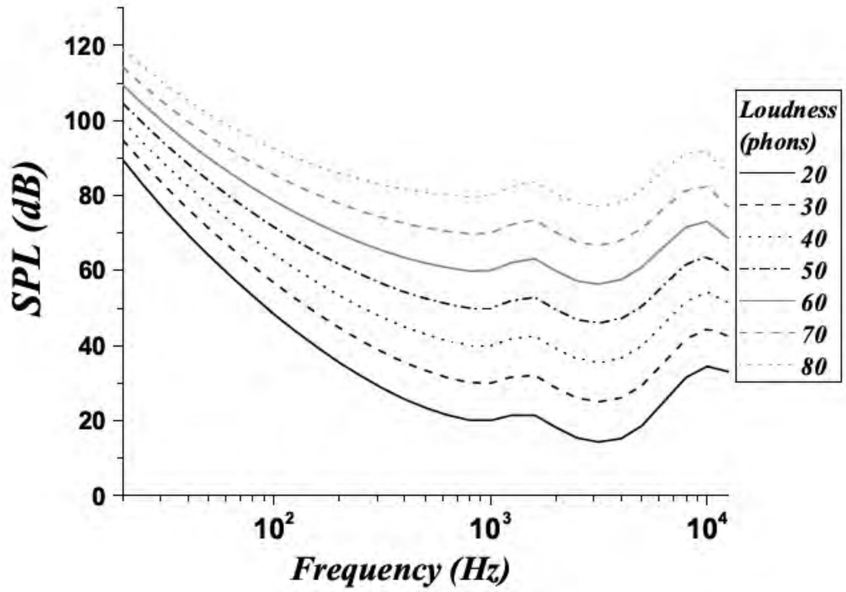

Figure 1-3.: Equal loudness curves for data between 20 Hz and 12500 Hz.

Another minimum appears beyond the region of 10 KHz, corresponding approximately to the third harmonic $(3f)$ of a closed cylindrical tube [33].

Two units of loudness commonly used are the *phon* and the *sone*. The *phon* is based in the decibels scale, and the *sone* was created in order to provide a linear scale for the perceived loudness.

In one hand, one *phon* is equivalent to 1 *dBspl* at 1 KHz, then if any sound is perceived to be as loud as X *dBspl* at 1 KHz, then this sound has X *phons*.

In the other hand, one *sone* is the loudness perceived by a 1000 Hz frequency with an intensity of $40dBspl$. A 10 *phons* increase in the sound level is usually perceived as a doubling in the loudness.

The conversion between loudness in *phons* (L_N) and loudness in *sones* (N), for $L_N > 40\,(N > 1)$, is given by

$$N = 2^{(0.1L_N - 4)};\;\; L_N = 40 + 10Log_2 N, \tag{1-10}$$

and for $L_N < 40\,(N < 1)$

$$N = \left(\frac{L_N}{40}\right)^{2.86} - 0.005;\;\; L_N = 40\,(N + 0.005)^{0.35}. \tag{1-11}$$

For a 1 KHz stimulus, the loudness N of a pure tone can be approximately related with the sound intensity $\mathcal{I}$ and the sound pressure Δp through Stevens Law [35, 36]:

$$N = \mathcal{Z}\mathcal{I}^{0.3} = \mathcal{Z}'\Delta p^{0.6}, \tag{1-12}$$

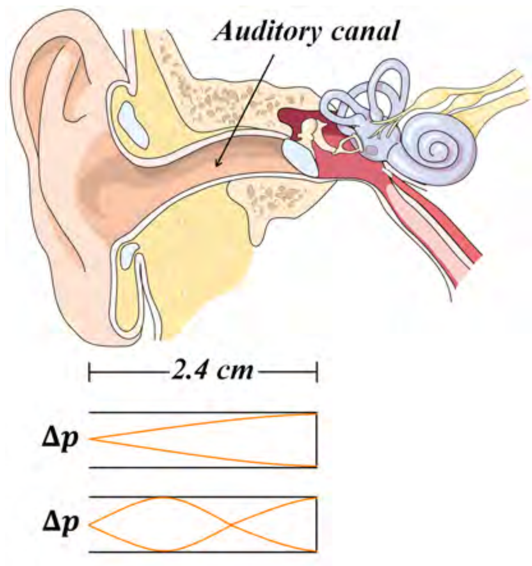

Figure 1-4.: Auditory canal anatomy modeled as a cylindrical closed tube [34].

where $\mathcal{Z}$ and $\mathcal{Z}'$ are constants.

Since the *sones* provides a linear scale for the perceived loudness, then this unit is useful to measure the loudness contribution to dissonance in the second Sethares model.

The Sethares approximation for loudness

In order to measure loudness, Sethares proposed a simple approximation based on Stevens Law [35] and the definition of *sone* for $N > 1$, given by (1-11). For a sinusoidal wave of amplitude a, the loudness N is approximately [32, p. 346].

$$l = \frac{1}{16} 2^{SPL/10}, \tag{1-13}$$

where SPL is the sound pressure level given by equation (1-9) with $\Delta p = a/\sqrt{2}$ [32, p. 346].

This approximation does not include the dependence of loudness with the frequency given by the equal loudness curves, but it gives a simple relation between the loudness and the amplitude: $l \approx c_o a^{0.60}$, (c_o constant) [37]. With this approximation l_{min} in (1-7) is found to be proportional to the minimum value between $a_i^{0.60}$ and $a_j^{0.60}$.

Figure **1-5** shows the loudness in *phons* as a function of the frequency for different values of the SPL. The loudness has been measured using the equal loudness curves (continuous lines) and the Sethares approximation (dashed lines). The range of frequencies ($20\,Hz - 6000\,Hz$) of Figure **1-5** corresponds to the range most frequently used in music. Notice that the approximation is better for high values of loudness. Besides, as this approximation is based in equation (1-11), constructed for $L_N > 40$ in the case of $1\,KHz$, the values of loudness for this frequency coincide with the values predicted by the equal loudness curves.

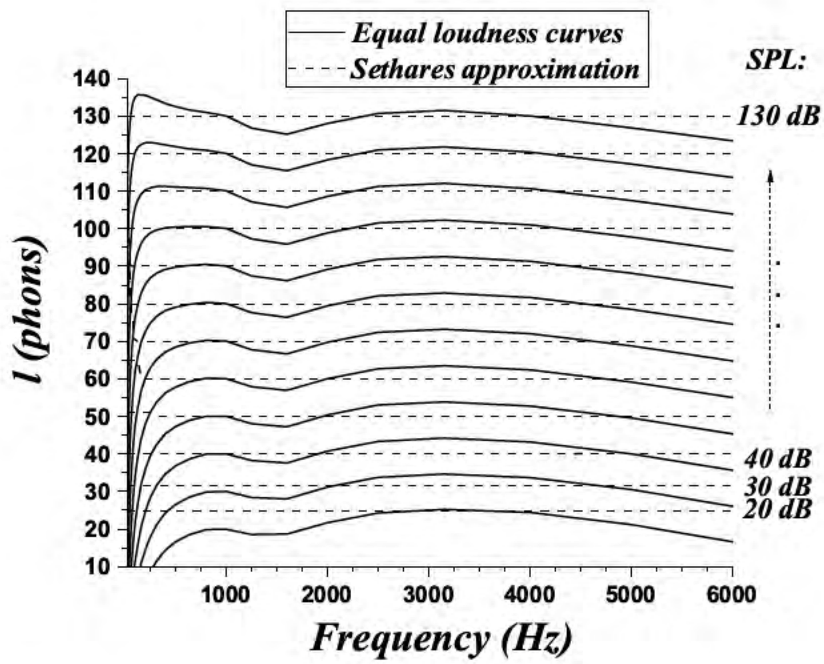

Figure 1-5.: Loudness l in *phons* estimated with the Sethares approximation (dashed lines) and with the equal loudness curves (continuous lines) for data between 20 *Hz* and 6000 *Hz*. In each case the sound pressure level differs by 10 dB.

1.1.7. Dissonance level of pairs of complex tones

Assuming that the main contribution to the perception of the timbre is contained in the the spectrum [21], and that a superposition principle for the measure of the dissonance in complex tones can be applied, then the following properties can inferred:

Dissonance level of an isolated complex tone

Let's suppose a complex tone F with m partials at frequencies $f_1, f_2, ..., f_m$, with $f_1 < f_2 < ... < f_m$, and f_1 the fundamental frequency. The total dissonance D_F can be calculated by superposing the individual dissonances for each possible pair of partials, it is

$$D_F = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\delta\left(f_i, f_j, x_i, x_j\right), \tag{1-14}$$

with $x_k = a_k$ for the first Sethares model [21, 25] and for the Vassilakis model, and $x_k = l_k$ for the second Sethares model [32].

Dissonance level of pairs of complex tones with the same timbre

Lets superpose two complex tones F and G with the same timbre. Each has m partials at frequencies $f_1, f_2, ..., f_m$, (with $f_1 < f_2 < ... < f_m$), and $g_1, g_2, ..., g_m$, (with $g_1 < g_2 < ... < g_m$). The number of partials is the same for the two sounds because we are considering the same timbre. If the pitch of the sound G is higher than the pitch of sound F, then there is a relation between the fundamental frequencies of the two sounds given by $g_1/f_1 = \alpha$, with $\alpha > 1$. Since the relative distances between the partial of sound F are equal to the relative distances between the partial of sound G, then there is also a relation for the overtones given by $g_x/f_x = \alpha$, with $x = 2, 3, ..., m$. In the case of the amplitudes, since timbre is assumed to be the same, then amplitudes $a_1, a_2, ..., a_m$ (or the loudnesses in the case of the second model of Sethares) of the sounds F and G can be taken as the same.

The total dissonance of the superposition of the two complex tones $D_F(\alpha)$ with the same timbre is given by [21, 25]

$$D_F(\alpha) = D_F + D_{\alpha F} + \sum_{i=1}^{m}\sum_{j=1}^{m} \delta\left(f_i, \alpha f_i, x_i, x_j\right), \tag{1-15}$$

where D_F and $D_{\alpha F}$ correspond to the total dissonance associated with the timbre of each individual complex tone, and the last term is due to the total dissonance generated by the interaction between the partials of the two complex tones. This equation can be used to identify the local minima of dissonance by scanning all possible values of α for the pair of fundamental frequencies f_1 and g_1. Usually the lowest fundamental frequency is taken as fixed, and the highest one varies changing the values of α. Fixing one base frequency, in the case of equal amplitudes for the fundamental frequency and the overtones, and for a normalized scale of dissonance in the interval [0, 1], the first Sethares model and the Vassilakis model predict the same results; additionally, if the Sethares approximation for loudness is used in the second Sethares model, then the three models are equivalent. Figure **1-6** shows the total dissonance $D_F(\alpha)$ (normalized to 1) as a function of the ratio of the fundamental frequencies α, and for six different base frequencies. The model used was the first Sethares model. This figure shows that the location of the minima is in agreement with the Pythagoras postulate in the sense that the location of the minima can be expressed in terms of two small natural numbers: $2/1 = 2.00$, $3/2 = 1.50$, $4/3 \approx 1.33$, $5/3 \approx 1.67$, $5/4 = 1.25$, $6/5 = 1.20$, $7/4 = 1.75$, $7/5 = 1.40$, $7/6 \approx 1.17$. Besides, Figure **1-6** shows that the dissonance level of two complex tones with the same fundamental frequency ratio α, depends on the location in the register of the complex tones, specifically lower registers (characterized by lower base frequencies) tends to be more dissonant, this phenomena is a well-known feature in music [2, 18].

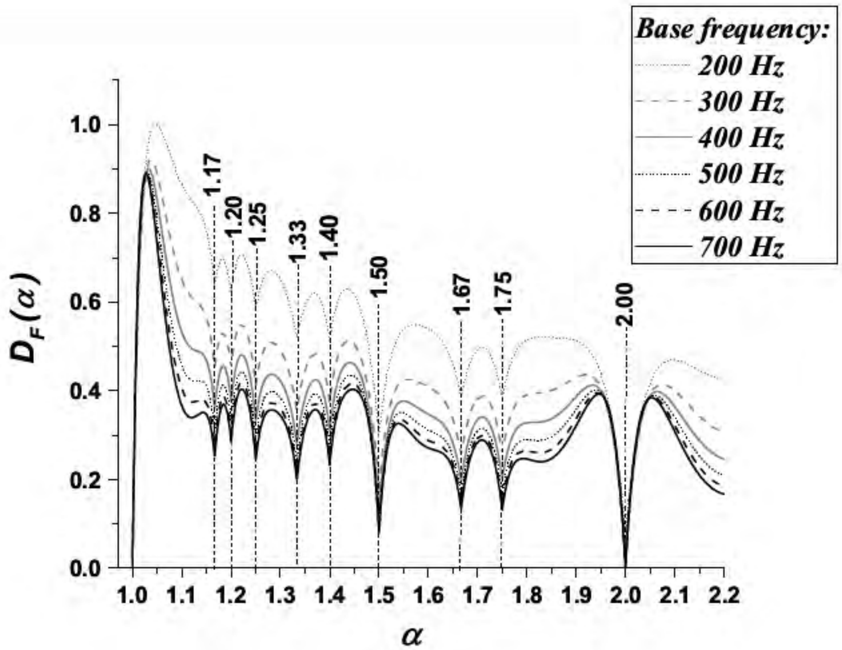

Figure 1-6.: Dissonance level $D_F(\alpha)$ for the superposition of two complex tones with the same harmonic timbre, as a function of the ratio of the fundamental frequencies α. A six harmonics spectrum with equal amplitudes has been consider for each sound. Six different base frequencies are consider. The first Sethares model has been used to measure the dissonance level, and the scale has been normalized to 1.

Dissonance level of pairs of complex tones with different timbre

Now, lets superpose two sounds F and G with different timbres. The sound F has partials at frequencies f_i with corresponding amplitudes a_i. The sound G has partials at frequencies g_j with corresponding amplitudes b_j. In the second Sethares model, quantities a_i and b_j correspond to the loudnesses of the partials instead of the amplitudes [32]. The ratio between the fundamental frequencies of the sounds will be taken as $g_1/f_1 = r$. If the pitch of the sound G is higher than the pitch of the sound F, then $r > 1$. Else, if the pitch of sound F is higher than the pitch of sound G, then $f_1/g_1 = y$, with $y > 1$. The relation between these two ratios is $r = 1/y$ [25, p. 117].

The total dissonance level of the superposition of these two sounds ($D_{F,G}$) is:
for the case $r > 1$

$$D_{F,G}(r) = D_F + D_{rG} + \sum_{i=1}^{m} \sum_{j=1}^{m'} \delta\left(f_i, g_j, a_i, b_j\right), \tag{1-16}$$

and for the case $y > 1$

$$D_{F,G}(y) = D_G + D_{yF} + \sum_{i=1}^{m} \sum_{j=1}^{m'} \delta\left(f_i, g_j, a_i, b_j\right). \tag{1-17}$$

If the pitch of the sound F is lower than the pitch of the sound G, then D_F and D_{rG} represent the total dissonance associated to the complex tones F and G, respectively. If the pitch of the sound G is lower than the pitch of the sound F, then D_G and D_{yF} represent the total dissonance associated to the complex tones G and F, respectively. The last term of equations (1-16) and (1-17) represents the total dissonance generated by the interaction between the partials of the two complex tones.

Notice that in the case of two different timbres there is a different contribution to the total dissonance depending of the selection of the timbre in the lower pitch, this phenomenon does not appear in the case of two complex tones with the same timbre because the contribution is independent of the selection of the lowest pitch.

Equations (1-16) and (1-17) can be used to identify the local minima of dissonance by scanning all possible values of r and y for the pair of fundamental frequencies f_1 and g_1. Figure **1-7** shows the dissonance level $D_{F,G}$ normalized to 1, as a function of the ratio of the fundamental frequencies r and y for the superposition of two complex tones with different timbres. The first sound has a timbre characterized by an spectrum with overtones at $1,7f_1$ and $2,84f_1$, the second pitch has an spectrum with overtones at $1.67g_1$ and $3.14g_1$. The loudnesses have been taken as 1 *sone*, 5 *sones*, and 5 *sones*, for the fundamental and the two overtones of both complex tones respectively [32, p. 126]. Figure **1-7** shows that the location of minima depends on the timbre of the lowest pitch.

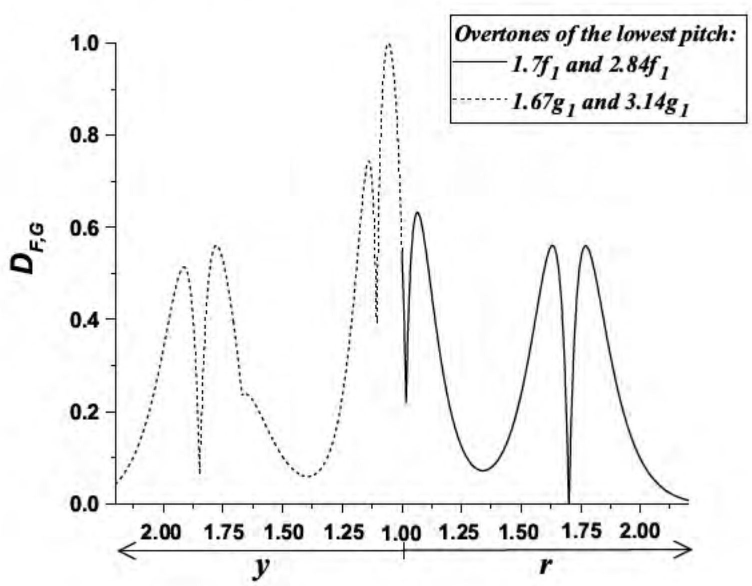

Figure 1-7.: Dissonance level $D_{F,G}$ as a function of the ratio of the fundamental frequencies r and y for two complex tones with different timbres. The second Sethares model has been used with loudnesses equal to 1 *sone* , 5 *sones*, and 5 *sones*, for the fundamental and the two overtones of both complex tones respectively. The dissonance level scale has been taken between 0 and 1. The base frequency has been taken as 300 Hz.

1.2. Musical scales, tuning and musical interval size

One of the most accepted approaches to the understanding of musical tunings and scales is based on consonance [27]. For several musical tunings the set of sounds is chosen in such a way as to yield a large number of consonant combinations when two or more elements are sounded together. Using this approach, Sethares posted that the best acoustical tuning of a musical instrument must be inferred from the local minima of dissonance associated to its timbre [21, 25, 32]. As it was explained above, considering the spectrum as the most important contribution to the perception of the timbre, a set of the minima of dissonance can be used in order to tune a musical instrument, in such a way that it produces the greatest number of consonance combinations of pairs of pitches [21, 29, 30, 32, 38].

In traditional Western Music, approximately before the second half of the 18th century, the most used tunings were the just and the Pythagorean. These tunings use one fundamental frequency as reference, and a set of fractions, given in the form of equation (1-1), in order to construct the other pitches. Nowadays the most used tuning is the 12-tone equal-tempered (12-TET), this tuning differs from the just and the Pythagorean in the sense that it does not verify the Pythagorean rule, as the frequency relation between pitches does not take rational values. Instead, the 12-TET is based on the division of a musical interval called the octave, that corresponds to a fraction of fundamental frequencies of 2, into equal "parts" in such a way that all semitones result to be equivalent.

1.2.1. Musical intervals

A musical interval is constituted by two pitches that belong to the same musical scale. If the two pitches sound simultaneously, then the interval is called *harmonic*; if the pitches sound successively, then the interval is called *melodic*. In the case of melodic intervals, there are three different possible cases: If the first pitch has a fundamental frequency smaller than the one of the second, then the interval is called *ascending*; if the first pitch has a fundamental frequency larger than the one of the second, then the interval is called *descending*; finally, if the both pitches have the same fundamental frequency, then the interval is called *unison*.

The size of a musical interval is defined as the distance in pitches between two of them [12], for example Figure **1-8** shows a music keyboard in which the size of the musical interval L in semitones is the number of steps between two keys. In this way we have the same music interval size between keys 1 and 7, and between keys 2 and 8 (size $L = 6$ in both cases).

1.2.2. Just, Pythagorean, and 12-TET scales

For the just scale, the relations between pitches are based in the major triads of a major scale called *tonic*, *subdominant* and *dominant*. Each one of these chords have a frequency relation between their successive pitches given by: $4 : 5 : 6$. Powers of these fractions are used

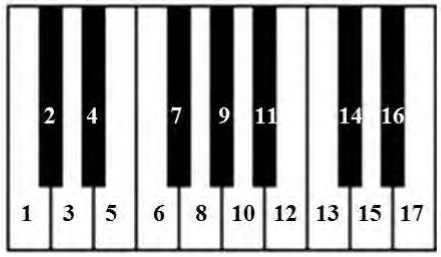

Figure 1-8.: Music keyboard. Each key has been numbered starting from the pitch C.

to construct all possible frequency relations between pitches starting from the fundamental frequency of reference.

For the Pythagorean scale, the fractions 3/2 (musical interval called *fifth*) and 4/3 (musical interval called *fourth*) are used to construct all possible relations between pitches multiplying powers of these fractions by the fundamental frequency of reference [1, 39].

For the 12-TET scale the fundamental frequencies are related by

$$f_i = f_0 \sqrt[12]{2^i}, \quad i \in \mathbb{Z}, \tag{1-18}$$

where f_0 is a reference fundamental frequency for constructing the others ones, and determines the *tuning* inside a particular musical scale, in most cases the reference frequency is taken as 440 Hz.

Table **1-1** summarizes the most common fundamental frequency ratios associated to the different musical intervals inside the octave for the just, Pythagorean, and 12-TET scales. Each fraction corresponds to a particular musical interval size. Notice that the just scale is the most appropriated for a harmonic timbre, in the sense that it uses many fractions that can be expressed in terms of two small natural numbers located at the minima of dissonance associated to the harmonic spectrum (see Figure **1-6**). Besides, notice that for the just, Pythagorean, and 12-TET scales, each musical interval size L corresponds to a particular fundamental frequency ratio α. As each α can be produced using different locations in the register (see Figure **1-6**), then the musical interval size is not enough to determine the level of dissonance because this quantity does not distinguish the location in the register of the pitches.

In the 12-TET scale, the partition of the octave in 12 equal parts generates the known names for pitches: A, A♯ or B♭, B, C or or B♯, C♯ or D♭, D, D♯ or E♭, E, F or E♯, F♯ or G♭, G, and G♯ or A♭. Notice that in this scale the sharps (♯) and the flats (♭) are equivalent because the partition has been created using equal parts, this feature is called *enharmonic equivalence*. This equivalence has the advantage that a musical piece can be played starting from different pitches without changing the fundamental frequency ratios of the intervals in the original location. Moving the same set of intervals of a musical piece to another location in the register conserving the same chronological order of occurrence, is a musical process called *transposition*. Up to now we have considered the partition of the octave in 12 pitches,

		Scale						
		Just			*Pythagorean*			*12-TET*
Interval name	*Size (semitones)*	n	m	n/m	n	m	n/m	f_j/f_i
Unison	0	1	1	1.000	1	1	1.000	1
Minor second	1	16	15	1.067	2187	2048	1.068	$2^{1/12}$
Major second	2	9	8	1.125	9	8	1.125	$2^{2/12}$
Minor third	3	6	5	1.200	32	27	1.185	$2^{3/12}$
Major third	4	5	4	1.250	81	64	1.266	$2^{4/12}$
Fourth	5	4	3	1.333	4	3	1.333	$2^{5/12}$
Tritone	6	45	32	1.406	729	512	1.424	$2^{6/12}$
Fifth	7	3	2	1.500	3	2	1.500	$2^{7/12}$
Minor sixth	8	8	5	1.600	6561	4096	1.602	$2^{8/12}$
Major sixth	9	5	3	1.667	27	16	1.688	$2^{9/12}$
Minor seventh	10	16	9	1.778	16	9	1.778	$2^{10/12}$
Major seventh	11	15	8	1.875	243	128	1.898	$2^{11/12}$
Octave	12	2	1	2.000	2	1	2.000	2

Table 1-1.: Frequency ratios used for generating the Pythagorean, the just, and the 12-TET scales. Interval size up to one octave (12 semitones). The frequency ratios for the just and the Pythagorean scales come from the Pythagorean rule: $f_j/f_i = n/m$. For all scales, the frequency ratios corresponding to intervals larger than one octave are obtained by multiplying the corresponding ratio of the previous octave by 2.

however intervals larger than the octave can be generated multiplying or dividing by powers of 2 the fundamental frequencies of the 12 pitches inside the octave. An interesting property emerges in this process, in the sense that the human being tends to appreciate in a similar way the pitches that differ in one or more octaves, this phenomenon is called *chroma* [2, p. 188] For this reason, the name of pitches differing by one or more octaves is the same. In the case of intervals larger than the octave, the *chroma* property of pitches states that the consonance values of these intervals can be measured by displacing the highest pitch to the next lower octave until the resulting interval is smaller than or equal to one octave [2, p. 188].

In the just and the Pythagorean scales, since the flats are not completely equivalent to the sharps, the transposition process affects the fundamental frequency ratios of the intervals; for this reason, the 12-TET scale is the most used scale in nowadays. This is an example in which the predicted minima of dissonance for an harmonic spectrum, that are largely took into account by the just scale, are sacrificed in order to facilitate a musical process, in this case the transposition.

There are many instruments tuned in the 12-TET that produced all the 12 pitches. The scale constituted by the 12 pitches is known as *chromatic* [12, p. 164]. However, in some cases there are musical instruments, also tuned in the 12-TET, that do not produce the 12 pitches, instead of this they produce a set of them. A common set is the one generated by the 7 pitches: C, D, E, F, G, A, and B, that produces a major scale on C, or a minor natural scale on A. This set of pitches, in the corresponding order, constitutes a *diatonic* scale [12, p. 231]. Transposing this sequence of intervals (conserving the order) to another initial pitch leads to another diatonic scales.

1.2.3. Equal temperament and isotonic scales

In some cases the octave is not divided into 12 equal parts, as in the 12-TET. Instead of this, the octave can be divided in a different number, this scales are known as $\mathcal{N}$-TET, with $\mathcal{N}$ the number of parts of the division. For a $\mathcal{N}$-TET scale the fundamental frequencies are related by

$$f_i = f_0 \sqrt[\mathcal{N}]{2^i}, \quad i \in \mathbb{Z}, \tag{1-19}$$

where f_0 is the reference fundamental frequency. Typically, the $\mathcal{N}$-TET scales assumes the octave as the interval to be partitioned, in this thesis we will follow this convention. In the case that the partition is carried out using a different musical interval $\mathcal{P}$, these scales are known as *isotonic* [40]

$$f_i = f_0 \sqrt[\mathcal{N}]{\mathcal{P}^i}, \quad i \in \mathbb{Z}, \tag{1-20}$$

with f_0 the reference fundamental frequency.

The partition of an isotonic scale using $\mathcal{N}$ parts is known as equi-$\mathcal{N}$tonic, for example for $\mathcal{N} = 5$ the name of the scale is equi-pentatonic, for $\mathcal{N} = 6$ equi-hexatonic, and for $\mathcal{N} = 7$ equi-heptatonic [40, 41].

1.3. Statistical analysis of musical intervals

Many quantitative analyses in music have been carried out using different elements as building blocks, or "units of context," which allow the message of a musical piece to be apprehensible at different time scales [42]. Common choices for these "units of context" are single pitches (ignoring or taking into account the *chroma* properties [43, 44]), single musical notes (i.e., pitch and rhythm values), pairs of pitches or musical intervals (either harmonic or melodic), triplets of pitches between contiguous notes, and chords [42, 43, 44, 45, 46, 47, 48, 49, 50].

In the case of musical intervals, quantitative analyses frequently employ parameters that describe their psychoacoustic properties, such as the sizes of intervals (commonly measured in semitones), the ratio of the fundamental frequencies of both pitches (commonly measured in units of *cents*), and the difference between the fundamental frequencies [2, 18].

Analyses based on statistical methods can capture information about musical features, such as the style of a musical piece, the composer, and even the emotions conveyed [42, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59]. Several statistical analyses employ successive pitches as units of context. George Kingsley Zipf studied the frequency of occurrences of melodic intervals in masterpieces of Western music, and he reported that the frequency of occurrences of ascending and descending intervals is almost inversely proportional to their size [45]. Vos and Trost studied music from 13 great composers of Western academic music, the Beatles, and folk music, finding that the proportion between musical interval sizes is too complex to be represented by a simple exponential or power law function. They also reported an asymmetry in the use of ascending and descending intervals [46]. Melodies tend to meander around a central pitch range [60, p. 76], and in many cultures an asymmetry emerges, in the sense that large melodic intervals are more likely to ascend than small ones [60, p. 75],[46]. Figure **1-9** illustrates this asymmetry: the vertical axis represents different pitches around a central pitch, located at zero, and the horizontal axis represents the time evolution. In this sketch, rhythm figures are all equal, with a time value of 1 s.

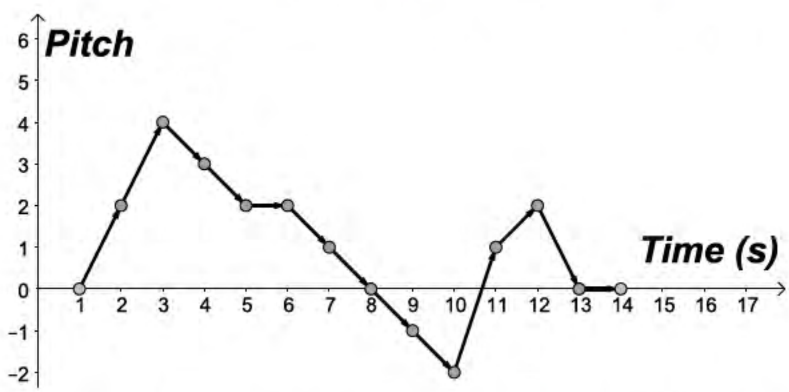

Figure 1-9.: Representation of the typical motion of pitches in a melody: large melodic intervals are more likely to ascend than small ones.

Gunnar Niklasson and Maria Niklasson studied the occurrences of melodic intervals as a function of their size, finding long-tailed Levy-stable distributions that they associated to a "music walk" between successive pitches, in analogy with a random walk [58]. In the framework of a network analysis, Liu, Small, and Tse studied the connectivity properties of complex networks representing the successive notes of musical pieces, finding scale-free behavior in the nodal degree for several sets of academic and popular music [49]. Entropy has been employed to measure the amount of information transmitted by music to a listener in a sequence of events organized in time [61]. One example is given by the study by

Güngör Gündüz and Ufuk Gündüz concerning the evolution of the entropy associated to the transitions between pitches during the progress of a melody. This study found that entropy increases up to a limiting value, which is smaller than the entropy of a random melody [62]. David Huron carried out a study of nearly 10,000 Western musical themes, finding that the average melodic interval size is slightly smaller in pieces written in a minor mode than in those written in a major mode. This result was interpreted by the author as a relation between sadness and small values for the average melodic interval size [47]. Huron also found that themes in a minor mode have slightly lower pitches on average in comparison with major ones [53], which suggests that the sizes of intervals and their locations in the register are important for conveying musical information.

2. Interplay between practices and tuning in the *marimba de chonta* music

2.1. On the *Marimba de chonta*

In the Pacific Coast of Colombia and Ecuador there is a marimba with bars made of the timber from a palm called *Chonta* (*Bactris jauari*), and tubular resonators made of a Bambuseae called *Guadua* (*Guadua angustifolia*) (Figure **2-1**). This marimba is called *marimba de chonta* and provides the melodic and harmonic contour for a traditional music of African descent.

Figure 2-1.: *Marimba de chonta* image: Bars made of *Chonta* and tubular resonators made of *Guadua*.

"The marimba music, traditional chants and dances from the Colombia South Pacific region and Esmeraldas Province of Ecuador" were inscribed by UNESCO in the list of Intangible Cultural Heritage of Humanity [13, 14, 15, 16].

Some instrument makers of the *marimba de chonta* use ancestral techniques for empirically tuning the instrument, resulting in tunings that do not conform to Western musical scales [13, 17, 40]. These tunings and the musical practices associated with this marimba remain largely unknown, and they are currently at risk of disappearing [13, 17]. In the Pacific coast of Colombia, the *marimba de chonta* with a traditional tuning is called "traditional marimba".

Traditional marimbas are played by one or two musicians, each using two percussion mallets that, commonly, simultaneously hit two different bars [17, 40, 63, 64, 65]. There are reports of the existence of several tunings [17, 40, 63] related to the voice of female singers of a particular territory [63], and with some features as for example the presence of low octaves and neutral thirds (neutral thirds are musical intervals between minor and major thirds) [40, 41].

The tunings of traditional marimbas do not follow any specific mathematical progression conserving musical intervals; however, the averages of the distance in cents between successive pairs of bars suggest the presence of equi-heptatonic scales [40, 41]. It has to be stressed that this tendency does not mean that these marimbas are equi-heptatonic, but that distances between pairs of successive bars meander around average values consistent with equi-heptatonic scales. Deviations in the tuning with respect to mathematical successions are also present in music of other cultures around the world, for example in the tuning of the gamelan, with deviations respect to equi-penthatonic scales [66].

The harmonic structure of this music can be summarized in geometrical schemes that contain the rules to play two bars simultaneously (harmonic interval) over a base that usually contains seven pitches that repeat periodically−−in a similar way to the octaves in a diatonic scale. In the case of seven pitches, there are two possible ways for grouping them: by discarding one bar, leading to a hexatonic scale; or discarding two bars, producing a pentatonic scale [17, 40, 63].

Figure **2-2** shows the harmony rules for the hexatonic and pentatonic scales: white bars are the discarded bars, and bars identified with grey or black colors can be played simultaneously when they have the same color. The largest harmonic interval used has six bars between the two sounded bars [63].

In this traditional music, female singers lead and the marimba interpreters have to accompany them finding out the best location in the register to play the corresponding succession of harmonic intervals. Since the harmony rules presented in Figure **2-2** are independent of the selection of the initial bar for any succession of harmonic intervals, then this practice corresponds to a transposition principle that preserves the same relative geometrical distances without conserving the same musical intervals [17, 40, 63, 67]. This practice is not well understood because transposition processes that keep the same relative geometrical distances must be carried out over isotonic scales, as for example the equi-heptatonic one, which is not the case of the traditional marimba. Instead, the tuning in which a musical piece is played in a traditional marimba just depends on the selection of the initial bar.

2.2. Methods

This section contains the theoretical and experimental procedures used to produce the dissonance level curves for the *marimba de chonta*, make the statistical analysis for the musical pieces, and compare the results obtained in this study with those from a previous one.

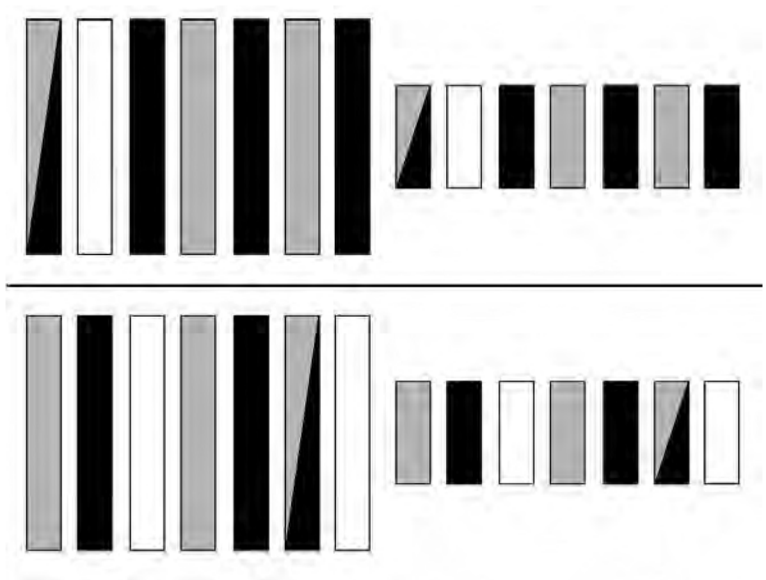

Figure 2-2.: Hexatonic (up) and Pentatonic (down) scales generated by an equi-heptatonic traditional marimba. Large bars and small bars indicate two different groups of seven bars. White bars are not used in each scale. Grey and black bars constitute two different harmonic families.

2.2.1. Theoretical

Measure of the dissonance level and tuning

We use three models to measure dissonance levels: The first Sethares model (equation (1-5)), the Vassilakis model (equation (1-6)), and the second Sethares model (equation (1-7)). The normalized amplitudes a_{norm} have been used in all cases. The approximation 1-13 was used to measure loudness in the second Sethares model; with this approximation l_{min} is found to be proportional to the minimum value between two amplitudes $a_1^{0.60}$ and $a_2^{0.60}$. Normalized loudness (l_{norm}), can be measured using the normalized amplitudes, leading to $l_{norm} = l/l_{max} = (a/a_{max})^{0.60} = (a_{norm})^{0.60}$.

In this work we measure the tuning of several *marimbas de chonta* using the ratio between the fundamental frequencies of pairs of bars. This quantity was selected in order to describe the consonance properties of the tunings, as the dissonance curves are presented traditionally using this frequency ratio.

For all cases, the frequency ratio α was measured with an uncertainty $\Delta\alpha$ given by

$$\Delta\alpha = |\alpha| \left(\frac{\Delta f_1}{f_1} + \frac{\Delta f_2}{f_2} \right), \tag{2-1}$$

where f_1 and f_2 are the fundamental frequencies measured with uncertainties Δf_1 and Δf_2, respectively.

Determination of the fundamental frequency for the Miñana study

In order to compare the results from this study with those from a previous one carried out in 1990 by Carlos Miñana [40], including 9 traditional marimbas made between 1950 and 1986, the fundamental frequency of each bar was obtained for all the marimbas as follows: Miñana produced a score with the approximate pitches associated with each marimba bar, including their corresponding increment or decrement in *cents* with respect to the 12-TET scale with A=440 Hz. Miñana measured with an uncertainty of ± 2.5 *cents*.

The approximate fundamental frequencies (f_{Real}) produced by these marimbas were found from the fundamental frequencies in the 12-TET scale (f_{Temp}), and using the relation between the number of cents e and the corresponding frequency ratio r: $f_{Real} \approx r \cdot f_{Temp}$; $r = c^e$, where c refers to a cent ($c = 2^{1/1200}$) [2, pp. 194-195]. Appendix A contains the score as presented by Miñana [40]. The marimbas presented in the study carried out by Miñana are identified with the subindex "M", the marimbas from the present study remain without subindex.

2.2.2. Experimental

In order to obtain evidence about the musical practices related to the traditional marimba, as well as about the tunings produced by instrument makers, an expedition to the heart of the *marimba de chonta* territories was carried out in 2015. The expedition was composed by musicians and researchers from the Conservatory of Music, the Department of Physics, and the School of Cinema and Media Arts of Universidad Nacional de Colombia. During the expedition the researchers interviewed instrument makers, female singers, and *marimba de chonta* interpreters, as well as music teachers from local schools. Additionally, the team attended several presentations by local musical groups.

Selection of the marimbas

This study includes 10 traditional marimbas, each one constructed by a different recognized instrument maker, and one diatonic *marimba de chonta* tuned in the 12-TET. Table **2-1** shows the instrument makers and the main features of the selected marimbas. Marimbas 1, 2, 3, 4, 6, 9, and 11 were made between 2014 and 2015. The remaining ones are undated and presumably old instruments; however, these marimbas are currently in use, and a common practice is to repair these instruments due to their natural fragility and the harsh conditions of use.

Procedure for the analysis of pitch and timbre

For all marimbas the sound of each bar with its respective resonator were recorded; in all cases the bars were struck in the geometrical center in agreement with the common style of traditional musicians. The microphone was located approximately at 40 cm directly over

Marimba number	Maker	Place	Bars	Kind
1	Baudilio Cuama	Buenaventura	22	Diatonic 12-TET
2	Baudilio Cuama	Buenaventura	16	Traditional
3	Jhon Jairo Cortés	Tumaco	24	Traditional
4	Francisco Tenorio	Tumaco	24	Traditional
5	Juan E. Sinisterra	Tumaco	18	Traditional
6	Silvino Mina	Guapi	20	Traditional
7	Dioselino Rodríguez	Guapi	17	Traditional
8	Guillermo Ríos	Guapi	17	Traditional
9	Genaro Torres	Guapi	21	Traditional
10	José Torres	Guapi	20	Traditional
11	Francisco Torres	Guapi	24	Traditional

Table 2-1.: Description of each marimba recorded for this study, including identification number, instrument maker, place, total number of bars, and tuning.

each bar, this direction was selected because it coincides with the position of the interpreter. Marimbas 1, 2 were recorded using TASCAM DR44WL with Behringer C-2 condenser microphone; marimbas 3, 4, 5, 6, 7, 9, 10 using ZOOM H6 handy recorder with X/Y microphone; marimba 8 using RCA VR5320R-A; marimba 11 using a Sony Cyber-shot DSC-H100 camera. The sample rates were: 96 KHz for marimbas 1, 2; 48 KHz for marimbas 4, 5, 6, 9, 10; 44.1 KHz for marimbas 3, 7; 8 KHz for marimbas 8, 11. The spectrum was obtained from samples covering at maximum the sound between the attack and the final release, and avoiding external sound sources. The samples recorded for this study are included in the "Audio Files 1". A Fast Fourier Transform over a Hanning window was used for finding the spectrum of each bar. The first 50 peaks with the largest amplitudes were identified, and the fundamental and the first 10 overtones with the largest amplitudes were included in the analysis. The sampling interval was determined from the average increment of the time sequence [68], and the peaks were identified using the Local Maximum Method up to the second nearest neighbors [69, 70]. For almost every bar, the fundamental frequency corresponds to the peak with the largest amplitude; however, for the traditional marimbas of the Torres family (marimbas 9, 10, and 11), some bars have overtones with larger amplitudes than that of the fundamental. In the sample of 223 bars, the fundamental frequency was not clearly identified for four bars (marked with "* " in Table **2-2**); these values were not used in the analysis. For all bars of each marimba the amplitudes were normalized for the analysis.

Procedure for the analysis of musical pieces

Seven musical pieces played by recognized traditional marimba musicians were recorded, with one musician and two mallets for each marimba (a maximum of two pitches can be sounded simultaneously in each musical piece). Musical scores with the closest transcriptions to the 12-TET scale with $A = 440Hz$ were generated. Traditional tunings do not follow a 12-TET system; however, this procedure allows the identification of the approximate size of harmonic intervals in order to measure their frequency of occurrence. The original recordings and musical scores are provided in "Audio Files 2" and Appendix B, respectively. The transcription of the musical scores was made by Kevin Pineda. In order to infer the most frequent harmonic intervals in traditional music, the musical scores were used to measure the probability of occurrence of each interval size. In order to distinguish between brief and lengthy intervals, a second analysis was carried out taking into account the time duration of each kind of interval, according to its size. The probability of occurrence was defined as proportional to the sum of the duration of each interval of a given size. For u different intervals with size z and F_z occurrences for each size the probability of a specific interval size is $p_z = T_z/T$, where $T_z = t_{1_z} + t_{2_z} + ... + t_{F_z}$ is the sum of the durations of all intervals with size z and total time $T = T_1 + T_2 + ... + T_u$. If all intervals have the same duration in a musical score, then the probability p_z is equal to the probability of occurrence.

2.3. Results and Discussion

The first part of this section is devoted to the tuning of the marimbas. The second part presents the use of harmonic intervals in this music. The third part is related to the consonance properties of the *marimba de chonta*.

2.3.1. Tunings

For all marimbas, the average frequency ratios r that result from the combination of all bars at different distances, the minimum and maximum values, and the standard deviation (σ) were found.

The distance s between bars is defined as the number of steps to reach the final bar starting from the initial one; hence, adjacent bars have a distance of 1 step.

Present study

For the marimbas of the present study, the fundamental frequency found for each bar coupled to its respective resonator is shown in Table **2-2**.

	Fundamental frequency (Hz)					
Bar	**Marimba 1**	**Marimba 2**	**Marimba 3**	**Marimba 4**	**Marimba 5**	**Marimba 6**
1	131.62 ± 0.59	270.85 ± 0.98	184.70 ± 0.48	167.91 ± 0.78	158.66 ± 0.55	171.93 ± 0.38
2	145.93 ± 0.63	305.01 ± 1.26	204.74 ± 0.43	188.53 ± 0.81	170.82 ± 0.54	175.40 ± 0.58
3	165.23 ± 0.54	341.11 ± 0.86	223.41 ± 0.58	209.54 ± 0.82	190.35 ± 0.81	200.89 ± 0.41
4	175.38 ± 0.53	367.60 ± 1.06	251.43 ± 0.65	219.15 ± 1.07	214.15 ± 0.69	221.17 ± 0.57
5	197.31 ± 0.50	409.85 ± 1.27	282.56 ± 1.02	241.62 ± 1.47	229.48 ± 1.32	250.50 ± 0.53
6	220.47 ± 0.43	450.50 ± 1.10	309.48 ± 1.18	275.01 ± 0.86	259.31 ± 0.91	267.41 ± 0.48
7	247.30 ± 0.43	496.49 ± 1.48	334.65 ± 0.97	298.57 ± 1.08	294.34 ± 1.58	292.42 ± 0.70
8	261.68 ± 0.41	536.73 ± 1.61	367.40 ± 0.95	331.03 ± 1.55	316.90 ± 1.04	334.85 ± 0.75
9	293.58 ± 0.58	609.06 ± 1.05	413.15 ± 1.68	369.93 ± 1.68	349.58 ± 0.71	360.21 ± 0.65
10	330.00 ± 0.45	687.68 ± 1.55	451.49 ± 1.28	413.35 ± 1.24	394.01 ± 0.85	387.14 ± 0.59
11	351.58 ± 0.52	747.17 ± 1.10	500.13 ± 1.05	440.39 ± 1.39	430.19 ± 0.84	438.80 ± 0.65
12	393.69 ± 0.51	818.48 ± 1.07	542.98 ± 1.32	486.81 ± 2.06	472.91 ± 0.93	481.43 ± 0.62
13	439.43 ± 0.50	910.90 ± 1.22	634.07 ± 1.29	539.29 ± 1.42	535.00 ± 1.42	518.89 ± 0.73
14	495.44 ± 0.47	994.77 ± 1.53	672.32 ± 1.57	585.53 ± 1.66	582.64 ± 1.02	580.70 ± 0.83
15	523.24 ± 0.57	1080.37± 1.30	740.70 ± 1.76	644.27 ± 1.82	650.65 ± 1.43	654.80 ± 0.93
16	588.65 ± 0.60	1240.32± 1.19	838.27 ± 1.98	737.47 ± 1.80	702.45 ± 1.53	705.79 ± 1.10
17	660.86 ± 0.54	——	896.38 ± 2.28	813.44 ± 1.74	798.79 ± 1.51	783.34 ± 1.06
18	700.72 ± 0.45	——	989.25 ± 1.71	876.62 ± 1.98	871.50 ± 2.25	859.39 ± 1.23
19	791.14 ± 0.42	——	1128.03± 2.14	986.17 ± 3.22	——	928.18 ± 1.18
20	878.25 ± 0.40	——	1248.11± 3.30	1131.23 ± 2.42	——	1011.77 ± 1.34
21	985.22 ± 0.49	——	1342.93± 3.09	1224.65 ± 3.22	——	——
22	1044.23 ± 0.54	——	1448.71± 3.29	1338.57 ± 3.36	——	——
23	——	——	1663.24± 4.04	1472.77 ± 4.36	——	——
24	——	——	1767.66± 4.58	1643.50 ± 4.83	——	——

	Fundamental frequency (Hz)				
Bar	**Marimba 7**	**Marimba 8**	**Marimba 9**	**Marimba 10**	**Marimba 11**
1	190.21 ± 1.79	145.71 ± 0.79	191.01 ± 1.68	130.00 ± 4.67*	169.39 ± 3.85*
2	210.60 ± 1.35	167.24 ± 0.80	205.53 ± 1.12	123.35 ± 2.68	143.81 ± 2.05*
3	226.28 ± 1.38	180.60 ± 0.84	227.84 ± 1.22	128.76 ± 3.39	209.20 ± 3.61*
4	249.61 ± 1.56	195.12 ± 0.84	247.56 ± 1.63	148.30 ± 3.09	153.34 ± 2.19
5	270.62 ± 1.37	207.33 ± 0.84	272.75 ± 0.99	147.50 ± 1.76	163.50 ± 2.82
6	305.05 ± 1.66	229.89 ± 0.85	287.25 ± 2.14	169.88 ± 1.57	184.31 ± 2.49
7	332.30 ± 1.33	252.85 ± 0.86	313.55 ± 2.75	187.36 ± 1.40	198.21 ± 1.38
8	365.02 ± 2.40	285.65 ± 0.88	349.14 ± 1.75	207.35 ± 0.56	207.13 ± 0.80
9	396.51 ± 1.42	315.69 ± 0.91	379.01 ± 2.92	219.98 ± 0.93	215.10 ± 1.47
10	449.56 ± 1.35	355.90 ± 0.85	409.60 ± 2.41	243.93 ± 0.58	233.14 ± 1.67
11	479.41 ± 1.53	390.45 ± 0.86	462.12 ± 2.16	262.04 ± 0.74	263.92 ± 2.75
12	543.40 ± 1.63	429.08 ± 0.94	495.17 ± 3.99	294.23 ± 0.78	283.19 ± 5.06
13	600.83 ± 1.39	477.37 ± 0.90	539.33 ± 2.43	311.65 ± 1.61	299.34 ± 1.64
14	655.81 ± 1.74	515.07 ± 0.87	603.41 ± 2.04	356.91 ± 0.76	312.88 ± 1.44
15	719.59 ± 2.50	572.42 ± 0.87	636.04 ± 2.65	381.38 ± 0.97	358.66 ± 1.52
16	800.23 ± 1.45	648.36 ± 0.89	722.67 ± 3.04	428.30 ± 1.37	378.97 ± 0.91
17	910.42 ± 1.50	722.11 ± 0.92	755.61 ± 2.33	475.78 ± 1.03	412.29 ± 0.82
18	——	——	874.57 ± 2.97	528.51 ± 1.18	431.40 ± 1.59
19	——	——	935.74 ± 2.35	563.27 ± 1.20	481.15 ± 1.69
20	——	——	1057.87± 2.08	608.65 ± 1.16	527.56 ± 1.88
21	——	——	1121.73± 3.17	——	556.13 ± 2.28
22	——	——	——	——	609.91 ± 2.54
23	——	——	——	——	657.09 ± 2.30
24	——	——	——	——	716.08 ± 2.28

Table 2-2.: Fundamental frequencies produced by each bar with its corresponding resonator. The uncertainty corresponds to one half of the distance between adjacent frequencies in the FFT analysis. The mark "*" refers to unconfident values for the fundamental frequencies.

The 12-TET marimba was found to be tuned in a major diatonic tempered scale over C with the traditional reference A=440 Hz. Figure **2-3(*a*)** shows the structure of tones and semitones found for the diatonic 12-TET marimba at different distances s. For example, adjacent bars present minor and major seconds, bars separated by a distance of 2 steps present minor and major thirds, and bars separated by a distance of 7 steps always produce octaves. Notice that the probability of finding a minor second, a major seventh or a tritone is small, a natural consequence of the structure of the major diatonic tempered scale. For this marimba, the average value of the frequency ratios at different distances (see Marimba 1 in Table **2-3**) corresponds to the average between the frequency ratios of the musical intervals belonging to the 12-TET scale. For example, for $s = 1$ the average value 1.104 results from the average for the minor seconds, with a frequency ratio of $2^{1/12}$, and major seconds, with a frequency ratio of $2^{2/12}$. Hence, for the 12-TET marimba the information about the average and the standard deviations σ values for each s is trivial, in the sense that this information only exhibits the internal structure of tones and semitones. However, in the case of the traditional marimbas, since the transposition practice allows to start a musical piece at any bar of the marimba, then this type of analysis shows the tolerance of the tuning for using this practice.

The tunings of the traditional marimbas of the present study were not found to follow a mathematical rule. Instead, the frequency ratios of pairs of bars separated the same distance s, were found to be distributed around the average frequency ratios. Figures **2-3(*b*)**,**(*c*)** show, for marimbas 2 and 5, the probability to find a particular frequency ratio generated by pairs of bars separated by different relative distances s. The average values r are indicated by dotted lines, and the ranges indicated as $r \pm \Delta$ represent uncertainties that contain the total deviation with respect to the average values.

In the traditional marimbas of the present study, three different behaviors of the average frequency ratios of the musical intervals were found: 7 marimbas, numbered from 2 up to 8, follow an equi-heptatonic scale, marimbas 9 and 10 follow an equi-octatonic scale, and the marimba number 11 follows an equi-enneatonic scale. This behavior was found for the averages, and it does not mean that these marimbas have isotonic tunings.

The marimbas with equi-heptatonic averages fulfill the condition that the average frequency ratio associated with two bars separated a distance s is $r_7^{s/7}$, with r_7 the average ratio of the fundamental frequencies for bars separated by a distance of 7 steps. Hence, knowing the value of r_7 for each marimba allows predicting the other average values. For the marimbas with equi-heptatonic averages, the values of r_7 follow $1.957 \leq r_7 \leq 2.043$, with an average $\overline{r_7} = 1.998 \pm 0.011$. The information about the frequency ratios generated at different distances in the case of the marimbas with equi-heptatonic averages is shown in Table **2-3**.

The marimbas with equi-octatonic averages fulfill the condition $r_8^{s/8}$ ($r_8 = 2.042 \pm 0.022$ and $r_8 = 2.078 \pm 0.026$); and the marimba number 11 follows an equi-enneatonic scale in the averages $r_9^{s/9}$, with $r_9 = 1.990 \pm 0.026$. In the case of these marimbas, r_8 and r_9 refer to the average ratios of the fundamental frequencies for bars separated by a distance of 8 and 9 steps, respectively. As in the previous case, knowing the values of r_8 and r_9 for each marimba allows predicting the other average values.

The information about the frequency ratios generated at different distances for the marimbas with equi-octatonic and equi-enneatonic averages is showed in Table **2-4**. These marimbas were made by members of the Torres family, who also made three of the marimbas studied by Miñana (marimbas 1_M, 2_M, 5_M).

The theoretical equi-heptatonic, equi-octatonic and equi-enneatonic scales constructed using r_7, r_8 and r_9 are presented in Table **2-6**. The predicted values have less than 1.00 % relative error (R.E.) with respect to the empirical values shown in Tables **2-3** and **2-4**.

Up to now, we have modeled with isotonic scales the relations between the average frequency ratios generated by bars separated by the same distance. The deviations with respect to the average values can be considered as contained inside an uncertainty (represented by "$\pm\Delta$" in Figure **2-3**). The use of uncertainties is useful in this case because the dispersion of the data inside each uncertainty does not follow any specific pattern, and it presents variations for different traditional marimbas, and even between different distances in the same traditional marimba. Despite of this, some features can be concluded:

Taking $\Delta = \sigma$, approximately 67 % of the frequency ratios are contained. Taking $\Delta = 2\sigma$, approximately 97 %. In order to take into account the mean intrinsic uncertainty associated to the measurement of each individual frequency ratio (equation (2-1)), a total uncertainty (T.U.) $\Delta = 2\sigma + \Delta\text{Avg.}$ can be considered as representative in order to capture most of the data. The values of the T.U. can be related with the average values r using a R.E. The mean R.E. associated to the traditional marimbas with equi-heptatonic averages is 5.3 %. In the case of the traditional marimbas with equi-octatonic and equi-enneatonic averages, the mean R.E. is 7.1 % and 7.7 %, respectively. For all marimbas, the values of ΔAvg., T.U., and R.E. associated to each distance between bars are presented in Tables **2-3** and **2-4**.

Summarizing, for the traditional marimbas with equi-heptatonic, equi-octatonic, and equi-enneatonic averages, most frequency ratios of musical intervals are contained inside the ranges $r \pm 0.053r$, $r \pm 0.071r$, and $r \pm 0.077r$, respectively, with r following isotonic scales.

The data presented in Tables **2-3** and **2-4** are also presented in units of *cents* at Appendix C.

Miñana study

For the traditional marimbas studied by Miñana, the average values for the frequency ratios of bars separated by the same geometrical distance are consistent with equi-heptatonic scales, stressing that this is only a tendency in the averages and does not mean that these marimbas are tuned using an equi-heptatonic scale.

Figure **2-3(*d*)** shows, for the case of the marimba 3_M, the probability to find a particular frequency ratio generated by pairs of bars separated by different relative distances s. The average values r are indicated by dotted lines, and the ranges indicated as $r \pm \Delta$ represent uncertainties that contain the deviations with respect to the average values. Notice the presence of low octaves, a behavior that contrasts with the findings of the present study.

For each marimba, the average ratio of the fundamental frequencies for bars separated by a distance of 7 steps (r_7) is slightly smaller than two, $1.903 \leq r_7 \leq 1.982$, with an average of 1.942 ± 0.006 (see Table **2-5**); hence, these marimbas present low octaves in the averages. The theoretical equi-heptatonic scales constructed using equation $r_7^{s/7}$ with the experimental values of r_7 are presented in Table **2-7**. The predicted values have less than 1.00 % R.E. with respect to the empirical values shown in Table **2-5**.

For the marimbas studied by Miñana, if the uncertainty of the frequency ratios is $\Delta = \sigma$, then it contains approximately 70 % of the frequency ratios. Taking $\Delta = 2\sigma$, then approximately 96 % of the data is contained. Taking the T.U. ($\Delta = 2\sigma + \Delta\text{Avg.}$) to contain most of the frequency ratios, the mean R.E. is 5.1 %. Hence, most frequency ratios of musical intervals are contained inside the range $r \pm 0.051r$, with the values of r following an equi-heptatonic scale.

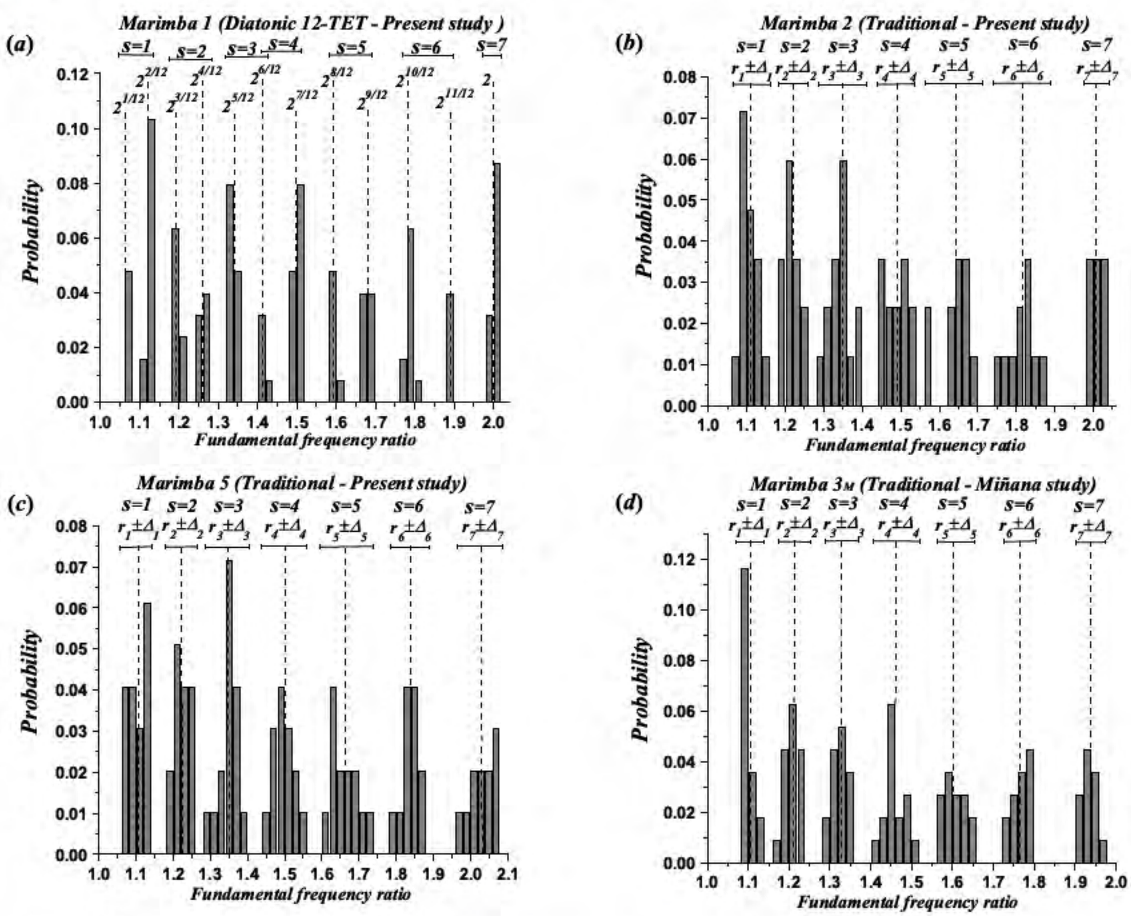

Figure 2-3.: Probability to find a particular frequency ratio generated from pairs of bars separated by different relative distances. (a) Diatonic 12-TET marimba. (b) Marimba 2 from the present study. (c) Marimba 5 from the present study. (d) Marimba 3M from the Miñana study. In all cases the bin width has been taken as 0.02.

The values of ΔAvg., T.U., and R.E. associated to each distance between bars are presented in Table **2-5**.

The data presented in Table **2-5** is also presented in units of *cents* at Appendix C.

		Marimba number - Present study				
s		**1**	**2**	**3**	**4**	**5**
1	**Min±ΔMin**	1.056 ± 0.002	1.078 ± 0.006	1.060 ± 0.005	1.046 ± 0.009	1.072 ± 0.010
	Max±ΔMax	1.132 ± 0.009	1.148 ± 0.002	1.168 ± 0.005	1.147 ± 0.006	1.137 ± 0.005
	Avg.±ΔAvg.	1.104 ± 0.004	1.107 ± 0.005	1.104 ± 0.006	1.105 ± 0.007	1.106 ± 0.007
	σ	0.029	0.021	0.028	0.024	0.023
	T. U. (R. E.%)	0.061 (5.57 %)	0.048 (4.31 %)	0.061 (5.54 %)	0.056 (5.06 %)	0.052 (4.74 %)
2	**Min±ΔMin**	1.187 ± 0.004	1.186 ± 0.003	1.161 ± 0.006	1.153 ± 0.012	1.188 ± 0.009
	Max±ΔMax	1.263 ± 0.002	1.281 ± 0.007	1.268 ± 0.005	1.290 ± 0.006	1.283 ± 0.014
	Avg.±ΔAvg.	1.221 ± 0.004	1.220 ± 0.005	1.219 ± 0.006	1.218 ± 0.008	1.225 ± 0.007
	σ	0.032	0.027	0.032	0.036	0.024
	T. U. (R. E.%)	0.069 (5.64 %)	0.060 (4.92 %)	0.069 (5.70 %)	0.079 (6.52 %)	0.056 (4.56 %)
3	**Min±ΔMin**	1.320 ± 0.001	1.310 ± 0.008	1.284 ± 0.005	1.282 ± 0.013	1.313 ± 0.006
	Max±ΔMax	1.410 ± 0.007	1.392 ± 0.006	1.404 ± 0.007	1.397 ± 0.007	1.381 ± 0.012
	Avg.±ΔAvg.	1.347 ± 0.004	1.346 ± 0.006	1.345 ± 0.007	1.343 ± 0.009	1.355 ± 0.008
	σ	0.028	0.025	0.030	0.033	0.018
	T. U. (R. E.%)	0.061 (4.53 %)	0.056 (4.15 %)	0.068 (5.02 %)	0.075 (5.62 %)	0.043 (3.18 %)
4	**Min±ΔMin**	1.414 ± 0.002	1.446 ± 0.004	1.414 ± 0.006	1.417 ± 0.008	1.446 ± 0.013
	Max±ΔMax	1.512 ± 0.002	1.526 ± 0.007	1.544 ± 0.007	1.534 ± 0.007	1.546 ± 0.015
	Avg.±ΔAvg.	1.489 ± 0.005	1.488 ± 0.007	1.485 ± 0.008	1.486 ± 0.010	1.499 ± 0.009
	σ	0.027	0.030	0.035	0.036	0.028
	T. U. (R. E.%)	0.058 (3.89 %)	0.067 (4.51 %)	0.077 (5.20 %)	0.083 (5.56 %)	0.064 (4.30 %)
5	**Min±ΔMin**	1.580 ± 0.002	1.571 ± 0.005	1.560 ± 0.006	1.559 ± 0.009	1.607 ± 0.012
	Max±ΔMax	1.695 ± 0.010	1.697 ± 0.007	1.726 ± 0.008	1.756 ± 0.009	1.723 ± 0.015
	Avg.±ΔAvg.	1.642 ± 0.005	1.643 ± 0.008	1.639 ± 0.008	1.646 ± 0.011	1.661 ± 0.010
	σ	0.046	0.040	0.042	0.050	0.035
	T. U. (R. E.%)	0.097 (5.88 %)	0.087 (5.27 %)	0.092 (5.64 %)	0.112 (6.78 %)	0.080 (4.84 %)
6	**Min±ΔMin**	1.773 ± 0.002	1.760 ± 0.013	1.728 ± 0.008	1.742 ± 0.013	1.783 ± 0.008
	Max±ΔMax	1.893 ± 0.005	1.871 ± 0.010	1.895 ± 0.009	1.932 ± 0.010	1.875 ± 0.014
	Avg.±ΔAvg.	1.814 ± 0.006	1.815 ± 0.008	1.810 ± 0.009	1.816 ± 0.012	1.840 ± 0.011
	σ	0.049	0.035	0.042	0.054	0.024
	T. U. (R. E.%)	0.103 (5.70 %)	0.079 (4.33 %)	0.093 (5.12 %)	0.120 (6.58 %)	0.059 (3.22 %)
7	**Min±ΔMin**	1.988 ± 0.012	1.982 ± 0.013	1.922 ± 0.012	1.946 ± 0.015	1.979 ± 0.014
	Max±ΔMax	2.012 ± 0.013	2.036 ± 0.005	2.077 ± 0.009	2.098 ± 0.010	2.070 ± 0.008
	Avg.±ΔAvg.	1.999 ± 0.006	2.011 ± 0.009	1.998 ± 0.010	2.004 ± 0.013	2.031 ± 0.012
	σ	0.007	0.018	0.036	0.047	0.030
	T. U. (R. E.%)	0.021 (1.03 %)	0.045 (2.24 %)	0.083 (4.17 %)	0.107 (5.35 %)	0.071 (3.52 %)

		Marimba number - Present study			
s		**6**	**7**	**8**	**Avg.*$_1$**
1	**Min±ΔMin**	1.020 ± 0.006	1.066 ± 0.007	1.063 ± 0.009	1.058 ± 0.007
	Max±ΔMax	1.145 ± 0.006	1.138 ± 0.004	1.148 ± 0.012	1.147 ± 0.006
	Avg.±ΔAvg.	1.098 ± 0.004	1.103 ± 0.009	1.105 ± 0.007	1.104 ± 0.006
	σ	0.031	0.022	0.023	0.025
	T. U. (R. E.%)	0.067 (6.06 %)	0.052 (4.75 %)	0.052 (4.70 %)	0.055 (5.02 %)
2	**Min±ΔMin**	1.156 ± 0.004	1.185 ± 0.015	1.148 ± 0.010	1.168 ± 0.008
	Max±ΔMax	1.262 ± 0.004	1.265 ± 0.006	1.262 ± 0.004	1.273 ± 0.006
	Avg.±ΔAvg.	1.211 ± 0.004	1.214 ± 0.010	1.218 ± 0.007	1.218 ± 0.007
	σ	0.034	0.023	0.034	0.030

	T. U. (R. E. %)	0.072 (5.92 %)	0.057 (4.70 %)	0.076 (6.25 %)	0.067 (5.51 %)
3	**Min±ΔMin**	1.286 ± 0.006	1.285 ± 0.015	1.240 ± 0.011	1.286 ± 0.009
	Max±ΔMax	1.428 ± 0.008	1.388 ± 0.006	1.408 ± 0.008	1.400 ± 0.008
	Avg.±ΔAvg.	1.334 ± 0.005	1.336 ± 0.011	1.342 ± 0.008	1.343 ± 0.008
	$\boldsymbol{\sigma}$	0.032	0.029	0.047	0.031
	T. U. (R. E. %)	0.069 (5.17 %)	0.069 (5.14 %)	0.103 (7.66 %)	0.069 (5.13 %)
4	**Min±ΔMin**	1.417 ± 0.004	1.423 ± 0.021	1.375 ± 0.012	1.420 ± 0.010
	Max±ΔMax	1.525 ± 0.008	1.515 ± 0.009	1.548 ± 0.009	1.534 ± 0.009
	Avg.±ΔAvg.	1.470 ± 0.005	1.472 ± 0.012	1.479 ± 0.009	1.482 ± 0.009
	$\boldsymbol{\sigma}$	0.033	0.027	0.055	0.035
	T. U. (R. E. %)	0.072 (4.91 %)	0.067 (4.57 %)	0.119 (8.03 %)	0.079 (5.30 %)
5	**Min±ΔMin**	1.545 ± 0.004	1.572 ± 0.014	1.512 ± 0.012	1.561 ± 0.009
	Max±ΔMax	1.691 ± 0.005	1.675 ± 0.008	1.717 ± 0.011	1.712 ± 0.009
	Avg.±ΔAvg.	1.616 ± 0.006	1.625 ± 0.014	1.638 ± 0.010	1.638 ± 0.009
	$\boldsymbol{\sigma}$	0.048	0.037	0.061	0.045
	T. U. (R. E. %)	0.102 (6.34 %)	0.087 (5.37 %)	0.132 (8.04 %)	0.099 (6.04 %)
6	**Min±ΔMin**	1.701 ± 0.008	1.733 ± 0.023	1.708 ± 0.013	1.736 ± 0.012
	Max±ΔMax	1.909 ± 0.011	1.899 ± 0.009	1.888 ± 0.010	1.895 ± 0.010
	Avg.±ΔAvg.	1.783 ± 0.006	1.790 ± 0.015	1.813 ± 0.011	1.810 ± 0.010
	$\boldsymbol{\sigma}$	0.050	0.045	0.060	0.044
	T. U. (R. E. %)	0.106 (5.92 %)	0.105 (5.85 %)	0.131 (7.23 %)	0.099 (5.46 %)
7	**Min±ΔMin**	1.922 ± 0.007	1.883 ± 0.019	1.888 ± 0.015	1.932 ± 0.013
	Max±ΔMax	2.054 ± 0.011	2.025 ± 0.009	2.077 ± 0.012	2.062 ± 0.009
	Avg.±ΔAvg.	1.964 ± 0.007	1.968 ± 0.017	2.009 ± 0.012	1.998 ± 0.011
	$\boldsymbol{\sigma}$	0.039	0.047	0.058	0.039
	T. U. (R. E. %)	0.084 (4.30 %)	0.111 (5.62 %)	0.127 (6.34 %)	0.090 (4.50 %)

Table 2-3.: Minimum, maximum, average, standard deviation, total uncertainty, and relative error of the frequency ratio for pairs of bars separated by different distances in the diatonic 12-TET marimba, and the traditional marimbas with equi-heptatonic averages recorded in the present study. "s" is the distance between bars in steps The symbol, "σ" is the standard deviation, "Avg." refers to the average, "Avg.$*_1$" refers to the average of the traditional marimbas 2, 3, 4, 5, 6, 7, and 8, "T. U." is the total uncertainty, and "R. E." corresponds to the "T. U." value expressed as a relative error.

s		Marimba number - Present study 9	10	Avg.*$_2$	11
1	**Min±ΔMin**	1.046 ± 0.008	0.995 ± 0.033	1.020 ± 0.020	1.038 ± 0.011
	Max±ΔMax	1.157 ± 0.008	1.152 ± 0.054	1.155 ± 0.031	1.146 ± 0.010
	Avg.±ΔAvg.	1.093 ± 0.011	1.093 ± 0.015	1.093 ± 0.013	1.081 ± 0.014
	σ	0.031	0.041	0.036	0.031
	T. U. (R. E. %)	0.073 (6.65 %)	0.097 (8.87 %)	0.085 (7.76 %)	0.077 (7.13 %)
2	**Min±ΔMin**	1.150 ± 0.014	1.145 ± 0.034	1.148 ± 0.024	1.085 ± 0.015
	Max±ΔMax	1.238 ± 0.007	1.270 ± 0.025	1.254 ± 0.016	1.227 ± 0.021
	Avg.±ΔAvg.	1.197 ± 0.012	1.199 ± 0.015	1.198 ± 0.014	1.168 ± 0.015
	σ	0.023	0.035	0.029	0.042
	T. U. (R. E. %)	0.057 (4.80 %)	0.085 (7.08 %)	0.071 (5.94 %)	0.100 (8.53 %)
3	**Min±ΔMin**	1.252 ± 0.008	1.196 ± 0.040	1.224 ± 0.024	1.167 ± 0.024
	Max±ΔMax	1.400 ± 0.007	1.406 ± 0.021	1.403 ± 0.014	1.318 ± 0.009
	Avg.±ΔAvg.	1.308 ± 0.014	1.313 ± 0.016	1.310 ± 0.015	1.259 ± 0.016
	σ	0.038	0.053	0.046	0.046
	T. U. (R. E. %)	0.089 (6.81 %)	0.123 (9.33 %)	0.106 (8.07 %)	0.107 (8.51 %)
4	**Min±ΔMin**	1.376 ± 0.019	1.377 ± 0.043	1.377 ± 0.031	1.265 ± 0.026
	Max±ΔMax	1.485 ± 0.009	1.527 ± 0.011	1.506 ± 0.010	1.414 ± 0.011
	Avg.±ΔAvg.	1.431 ± 0.015	1.445 ± 0.017	1.438 ± 0.016	1.355 ± 0.017
	σ	0.036	0.039	0.038	0.034
	T. U. (R. E. %)	0.087 (6.07 %)	0.095 (6.58 %)	0.091 (6.33 %)	0.085 (6.28 %)
5	**Min±ΔMin**	1.502 ± 0.014	1.483 ± 0.037	1.493 ± 0.026	1.403 ± 0.030
	Max±ΔMax	1.663 ± 0.010	1.696 ± 0.013	1.680 ± 0.011	1.538 ± 0.012
	Avg.±ΔAvg.	1.559 ± 0.017	1.585 ± 0.019	1.572 ± 0.018	1.464 ± 0.019
	σ	0.044	0.058	0.051	0.041
	T. U. (R. E. %)	0.105 (6.72 %)	0.134 (8.45 %)	0.119 (7.60 %)	0.100 (6.84 %)
6	**Min±ΔMin**	1.635 ± 0.013	1.645 ± 0.038	1.640 ± 0.025	1.510 ± 0.019
	Max±ΔMax	1.766 ± 0.020	1.816 ± 0.009	1.791 ± 0.015	1.686 ± 0.014
	Avg.±ΔAvg.	1.708 ± 0.018	1.734 ± 0.021	1.721 ± 0.019	1.585 ± 0.021
	σ	0.045	0.052	0.049	0.059
	T. U. (R. E. %)	0.109 (6.39 %)	0.125 (7.22 %)	0.117 (6.81 %)	0.140 (8.82 %)
7	**Min±ΔMin**	1.798 ± 0.020	1.767 ± 0.042	1.782 ± 0.031	1.579 ± 0.018
	Max±ΔMax	1.961 ± 0.013	2.017 ± 0.010	1.989 ± 0.011	1.777 ± 0.015
	Avg.±ΔAvg.	1.866 ± 0.020	1.900 ± 0.024	1.883 ± 0.022	1.712 ± 0.023
	σ	0.046	0.076	0.061	0.060
	T. U. (R. E. %)	0.111 (5.95 %)	0.176 (9.27 %)	0.144 (7.62 %)	0.142 (8.32 %)
8	**Min±ΔMin**	1.977 ± 0.016	1.977 ± 0.048	1.977 ± 0.032	1.698 ± 0.031
	Max±ΔMax	2.136 ± 0.021	2.167 ± 0.010	2.152 ± 0.016	1.949 ± 0.017
	Avg.±ΔAvg.	2.042 ± 0.022	2.078 ± 0.026	2.060 ± 0.024	1.846 ± 0.025
	σ	0.056	0.064	0.060	0.060
	T. U. (R. E. %)	0.134 (6.56 %)	0.154 (7.41 %)	0.144 (6.99 %)	0.144 (7.81 %)
9	**Min±ΔMin**	—	—	—	1.912 ± 0.018
	Max±ΔMax	—	—	—	2.100 ± 0.017
	Avg.±ΔAvg.	—	—	—	1.990 ± 0.026
	σ	—	—	—	0.057
	T. U. (R. E. %)	—	—	—	0.141 (7.06 %)

Table 2-4.: Minimum, maximum, average, standard deviation, total uncertainty, and relative error of the frequency ratio for pairs of bars separated by different distances in the traditional marimbas with equi-octatonic and equi-enneatonic averages recorded in the present study. "s" is the distance between bars in steps. The symbol, "σ" is the standard deviation, "Avg." refers to the average, "Avg.*$_2$" refers to the average of the traditional marimbas 9 and 10, "T. U." is the total uncertainty, and "R. E." corresponds to the "T. U." value expressed as a relative error.

		Marimba number - Miñana study				
s		**1_M**	**2_M**	**3_M**	**4_M**	**5_M**
1	**Min±ΔMin**	1.041 ± 0.003	1.059 ± 0.003	1.081 ± 0.003	1.081 ± 0.003	1.066 ± 0.003
	Max±ΔMax	1.162 ± 0.003	1.155 ± 0.003	1.136 ± 0.003	1.116 ± 0.003	1.129 ± 0.003
	Avg.±ΔAvg.	1.105 ± 0.003	1.103 ± 0.003	1.098 ± 0.003	1.100 ± 0.003	1.096 ± 0.003
	σ	0.031	0.029	0.015	0.008	0.015
	T. U. (R. E. %)	0.064 (5.83 %)	0.061 (5.49 %)	0.033 (3.00 %)	0.020 (1.81 %)	0.032 (2.95 %)
2	**Min±ΔMin**	1.179 ± 0.003	1.189 ± 0.003	1.179 ± 0.003	1.189 ± 0.003	1.142 ± 0.003
	Max±ΔMax	1.297 ± 0.004	1.275 ± 0.004	1.235 ± 0.004	1.231 ± 0.004	1.238 ± 0.004
	Avg.±ΔAvg.	1.217 ± 0.004	1.218 ± 0.004	1.207 ± 0.003	1.209 ± 0.003	1.203 ± 0.003
	σ	0.029	0.028	0.018	0.011	0.025
	T. U. (R. E. %)	0.062 (5.06 %)	0.059 (4.83 %)	0.039 (3.25 %)	0.025 (2.10 %)	0.054 (4.49 %)
3	**Min±ΔMin**	1.320 ± 0.004	1.267 ± 0.004	1.293 ± 0.004	1.304 ± 0.004	1.238 ± 0.004
	Max±ΔMax	1.418 ± 0.004	1.406 ± 0.004	1.350 ± 0.004	1.358 ± 0.004	1.358 ± 0.004
	Avg.±ΔAvg.	1.345 ± 0.004	1.340 ± 0.004	1.326 ± 0.004	1.329 ± 0.004	1.322 ± 0.004
	σ	0.029	0.039	0.031	0.014	0.032
	T. U. (R. E. %)	0.061 (4.56 %)	0.082 (6.15 %)	0.066 (4.97 %)	0.032 (2.40 %)	0.067 (5.08 %)
4	**Min±ΔMin**	1.414 ± 0.004	1.456 ± 0.004	1.418 ± 0.004	1.431 ± 0.004	1.358 ± 0.004
	Max±ΔMax	1.597 ± 0.005	1.516 ± 0.004	1.507 ± 0.004	1.498 ± 0.004	1.481 ± 0.004
	Avg.±ΔAvg.	1.483 ± 0.004	1.482 ± 0.004	1.458 ± 0.004	1.461 ± 0.004	1.456 ± 0.004
	σ	0.048	0.023	0.022	0.017	0.032
	T. U. (R. E. %)	0.100 (6.76 %)	0.050 (3.40 %)	0.049 (3.35 %)	0.037 (2.55 %)	0.067 (4.62 %)
5	**Min±ΔMin**	1.569 ± 0.005	1.569 ± 0.005	1.560 ± 0.005	1.578 ± 0.005	1.533 ± 0.004
	Max±ΔMax	1.711 ± 0.005	1.682 ± 0.005	1.643 ± 0.005	1.639 ± 0.005	1.634 ± 0.005
	Avg.±ΔAvg.	1.631 ± 0.005	1.632 ± 0.005	1.603 ± 0.005	1.605 ± 0.005	1.603 ± 0.005
	σ	0.042	0.041	0.026	0.017	0.028
	T. U. (R. E. %)	0.089 (5.45 %)	0.087 (5.35 %)	0.057 (3.58 %)	0.038 (2.40 %)	0.060 (3.76 %)
6	**Min±ΔMin**	1.741 ± 0.005	1.751 ± 0.005	1.721 ± 0.005	1.731 ± 0.005	1.682 ± 0.005
	Max±ΔMax	1.910 ± 0.006	1.866 ± 0.005	1.787 ± 0.005	1.792 ± 0.005	1.803 ± 0.005
	Avg.±ΔAvg.	1.798 ± 0.005	1.799 ± 0.005	1.763 ± 0.005	1.764 ± 0.005	1.760 ± 0.005
	σ	0.054	0.041	0.021	0.019	0.034
	T. U. (R. E. %)	0.113 (6.27 %)	0.087 (4.86 %)	0.048 (2.72 %)	0.043 (2.43 %)	0.072 (4.12 %)
7	**Min±ΔMin**	1.888 ± 0.005	1.910 ± 0.006	1.904 ± 0.005	1.899 ± 0.005	1.834 ± 0.005
	Max±ΔMax	2.125 ± 0.006	2.047 ± 0.006	1.966 ± 0.006	1.966 ± 0.006	1.988 ± 0.006
	Avg.±ΔAvg.	1.976 ± 0.006	1.976 ± 0.006	1.935 ± 0.006	1.939 ± 0.006	1.934 ± 0.006
	σ	0.066	0.051	0.019	0.023	0.042
	T. U. (R. E. %)	0.137 (6.95 %)	0.108 (5.46 %)	0.044 (2.27 %)	0.051 (2.65 %)	0.091 (4.68 %)

		Marimba number - Miñana study				
s		**6_M**	**7_M**	**8_M**	**9_M**	**Avg.***
1	**Min±ΔMin**	1.041 ± 0.003	1.066 ± 0.003	1.063 ± 0.003	1.047 ± 0.003	1.061 ± 0.003
	Max±ΔMax	1.122 ± 0.003	1.149 ± 0.003	1.149 ± 0.003	1.169 ± 0.003	1.143 ± 0.003
	Avg.±ΔAvg.	1.091 ± 0.003	1.102 ± 0.003	1.097 ± 0.003	1.095 ± 0.003	1.098 ± 0.003
	σ	0.026	0.019	0.022	0.028	0.021
	T. U. (R. E. %)	0.054 (4.99 %)	0.041 (3.75 %)	0.047 (4.28 %)	0.059 (5.42 %)	0.046 (4.17 %)
2	**Min±ΔMin**	1.110 ± 0.003	1.159 ± 0.003	1.149 ± 0.003	1.110 ± 0.003	1.156 ± 0.003
	Max±ΔMax	1.253 ± 0.004	1.253 ± 0.004	1.264 ± 0.004	1.264 ± 0.004	1.256 ± 0.004
	Avg.±ΔAvg.	1.194 ± 0.003	1.214 ± 0.004	1.203 ± 0.003	1.200 ± 0.003	1.207 ± 0.003
	σ	0.039	0.022	0.031	0.034	0.026
	T. U. (R. E. %)	0.082 (6.87 %)	0.048 (3.92 %)	0.066 (5.45 %)	0.072 (6.02 %)	0.056 (4.66 %)
3	**Min±ΔMin**	1.176 ± 0.003	1.293 ± 0.004	1.264 ± 0.004	1.217 ± 0.004	1.264 ± 0.004
	Max±ΔMax	1.398 ± 0.004	1.382 ± 0.004	1.374 ± 0.004	1.410 ± 0.004	1.384 ± 0.004
	Avg.±ΔAvg.	1.308 ± 0.004	1.335 ± 0.004	1.324 ± 0.004	1.314 ± 0.004	1.327 ± 0.004
	σ	0.054	0.025	0.034	0.046	0.034

	T. U. (R. E. %)	0.113 (8.61 %)	0.054 (4.04 %)	0.073 (5.50 %)	0.095 (7.23 %)	0.071 (5.38 %)
4	**Min±ΔMin**	1.312 ± 0.004	1.414 ± 0.004	1.374 ± 0.004	1.320 ± 0.004	1.389 ± 0.004
	Max±ΔMax	1.516 ± 0.004	1.524 ± 0.004	1.529 ± 0.004	1.556 ± 0.004	1.525 ± 0.004
	Avg.±ΔAvg.	1.440 ± 0.004	1.471 ± 0.004	1.455 ± 0.004	1.443 ± 0.004	1.461 ± 0.004
	σ	0.052	0.033	0.043	0.061	0.037
	T. U. (R. E. %)	0.108 (7.52 %)	0.070 (4.76 %)	0.091 (6.24 %)	0.126 (8.7 %)	0.078 (5.31 %)
5	**Min±ΔMin**	1.473 ± 0.004	1.551 ± 0.004	1.529 ± 0.004	1.464 ± 0.004	1.536 ± 0.004
	Max±ΔMax	1.662 ± 0.005	1.682 ± 0.005	1.696 ± 0.005	1.682 ± 0.005	1.670 ± 0.005
	Avg.±ΔAvg.	1.585 ± 0.005	1.619 ± 0.005	1.596 ± 0.005	1.586 ± 0.005	1.607 ± 0.005
	σ	0.054	0.036	0.050	0.068	0.040
	T. U. (R. E. %)	0.112 (7.09 %)	0.076 (4.69 %)	0.104 (6.50 %)	0.140 (8.84 %)	0.085 (5.29 %)
6	**Min±ΔMin**	1.643 ± 0.005	1.716 ± 0.005	1.662 ± 0.005	1.606 ± 0.005	1.695 ± 0.005
	Max±ΔMax	1.845 ± 0.005	1.861 ± 0.005	1.845 ± 0.005	1.855 ± 0.005	1.840 ± 0.005
	Avg.±ΔAvg.	1.742 ± 0.005	1.784 ± 0.005	1.753 ± 0.005	1.745 ± 0.005	1.767 ± 0.005
	σ	0.056	0.044	0.053	0.081	0.045
	T. U. (R. E. %)	0.117 (6.74 %)	0.094 (5.26 %)	0.110 (6.29 %)	0.167 (9.59 %)	0.095 (5.36 %)
7	**Min±ΔMin**	1.782 ± 0.005	1.866 ± 0.005	1.861 ± 0.005	1.751 ± 0.005	1.855 ± 0.005
	Max±ΔMax	2.012 ± 0.006	2.041 ± 0.006	1.994 ± 0.006	2.053 ± 0.006	2.021 ± 0.006
	Avg.±ΔAvg.	1.909 ± 0.006	1.965 ± 0.006	1.923 ± 0.006	1.923 ± 0.006	1.942 ± 0.006
	σ	0.067	0.049	0.042	0.088	0.050
	T. U. (R. E. %)	0.139 (7.26 %)	0.105 (5.32 %)	0.090 (4.68 %)	0.181 (9.39 %)	0.105 (5.41 %)

Table 2-5.: Minimum, maximum, average, standard deviation, total uncertainty, and relative error of the frequency ratio for pairs of bars separated by different distances in the traditional marimbas recorded by Miñana. "s" is the distance between bars in steps The symbol, "σ" is the standard deviation, "Avg." refers to the average, "T. U." is the total uncertainty, "R. E." corresponds to the "T. U." value expressed as a relative error, and "Avg.*" refers to the average of all marimbas.

s	**Marimba number - Present study**											
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Avg.*$_1$**	**9**	**10**	**Avg.*$_2$**	**11**
1	1.104	1.105	1.104	1.104	1.107	1.101	1.102	1.105	1.104	1.093	1.096	1.095	1.079
2	1.219	1.221	1.219	1.220	1.224	1.213	1.213	1.221	1.219	1.195	1.201	1.198	1.165
3	1.346	1.349	1.345	1.347	1.355	1.336	1.336	1.348	1.345	1.307	1.316	1.311	1.258
4	1.486	1.491	1.485	1.488	1.499	1.471	1.472	1.490	1.485	1.429	1.442	1.435	1.358
5	1.640	1.647	1.639	1.643	1.659	1.620	1.622	1.646	1.639	1.563	1.580	1.571	1.466
6	1.811	1.820	1.810	1.814	1.836	1.784	1.786	1.818	1.810	1.709	1.731	1.720	1.582
7	1.999	2.011	1.998	2.004	2.031	1.964	1.968	2.009	1.998	1.868	1.897	1.882	1.708
8	—	—	—	—	—	—	—	—	—	2.042	2.078	2.060	1.844
9	—	—	—	—	—	—	—	—	—	—	—	—	1.990

Table 2-6.: Theoretical equi-heptatonic, equi-octatonic, and equi-enneatonic scales for the traditional marimbas recorded for the present study. "s" is the distance between bars in steps. "Avg.*$_1$" refers to the theoretical scale constructed for the average of the average values from the marimbas 2,3,4,5,6,7 and 8. "Avg.*$_2$" refers to the theoretical scale constructed for the average of the averages values of the marimbas 9 and 10. In all cases, the values predicted by the theoretical scales have a relative error $\leq 1.00\,\%$ with respect to the experimental values presented in Table **2-3**.

s	Marimba number - Miñana study									
	1_M	2_M	3_M	4_M	5_M	6_M	7_M	8_M	9_M	Avg.*
1	1.102	1.102	1.099	1.099	1.099	1.097	1.101	1.098	1.098	1.099
2	1.215	1.215	1.208	1.208	1.207	1.203	1.213	1.205	1.205	1.209
3	1.339	1.339	1.327	1.328	1.327	1.319	1.336	1.323	1.323	1.329
4	1.476	1.476	1.458	1.460	1.458	1.447	1.471	1.453	1.453	1.461
5	1.627	1.627	1.602	1.605	1.602	1.587	1.620	1.595	1.595	1.607
6	1.793	1.793	1.761	1.764	1.760	1.740	1.784	1.751	1.751	1.766
7	1.976	1.976	1.935	1.939	1.934	1.909	1.965	1.923	1.923	1.942

Table 2-7.: Theoretical equi-heptatonic scales for the traditional marimbas recorded by Miñana. "s" is the distance between bars in steps. "Avg.*" refers to the theoretical scale constructed for the average of the average values from the marimbas 1_M, 2_M, 3_M, 4_M, 5_M, 6_M, 7_M, 8_M, and 9_M. In all cases, the values predicted by the theoretical scales have a relative error $\leq 1.00\,\%$ with respect to the experimental values presented in Table **2-5**.

Discussion about tunings

Based on the interviews with the instrument makers, the systematic deviations in the frequency ratios found in the traditional marimbas might be due to the variations in the voice preferences of female singers in a region or territory. Since in each territory there are several singers that are accompanied by musicians playing the same marimba, then it is impossible to define only one fundamental frequency as a reference to construct the scale. Hence, the marimbas are constructed by adjusting the tuning as close as possible to the different vocal preferences, but trying to preserve an isotonic tendency in order to approximately preserve the transposition practice keeping geometrical distances.

The results of this section show that traditional tunings have changed with time in the last three decades, losing the low octaves feature in favor of octaves usually closer to the just interval. This change may be the result of the influence of Western music, which is characterized by the just octave. It has been argued that the globalization of Western music, and the commercial value of new genres that combine local rhythms and timbres, have led to a broad use of 12-TET *marimbas de chonta*, modifying the taste of African descendants in the Pacific Coast of Colombia [64]. Frequently the traditional marimbas are referred to as "badly made marimbas" or "wrongly tuned marimbas". The most important cultural event of music from this region, the Petronio Alvarez Festival [64], promoted the use of chromatic and diatonic 12-TET marimbas for several years. It was not until 2015 that the festival opened a category for music played with traditional marimbas. It is a common practice of the Ministry of Culture of Colombia to supply the music schools with 12-TET marimbas, where music is taught using the logics of diatonic and chromatic scales with tones and semitones.

2.3.2. Use of harmonic intervals in the musical practices

Statistical study

According to Miñana [13], neutral thirds are among the most used harmonic intervals in this traditional music. Frequency ratios covering the region of the thirds are commonly generated using distances of 2 and 3 steps between bars (see Tables **2-3**, **2-4**, and **2-5**.). In order to check the frequency of use of intervals, their probability of occurrence was obtained for seven performances of traditional marimba pieces by applying the procedure described in the Methods section. If one of the most used intervals is the neutral third, then minor and major thirds are expected to occur with high probability in the scores.

Table **2-8** shows the probability of occurrence of each harmonic interval, and the probability of occurrence proportional to the total time of duration. Intervals between 3 and 4 semitones are present with high probability in all pieces. Intervals containing sizes between the fourths and fifths, in most cases generated by pairs of bars separated by a distance of 3 and 4 steps, are also frequent in some pieces; and those intervals near to the sixths, in most cases generated by 5 steps, appear less frequently in performances. The region of the seconds, generated by a distance of 1 step, is completely avoided in all musical pieces, and intervals larger than one octave are almost absent. The region of the sevenths, in most cases generated by a distance of 6 steps, is avoided in all pieces with the exception of one, where it has a small incidence, and the region of the octaves is used with small probability of occurrence.

Interval	**Probability**															
	Frequency of occurrence								**Total duration**							
	Musical piece								**Musical piece**							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**Avg.**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**Avg.**
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	0.24	0.59	0.53	0.13	0.55	0.23	0.55	0.40	0.19	0.60	0.52	0.13	0.56	0.23	0.54	0.39
4	0.08	0.41	0.47	0.12	0.44	0.26	0.37	0.31	0.08	0.40	0.48	0.12	0.43	0.26	0.35	0.30
5	0.23	0.00	0.00	0.30	0.00	0.34	0.07	0.13	0.26	0.00	0.00	0.33	0.00	0.34	0.07	0.14
6	0.00	0.00	0.00	0.01	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.03	0.01
7	0.07	0.00	0.00	0.17	0.01	0.15	0.00	0.06	0.05	0.00	0.00	0.20	0.00	0.15	0.00	0.06
8	0.21	0.00	0.00	0.03	0.00	0.00	0.00	0.03	0.17	0.00	0.00	0.02	0.00	0.00	0.00	0.03
9	0.13	0.00	0.00	0.02	0.00	0.00	0.00	0.02	0.20	0.00	0.00	0.02	0.00	0.00	0.00	0.03
10	0.00	0.00	0.00	0.15	0.00	0.00	0.00	0.02	0.01	0.00	0.00	0.12	0.00	0.00	0.00	0.02
11	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12	0.05	0.00	0.00	0.07	0.00	0.02	0.00	0.02	0.04	0.00	0.00	0.06	0.00	0.02	0.00	0.02
>12	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00

Table 2-8.: Probability of occurrence of harmonic intervals as a function of their size in semitones. Interval in semitones. "Avg." refers to the average of the seven musical pieces. The probability has been taken to be proportional to the frequency of occurrence and to the total time of duration of each size of harmonic interval.

Geometrical schemes analysis

The geometrical schemes shown in Figure **2-2** can be applied for the traditional marimbas with equi-heptatonic averages, the most common ones found in the present study, and in the studies of Miñana. From these schemes it is possible to infer that the harmonic intervals with distances of 1 and 6 steps between bars are the least frequent.

Table **2-9** shows the total number of different combinations of pairs of bars generating the same distance between them for different marimba sizes. Notice that, independently of the type of scale (hexatonic or pentatonic), the adjacent bars and pairs of bars separated by 6 steps are highly uncommon, this is due to the harmony rules of the geometrical schemes and incidence of the discarded bars, the so called "bad bars". In the marimbas with equi-heptatonic averages, the regions of adjacent bars and 6 steps distance correspond to intervals around the seconds (frequency ratios between or near to the range $[1.06, 1.12]$) and the sevenths (frequency ratios between or near to the range $[1.78, 1.89]$) (see Tables **2-3** and **2-5**).

For the same distance between bars, the marimbas of the Torres family analyzed in the present study (marimbas 9, 10, and 11) have lower values in the tunings in comparison with the other ones (see Tables **2-3** and **2-4**), this feature is more evident for larger distances than for smaller ones (a natural consequence of the presence of equi-octatonic and equi-enneatonic

		Marimba size (bars)											
		14	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	
Distance (steps)	**1**	1	2	2	2	2	2	2	2	3	3	3	**Hexatonic**
	2	9	10	10	11	11	12	13	14	15	15	16	
	3	4	5	5	6	7	7	7	7	8	8	9	
	4	5	6	6	6	6	7	7	8	9	9	9	
	5	6	7	7	8	9	10	11	11	12	12	13	
	6	2	2	2	2	2	2	2	3	3	3	3	
	7	6	7	7	8	9	10	11	12	13	13	14	
Distance (steps)	**1**	2	2	2	2	2	2	3	3	3	3	3	**Pentatonic**
	2	3	4	4	4	4	4	5	5	6	6	6	
	3	5	5	6	6	7	8	8	8	8	9	9	
	4	4	5	6	6	6	6	7	7	8	9	9	
	5	3	3	3	3	4	4	5	5	5	5	5	
	6	1	1	1	1	1	2	2	2	2	2	2	
	7	5	6	7	7	8	9	10	10	11	12	12	

Table 2-9.: Total number of different combinations of pairs of bars generating the same distance between them for different marimba sizes (from 14 bars up to 24 bars).

scales). For marimbas 9 and 10 (see Table **2-4**), intervals near to the sevenths and the octaves are better generated using 7 and 8 steps, respectively. For marimba 11 (see Table **2-4**), the same intervals are better generated using 8 and 9 steps, respectively.

Discussion about the use of harmonic intervals

From the statistical analysis as well as from the analysis of the geometrical schemes, it is possible to infer that regions around the seconds and the sevenths seem to be deliberately avoided in this music. Since octaves present a small probability of occurrence in the statistical analysis (in average over all the musical pieces, octaves are 2 % of all intervals), we infer that the most used region of the traditional marimba music is between the thirds and the sixths (Table **2-8**). For the traditional marimbas of the present study following equi-heptatonic averages, and from the study by Miñana, this region is generated using bars separated between 2 and 5 steps; from the average values of the minima and maxima of both studies, this region is approximately between $1.16 \leq \alpha \leq 1.72$.

2.3.3. Theoretical dissonance level curves

In this subsection the theoretical dissonance curves associated with the sound emitters of the *marimba de chonta* are presented. In all cases the dissonance scale is taken between 0 and 1.

Isolated bars

In the tuning process, some makers construct the bars before the resonators and test the tuning within sets of three, four, or five adjacent bars. In order to analyze this procedure, the dissonance curve for an isolated bar was obtained theoretically. The range used by the instrument makers to test the tuning include fundamental frequency ratios up to $\alpha = 1.53$, which is the maximum value among the average of the maxima found for 5 successive bars (4 steps) in both studies (see Tables **2-3**, **2-4**, and **2-5**).

Assuming that the main contributions to the vibration of a bar with free ends come from the transverse modes, then the allowed frequencies (in Hz) are [71, p. 62], [72, p. 922]:

$$f_n = \frac{\pi \mathcal{K}}{8\mathcal{B}^2}\sqrt{\frac{E}{\rho}}\left[3.011^2, 5^2, 7^2, ..., (2n+1)^2\right], n \in \mathbb{N}, n \geq 2, \tag{2-2}$$

where E is Young's modulus, ρ is the density of the material, $\mathcal{B}$ is the length of the bar, and $\mathcal{K}$ is the radius of gyration of the cross section. The fundamental frequency f_1 is given by $[\pi \mathcal{K}\sqrt{E}(3.011^2)]/(8\mathcal{B}^2\sqrt{\rho})$, and the first five overtones are given approximately by $\{2.758 f_1$, $5.405 f_1$, $8.934 f_1$, $13.346 f_1$, $18.641 f_1\}$ [71, p. 63], [73, p. 85], [21].

Up to know, we have suppose that the bar has a constant cross section, that it is homogeneous, untwisted (the principal axes of elasticity of all sections are equally directed in space), and that its axis corresponds to a straight line [74, p. 53]. The specific shape of the bar is taken into account in the fundamental frequency through the parameters $\mathcal{B}$ and $\mathcal{K}$. For the case of a rectangular cross section $\mathcal{K} = \mathcal{W}/\sqrt{12}$, with $\mathcal{W}$ the width of the bar [71, p. 53]. In our analysis it was not necessary to assume a specific shape for the bar, because this information does not change the location of the overtones with respect to the fundamental frequency.

This theoretical model has been used in previous analyses, as for example for the gamelan bars in order to approximate the modes of vibration [66]. Figure **2-4($\boldsymbol{a}$)** presents the dissonance curve for this case inside one octave, taking equal values of the amplitudes for the fundamental and the overtones in order to appreciate the contributions of all overtones, in agreement with the procedure carried out by Sethares [21, 25, 32]. This curve was reported by Sethares [21] in order to expose that if the tuning of instruments is carried out according to the consonance properties of their timbre, then bar instruments must be tuned in a different way than instruments with harmonic timbre, such as strings and pipes.

Local minima of dissonance were found at $\alpha = 1.26, 1.40$ and 1.49, as illustrated in Figure **2-4($\boldsymbol{a}$)**. The minimum at $\alpha = 1.40$ does not correspond to any average in the experimental tunings (see Tables **2-3**, **2-4**, and **2-5**). The minimum at $\alpha = 1.49$ (close to the just fifth and the 12-TET fifth) is near the average value for the distance of 4 steps in the case of the equi-heptatonic marimbas in the present study ($\alpha \approx 1.48$), and slightly farther from the average for the marimbas studied by Miñana ($\alpha \approx 1.46$). The minimum located at 1.26 is broader than the other minima and covers the region around the thirds and the fourths ($1.16 < \alpha < 1.37$, grey box in Figure **2-4($\boldsymbol{a}$)**). The minimum at 1.26 is less dissonant than the ones found in the same region for the case of a harmonic spectrum (Figure **2-4($\boldsymbol{a}$)**), and it is placed in an important range of use of harmonic intervals in traditional marimba music (see 3, 4, 5 semitones in Table **2-8**).

Figure **2-4($\boldsymbol{b}$)** shows the differentiation of the dissonance level $D_F(\alpha)$ with respect to the α parameter, $dD_F(\alpha)/d\alpha$, illustrating effects due to the shapes of local minima. In the region of the thirds and the fourths the level of dissonance for bars changes smoothly when compared to the curve for emitters with harmonic spectrum.

Bars and resonators

In order to include effects due to the tubular resonator located under each bar, it is modeled as a cylindrical closed pipe having the same fundamental frequency as the bar. Since closed pipes only produce odd harmonics [71], the first thirteen overtones for the bar-resonator system used in the analysis are $\{2.758f_1$, $3.000f_1$, $5.000f_1$, $5.405f_1$, $7.000f_1$, $8.934f_1$, $9.000f_1$, $11.000f_1$, $13.000f_1$, $13.346f_1$, $15.000f_1$, $17.000f_1$, $18.641f_1\}$.

The amplitudes for the partials are assumed to diminish as $a_n = a_0(0.16)exp[(-1/5)(n - 2.758)]$, where n is the ratio of the frequency of the corresponding overtone to the fundamental ($n = 2.758, 3.000, 5.000, ...$), and a_0 is the amplitude for the fundamental. Figure **2-5** shows

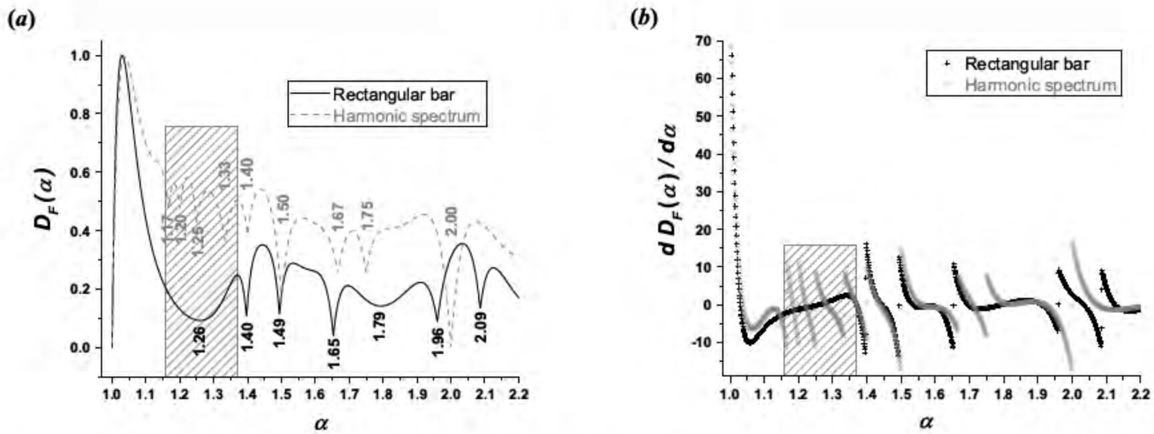

Figure 2-4.: (*a*) Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α, for a bar free to vibrate at both ends, and for the case of a six harmonics spectrum with equal amplitudes. The location of the minima for the harmonic spectrum corresponds to the ratio of small natural numbers posted by Pythagoras: $2/1 = 2.00$, $3/2 = 1.50$, $4/3 = 1.33$, $5/3 = 1.67$, $5/4 = 1.25$, $6/5 = 1.20$, $7/4 = 1.75$, $7/5 = 1.40$, $7/6 = 1.17$. (*b*) Differentiation of the dissonance level $D_F(\alpha)$ with respect to the α parameter, as a function of the ratio between the fundamental frequencies α. The grey box indicates the region around the thirds and the fourths. In both graphics $300Hz$ has been taken as the lowest fundamental frequency, and the dissonance level scale has been normalized to 1.

the corresponding spectrum.

According to Sethares, the precise value of the amplitude for each overtone can be somewhat arbitrary. However, these values must reflect the average properties of the samples [32]. In this sense, the assumption that the amplitudes a_n diminish exponentially is in agreement with the most common tendency of high overtones having smaller amplitudes than lower ones, and it is in agreement with the average amplitude proportions between the fundamental and the main modes measured experimentally for a set of traditional marimbas (Figure **2-19**). This is discussed in more detail in the following section.

The dissonance curves produced using the three models [21, 30, 32] are shown in Figure **2-6**. The three models predict a peak of maximum dissonance located in the range [1.00,1.15], and three minima of low dissonance at $\alpha = 1.67$, 1.81, and 1.96. From the second model of Sethares and from the Vassilakis model, the surrounding of the peaks $\alpha < 1.15$, $\alpha = 1.67$, and $\alpha = 1.81$ are the regions in which small variations in the α parameter produce the largest changes in the dissonance level.

As a feature of theses curves, when we use small amplitude values for the overtones then the first Sethares model predicts only a small effect in comparison with the Vassilakis and the second Sethares models, in which the contribution of the amplitudes have been improved. The Vassilakis and the second Sethares models predict that there is an important contribution from the overtones to the dissonance curves, even when their amplitudes are small in comparison with those of the fundamental frequencies.

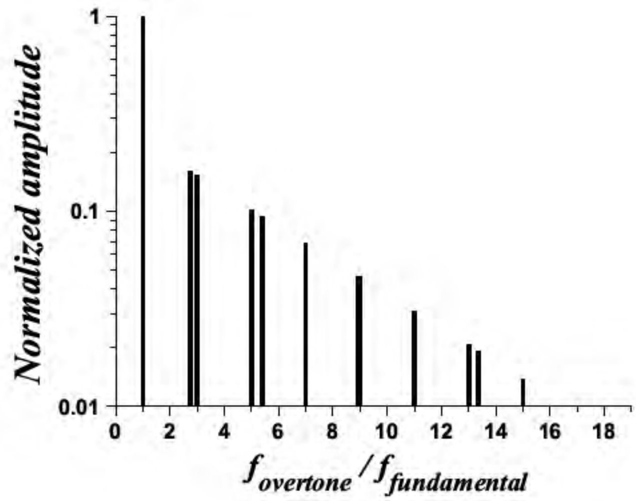

Figure 2-5.: Spectrum corresponding to a bar free to vibrate at both ends with a cylindrical tubular resonator. The amplitude of the first overtone is taken to be 0.16 of the fundamental frequency, and the amplitude decays exponentially for the remaining successive overtones.

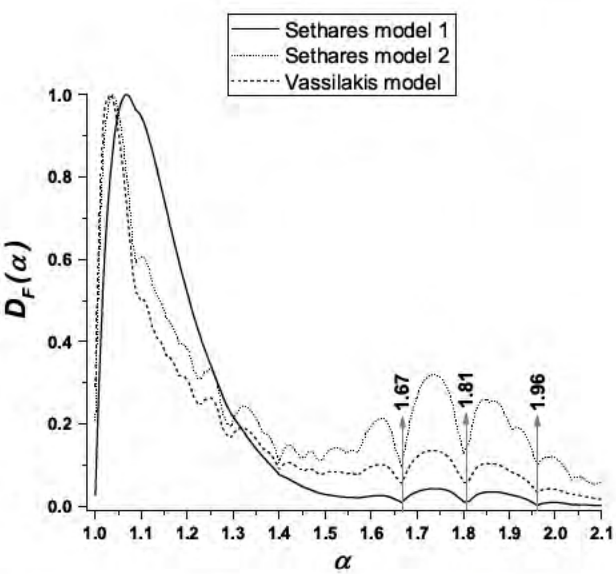

Figure 2-6.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α for a bar free to vibrate at both ends with a cylindrical tubular resonator. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. Amplitudes have been assumed to decay in the form presented in Figure **2-5**. The dissonance level scale has been normalized to 1. The lowest fundamental frequency is taken to be $300\,Hz$.

2.3.4. Experimental dissonance level curves for the marimbas of the present study

The experimental analysis uses the recordings of the sound produced by each bar coupled with its resonator. The fundamental and the first 10 overtones with the largest amplitudes were found for the marimbas of the present study; this is shown graphically for all bars of each marimba in Figures **2-7** to **2-17**.

Figure **2-18** shows, for each marimba, the average normalized amplitude as a function of the ratio between the frequency of the corresponding overtone and the fundamental frequency. The amplitude has been averaged over all the bars of a particular marimba. Traditional marimbas 2, 3, 4, 5, 9, 10, and 11 exhibit a common pattern in the spectrum. Specifically, there are important incidences of overtones around $2.4 < n < 3.0$ and $4.6 < n < 5.5$, indicating that these regions group the main contributions coming from the overtones produced by the different bars and resonators of these marimbas. The normalized amplitudes obtained from superposition of the samples from these marimbas are shown in Figure **2-19**; the most important overtones are located near $n = 2.7 \pm 0.2$, 5.0 ± 0.2, and 5.4 ± 0.2. The values of these overtones are consistent inside the uncertainties with those from the theoretical model of the first and the second transverse modes of a bar free to vibrate at both ends and the second mode of a closed pipe. Other modes are absent or have low amplitudes in almost all cases, such as the unexpected peaks located at $n = 2.0$ that we associate with the first overtone of broken resonators functioning as open pipes.

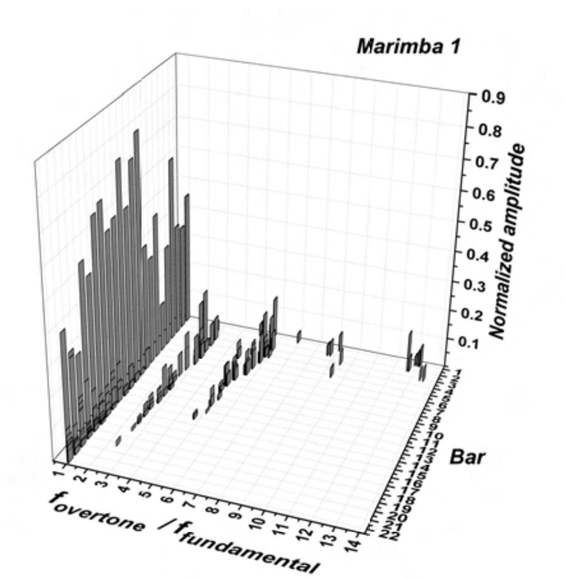

Figure 2-7.: Normalized amplitude of the Marimba 1 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

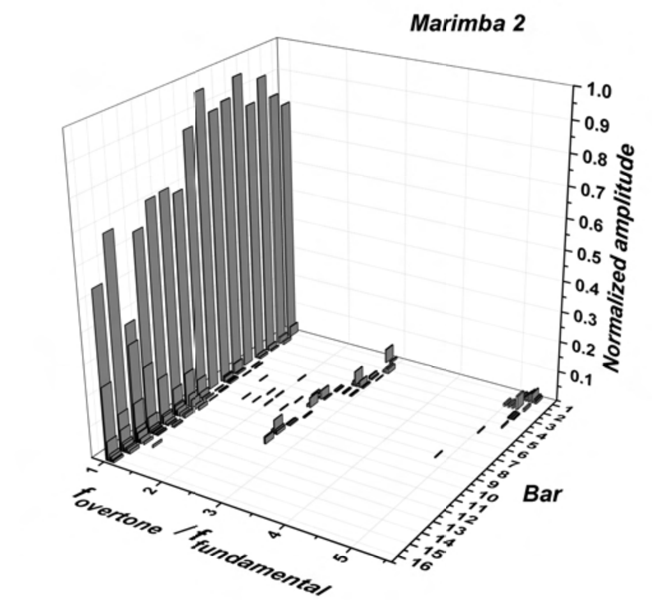

Figure 2-8.: Normalized amplitude of the Marimba 2 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

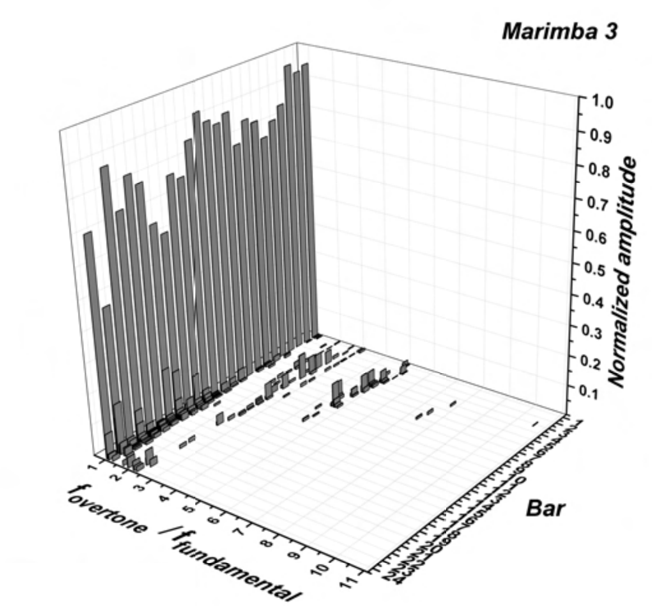

Figure 2-9.: Normalized amplitude of the Marimba 3 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

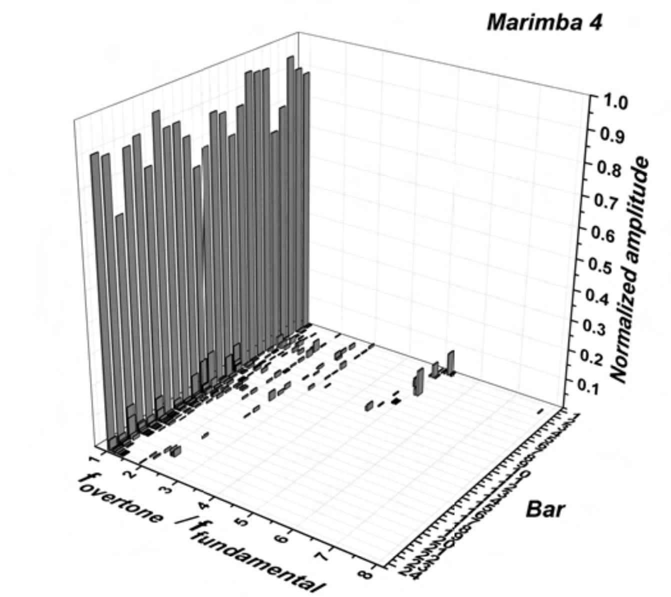

Figure 2-10.: Normalized amplitude of the Marimba 4 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

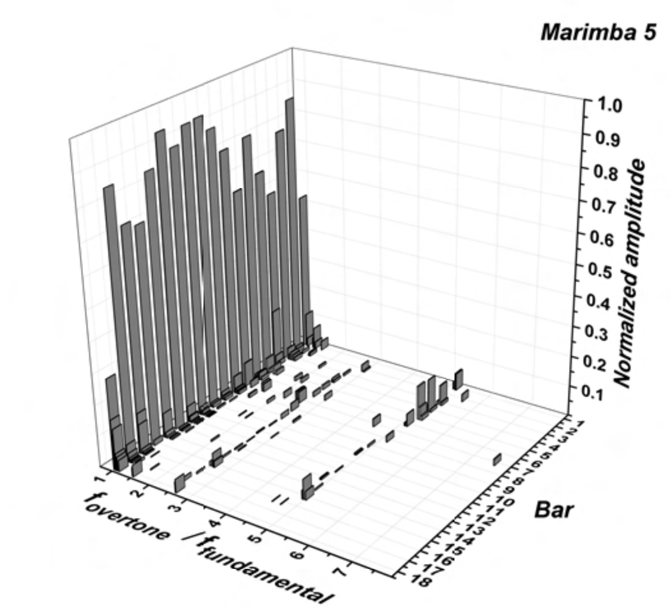

Figure 2-11.: Normalized amplitude of the Marimba 5 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

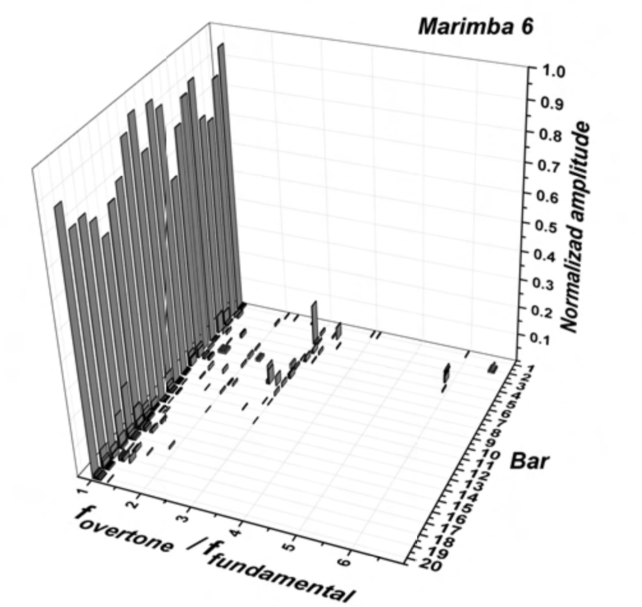

Figure 2-12.: Normalized amplitude of the Marimba 6 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

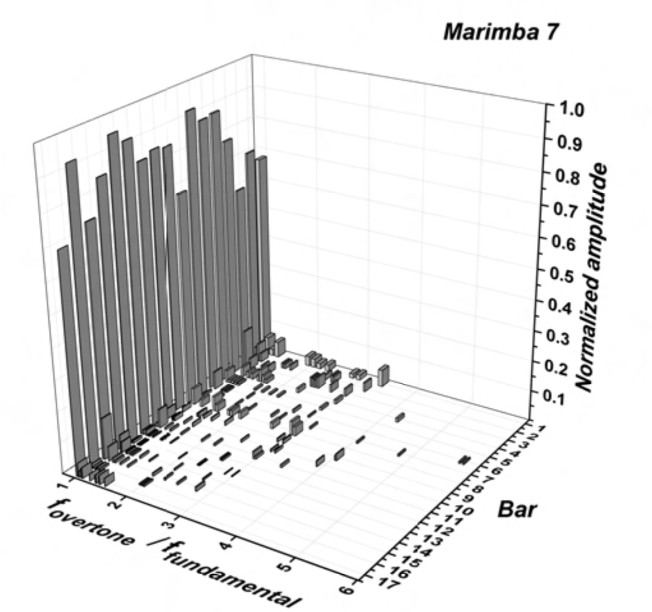

Figure 2-13.: Normalized amplitude of the Marimba 7 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

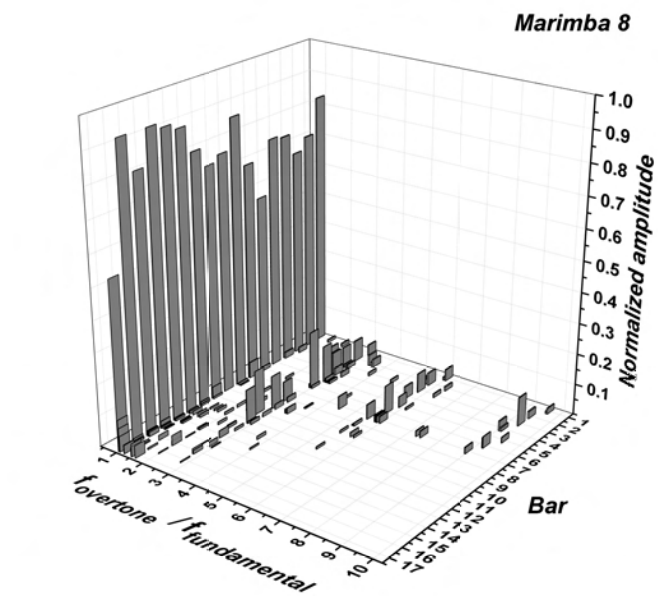

Figure 2-14.: Normalized amplitude of the Marimba 8 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

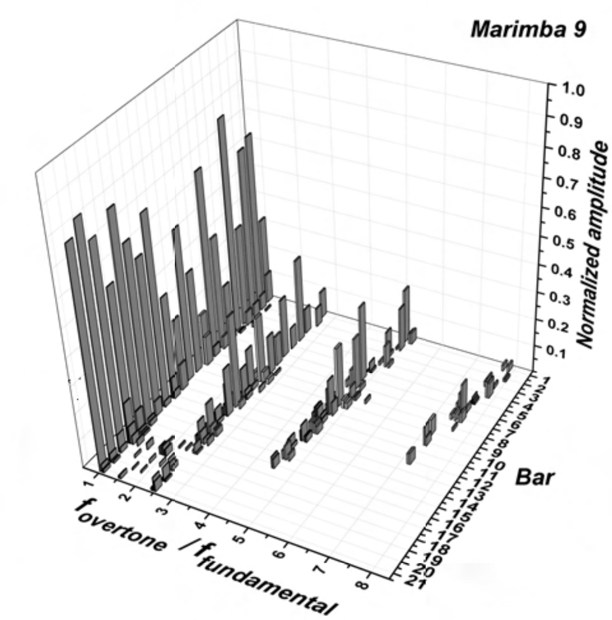

Figure 2-15.: Normalized amplitude of the Marimba 9 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

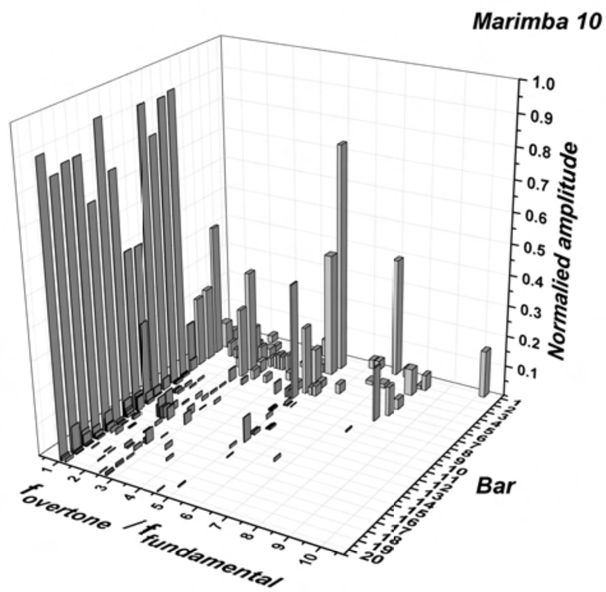

Figure 2-16.: Normalized amplitude of the Marimba 10 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

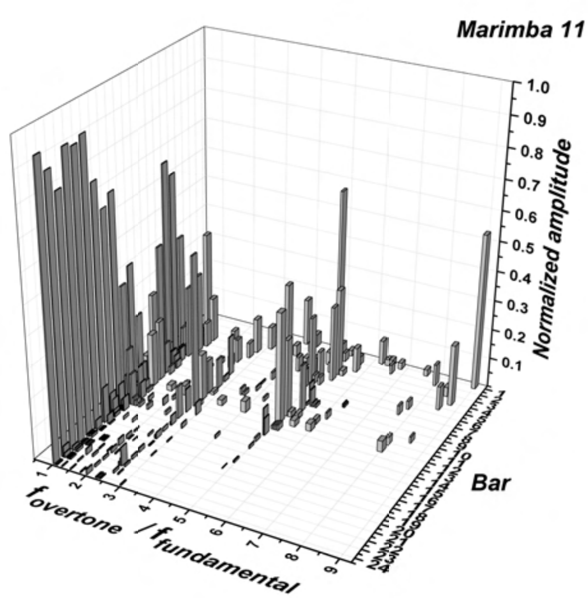

Figure 2-17.: Normalized amplitude of the Marimba 11 for the first 10 overtones produced by each bar with its corresponding resonator. The first bar of each marimba corresponds to the lowest frequency and the last bar to the highest one.

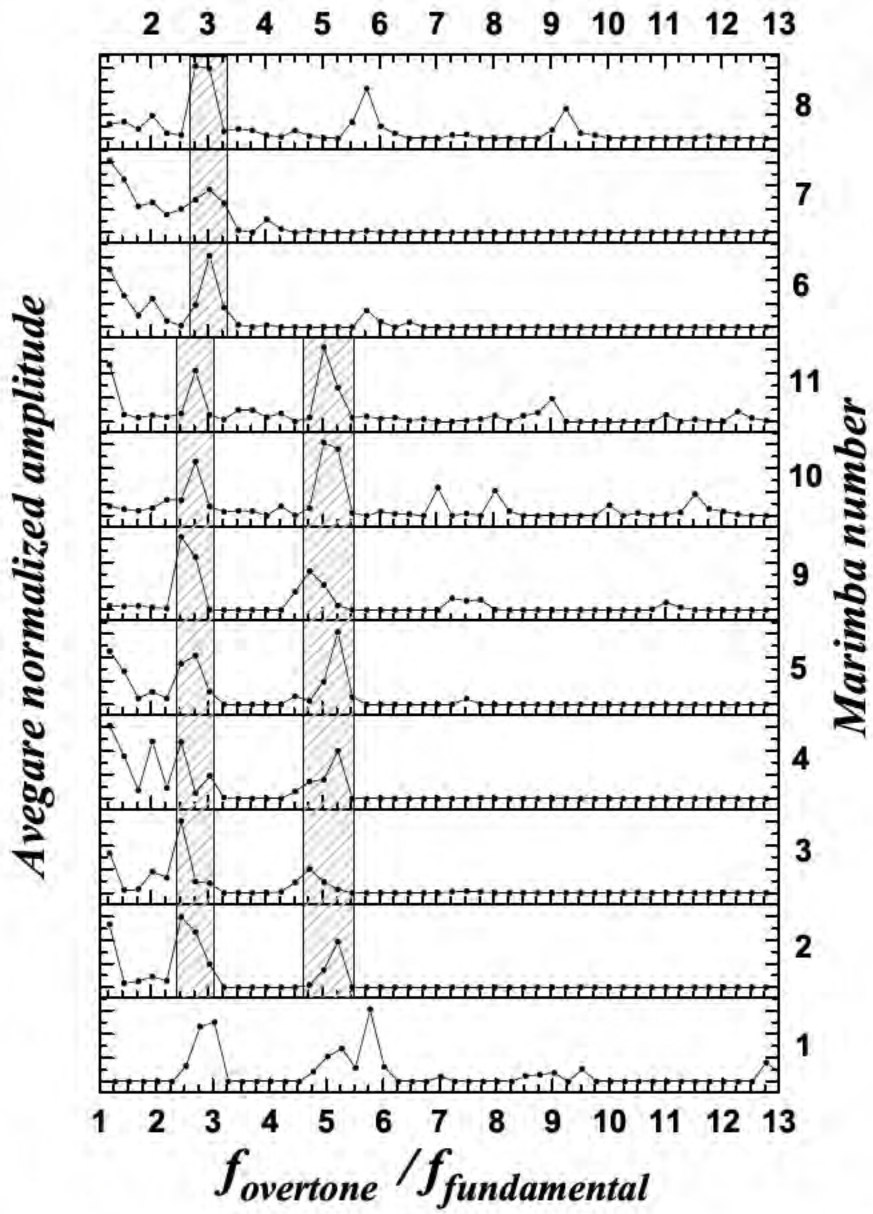

Figure 2-18.: Average normalized amplitude for each marimba. The amplitudes have been averaged using a bin width of 0.25 for the ratio between the frequency of the overtones and the fundamental frequency.

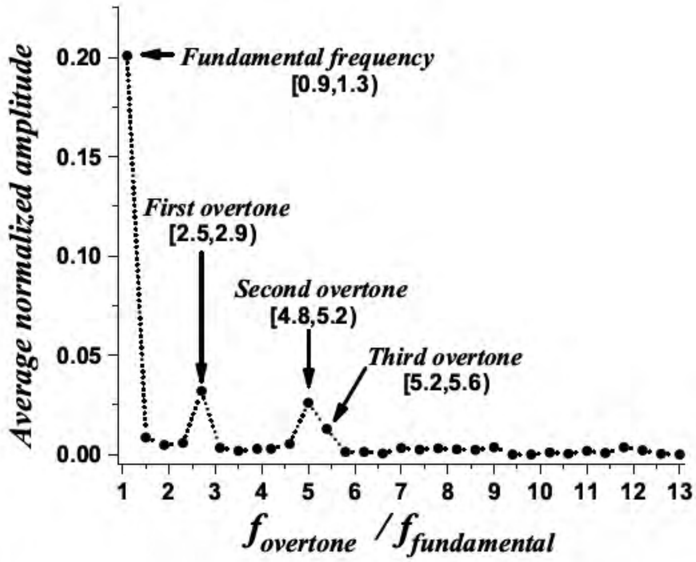

Figure 2-19.: Average normalized amplitudes obtained from superposition of the samples from traditional marimbas 2, 3, 4, 5, 9, 10, and 11 of the present study. Bin width of 0.4 for the ratio between the frequency of the overtones and the fundamental frequency.

In most of the studied marimbas the fundamental has the largest amplitude; however, in the case of the Torres Family (Figures **2-15**, **2-16**, and **2-17**) some overtones have larger amplitudes than the fundamental.

The experimental values for the ratios of the heights of the peaks in Figure **2-19** are $a_1/a_0 \approx 0.16$, $a_2/a_0 \approx 0.13$, and $a_3/a_0 \approx 0.06$, where a_0, a_1, a_2, and a_3 are the amplitudes of the fundamental, first overtone, second overtone, and third overtone, respectively. These ratios of the amplitudes are approximately reproduced using the theoretical model proposed for the fundamental frequency and overtones $n = 2.758$, 5.000, and 5.406 (see Figure **2-5**).

In order to identify the approximate positions of the local minima of dissonance for traditional marimbas 2, 3, 4, 5, 9, 10, and 11, the overtones $n = 2.758$, 5.000, and 5.405 were taken as representative of the most important contributions found experimentally. The corresponding dissonance curves were generated for both the Sethares and Vassilakis models (Figure **2-20(*a*)**) using the values of the normalized amplitudes presented previously, taking $a_0 = 1.0$. The case of equal amplitudes is also presented; as in the low registers of the marimbas made by the Torres family, the amplitudes of the overtones are comparable to those of the fundamental (see Figures **2-15**, **2-16**, and **2-17**).

Comparing the experimental result (Figure **2-20(*a*)**) with the theoretical model for a bar and its tubular resonator (Figure **2-6**), illustrates that the dissonance curves have similar shapes with a large peak of dissonance, for $\alpha < 1.15$, and two narrow peaks of low dissonance, at $\alpha = 1.81$ and $\alpha = 1.96$. The most notable differences between the curves are smoothness, and the presence of a low dissonance narrow peak at $\alpha = 1.67$ in the theoretical model.

Figure **2-20**(*b*) shows the differentiation $dD_F(\alpha)/d\alpha$. There are two regions in which small variations in the α parameter produce large changes in the dissonance level $D_F(\alpha)$. These regions contain the musical intervals around the seconds, for $\alpha < 1.15$, and the sevenths, $\alpha \approx 1.81$, that are commonly avoided in the traditional marimba music. The gray box in Figure **2-20** corresponds to the most used region for the traditional marimba music, notice that this region exclude the large changes in the dissonance level.

The peak located at $\alpha \approx 1.67$ in the dissonance curve from the theoretical analysis of the bar coupled to the resonator (Figure **2-6**) is due to the first overtone generated by the vibration of the tubular resonator $n = 3.0$. For the most traditional marimbas this overtone presents insignificant values of amplitude, however for the marimbas 6, 7, and 8, this overtone contributes significantly (see Figure **2-18**). The ratio 1.67 is very close to the major sixths in the 12-TET scale ($\alpha \approx 1.68$); the statistical analysis finds that this interval occurs in only 2-3 %, on average, of musical pieces (Table **2-8**), indicating that the region around this peak is not commonly used in the *marimba de chonta* music.

The peak located at $\alpha \approx 1.96$ is due to the overtone $n = 5.405$. Since the amplitude for this overtone is smaller than for overtones $n = 2.758$, and $n = 5.000$, the dissonance curve changes less abruptly in the region of the octaves (specially for the case of the Vassilakis model), when compared to those in the region of the sevenths. Perhaps this is the reason the octaves are not suppressed in the geometrical schemes of Figure **2-2**.

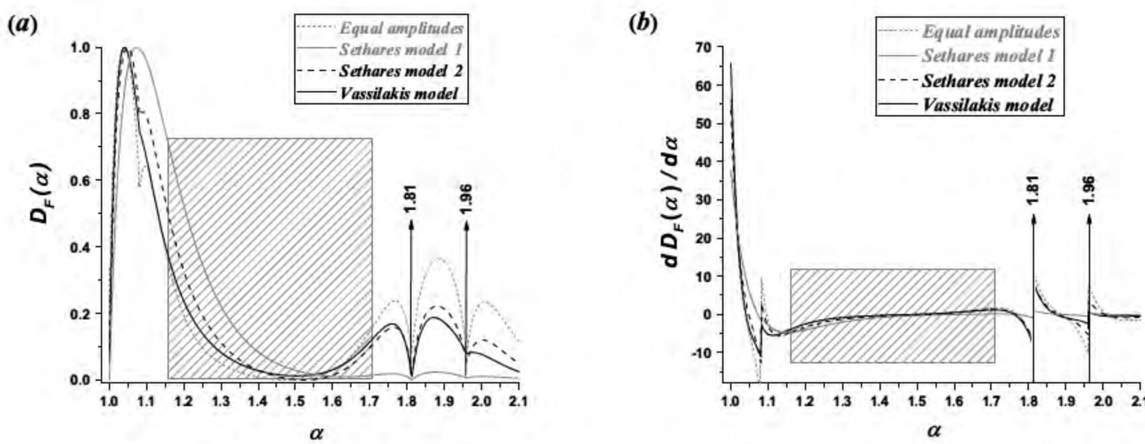

Figure 2-20.: (a) Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α for the most important components of the experimental spectrum. (b) Differentiation of the dissonance level with respect to the fundamental frequency ratio $dD_F(\alpha)/d\alpha$ as a function of the fundamental frequency ratio α, for the most important components of the experimental spectrum. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The dissonance level scale has been normalized to 1. The gray box corresponds to the most used region for the traditional marimba music. The lowest fundamental frequency has been taken to be $300Hz$.

2.3.5. The transposition practice

The transposition of a musical piece involves rewriting a composition in a different key, a practice that is common when accommodating different voice ranges [12]. Since in an equal tempered scale the frequency ratios are preserved, the dissonance sequence in a transposition process remains almost unchanged conserving similar tension-relaxation sequences. In the isotonic scales, the transposition process keeps the same frequency ratios for the same geometrical distances. In traditional marimbas the average frequency ratios are the ones that follow isotonic scales, with deviations with respect to the mean, suggesting that similar successive dissonance changes can be preserved if the regions with large changes in dissonance are avoided.

Up to now, the dissonance curves have been constructed over a fixed value for one of the fundamental frequencies (300 Hz). In general, the form of the dissonance curves is similar for different values of the fixed fundamental frequency. For example, Figures **2-21(*a*),(*c*)** show the dissonance curves, for the experimental spectrum shown in Figure **2-19**, in the case of the Vassilakis and the second Sethares models. The set of fundamental frequencies was obtained using the ratio 1.104, coming from the average ratios of the fundamental frequencies found for adjacent bars of the equi-heptatonic traditional marimbas of the present study (see Table **2-3**). Hence this figure represents the dissonance curves starting from different bars of the marimba. This figure also shows that the same musical interval played in a lower part of the register tends to be more dissonant than when played in a higher part, a well-known property of musical intervals [2, 18].

In the performance of a musical piece on a traditional marimba different simultaneous pairs of bars are hit to construct successive dissonance levels. Since the melodic motion is produced using bars near to each other [40], this practice results in jumps between different dissonance curves near to each other (see Figures **2-21(*b*),(*d*)**). In the region of the dissonance curves of traditional marimbas that excludes the seconds, the sevenths, and the octaves, transitions or jumps made between near or adjacent dissonance curves lead to small changes in the dissonance sequences. For example, for the traditional marimbas consistent with the spectrum shown in Figure **2-19**, Figures **2-21(*b*),(*d*)** show that the most dissonant elements in this music are usually due to intervals with a distance of 2 steps, and the most consonant are usually due to intervals with distances of 4 or 5 steps. These features do not change significantly due to jumps between near or adjacent dissonance curves.

A different situation occurs for 6, 7, and 8 step distances in the equi-heptatonic, equi-octatonic, and equi-enneatonic scales, respectively, in which narrow peaks can lead to changes from consonant to dissonant values for small changes in α, or vice-versa, especially in the region of sevenths.

Regarding the transposition process, the bar used to start the performance of a piece determines the initial dissonance curve for the sequence of steps. If two initial bars, associated to a transposition process, are adjacent or close to each other, then the same sequence of harmonic intervals generated using distances of 2, 3, 4, and 5 steps in the case of the equi-heptatonic marimbas (2, 3, 4, 5, and 6 for equi-octatonic; 2, 3, 4, 5, 6, and 7 for the equi-enneatonic) will produce similar sequences of dissonance changes, independently of which of the dissonance curves is the initial one. On the other hand, avoiding intervals in the regions with large variations in the dissonance level (1, 6, and 7 distance steps for equi-heptatonic marimbas; 1, 7, and 8 for equi-octatonic; 1,8, and 9 for equi-enneatonic) allows the preservation of geometrical distances of the sequence of intervals in the transposition process.

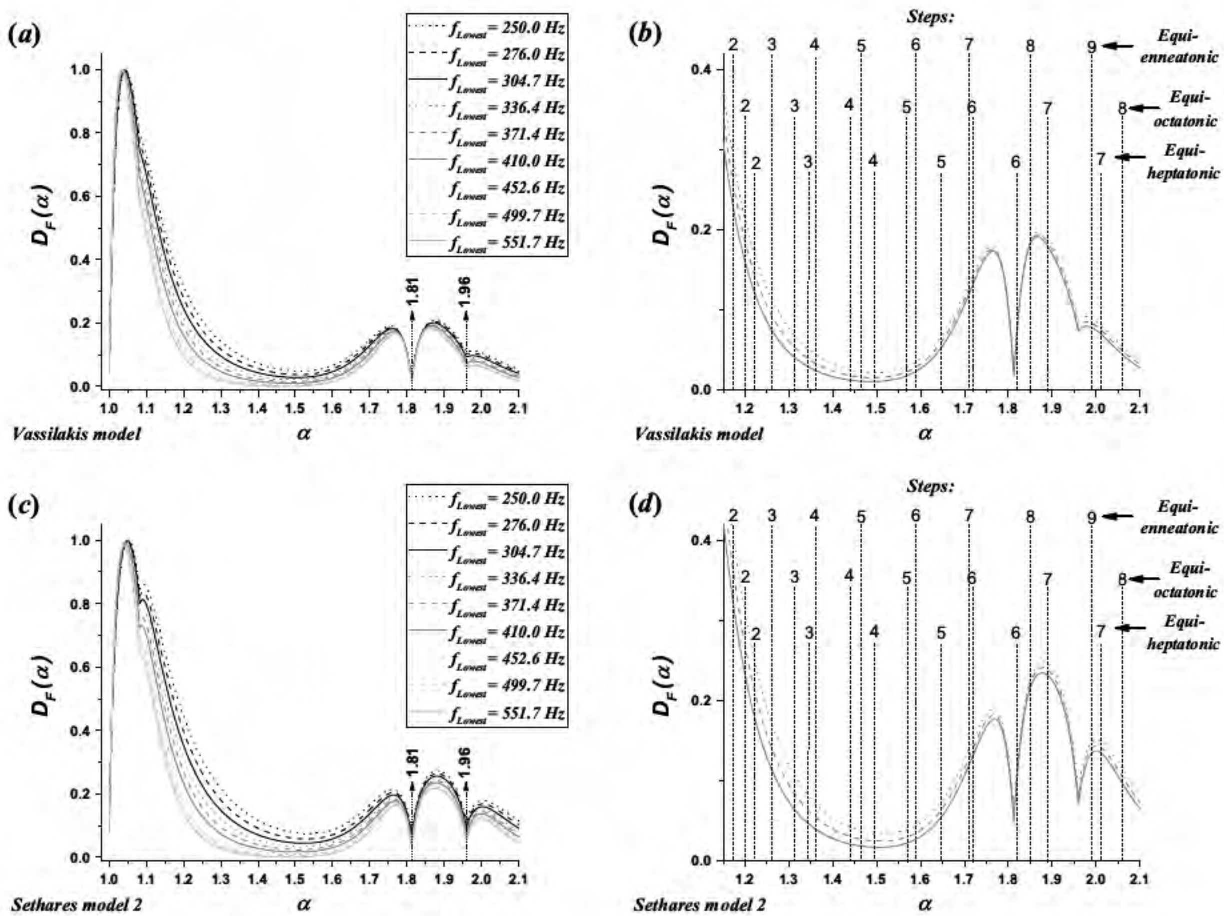

Figure 2-21.: (a,c) Dependence of the dissonance level $D_F(\alpha)$ with the lowest fundamental frequency for the most important components of the experimental spectrum for the case of exponentially decaying amplitudes. The dissonance level scale has been normalized to 1. (b,d) Close up of the region used in the *marimba de chonta* music for three successive bars (f_{Lowest} = $336.4Hz$,$371.4Hz$,$410.0Hz$). Vertical dash lines represent the average tuning as a function of the distance between bars in steps for the case of the equi-heptatonic marimbas $\{2, 3, 4, 5\}$, the equi-octatonic marimbas $\{9, 10\}$, and the equi-enneatonic marimba 11. Figures (a) and (b) have been produced using the Vassilakis model, and figures (c) and (d) using the second Sethares model.

Up to now $n = 2.758$, 5.000, and 5.405 have been used as representative values for the overtones $n = 2.7 \pm 0.2$, 5.0 ± 0.2, and 5.4 ± 0.2. We explore the effect of selecting other values inside the uncertainties of these overtones in Figures **2-22** to **2-31**, and the results show that the two narrow local minima of dissonance move inside the region of the sevenths, the octaves, intervals larger than the octaves, and eventually can reach the region of the major sixths (that are not specially used in this music), however, even for these cases, the broad minimum of dissonance in the most important region of use of the *marimba de chonta* is preserved.

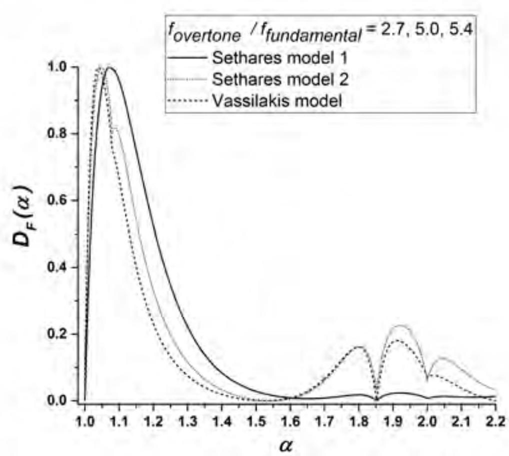

Figure 2-22.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The overtones selected have a ratio between the frequency of the overtone and the frequency of the fundamental given by 2.7, 5.0, and 5.4

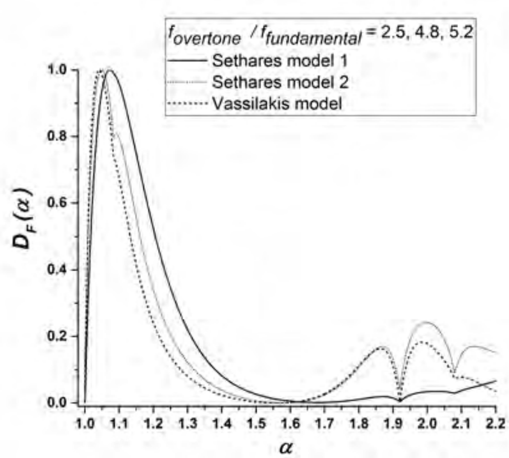

Figure 2-23.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The overtones selected have a ratio between the frequency of the overtone and the frequency of the fundamental given by 2.5, 4.8, and 5.2

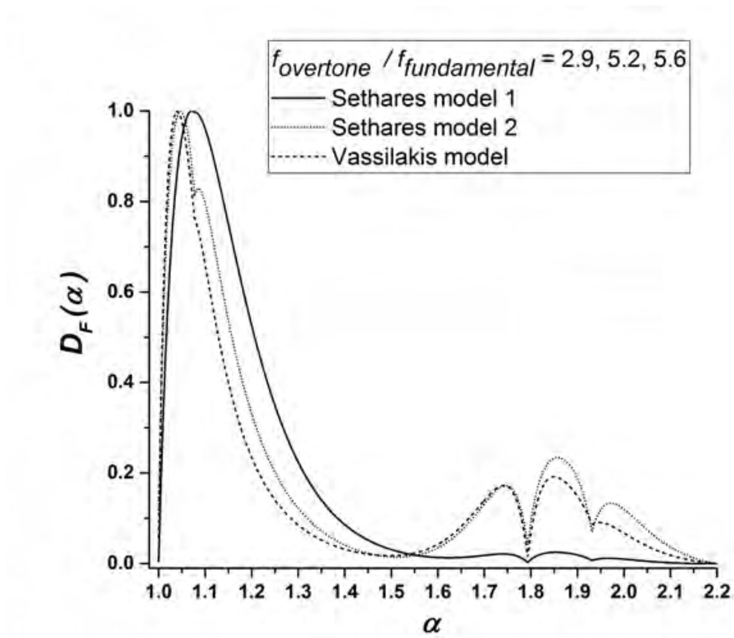

Figure 2-24.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The overtones selected have a ratio between the frequency of the overtone and the frequency of the fundamental given by 2.9, 5.2, and 5.6

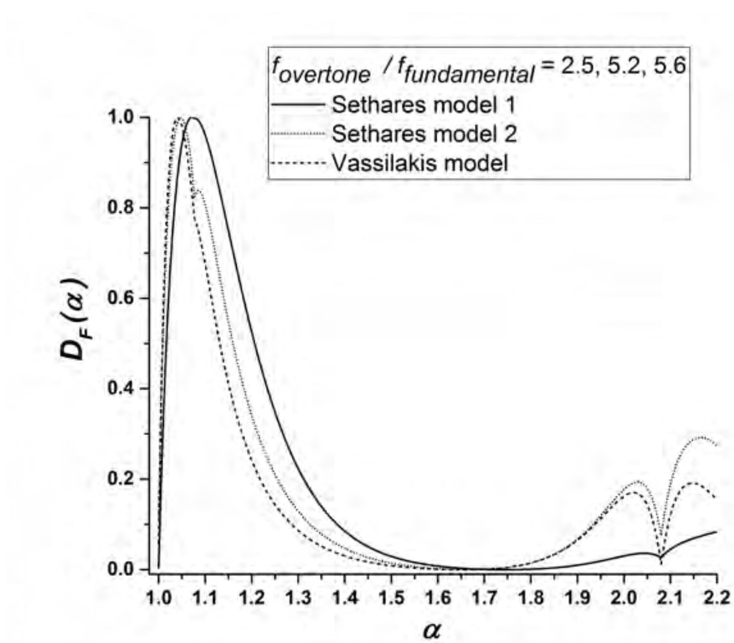

Figure 2-25.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The overtones selected have a ratio between the frequency of the overtone and the frequency of the fundamental given by 2.5, 5.2, and 5.6

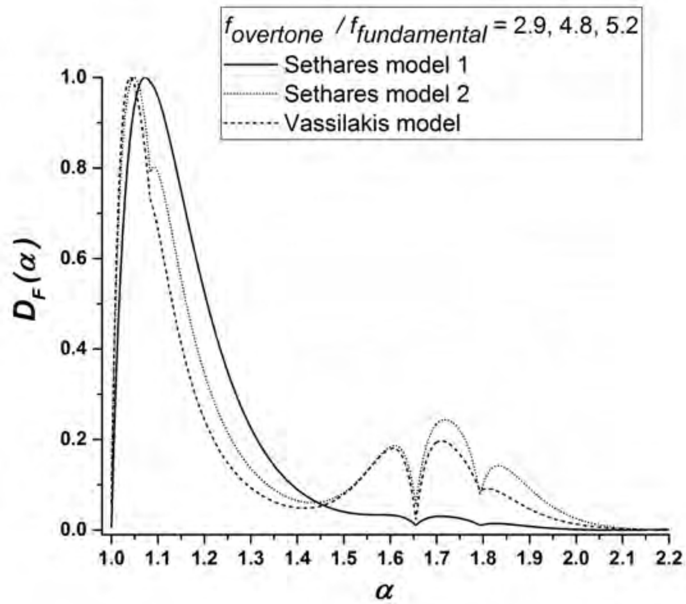

Figure 2-26.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The overtones selected have a ratio between the frequency of the overtone and the frequency of the fundamental given by 2.9, 4.8, and 5.2

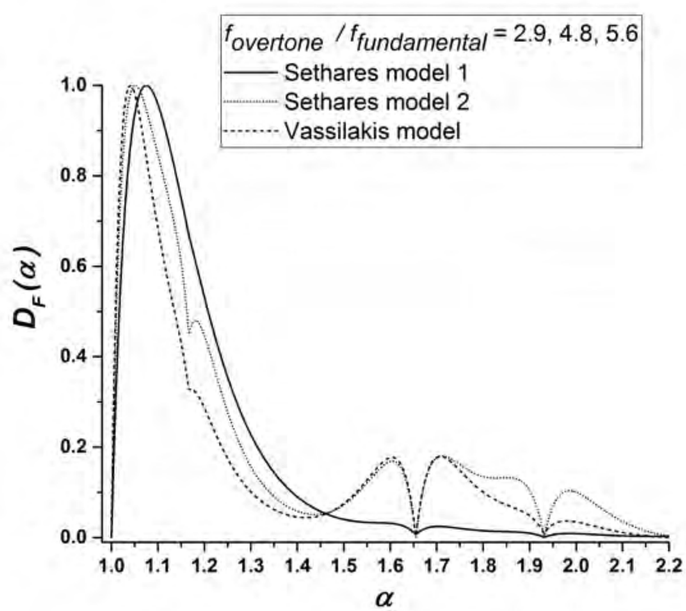

Figure 2-27.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The overtones selected have a ratio between the frequency of the overtone and the frequency of the fundamental given by 2.9, 4.8, and 5.6

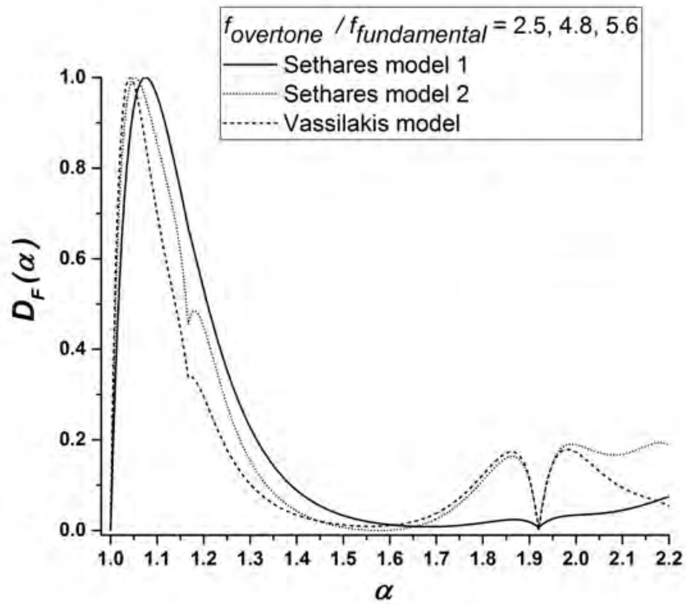

Figure 2-28.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The overtones selected have a ratio between the frequency of the overtone and the frequency of the fundamental given by 2.5, 4.8, and 5.6

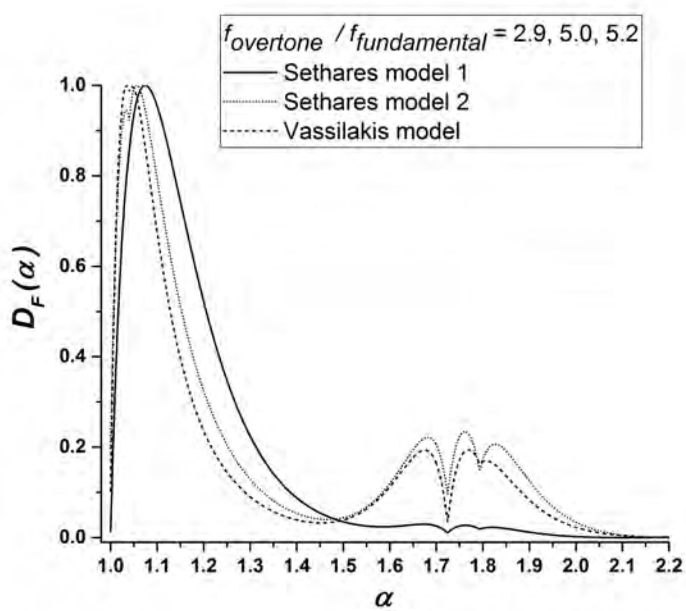

Figure 2-29.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The overtones selected have a ratio between the frequency of the overtone and the frequency of the fundamental given by 2.9, 5.0, and 5.2

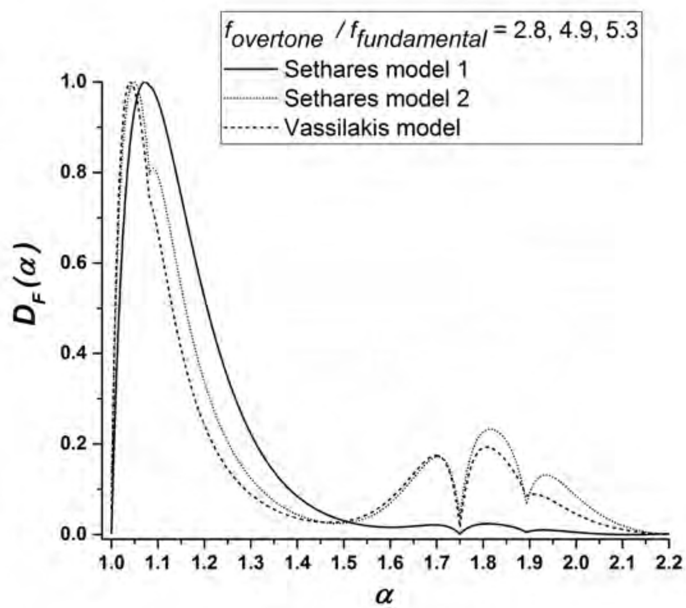

Figure 2-30.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The overtones selected have a ratio between the frequency of the overtone and the frequency of the fundamental given by 2.8, 4.9, and 5.3

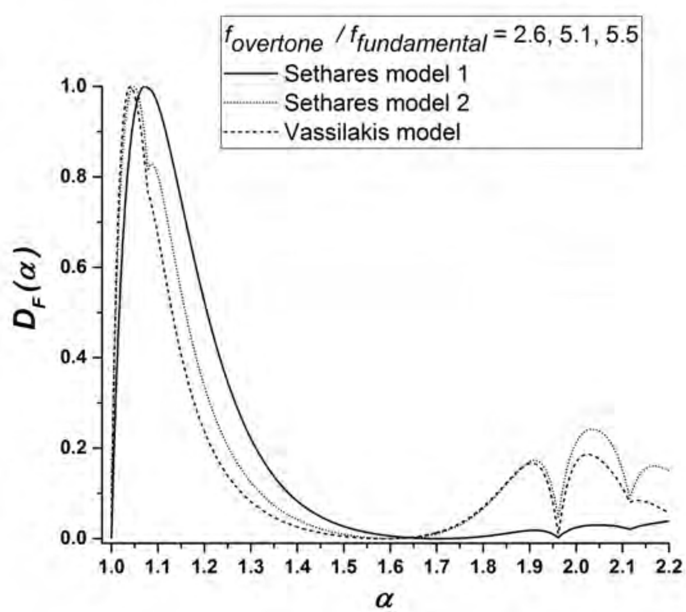

Figure 2-31.: Dissonance level $D_F(\alpha)$ as a function of the ratio of the fundamental frequencies α. Dissonance curves produced using the first and the second Sethares model (model 1 and 2 respectively), and the Vassilakis model. The overtones selected have a ratio between the frequency of the overtone and the frequency of the fundamental given by 2.6, 5.1, and 5.5

3. Generalization of the interval size and its application to melody

This chapter is organized as follows. The first section presents the microscopic representation of intervals and describes how to construct their macroscopic observables. The second section describes how to measure levels of tonal consonance using the representation proposed. The third section introduces macroscopic observables in melody. The fourth and the fifth sections present an application to real melodic lines and a statistical model that reproduces the main experimental findings, and the final section presents conclusions.

3.1. Microscopic representation and macroscopic observables of intervals

This section presents the microscopic representation of musical intervals using physical quantities, the expected values of the relevant quantities, the mathematical description of transposition processes in this representation, and an analysis of distinguishability of musical intervals.

3.1.1. Interval size and its relation to the fundamental frequency of pitches

Many musical systems employ discrete sets of sounds produced by musical instruments, which are usually grouped into musical scales. Two well-known scales based on the Pythagoras rule are the just and the Pythagorean [1, 2].

Ordering the R pitches produced by a musical instrument tuned to a particular musical scale from the lowest to the highest fundamental frequency leads to a collection of pitches $\{f_1, f_2, ..., f_i, ..., f_R\}$ with $f_1 < f_2 <, ... < f_i < ... < f_R$. The interval size L associated to a pair of pitches f_i and f_j is defined for many musical scales as $L \equiv L(f_i, f_j) = j - i$. The magnitude of L determines the plain distance between pitches, and its sign is meaningful for successive pitches, distinguishing the chronological order of their appearances. Intervals with the same size L can be produced in different locations of the register, and this quantity can be considered as degenerated with a value that is equal to the total number of such intervals. For complex tones, such as the sounds produced by musical instruments that can be described as a superposition of several pure tones, Plomp and Levelt found that an interval with a given frequency ratio f_j/f_i might be more or less consonant depending on its location in the register [18]. In many musical cases, there is a one-to-one correspondence between L and f_j/f_i, for example in an equal-tempered system.

The Pythagoras rule can be expressed as the frequency difference

$$f_j - f_i = [(n-m)/(n+m)](f_j + f_i), \tag{3-1}$$

where for the just and Pythagorean scales the quantity $(n-m)/(n+m)$ depends on the size of the interval L (see Figure **3-1**).

The 12-tone equal-tempered (12-TET) scale belongs to the equal tempered system, and has been widely utilized in Western tonal music. This system is based on a different mathematical rule, $f_i = f_1 \sqrt[h]{2^i}$, where h is a natural number ($h = 12$ for the 12-TET) and f_1 is a reference frequency. In this system, the frequency ratio is given by

$$f_j/f_i = \sqrt[h]{2^{j-i}} = \sqrt[h]{2^L}, \tag{3-2}$$

and an equivalent expression to (3-1) is

$$f_j - f_i = \frac{2^{L/h} - 1}{2^{L/h} + 1}(f_j + f_i). \tag{3-3}$$

Equation (3-3) approximately holds for the just and Pythagorean scales, taking $(n-m)/(n+m) = (2^{L/b} - 1)/(2^{L/b} + 1)$ and using the most common values of n and m related to each musical interval size L in the just and Pythagorean scales (see Table **1-1**) [1], then for a register with 88 pitches the obtained fit parameters are as follows:

- In the just scale, $b = 12.0040 \pm 6.8 \times 10^{-3}$ with a determination coefficient $R^2 \approx 1$.
- In the Pythagorean scale, $b = 11.9767 \pm 4.9 \times 10^{-3}$ with $R^2 \approx 1$.
- In the 12-TET, $b = 12$.

The expression $(2^{L/b} - 1)/(2^{L/b} + 1)$ can be written as a linear function of L in a broad region: see Figure **3-1**. The second-order term of the Taylor expansion around $L = 0$ vanishes, and the first-order term leads to $(2^{L/b} - 1)/(2^{L/b} + 1) \approx cL$, with $c = (ln2)/(2b)$.

In many musical cases, the sizes of intervals are smaller than or equal to two octaves, such as in the case of melodic intervals in typical melodic lines [9]. For the case that $-24 \leq L \leq 24$, the fit parameters are given as follows:

- For the just scale, $c = 2.632 \times 10^{-2} \pm 1.52 \times 10^{-4}$ with a determination coefficient $R^2 = 0.998$.
- For the Pythagorean scale, $c = 2.642 \times 10^{-2} \pm 1.55 \times 10^{-4}$ with $R^2 = 0.998$.

- For the 12-TET scale, $c = 2.635 \times 10^{-2} \pm 1.48 \times 10^{-4}$ with $R^2 = 0.998$.

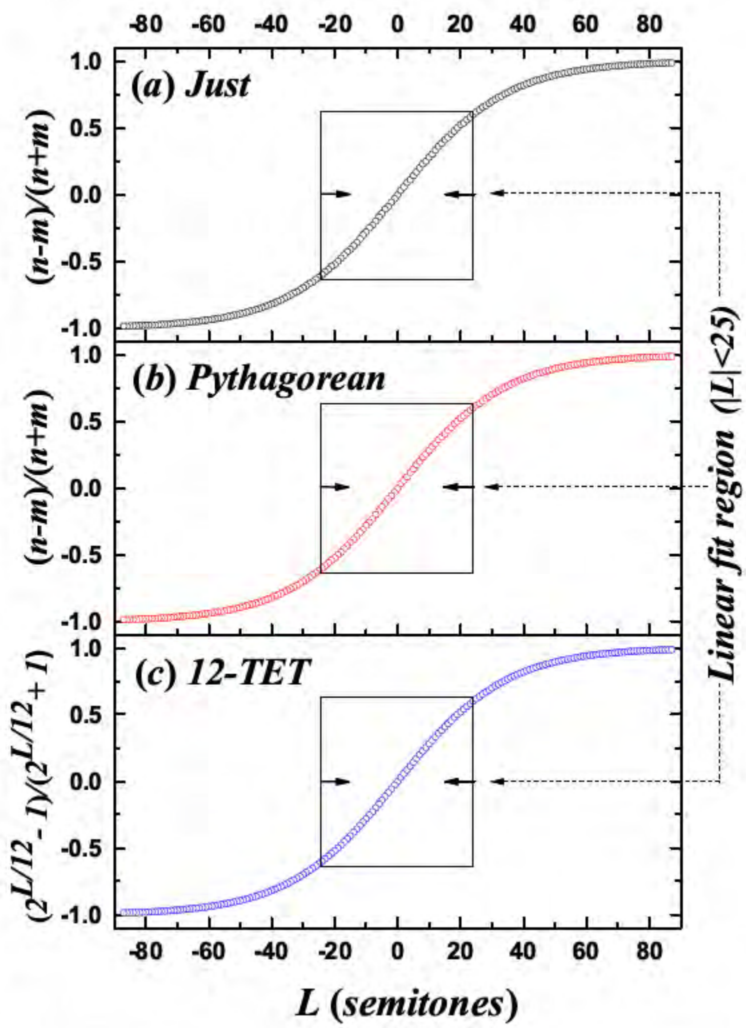

Figure 3-1.: Relation between musical scale parameters and the interval size for the (a) just, (b) Pythagorean, and (c) 12-TET scales, with an interval size from -87 to 87 semitones (representing a typical piano). The linear fit corresponds to interval sizes between -24 and 24 semitones.

With these results, equations (3-1) and (3-3) can be expressed as

$$f_j - f_i \approx cL(f_j + f_i) = 2cLX, \tag{3-4}$$

where $X = (f_j + f_i)/2$ is the center frequency, which provides information about the location of an interval in the register [2]. Then, $f_j - f_i$ is proportional to the product of the interval size L and its corresponding location in the register X, lifting the degeneration associated to the fact that intervals of the same size might be produced in different locations of the register.

3.1.2. Expected values with musical meaning

Let us suppose that in a musical piece, the probability associated to the frequency of occurrence of each interval of size L is known to be $\{p_L\}$ with $\sum_L p_L = 1$. If the probability p_L is related to simultaneous pitches, then L can be defined as $|L| \equiv |L(f_i, f_j)| = |j - i|$.

Probability distributions (PDs) allow us to obtain macroscopic quantities related to specific properties of musical pieces. For example, the average magnitude of the interval size is given by

$$\langle |L| \rangle = \sum_{L=L_{min}}^{L_{max}} |L| p_L . \tag{3-5}$$

Frequently, different musical instruments have different registers. However, equation (3-5) does not capture this information, for example in a transposition process that moves a set of intervals from one part of the register to another. The expected value of the frequency difference captures information about the locations of intervals in the register:

$$\begin{aligned} \langle |f_j - f_i| \rangle &= \frac{|f_{j_1} - f_{i_1}| + |f_{j_2} - f_{i_2}| + ... + |f_{j_N} - f_{i_N}|}{N} \\ &= \frac{\sum_{i',j'} |f_{j'} - f_{i'}|\Big|_{L_{min}} + ... + \sum_{i'',j''} |f_{j''} - f_{i''}|\Big|_{L_{max}}}{N} \\ &= \frac{N_{L_{min}} \langle |f_{j'} - f_{i'}| \rangle_{L_{min}}}{N} + ... + \frac{N_{L_{min}} \langle |f_{j''} - f_{i''}| \rangle_{L_{max}}}{N}, \end{aligned} \tag{3-6}$$

where N is the total number of intervals, $\sum_{i',j'} |f_{j'} - f_{i'}|\Big|_{L_{min}} + ... + \sum_{i'',j''} |f_{j''} - f_{i''}|\Big|_{L_{max}}$ is the sum of the frequency differences of intervals grouped by their size, and $N_{L_{min}}, ..., N_{L_{max}}$ are the total numbers of intervals of each size L. Taking $p_L = N_L / N$ as the probability of finding an interval of size L, the expected value is

$$\langle |f_j - f_i| \rangle = \sum_{L=L_{min}}^{L_{max}} \langle |f_{j'} - f_{i'}| \rangle_L p_L , \tag{3-7}$$

where $\langle |f_{j'} - f_{i'}| \rangle_L$ is the mean value of the frequency differences for a set of intervals of size L. The linear approximation leads to

$$\langle |f_j - f_i| \rangle \approx 2c \sum_{L=L_{min}}^{L_{max}} |L| \langle X \rangle_L p_L = 2c \sum_{L=L_{min}}^{L_{max}} |\mathbb{L}| p_L , \tag{3-8}$$

where $\mathbb{L} = L \langle X \rangle_L$ is an effective size containing information about the contribution of the average location in the register. Equation (3-8) can be considered as an extension of equation (3-5) when the average position in the register of each type of interval size $\langle X \rangle_L$ is taken into account. Notice that if all intervals have the same average position in the register X_C, then the expected value $|\langle f_j - f_i\rangle|$ is proportional to the expected value $\langle |L|\rangle$, being given by $\langle |f_j - f_i|\rangle \approx 2cX_C\langle |L|\rangle$. Equation (3-8) shows that the expected value associated to the frequency differences takes into account the mean location in the register of intervals. However, the diversity of locations in the register for the same interval does not contribute to the expression (3-8). A quantity that takes into account this diversity can be constructed from equation (3-4) as

$$f_j^2 - f_i^2 = (f_j - f_i)(f_j + f_i) \approx 4cL\left(\frac{f_j + f_i}{2}\right)^2 = 4cLX^2. \tag{3-9}$$

From the physics perspective, this quantity is proportional to the difference in the average energy densities $\epsilon_j - \epsilon_i$ for two harmonic waves with equal amplitudes a propagating in a medium with density ρ [75]:

$$\epsilon_j - \epsilon_i = 2\pi^2\rho a(f_j^2 - f_i^2)\ . \tag{3-10}$$

The expected value of the quantity $f_j^2 - f_i^2$ in equation (3-9) can be written as

$$\begin{aligned}\langle |f_j^2 - f_i^2|\rangle &= \sum_{L=L_{min}}^{L_{max}} \langle |f_{j'}^2 - f_{i'}^2|\rangle_L p_L \approx 4c \sum_{L=L_{min}}^{L_{max}} |L|\langle X^2\rangle_L p_L \\ &= 4c \sum_{L=L_{min}}^{L_{max}} |L|\left(\langle X\rangle_L^2 + \sigma_L^2\right) p_L = 4c \sum_{L=L_{min}}^{L_{max}} |\mathfrak{L}| p_L,\end{aligned} \tag{3-11}$$

where the term ${\sigma_L}^2$ represents the dispersion of the intervals of size L in the register (measured as a variance) with respect to the average position $\langle X\rangle_L$, and $\mathfrak{L} = L\left(\langle X\rangle_L^2 + \sigma_L^2\right)$ is an effective size that takes into account the contribution of the average location of intervals in the register as well as their dispersion. Equation (3-11) can be considered as an extension of equation (3-8), when the contribution from the dispersion in the locations of the intervals is taken into account. In the case of just one possible location in the register for each kind of interval of size L, ${\sigma_L}^2 = 0$. In addition, if the average positions in the register for intervals of different sizes are close to each other and they are located around the position X_C, then the first-order term of the Taylor expansion around X_C leads to $\langle X\rangle_L^2 \approx 2X_C\langle X\rangle_L - X_C^2 \approx X_C\langle X\rangle_L$. Hence, these approximations lead to $\langle |f_j^2 - f_i^2|\rangle \approx 2X_C\langle |f_j - f_i|\rangle$.

3.1.3. Transposition process

In a transposition process, the set of probabilities $\{p_L\}$ remains unvaried when the location of the intervals in the register is moved from the original one $\langle X\rangle_L^O$ to a new one $\langle X\rangle_L^N$. These locations are related as

$$\langle X\rangle_L^N = w\langle X\rangle_L^O; \quad w = f_N/f_O, \tag{3-12}$$

where f_O refers to any fundamental frequency in the original location, f_N is the corresponding frequency in the new location, and w is the interval of the transposition. While the observable $\langle|L|\rangle$ remains unchanged after the transposition process, $\langle|f_j - f_i|\rangle$ changes as follows:

$$\langle|f_j - f_i|\rangle_N = w\langle|f_{j'} - f_{i'}|\rangle_O, \tag{3-13}$$

where $\langle|f_j - f_i|\rangle_O$ and $\langle|f_j - f_i|\rangle_N$ denote to the expected values in the original and new locations of the register, respectively.

In the case of an observable $\langle|f_j^2 - f_i^2|\rangle$, the variance in the new location $(\sigma^2_{_L})_N$ changes with respect to the variance in the original location $(\sigma^2_{_L})_O$ by the square of the interval of the corresponding transposition w^2,

$$(\sigma^2_{_L})_N = w^2[(\sigma^2_{_L})_O]. \tag{3-14}$$

Because $\langle X\rangle^2_{_L}$ also scales with w^2, in a transposition process the ratio $\langle X\rangle^2_{_L}/\sigma^2_{_L}$ remains unchanged, and the expected value $\langle|f_j^2 - f_i^2|\rangle$ scales as

$$\langle|f_j^2 - f_i^2|\rangle_N = w^2\langle|f_{j'}^2 - f_{i'}^2|\rangle_O, \tag{3-15}$$

where $\langle|f_j^2 - f_i^2|\rangle_O$ and $\langle|f_j^2 - f_i^2|\rangle_N$ are the expected values in the original and new locations, respectively.

3.1.4. Distinguishability of pairs of pitches

So far, it has been shown that the quantities $f_j - f_i$ and $f_j^2 - f_i^2$ distinguish between intervals of the same size in different locations in the register (equations (3-4) and (3-9)). In order to understand whether the values of the quantities $f_j - f_i$ and $f_j^2 - f_i^2$ can be used to distinguish each possible pair of pitches in the just, Pythagorean, and 12-TET musical scales, the general problem of distinguishability is treated here, including intervals of different sizes.

Figure **3-2** illustrates the dependence of $f_j - f_i$ and $f_j^2 - f_i^2$ on the magnitude of the interval size $|L|$ for the 12-TET scale tuned with $A = 440Hz$. Considering the orders of magnitude of the values and the relative separations between branches, this figure indicates that the quantity $f_j^2 - f_i^2$ has a better resolution than $f_j - f_i$ for distinguishing intervals of equal size in different locations of the register.

The general distinguishability problem for pairs of pitches can be formulated independently of the musical scale and the particular tuning as follows: If two pairs of different pitches $\{f_i, f_j\}$ and $\{f_r, f_s\}$ produce the same frequency difference or the same difference in the squares of the frequencies, then

$$f_j - f_i = f_s - f_r \;;\; f_j^2 - f_i^2 = f_s^2 - f_r^2 \;;\; \text{for } f_j > f_i (\text{i.e. } \; j > i) \text{ and } f_s > f_r (\text{i.e. } s > r). \tag{3-16}$$

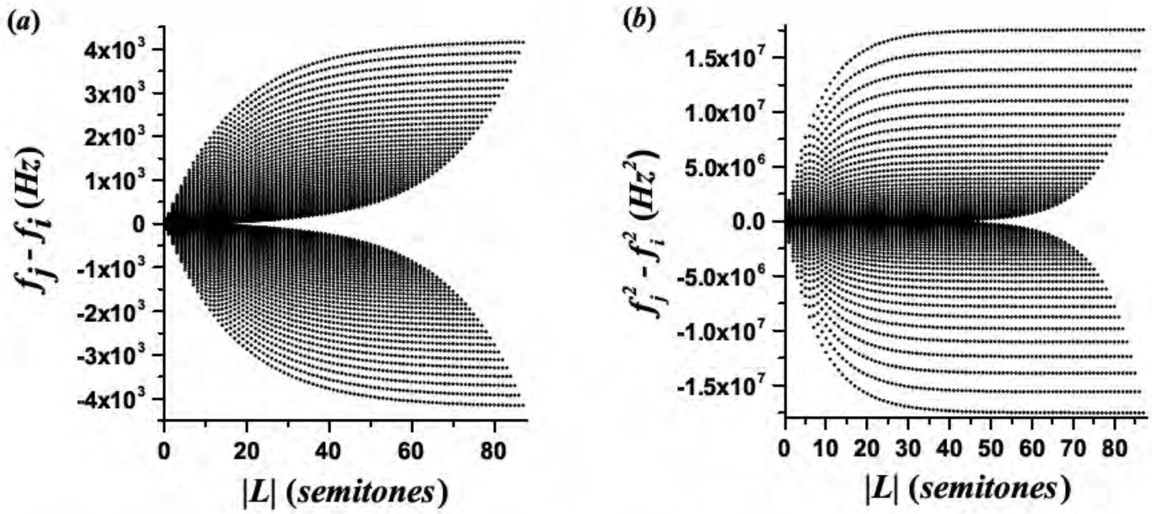

Figure 3-2.: Relation between the quantities $f_j - f_i$ and $f_j^2 - f_i^2$ and the magnitude of the interval size $|L|$ in semitones for $f_j > f_i$, shown in panels (a) and (b), respectively. The register corresponds to a typical 88 key piano. The upper branch comes from $j = 88$ (highest pitch) and i varies from 88 to 1. The tuning comes from the frequency relation for the 12-TET scale with $A = 440\,Hz$. Figure (b) was first presented in [39].

When $f_j < f_i$ and $f_s < f_r$, equation (3-16) can be transformed into a positive equality by changing the index order $i \leftrightarrow j; s \leftrightarrow r$. Then, both cases are equivalent.

The fundamental frequency ratios $\alpha_{(j-i)}$ can be used to relate the fundamental frequencies in a given musical scale, $f_j = \alpha_{(j-i)} f_i$ and $f_s = \alpha_{(s-r)} f_r$, with $\alpha_{(j-i)} > 1$ and $\alpha_{(s-r)} > 1$. Then, equations (3-16) can be written as

$$\frac{f_i}{f_r} = \frac{\alpha_{(s-r)} - 1}{\alpha_{(j-i)} - 1} \qquad \text{and} \qquad \frac{f_i^2}{f_r^2} = \frac{\alpha_{(s-r)}^2 - 1}{\alpha_{(j-i)}^2 - 1}. \tag{3-17}$$

Then, for $i > r$ and $i < r$, $f_i = \alpha_{(i-r)} . f_r$ and $f_r = \alpha_{(r-i)} . f_i$, respectively. Therefore, equations (3-17) can be written for the frequency difference as

$$\frac{\alpha_{(s-r)} - 1}{\alpha_{(j-i)} - 1} = \alpha_{(i-r)} \quad \text{for } i > r \qquad \text{and} \qquad \frac{\alpha_{(j-i)} - 1}{\alpha_{(s-r)} - 1} = \alpha_{(r-i)} \quad \text{for } i < r\,, \tag{3-18}$$

and for the difference between the squares of the frequencies as

$$\frac{\alpha_{(s-r)}^2 - 1}{\alpha_{(j-i)}^2 - 1} = \alpha_{(i-r)}^2 \quad \text{for } i > r \qquad \text{and} \qquad \frac{\alpha_{(j-i)}^2 - 1}{\alpha_{(s-r)}^2 - 1} = \alpha_{(r-i)}^2 \quad \text{for } i < r\,. \tag{3-19}$$

Given an uncertainty $\triangle\alpha$, there is a specific number of combinations of pairs (i, j) and (s, r) that satisfy equation (3-18). In a similar manner, an uncertainty $\triangle\alpha^2$ gives a specific number of combinations that satisfy equation (3-19). All possible combinations of the α

ratios ($1 \leq s-r \leq L_{max}$; $1 \leq j-i \leq L_{max}$; $1 \leq |i-r| < L_{max}$) are generated in order to find the number of times that the degeneracy equations ((3-18) and (3-19)) are satisfied as a function of the precision in the decimal places. This procedure is equivalent to checking if for each possible α ratio, the following equations are verified

$$\frac{\alpha_1 \pm \Delta\alpha_1 - 1}{\alpha_2 \pm \Delta\alpha_2 - 1} = \alpha_3 \pm \Delta\alpha_3 \quad ; \quad \frac{\alpha_1^2 \pm \Delta\alpha_1^2 - 1}{\alpha_2^2 \pm \Delta\alpha_2^2 - 1} = \alpha_3^2 \pm \Delta\alpha_3^2. \tag{3-20}$$

Assuming equal uncertainties for the three ratios, $\Delta\alpha_1 = \Delta\alpha_2 = \Delta\alpha_3 = \Delta\alpha$, given by the number of decimal places d, this is $\Delta\alpha = (1 \times 10^{-d})/2$, , and using the property $\Delta(\alpha^2) \approx 2\alpha\Delta\alpha$, then the previous equations can be rewritten as:

$$\frac{\alpha_1 - 1 \pm \Delta\alpha}{\alpha_2 - 1 \pm \Delta\alpha} = \alpha_3 \pm \Delta\alpha \quad ; \quad \frac{\alpha_1^2 - 1 \pm 2\alpha_1\Delta\alpha}{\alpha_2^2 - 1 \pm 2\alpha_2\Delta\alpha} = \alpha_3^2 \pm 2\alpha_3\Delta\alpha. \tag{3-21}$$

The minimum and the maximum values that can take the fractions in the previous equations, for the frequency differences (min_α, max_α) and for the difference in the squares of the frequencies ($min_{\alpha^2}, max_{\alpha^2}$) are given by:

Minimum values:

$$min_\alpha = \frac{\alpha_1 - 1 - \Delta\alpha}{\alpha_2 - 1 + \Delta\alpha} \quad ; \quad min_{\alpha^2} = \frac{\alpha_1^2 - 1 - 2\alpha_1\Delta\alpha}{\alpha_2^2 - 1 + 2\alpha_2\Delta\alpha} \tag{3-22}$$

Maximum values:

$$max_\alpha = \frac{\alpha_1 - 1 + \Delta\alpha}{\alpha_2 - 1 - \Delta\alpha} \quad ; \quad max_{\alpha^2} = \frac{\alpha_1^2 - 1 + 2\alpha_1\Delta\alpha}{\alpha_2^2 - 1 - 2\alpha_2\Delta\alpha} \tag{3-23}$$

For the frequency difference, if at least one value of α in the interval $[\alpha_3 - \Delta\alpha, \alpha_3 + \Delta\alpha]$ is also in the interval $[min_\alpha, max_\alpha]$, or vice versa, then the degeneracy equation is satisfied for the corresponding values α_1, α_2, and α_3 with a particular uncertainty $\Delta\alpha$.

For the difference in the squares of the frequencies, if at least one value in the interval $[\alpha_3^2 - 2\alpha_3\Delta\alpha, \alpha_3^2 + 2\alpha_3\Delta\alpha]$ is comprehended in the interval $[min_{\alpha^2}, max_{\alpha^2}]$ or vice versa, the degeneracy equation is satisfied.

The values of α for the Pythagorean, the just, and the 12-TET scales are calculated from the values n/m and f_j/f_i presented in Table **1-1**. For all scales, the α coefficients corresponding to intervals larger than one octave are obtained by multiplying the corresponding coefficient of the previous octave by 2.

Table **3-1** shows the number of combinations of α ratios satisfying the degeneracy equations as a function of the number of decimal places d used to measure these ratios ($1 \leq d \leq 10$). Two possible situations are considered: intervals up to two octaves, for which is possible to interpret the quantities $f_j - f_i$ and $f_j^2 - f_i^2$ as proportional to the interval sizes; and the case of all possible intervals on an 88-pitch musical instrument, such as a traditional piano.

For intervals with size L up to two octaves ($L_{max} = 24$ semitones), the number of possible combinations of the α and α^2 ratios is $24 \times 24 \times 23 = 13248$. For intervals with sizes up

Scale	Up to 24 semitones		Up to 87 semitones	
	$f_j - f_i$	$f_j^2 - f_i^2$	$f_j - f_i$	$f_j^2 - f_i^2$
Just	52 for $d \geq 4$	2 for $d \geq 5$	208 for $d \geq 4$	5 for $d \geq 8$
Pythagorean	8 for $d \geq 4$	0 for $d \geq 5$	47 for $d \geq 5$	2 for $d \geq 8$
12-TET	0 for $d \geq 5$	0 for $d \geq 4$	0 for $d \geq 5$	0 for $d \geq 8$

Table 3-1.: Number of combinations of the α ratios that satisfy the degeneracy equations (3-18) and (3-19), as a function of their precision given in terms of the utilized number of decimal places d. Results for $1 \leq d \leq 10$.

to 87 semitones (corresponding to an 88-pitch musical instrument) the number of possible combinations is $87 \times 87 \times 86 = 650934$.

In the 12-TET scale, the quantity $f_j - f_i$ distinguishes each pair of different pitches when the degeneracy is lifted by rounding the value of the α ratio to $d \geq 5$ for the 24 and 87 semitones cases. The quantity $f_j^2 - f_i^2$ lifts the degeneracy for $d \geq 4$ in the case of 24 semitones, and for $d \geq 8$ in the case of 87 semitones (see Table **3-1**). In the Pythagorean scale, the degeneracy of the quantity $f_j^2 - f_i^2$ can only be lifted for the 24 semitones case, taking $d \geq 5$, and the degeneracy of $f_j - f_i$ cannot be lifted with up to 10 decimal places ($d = 10$) (see Table **3-1**). In the just scale, the degeneracy remains up to $d = 10$ for both quantities and in both cases (24 and 87 semitones) (see Table **3-1**).

In some cases, the degeneracy equations are satisfied independently of the precision used to measure the α ratios. For example, in the case of the quantity $f_j^2 - f_i^2$ for the just scale, the combination of $\alpha_{(s-r)} = 5/3$ and $\alpha_{(j-i)} = 5/4$ produces $\alpha_{(i-r)} = 16/9$, *i.e.*, the major thirds (5/3) produce equal values to major sixths (5/4) when the lowest pitches of each of these intervals generate minor sevenths (16/9). Whenever it is possible to lift the degeneracy, this can be achieved by controlling the level of distinguishability between pairs of pitches through the selection of the precision of the α ratios, determined by the uncertainty $\Delta\alpha$. This can be done using the uncertainty in the measurement of the frequencies Δf and the squares of the frequencies Δf^2 in order to construct the quantities $f_j - f_i$ and $f_j^2 - f_i^2$.

Defining $\alpha = (f_j \pm \Delta f_j)/(f_i \pm \Delta f_i)$, with $f_j > f_i$ (i.e. $\alpha > 1$), the uncertainty for the quotient is

$$\Delta\alpha = \alpha \left(\frac{\Delta f_j}{f_j} + \frac{\Delta f_i}{f_i} \right). \tag{3-24}$$

Assuming equal uncertainty values for the frequencies $\Delta f_i = \Delta f_j = \Delta f$:

$$\Delta\alpha = \alpha \left(\frac{1}{f_j} + \frac{1}{f_i} \right) \Delta f, \tag{3-25}$$

then

$$\Delta f = \frac{\Delta\alpha f_j f_i}{\alpha(f_j + f_i)}. \tag{3-26}$$

Equation (3-26) relates the uncertainty in the measure of the α ratios ($\Delta\alpha$) with the uncertainty in the measure of the frequency Δf.

For the case of the square of the frequencies: $\alpha^2 = [f_j^2 \pm \Delta(f_j^2)]/[f_i^2 \pm \Delta(f_i^2)]$; $f_j > f_i$, $\alpha > 1$, and using the property $\Delta(\alpha^2) \approx 2\alpha\Delta\alpha$ we have:

$$\Delta(\alpha^2) = 2\alpha\Delta\alpha = \alpha^2 \left[\frac{\Delta(f_j^2)}{f_j^2} + \frac{\Delta(f_i^2)}{f_i^2}\right]. \tag{3-27}$$

Assuming equal uncertainty values for the squares of the frequencies: $\Delta(f_i^2) = \Delta(f_j^2) = \Delta f^2$, then

$$\Delta\alpha = \frac{\alpha}{2}\left(\frac{1}{f_j^2} + \frac{1}{f_i^2}\right)\Delta f^2, \tag{3-28}$$

and finally:

$$\Delta f^2 = \frac{2\Delta\alpha f_j^2 f_i^2}{\alpha(f_j^2 + f_i^2)}. \tag{3-29}$$

As an example, to distinguish each possible pair of pitches in a range of up to 24 semitones ($1 \leq \alpha \leq 4$) in the 12-TET scale, the quantity $f_j^2 - f_i^2$ must be measured with a precision in f^2, i.e., Δf^2, that is consistent with an uncertainty $\Delta\alpha = 5 \times 10^{-5}$, corresponding to rounding α to 4 decimal places.

3.2. Connection with tonal consonance

This section shows the connection between the representation of musical intervals previously presented and the tonal consonance formalism.

3.2.1. Measuring the dissonance levels of intervals

Musical instruments produce complex tones composed of pure tones. The dissonance level D of two simultaneous complex tones with the same timbre, as in the case of a harmonic interval, can be calculated using models as for example the Sethares or the Vassilakis ones. Taking into account the contributions of all individual dissonances δ generated from all possible combinations of pure tones in the superposition of the spectra.

As it was explained in Chapter 1, the frequency difference has been utilized to determine the dissonance levels of pairs of pure tones being sounded together [18, 21, 25, 30, 31, 32].

Figure **3-3** presents the dissonance curves generated using the Vassilakis model (equation (1-6)) for the intervals within the octave in the case of the 12-TET scale. This figure has been generated fixing a particular frequency ratio α and then varying the value of the frequency f_{min} in equation (1-6). The spectrum of each complex tone corresponds to six harmonics, with amplitudes falling at a rate of 0.88, as proposed by Sethares [21]. Explicitly, this is $A_n = A_0(0.88)^n$, where A_0 is the amplitude of the fundamental and A_n is that of the corresponding harmonic $n = 1, 2, 3, 4, 5, 6$.

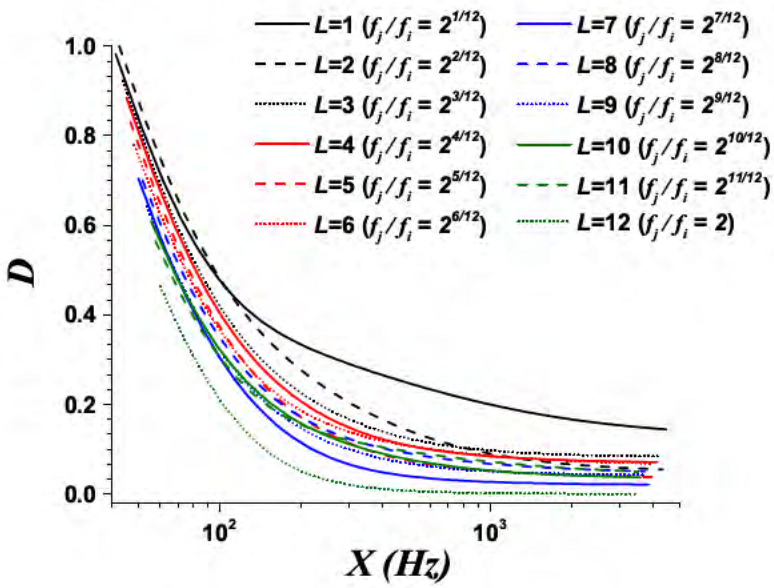

Figure 3-3.: Relation between the dissonance level D and the locations of harmonic intervals in the register $X = (f_j + f_i)/2$ for the 12-TET scale. The spectrum of each complex tone contains six harmonics with amplitudes falling at a rate of 0.88. Each possible size L corresponds to a particular frequency ratio inside the octave in the 12-TET scale. The dissonance level has been normalized to 1 for the typical register of an 88 key piano.

Figure **3-3** shows that the same interval of size L is less dissonant in the middle part of the register than in the lowest part, which is a well-known property of intervals [2, 18].

For each interval size L inside the octave, the dissonance level depends on its corresponding location in the register $X = (f_j + f_i)/2$. The fit to exponential functions is

$$D = F(X) = A_1 exp(-X/\gamma_1) + A_2 exp(-X/\gamma_2) + A_3, \tag{3-30}$$

with fit parameters A_1, A_2, A_3, γ_1, and γ_2. The values for the fit parameters of each interval size, and the corresponding determination coefficients R^2, are presented in Table **3-2**.

In the case of intervals larger than the octave, the *chroma* property of pitches states that the consonance values of these intervals can be measured by displacing the highest pitch to the next lower octave until the resulting interval is smaller than or equal to one octave [2]. With this property, the plots shown in Figure **3-3** can be employed to measure the tonal consonance levels of all possible interval sizes located at any part of the register.

		A_1	A_2	A_3	γ_1 (Hz)	γ_2 (Hz)	R^2
Interval size (semitones)	1	1.6528 ± 0.0230	0.2087 ± 0.0018	0.1493 ± 0.0004	41.5591 ± 0.4704	723.8846 ± 9.1151	0.9972
	2	1.5586 ± 0.0193	0.2818 ± 0.0042	0.0595 ± 0.0003	49.2216 ± 0.6564	435.4327 ± 6.2571	0.9977
	3	1.7258 ± 0.0171	0.2461 ± 0.0046	0.0866 ± 0.0002	42.2367 ± 0.4526	276.1143 ± 3.9790	0.9988
	4	1.6806 ± 0.0177	0.2182 ± 0.0051	0.0740 ± 0.0002	44.5340 ± 0.5252	284.3376 ± 5.1389	0.9985
	5	1.6461 ± 0.0172	0.2037 ± 0.0052	0.0414 ± 0.0002	46.3160 ± 0.5511	290.0944 ± 5.6755	0.9986
	6	1.5680 ± 0.0174	0.1350 ± 0.0031	0.0714 ± 0.0002	47.7470 ± 0.5118	414.0880 ± 9.1128	0.9981
	7	1.6566 ± 0.0146	0.1824 ± 0.0059	0.0226 ± 0.0001	43.7348 ± 0.4685	215.1966 ± 4.4173	0.9992
	8	1.4392 ± 0.0170	0.1345 ± 0.0031	0.0533 ± 0.0002	50.1781 ± 0.5704	424.4440 ± 9.2791	0.9983
	9	1.4719 ± 0.0162	0.1750 ± 0.0044	0.0444 ± 0.0001	45.1189 ± 0.5146	260.5915 ± 4.6645	0.9991
	10	1.3545 ± 0.0182	0.1550 ± 0.0031	0.0397 ± 0.0002	48.0988 ± 0.5862	375.0375 ± 6.6909	0.9987
	11	1.3514 ± 0.0171	0.1045 ± 0.0015	0.0523 ± 0.0002	48.2632 ± 0.4694	601.3901 ± 11.1094	0.9986
	12	1.5831 ± 0.0107	0.1117 ± 0.0044	0.0006 ± 0.0001	42.1740 ± 0.3228	174.6484 ± 3.6890	0.9998

Table 3-2.: Fitting parameters and determination coefficients R^2 for the dissonance curves of musical interval sizes inside the octave. The fit parameters correspond to the function (3-30).

3.2.2. Expected values of the dissonance levels associated to intervals

Suppose that in a musical piece the probability associated to the frequency of occurrence of each harmonic interval size L is known, as $\{p_L\}$ with $\sum_L p_L = 1$. The average dissonance associated to harmonic intervals can be defined as

$$\langle D \rangle = \frac{1}{H} \sum_j D_j, \tag{3-31}$$

where H is the total number of harmonic intervals in the musical score. Grouping by intervals of equal size, as in equation (3-6), we have that

$$\langle D \rangle = \frac{\sum_i D_i \Big|_{L_{min}} + ... + \sum_{i'} D_{i'} \Big|_{L_{max}}}{N} = \frac{N_{L_{min}} \langle D \rangle_{L_{min}}}{N} + ... + \frac{N_{L_{max}} \langle D \rangle_{L_{max}}}{N}, \tag{3-32}$$

and taking $p_{L_i} = N_{L_i}/N$ as the probability of finding an interval of size L_i the expected value of dissonance in a musical piece owing to the contribution of harmonic intervals is

$$\langle D \rangle = \sum_{L=L_{min}}^{L_{max}} \langle D \rangle_L p_L. \tag{3-33}$$

If all harmonic intervals have the same timbre and D can be expressed as in equation (3-30), then the average dissonance for each kind of interval size $\langle D \rangle_L$ can be approximately obtained by expanding equation (3-30) in a Taylor series around the mean position in the register

First order approximation

Taking the first order approximation around a point q:

$$D = F(X) \approx F(q) + F'(q)(X - q). \tag{3-34}$$

If we have M musical intervals of equal size L in the musical piece, then we can take q as the mean position in the register $\langle X \rangle = (1/M) \sum_{e=1}^{M} X_e$, then

$$D = F(X) \approx F(\langle X \rangle) + XF'(\langle X \rangle) - \langle X \rangle F'(\langle X \rangle). \tag{3-35}$$

The mean dissonance level is given by $\langle D \rangle = (1/M) \sum_{y=1}^{M} D_y$, then taking the average of the contribution of all musical intervals we have

$$\langle D \rangle \approx F(\langle X \rangle) \frac{1}{M} \sum_{y=1}^{M} 1 + F'(\langle X \rangle) \frac{1}{M} \sum_{z=1}^{M} X_z + \langle X \rangle F'(\langle X \rangle) \frac{1}{M} \sum_{w=1}^{M} 1 \tag{3-36}$$

thus

$$\langle D \rangle \approx F(\langle X \rangle) + \langle X \rangle F'(\langle X \rangle) - \langle X \rangle F'(\langle X \rangle) = F(\langle X \rangle). \tag{3-37}$$

As there are different musical interval sizes L, for each one we have

$$\langle D \rangle_L \approx F(\langle X \rangle_L). \tag{3-38}$$

Second order approximation

In the second order approximation, we have and additional contribution

$$D = F(X) \approx F(q) + XF'(q) - qF'(q) + \frac{1}{2}(X - q)^2 F''(q). \tag{3-39}$$

Taking q as the mean location in the register $\langle X \rangle$, and taking the average value of the dissonance level $\langle D \rangle$ as in the case of the first order approximation, then

$$\langle D \rangle \approx F(\langle X \rangle) + \frac{1}{2} F''(\langle X \rangle) \frac{1}{M} \sum_{y=1}^{M} \left(X^2 - 2X\langle X \rangle + \langle X \rangle^2 \right)_y \tag{3-40}$$

thus

$$\langle D \rangle \approx F(\langle X \rangle) + \frac{1}{2} F''(\langle X \rangle) \frac{1}{M} \sum_{y=1}^{M} X_y^2 - \langle X \rangle F''(\langle X \rangle) \frac{1}{M} \sum_{z=1}^{M} X_z + \frac{1}{2} \langle X \rangle^2 F''(\langle X \rangle) \frac{1}{M} \sum_{w=1}^{M} 1. \tag{3-41}$$

Since $(1/M) \sum_{z=1}^{M} X_z = \langle X \rangle$, then

$$\langle D \rangle \approx F(\langle X \rangle) + \frac{1}{2} F''(\langle X \rangle) \frac{1}{M} \sum_{y=1}^{M} X_y^2 - \frac{1}{2} \langle X \rangle^2 F''(\langle X \rangle). \tag{3-42}$$

Finally, since $(1/M)\sum_{y=1}^{M} X_y^2 = \langle X^2 \rangle$ and the expression for the variance is $\sigma^2 = \langle X^2 \rangle - \langle X \rangle^2$, then

$$\langle D \rangle \approx F(\langle X \rangle) + \frac{1}{2}\sigma^2 F''(\langle X \rangle). \tag{3-43}$$

As there are different musical interval sizes L, for each one we have

$$\langle D \rangle_L \approx F(\langle X \rangle_L) + \frac{1}{2}\sigma_L^2 F''(\langle X \rangle_L). \tag{3-44}$$

The first term in equation (3-44) indicates that the mean location in the register for each kind of interval size $\langle X \rangle_L$ corresponds to the most important contribution to measuring the mean dissonance. The second term in equation (3-44) indicates that the dispersion of each interval size σ_L^2 is necessary to more precisely measure the mean dissonance $\langle D \rangle$.

To summarize, by knowing $\langle X \rangle_L$ and the set of probabilities $\{p_L\}$ it is possible to measure $\mathbb{L}$, the expected value of $\langle |f_j - f_i| \rangle$, and approximate the mean dissonance level $\langle D \rangle$. On the other hand, by knowing $\langle X \rangle_L$, σ_L^2, and $\{p_L\}$ it is possible to measure $\mathfrak{L}$, the expected value $\langle |f_j^2 - f_i^2| \rangle$, and the mean dissonance level $\langle D \rangle$ with greater precision.

3.3. Melody and expected values of melodic intervals

This section presents some concepts about melody, and the expected values associated to the asymmetry in the use of ascending and descending intervals in melodic lines.

3.3.1. Concerning melody

Melody is defined in the New Grove Dictionary of Music and Musicians as "pitched sounds arranged in musical time in accordance with given cultural conventions and constraints" [12]. A definition that encompasses music and speech was given by Aniruddh Patel as "an organized sequence of pitches that conveys a rich variety of information to a listener" [9].

So far, the sign of the interval size L has not been considered, as pitches in harmonic intervals are played simultaneously. However, in the case of melody, pitches are ordered chronologically (melodic intervals). For $f_i = f_i(t)$ and $f_j = f_j(t+1)$, there are three possible cases: if $f_j > f_i$ then $L = j - i > 0$ (ascending interval), if $f_j < f_i$ then $L = j - i < 0$ (descending interval), and if $f_i = f_j$ then $L = 0$ (unison). Therefore, the sign of L distinguishes the chronological order of a pair of pitches.

For the case of the quantities $f_j - f_i$ and $f_j^2 - f_i^2$, the following notation will be employed: if $\{t_z\}$ represents a collection of times, at each of which one pitch is played in a melody (without rests), then the quantities $f_{t_{(z+1)}} - f_{t_z} \equiv f_{t+1} - f_t$ and $f_{t_{(z+1)}}^2 - f_{t_z}^2 \equiv f_{t+1}^2 - f_t^2$ symbolize melodic intervals, with the sign distinguishing between ascending $(f_{t+1} > f_t)$ and descending $(f_{t+1} < f_t)$ intervals.

The case of $f_j > f_i$ and $f_s > f_r$ was analyzed in the section on the distinguishability of pairs of pitches, which corresponds to ascending intervals, and it was shown that the case with $f_j < f_i$ and $f_s < f_r$, which corresponds here to descending intervals, was shown to be equivalent (see equation (3-16)).

3.3.2. Expected values of melodic intervals

In the case of melody, there are three kinds of melodic intervals, ascending, descending, and unisons, and the normalization constraint may be stated as $\tilde{p}_a + \tilde{p}_d + \tilde{p}_u = 1$, where $\tilde{p}_a$ is the probability of ascending intervals, $\tilde{p}_d$ is the probability of descending ones, and $\tilde{p}_u$ is the probability of unisons. The average magnitude of the melodic interval size contains the contributions of positive, negative, and zero values of L in the sum, $L \in [L_{min}, L_{max}]$, and the expression (3-5) remains unaltered. The average magnitude of the melodic interval size taking into account the mean location in the register $\langle X \rangle_L$ and the dispersion σ_L^2 leads to the same expressions given previously (equations (3-8) and (3-11)). However, now these contain the contributions of the ascending, descending, and unison intervals. These expected values include the average magnitude of the melodic intervals, but do not discriminate between ascending and descending intervals. The average magnitudes of ascending and descending intervals, $\langle L_{>0} \rangle$ and $\langle L_{<0} \rangle$, respectively, can be measured by

$$\langle L_{>0} \rangle = \frac{1}{\tilde{p}_a} \sum_{L=1}^{L_{max}} L p_{_L} \quad ; \quad \langle L_{<0} \rangle = \frac{1}{\tilde{p}_d} \sum_{L=L_{min}}^{-1} L p_{_L}, \tag{3-45}$$

where the ratio $p_i/\tilde{p}_a$ ($p_i/\tilde{p}_d$) refers to the probability of the occurrence of an interval of size L_i in the ascending (descending) intervals of a musical piece.

The asymmetry in the total number of intervals is $\tilde{p}_a - \tilde{p}_d$, and the asymmetry between the average magnitudes of ascending and descending intervals can be obtained as $\langle L_{>0} \rangle + \langle L_{<0} \rangle$, where $\langle L_{<0} \rangle < 0$. Because the existing literature reports that in many cultures large melodic intervals are more likely to ascend than small ones, and that melodies tend to meander around a central pitch range (See Figure **1-9**), the quantity $\tilde{p}_a - \tilde{p}_d$ is expected to be negative, and the quantity $\langle L_{>0} \rangle + \langle L_{<0} \rangle$ is expected to be positive, for melodic lines of several musical pieces. See Figure **1-9**.

The asymmetry in the average magnitudes of ascending and descending intervals, taking into account the mean position in the register $\langle X \rangle_L$ and the dispersion of the intervals σ_L^2, can be measured using $\langle (f_j - f_i)_{>0} \rangle + \langle (f_j - f_i)_{<0} \rangle$ and $\langle (f_j^2 - f_i^2)_{>0} \rangle + \langle (f_j^2 - f_i^2)_{<0} \rangle$. These expressions take the form

$$\langle (f_{t+1} - f_t)_{>0} \rangle + \langle (f_{t'+1} - f_{t'})_{<0} \rangle = \frac{1}{\tilde{p}_a} \sum_{L=1}^{L_{max}} \langle f_{\tau+1} - f_\tau \rangle_L p_L + \frac{1}{\tilde{p}_d} \sum_{L=L_{min}}^{-1} \langle f_{\tau'+1} - f_{\tau'} \rangle_L p_L$$
$$\approx 2c \left(\frac{1}{\tilde{p}_a} \sum_{L=1}^{L_{max}} \mathbb{L} p_L + \frac{1}{\tilde{p}_d} \sum_{L=L_{min}}^{-1} \mathbb{L} p_L \right) \quad (3\text{-}46)$$

and

$$\langle (f^2_{t+1} - f^2_t)_{>0} \rangle + \langle (f^2_{t'+1} - f^2_{t'})_{<0} \rangle = \frac{1}{\tilde{p}_a} \sum_{L=1}^{L_{max}} \langle f^2_{\tau+1} - f^2_\tau \rangle_L p_L + \frac{1}{\tilde{p}_d} \sum_{L=L_{min}}^{-1} \langle f^2_{\tau'+1} - f^2_{\tau'} \rangle_L p_L$$
$$\approx 4c \left(\frac{1}{\tilde{p}_a} \sum_{L=1}^{L_{max}} \mathfrak{L} p_L + \frac{1}{\tilde{p}_d} \sum_{L=L_{min}}^{-1} \mathfrak{L} p_L \right). \quad (3\text{-}47)$$

Traditionally, consonance properties have been associated with simultaneous sounds. However, in music theory, consonance sensations have also been related to successive sounds [11]. It has been observed that musicians tend to transpose their knowledge about the consonance levels of harmonic intervals to judge melodic ones [20]. In addition, a possible mechanism for the creation of consonance sensations in the case of successive pitches is the short-term persistence of pitch in auditoriums [20]. Therefore, the connection between tonal consonance and simultaneous pitches made in the previous section can be assumed to hold for successive pitches, at least in the case of musicians. For this case, the expression (3-33) takes into account the contributions of ascending and descending intervals and melodic unisons.

For the consonance analysis of melodic intervals, the sign of L is irrelevant: only its magnitude is important. Then, Figure **3-3** can be utilized for ascending intervals as well as descending ones.

3.4. An application to melodic lines

This section shows the analysis of a set of melodic lines using the representation of musical intervals proposed, the procedures followed to obtain their corresponding probability and cumulative distributions, and the corresponding results.

3.4.1. Selection of melodic lines

Twenty melodic lines from seven vocal and instrumental masterpieces of the Baroque and Classical periods were analyzed. The base data was taken from [39]. The selected pieces contain melodic lines characterized by their considerable length, internal coherence, and rich variety of instruments and registers. The collection of pieces is as follows:

- *Brandenburg Concerto No. 3 in G Major BWV 1048*. Johann Sebastian Bach: Polyphonic concerto for 11 musical instruments (three violins, three violas, three cellos, violone, and harpsichord).
- *Missa Super Dixit Maria*. Hans Leo Hassler: Polyphonic composition for four voices (soprano, contralto, tenor, and bass).
- *First movement of the Partita in A Minor BWV 1013*. Johann Sebastian Bach: This piece has just one melodic line for a flute.
- *Piccolo Concerto RV444*. Antonio Vivaldi (arrangement by Gustav Anderson): We selected the piccolo melodic line, owing to its rich melodic content.
- *Sonata KV 545*. Wolfgang Amadeus Mozart: We selected the melodic line for the right hand of this piano sonata, assuming that it drives the melodic content.
- *Suite No. 1 in G Major BWV 1007* and *Suite No. 2 in D Minor BWV 1008*. Johann Sebastian Bach: The melodic lines of these pieces written for cello contain mainly successive pitches. In the cases of the few simultaneous pitches, the continuation of the melodic lines was assumed in the direction of the highest pitch.

3.4.2. Procedure to obtain the probability and the cumulative distributions

The PDs for the quantities $f_{t+1} - f_t$ and $f_{t+1}^2 - f_t^2$ were obtained for each melodic line in order to gather information concerning the selections of melodic intervals made by the composers. The procedure for the analysis of melodic lines was as follows:

- The simplified MIDI files were generated from scores [49]. Only successive pitches without rests between them were considered.
- The MIDI information was transformed into frequencies using the 12-TET scale with $A = 440\,Hz$.
- The PDs were obtained in three different cases:

  - Case 1: $|f_{t+1} - f_t|$ and $|f_{t+1}^2 - f_t^2|$ not distinguishing between ascending and descending intervals. The complementary cumulative distribution (CCD) was also obtained.

  - Case 2: $|f_{t+1} - f_t|$ and $|f_{t+1}^2 - f_t^2|$ for two different sets of intervals: ascending and unisons, and descending and unisons. The CCD was also obtained for each set.

  - Case 3: $f_{t+1}^2 - f_t^2$ for the set of ascending, descending, and unison intervals together. In this case, the sign of the descending intervals was considered as negative. The reason for only using the quantity $f_{t+1}^2 - f_t^2$ is the quality of the experimental fits obtained in the two previous analyses for both quantities, and even more relevantly that the distinguishability analysis shows that $f_{t+1}^2 - f_t^2$ has the best resolution properties for the case of 24 semitones in the 12-TET scale, which is the relevant range for melodic intervals in the analyzed melodic lines. The CCD was employed for the branch of the PD that contains the ascending intervals, and the cumulative distribution (CD) was utilized for the branch that contains the descending intervals.

Some clarifications are required in order to implement the sketch described above.

- Because the number of melodic intervals in the studied melodic lines is at most one order of magnitude larger than the total number of possible pairs of successive pitches generated by the same *ambitus* (the range between the lowest and highest pitches) of the original melodic line, the PDs were constructed using histograms, in order to capture significant probabilities. Table **3-3** shows the number of intervals of each melodic line, the number of ascending intervals, descending ones, and unisons, and the corresponding *ambitus*.

- As the number of possible melodic intervals for any melodic line is finite, independently of its length, the bin width in the histograms will be moderately dependent on the number of melodic intervals. This condition is satisfied by the Sturges criterion [76], and thus this criterion was used to determine the bin width.

- In the third case, when ascending and descending PDs were combined in the same distribution for the quantity $f_{t+1}^2 - f_t^2$, the bin width was taken as the average of those obtained separately using the Sturges criterion for ascending and descending distributions. The average bins were symmetrically located to the left and right, starting from the point $f_{t+1}^2 - f_t^2 = 0$.

- In the experimental analysis, the contribution of unisons in the histograms is important for ascending intervals as well as descending ones, with different right-hand and left-hand limits at 0. In addition, if we attempt to split the unisons into the ascending and descending parts, this procedure reduces the determination coefficient R^2 of the fits for the histograms to an exponential function [39]. Hence, all unisons were included in the ascending part as well as the descending one, and then a correction of this double count was carried out in the procedure to obtain the expected values. In the histograms, the descending intervals are contained inside the bins labeled from 1 to $N/2$ (from left to right), and the ascending ones inside those labeled from $N/2 + 1$ to N (from left to right). Hence, all unisons have been taken into account inside the bin labeled $N/2$ as well as that labelled $N/2 + 1$. Note that N is an even number.

		Intervals	Ascending	Descending	Unisons	$\tilde{p}_u$	$\tilde{p}_a - \tilde{p}_d$	***Ambitus*(*Hz*)**	
T. B. C.	Violin 1	3076	1253	1467	356	0.1157	-0.0696	Min.	195.997
								Max.	1244.507
	Violin 2	2589	1111	1124	354	0.1367	-0.0050	Min.	195.997
								Max.	1174.659
	Violin 3	2404	989	1050	365	0.1518	-0.0254	Min.	195.997
								Max.	1174.659
	Viola 1	2641	1022	1100	519	0.1965	-0.0295	Min.	146.832
								Max.	698.456
	Viola 2	2527	973	1037	517	0.2046	-0.0253	Min.	146.832
								Max.	659.255
	Viola 3	2429	922	985	522	0.2149	-0.0259	Min.	130.812
								Max.	659.255
	Cello 1	2138	895	1040	203	0.0949	-0.0678	Min.	65.406
								Max.	329.627
	Cello 2	2134	893	1038	203	0.0951	-0.0679	Min.	65.406
								Max.	329.627
	Cello 3	2132	890	1040	202	0.0947	-0.0704	Min.	65.406
								Max.	329.627
	Violone	2120	849	942	329	0.1552	-0.0439	Min.	32.703
								Max.	164.813
	Harpsichord	2120	849	942	329	0.1552	-0.0439	Min.	65.406
								Max.	329.627
M. D. M.	Soprano	1768	608	789	371	0.2098	-0.1024	Min.	261.625
								Max.	698.456
	Contralto	2170	818	912	440	0.2028	-0.0433	Min.	174.614
								Max.	523.251
	Tenor	2215	866	931	418	0.1887	-0.0293	Min.	164.813
								Max.	349.228
	Bass	1773	725	778	270	0.1523	-0.0299	Min.	97.998
								Max.	293.664
P. M.	Suite 1	3695	1749	1856	90	0.0244	-0.0290	Min.	65.406
								Max.	391.995
	Suite 2	3601	1713	1829	59	0.0164	-0.0322	Min.	65.406
								Max.	391.995
	Mozart sonata	3991	1613	2141	237	0.0594	-0.1323	Min.	195.997
								Max.	1864.655
	First mov. Partita	1020	472	546	2	0.0020	-0.0725	Min.	293.664
								Max.	1760.000
	Piccolo concerto	2740	1314	1195	231	0.0843	0.0434	Min.	698.456
								Max.	2793.825

Table 3-3.: Total number of melodic intervals, *ambitus*, and asymmetry between the number of ascending and descending intervals for each melodic line. T. B. C. refers to the third Brandenburg concerto, M. D. M. to the Missa Dixit Maria, and P. M. to a piece or movement. Intervals: Total number of melodic intervals. Ascending: Number of ascending intervals. Descending: Number of descending intervals. Unisons: Number of unisons. $\tilde{p}_u$: Probability of unisons in the melodic line. $\tilde{p}_a - \tilde{p}_d$: Asymmetry in the total number of intervals given in terms of the difference between the probabilities of ascending $\tilde{p}_a$ and descending $\tilde{p}_d$ intervals. *Ambitus*: Range between the lowest and the highest pitches of the melodic line.

3.4.3. Experimental results and analysis

For the first and the second cases, the histograms and CCD for both quantities ($|f_{t+1} - f_t|$ and $|f_{t+1}^2 - f_t^2|$) fit to exponential functions. Table **3-4** shows, for each melodic line in the first and the second case, the determination coefficient R^2 for the fits to exponential functions in histograms and CCD. The average $\overline{R^2}$ of the CCD is $\overline{R^2} \approx 0.99$, with a standard deviation (SD) of ≈ 0.01. Usually, the cumulative probability associated to the unison in the CCD is larger than the value predicted by the exponential behavior. This is not surprising, as the value 0 is degenerated, and represents more than one possible pair of pitches. For histograms, the highest $\overline{R^2}$ is for the quantity $|f_{t+1}^2 - f_t^2|$ with ascending and descending intervals taken separately. For ascending intervals, $\overline{R^2} = 0.987$ with $SD = 0.009$, and for descending ones $\overline{R^2} = 0.986$ with $SD = 0.016$.

For the third case, with the left and right branches of the PD combined in the same histogram, the PD can be written as

$$P(\varepsilon) = \begin{cases} F_+^H e^{-\varepsilon/G_+^H} & \text{for } \varepsilon > 0 \\ F_-^H e^{\varepsilon/G_-^H} & \text{for } \varepsilon < 0 \end{cases}, \tag{3-48}$$

where the notation ε emphasizes that these distributions are constructed over bins.

In the case of the cumulative distributions, the CCD and CD conserve the same functional form of the PD (as the PDs are exponential):

$$P(f_{t+1}^2 - f_t^2) = \begin{cases} F_+^C e^{-(f_{t+1}^2 - f_t^2)}/G_+^C & \text{for } (f_{t+1}^2 - f_t^2) > 0 \\ F_-^C e^{(f_{t+1}^2 - f_t^2)}/G_-^C & \text{for } (f_{t+1}^2 - f_t^2) < 0 \end{cases}. \tag{3-49}$$

Table **3-5** contains the values of $F_+^H, F_-^H, G_+^H, G_-^H, F_+^C, F_-^C, G_+^C, G_-^C$, and R^2 for the fits. These PDs resemble the asymmetric Laplace PD, with different amplitudes for positive and negative branches leading to a discontinuity at the origin (Figure **3-4**) [77].

		CCD						Histograms					
		$\vert f_{t+1}-f_t\vert$	$\vert f^2_{t+1}-f^2_t\vert$	$\vert f_{t+1}-f_t\vert+$	$\vert f_{t+1}-f_t\vert-$	$\vert f^2_{t+1}-f^2_t\vert+$	$\vert f^2_{t+1}-f^2_t\vert-$	$\vert f_{t+1}-f_t\vert$	$\vert f^2_{t+1}-f^2_t\vert$	$\vert f_{t+1}-f_t\vert+$	$\vert f_{t+1}-f_t\vert-$	$\vert f^2_{t+1}-f^2_t\vert+$	$\vert f^2_{t+1}-f^2_t\vert-$
T. B. C.	Violin 1	0.99613	0.99719	0.99822	0.99482	0.99549	0.99532	0.97461	0.99671	0.99771	0.96960	0.99860	0.99050
	Violin 2	0.99720	0.99522	0.99682	0.99183	0.99357	0.98828	0.99202	0.99738	0.99497	0.99419	0.98618	0.99115
	Violin 3	0.99826	0.99458	0.99590	0.99296	0.99122	0.98703	0.99588	0.98979	0.99870	0.99663	0.99138	0.99695
	Viola 1	0.99690	0.99616	0.99339	0.99146	0.98750	0.98683	0.98058	0.97628	0.98559	0.99433	0.97761	0.99361
	Viola 2	0.99572	0.99312	0.98949	0.98226	0.98216	0.97975	0.98669	0.98637	0.97808	0.99513	0.98102	0.99427
	Viola 3	0.99589	0.99277	0.98768	0.98067	0.98114	0.97855	0.99322	0.98978	0.97801	0.99527	0.98090	0.99612
	Cello 1	0.98868	0.99195	0.99468	0.98246	0.99126	0.98999	0.99376	0.99218	0.99733	0.99322	0.99147	0.99721
	Cello 2	0.98887	0.99223	0.99488	0.98284	0.99147	0.99035	0.99365	0.99235	0.99736	0.99381	0.99181	0.99745
	Cello 3	0.98832	0.99142	0.99466	0.98192	0.99132	0.98876	0.99364	0.99678	0.99783	0.99265	0.99236	0.99711
	Violone	0.98718	0.99079	0.99307	0.97590	0.98629	0.98348	0.99226	0.99089	0.99587	0.99295	0.98961	0.99661
	Harpsichord	0.98718	0.99079	0.99307	0.97590	0.98629	0.98348	0.99226	0.99089	0.99587	0.99404	0.98961	0.99661
M. D. M.	Soprano	0.96450	0.97219	0.99292	0.97157	0.98935	0.97597	0.80746	0.79343	0.99230	0.83315	0.96212	0.95440
	Contralto	0.96977	0.98016	0.98716	0.97415	0.98414	0.98051	0.94996	0.96966	0.99846	0.92906	0.99367	0.94612
	Tenor	0.97258	0.97930	0.99171	0.98456	0.99114	0.99017	0.88303	0.95319	0.99527	0.95344	0.97790	0.97827
	Bass	0.98360	0.99073	0.98565	0.98553	0.98849	0.98765	0.84540	0.86061	0.93378	0.97730	0.97005	0.97445
P. M.	Suite 1	0.99224	0.99144	0.98835	0.98641	0.99272	0.98976	0.99733	0.99441	0.93554	0.99744	0.99272	0.98976
	Suite 2	0.99323	0.99578	0.99168	0.99395	0.99328	0.99681	0.97859	0.99863	0.99781	0.99503	0.99328	0.99681
	Mozart sonata	0.98566	0.99485	0.99137	0.97616	0.99842	0.95533	0.86825	0.99834	0.97652	0.80946	0.99842	0.95533
	First mov. Partita	0.99620	0.99487	0.99018	0.99331	0.99342	0.99729	0.96832	0.99709	0.92956	0.98097	0.99342	0.99729
	Piccolo concerto	0.99132	0.99408	0.99283	0.99295	0.98227	0.98902	0.91862	0.96318	0.93085	0.97725	0.98227	0.98902
	Average	0.988	0.991	0.992	0.985	0.990	0.986	0.955	0.971	0.980	0.968	0.987	0.986
	SD	0.010	0.006	0.003	0.007	0.005	0.009	0.058	0.052	0.026	0.053	0.009	0.016

Table 3-4.: Determination coefficient R^2 for the fits of complementary cumulative distributions (CCDs) and histograms to exponential functions, for the quantities $|f_{t+1}-f_t|$ and $|f^2_{t+1}-f^2_t|$. The exponential functions have the form: $P(z)=\mathcal{C}exp(-|z|/\mathcal{D})$, with $\mathcal{C}$, and $\mathcal{D}$ parameters of the fit, and z refers to the quantities $f_{t+1}-f_t$ and $f^2_{t+1}-f^2_t$. Ascending transitions are identified by the sign "+", descending transitions are identified by the sign "−", and their combination is left without sign. T. B. C. refers to the third Brandenburg concerto, M. D. M. to the Missa Dixit Maria, and P. M. to a piece or movement. SD is the standard deviation.

		Cumulative probability			Histograms			
		R^2	$F^C_\pm$	$G^C_\pm$ (Hz^2)	Bin (Hz^2)	R^2	$F^H_\pm$	$G^H_\pm$ (Hz^2)
T. B. C.	Violin 1 +	0.995	0.447 ± 0.002	121580 ± 1068	82605	0.996	0.468 ± 0.012	90831 ± 3110
	Violin 1 -	0.995	0.542 ± 0.003	103251 ± 969		0.999	0.549 ± 0.006	90393 ± 1375
	Violin 2 +	0.994	0.475 ± 0.003	115448 ± 1367	83037	0.991	0.533 ± 0.021	87434 ± 4540
	Violin 2 -	0.988	0.481 ± 0.004	104910 ± 1613		0.996	0.618 ± 0.017	75333 ± 2656
	Violin 3 +	0.991	0.456 ± 0.004	100904 ± 1452	83663	0.992	0.629 ± 0.026	74761 ± 3864
	Violin 3 -	0.987	0.489 ± 0.005	86160 ± 1441		0.998	0.753 ± 0.016	66113 ± 1704
	Viola 1 +	0.988	0.460 ± 0.005	56383 ± 914	26833	0.986	0.422 ± 0.019	35203 ± 2082
	Viola 1 -	0.987	0.501 ± 0.006	49211 ± 853		0.994	0.478 ± 0.014	33187 ± 1296
	Viola 2 +	0.982	0.431 ± 0.005	53005 ± 1060	26958	0.979	0.546 ± 0.035	27153 ± 2246
	Viola 2 -	0.980	0.472 ± 0.006	43020 ± 929		0.995	0.618 ± 0.021	25882 ± 1110
	Viola 3 +	0.981	0.426 ± 0.005	52089 ± 1033	25898	0.982	0.578 ± 0.035	24507 ± 1864
	Viola 3 -	0.979	0.466 ± 0.006	40934 ± 903		0.996	0.672 ± 0.019	23151 ± 835
	Cello 1 +	0.991	0.460 ± 0.004	9665 ± 147	8398	0.991	0.612 ± 0.033	6869 ± 445
	Cello 1 -	0.990	0.540 ± 0.005	8313 ± 128		0.997	0.799 ± 0.030	6120 ± 266
	Cello 2 +	0.991	0.460 ± 0.004	9550 ± 143	8400	0.992	0.617 ± 0.033	6840 ± 436
	Cello 2 -	0.990	0.541 ± 0.005	8282 ± 124		0.997	0.799 ± 0.029	6137 ± 261
	Cello 3 +	0.991	0.461 ± 0.004	9328 ± 142	8401	0.992	0.627 ± 0.032	6739 ± 419
	Cello 3 -	0.989	0.542 ± 0.005	8136 ± 134		0.996	0.827 ± 0.033	5939 ± 270
	Violone +	0.986	0.454 ± 0.005	2280 ± 43	1969	0.990	0.738 ± 0.044	1468 ± 103
	Violone -	0.983	0.509 ± 0.006	1901 ± 42		0.995	0.993 ± 0.055	1207 ± 71
	Harpsichord +	0.986	0.454 ± 0.005	9122 ± 172	7877	0.990	0.738 ± 0.044	5873 ± 414
	Harpsichord -	0.983	0.509 ± 0.006	7605 ± 167		0.995	0.993 ± 0.055	4829 ± 283
M. D. M.	Soprano +	0.989	0.540 ± 0.011	37543 ± 1089	21403	0.972	0.335 ± 0.023	36194 ± 3340
	Soprano -	0.976	0.683 ± 0.021	34985 ± 1352		0.904	0.354 ± 0.048	42974 ± 8003
	Contralto +	0.984	0.557 ± 0.012	23451 ± 693	13797	0.989	0.388 ± 0.016	20716 ± 1147
	Contralto -	0.981	0.622 ± 0.015	22286 ± 784		0.949	0.403 ± 0.046	22204 ± 3485
	Tenor +	0.991	0.588 ± 0.011	12664 ± 285	7005	0.968	0.302 ± 0.020	13946 ± 1298
	Tenor -	0.990	0.628 ± 0.012	12413 ± 293		0.961	0.321 ± 0.024	13872 ± 1431
	Bass +	0.988	0.537 ± 0.009	9807 ± 258	4669	0.974	0.313 ± 0.018	8379 ± 663
	Bass -	0.988	0.565 ± 0.010	9982 ± 265		0.978	0.324 ± 0.017	8414 ± 608
P. M.	Suite 1 +	0.992	0.557 ± 0.004	13559 ± 163	11115	0.993	0.369 ± 0.013	15314 ± 729
	Suite 1 -	0.992	0.581 ± 0.004	13211 ± 163		0.990	0.392 ± 0.016	15521 ± 879
	Suite 2 +	0.994	0.535 ± 0.003	13356 ± 138	8893	0.996	0.304 ± 0.007	14493 ± 424
	Suite 2 -	0.997	0.558 ± 0.002	12918 ± 96		0.997	0.327 ± 0.006	14527 ± 384
	Mozart sonata +	0.992	0.469 ± 0.004	111508 ± 1788	104025	0.998	0.550 ± 0.008	86724 ± 1533
	Mozart sonata -	0.994	0.641 ± 0.005	116151 ± 1644		0.997	0.589 ± 0.009	107964 ± 2230
	First mov. Partita +	0.996	0.503 ± 0.003	215382 ± 2152	148060	0.995	0.283 ± 0.007	254793 ± 8407
	First mov. Partita -	0.990	0.596 ± 0.005	159508 ± 2517		0.998	0.496 ± 0.009	160794 ± 3881
	Piccolo concerto +	0.995	0.583 ± 0.004	952916 ± 10349	579759	0.987	0.325 ± 0.015	1054820 ± 66251
	Piccolo concerto -	0.994	0.519 ± 0.004	1022650 ± 12184		0.982	0.299 ± 0.016	1032690 ± 75104

Table 3-5.: Fit parameters for the discontinuous asymmetric Laplace distribution function: Real melodic lines. The sign "+" identifies the ascending intervals and the sign "-" the descending ones. $F^C_\pm$, $G^C_\pm$, $F^H_\pm$, and $G^H_\pm$ are the fit parameters for cumulative distributions and histograms. For melodic lines with the sign "+" the fit parameters are F^C_+, G^C_+, F^H_+, and G^H_+. For melodic lines with the sign "-" the fit parameters are F^C_-, G^C_-, F^H_-, and G^H_-. R^2 is the determination coefficient. Bin: Bin width of each histogram in Hz^2.

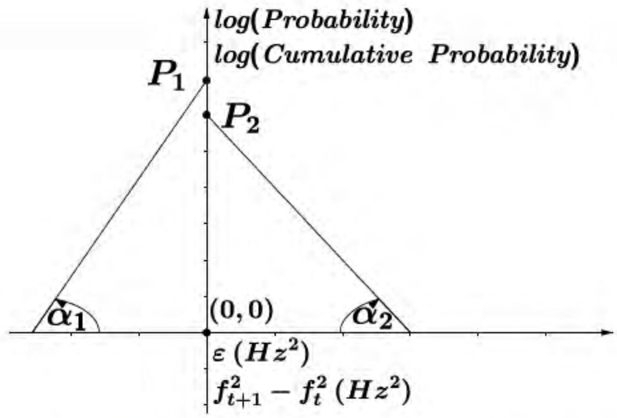

Figure 3-4.: General forms of the probability and cumulative distributions for the melodic lines $P(\varepsilon)$ and $P(f_{t+1}^2 - f_t^2)$, respectively. In the symmetric case, $P_1 = P_2$ and $\alpha_1 = \alpha_2$.

Figure **3-5**(a) shows the histogram of the PD for the first movement of the *Partita in A minor BWV 1013*, as well as the PD for the bin degeneration in the corresponding *ambitus*, which originates from the structure of the musical scale, and represents the melodic line with the highest diversity of melodic intervals in different locations of the register. Notice that the distance in Hz^2 between pairs of differences $f_j^2 - f_i^2$ for the 12-TET scale varies in such a manner that the number of differences inside an arbitrary bin ε, representing its degeneracy, decreases when $|f_j^2 - f_i^2|$ increases.

In order to understand the musical meaning of the bin degeneration PD, a connection with a random melody is proposed. In a random melody with $M \to \infty$ pitches $(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, ..., \mathfrak{n}_M)$, taken randomly from a set of N different pitches, each pitch will have a probability $1/N$ to be a specific one. Now, translating the sequence of pitches into a sequence of melodic intervals, we have: $(\mathfrak{n}_1, \mathfrak{n}_2), (\mathfrak{n}_2, \mathfrak{n}_3), ..., (\mathfrak{n}_{M-1}, \mathfrak{n}_M)$. Since successive pitches are uncorrelated, then each pair of successive pitches has a probability $(1/N)(1/N) = (1/N^2)$ to take a particular different combination of pitches (including unisons).

Now, if we have N possible pitches, and we want to construct all possible combinations of them including unisons, we have N^2 possibilities. Then, if we want to construct a melodic line with the highest diversity of combinations of pitches we must assign to each pair of pitches a probability of occurrence given by $1/N^2$. In this way, the probability distributions coming from a long random melodic line will be quite similar to the corresponding one coming from the the highest diversity of melodic intervals. An example is showed in Figure **3-6** (this figure was first presented in [39]).

The comparison between the distributions of real melodic lines and those from bin degeneration for the corresponding *ambitus* indicates that the scale contributes to the observed results, but does not explain them. In addition, the PD for bin degeneration fits better to a power law function ($\overline{R^2} = 0.963$) than to an exponential function ($\overline{R^2} = 0.934$). Table **3-6** contains the determination coefficient R_2 for the fit to a power law and an exponential function, in the case of each melodic line.

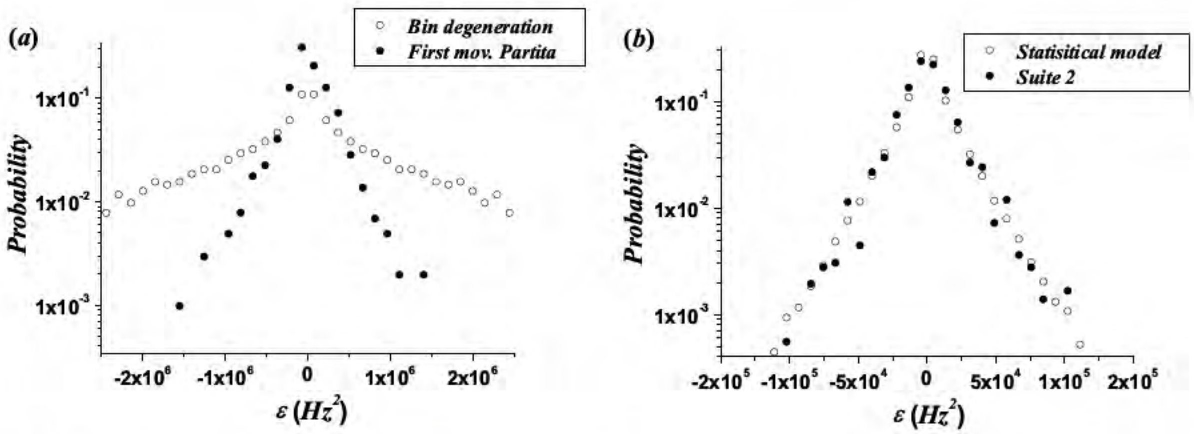

Figure 3-5.: (a) Comparison between the PDs for the real melodic line of the first movement of the *Partita in A minor BWV 1013* by J. S. Bach and for the corresponding bin degeneration for the same *ambitus*. (b) Comparison between histogram for the melodic line of *Suite No. 2 BWV 1008* by J. S. Bach and that produced by the statistical model.

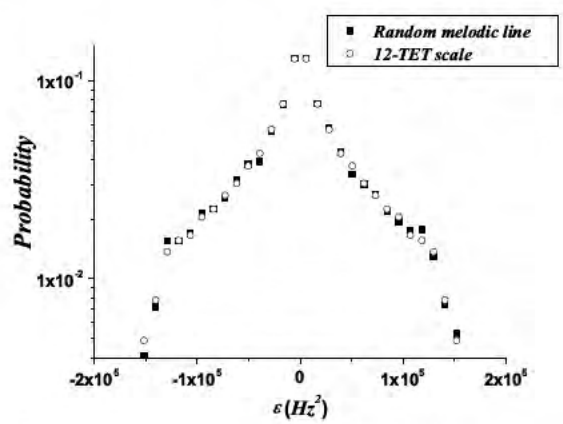

Figure 3-6.: Comparison between histograms for a random melodic line played in the 12-TET scale ($A = 440\,Hz$) and the bin degeneracy for the same scale. The random melodic line contains 10000 pitches without rests (9999 transitions) and was generated in the same *ambitus* as the Suite No. 1 in G Major BWV 1007 by Johann Sebastian Bach (65.406 Hzâı391.995 Hz). The bin width is 11242 Hz^2. This figure was first presented in [39]

The quantitative difference between the PD for a real melodic line and its corresponding random one (the bin degeneration PD) provides information on the order introduced into the system by the composer, stemming from the selection of successive pairs of pitches. A mathematical tool for comparing two PDs is provided by the Kullback–Leibler divergence, or relative entropy [78],

$$D_{KL} = \sum_{k=1}^{N} p_k \ln\left(\frac{p_k}{q_k}\right), \tag{3-50}$$

where p_k is the PD for the real melodic line to be compared with the *a priori* distribution

q_k coming from the degeneration of the k^{th} bin, and N is the number of bins in the *ambitus* with $N/2$ bins for each branch (ascending and descending). The PD q_k has been formally related to the probability associated with the number of distinguishable subcategories in the category k, representing its degeneracy [79].

The minimization of the relative entropy under constraints will be useful to describe the form of the PD, as will be explained in the next section.

		12-TET scale	
		Exponential	**Power law**
		R^2	R^2
T. B. C.	Violin 1 +	0.9026	0.9792
	Violin 1 -	0.9026	0.9792
	Violin 2 +	0.9145	0.9753
	Violin 2 -	0.9145	0.9753
	Violin 3 +	0.9080	0.9771
	Violin 3 -	0.9080	0.9771
	Viola 1 +	0.9232	0.9656
	Viola 1 -	0.9232	0.9656
	Viola 2 +	0.9275	0.9637
	Viola 2 -	0.9275	0.9637
	Viola 3 +	0.9263	0.9644
	Viola 3 -	0.9263	0.9644
	Cello 1 +	0.9708	0.9712
	Cello 1 -	0.9708	0.9712
	Cello 2 +	0.9708	0.9712
	Cello 2 -	0.9708	0.9712
	Cello 3 +	0.9708	0.9712
	Cello 3 -	0.9708	0.9712
	Violone +	0.9349	0.9691
	Violone -	0.9349	0.9691
	Harpsichord +	0.9349	0.9691
	Harpsichord -	0.9349	0.9691
M. D. M.	Soprano +	0.9308	0.9334
	Soprano -	0.9308	0.9334
	Contralto +	0.9536	0.9332
	Contralto -	0.9536	0.9332
	Tenor +	0.9505	0.9064
	Tenor -	0.9505	0.9064
	Bass +	0.9536	0.9344
	Bass -	0.9536	0.9344
P. M.	Suite 1 +	0.9709	0.9811
	Suite 1 -	0.9709	0.9811
	Suite 2 +	0.9099	0.9778
	Suite 2 -	0.9099	0.9778
	Mozart sonata +	0.8488	0.9831
	Mozart sonata -	0.8488	0.9831
	First mov. Partita +	0.8966	0.9736
	First mov. Partita -	0.8966	0.9736
	Piccolo concerto +	0.9724	0.9626
	Piccolo concerto -	0.9724	0.9626
	Average	**0.934**	**0.963**

Table 3-6.: Determination coefficient R^2 for the fit of the bin degeneracy distribution to a power law and to an exponential function. The exponential function has the form: $P = \mathcal{C}'exp(-|\varepsilon|/\mathcal{D}')$, and the power law function has the form: $P = \mathcal{E}'|\varepsilon|^{\mathcal{F}'}$, where P is the probability and ε refers to the quantity $f_{t+1}^2 - f_t^2$ measure in bins. The bin width was taken as the average of those obtained separately using the Sturges criterion for ascending and descending distributions. $\mathcal{C}'$, $\mathcal{D}'$, $\mathcal{E}'$ and $\mathcal{F}'$ are the parameters of the fit. T. B. C. refers to the third Brandenburg concerto, M. D. M. to the Missa Dixit Maria, and P. M. to a piece or movement.

3.5. A statistical model for melodic lines

This section shows the Shannon entropy evolution in time for each melodic line, a statistical model based on the minimization of the Kullback-Leibler divergence that reproduce the main features of the experimental results, and the connection between the parameters of the statistical model with the transposition processes, the asymmetry between ascending and descending intervals, and the mean dissonance level of the studied melodic lines.

3.5.1. Entropy evolution in melodic lines

Assuming that each possible melodic interval generated from the *ambitus* of a melodic line corresponds to a possible state, an analysis of the entropy evolution in the progression of the melodic line can be performed in a similar manner as in the work by G. Gündüz and U. Gündüz [62]. For the $\mathcal{A}$ different pitches inside the *ambitus* of a melodic line, the number of different melodic intervals is $\mathcal{A}^2$. Following [62] we use the Shannon entropy

$$S\,(bits) = -\sum_{m=1}^{M} p_m log_2 p_m, \tag{3-51}$$

where M refers to the final melodic interval appearing in the progression of the melodic line, and p_m is the probability that the interval m has already appeared in the sequence. The final Shannon entropy S_f is reached when M is equal to the total number of melodic intervals in the melodic line.

Figure **3-7(*a*)** illustrates the Shannon entropy evolution for some of the analyzed melodic lines. The remaining ones exhibit similar behavior. Figure **3-7(*b*)** illustrates the Shannon entropy evolution of the melodic lines for the soprano of the *Missa Super Dixit Maria* and the *Suite No. 2 BWV 1008*, with their corresponding random melodies constructed using the same *ambitus*. The maximum Shannon entropy S_{max} corresponds to the maximum possible value of the Shannon entropy in a long random melodic line with the same *ambitus* as the original one, namely $S_{max} = log_2(\mathcal{A}^2)$.

Figures **3-7(*a*)** and **3-7(*b*)** show that the Shannon entropy increases with each new melodic interval in the progression until it reaches a limiting value, which is smaller than the Shannon entropy of a random melodic line with the same *ambitus*. Some fluctuations appear in this process. However, the Shannon entropy tends to be stabilized at the final section of the melodic line. This result is similar to the findings of G. Gündüz and U. Gündüz analyzing the entropy evolution associated to the connectivity of pitches in different melodies [62].

For each melodic line, Table **3-7** presents the final Shannon entropy S_f, the maximum Shannon entropy reached by the melodic line S^*_{max}, and the maximum Shannon entropy generated by the *ambitus* of the corresponding melodic line S_{max}.

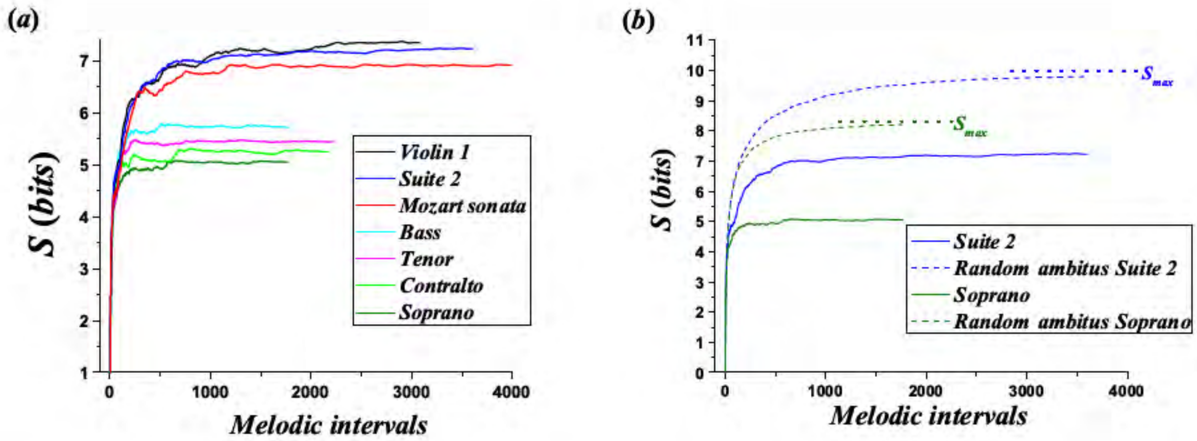

Figure 3-7.: (a) Shannon entropy evolution for different melodic lines. (b) Shannon entropy evolution for the melodic lines of the *Soprano* and *Suite 2 BWV 1008* with the corresponding random melodies constructed using the same *ambitus*. The maximum Shannon entropy S_{max} corresponds to the maximum possible value of the entropy in a long random melodic line with the same *ambitus* as the original one.

3.5.2. Relative entropy minimization under macroscopic constraints

From the previously presented definitions of melody [9, 12] and the results in Figures **3-7**(a) and **3-7**(b) and Table **3-7**, we infer that the composer creates a melodic line among the richest in terms of the use of melodic intervals, but in accordance with musical constraints. Because each melodic interval in the 12-TET scale corresponds to a particular value of $f_{t+1}^2 - f_t^2$ (except for unisons), and the expected value of this quantity contains musical information, the work carried out by the composer can be modeled as a procedure in which the relative entropy is minimized (the closest p_k to q_k) under constraints with musical meaning.

Different musical constraints can be proposed in order to reduce the entropy value of a melodic line away from that of a random one, and we propose the following ones.

Assuming that the total numbers of ascending and descending intervals and unisons are known, the first two constraints measured from histograms are

$$\tilde{p}_d + \tilde{p}_u = \sum_{k=1}^{N/2} p_k \quad \text{and} \quad \tilde{p}_a + \tilde{p}_u = \sum_{k=(\frac{N}{2}+1)}^{N} p_k, \tag{3-52}$$

where $\tilde{p}_a$ is the probability of an ascending interval, $\tilde{p}_d$ is that of a descending one, $\tilde{p}_u$ is the probability of a unison, and $\tilde{p}_a + \tilde{p}_d + \tilde{p}_u = 1$. Here, the unisons contribute to the ascending part as well as the descending part, as was explained in the procedure section.

The next constraint comes from the best estimation of the average magnitude of the melodic intervals using histograms (equation (3-11)):

$$\langle|\varepsilon|\rangle = \sum_{k=1}^{N} p_k \cdot |\varepsilon_k| - \frac{1}{2}[\tilde{p}_u|\varepsilon_{N/2}| + \tilde{p}_u|\varepsilon_{(N/2)+1}|] = \sum_{k=1}^{N} p_k \cdot |\varepsilon_k| - \tilde{p}_u|\varepsilon_{N/2}|, \tag{3-53}$$

Melodic line	S_f	S^*_{max}	S_{max}	$\lambda_1(\times 10^{-5})$ $\left[Hz^{-2}\right]$	$\lambda_2(\times 10^{-7})$ $\left[Hz^{-2}\right]$	$\langle D\rangle$ $(\times 10^{-1})$	$\langle D\rangle^*$ $(\times 10^{-1})$
Violin 1	7.358	7.378	10.089	0.550	-1.870	1.282	1.278
Violin 2	7.213	7.234	10.000	0.570	-0.189	1.215	1.211
Violin 3	7.253	7.285	10.000	0.660	-0.895	1.242	1.240
Viola 1	6.941	6.953	9.615	1.330	-1.860	1.339	1.333
Viola 2	6.935	6.944	9.510	1.500	-1.280	1.381	1.375
Viola 3	7.022	7.053	9.716	1.540	-2.200	1.364	1.357
⋆Cello 1	6.888	6.904	9.716	6.300	-18.700	2.795	2.788
⋆Cello 2	6.884	6.899	9.716	6.400	-17.200	2.797	2.790
⋆Cello 3	6.862	6.879	9.716	6.500	-15.100	2.816	2.812
Violone	6.779	6.796	9.716	30.000	-34.000	4.900	4.917
⋆Harpsichord	6.779	6.796	9.716	7.400	-4.200	2.596	2.598
Soprano	5.055	5.082	8.340	1.940	-2.850	1.470	1.470
Contralto	5.247	5.313	8.644	3.250	-6.800	1.591	1.591
Tenor	5.443	5.491	7.615	5.100	-6.500	1.893	1.893
Bass	5.723	5.787	8.644	7.300	6.450	2.219	2.218
⋆Suite 1	7.069	7.073	10.000	3.500	-5.100	2.528	2.509
⋆Suite 2	7.235	7.248	10.000	3.700	-5.800	2.653	2.631
Mozart sonata	6.923	6.935	10.644	0.490	-1.520	1.353	1.357
First mov. Partita	7.145	7.145	10.000	0.295	-1.760	1.293	1.294
⋆Piccolo concerto	7.087	7.182	9.288	0.056	0.175	0.749	0.747

Table 3-7.: For each melodic line: Final Shannon entropy S_f, maximum Shannon entropy reached S^*_{max}, maximum Shannon entropy generated by the *ambitus* of the corresponding melodic line S_{max}, Lagrange multipliers λ_1 and λ_2, mean dissonance level $\langle D\rangle$, and mean dissonance level approximated using the Taylor expansion up to second order (equation (3-44)) $\langle D\rangle^*$. Melodic lines marked with "⋆" do not satisfy a linear relation between λ_1 and $\langle D\rangle$.

where the quantity $-\tilde{p}_u|\varepsilon_{N/2}|$ corrects the double counting of unisons.

The asymmetry in the magnitudes of ascending and descending intervals is the final constraint. This asymmetry is present in the difference between the coefficients for the left and right branches in equations (3-48) and (3-49). Using histograms, the best estimate that we can obtain for the expression (3-47) is

$$\langle \varepsilon_{>0} \rangle + \langle \varepsilon_{<0}\rangle = \frac{1}{\tilde{p}_d}\sum_{k=1}^{N/2} p_k\varepsilon_k + \frac{1}{\tilde{p}_a}\sum_{k=N/2+1}^{N} p_k\varepsilon_k + |\varepsilon_{N/2}|\left(\frac{\tilde{p}_u}{\tilde{p}_d} - \frac{\tilde{p}_u}{\tilde{p}_a}\right), \tag{3-54}$$

where the quantity $|\varepsilon_{N/2}|\left(\frac{\tilde{p}_u}{\tilde{p}_d} - \frac{\tilde{p}_u}{\tilde{p}_a}\right)$ removes the contribution from unisons.

Table **3-8** contains the values of the quantities shown in equations (3-11) and (3-47) and their corresponding approximations using histograms through the equations (3-53) and (3-54).

		Melodic lines				Statistical model	
		$\langle \lvert f_{t+1}^2 - f_t^2 \rvert \rangle (Hz^2)$	$\langle (f_{t+1}^2 - f_t^2)_{>0} \rangle + \langle (f_{t'+1}^2 - f_{t'}^2)_{<0} \rangle (Hz^2)$	$\langle \lvert\varepsilon\rvert \rangle$ (Hz^2)	$\langle \lvert\varepsilon_{>0}\rvert \rangle + \langle \lvert\varepsilon_{<0}\rvert \rangle$ (Hz^2)	$\langle \lvert\varepsilon\rvert \rangle$ (Hz^2)	$\langle \lvert\varepsilon_{>0}\rvert \rangle + \langle \lvert\varepsilon_{<0}\rvert \rangle$ (Hz^2)
T. B. C.	Violin 1	114084.152	16873.862	121034.047	18860.204	121173.179	18844.969
	Violin 2	110187.223	1291.898	118878.577	1785.604	118511.515	1792.433
	Violin 3	95961.137	5615.954	106840.855	5575.756	106771.519	5574.559
	Viola 1	50149.693	4616.013	52248.657	4570.776	52052.875	4557.572
	Viola 2	44896.866	3216.003	47147.163	2805.660	47101.591	2812.232
	Viola 3	43534.553	3887.765	45809.287	4128.019	45616.426	4138.110
	Cello 1	10057.834	1260.495	10766.566	1285.729	10787.348	1284.776
	Cello 2	9980.889	1155.726	10686.973	1170.239	10667.455	1167.694
	Cello 3	9840.929	1024.148	10552.476	1027.253	10546.138	1026.550
	Violone	2253.971	186.771	2430.600	158.407	2435.305	158.986
	Harpsichord	9015.882	747.084	9723.636	633.709	9734.594	635.298
M. D. M.	Soprano	39232.256	6380.934	41086.981	6893.717	41195.816	6886.436
	Contralto	23856.749	3184.718	24643.858	3465.858	24640.210	3461.540
	Tenor	17292.851	974.011	14663.062	1129.207	14662.207	1126.877
	Bass	9838.312	9.985	9905.495	-196.693	9856.042	-196.548
P. M.	Suite 1	15911.774	985.541	16392.745	697.399	16412.938	696.897
	Suite 2	14848.605	996.530	15431.244	712.161	15393.671	714.053
	Mozart sonata	134286.653	13515.327	134195.113	17101.418	133863.864	17127.414
	First mov. Partita	209509.526	35940.692	215557.941	38563.899	215398.288	38711.806
	Piccolo concerto	1042526.831	-113889.610	1068746.164	-113752.187	1069401.290	-113358.307

Table 3-8.: Relevant expected values for real melodic lines and the statistical model results. For histograms, $\langle |\varepsilon| \rangle$ and $\langle |\varepsilon_{>0}| \rangle + \langle |\varepsilon_{<0}| \rangle$ are approximations of the quantities $\langle |f_{t+1}^2 - f_t^2| \rangle$ and $\langle (f_{t+1}^2 - f_t^2)_{>0} \rangle + \langle (f_{t'+1}^2 - f_{t'}^2)_{<0} \rangle$, respectively .

Minimizing the relative entropy subject to the constraints (3-52), (3-53), and (3-54) (in a similar procedure to that shown in [77]) produces the following PD:

$$p_k = \begin{cases} \dfrac{(\tilde{p}_d + \tilde{p}_u) q_k e^{\left(-\lambda_1 |\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_d}\varepsilon_k\right)}}{\sum\limits_{m=1}^{N/2} \left[q_m e^{\left(-\lambda_1 |\varepsilon_m| - \frac{\lambda_2}{\tilde{p}_d}\varepsilon_m\right)} \right]} & \text{for } k \in [1, N/2] \\ \\ \dfrac{(\tilde{p}_a + \tilde{p}_u) q_k e^{\left(-\lambda_1 |\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_a}\varepsilon_k\right)}}{\sum\limits_{m=\frac{N}{2}+1}^{N} \left[q_m e^{\left(-\lambda_1 |\varepsilon_m| - \frac{\lambda_2}{\tilde{p}_a}\varepsilon_m\right)} \right]} & \text{for } k \in [\frac{N}{2}+1, N], \end{cases} \tag{3-55}$$

where λ_1 and λ_2 are the Lagrange multipliers for the constraints (3-53) and (3-54), respectively. Appendix D contains the detailed procedure of the minimization.

The values of λ_1 and λ_2 were obtained using the expected values $\langle|\varepsilon|\rangle$ and $\langle\varepsilon\rangle$ from the histograms of the empirical distributions for the selected melodic lines, and allowing the relative error between the expected values from the statistical model and those from the real data to be smaller than 1.0 %. Table **3-8** contains the expected values used in the statistical model, and Table **3-7** presents the values of the Lagrange multipliers generated from them. While the values of λ_1 are positive, those of λ_2 can be positive or negative, exhibiting possible asymmetries in the use of ascending and descending intervals. In addition, λ_1 is between one and two orders of magnitude larger than λ_2.

Figure **3-5(*b*)** presents a comparison between the statistical model and the empirical results in the case of *Suite No. 2 BWV 1008*. Some differences between the empirical data and the results from the statistical model are expected, because there are patterns in real melodic lines that cannot be captured by this simple model.

The CCD (ascending branch) and CD (descending branch) can be utilized to compare different melodic lines that are either experimental or obtained from the statistical model. The CCD and CD were obtained from the histograms produced by the statistical model, randomly distributing the probability assigned to a bin between all the possible melodic intervals inside it, which were generated using the *ambitus* of the corresponding melodic line. Because $\tilde{p}_u$ is known, the probability assigned to 0 inside the bins containing unisons was taken as $\tilde{p}_u$, and the remaining probability of the bin was distributed randomly in the other possible melodic intervals. Figure **3-8** depicts the CCD and CD for the empirical data and the corresponding results from the statistical model for most melodic lines. In this figure, and taking into account the values in Table **3-7**, the following features can be inferred:

1. Different registers of musical instruments and human voices can be distinguished using the Lagrange multiplier λ_1, allowing, for example, to discriminate between the same melodic line played in different parts of the register (a transposition). An example of a transposition is given in the *Brandenburg Concerto No. 3 BWV 1048* by J. S. Bach, in which the harpsichord plays the same melodic line as the violone but transposed one octave higher (the fundamental

frequency ratio of the transposition is equal to 2): while the entropy evolution of these melodic lines is the same, there is a change in the exponential decay parameters, characterized by the values of the Lagrange multipliers (see Table **3-7**), and the numerical values of the expected values are related as

$$\begin{aligned} \langle|\varepsilon|\rangle_{Harpsichord} &= 2^2\langle|\varepsilon|\rangle_{Violone} \\ \langle|f_{t+1}^2 - f_t^2|\rangle_{Harpsichord} &= 2^2\langle|f_{t+1}^2 - f_t^2|\rangle_{Violone} \\ [\langle\varepsilon_{>0}\ \rangle + \langle\varepsilon_{<0}\rangle]_{Harpsichord} &= 2^2\,[\langle\varepsilon_{>0}\ \rangle + \langle\varepsilon_{<0}\rangle]_{Violone} \\ \left[\langle(f_{t+1}^2 - f_t^2)_{>0}\rangle + \langle(f_{t'+1}^2 - f_{t'}^2)_{<0}\rangle\right]_{Harpsichord} &= 2^2\left[\langle(f_{t+1}^2 - f_t^2)_{>0}\rangle + \langle(f_{t'+1}^2 - f_{t'}^2)_{<0}\rangle\right]_{Violone}, \end{aligned} \tag{3-56}$$

in agreement with the properties derived above for transposition processes (equation (3-15)).

2. With respect to the quantitative results of the model, the orders of magnitude of the fit parameters of the statistical model are in agreement with the corresponding results of the experimental fits. For each melodic line, Table **3-9** contains the fit parameters to discontinuous asymmetric Laplace distributions, generated from the statistical model results. The average relative error in the histograms for the amplitude of the exponential distributions is 17.1 %, and that for the decay coefficient is 20.6 %. In the cases of the CD and CCD, the average relative errors of the amplitude and the decay coefficient are 7.2 % and 11.8 %, respectively. Table **3-10** contains the values of these errors for each melodic line.

3. In most cases (90 % of the melodic lines), the constraint (3-54) takes positive values (corresponding to negative values of λ_2), and $\tilde{p}_a - \tilde{p}_d$ takes negative values (see Table **3-3**). This behavior is consistent with the asymmetry represented in Figure **1-9**, in the sense that the magnitudes of ascending intervals are expected to be larger than those of descending ones, and the total number of descending intervals must be larger than that of ascending ones. Negative values of $\tilde{p}_a - \tilde{p}_d$ and λ_2 lead to different decay coefficients and different intercept points with the ordinate axis for the ascending and descending branches, which can be observed in the experimental fits of the CD and CCD through the comparison of the corresponding coefficients, $F_+^C < F_-^C$ and $G_+^C > G_-^C$ (see Table **3-5**). Figure **3-4** was created with the purpose of magnifying these particular asymmetries: $P_1 > P_2$ and $\alpha_1 > \alpha_2$ (implying that $\lambda_2 < 0$). The two exceptions are the *Piccolo Concerto RV444* of Antonio Vivaldi, where $\lambda_2 > 0$ and $\tilde{p}_a - \tilde{p}_d > 0$, and the melodic line of the tenor voice in *Missa Super Dixit Maria*, where $\lambda_2 > 0$ and $\tilde{p}_a - \tilde{p}_d < 0$.

4. Because the difference between λ_1 and λ_2 is between one and two orders of magnitude (i.e., the decay coefficients have the same order of magnitude), and the bin width selection affects the measure of the decay parameters, the asymmetry in the values of the decay coefficients is better observed in the cumulative distributions than in the histograms.

5. Because in Figure **3-4** the limit P_1 of the CD (constructed for descending intervals) when $f_{t+1}^2 - f_t^2 \to 0^-$ represents the probability of a value slightly smaller than 0, and in the CCD

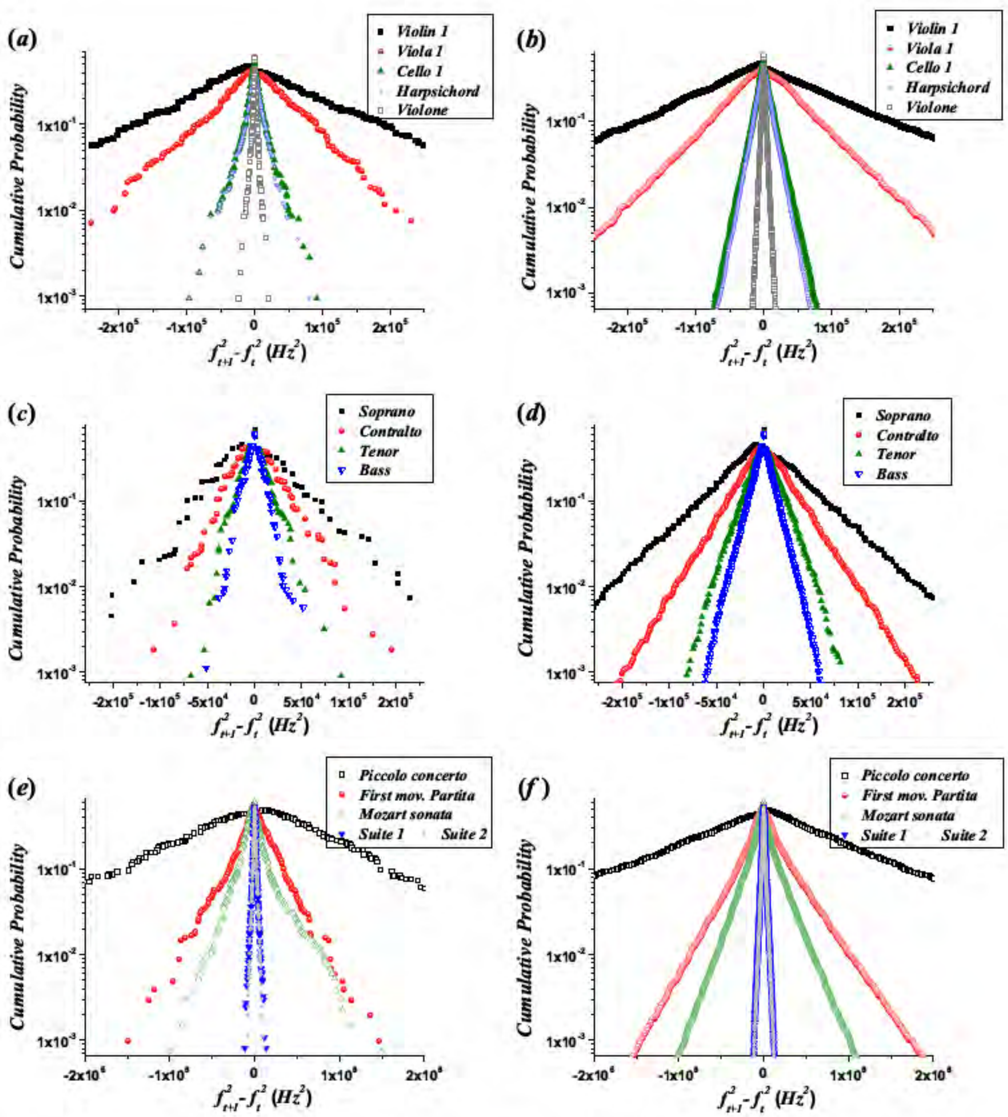

Figure 3-8.: CCD (ascending branches) and CD (descending branches) for the empirical distributions (a, c, e) and the corresponding statistical model results (b, d, f). (a, b) *Brandenburg Concerto No. 3 in G Major BWV 1048* by J. S. Bach, (c, d) *Missa Super Dixit Maria* by Hans Leo Hassler, and (e, f) *Piccolo Concerto RV444* by Antonio Vivaldi; *First movement of the Partita in A Minor BWV 1013* by J. S. Bach; *Sonata KV 545* by W. A. Mozart; *Suite No. 1 in G Major BWV 1007* by J. S. Bach and *Suite No. 2 in D Minor BWV 1008* by J. S. Bach.

(constructed for ascending intervals) P_2 when $f_{t+1}^2 - f_t^2 \to 0^+$ represents the probability of a value slightly larger than 0, the asymmetry $\tilde{p}_a - \tilde{p}_d \approx P_2 - P_1$. This result can be observed in Figure **3-8**, and represents the difference in the amplitudes of the exponential decay for the CD and CCD. In most cases, except for the *Piccolo Concerto RV444*, $\tilde{p}_a < \tilde{p}_d$, implying that $P_1 > P_2$. In the case of the *Piccolo Concerto RV444*, it holds that $\tilde{p}_a > \tilde{p}_d$, implying that $P_1 < P_2$.

		Cumulative probability			Bin (Hz^2)	Histograms		
		R^2	$F^C_\pm$	$G^C_\pm$ (Hz^2)		R^2	$F^H_\pm$	$G^H_\pm$ (Hz^2)
T. B. C.	Violin 1 +	0.998	0.425 ± 0.001	131218 ± 446	82605	0.996	0.466 ± 0.011	90676 ± 2899
	Violin 1 -	0.999	0.499 ± 0.001	115344 ± 373		0.997	0.587 ± 0.013	82788 ± 2260
	Violin 2 +	0.998	0.463 ± 0.001	121367 ± 453	83037	0.997	0.546 ± 0.012	85291 ± 2442
	Violin 2 -	0.998	0.465 ± 0.001	119871 ± 430		0.997	0.557 ± 0.012	84529 ± 2382
	Violin 3 +	0.998	0.447 ± 0.001	110014 ± 457	83663	0.998	0.612 ± 0.013	77314 ± 2096
	Violin 3 -	0.998	0.463 ± 0.001	108806 ± 457		0.998	0.653 ± 0.012	76248 ± 1812
	Viola 1 +	0.992	0.436 ± 0.002	58537 ± 500	26833	0.995	0.415 ± 0.011	36331 ± 1286
	Viola 1 -	0.994	0.464 ± 0.002	54312 ± 395		0.996	0.458 ± 0.011	34829 ± 1163
	Viola 2 +	0.993	0.441 ± 0.002	51768 ± 432	26958	0.996	0.470 ± 0.012	32931 ± 1155
	Viola 2 -	0.994	0.464 ± 0.002	49281 ± 384		0.996	0.506 ± 0.013	32076 ± 1083
	Viola 3 +	0.991	0.430 ± 0.002	51448 ± 443	25898	0.996	0.470 ± 0.012	31890 ± 1044
	Viola 3 -	0.993	0.453 ± 0.002	48076 ± 376		0.997	0.515 ± 0.012	30530 ± 932
	Cello 1 +	0.998	0.446 ± 0.001	11203 ± 42	8398	0.997	0.509 ± 0.013	8458 ± 276
	Cello 1 -	0.999	0.524 ± 0.001	9821 ± 33		0.998	0.632 ± 0.014	7808 ± 222
	Cello 2 +	0.999	0.441 ± 0.001	11075 ± 37	8400	0.998	0.517 ± 0.013	8344 ± 266
	Cello 2 -	0.999	0.522 ± 0.001	9641 ± 30		0.998	0.637 ± 0.014	7759 ± 218
	Cello 3 +	0.998	0.459 ± 0.001	10643 ± 39	8401	0.998	0.524 ± 0.013	8222 ± 255
	Cello 3 -	0.999	0.521 ± 0.001	9805 ± 32		0.998	0.642 ± 0.014	7721 ± 215
	Violone +	0.997	0.451 ± 0.001	2481 ± 13	1969	0.998	0.606 ± 0.013	1827 ± 48
	Violone -	0.997	0.475 ± 0.001	2417 ± 12		0.999	0.680 ± 0.014	1767 ± 44
	Harpsichord +	0.997	0.440 ± 0.001	10172 ± 53	7877	0.998	0.607 ± 0.013	7305 ± 193
	Harpsichord -	0.997	0.464 ± 0.001	9961 ± 52		0.998	0.656 ± 0.013	7184 ± 185
M. D. M.	Soprano +	0.991	0.464 ± 0.005	42647 ± 572	21403	0.991	0.399 ± 0.013	28530 ± 1271
	Soprano -	0.996	0.587 ± 0.004	37537 ± 307		0.992	0.502 ± 0.016	27003 ± 1141
	Contralto +	0.992	0.473 ± 0.004	26022 ± 303	13797	0.995	0.430 ± 0.011	18121 ± 622
	Contralto -	0.995	0.533 ± 0.004	22881 ± 205		0.996	0.502 ± 0.012	16810 ± 525
	Tenor +	0.994	0.515 ± 0.005	14196 ± 177	7005	0.994	0.359 ± 0.011	10954 ± 438
	Tenor -	0.994	0.547 ± 0.005	13700 ± 171		0.995	0.396 ± 0.011	10484 ± 398
	Bass +	0.997	0.478 ± 0.002	9648 ± 70	4669	0.993	0.364 ± 0.012	6917 ± 297
	Bass -	0.997	0.515 ± 0.002	9844 ± 64		0.992	0.371 ± 0.012	7120 ± 315
P. M.	Suite 1 +	0.997	0.483 ± 0.001	15626 ± 73	11115	0.995	0.434 ± 0.014	12379 ± 536
	Suite 1 -	0.997	0.505 ± 0.001	15282 ± 65		0.995	0.477 ± 0.015	11979 ± 493
	Suite 2 +	0.996	0.467 ± 0.001	15217 ± 80	8893	0.994	0.360 ± 0.011	11625 ± 466
	Suite 2 -	0.996	0.514 ± 0.002	14025 ± 76		0.995	0.400 ± 0.011	11220 ± 427
	Mozart sonata +	0.998	0.416 ± 0.001	134981 ± 471	104025	0.997	0.466 ± 0.009	102225 ± 2406
	Mozart sonata -	0.998	0.559 ± 0.001	119664 ± 370		0.998	0.659 ± 0.011	94399 ± 1940
	First mov. Partita +	0.995	0.486 ± 0.002	211019 ± 1275	148060	0.995	0.374 ± 0.010	177911 ± 6009
	First mov. Partita -	0.998	0.570 ± 0.002	177953 ± 759		0.997	0.513 ± 0.011	153537 ± 4135
	Piccolo concerto +	0.998	0.512 ± 0.001	1017750 ± 3967	579759	0.994	0.411 ± 0.013	770520 ± 33560
	Piccolo concerto -	0.999	0.472 ± 0.001	1119390 ± 4155		0.993	0.348 ± 0.012	834946 ± 39978

Table 3-9.: Fit parameters to discontinuous asymmetric Laplace distribution functions: Statistical model results. The sign "+" identifies the ascending intervals and the sign "-" the descending ones. $F^C_\pm$, $G^C_\pm$, $F^H_\pm$, and $G^H_\pm$ are the fit parameters for cumulative distributions and histograms. For melodic lines with the sign "+" the fit parameters are F^C_+, G^C_+, F^H_+, and G^H_+. For melodic lines with the sign "-" the fit parameters are F^C_-, G^C_-, F^H_-, and G^H_-. R^2 is the determination coefficient. Bin: Bin width of each histogram in Hz^2.

		Relative error (%)			
		Cumulatives		Histograms	
		$F_{\pm}^{C}$	$G_{\pm}^{C}$	$F_{\pm}^{H}$	$G_{\pm}^{H}$
T. B. C.	Violin 1 +	4.87	7.93	0.42	0.17
	Violin 1 -	7.85	11.71	7.08	8.41
	Violin 2 +	2.59	5.13	2.48	2.45
	Violin 2 -	3.38	14.26	9.84	12.21
	Violin 3 +	2.16	9.03	2.60	3.42
	Violin 3 -	5.30	26.28	13.28	15.33
	Viola 1 +	5.25	3.82	1.59	3.21
	Viola 1 -	7.35	10.37	4.17	4.95
	Viola 2 +	2.13	2.33	13.78	21.28
	Viola 2 -	1.76	14.55	18.16	23.93
	Viola 3 +	1.07	1.23	18.77	30.12
	Viola 3 -	2.68	17.45	23.24	31.87
	Cello 1 +	2.98	15.92	16.86	23.12
	Cello 1 -	2.98	18.15	20.92	27.57
	Cello 2 +	4.14	15.97	16.16	21.99
	Cello 2 -	3.37	16.40	20.22	26.44
	Cello 3 +	0.46	14.10	16.30	22.00
	Cello 3 -	3.82	20.52	22.41	30.00
	Violone +	0.83	8.78	17.84	24.47
	Violone -	6.57	27.12	31.53	46.37
	Harpsichord +	3.10	11.51	17.78	24.38
	Harpsichord -	8.89	30.99	34.01	48.75
M. D. M.	Soprano +	14.20	13.60	19.19	21.17
	Soprano -	14.06	7.29	41.85	37.17
	Contralto +	15.06	10.96	10.91	12.52
	Contralto -	14.29	2.67	24.75	24.29
	Tenor +	12.46	12.09	19.23	21.45
	Tenor -	12.93	10.37	23.41	24.42
	Bass +	11.01	1.62	16.22	17.44
	Bass -	8.76	1.38	14.69	15.38
P. M.	Suite 1 +	13.40	15.24	17.62	19.17
	Suite 1 -	13.16	15.68	21.87	22.82
	Suite 2 +	12.67	13.94	18.48	19.79
	Suite 2 -	7.75	8.57	22.14	22.76
	Mozart sonata +	11.27	21.05	15.37	17.87
	Mozart sonata -	12.79	3.03	11.82	12.56
	First mov. Partita +	3.45	2.03	32.20	30.17
	First mov. Partita -	4.42	11.56	3.48	4.51
	Piccolo concerto +	12.24	6.80	26.65	26.95
	Piccolo concerto -	8.96	9.46	16.45	19.15
	Average	7.16	11.77	17.14	20.55

Table 3-10.: Relative error of the fit parameters for the statistical model with respect to those of the real melodic lines. The sign "+" identifies the ascending intervals and the sign "-" the descending ones. $F_{\pm}^{C}$, $G_{\pm}^{C}$, $F_{\pm}^{H}$, and $G_{\pm}^{H}$ are the fit parameters for cumulative distributions and histograms. For melodic lines with the sign "+" the fit parameters are F_{+}^{C}, G_{+}^{C}, F_{+}^{H}, and G_{+}^{H}. For melodic lines with the sign "-" the fit parameters are F_{-}^{C}, G_{-}^{C}, F_{-}^{H}, and G_{-}^{H}. R^2 is the determination coefficient. Bin: Bin width of each histogram in Hz^2.

3.5.3. Transposition processes and mean dissonance level of melodic lines

As explained in the section on melody, tonal consonance properties can be formally associated to melodic intervals in the case of musicians. Because the musical instruments analyzed in this study use vibrating strings and air columns, the main consonance properties may be captured using the model of the harmonic spectrum presented in the tonal consonance section.

For each melodic line, the mean dissonance level $\langle D \rangle$ was measured using the curves shown in Figure **3-3** for intervals inside the octave, and the *chroma* properties of pitch for intervals wider than one octave. Table **3-7** lists the values of the mean dissonance $\langle D \rangle$ and their corresponding approximations $\langle D \rangle^*$ using $\langle X \rangle_L$ and σ_L^2 in equation (3-44). Comparing $\langle D \rangle^*$ with $\langle D \rangle$, the observed relative error is less than 1.0 % for all melodic lines.

From the results in Table **3-7**, melodic lines tend to be more dissonant for instruments with lower registers, which is a well-known phenomenon in music theory [2]. An interesting case is that of transposition, as the same melodic lines played in different parts of the register have different dissonance levels. For example, the melodic line of the *violone* in the *Brandenburg Concerto BWV 1048* is perceived as more dissonant than that of the *harpsichord*.

Low registers are associated with small values of $\mathfrak{L}$, and therefore of $\langle |f_{t+1}^2 - f_t^2| \rangle$ and consequently also $\langle |\varepsilon| \rangle$. For all melodic lines, a power law relation was observed between the quantity $\langle |\varepsilon| \rangle$ and the Lagrange multiplier λ_1 (see Figure **3-9(*a*)**):

$$\lambda_1 = \mathbb{A} \langle |\varepsilon| \rangle^{\mathbb{B}}, \tag{3-57}$$

where the magnitude of $\mathbb{A}$ is $9.423 \times 10^{-1} \pm 9.76 \times 10^{-2}$, and $\mathbb{B} = -1.033 \pm (1.26 \times 10^{-2})$, with $R^2 = 0.998$. If $\mathbb{B}$ is taken as -1, then $\mathbb{A}$ is dimensionless. Low values of $\langle |\varepsilon| \rangle$ correspond to high values of λ_1, and vice versa, and λ_1 scales in a transposition process as

$$\lambda_1^N \approx \omega^{2\mathbb{B}} \lambda_1^O, \tag{3-58}$$

where λ_1^O and λ_1^N denote the first Lagrange multiplier in the original and new locations of the register, respectively. For the transposition between the *violone* and the *harpsichord*, $\lambda_1^{Harpsichord} \approx \omega^{2(-1.033)} \lambda_1^{Violone}$, with a 3 % relative error (see Table **3-7**).

For 13 of the melodic lines studied, a linear relation was observed between the mean dissonance levels of melodic lines and the first Lagrange multiplier (see Figure **3-9(*b*)**):

$$\langle D \rangle = \mathbb{C} + \mathbb{D} \lambda_1, \tag{3-59}$$

where $\mathbb{C} = 1.122 \times 10^{-1} \pm 1.7 \times 10^{-3}$ and $\mathbb{D} = (1236.29 \pm 19.81) Hz^2$, with $R^2 = 0.997$. The Lagrange multiplier λ_1 locates the approximate region of exponential decay, and for these 13 melodic lines this geometrical parameter can be employed as an indicator of the mean dissonance properties. Strong exponential decays correspond to low registers with high dissonance levels, and vice versa. The seven pieces that do not follow a linear relation (marked with "⋆" in Table **3-7** and dot circles in Figure **3-9(*b*)**) correspond to five *cellos* and a *harpsichord*, characterized by mean dissonance values between 0.25 and 0.30, and the *piccolo* of *Concerto RV444* with a mean dissonance level of 0.0749.

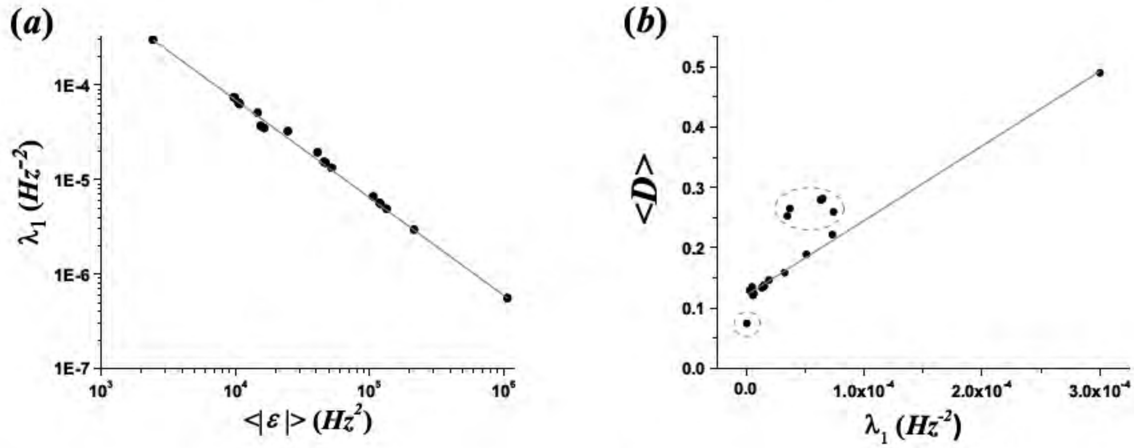

Figure 3-9.: (a) Power law relation between the quantity $\langle|\varepsilon|\rangle$ and the Lagrange multiplier λ_1. (b) Relation between the mean dissonance $\langle D\rangle$ and the Lagrange multiplier λ_1. For 13 of the 20 melodic lines a linear relation was observed.

4. The statistical analysis of chords in *secco recitatives*

The results presented in this chapter constitute an exploration beyond the main objectives of this thesis. The analysis of harmonic sequences of *secco recitatives* of eight operas was done. Frequently, *secco recitatives* were written using figured bass or functional notation, however in this analysis editions with an explicit realization of the chords was selected in order to avoid ambiguities. The analyses were made in two directions: the statistical analysis of chords, and the statistical analysis of transitions between chords. Additionally, an analysis of the co-occurrence of chords in different *secco recitatives* of the same opera is made.

4.1. On the *secco recitative*

The *recitative* is a vocal style which imitates the inflections of the human voice when speaking. In the opera it normally fulfills the function of transferring the action of one aria to another [12, p. 718]. During the 17th century this spoken style was opened in different directions, one of them was the *secco recitative*, which was not regularly integrated to the opera until the 18th century [12, p. 718]. This type of *recitative* is constituted by a continuous bass that accompanies the voice, which is interpreted by a keyboard instrument such as a harpsichord or a pianoforte. Sometimes the keyboard and the voice are accompanied by a cello that performs the bass line.

The *secco recitative* flourished mainly in 18th century opera, while in the 19th century it was mainly used in the Rossini operas [80], [81, pp. 1-6], [12, pp. 718-719]. Much of the information on how recitatives were accompanied is little known since there are few sources that talk about this [19].

During the 18th century, the Italian opera was one of the most representative sources of *secco recitatives* [12, p. 718]. Typical Italian recitatives of this century were organized using the following harmonic schemes [81, pp. 1-6]:

- Secondary dominant chains (as for example V/V).
- Chains interlocking the I-IV-V-I progressions, for which the cadence V-I is used as the progression I-IV of the next sequence.

- Juxtaposition of major/minor harmonies between tonalities with relative keys between them [12, p. 453].

4.2. *Secco recitatives* selection

In order to analyze a set of *secco recitatives* of different composers, a selection of eight operas was done from editions with an explicit realization of the chords. Each opera contains several *secco recitatives*, Table **4-1** contains the information of the operas. The appendix section contains the *secco recitatives* analyzed for each opera with their corresponding page ranges and number of bars.

ID	Opera	Composer	Source/Editor	Recitatives	Appendix
1	The Coronation of Poppea	C. Monteverdi (1567-1643)	Leningrad: Muzyka (imslp) [82]	17	E
2	Acis and Galatea	F. Handel (1685-1759)	Friedrich Chrysander (imslp) [83]	7	F
3	The Marriage of Hercules and Hebe	C. Gluck (1714-1787)	Breitkopf und Härtel [84]	12	G
4	Mithridates, King of Pontus	W. A. Mozart (1756-1791)	Mozarteum [85], Serie V [86]	19	H
5	Apollo and Hyacinthus	W. A. Mozart (1756-1791)	Mozarteum [85]	10	I
6	The marriage of Figaro	W. A. Mozart (1756-1791)	Dover Publications, Inc. [87]	27	J
7	Cinderella	G. Rossini (1792-1868)	Ricordi (imslp) [88]	15	K
8	The Barber of Seville	G. Rossini (1792-1868)	Dover Publications, Inc. [89]	17	L

Table 4-1.: Selected operas containing *secco recitatives*. Information about the composer, edition, source, number of recitatives analyzed, and the corresponding appendix containing the detailed list of chords.

4.3. Methodology for the extraction of chords

Chords were extracted manually from the score of the *continuo*. In order to determine the points in the score in which the chords must be analyzed, the natural beats of the bar were used as the minimum unit of time to follow the harmony, for example, in a measure of 4/4, the quarter note was considered as the minimum unit of time to report the chords. Although the natural beats of the bar were used as the minimum unit of time, it does not means that a new chord appears every minimum unit of time, only the occurrence of new sounds located in the natural beats of the bar contribute to the appearance of new chords. In some few cases, there is not bar in some parts of the recitatives, for these cases the quarter note was used as the minimum temporal unit to follow the harmony.

Each chord was coded between the different types of chords with structure by thirds. This choice is based in the harmonic structure used at that time and the geographical location.

The chords were coded using four features:

1. *Root of the chord*: 12 different pitches were considered. The musical scale was assumed to be the 12-TET, for which flats and sharps are equivalent (enharmonic equivalence). The standard notation for the pitch is the set of letters from A to G.

2. *Triad*: Three types were considered: Major, Minor, and Diminished. This election is based on the structure of the chords used in that period of time.

3. *Alterations of the triad*: Triads can contain sevenths or can appear without the fifth (-5). The sevenths were identified among major (7Ma), minor (7m), or diminished (7°). Combinations of sevenths with or without the fifth (for example 7m,-5) were also distinguished. In some few cases, it was not possible to characterize chords with any of the alterations (or any of the combinations of alterations) proposed. In the statistical analysis, these few cases where all unified in a category called "other". If a triad is not altered, the number "0" is used to describe this case.

4. *Inversion of the chords*: For chords without sevenths, the root position and the inversions were considered. The root position, and the first and second inversions were identified, using the standard notation 53, 63, and 64, respectively. For the chords with sevenths, the root position and three inversions were considered. The root position, and the first, second and third inversions were identified using the standard notation 7, 65, 43, and 2, respectively. For the diminished chords with diminished seventh, since this chord is invariant against inversions, it was always considered in the root position (7) on the bass note.

Table **4-2** summarizes the categories taken into account in the chord analysis. The notation "0 0 0 0" (this means "0" in all categories) was used to represent the occurrence of a rest.

Root	**Triad**	**Alteration**	**Inversion**
A	Major	0	53
A♯, B♭	Minor	7m	63
B	Diminished	7°	64
C, B♯		7Ma	7
C♯, D♭		-5	65
D		7m,-5	43
D♯, E♭		7°,-5	2
E		7Ma,-5	
F, E♯		Other	
F♯, G♭			
G			
G♯, A♭			

Table 4-2.: Categories to classify the properties of a chord by thirds.

The complete list of chords for each selected recitative of each opera is showed in the appendixes E, F, G, H, I, J, K, and L.

In some cases, it was necessary to analyze the pitch of the voice in order to determine the nature of the chord. In other cases, there were some chords reported in places that do not correspond to the natural beats of the bar, for example to take correctly into account the harmonic changes of the cadential six-four [11, pp. 140-149]. The cases that required special treatment are described in the appendixes at the end of the tables containing each recitative.

4.4. Rank analysis of chords

Taking the constitutive elements of a recitative as the different types of chords, coming from the combinations of the four categories posted above, the rank against the probability of occurrence plot were obtained. In this analysis the rests have been excluded.

Figures **4-1** to **4-8** show the probability of occurrence against the rank for the chords of each studied opera. Figure **4-9** shows the superposition of all chords from the studied operas. For all cases, experimental data fit to a function of the form

$$P(r) = e - \frac{b}{(1 + cr)^{(1/d)}}, \tag{4-1}$$

where r is the rank, $P(r)$ is the probability of occurrence, and e, b, c, d correspond to the fitting parameters.

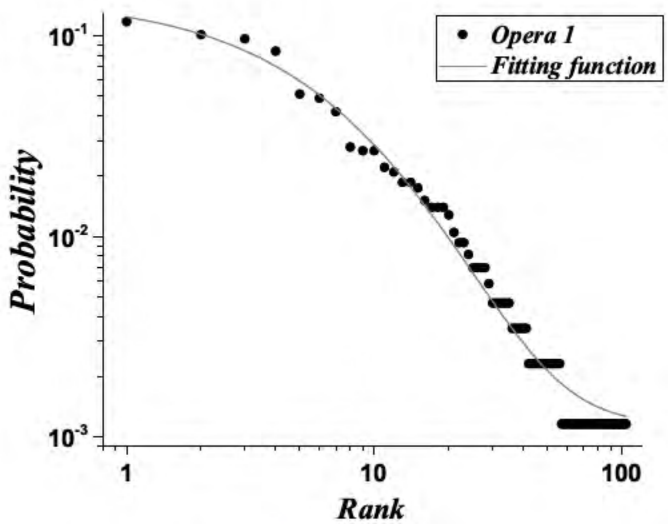

Figure 4-1.: Probability of occurrence *vs* rank for the chords of the *secco recitatives* from the opera the Coronation of Poppea.

Table **4-3** shows the fit parameters for each opera and the superposition of all of them, with the corresponding determination coefficient R^2 of the fit.

Table **4-3** shows that, excluding the opera N°2, for each opera as well as for the superposition of all of them, the contribution to the probability $P(r)$ of the parameter e in equation (4-1) is relatively small, mainly for the operas N°5, N°6, and "All". Neglecting the parameter e in (4-1), the form of the probability distribution $P(r)$ is similar to the function that

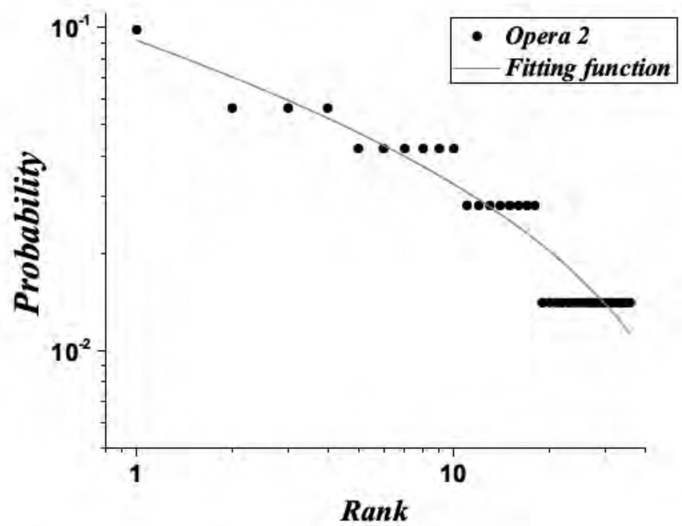

Figure 4-2.: Probability of occurrence *vs* rank for the chords of the *secco recitatives* from the opera Acis and Galatea.

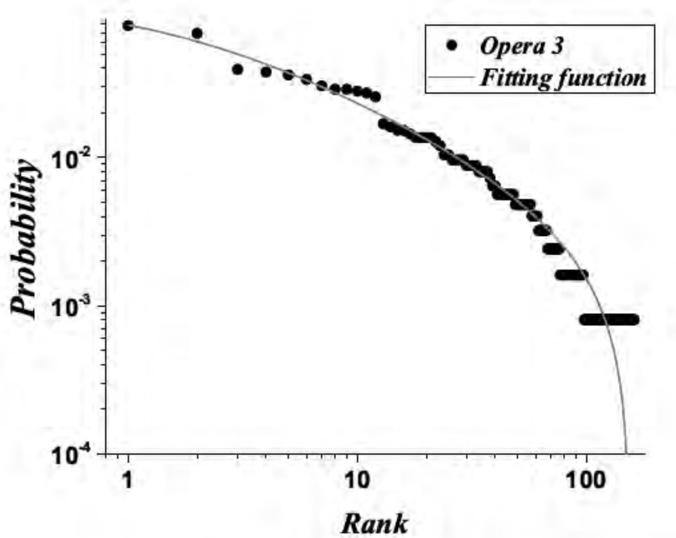

Figure 4-3.: Probability of occurrence *vs* rank for the chords of the *secco recitatives* from the opera the Marriage of Hercules and Hebe.

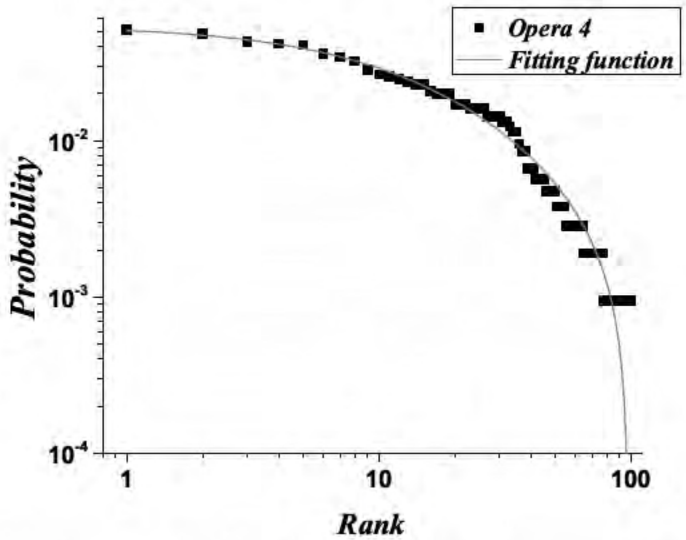

Figure 4-4.: Probability of occurrence *vs* rank for the chords of the *secco recitatives* from the opera Mithridates, King of Pontus.

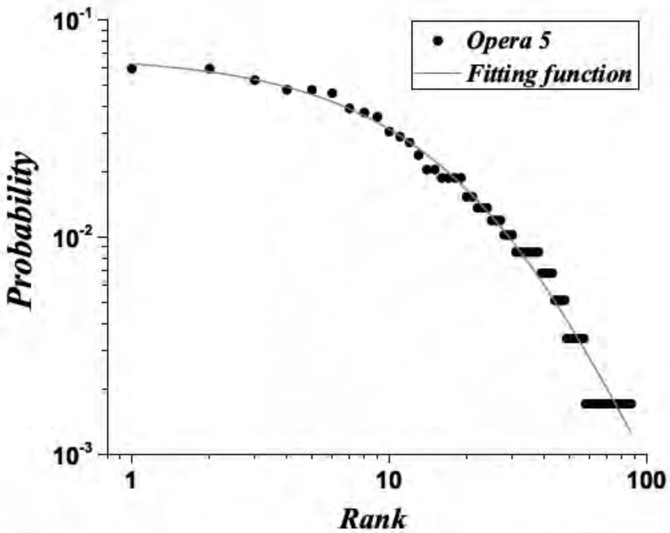

Figure 4-5.: Probability of occurrence *vs* rank for the chords of the *secco recitatives* from the opera Apollo and Hyacinthus.

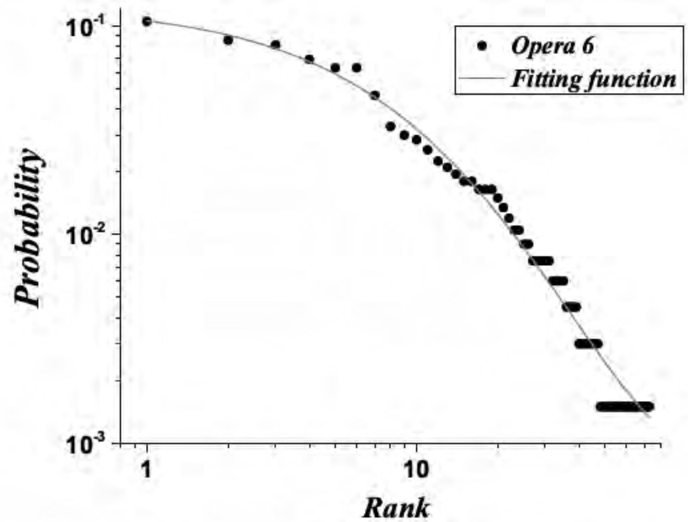

Figure 4-6.: Probability of occurrence *vs* rank for the chords of the *secco recitatives* from the opera the marriage of Figaro.

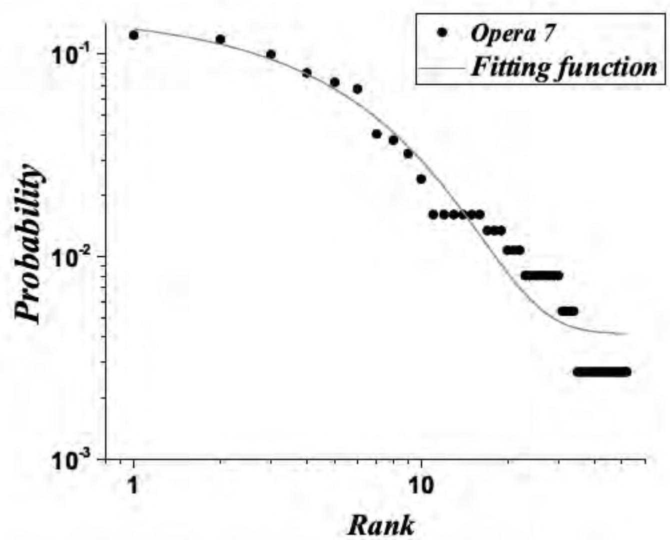

Figure 4-7.: Probability of occurrence *vs* rank for the chords of the *secco recitatives* from the opera Cinderella.

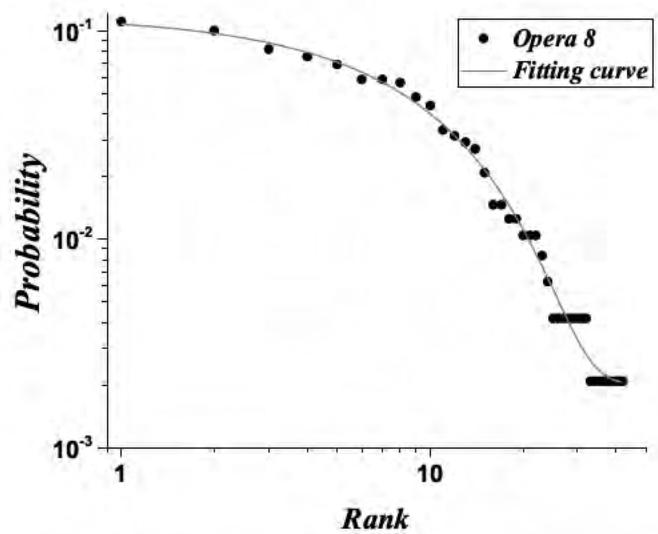

Figure 4-8.: Probability of occurrence *vs* rank for the chords of the *secco recitatives* from the opera the Barber of Seville.

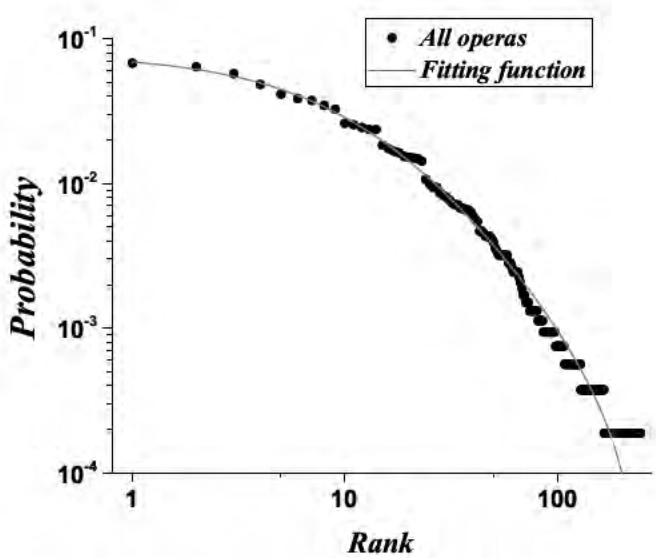

Figure 4-9.: Probability of occurrence *vs* rank for the superposition of all chords of the *secco recitatives* from the studied operas.

Opera	e	b	c	d	R^2
1	0.0012 ± 0.0004	-0.1519 ± 0.0044	0.0612 ± 0.0158	0.2785 ± 0.0549	0.9848
2	-0.0518 ± 0.0964	-0.8977 ± 6.6151×10^2	3.0602×10^3 ± 9.8862×10^6	4.3728 ± 5.6062	0.9293
3	-0.0041 ± 0.0008	-0.1190 ± 0.0071	0.6489 ± 0.1477	1.3704 ± 0.1050	0.9808
4	-0.0045 ± 0.0014	-0.0587 ± 0.0022	0.0434 ± 0.0116	0.6420 ± 0.1441	0.9913
5	0.0002 ± 0.0006	-0.0682 ± 0.0014	0.0233 ± 0.0066	0.2680 ± 0.0660	0.9939
6	0.0006 ± 0.0008	-0.1239 ± 0.0037	0.0506 ± 0.0157	0.2983 ± 0.0742	0.9889
7	0.0042 ± 0.0009	-0.1537 ± 0.0058	-0.0020 ± 0.0145	-0.0110 ± 0.0825	0.9829
8	0.0021 ± 0.0009	-0.1170 ± 0.0027	-0.0193 ± 0.0054	-0.1906 ± 0.0611	0.9936
All	-0.0002 ± 0.0001	-0.0784 ± 0.0006	0.0574 ± 0.0032	0.4521 ± 0.0168	0.9969

Table 4-3.: Fit parameters e, b, c, d, and determination coefficient R^2 for the rank distribution of chords of each opera and the superposition of all of them.

describes the frequency of occurrence $n(r)$ against rank r, for the case of the statistics of musical notes in some musical compositions, which is given by [42]:

$$n(r) = \frac{1}{(a' + b'r)^z}, \tag{4-2}$$

with a', b', and z the fitting parameters coming from experimental data. Taking $e = 0$, it is possible to appreciate that the parameter b in equation (4-1) changes if the frequency of occurrence is measured instead of the probability. In this sense, this parameter contains information about the total number of chords.

Simon's model (with two additional conditions) can produce a similar form for the frequency of occurrence given by equation (4-2). In language, Simon's model considers that the probability of occurrence of a word is proportional to its previous number of occurrences, the words that have not been used are added to the text with a constant rate [42, 90]. One of the conditions added to Simon's model is that the rate of appearance of new words changes with the length of the text [91]. The other condition is that there is a maximum number of occurrences for a single word [42]. Future analysis can be performed in order to considering an analogy between a word in a text and a chord in a musical composition, and to explore the possibility that this type of process can also be used to reproduce experimental data, as in the case of the *secco recitatives* harmony.

In the case of the opera N°2, the limited number of chords that appear in the *secco recitatives* of the musical score does not allow a good statistical description of the system, this can be observed in Figure **4-2**, and from the fit error values for the parameters a, b, c and d, and from the R^2 of the fit.

4.5. Analysis of roots

PDs for the root of the chords of each opera were made for the three types of triads considered. The PDs were constructed using numbers to represent each root. The distance between adjacent roots was considered as a fifth, this election is based on the relatedness of different chords in the musical context, in the sense that for the circle of fifths the distance between adjacent musical keys differs only in terms of one pitch class [9, pp. 250-252].

For each root of a chord, the corresponding number assigned is: F$\sharp \to 1$, C$\sharp \to 2$, G$\sharp \to 3$, D$\sharp \to 4$, A$\sharp \to 5$, F$\to 6$, C$\to 7$, G$\to 8$, D$\to 9$, A$\to 10$, E$\to 11$, and B$\to 12$.

This election was made in order to put the root C in the center of the graphic because this is one of the most used in the *secco recitatives*.

Figures **4-10**, **4-11**, and **4-12** show the PD for the roots of the major, minor, and diminished chords in the the *secco recitatives*.

Major chords tend to follow a sinusoidal function of the form

$$P(x) = a'' + b'' sin \left[\pi \left(\frac{x - c''}{d''} \right) \right], \tag{4-3}$$

where x refers to the root of the chord, $P(x)$ refers to the probability of occurrence of a given root x, and a'', b'', c'', and d'' refer to the fit parameters of each opera. Table **4-4** shows the fit parameters for each opera and the superposition of all of them, and the corresponding determination coefficient R^2 of each fit.

Figure **4-10** shows that major chords tend to be organized around a tonal center (near to C, G, or D) with smooth variations when we move away from it. From the probability values showed in Figure **4-11**, it is possible to observe that, in almost all operas, the minor chords have significant less occurrences than major ones, an exception to this behavior is found at the operas N° 1 and N° 2. For the minor chords, the most probable roots are displaced with respect to the roots of the major chords, in this case roots are near G, D, and A. The relative large occurrence of A minor in some operas can be explained because this chord is the relative minor of C major, and the relative minor of a key is highly used in the tonal music harmony, since C major and A minor are close in the psychological distance between keys [9, p. 252].

Figure **4-12** shows that diminished chords have very small probabilities of occurrence in comparison with major and minor chords. Besides, the roots of these chords are far (in the circle of fifths) from the roots of the major and minor chords.

4.6. Transitions between roots

For each *secco recitative*, the transitions between the roots of the chords were studied. The rests were omitted in all cases. Figure **4-13** shows the way in which the transitions were measured in clockwise direction, for example a typical cadence V-I starting in the root G and finishing in the root C corresponds to 5 steps (measured in semitones).

Since the incidence of diminished triads in the *secco recitatives* is low (see Figure **4-12**), then most transitions between chords can be classified in one of the following combinations: major-major, major-minor, minor-major, and minor-minor.

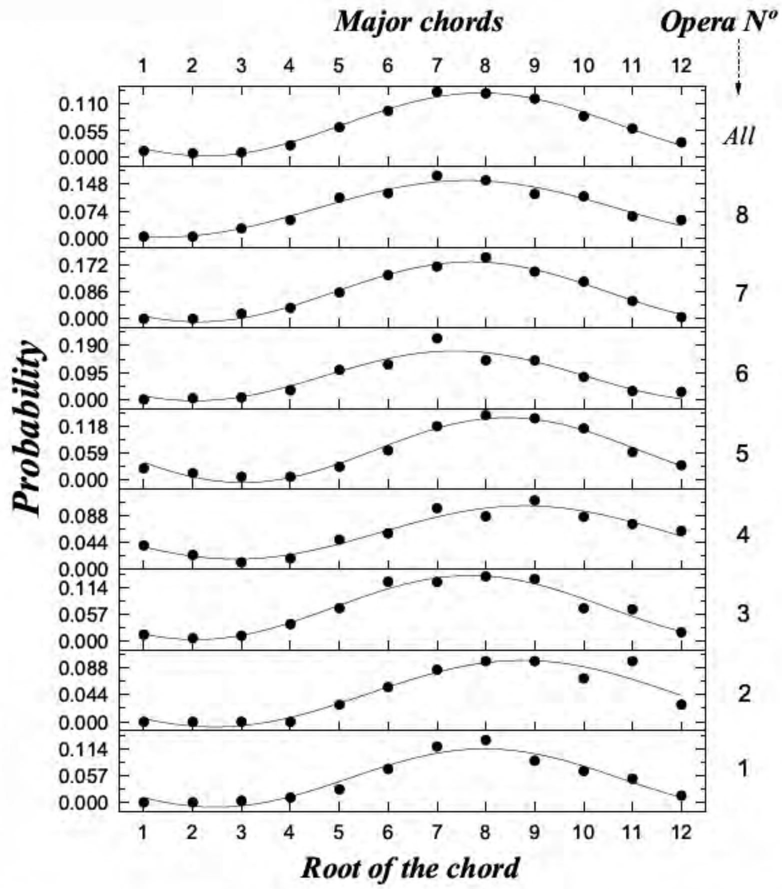

Figure 4-10.: Probability distribution for the roots of the major chords from the *secco recitatives* of each studied opera, and the superposition of all of them ("*All*").

	a''	b''	c''	d''	R^2
1	0.052±0.004	0.062±0.006	5.219±0.190	5.513±0.286	0.911
2	0.046±0.005	0.054±0.006	5.597±0.258	6.298±0.454	0.884
3	0.071±0.004	0.068±0.005	4.877±0.168	5.622±0.265	0.948
4	0.060±0.003	0.043±0.004	5.837±0.186	5.914±0.302	0.909
5	0.065±0.003	0.071±0.004	5.732±0.100	5.403±0.143	0.964
6	0.084±0.008	0.088±0.010	4.706±0.251	5.319±0.395	0.881
7	0.086±0.004	0.095±0.004	4.907±0.107	5.587±0.168	0.977
8	0.081±0.006	0.077±0.006	4.512±0.241	6.216±0.412	0.948
All	0.068±0.002	0.065±0.003	5.085±0.088	5.637±0.136	0.983

Table 4-4.: Fit parameters a'', b'', c'', d'', and the determination coefficient R^2 for the fit of the roots of the major chords to equation (4-3) in the case of each opera and the superposition of all of them ("*All*").

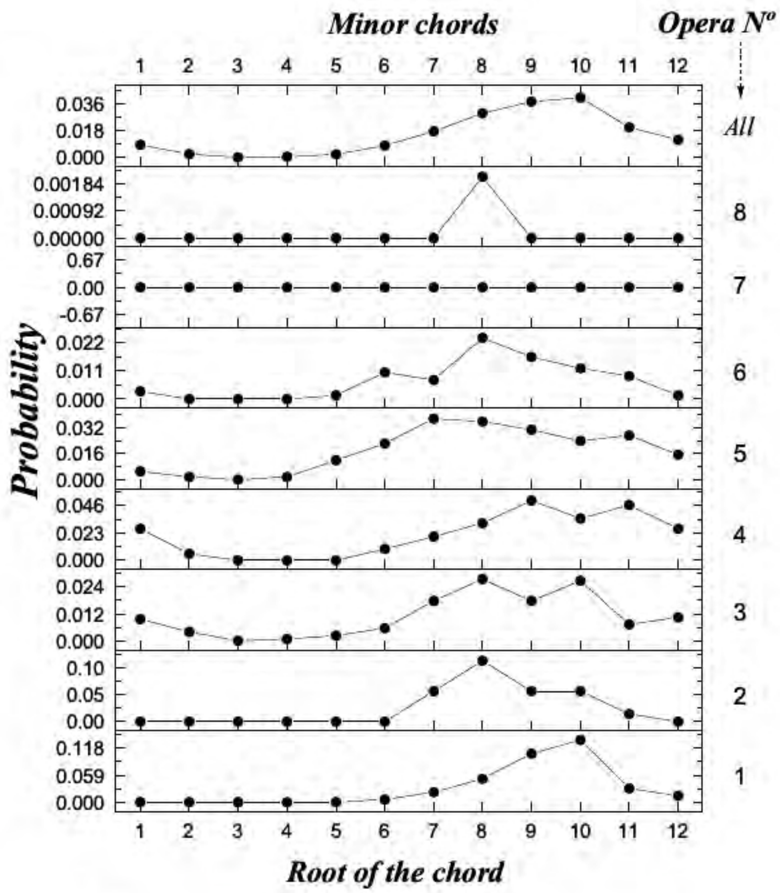

Figure 4-11.: Probability distribution for the roots of the minor chords from the *secco recitatives* of each studied opera, and the superposition of all of them ("*All*").

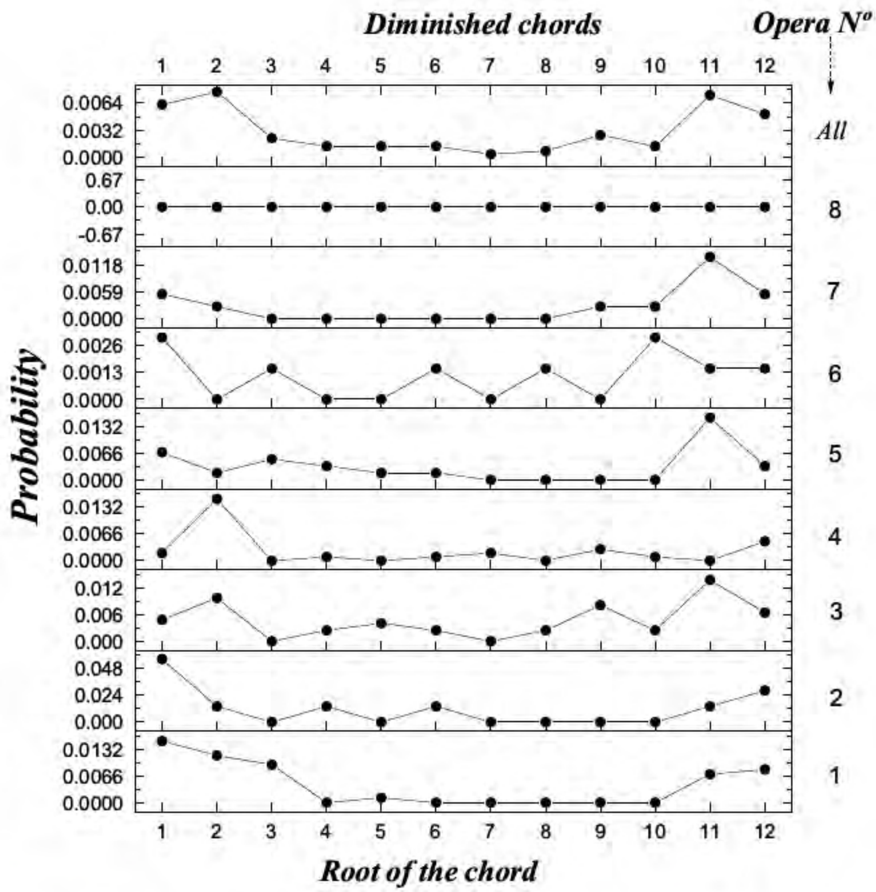

Figure 4-12.: Probability distribution for the roots of the diminished chords from the *secco recitatives* of each studied opera, and the superposition of all of them ("*All*").

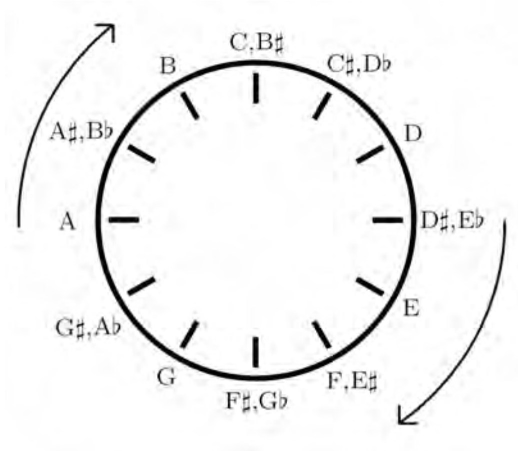

Figure 4-13.: The 12 pitches of the chromatic scale, and the transitions between them in clockwise direction.

Figures **4-14** to **4-17** show the probability to find a particular transition of roots in the operas studied and in their superposition, for the four transitions proposed.

In the major-major case, the most important contributions are those from transitions between the same root, frequently made using the inversions of the chord, and the transitions of five semitones between roots. Transitions of 2 semitones are also important. In the case of transitions containing the tonic chord (I), a 5 semitones step corresponds to the progression tonic-subdominant (I-IV) or dominant-tonic (V-I). Transitions of 2 semitones can occur in progressions between chords with functions of subdominant and dominant, as for example IV-V.

In the major-minor case, most transitions correspond to 5 semitones. In the case of transitions containing the tonic chord (i), a 5 semitones step is frequently related to progressions of the form dominant-tonic (V-i). In the opera N°8, this feature is absent, as most relevant transition correspond to 11 semitones, however this transition has a very small probability in the opera. In some operas, the 9 semitones transition also has some importance, this transition (in the major-minor case) corresponds to chords related by a *relative* relation, that is characterized by the transition between two chords in different modes, and with roots related as follows: the minor chord must have a root three semitones below the root of the major chord (9 steps starting from the major chord in Figure **4-13**). Two keys with a relative relation between them have the particularity that they have the same pitch classes [9, pp. 251-252].

In the minor-major case, most transitions correspond to 2 semitones. Transitions of 2 semitones can occur in progressions between chords with functions of subdominant and dominant, as for example iv-V. The contribution of 0 semitones is also important, this last transition corresponds to chords related by a *parallel* relation, that is characterized by the transition between two chords in different modes, but with the same root: one in minor mode and the other one in major mode (or vice versa). In a perceptual space with a psychological distance between keys proportional to the geometrical distance, the parallel and relative relations constitute some of the closest distances [9, p. 252].

For this case, the transition of 9 semitones does not contribute significantly as in the case of the major-minor progression. The transitions of 3, 5, 7, 8 and 10 semitones also appear with some probability, showing an important variety of combinations for the minor-major transitions.

Some asymmetries emerges in the transitions between chords in major and minor modes: the transition of 0 semitones (parallel relation) contributes significantly to the minor-major transitions, but this is not especially important in the transitions major-minor. Additionally, the transition of 9 semitones (relative relation in the mayor-minor case) contributes significantly to the transition major-minor in the operas N°1, 3, and 4, but this is not especially important in the minor-major transitions of the same set of operas.

In the minor-minor case, most transitions correspond to 0 and 5 semitones. A 0 semitones step is associated to inversions of the same chord. For the case of transitions containing the tonic chord (i), a 5 semitones step is associated to the progression i-iv (tonic-subdominant).

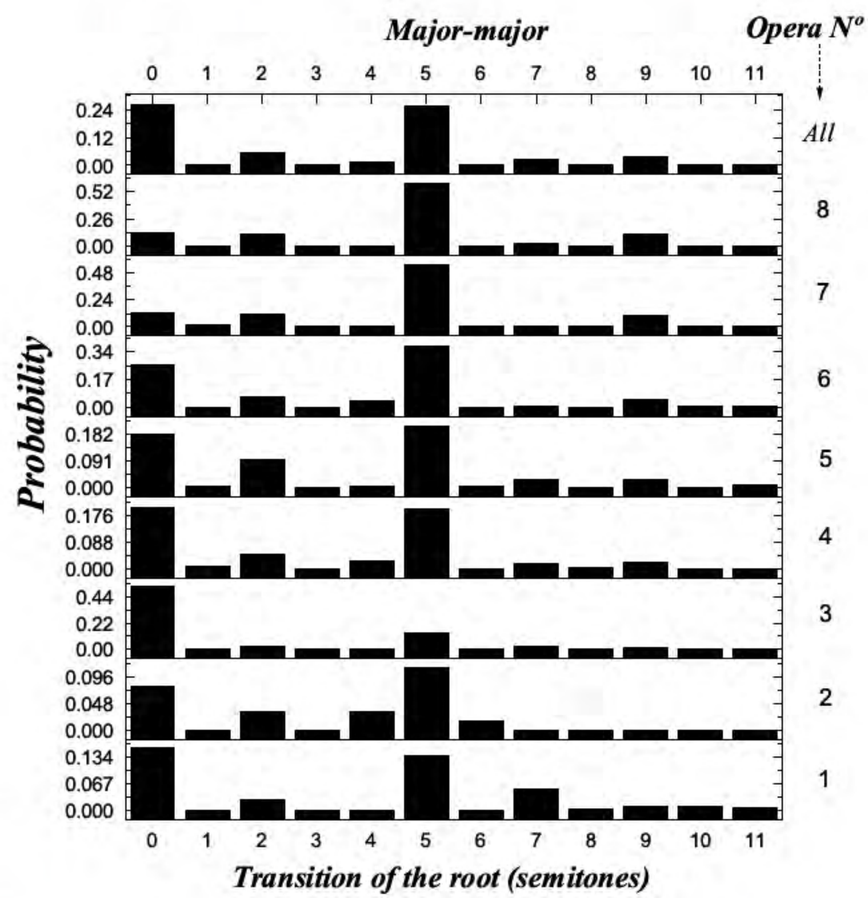

Figure 4-14.: Probability distribution for the transitions between the roots of two major chords from the *secco recitatives* of each studied opera, and the superposition of all of them ("*All*").

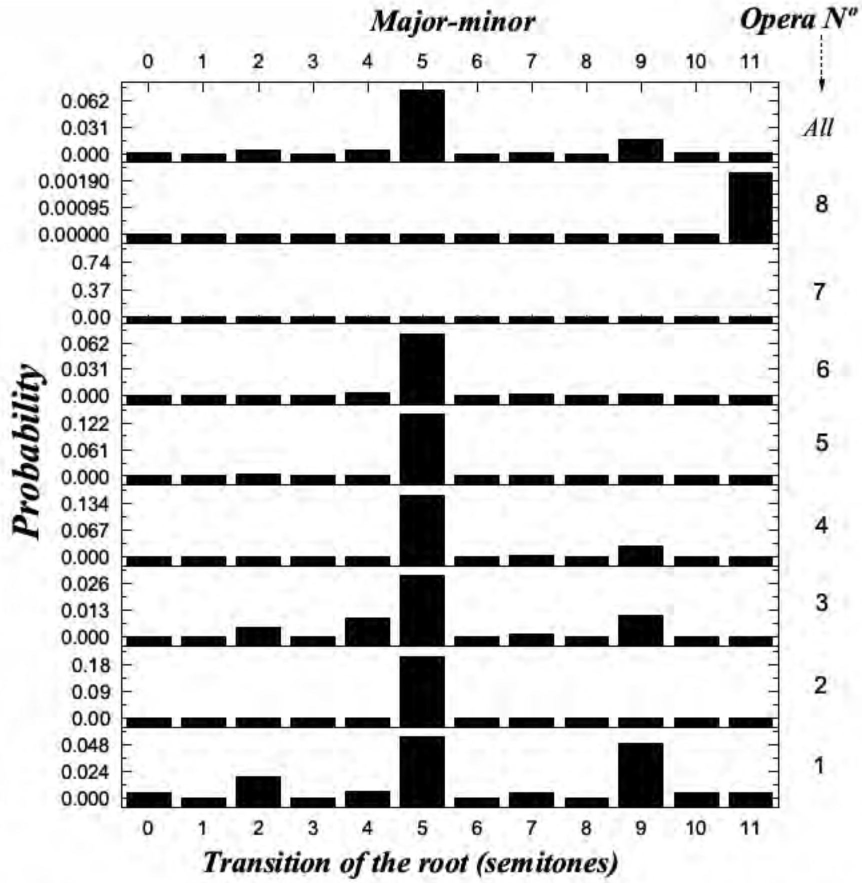

Figure 4-15.: Probability distribution for the transitions between the roots of two chords (major-minor) from the *secco recitatives* of each studied opera, and the superposition of all of them ("*All*").

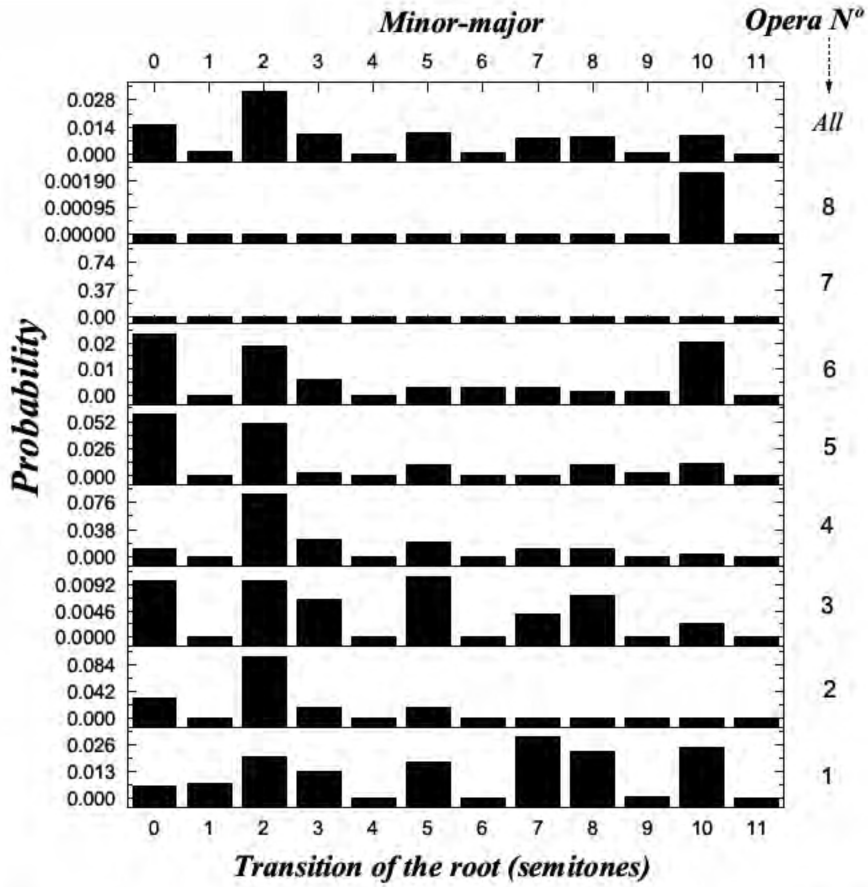

Figure 4-16.: Probability distribution for the transitions between the roots of two chords (minor-major) from the *secco recitatives* of each studied opera, and the superposition of all of them ("*All*").

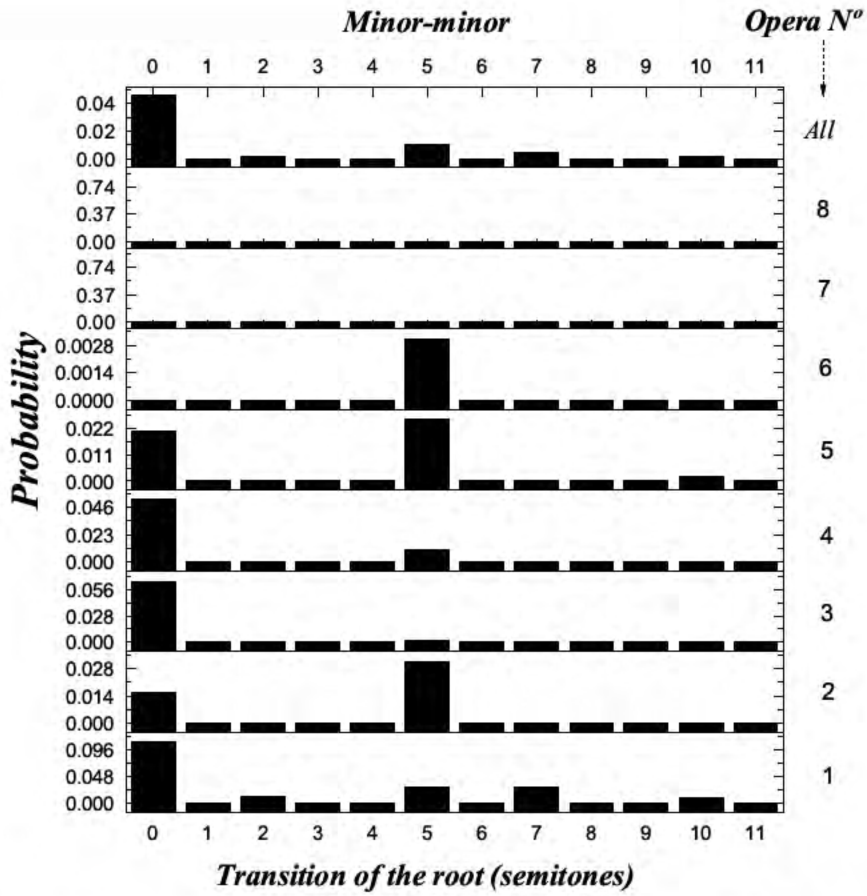

Figure 4-17.: Probability distribution for the transitions between the roots of two minor chords from the *secco recitatives* of each studied opera, and the superposition of all of them ("*All*").

4.7. Final transitions

Due to the importance of the final cadence in a musical piece, the final transition of each recitative was also studied. Table **4-5** shows, for the set of all operas, the number of final transitions according to the jump of the root in semitones for the different types of transitions: major-major, major-minor, minor-major, minor.minor, and diminished-minor. A total of 127 transitions were classified, corresponding to 124 *secco recitatives*. The three additional transitions emerge from the separations of the *secco recitatives* N° 4 in the opera Acis and Galatea (see Appendix F), and N° 10 and 19 in the opera The marriage of Figaro (see Appendix G).

Table **4-5** shows that most final transitions correspond to 5 semitones in the major-major and major-minor successions. About the inversions, the total number of final transitions according to their inversions are shown in Table **4-6**, notice that most transitions are made between the roots of the chords.

Assuming that the transitions go to the tonic chord, the major-major and minor-major

Transition of the root (semitones)	Major-major	Major-minor	Minor-major	Minor-Minor	Diminished-Minor
0	1	0	0	0	0
1	2	0	0	0	1
2	0	0	3	0	0
3	0	0	0	0	0
4	0	0	0	0	0
5	90	26	0	1	0
6	0	0	0	0	0
7	0	0	0	0	0
8	0	0	0	0	0
9	0	1	0	0	0
10	0	0	0	0	0
11	1	0	0	0	0
12	0	0	0	0	0

Table 4-5.: Number of final transitions in the *secco recitatives* according to the jump of the root in semitones, and the type of transition: major-major, major-minor, minor-major, minor-minor, and diminished-minor.

transitions between two chords in root position separated by 5 semitones between them (measuring the distance in agreement with Figure **4-13**) correspond to the dominant-tonic movements (V-I or V-i), and they are consistent with the *perfect authentic* and *imperfect authentic* cadences [12, p. 119]. These types of transitions correspond to a strong cadence, commonly found in the closure of musical pieces.

Inversions	Number
53→ 53	83
7→ 53	24
63→ 53	7
65→ 53	6
2→ 53	4
53→ 63	2
2→ 63	1

Table 4-6.: Number of final transitions in the *secco recitatives* according to their inversions.

4.8. Transitions between basses

The frequency of occurrence of the transitions between the lowest pitches of each chord (basses) were studied. Since most final transitions are used with closure purposes with a movement of 5 semitones between two chords in the root position, then the final transition of each *secco recitative* was not taken into account in this analysis in order to avoid redundant information.

Since the bass of each chord can be located up or down with respect to the bass of the previous chord, we assume that the jump of the bass corresponds to the minimum distance between the ascending and the descending transition (taking into account the most frequent movements found in the *secco recitatives*). This decision corresponds to measuring the distance in Figure **4-13** in the clockwise and counterclockwise directions, and to take the minimum distance between both directions. As a consequence of this procedure, the maximum possible distance between basses corresponds to 6 semitones. Figure **4-18** shows, for each opera and the superposition of all of them, the probability of occurrence according to the transition between basses measured in semitones. One of the most important contributions is the 0, 1, and 2 semitones, suggesting that the composers tend to minimize the bass movement. An extreme example of this behavior is found in the opera N°3, for which the location of the bass remains fixed in many transitions.

Another important contribution to the transition between basses is for 5 semitones. For the case of transitions containing the tonic chord, a 5 semitones step in the basses is related to progressions of the form tonic-subdominant (I-IV and i-iv) and dominant-tonic (V-I and V-i) with the chords in root position. Another possibility is the transition between the same chord in root position and in the second inversion, for example I-I_{64}, IV-IV_{64}, and V-V_{64}.

4.9. Betweenness centrality in *secco recitatives* networks

In this section we use graph theory in order to represent each individual *secco recitative* of an opera as a *cluster* in an indirect binary network. A *clique* is a complete subgraph in which all links are present, and all nodes are adjacent, this is the hardest definition of a *cluster* [92, p. 102]. The different chords used in each recitative are represented as nodes, and the links between them represent the occurrence of these chords in a same *secco recitative*. Figure **4-19** represents two recitatives for which only one chord is common between them. The size of nodes is proportional to the Freeman betweenness centrality C_B, but conserving a minimum size for the nodes in the case of $C_B = 0$. For a given node x_i, the centrality C_B is defined as [93]

$$C_B(x_i) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{g_{jk}(x_i)}{g_{jk}}; j \neq k \neq i; j < k, \tag{4-4}$$

where g_{jk} represents the total number of shortest paths from node j to note k, and g_{jk} represents the number of those paths that contain the node x_i.

This particular centrality measure captures information about the chords with more occurrences in different recitatives of the same opera, because in the case of a node $x_{i'}$ that only belong to one cluster $C_B(x_{i'}) = 0$ all shortest paths $g_{jk}(x_{i'})$ do not include the node $x_{i'}$.

The Freeman betweenness centrality can be normalized using the maximum possible centrality of a node in a graph with n nodes, this is

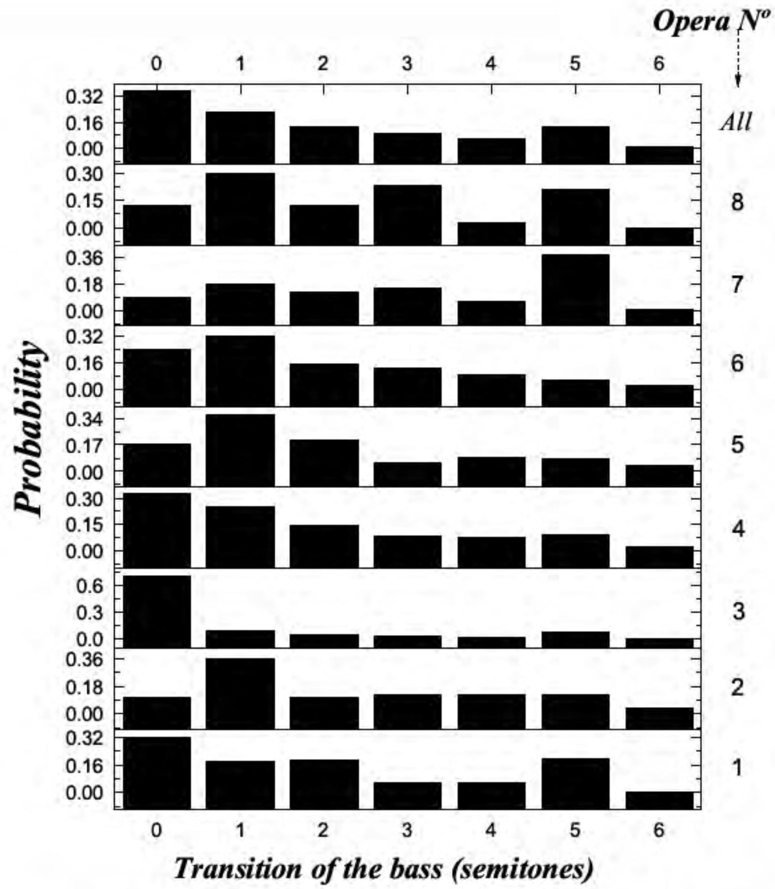

Figure 4-18.: Probability distribution for the transitions between the basses of the *secco recitatives* for each studied opera and the superposition of all of them ("*All*").

$$C'_B(x_i) = \frac{2C_B(x_i)}{n^2 - 3n + 2}. \tag{4-5}$$

The quantity C'_B allows to compare between different nodal indices or between different networks [92, p. 190].

Figures **4-20** to **4-27** show the networks of each studied opera taking the nodes as the different chords that appear in each opera, and the links representing the occurrence of these chords in a same *secco recitative*. As in the case of Figure **4-19**, the size of the nodes have been taken as proportional to the Freeman betweenness centrality C_B. Notice that in almost all operas, there is only one visible cluster, indicating that most chords appear in many secco recitatives of the same opera. An exception to this behavior is found in the opera N°7 for which there are two visible different clusters, indicating that there is at least one recitative with many chords that are not used in other *secco recitatives* of the opera.

Table **4-7** shows the nodes of each opera, each one describing a chord constituted by the four properties exposed in Table **4-2**, and with their respective values of Freeman betweenness

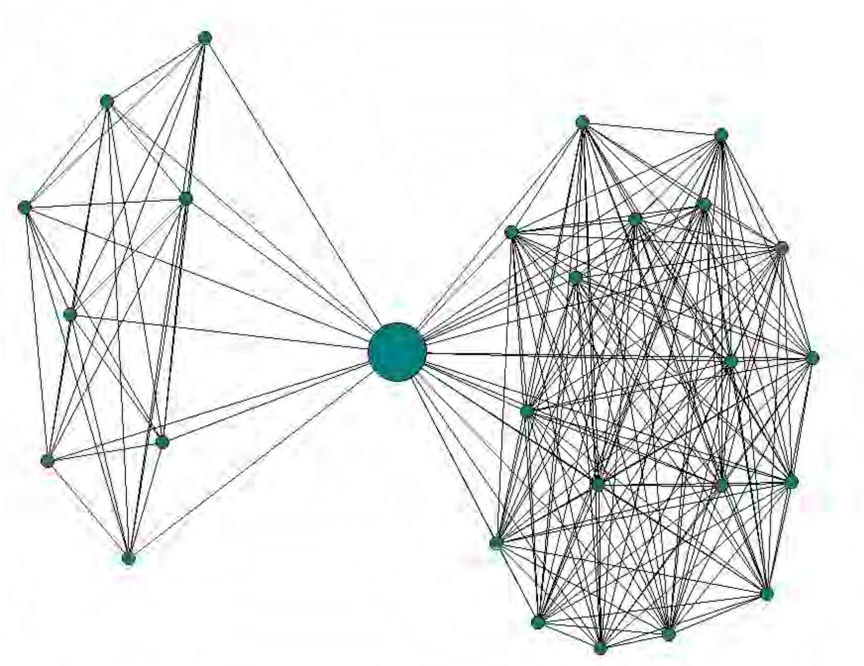

Figure 4-19.: Representation of two *secco recitatives* as clusters in a network. The nodes correspond to different chords, and the links between them to the occurrence of two of these chords inside a same recitative.

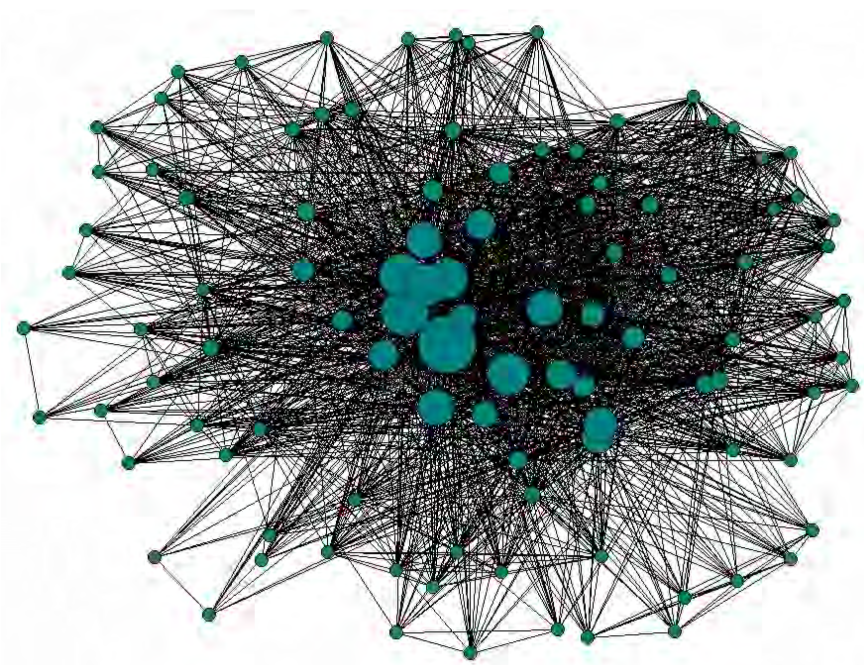

Figure 4-20.: Graph of the opera the Coronation of Poppea with clusters representing different *secco recitatives*. The nodes correspond to different chords, and the links between them to the co-occurrence of two of these chords inside a same *secco recitative*.

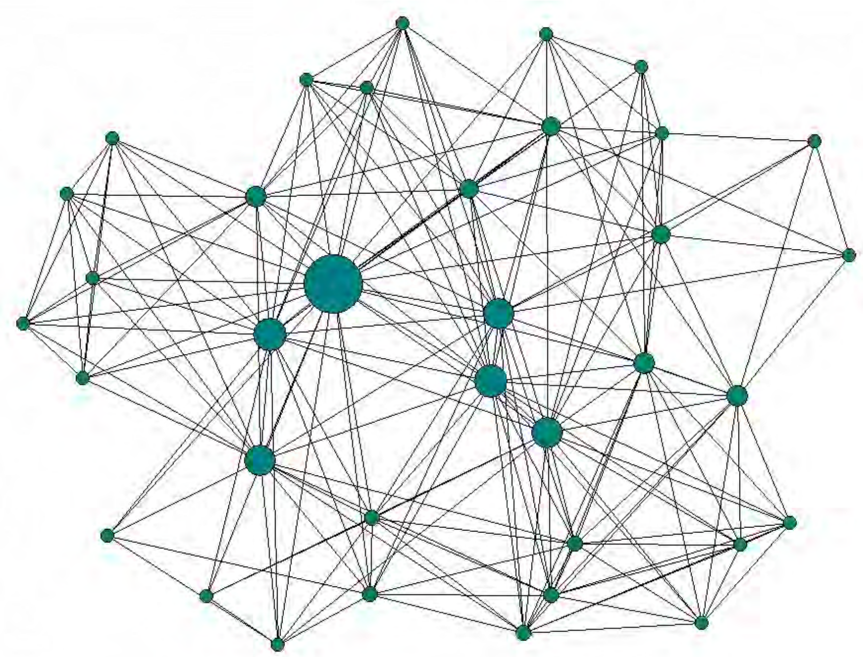

Figure 4-21.: Graph of the opera Acis and Galatea with clusters representing different *secco recitatives*. The nodes correspond to different chords, and the links between them to the co-occurrence of two of these chords inside a same *secco recitative*.

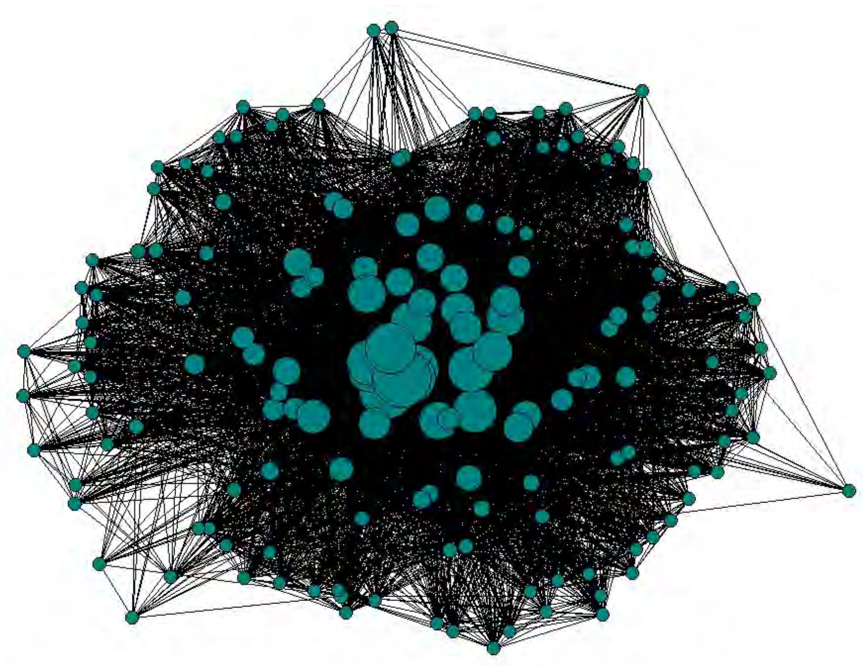

Figure 4-22.: Graph of the opera the Marriage of Hercules and Hebe with clusters representing different *secco recitatives*. The nodes correspond to different chords, and the links between them to the co-occurrence of two of these chords inside a same *secco recitative*.

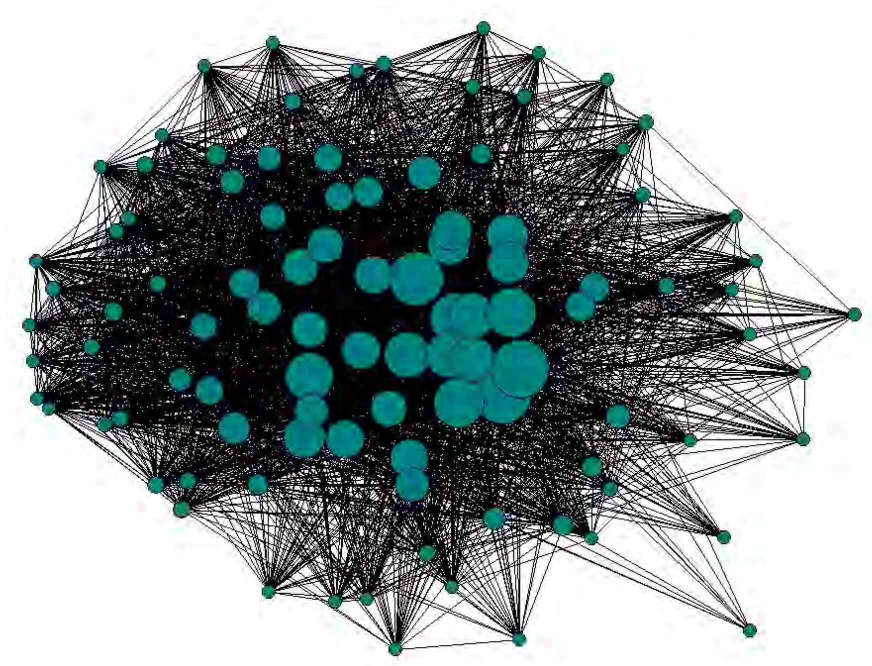

Figure 4-23.: Graph of the opera Mithridates, King of Pontus with clusters representing different *secco recitatives*. The nodes correspond to different chords, and the links between them to the co-occurrence of two of these chords inside a same *secco recitative*.

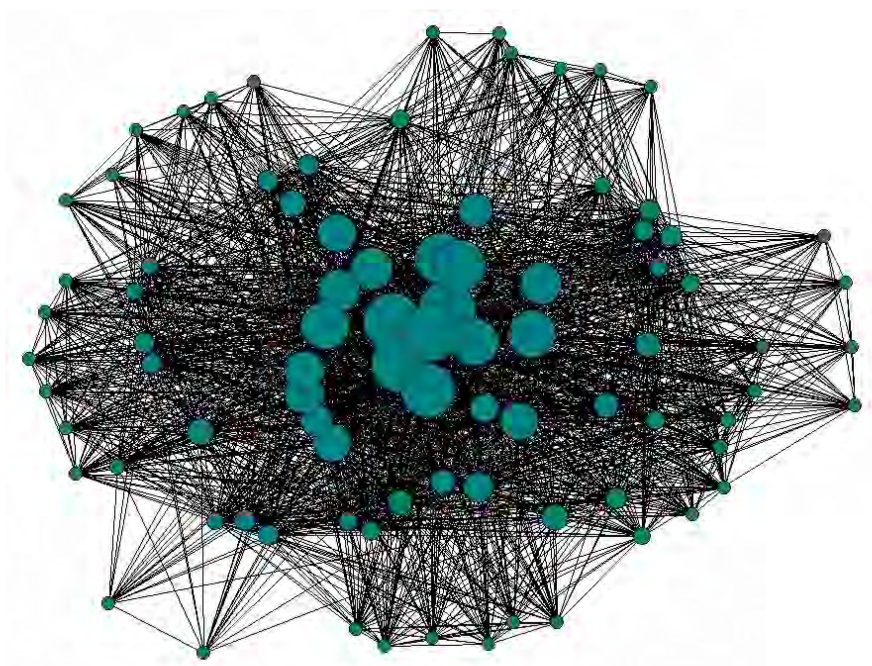

Figure 4-24.: Graph of the opera Apollo and Hyacinthus with clusters representing different *secco recitatives*. The nodes correspond to different chords, and the links between them to the co-occurrence of two of these chords inside a same *secco recitative*.

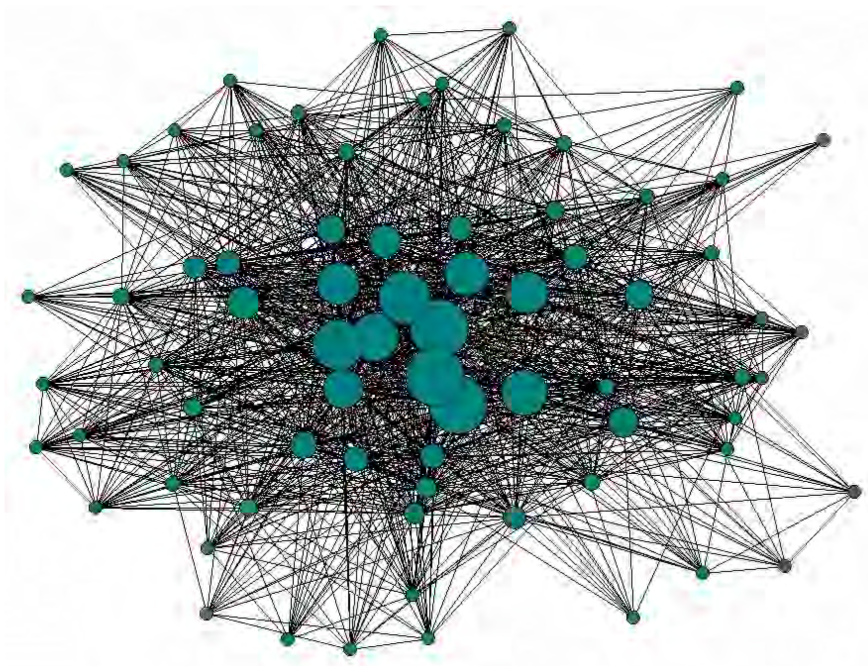

Figure 4-25.: Graph of the opera the marriage of Figaro with clusters representing different *secco recitatives*. The nodes correspond to different chords, and the links between them to the co-occurrence of two of these chords inside a same *secco recitative*.

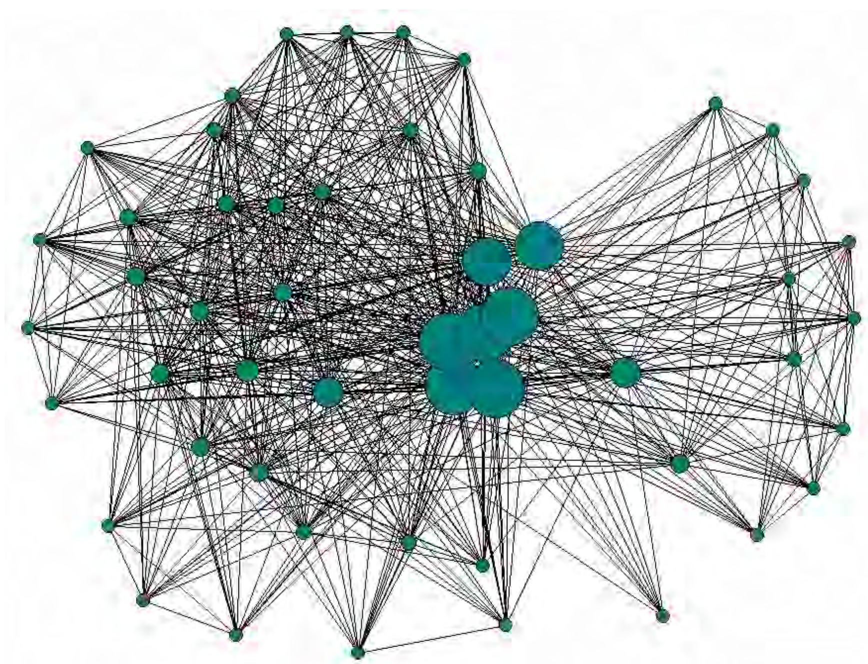

Figure 4-26.: Graph of the opera Cinderella with clusters representing different *secco recitatives*. The nodes correspond to different chords, and the links between them to the co-occurrence of two of these chords inside a same *secco recitative*.

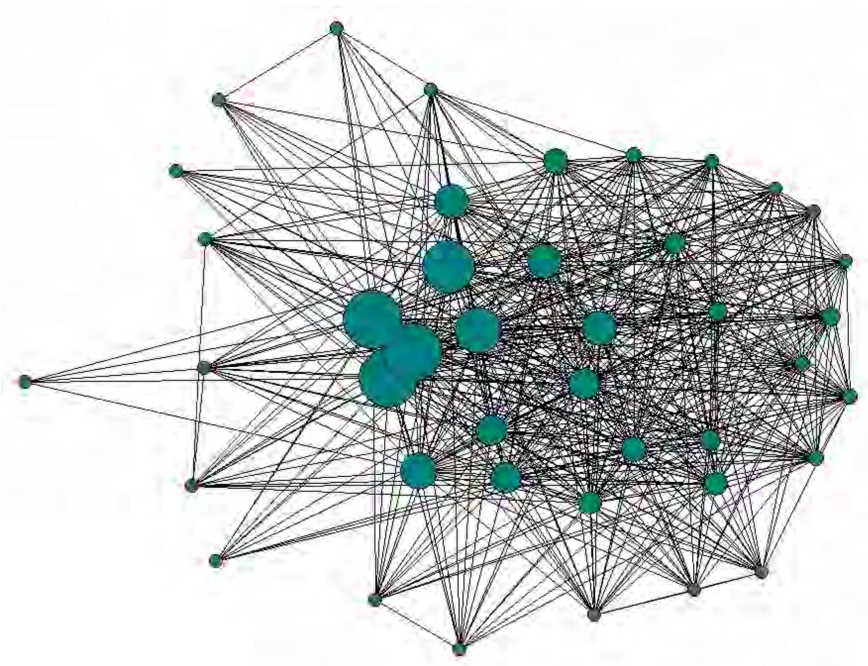

Figure 4-27.: Graph of the opera the Barber of Seville with clusters representing different *secco recitatives*. The nodes correspond to different chords, and the links between them to the co-occurrence of two of these chords inside a same *secco recitative*.

centrality C_B and normalized betweenness centrality C'_B. Nodes have been ordered from the highest to the lowest values of C_B (and hence C'_B). In 7 of the 8 operas, the five nodes with the greatest values of Freeman betweenness centrality always include the major triads of the roots C, G, D. The exception to this behavior is the opera N°2, for which the nodes associated to the root C do not appear between the ones with the highest values of C_B. For all operas, the nodes with the highest values of C_B correspond to chords with major or minor triads in root position (53) or first inversion (63), however an interesting phenomenon occurs in opera N°2, for which a diminished chord appears as the second node with highest betweenness centrality, indicating that harmony in the *secco recitatives* of this opera is different of the other ones.

Figure **4-28** shows the normalized Freeman betweenness centrality of each node C'_B ordered in rank (from the highest to the lowest value) for each studied opera. There is a slow decay for the operas N° 3, 4, and 5, a moderate decay for the operas N° 1 and 6, and a fast decay for the operas N° 2, 7, and 8. On one hand, a slow decay means that there are many chords that appear in most *secco recitatives* of the opera. On the other hand, an strong decay indicates that only a few chords are common to most *secco recitatives* of the opera, these few chords are very important, in such a way that if they are removed, the harmony of the entire opera is strongly affected.

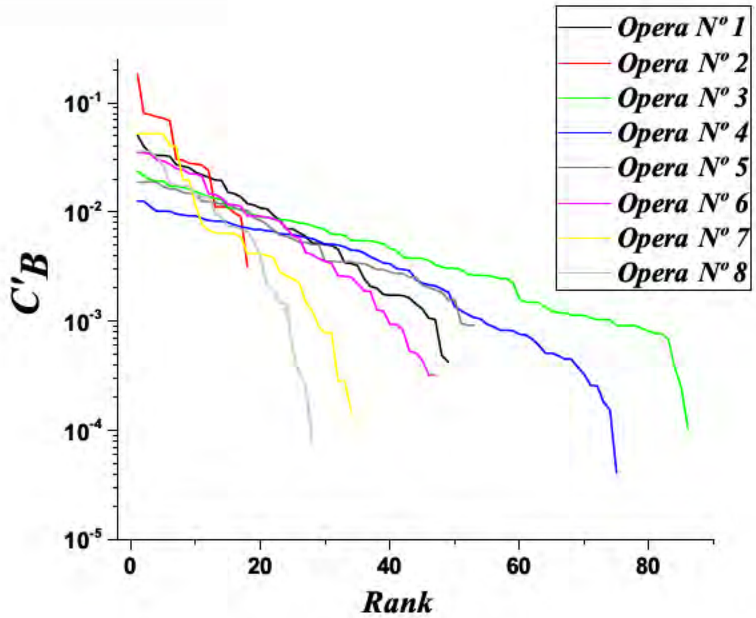

Figure 4-28.: Normalized Freeman betweenness centrality of each node C'_B ordered in rank (from the highest to the lowest value) for all operas.

Opera 1					
Root	**Triad**	**Alteration**	**Inversion**	C_B	C'_B
A	Minor	0	53	264.11	0.0503
C	Major	0	53	210.61	0.0401
G	Major	0	53	180.24	0.0343
D	Major	0	53	173.82	0.0331
D	Minor	0	53	173.48	0.0330
F	Major	0	53	169.69	0.0323
E	Minor	0	53	142.83	0.0272
E	Major	0	63	138.32	0.0263
E	Major	0	53	134.45	0.0256
A	Major	0	53	121.64	0.0232
A	Major	0	63	114.79	0.0219
D	Major	0	63	108.99	0.0207
F	Major	0	63	103.58	0.0197
G	Major	0	63	102.15	0.0194
G	Minor	0	53	79.66	0.0152
D	Minor	0	63	77.20	0.0147

F♯	Diminished	0	63	72.27	0.0138
G	Minor	0	63	62.67	0.0119
B	Major	0	53	60.96	0.0116
C	Minor	0	53	57.00	0.0109
A♯	Major	0	53	56.23	0.0107
E	Minor	0	63	48.95	0.0093
D♯	Major	0	53	44.30	0.0084
C	Major	0	63	36.79	0.0070
E	Diminished	0	63	36.79	0.0070
A	Minor	0	63	33.96	0.0065
G	Minor	7m	7	31.30	0.0060
C♯	Diminished	0	63	27.30	0.0052
B	Minor	0	53	27.24	0.0052
A♯	Major	0	63	26.43	0.0050
C	Major	-5	53	26.10	0.0050
B	Diminished	0	63	23.95	0.0046
B	Diminished	0	53	18.08	0.0034
C	Major	7Ma	2	17.73	0.0034
D	Minor	0	64	16.92	0.0032
A	Major	7m	65	12.80	0.0024
A	Minor	Other	53	10.94	0.0021
D	Major	-5	53	10.53	0.0020
F	Minor	0	63	9.25	0.0018
B	Minor	0	63	9.09	0.0017
D	Major	7m	7	9.09	0.0017
G	Major	-5	53	9.03	0.0017
G	Major	7m	2	8.56	0.0016
A	Minor	7m,-5	7	7.56	0.0014
G	Major	7m,-5	7	6.82	0.0013
C♯	Diminished	0	53	5.60	0.0011
G♯	Major	0	53	5.47	0.0010
A	Minor	7m	7	2.56	0.0005
A	Major	7m,-5	7	2.22	0.0004
D	Minor	7m,-5	7	0.00	0.0000
E	Major	7m,-5	2	0.00	0.0000
A	Major	-5	63	0.00	0.0000

A	Major	7m,-5	65	0.00	0.0000
G	Minor	-5	53	0.00	0.0000
G	Major	Other	7	0.00	0.0000
E	Major	-5	63	0.00	0.0000
E	Minor	7m	65	0.00	0.0000
D	Minor	Other	64	0.00	0.0000
G♯	Diminished	0	63	0.00	0.0000
G	Major	7m,-5	2	0.00	0.0000
A	Minor	0	64	0.00	0.0000
F	Major	7Ma	43	0.00	0.0000
C	Major	7m	43	0.00	0.0000
D	Major	Other	43	0.00	0.0000
A♯	Major	7Ma	2	0.00	0.0000
G♯	Major	0	63	0.00	0.0000
C	Minor	Other	53	0.00	0.0000
G	Minor	-5	63	0.00	0.0000
C	Major	7Ma,-5	7	0.00	0.0000
E	Major	7m	65	0.00	0.0000
C	Major	7m,-5	7	0.00	0.0000
A	Major	Other	7	0.00	0.0000
D	Major	7m	2	0.00	0.0000
E	Minor	0	64	0.00	0.0000
B	Minor	-5	53	0.00	0.0000
D♯	Major	7Ma,-5	7	0.00	0.0000
C	Minor	0	63	0.00	0.0000
A♯	Major	7Ma,-5	7	0.00	0.0000
G	Minor	7m,-5	7	0.00	0.0000
D	Minor	-5	53	0.00	0.0000
D	Major	7m,-5	63	0.00	0.0000
F♯	Diminished	7m	65	0.00	0.0000
C	Minor	7m,-5	2	0.00	0.0000
A	Minor	7m	2	0.00	0.0000
F	Major	7Ma	2	0.00	0.0000
A♯	Major	7Ma,-5	53	0.00	0.0000
C	Major	7m	65	0.00	0.0000
A♯	Diminished	0	53	0.00	0.0000

C	Major	7Ma	7	0.00	0.0000
D	Minor	7m	65	0.00	0.0000
F	Major	-5	53	0.00	0.0000
B	Major	0	63	0.00	0.0000
B	Major	7m	2	0.00	0.0000
F	Minor	0	53	0.00	0.0000
B	Major	Other	53	0.00	0.0000
C	Major	0	64	0.00	0.0000
E	Major	-5	53	0.00	0.0000
F	Major	7Ma	7	0.00	0.0000
D	Major	-5	63	0.00	0.0000
A	Major	-5	53	0.00	0.0000
F	Minor	7m	7	0.00	0.0000
G♯	Diminished	-5	53	0.00	0.0000
G♯	Diminished	7^o,-5	7	0.00	0.0000
E	Minor	7m	7	0.00	0.0000
Opera 2					
Root	**Triad**	**Alteration**	**Inversion**	C_B	C'_B
G	Minor	0	53	109.47	0.1840
F♯	Diminished	7^o	7	47.72	0.0802
D	Major	0	63	46.36	0.0779
F	Major	0	53	44.40	0.0746
A	Minor	0	53	43.13	0.0725
D	Major	0	53	40.70	0.0684
A	Major	0	63	18.19	0.0306
C	Major	0	63	17.70	0.0297
E	Major	0	2	16.67	0.0280
E	Major	0	53	16.41	0.0276
D	Minor	0	53	16.02	0.0269
A	Major	0	53	14.27	0.0240
B	Diminished	0	63	6.61	0.0111
C	Major	0	53	6.61	0.0111
E	Major	0	63	6.61	0.0111
G	Major	0	63	5.83	0.0098
G	Major	0	53	5.43	0.0091
C	Minor	0	63	1.88	0.0032

A♯	Major	0	53	0.00	0.0000
C	Major	0	2	0.00	0.0000
E	Diminished	0	53	0.00	0.0000
G	Major	0	2	0.00	0.0000
C♯	Diminished	7º	7	0.00	0.0000
D	Minor	0	63	0.00	0.0000
E	Major	7m	65	0.00	0.0000
F	Diminished	0	63	0.00	0.0000
A	Major	0	2	0.00	0.0000
G	Minor	0	63	0.00	0.0000
A♯	Major	0	63	0.00	0.0000
D♯	Diminished	7º	7	0.00	0.0000
B	Major	0	63	0.00	0.0000
B	Major	0	65	0.00	0.0000
E	Minor	0	53	0.00	0.0000
A	Minor	0	63	0.00	0.0000
C	Minor	0	53	0.00	0.0000
D	Major	0	7	0.00	0.0000
Opera 3					
Root	**Triad**	**Alteration**	**Inversion**	C_B	C'_B
C	Major	0	53	303.08	0.0235
D	Major	0	53	278.08	0.0216
A♯	Major	0	63	253.13	0.0197
G	Major	0	53	246.60	0.0191
A	Major	0	63	246.60	0.0191
C	Major	0	63	221.02	0.0172
F	Major	0	53	221.02	0.0172
D	Major	0	63	217.59	0.0169
G	Major	0	63	209.34	0.0163
D	Major	7m	7	198.27	0.0154
C	Major	7m	7	187.69	0.0146
D	Major	7m	65	177.80	0.0138
A	Major	0	53	171.96	0.0134
G	Minor	0	53	168.71	0.0131
E	Major	0	53	140.18	0.0109
A	Major	7m	65	139.66	0.0108

C	Major	7m	65	135.31	0.0105
A♯	Major	0	53	128.04	0.0099
D♯	Major	0	53	122.37	0.0095
A	Minor	0	63	115.98	0.0090
E	Major	7m	7	115.98	0.0090
B	Major	0	63	110.92	0.0086
F	Major	7m	2	110.36	0.0086
G	Major	7m	7	108.88	0.0085
A	Major	7m	2	106.57	0.0083
A♯	Major	7m	65	102.45	0.0080
D	Minor	0	63	101.29	0.0079
F	Major	7m	7	96.77	0.0075
G	Major	7m	2	92.90	0.0072
C	Major	-5	53	88.56	0.0069
G♯	Major	0	63	80.87	0.0063
F	Major	0	63	80.50	0.0063
D	Major	-5	63	79.02	0.0061
B	Major	7m	65	70.91	0.0055
A	Minor	0	64	70.89	0.0055
D	Minor	0	53	70.13	0.0054
E	Major	0	63	69.83	0.0054
C♯	Diminished	7º	7	68.50	0.0053
C	Major	-5	63	66.15	0.0051
D	Major	7m	2	59.49	0.0046
G	Minor	0	63	57.73	0.0045
G	Major	7m	65	49.76	0.0039
E	Diminished	0	53	48.94	0.0038
E	Diminished	7º	7	48.94	0.0038
D♯	Major	0	63	48.46	0.0038
F♯	Minor	0	53	46.46	0.0036
A♯	Major	7m,-5	7	42.18	0.0033
E	Major	7m	65	40.42	0.0031
E	Major	7m	2	39.27	0.0030
B	Minor	0	63	39.27	0.0030
D	Major	0	64	38.43	0.0030
A	Major	7m	7	34.65	0.0027

A♯	Major	-5	63	34.04	0.0026
F♯	Major	7m	7	33.88	0.0026
D	Diminished	7º	7	33.88	0.0026
C	Minor	0	63	32.62	0.0025
F♯	Major	0	53	31.69	0.0025
E	Major	7m	53	31.60	0.0025
F♯	Diminished	7º	7	29.21	0.0023
C	Minor	0	53	20.71	0.0016
A♯	Major	7m	7	19.23	0.0015
F	Minor	0	53	19.23	0.0015
F♯	Major	7m	2	19.20	0.0015
E	Minor	-5	53	17.34	0.0013
B	Minor	0	53	15.88	0.0012
C♯	Major	0	63	15.45	0.0012
E	Major	Other	7	15.34	0.0012
A♯	Diminished	7º	7	14.66	0.0011
A	Minor	0	53	14.66	0.0011
G	Diminished	7º	7	14.66	0.0011
C	Major	7m	2	13.74	0.0011
A	Major	7m,-5	7	13.31	0.0010
C♯	Diminished	0	53	13.31	0.0010
B	Major	0	53	13.31	0.0010
A	Minor	-5	63	11.75	0.0009
A	Minor	7m,-5	2	11.75	0.0009
D♯	Diminished	7º	7	11.71	0.0009
E	Minor	0	53	11.71	0.0009
C	Minor	7m	65	11.00	0.0009
D	Major	7m,-5	7	10.44	0.0008
G	Major	7m	43	9.91	0.0008
D	Major	-5	53	9.90	0.0008
D	Major	7m,-5	2	8.77	0.0007
C♯	Minor	-5	53	4.80	0.0004
G	Major	0	64	3.16	0.0002
D	Diminished	0	53	1.32	0.0001
E	Diminished	0	64	0.00	0.0000
E	Major	Other	2	0.00	0.0000

A	Major	-5	53	0.00	0.0000
A	Major	Other	7	0.00	0.0000
A	Diminished	7°	7	0.00	0.0000
B	Major	7m	7	0.00	0.0000
A♯	Major	7Ma,-5	2	0.00	0.0000
F	Major	7m	65	0.00	0.0000
B	Diminished	0	53	0.00	0.0000
G	Major	7m	63	0.00	0.0000
D	Major	0	7	0.00	0.0000
D	Major	0	2	0.00	0.0000
A	Major	7m,-5	65	0.00	0.0000
B	Minor	7m,-5	63	0.00	0.0000
E	Major	0	64	0.00	0.0000
E	Major	7m,-5	7	0.00	0.0000
A	Diminished	0	63	0.00	0.0000
A♯	Major	7m,-5	65	0.00	0.0000
F	Diminished	0	64	0.00	0.0000
G	Minor	-5	53	0.00	0.0000
F	Major	0	64	0.00	0.0000
F	Major	7m,-5	7	0.00	0.0000
C	Major	7m,-5	65	0.00	0.0000
C	Major	7m,-5	7	0.00	0.0000
G	Minor	7m,-5	2	0.00	0.0000
F	Diminished	0	63	0.00	0.0000
F	Diminished	7°	7	0.00	0.0000
F♯	Minor	-5	53	0.00	0.0000
C	Minor	-5	63	0.00	0.0000
E	Diminished	7m	7	0.00	0.0000
B	Diminished	0	64	0.00	0.0000
B	Minor	7m,-5	2	0.00	0.0000
C♯	Major	7m	2	0.00	0.0000
C♯	Major	7m	7	0.00	0.0000
F♯	Diminished	Other	64	0.00	0.0000
C	Minor	7m	7	0.00	0.0000
C	Minor	Other	7	0.00	0.0000
E	Diminished	0	63	0.00	0.0000

D	Minor	-5	53	0.00	0.0000
B	Diminished	7^o	7	0.00	0.0000
C	Major	7m,-5	2	0.00	0.0000
A	Minor	-5	53	0.00	0.0000
A	Major	-5	63	0.00	0.0000
E	Major	0	2	0.00	0.0000
D♯	Minor	-5	53	0.00	0.0000
C♯	Major	7m	65	0.00	0.0000
D	Major	7Ma,-5	65	0.00	0.0000
G	Major	0	2	0.00	0.0000
B	Minor	-5	63	0.00	0.0000
B	Diminished	0	63	0.00	0.0000
D♯	Major	-5	53	0.00	0.0000
D♯	Major	7m	2	0.00	0.0000
G	Diminished	0	64	0.00	0.0000
D♯	Major	7m,-5	2	0.00	0.0000
D	Minor	7m	7	0.00	0.0000
D	Minor	0	64	0.00	0.0000
C	Major	0	7	0.00	0.0000
D	Major	7^o	7	0.00	0.0000
F♯	Major	7m,-5	7	0.00	0.0000
G	Major	-5	63	0.00	0.0000
A	Diminished	0	64	0.00	0.0000
D♯	Major	0	64	0.00	0.0000
D♯	Major	-5	63	0.00	0.0000
A♯	Major	-5	53	0.00	0.0000
D	Diminished	0	64	0.00	0.0000
G	Minor	Other	53	0.00	0.0000
C	Major	Other	7	0.00	0.0000
G	Minor	-5	63	0.00	0.0000
G♯	Major	0	53	0.00	0.0000
C♯	Minor	0	53	0.00	0.0000
F♯	Minor	0	64	0.00	0.0000
F♯	Diminished	0	64	0.00	0.0000
C	Major	0	64	0.00	0.0000
G	Major	7m,-5	2	0.00	0.0000

E	Minor	0	63	0.00	0.0000
A♯	Minor	0	53	0.00	0.0000
Opera 4					
Root	**Triad**	**Alteration**	**Inversion**	C_B	C'_B
B	Major	0	63	60.75	0.0125
D	Major	0	53	60.75	0.0125
G	Major	0	53	54.49	0.0112
E	Minor	0	53	49.37	0.0102
C	Major	0	63	49.37	0.0102
A	Minor	0	63	49.37	0.0102
C	Major	0	53	46.43	0.0096
D	Minor	0	53	45.73	0.0094
D	Major	0	63	44.65	0.0092
D	Minor	0	63	44.65	0.0092
C	Major	7m	65	42.54	0.0088
A	Major	0	63	40.43	0.0083
C♯	Diminished	0	63	39.97	0.0082
E	Major	7m	2	39.97	0.0082
F	Major	0	53	38.91	0.0080
A	Major	0	53	36.91	0.0076
A♯	Major	0	63	35.47	0.0073
F♯	Minor	0	53	34.69	0.0072
G	Major	0	63	33.58	0.0069
D	Major	7m	65	33.47	0.0069
E	Major	0	53	32.61	0.0067
B	Major	7m	65	32.50	0.0067
A♯	Major	0	53	30.63	0.0063
F♯	Major	7m	2	30.41	0.0063
C♯	Major	0	63	30.11	0.0062
D♯	Major	0	53	29.18	0.0060
G	Major	7m	2	28.26	0.0058
G	Minor	0	63	27.84	0.0057
G	Minor	0	53	26.01	0.0054
C	Minor	0	63	24.59	0.0051
A	Major	7m	65	24.58	0.0051
F♯	Major	0	53	24.22	0.0050

B	Minor	0	63	23.30	0.0048
A	Minor	0	53	21.65	0.0045
F	Minor	0	53	21.54	0.0044
E	Major	0	63	20.51	0.0042
A♯	Major	7m	65	18.80	0.0039
C♯	Major	7m	65	17.77	0.0037
E	Minor	0	63	16.59	0.0034
B	Major	0	53	16.29	0.0034
F♯	Major	0	63	14.73	0.0030
B	Diminished	0	63	14.47	0.0030
D	Major	7m	2	14.11	0.0029
B	Major	7m	2	11.60	0.0024
F	Major	7m	2	10.82	0.0022
F	Major	0	63	10.41	0.0021
G	Major	7m	65	10.25	0.0021
A	Major	7m	2	9.54	0.0020
C	Minor	0	53	8.93	0.0018
C	Major	7m	7	6.58	0.0014
A	Major	-5	53	6.04	0.0012
E	Major	7m	65	5.63	0.0012
B	Minor	0	53	5.24	0.0011
G♯	Major	0	53	5.09	0.0010
C	Major	7m	2	4.54	0.0009
G♯	Major	7m	2	4.32	0.0009
F♯	Minor	0	63	4.00	0.0008
D	Minor	-5	53	4.00	0.0008
C	Major	-5	53	3.99	0.0008
E	Major	7m	7	3.67	0.0008
D♯	Major	0	63	3.63	0.0007
A♯	Major	7m	7	3.31	0.0007
C♯	Minor	0	63	2.90	0.0006
F	Major	7m	7	2.47	0.0005
A♯	Major	-5	53	2.47	0.0005
D	Major	7m,-5	7	2.35	0.0005
F♯	Major	7m	65	2.18	0.0004
C	Diminished	7^o	7	2.18	0.0004

F	Major	7Ma,-5	7	1.84	0.0004
C♯	Major	0	53	1.58	0.0003
G♯	Major	0	63	1.25	0.0003
G	Major	7m	7	1.24	0.0003
F♯	Minor	-5	53	0.88	0.0002
D	Diminished	7^{o}	7	0.75	0.0002
F♯	Diminished	7^{o}	7	0.20	0.0000
C♯	Major	-5	53	0.00	0.0000
C♯	Major	7m,-5	7	0.00	0.0000
A	Major	7m	43	0.00	0.0000
F	Major	7m	65	0.00	0.0000
F	Minor	0	63	0.00	0.0000
A	Diminished	7Ma	43	0.00	0.0000
F	Diminished	7^{o}	7	0.00	0.0000
D♯	Diminished	7^{o}	7	0.00	0.0000
D	Diminished	0	53	0.00	0.0000
D	Minor	0	64	0.00	0.0000
C♯	Major	7m	2	0.00	0.0000
C	Major	7m,-5	7	0.00	0.0000
E	Minor	7m,-5	7	0.00	0.0000
A	Major	7m	7	0.00	0.0000
D	Major	7m	7	0.00	0.0000
D♯	Major	-5	53	0.00	0.0000
D♯	Major	7m	7	0.00	0.0000
B	Major	7m	43	0.00	0.0000
A♯	Major	7m	43	0.00	0.0000
G	Major	7Ma,-5	7	0.00	0.0000
C♯	Diminished	7^{o}	7	0.00	0.0000
E	Minor	-5	53	0.00	0.0000
D♯	Major	7Ma,-5	7	0.00	0.0000
E	Major	7m,-5	7	0.00	0.0000
A	Major	7m,-5	7	0.00	0.0000
Opera 5					
Root	**Triad**	**Alteration**	**Inversion**	C_B	C'_B
D	Major	0	63	68.98	0.0189
D	Major	0	53	68.98	0.0189

G	Major	0	53	68.98	0.0189
C	Major	0	63	68.98	0.0189
G	Major	0	63	62.64	0.0171
A	Major	0	63	59.35	0.0162
G	Major	7m	2	59.35	0.0162
F	Major	0	53	55.24	0.0151
A	Minor	0	53	54.13	0.0148
C	Major	0	53	53.40	0.0146
B	Major	0	63	45.50	0.0124
E	Minor	0	53	45.50	0.0124
G	Minor	0	53	44.51	0.0122
A	Major	7m	2	43.36	0.0119
A	Major	0	53	43.36	0.0119
D	Minor	0	53	40.13	0.0110
C	Minor	0	53	38.24	0.0105
E	Major	0	63	37.95	0.0104
C	Minor	0	63	32.13	0.0088
E	Major	0	53	30.71	0.0084
D	Major	7m	2	28.20	0.0077
D	Minor	0	63	24.99	0.0068
D	Major	7m	65	22.47	0.0061
B	Minor	0	63	21.96	0.0060
A♯	Major	0	53	20.75	0.0057
A	Minor	0	63	19.35	0.0053
A♯	Major	0	63	19.29	0.0053
E	Major	7m	2	18.43	0.0050
F	Major	0	63	18.40	0.0050
B	Major	0	53	12.94	0.0035
B	Major	7m	2	12.94	0.0035
E	Diminished	7^o	7	12.78	0.0035
F	Minor	0	53	12.78	0.0035
F	Major	7m	2	12.78	0.0035
F♯	Major	0	53	12.32	0.0034
C♯	Major	0	63	12.32	0.0034
F	Minor	0	63	11.58	0.0032
B	Minor	0	53	10.91	0.0030

F♯	Major	0	63	10.91	0.0030
E	Minor	0	63	10.37	0.0028
A♯	Minor	0	63	9.97	0.0027
C	Major	7m	65	9.97	0.0027
E	Major	7m	65	9.21	0.0025
B	Diminished	7º	7	9.21	0.0025
D♯	Diminished	7º	7	7.85	0.0021
F♯	Minor	0	53	7.42	0.0020
D♯	Major	0	63	7.10	0.0019
F♯	Diminished	7º	7	6.72	0.0018
F♯	Major	7m	65	5.75	0.0016
D	Major	7m	7	5.75	0.0016
F♯	Major	7m	2	3.48	0.0010
C	Major	7m	2	3.37	0.0009
G	Minor	0	63	3.37	0.0009
F	Diminished	0	64	0.00	0.0000
C♯	Major	0	53	0.00	0.0000
A	Major	7m	65	0.00	0.0000
B	Major	7m	7	0.00	0.0000
B	Major	7m	65	0.00	0.0000
A	Major	0	2	0.00	0.0000
G	Major	7m	7	0.00	0.0000
C	Minor	7m	65	0.00	0.0000
C♯	Major	7m	65	0.00	0.0000
D	Major	7m,-5	7	0.00	0.0000
C♯	Major	7m	2	0.00	0.0000
F♯	Minor	0	63	0.00	0.0000
F♯	Major	0	2	0.00	0.0000
G	Major	7m	65	0.00	0.0000
G	Major	7m,-5	7	0.00	0.0000
C♯	Diminished	7º	7	0.00	0.0000
A♯	Minor	0	53	0.00	0.0000
A♯	Diminished	7º	7	0.00	0.0000
D♯	Minor	0	63	0.00	0.0000
C	Major	7m,-5	2	0.00	0.0000
A♯	Major	-5	53	0.00	0.0000

E	Diminished	0	63	0.00	0.0000
G♯	Diminished	7º	7	0.00	0.0000
G♯	Major	0	63	0.00	0.0000
D	Minor	7m,-5	2	0.00	0.0000
G♯	Major	7m,-5	53	0.00	0.0000
D♯	Major	0	53	0.00	0.0000
E	Diminished	7º,-5	7	0.00	0.0000
E	Major	7m	7	0.00	0.0000
G♯	Major	7m	2	0.00	0.0000
G♯	Major	0	53	0.00	0.0000
C♯	Minor	0	63	0.00	0.0000
A♯	Major	7m	2	0.00	0.0000
G♯	Diminished	0	64	0.00	0.0000
Opera 6					
Root	**Triad**	**Alteration**	**Inversion**	C_B	C'_B
C	Major	0	63	89.03	0.0348
D	Major	0	63	88.82	0.0347
G	Major	0	53	87.34	0.0342
A♯	Major	0	63	76.84	0.0301
F	Major	0	53	74.88	0.0293
C	Major	7m	65	70.64	0.0276
A	Major	7m	2	63.98	0.0250
D	Major	7m	65	62.89	0.0246
A	Major	0	63	57.61	0.0225
G	Major	0	63	56.97	0.0223
G	Major	7m	2	54.47	0.0213
G	Minor	0	53	37.61	0.0147
A♯	Major	0	53	36.94	0.0145
E	Major	0	63	34.29	0.0134
A♯	Major	7m	65	29.93	0.0117
F	Major	0	63	29.80	0.0117
C	Major	0	53	28.75	0.0112
B	Major	0	63	23.48	0.0092
B	Major	7m	65	23.48	0.0092
F	Major	7m	2	23.48	0.0092
D♯	Major	0	53	22.89	0.0090

A	Major	0	53	22.06	0.0086
F	Minor	0	53	21.08	0.0082
A	Minor	0	53	18.22	0.0071
C	Minor	0	63	15.46	0.0060
D	Minor	0	53	14.81	0.0058
A	Major	7m	65	10.93	0.0043
E	Minor	0	53	10.38	0.0041
D	Major	0	53	9.59	0.0038
D	Minor	0	63	9.04	0.0035
D♯	Major	0	63	8.65	0.0034
E	Major	7m	65	6.66	0.0026
A	Major	7m	7	6.58	0.0026
E	Major	0	53	6.47	0.0025
G	Minor	0	63	5.61	0.0022
F	Major	7m	7	4.85	0.0019
G	Major	7m	7	4.79	0.0019
F♯	Minor	0	53	3.23	0.0013
C	Major	7m	7	3.17	0.0012
G	Major	7m	65	2.39	0.0009
C♯	Major	0	63	2.39	0.0009
G♯	Major	0	63	2.12	0.0008
G	Minor	-5	53	1.36	0.0005
D♯	Major	7m	2	1.30	0.0005
B	Major	7m	7	1.13	0.0004
E	Major	7m	2	0.82	0.0003
A	Minor	0	63	0.82	0.0003
C♯	Major	7m	65	0.00	0.0000
A	Diminished	0	63	0.00	0.0000
A	Diminished	0	53	0.00	0.0000
B	Diminished	0	63	0.00	0.0000
E	Minor	7m	7	0.00	0.0000
F	Diminished	7º	7	0.00	0.0000
F	Major	7m	65	0.00	0.0000
A	Major	0	65	0.00	0.0000
F♯	Diminished	7º	7	0.00	0.0000
F♯	Diminished	0	63	0.00	0.0000

C	Major	-5	53	0.00	0.0000
G♯	Diminished	0	63	0.00	0.0000
A	Minor	-5	53	0.00	0.0000
E	Diminished	7º	7	0.00	0.0000
D♯	Major	7m	65	0.00	0.0000
G♯	Major	0	53	0.00	0.0000
G	Major	0	2	0.00	0.0000
C	Major	0	65	0.00	0.0000
B	Minor	-5	53	0.00	0.0000
B	Major	7m	2	0.00	0.0000
D	Major	7m	7	0.00	0.0000
D	Major	7m	2	0.00	0.0000
A♯	Minor	0	63	0.00	0.0000
A	Major	7m	43	0.00	0.0000
G	Diminished	7º	7	0.00	0.0000
C	Major	7m	2	0.00	0.0000
Opera 7					
Root	**Triad**	**Alteration**	**Inversion**	C_B	C'_B
C	Major	0	63	66.15	0.0519
D	Major	0	63	66.15	0.0519
G	Major	0	63	66.15	0.0519
F	Major	0	53	66.15	0.0519
A♯	Major	0	63	66.15	0.0519
A	Major	0	63	52.20	0.0409
F	Major	0	63	50.35	0.0395
G	Major	7m	2	25.34	0.0199
G	Major	0	53	24.85	0.0195
D	Major	0	53	14.66	0.0115
F	Major	7m	7	9.79	0.0077
A	Major	7m	2	8.74	0.0069
E	Diminished	0	53	8.21	0.0064
D♯	Major	0	63	8.15	0.0064
C	Major	0	53	8.12	0.0064
D♯	Major	0	53	8.09	0.0063
A♯	Major	7m	7	7.45	0.0058
E	Major	7m	7	5.29	0.0041

G♯	Major	0	63	5.29	0.0041
E	Major	7m	2	5.29	0.0041
C	Major	7m	43	5.09	0.0040
A	Major	0	53	4.88	0.0038
F	Major	7m	2	3.62	0.0028
E	Major	0	63	3.33	0.0026
A	Major	-5	53	3.12	0.0024
G♯	Major	0	53	2.87	0.0022
C	Major	-5	53	1.90	0.0015
D	Major	7m	2	1.59	0.0012
A♯	Major	-5	53	1.14	0.0009
G	Major	7m	7	1.00	0.0008
C	Major	7m	65	1.00	0.0008
B	Major	0	63	0.36	0.0003
E	Major	0	53	0.36	0.0003
A	Major	7m,-5	7	0.19	0.0002
G	Major	0	7	0.00	0.0000
B	Diminished	0	53	0.00	0.0000
B	Diminished	0	64	0.00	0.0000
F♯	Diminished	0	53	0.00	0.0000
F	Major	7m	65	0.00	0.0000
C♯	Diminished	0	53	0.00	0.0000
D	Major	7m	65	0.00	0.0000
A	Diminished	0	64	0.00	0.0000
A	Major	7m	7	0.00	0.0000
F♯	Diminished	0	64	0.00	0.0000
E	Diminished	0	63	0.00	0.0000
D	Diminished	0	64	0.00	0.0000
D♯	Major	-5	53	0.00	0.0000
D♯	Major	0	64	0.00	0.0000
D	Major	7m	7	0.00	0.0000
A♯	Major	7m,-5	7	0.00	0.0000
F	Major	0	64	0.00	0.0000
F	Major	7m	43	0.00	0.0000

Opera 8

Root	**Triad**	**Alteration**	**Inversion**	C_B	C'_B

G	Major	0	53	29.31	0.0357
C	Major	0	63	29.31	0.0357
D	Major	0	63	29.31	0.0357
D	Major	0	53	25.99	0.0317
A♯	Major	0	63	21.83	0.0266
G	Major	0	63	15.12	0.0184
G♯	Major	0	63	14.55	0.0177
F	Major	0	63	14.31	0.0174
A♯	Major	0	53	14.07	0.0172
A	Major	0	63	11.71	0.0143
C	Major	0	53	11.26	0.0137
F	Major	0	53	11.26	0.0137
D♯	Major	0	53	8.09	0.0099
D♯	Major	0	63	7.36	0.0090
B	Major	0	63	6.04	0.0074
E	Major	0	53	6.04	0.0074
A	Major	7m	2	5.67	0.0069
G	Major	7m	2	5.43	0.0066
A	Major	0	53	4.38	0.0053
C	Major	7m	65	3.25	0.0040
B	Major	0	53	1.73	0.0021
F	Major	7m	2	1.60	0.0020
C	Major	7m	7	1.20	0.0015
E	Major	0	63	1.20	0.0015
A♯	Major	7m	7	0.44	0.0005
G	Major	7m	65	0.27	0.0003
E	Major	7m	2	0.21	0.0003
A♯	Major	7m	65	0.06	0.0001
C♯	Major	0	53	0.00	0.0000
C♯	Major	7m	7	0.00	0
G♯	Major	0	53	0.00	0
A♯	Major	7m	2	0.00	0
D	Major	7m	65	0.00	0
B	Major	7m	2	0.00	0
D♯	Major	7m	65	0.00	0
G	Minor	0	63	0.00	0

C	Major	7m	2	0.00	0
B	Major	7m	65	0.00	0
D	Major	7m	2	0.00	0
F♯	Major	7m	2	0.00	0
F♯	Major	0	53	0.00	0
G♯	Major	7m	65	0.00	0

Table 4-7.: Nodes of each opera with their respective values of Freeman betweenness centrality C_B and normalized betweenness centrality C'_B. The nodes have been ordered from the highest to the lowest betweenness centrality.

5. Conclusions

The conclusions are divided in those related with the results of each chapter of the research, and the general conclusions.

On the practices and tuning in the marimba de chonta music

We analyzed the theoretical and experimental spectra of the *marimba de chonta* in order to understand the tuning and the musical practices carried out with this instrument. The harmony employed in the *marimba de chonta* practices is based on relative geometrical distances between bars, generating a transposition principle that keeps the geometrical distances.

We find that the tunings of the musical intervals produced at different relative distances do not follow any specific mathematical rule. Rather, the tunings are distributed around average values that follow equi-heptatonic, equi-octatonic and equi-enneatonic scales. The average values correspond to the averages of the frequency ratios between bars that are separated by the same geometrical distance, while the deviation of the tunings with respect to the average values are expressed in terms of an uncertainty. For traditional marimbas following equi-heptatonic, equi-octatonic, and equi-enneatonic averages in the musical intervals, the values of these uncertainties are approximately 5.3 %, 7.1 %, and 7.7 % of relative error with respect to the average values, respectively.

A comparison with a previous study carried out using data collected in the 1980's, indicates that the tuning of the *marimba de chonta* with musical intervals that follow equi-heptatonic averages has changed. The average values that define the octaves move from low octaves, with frequency ratios strictly smaller than 2.00, to octaves with frequency ratios close to 2.00. This quantitative mathematical analysis, together with the first-hand testimonies of musicians and instrument makers, suggests that this change is due to the influence of Western music.

One of the main approaches for understanding tuning is based on the acoustical properties of the musical instrument: the tuning must be done in the positions of the local minima of dissonance generated by the spectrum of the timbre of the musical instrument. We analyzed the dissonance curves coming from the spectrum of the *marimba de chonta* using three different dissonance-measuring models. We found that the tunings of the musical intervals approximately follow this principle, as the most used harmonic intervals are placed in a broad minimum of dissonance. However, the sharp minima of dissonance, characterized by narrow peaks, are located in regions with large variations in dissonance. These regions are avoided in *marimba de chonta* music. As the transposition practice implies deviations in the tuning of musical intervals, the use of these regions would lead to large changes in the dissonance level that would modify the tension-relaxation sequences. This rational is reinforced by the observed suppression of intervals close to the seconds in the musical practices of the *marimba de chonta*, as they are placed in a region with a large slope peak with a maximum of dissonance.

The main relevance of the broad minimum of dissonance is that two bars separated by a specific geometric distance in two regions close to each other in the register, keeps similar levels of dissonance. This feature allows the transposition of full arrays of intervals maintaining the main harmonic and melodic information in the tension-relaxation sequences. A general conclusion beyond the case of *marimba de chonta* music, is that if a transposition practice involves deviations in the tuning of musical intervals, regions with large changes in the dissonance level for small variations in the tuning must be avoided, even if they correspond to minima of dissonance.

On the generalization of the interval size and its application to melody

The concept of the musical interval size was extended using two physical quantities: the difference between the fundamental frequencies of pitches and the difference in the squares of the fundamental frequencies. We explored the characteristics of these quantities in three different musical scales: the just, Pythagorean, and 12-TET. We found that both quantities contain information on the size of the interval and its location in the register, owing to the existence of a relationship between the construction rules of the scales and the sizes of intervals, which becomes linear in the most relevant regime for utilization in music. These quantities can be measured with different precision levels, allowing us in many cases to lift a degeneracy associated with the traditional musical interval size concept, in the sense that it cannot distinguish intervals of the same size located in different locations of the register.

The expected values of the two physical quantities were shown to be macroscopic quantities that contain relevant musical information. Specifically, they correspond to a generalization of the traditional mean musical interval size, as the expected values also take into account the mean location and the dispersion of the intervals in the register.

A link between the theory of tonal consonance and the expected values of the two considered physical quantities was developed. Specifically, knowing the mean location of musical intervals with a given size in the register, and the corresponding variance, it is possible to measure both the expected values and the mean dissonance properties of a musical piece, owing to the use of musical intervals produced by an instrument with a particular timbre.

In order to verify the usefulness of this formalism, it was applied to melodies. The frequency of occurrences of melodic intervals in 20 melodic lines from seven masterpieces of Western tonal music was measured, and the probability distributions of both quantities were obtained. In all cases we obtained non-continuous asymmetric Laplace distributions. In addition, the entropy associated with the appearances of melodic intervals during the progression of a melodic line increases up to a limiting value, which is smaller than the entropy of a random composition. In order to explain these empirical findings, a statistical model based on the minimization of the relative entropy under constraints was proposed for the difference in the squares of the fundamental frequencies. Two constraints are associated with the number of ascending, descending, and unison intervals, and the two other constraints correspond to expected values arising from the average magnitude of the physical quantity, and the asymmetry in the magnitudes of ascending and descending intervals. The model includes two Lagrange multipliers. The first locates the region in the register where the melody is played, giving information on musical processes such as transposition. The second captures asymmetry patterns between ascending and descending intervals. For 13 of the 20 studied melodic lines, the first Lagrange multiplier is related to the mean dissonance level of the melodic line, connecting macroscopic statistical properties with psychoacoustic features of the system.

The presented findings show that for the studied musical pieces the selection of melodic intervals made by the composers, including their locations in the register, can be modeled as a tight compromise between order and disorder, with a principle of entropy extremalization constrained by macroscopic quantities with musical meanings, which embed microscopic musical rules, as well as the composer's preferences. While many complex systems exhibit emergent properties associated to non-physical quantities, this work employs physical parameters to trace a connection between the properties of a musical piece as a whole, and the psychoacoustic properties of its individual elements.

On the statistical analysis of chords in *secco recitatives*

The statistical analysis of *secco recitatives* from eight operas was carried out defining chords as the constituent elements. Amongst the most interesting findings are: First, rank distributions for the use of chords are similar to those reported for single musical notes, second, roots of the major chords exhibit a sinusoidal behavior in a space in which adjacent pitches are organized by fifths, third, the presence of an asymmetry in the transitions between major and minor chords (and vice versa), and fourth, the transitions between the basses of the chords suggests that composers tend to minimize the bass movement. From the exploration using graphs, the rank distribution of the Freeman betweenness centrality describes the way in which composers use chords inside the *secco recitatives* of an opera. Fast decays in this distribution means that only few chords are common to most *secco recitatives* of the opera, indicating that these chords are essential for the harmony of the entire piece. On the other hand, slow decays indicate that there are several chords that appear in many *secco recitatives* of the opera. Fast decays were found for the operas Acis and Galatea, Cinderella, and the Barber of Seville, moderate decays for The Coronation of Poppea and The Marriage of Figaro, and slow decays for The Marriage of Hercules and Hebe, Mithridates, King of Pontus, and Apollo and Hyacinthus. These results give new information about *secco recitatives*, which interpretation practices were poorly registered at the time, and now days remain largely unknown. Additionally, they show that the methodology developed allows to capture new features about the harmony of musical pieces.

General conclusions

This study explored complexity in music at different scales, from a pure tone to complete musical pieces.

The first issue is about the microscopic elements found in music. The course of this exploration involved the use of several representations for the constituent elements of musical pieces. First there was one pure tone, then the superposition of two pure tones, followed by complex tones made of many pure tones, and finally to sets of complex tones associated to tunings, musical intervals and chords.

The second issue is the rules governing constituent elements of musical pieces. The rules of music vary in nature; some come from the physical properties of sound and how we perceive consonance and dissonance as pleasant or unpleasant sensations, and some come from cultural constraints and the creativity of the composer. These rules define different levels of complexity, from the main constituent elements defined below to the whole piece.

From the study of the *marimba de chonta* music, we found that the tunings of this traditional marimba music are different from those found in Western music, and that they cannot be inferred by a simple exploration of the minima of dissonance without taking into account the musical practices. One of the most relevant characteristics of the *marimba de chonta* music is a transposition practice that keeps the same relative distance between bars. The transposition practice allows the coupling of the marimba music with the vocal preferences of female singers. An important result is the understanding that this practice and those that define the pairs of bars that can be played simultaneously, are related to the consonance properties of the *marimba de chonta*.

The *marimba de chonta* music is considered by UNESCO as a “Heritage of Humanity.” We found that the low octaves of this music (intervals with mean fundamental frequency ratios slightly smaller than 2.00), that were observed in previous studies, have been lost in favor of octaves with mean frequency ratios close to 2.00, which implies a severe modification of this music. Some results of this study are intended to be used in public policy making, as currently some programs of the Colombian government are enhancing the risk of disappearance of this music. Specifically, the Government promotes musical education in the regions where *marimba de chonta* music is present, the instruments provided are commonly tuned using diatonic 12-TET marimbas, and the courses emphasize in the Western music theory associated with tempered instruments. Additionally, the methodology used in this study can compare an unknown marimba with a set of traditional marimbas, which has the potential to prevent some alterations of the traditional tuning.

The study of melodic lines provides evidence of a connection between the mean dissonance level associated with a melodic line, and a macroscopic quantity coming from the organizational features of a musical piece as a whole. The organizational features are captured in a model that extremalizes entropy subject to microscopic and macroscopic constraints, which approximately reproduces the final selection made by composers of musical intervals in the melodic lines studied. The microscopic constraints are due to the tuning of the musical instrument. The macroscopic constraints are related to the mean position and the dispersion in the register of melodic intervals.

The study of the *secco recitatives* of eight operas shows that chords have similar organizational features than single musical notes. Besides, the roots of the major chords follow a sinusoidal behavior, and the bass movement of the chords suggests that composers tend to minimize the bass movement. Additionally, an asymmetry associated to the transitions between major and minor chords (and vice versa) was found. Finally, from the exploration using graphs, the rank distribution of the Freeman betweenness centrality was found to be useful in order to capture the way in which composers use chords inside the *secco recitatives* of an opera.

Two last statements:

- The phenomenon of tonal consonance is present in timbre, tuning, and several rules at different levels of complexity in music.

- Understanding melody as a tension between order and disorder, where rules constraint the system and the dynamics is ruled by the maximization of entropy, recalls the tension between Will and Representation, as posted by Schopenhauer [6].

Appendix A. *Marimba de chonta* tunings by Carlos Miñana

*In the analysis this pitch has been taken as A, ignoring the symbol "?".


24
Marimba 6м
30
Marimba 7м
36
Marimba 8м
42
Marimba 9м

Appendix B. Scores of the musical pieces played in *Marimba de chonta*

Musical piece Nº	**Musician**
1	Dioselino Rodríguez
2	
3	
4	
5	
6	Genaro Torres
7	Francisco Torres

Musical piece # 1


59
2
2
2
2
69
77
2
2
2
85
2
2
93
2
2
2
2
2
105
2

Musical piece # 2

Musical piece # 3


49
55
61
67
73
79
85
91
rit.
a tempo

Musical piece # 4


61
67
75
83
89
97
105
111
119


125
131
2
2
139
2
2
147
2
153

Musical piece # 5


54
60
66
72
rit.

Musical piece # 6


69
2
2
2
77
2
2
2
85
2
2
93
2

Musical piece # 7

Appendix C. Data in *cents* of the tuning of the *marimbas de chonta* in the Miñana study and the present one

		Marimba number - Present study				
s		**1**	**2**	**3**	**4**	**5**
1	**Min±ΔMin**	95 ± 4	130 ± 9	101 ± 8	78 ± 15	120 ± 16
	Max±ΔMax	215 ± 13	239 ± 4	268 ± 8	238 ± 9	222 ± 7
	Avg.±ΔAvg.	171 ± 6	176 ± 8	171 ± 9	172 ± 12	174 ± 10
	σ	45	33	43	38	36
	T. U.	97	75	96	88	82
2	**Min±ΔMin**	297 ± 6	295 ± 4	258 ± 9	247 ± 17	298 ± 13
	Max±ΔMax	404 ± 3	429 ± 9	411 ± 7	441 ± 8	431 ± 19
	Avg.±ΔAvg.	345 ± 6	345 ± 8	343 ± 9	342 ± 12	352 ± 10
	σ	46	39	45	51	34
	T. U.	98	85	99	113	79
3	**Min±ΔMin**	480 ± 2	467 ± 11	433 ± 7	429 ± 18	471 ± 8
	Max±ΔMax	595 ± 8	573 ± 8	588 ± 8	579 ± 8	559 ± 16
	Avg.±ΔAvg.	515 ± 5	514 ± 8	513 ± 9	511 ± 12	525 ± 10
	σ	37	32	39	43	22
	T. U.	79	72	87	97	55
4	**Min±ΔMin**	600 ± 3	638 ± 5	599 ± 8	603 ± 10	639 ± 16
	Max±ΔMax	716 ± 3	732 ± 8	752 ± 8	741 ± 8	755 ± 17
	Avg.±ΔAvg.	689 ± 5	688 ± 8	684 ± 9	685 ± 12	701 ± 10
	σ	31	35	40	42	32
	T. U.	67	78	90	96	75
5	**Min±ΔMin**	792 ± 2	782 ± 6	770 ± 7	768 ± 10	821 ± 13
	Max±ΔMax	913 ± 11	916 ± 8	945 ± 8	975 ± 9	942 ± 15
	Avg.±ΔAvg.	859 ± 5	859 ± 8	855 ± 9	862 ± 12	879 ± 10
	σ	48	42	44	53	37
	T. U.	102	91	98	118	84
6	**Min±ΔMin**	991 ± 2	978 ± 12	947 ± 8	960 ± 13	1001 ± 8
	Max±ΔMax	1105 ± 4	1084 ± 9	1106 ± 9	1140 ± 9	1088 ± 13
	Avg.±ΔAvg.	1031 ± 5	1032 ± 8	1027 ± 9	1033 ± 12	1056 ± 10
	σ	47	34	40	51	23
	T. U.	99	75	89	114	56
7	**Min±ΔMin**	1190 ± 10	1184 ± 11	1131 ± 10	1153 ± 13	1182 ± 12
	Max±ΔMax	1210 ± 11	1231 ± 5	1266 ± 7	1282 ± 8	1259 ± 7
	Avg.±ΔAvg.	1199 ± 6	1209 ± 8	1198 ± 9	1203 ± 12	1227 ± 10
	σ	6	15	32	41	25
	T. U.	18	39	72	93	61

		Marimba number - Present study			
s		**6**	**7**	**8**	**Avg.*$_1$**
1	**Min±ΔMin**	35 ± 10	111 ± 11	105 ± 14	97 ± 12
	Max±ΔMax	235 ± 9	223 ± 6	238 ± 18	238 ± 9
	Avg.±ΔAvg.	162 ± 6	170 ± 15	174 ± 10	171 ± 10
	σ	49	34	36	38
	T. U.	105	82	81	87
2	**Min±ΔMin**	251 ± 7	294 ± 22	239 ± 15	269 ± 12
	Max±ΔMax	403 ± 5	407 ± 9	402 ± 5	418 ± 9
	Avg.±ΔAvg.	331 ± 6	335 ± 15	342 ± 10	341 ± 10
	σ	48	33	49	43
	T. U.	103	81	108	95
3	**Min±ΔMin**	436 ± 8	434 ± 20	372 ± 15	435 ± 12
	Max±ΔMax	617 ± 9	568 ± 7	592 ± 10	582 ± 10
	Avg.±ΔAvg.	499 ± 6	502 ± 15	509 ± 10	510 ± 10
	σ	42	37	61	39
	T. U.	90	89	133	89
4	**Min±ΔMin**	604 ± 5	610 ± 25	551 ± 15	607 ± 12
	Max±ΔMax	730 ± 9	719 ± 10	757 ± 11	741 ± 10
	Avg.±ΔAvg.	667 ± 6	669 ± 15	678 ± 10	682 ± 10

	σ	39	32	65	41
	T. U.	85	79	139	92
5	**Min±ΔMin**	753 ± 5	783 ± 15	716 ± 14	771 ± 10
	Max±ΔMax	910 ± 5	893 ± 8	935 ± 11	931 ± 9
	Avg.±ΔAvg.	831 ± 6	840 ± 15	854 ± 10	854 ± 10
	σ	52	39	65	47
	T. U.	110	93	139	105
6	**Min±ΔMin**	919 ± 8	952 ± 22	927 ± 14	955 ± 12
	Max±ΔMax	1119 ± 10	1110 ± 8	1100 ± 9	1107 ± 9
	Avg.±ΔAvg.	1001 ± 6	1007 ± 15	1030 ± 10	1027 ± 10
	σ	48	43	58	42
	T. U.	103	101	125	95
7	**Min±ΔMin**	1131 ± 6	1095 ± 17	1100 ± 13	1140 ± 12
	Max±ΔMax	1246 ± 9	1222 ± 8	1265 ± 10	1253 ± 8
	Avg.±ΔAvg.	1169 ± 6	1172 ± 15	1208 ± 10	1198 ± 10
	σ	34	41	50	34
	T. U.	74	97	110	78

Table C-1.: Minimum, maximum, average, standard deviation, and total uncertainty of the frequency ratio for pairs of bars separated by different distances in the diatonic 12-TET marimba, and the traditional marimbas with equi-heptatonic averages, recorded in the present study. Data in *cents*. "s" is the distance between bars in steps The symbol, "σ" is the standard deviation, "Avg." refers to the average, "Avg.$*_1$" refers to the average of the traditional marimbas 2, 3, 4, 5, 6, 7, and 8, and "T. U." is the total uncertainty.

		Marimba number - Present study			
s		**9**	**10**	**Avg.*$_2$**	**11**
1	**Min±ΔMin**	77 ± 13	-9 ± 57	34 ± 34	65 ± 19
	Max±ΔMax	253 ± 11	245 ± 82	249 ± 46	236 ± 15
	Avg.±ΔAvg.	154 ± 18	155 ± 23	154 ± 20	134 ± 23
	σ	49	65	57	50
	T. U.	115	154	135	124
2	**Min±ΔMin**	241 ± 21	235 ± 52	238 ± 37	142 ± 24
	Max±ΔMax	370 ± 10	414 ± 34	392 ± 22	354 ± 30
	Avg.±ΔAvg.	311 ± 18	314 ± 22	312 ± 20	268 ± 23
	σ	33	50	42	63
	T. U.	83	123	103	148
3	**Min±ΔMin**	389 ± 11	309 ± 58	350 ± 34	267 ± 35
	Max±ΔMax	582 ± 9	590 ± 25	586 ± 17	478 ± 11
	Avg.±ΔAvg.	464 ± 18	471 ± 21	468 ± 19	398 ± 22
	σ	50	71	60	63
	T. U.	118	162	140	148
4	**Min±ΔMin**	553 ± 24	554 ± 54	553 ± 39	407 ± 36
	Max±ΔMax	684 ± 10	732 ± 13	708 ± 11	599 ± 14
	Avg.±ΔAvg.	620 ± 18	637 ± 20	629 ± 19	526 ± 22
	σ	44	47	45	43
	T. U.	105	114	110	109
5	**Min±ΔMin**	704 ± 16	683 ± 43	693 ± 30	586 ± 37
	Max±ΔMax	881 ± 11	914 ± 13	898 ± 12	746 ± 14
	Avg.±ΔAvg.	769 ± 18	797 ± 20	783 ± 19	660 ± 23
	σ	49	63	56	48
	T. U.	117	147	132	119
6	**Min±ΔMin**	851 ± 13	861 ± 40	856 ± 27	714 ± 22
	Max±ΔMax	985 ± 20	1033 ± 9	1009 ± 14	904 ± 14
	Avg.±ΔAvg.	927 ± 18	953 ± 21	940 ± 19	797 ± 23
	σ	46	52	49	65
	T. U.	111	125	118	153
7	**Min±ΔMin**	1015 ± 19	985 ± 41	1000 ± 30	790 ± 20
	Max±ΔMax	1166 ± 11	1215 ± 9	1191 ± 10	996 ± 15
	Avg.±ΔAvg.	1080 ± 18	1111 ± 21	1096 ± 20	930 ± 23
	σ	42	70	56	60
	T. U.	103	161	132	144
8	**Min±ΔMin**	1180 ± 14	1180 ± 42	1180 ± 28	916 ± 31
	Max±ΔMax	1314 ± 17	1338 ± 8	1326 ± 13	1155 ± 15
	Avg.±ΔAvg.	1236 ± 18	1266 ± 22	1251 ± 20	1061 ± 23
	σ	48	53	50	56
	T. U.	114	128	121	135
9	**Min±ΔMin**	—	—	—	1122 ± 16
	Max±ΔMax	—	—	—	1284 ± 14
	Avg.±ΔAvg.	—	—	—	1191 ± 22
	σ	—	—	—	50
	T. U.	—	—	—	122

Table C-2.: Minimum, maximum, average, standard deviation, and total uncertainty of the frequency ratio for pairs of bars separated by different distances in the traditional marimbas with equi-octatonic and equi-enneatonic averages recorded in the present study. Data in *cents*. "s" is the distance between bars in steps The symbol, "σ" is the standard deviation, "Avg." refers to the average, "Avg.*$_2$" refers to the average of the traditional marimbas 9 and 10, and "T. U." is the total uncertainty.

		Marimba number - Miñana study									
s		$\mathbf{1}_M$	$\mathbf{2}_M$	$\mathbf{3}_M$	$\mathbf{4}_M$	$\mathbf{5}_M$	$\mathbf{6}_M$	$\mathbf{7}_M$	$\mathbf{8}_M$	$\mathbf{9}_M$	**Avg.***
	Min (±5 *cents*)	70	100	135	135	110	70	110	105	80	102
	Max (±5 *cents*)	260	250	220	190	210	200	240	240	270	231
1	**Avg. (±5 *cents*)**	173	170	161	164	159	150	168	161	156	163
	σ	48	45	23	13	23	41	30	35	44	34
	T. U.	101	95	52	31	51	86	65	74	94	72
	Min (±5 *cents*)	285	300	285	300	230	180	255	240	180	251
	Max (±5 *cents*)	450	420	365	360	370	390	390	405	405	395
	Avg. (±5 *cents*)	340	341	325	328	320	307	336	321	316	326
	σ	41	39	26	16	36	57	31	45	50	38
2	**T. U.**	88	84	56	36	78	119	68	94	104	81
	Min (±5 *cents*)	480	410	445	460	370	280	445	405	340	405
	Max (±5 *cents*)	605	590	520	530	530	580	560	550	595	562
	Avg. (±5 *cents*)	514	507	489	492	484	465	500	486	472	490
	σ	37	51	41	18	41	72	32	45	60	44
	T. U.	79	107	86	42	88	149	70	95	125	93
	Min (±5 *cents*)	600	650	605	620	530	470	600	550	480	568
	Max (±5 *cents*)	810	720	710	700	680	720	730	735	765	730
4	**Avg. (±5 *cents*)**	682	681	653	657	650	632	668	650	635	656
	σ	56	27	26	20	38	63	39	51	73	43
	T. U.	117	59	58	44	80	130	82	108	151	92
	Min (±5 *cents*)	780	780	770	790	740	670	760	735	660	743
	Max (±5 *cents*)	930	900	860	855	850	880	900	915	900	888
5	**Avg. (±5 *cents*)**	847	848	817	819	817	797	834	810	798	821
	σ	45	44	29	18	30	59	38	54	74	43
	T. U.	94	93	62	42	65	123	81	113	153	92
	Min (±5 *cents*)	960	970	940	950	900	860	935	880	820	913
	Max (±5 *cents*)	1120	1080	1005	1010	1020	1060	1075	1060	1070	1056
6	**Avg. (±5 *cents*)**	1016	1017	981	983	978	961	1002	972	963	986
	σ	52	40	21	19	33	56	43	52	81	44
	T. U.	109	84	47	42	71	117	91	109	167	93
	Min (±5 *cents*)	1100	1120	1115	1110	1050	1000	1080	1075	970	1070
	Max (±5 *cents*)	1305	1240	1170	1170	1190	1210	1235	1195	1245	1218
7	**Avg. (±5 *cents*)**	1179	1179	1143	1146	1142	1119	1170	1132	1132	1149
	σ	58	45	17	20	38	60	44	38	79	44
	T. U.	121	95	39	46	81	126	92	81	163	94

Table C-3.: Minimum, maximum, average, standard deviation and total uncertainty of the frequency ratio for pairs of bars separated by different distances in the traditional marimbas recorded by Miñana. Data in *cents*. "s" is the distance between bars in steps The symbol, "σ" is the standard deviation, "Avg." refers to the average, "T. U." is the total uncertainty, and "Avg.*" refers to the average of all marimbas.

Appendix D. Minimization of the relative entropy subject to constraints

Extremalization process

In order to carry out the extremalization process of the quantity $\sum_{k=1}^{N} p_k ln\,(p_k/q_k)$ subject to constraints, with N an even number, we used the Lagrange multipliers method. In this method, we want to know the extremes of a function $f(p_1, p_2, ..., p_N)$ subject to l constraints: $g_1(p_1, p_2, ..., p_N) = 0$, $g_2(p_1, p_2, ..., p_N) = 0$,..., $g_l(p_1, p_2, ..., p_N) = 0$.

In our case:

$$f(p_1, p_2, ..., p_N) = \sum_{k=1}^{N} p_k ln(p_k/q_k) \tag{D-1}$$

$$g_1(p_1, p_2, ..., p_N) = \sum_{k=1}^{N/2} p_k - (\tilde{p}_d + \tilde{p}_u) = 0 \tag{D-2}$$

$$g_2(p_1, p_2, ..., p_N) = \sum_{k=\frac{N}{2}+1}^{N} p_k - (\tilde{p}_a + \tilde{p}_u) = 0 \tag{D-3}$$

$$g_3(p_1, p_2, ..., p_N) = \sum_{k=1}^{N} p_k|\varepsilon_k| - \tilde{p}_u|\varepsilon_{N/2}| - \langle|\varepsilon|\rangle = 0 \tag{D-4}$$

$$g_4(p_1, p_2, ..., p_N) = \frac{1}{\tilde{p}_d}\sum_{k=1}^{N/2} p_k\varepsilon_k + \frac{1}{\tilde{p}_a}\sum_{k=\frac{N}{2}+1}^{N} p_k\varepsilon_k + |\varepsilon_{N/2}|\left(\frac{\tilde{p}_u}{\tilde{p}_d} - \frac{\tilde{p}_u}{\tilde{p}_a}\right) - \langle\varepsilon\rangle = 0, \tag{D-5}$$

with $\langle\varepsilon\rangle \equiv \langle\varepsilon_{>0}\,\rangle + \langle\varepsilon_{<0}\rangle$ in equation (D-5).

Then, we construct the auxiliary function L containing the Lagrange multipliers λ_-, λ_+, λ_1, and λ_2 (one per constraint):

$$\begin{aligned} L(p_1, p_2, ..., p_N, \lambda_-, \lambda_+, \lambda_1, \lambda_2) = f(p_1, p_2, ..., p_N) + (\lambda_-)g_1(p_1, p_2, ..., p_N) + \\ (\lambda_+)g_2(p_1, p_2, ..., p_N) + (\lambda_1)g_3(p_1, p_2, ..., p_N) + (\lambda_2)g_4(p_1, p_2, ..., p_N). \end{aligned} \tag{D-6}$$

In order to find the extremes, we have to solve the equations system for the set of N independent probabilities p_k and the Lagrange multipliers:

$$\frac{\partial L}{\partial p_k} = 0, (k \in [1, N]); \quad \frac{\partial L}{\partial \lambda_-} = 0; \quad \frac{\partial L}{\partial \lambda_+} = 0; \quad \frac{\partial L}{\partial \lambda_1} = 0; \quad \frac{\partial L}{\partial \lambda_2} = 0. \tag{D-7}$$

In the case of the differentiation with respect to the Lagrange multipliers, we obtain the constraints:

$$\begin{gathered}\frac{\partial L}{\partial \lambda_-} = g_1(p_1, p_2, ..., p_N) = 0; \quad \frac{\partial L}{\partial \lambda_+} = g_2(p_1, p_2, ..., p_N) = 0; \\ \frac{\partial L}{\partial \lambda_1} = g_3(p_1, p_2, ..., p_N) = 0; \quad \frac{\partial L}{\partial \lambda_2} = g_4(p_1, p_2, ..., p_N) = 0.\end{gathered} \tag{D-8}$$

In the case of the probabilities, $\frac{\partial L}{\partial p_k} = 0$, we have:

$$\begin{gathered}L = \sum_{k=1}^{N} p_k ln\left(\frac{p_k}{q_k}\right) + \lambda_-\left[\sum_{k=1}^{N/2} p_k - (\tilde{p}_d + \tilde{p}_u)\right] + \lambda_+\left[\sum_{k=\frac{N}{2}+1}^{N} p_k - (\tilde{p}_a + \tilde{p}_u)\right] + \\ \lambda_1\left[\sum_{k=1}^{N} p_k|\varepsilon_k| - \tilde{p}_u|\varepsilon_{N/2}| - \langle|\varepsilon|\rangle\right] + \\ \lambda_2\left[\frac{1}{\tilde{p}_d}\sum_{k=1}^{N/2} p_k\varepsilon_k + \frac{1}{\tilde{p}_a}\sum_{k=\frac{N}{2}+1}^{N} p_k\varepsilon_k + |\varepsilon_{N/2}|\left(\frac{\tilde{p}_u}{\tilde{p}_d} - \frac{\tilde{p}_u}{\tilde{p}_a}\right) - \langle\varepsilon\rangle\right],\end{gathered} \tag{D-9}$$

then

$$\begin{gathered}L = \sum_{k=1}^{N}\left[p_k ln\left(\frac{p_k}{q_k}\right) + \lambda_1 p_k|\varepsilon_k|\right] + \sum_{k=1}^{N/2}\left(\lambda_- p_k + \frac{\lambda_2}{\tilde{p}_d} p_k\varepsilon_k\right) + \\ \sum_{k=\frac{N}{2}+1}^{N}\left(\lambda_+ p_k + \frac{\lambda_2}{\tilde{p}_a} p_k\varepsilon_k\right) - \lambda_-(\tilde{p}_d + \tilde{p}_u) - \lambda_+(\tilde{p}_a + \tilde{p}_u) - \lambda_1\tilde{p}_u|\varepsilon_{N/2}| \\ -\lambda_1\langle|\varepsilon|\rangle - \lambda_2\langle\varepsilon\rangle + \lambda_2|\varepsilon_{N/2}|\left(\frac{\tilde{p}_u}{\tilde{p}_d} - \frac{\tilde{p}_u}{\tilde{p}_a}\right).\end{gathered} \tag{D-10}$$

Then, for $k \in [1, N/2]$

$$\begin{aligned}\frac{\partial L}{\partial p_k} &= ln\left(\frac{p_k}{q_k}\right) + \left(\frac{p_k}{q_k}\right)\left(\frac{q_k}{p_k}\right) + \lambda_1|\varepsilon_k| + \lambda_- + \frac{\lambda_2}{\tilde{p}_d}\varepsilon_k \\ &= ln\left(\frac{p_k}{q_k}\right) + \lambda_1|\varepsilon_k| + \frac{\lambda_2}{\tilde{p}_d}\varepsilon_k + \lambda_-^* = 0; \quad \lambda_-^* \equiv 1 + \lambda_-.\end{aligned} \tag{D-11}$$

Now, for $k \in [(N/2)+1, N]$

$$\begin{aligned} \frac{\partial L}{\partial p_k} &= ln\left(\frac{p_k}{q_k}\right) + \left(\frac{p_k}{q_k}\right)\left(\frac{q_k}{p_k}\right) + \lambda_1|\varepsilon_k| + \lambda_+ + \frac{\lambda_2}{\tilde{p}_a}\varepsilon_k \\ &= ln\left(\frac{p_k}{q_k}\right) + \lambda_1|\varepsilon_k| + \frac{\lambda_2}{\tilde{p}_a}\varepsilon_k + \lambda_+^* = 0; \quad \lambda_+^* \equiv 1 + \lambda_+. \end{aligned} \tag{D-12}$$

Solving, we have:

$$p_k = \begin{cases} q_k exp\left(-\lambda_1|\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_d}\varepsilon_k - \lambda_-^*\right) & \text{for } k \in [1, N/2] \\ \\ q_k exp\left(-\lambda_1|\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_a}\varepsilon_k - \lambda_+^*\right) & \text{for } k \in [\frac{N}{2}+1, N]. \end{cases} \tag{D-13}$$

Using the constraints $g_1(p_1, p_2, ..., p_N)$ and $g_2(p_1, p_2, ..., p_N)$, we obtain the quantities $exp(-\lambda_-^*)$ and $exp(-\lambda_+^*)$.

Since

$$\sum_{m=1}^{N/2} q_m exp(-\lambda_-^*) exp\left(-\lambda_1|\varepsilon_m| - \frac{\lambda_2}{\tilde{p}_d}\varepsilon_k\right) = \tilde{p}_d + \tilde{p}_u, \tag{D-14}$$

then

$$exp(-\lambda_-^*) = \frac{\tilde{p}_d + \tilde{p}_u}{\sum_{m=1}^{N/2} q_m exp\left(-\lambda_1|\varepsilon_m| - \frac{\lambda_2}{\tilde{p}_d}\varepsilon_m\right)} \quad \text{for } m \in [1, N/2], \tag{D-15}$$

and since

$$\sum_{m=\frac{N}{2}+1}^{N} q_m exp(-\lambda_+^*) exp\left(-\lambda_1|\varepsilon_m| - \frac{\lambda_2}{\tilde{p}_a}\varepsilon_m\right) = \tilde{p}_a + \tilde{p}_u, \tag{D-16}$$

then

$$exp(-\lambda_+^*) = \frac{\tilde{p}_a + \tilde{p}_u}{\sum_{m=\frac{N}{2}+1}^{N} q_m exp\left(-\lambda_1|\varepsilon_m| - \frac{\lambda_2}{\tilde{p}_a}\varepsilon_m\right)} \quad \text{for } m \in [(N/2)+1, N]. \tag{D-17}$$

Finally, the probabilities p_k are given by

$$p_k = \begin{cases} \dfrac{(\tilde{p}_d + \tilde{p}_u) q_k e^{\left(-\lambda_1 |\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_d} \varepsilon_k\right)}}{\sum\limits_{m=1}^{N/2} \left[q_m e^{\left(-\lambda_1 |\varepsilon_m| - \frac{\lambda_2}{\tilde{p}_d} \varepsilon_m\right)} \right]} & \text{for } k \in [1, N/2] \\ \\ \dfrac{(\tilde{p}_a + \tilde{p}_u) q_k e^{\left(-\lambda_1 |\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_a} \varepsilon_k\right)}}{\sum\limits_{m=\frac{N}{2}+1}^{N} \left[q_m e^{\left(-\lambda_1 |\varepsilon_m| - \frac{\lambda_2}{\tilde{p}_a} \varepsilon_m\right)} \right]} & \text{for } k \in [\frac{N}{2}+1, N]. \end{cases} \tag{D-18}$$

The solution corresponds to a minimum

Now, in order to prove that the solution p_k corresponds to a minimum, we suppose that there is a $P_k \neq p_k$ that also satisfies the constraints, then

$$\sum_{k=1}^{N/2} p_k = \sum_{k=1}^{N/2} P_k = \tilde{p}_d + \tilde{p}_u \tag{D-19}$$

$$\sum_{k=\frac{N}{2}+1}^{N} p_k = \sum_{k=\frac{N}{2}+1}^{N} P_k = \tilde{p}_a + \tilde{p}_u \tag{D-20}$$

$$\sum_{k=1}^{N} p_k |\varepsilon_k| - \tilde{p}_u |\varepsilon_{N/2}| = \sum_{k=1}^{N} P_k |\varepsilon_k| - \tilde{p}_u |\varepsilon_{N/2}| = \langle |\varepsilon| \rangle \tag{D-21}$$

$$\begin{aligned} &\frac{1}{\tilde{p}_d} \sum_{k=1}^{N/2} p_k \varepsilon_k + \frac{1}{\tilde{p}_a} \sum_{k=\frac{N}{2}+1}^{N} p_k \varepsilon_k + |\varepsilon_{N/2}| \left(\frac{\tilde{p}_u}{\tilde{p}_d} - \frac{\tilde{p}_u}{\tilde{p}_a} \right) \\ = &\frac{1}{\tilde{p}_d} \sum_{k=1}^{N/2} P_k \varepsilon_k + \frac{1}{\tilde{p}_a} \sum_{k=\frac{N}{2}+1}^{N} P_k \varepsilon_k + |\varepsilon_{N/2}| \left(\frac{\tilde{p}_u}{\tilde{p}_d} - \frac{\tilde{p}_u}{\tilde{p}_a} \right) = \langle \varepsilon \rangle. \end{aligned} \tag{D-22}$$

Now, we construct two functions $D(p_k, q_k)$ and $D(P_k, q_k)$

$$D(p_k, q_k) = \sum_{k=1}^{N} p_n ln \left(\frac{p_k}{q_k} \right) \tag{D-23}$$

and

$$D(P_k, q_k) = \sum_{k=1}^{N} P_k ln \left(\frac{P_k}{q_k} \right). \tag{D-24}$$

Next, if we show that the quantity $D(P_k, q_k) - D(p_k, q_k)$ is always greater or equal to 0 for all possible P_k that satisfy the constraints, then p_k corresponds to a minimum, thus

$$
\begin{aligned}
D(P_k,q_k)-D(p_k,q_k) &= \sum_{k=1}^{N} P_k ln\left(\frac{P_k}{q_k}\right) - \sum_{k=1}^{N} p_k ln\left(\frac{p_k}{q_k}\right) \\
&= \sum_{k=1}^{N} P_k ln\left(\frac{P_k}{q_k}\right) - \left[\sum_{k=1}^{N/2} p_k ln\left(\frac{p_k}{q_k}\right) + \sum_{k=\frac{N}{2}+1}^{N} p_k ln\left(\frac{p_k}{q_k}\right)\right].
\end{aligned}
\tag{D-25}
$$

Using equation (D-13), then

$$
ln\left(\frac{p_k}{q_k}\right) = \begin{cases} -\lambda_1|\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_d}\varepsilon_k - \lambda_-^* & \text{for } k \in [1, N/2] \\ \\ -\lambda_1|\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_a}\varepsilon_k - \lambda_+^* & \text{for } k \in [\frac{N}{2}+1, N]. \end{cases}
\tag{D-26}
$$

Thus

$$
\begin{aligned}
D(P_k,q_k)-D(p_k,q_k) = \sum_{k=1}^{N} P_k ln\left(\frac{P_k}{q_k}\right) - \Bigg[\sum_{k=1}^{N/2} p_k \left(\lambda_1|\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_d}\varepsilon_k - \lambda_-^*\right) \\
+ \sum_{k=\frac{N}{2}+1}^{N} p_k \left(-\lambda_1|\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_a}\varepsilon_k - \lambda_+^*\right)\Bigg]
\end{aligned}
\tag{D-27}
$$

.

$$
\begin{aligned}
= \sum_{k=1}^{N} P_k ln\left(\frac{P_k}{q_k}\right) - \Bigg[-\lambda_-^* \sum_{k=1}^{N/2} p_k - \lambda_+^* \sum_{k=\frac{N}{2}+1}^{N} p_k - \lambda_1 \sum_{k=1}^{N} p_k|\varepsilon_k| \\
-\lambda_2 \left(\frac{1}{\tilde{p}_d}\sum_{k=1}^{N/2} p_k\varepsilon_k + \frac{1}{\tilde{p}_a}\sum_{k=\frac{N}{2}+1}^{N} p_k\varepsilon_k\right)\Bigg]
\end{aligned}
\tag{D-28}
$$

$$
\begin{aligned}
= \sum_{k=1}^{N} P_k ln\left(\frac{P_k}{q_k}\right) - \Bigg\{ - \lambda_-^*(\tilde{p}_d+\tilde{p}_u) - \lambda_+^*(\tilde{p}_a+\tilde{p}_u) - \lambda_1\left(\langle|\varepsilon|\rangle + \tilde{p}_u|\varepsilon_{N/2}|\right) \\
-\lambda_2\left[\langle\varepsilon\rangle + |\varepsilon_{N/2}|\left(\frac{\tilde{p}_u}{\tilde{p}_d} - \frac{\tilde{p}_u}{\tilde{p}_a}\right)\right]\Bigg\}
\end{aligned}
\tag{D-29}
$$

Since P_k also satisfies the constraints, then

$$
\begin{aligned}
D(P_k,q_k)-D(p_k,q_k) = \sum_{k=1}^{N}\left[P_k ln\left(\frac{P_k}{q_k}\right)\right] - \Bigg[-\lambda_-^* \sum_{k=1}^{N/2} P_k - \lambda_+^* \sum_{k=\frac{N}{2}+1}^{N} P_k \\
-\lambda_1 \sum_{k=1}^{N} P_k|\varepsilon_k| - \lambda_2\left(\frac{1}{\tilde{p}_d}\sum_{k=1}^{N/2} P_k\varepsilon_k + \frac{1}{\tilde{p}_a}\sum_{k=\frac{N}{2}+1}^{N} P_k\varepsilon_k\right)\Bigg].
\end{aligned}
\tag{D-30}
$$

$$= \sum_{k=1}^{N}\left[P_k ln\left(\frac{P_k}{q_k}\right)\right] - \left[\sum_{k=1}^{N/2} P_k\left(-\lambda_1|\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_d}\varepsilon_k - \lambda_-^*\right)\right.$$
$$\left. + \sum_{k=\frac{N}{2}+1}^{N} P_k\left(-\lambda_1|\varepsilon_k| - \frac{\lambda_2}{\tilde{p}_a}\varepsilon_k - \lambda_+^*\right)\right]. \tag{D-31}$$

Using equation (D-26):

$$D(P_k, q_k) - D(p_k, q_k) = \sum_{k=1}^{N}\left[P_k ln\left(\frac{P_k}{q_k}\right)\right] - \sum_{k=1}^{N}\left[P_k ln\left(\frac{p_k}{q_k}\right)\right]$$
$$= \sum_{k=1}^{N} P_k\left[ln\left(\frac{P_k}{q_k}\right) - ln\left(\frac{p_k}{q_k}\right)\right] = \sum_{k=1}^{N}\left[P_k ln\left(\frac{P_k}{p_k}\right)\right] \tag{D-32}$$

Now, since P_k and p_k do not satisfy a normalization constraint due to the double count of unisons, then we can construct P_k^* and p_k^* multiplying P_k and p_k by a constant factor

$$\sum_{k=1}^{N} P_k = \sum_{k=1}^{N} p_k = \tilde{p}_d + 2\tilde{p}_u + \tilde{p}_a = 1 + \tilde{p}_u; \tag{D-33}$$

$$P_k^* = \frac{P_k}{1+\tilde{p}_u} \quad ; \quad p_k^* = \frac{p_k}{1+\tilde{p}_u}, \tag{D-34}$$

then

$$\sum_{k=1}^{N} P_k^* = \sum_{k=1}^{N} p_k^* = 1. \tag{D-35}$$

Thus

$$D(P_k, q_k) - D(p_k, q_k) = (1+\tilde{p}_u)\sum_{k=1}^{N}\frac{P_k}{1+\tilde{p}_u} ln\left(\frac{\frac{P_k}{1+\tilde{p}_u}}{\frac{p_k}{1+\tilde{p}_u}}\right)$$
$$= (1+\tilde{p}_u)\sum_{k=1}^{N} P_k^* ln\left(\frac{P_k^*}{p_k^*}\right). \tag{D-36}$$

Since the quantities P_k^* and p_k^* are normalized, then the quantity $\sum_{k=1}^{N} P_k^* ln(P_k^*/p_k^*)$ corresponds to a Kullback–Leibler divergence. One of the properties of this divergence is that is always greater or equal to 0 (equal to 0 only in the case $P_k^* = p_k^*$), and as $(1+\tilde{p}_u) \geq 1$, then

$$D(P_k, q_k) - D(p_k, q_k) \geq 0, \tag{D-37}$$

showing that p_k corresponds to a minimum.

Appendix E. The Coronation of Poppea

N°1: *Caro tetto caro tetto* (pp.21-22, 11 bars)			
Root	**Triad**	**Alteration**	**Inversion**
A	Minor	0	53
0	0	0	0
A	Major	0	63
D	Minor	0	53
G	Minor	0	63
[1]A	Major	0	53
E	Major	0	63
A	Major	0	53
F	Major	0	63
[2]E	Minor	0	63
[2]D	Minor	0	63
[3]C	Major	0	63
D	Minor	7m,-5	7
B	Diminished	0	63
[4]E	Major	0	53
E	Major	7m,-5	2
A	Major	0	63
A	Major	-5	63
A	Major	7m,-5	65
[5]D	Minor	0	53
D	Minor	0	53
G	Minor	-5	53
E	Diminished	0	63
G	Major	7m,-3	7
A	Major	0	53
[6]D	Minor	0	53

[1]Bar 3: The fifth of the chord is made by the voice.
[2]Bar 6: The root of the chord arrives late.
[3]Bar 7: The root of the chord arrives late.
[4]Bar 7: The fifth of the chord is made by the voice.
[5]Bar 9: The third of the chord is made by the voice previously.
[6]Bar 11: The chord is assumed as minor.

N°2: *Ma che veggio, infelice?* (pp. 27-31, 48 bars)			
Root	**Triad**	**Alteration**	**Inversion**
A	Minor	0	53
A	Minor	0	53
E	Major	0	63
E	Major	0	63
A	Minor	0	53
0	0	0	0
A	Minor	0	53
G	Major	0	63
C	Major	0	53
F♯	Diminished	0	63
G	Major	-5	53
0	0	0	0
D	Major	0	63
G	Major	0	53
E	Minor	0	53
C♯	Diminished	0	63
D	Major	0	63
G	Major	0	53
E	Major	-5	63
[1]A	Minor	0	53
A	Minor	0	53
F♯	Diminished	0	63
B	Minor	0	53
G	Major	0	63
C♯	Diminished	0	53
D	Minor	0	53
D	Minor	0	53
B♭	Major	0	53
0	0	0	0
D	Major	0	63
G	Major	0	53
E	Minor	7m	65
A	Major	0	53
G	Minor	7m	7
[2]D	Minor	-3	64
D	Minor	0	53
C	Major	0	53
C	Major	0	53
A	Minor	0	53
F	Major	0	53
D	Minor	0	63
[3]C♯	Diminished	0	63
D	Minor	0	53
0	0	0	0
D	Minor	0	53
D	Minor	0	53
A	Minor	0	53
E	Major	0	63
F♯	Diminished	0	63
B	Major	0	53
E	Minor	0	53
A	Minor	0	53
0	0	0	0
[4]G♯	Diminished	0	63
A	Minor	0	53
A	Minor	0	53
G♯	Diminished	0	63
G♯	Diminished	0	63
A	Minor	0	53
0	0	0	0
A	Minor	0	53
G♯	Diminished	0	63
0	0	0	0
A	Minor	0	53
0	0	0	0
A	Minor	0	53
F	Major	0	63
E	Major	0	63
F	Major	0	63
E	Major	0	63
F	Major	0	63
E	Major	0	63
A	Minor	0	53
A	Minor	0	53
D	Minor	0	53
B	Diminished	0	53

F♯	Diminished	0	63
G	Major	-5	53
G	Major	7m,-5	2
E	Major	0	53
A	Major	0	63
D	Major	0	53
G	Major	0	53
A	Major	0	63
A	Minor	0	64
A	Minor	0	53
A	Minor	0	53
A	Minor	0	53
G♯	Diminished	0	63
A	Minor	0	53
D	Major	0	53
G	Major	0	53
C	Major	0	53
G♯	Diminished	0	63
A	Minor	0	53
[5]E	Major	0	53
A	Minor	0	53
0	0	0	0

[1]Bar 7: The fifth of the chord is made by the voice.
[2]Bar 13: The note E is omitted in the analysis.
[3]Bar 18: The root of the chord arrives late.
[4]Bar 23: The root of the chord arrives late.
[5]Bar 48: The third of the chord arrives late.

N°3: ***Camerata, camerata*** (pp. 32-34, 17 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	53
F	Major	0	53
0	0	0	0
F	Major	0	53
B♭	Major	0	53
B♭	Major	0	53
F	Major	0	63
G	Major	0	53
0	0	0	0
G	Major	0	53
C	Major	0	53
C	Major	0	53
G	Major	0	53
C	Major	0	53
0	0	0	0
C	Major	0	53
A	Minor	0	53
A	Minor	0	53
A	Minor	0	53
E	Major	0	53
E	Major	0	63
A	Major	0	53
A	Minor	0	53
A	Minor	0	53
A	Minor	0	63
*G	Minor	0	63
A	Minor	0	53
A	Minor	0	53
D	Major	0	53
D	Major	0	53
A	Major	0	53
A	Major	0	53
D	Minor	0	53

*Bar 14: The root of the chord arrives late.

N°4: ***Di pur di pur che il Prence*** (pp. 36-37, 25 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
A	Minor	0	53
C	Major	0	63
E	Minor	0	53
A	Minor	0	53
0	0	0	0
G	Minor	0	53
D	Minor	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	53
F	Major	0	53
[1]G	Major	0	53
C	Major	0	53
F	Major	0	53
B♭	Major	0	53
G	Minor	7m	7
G	Minor	0	53
A	Minor	0	53
D	Minor	0	53
D	Minor	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
F	Major	0	53
D	Minor	0	53
A	Minor	0	53
D	Minor	0	53
D	Major	0	53
G	Major	0	63
F	Major	0	63
G	Major	0	53
G	Major	0	53
C	Major	0	53
A	Minor	0	53
F	Major	0	53
B♭	Major	0	53
F♯	Diminished	0	63
G	Minor	0	53
G	Minor	0	53
D	Major	0	53
D	Major	0	53
G	Minor	0	63
[2]D	Major	0	53
G	Major	0	53

[1]Bar 7: The third of the chord arrives late.
[1]Bar 25: The third of the chord arrives late.

N°5: ***Signor, signor deh, non partire*** (pp.40-44, 49 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Minor	0	53
D	Minor	0	53
A	Major	0	63
D	Minor	0	53
A	Major	0	53
0	0	0	0
[1]A	Major	0	53
A	Major	7m	65
D	Minor	0	53
D	Minor	0	53
F	Major	7Ma	43
B♭	Major	0	53
F	Major	0	63

G	Minor	7m	7
[2]G	Minor	0	53
A	Major	0	53
D	Minor	0	53
D	Minor	0	53
A	Minor	0	53
0	0	0	0
F	Major	0	53
B♭	Major	0	53
C	Major	0	53
F	Major	0	53
0	0	0	0
D	Major	0	63
D	Major	0	63
G	Minor	0	53
D	Major	0	53
D	Minor	0	53
[3]G	Minor	0	63
A	Minor	0	53
A	Minor	0	53
A	Minor	0	53
A	Minor	0	53
G	Minor	0	53
F	Major	0	53
0	0	0	0
A	Minor	0	53
C	Major	7m	43
F	Major	0	53
F	Major	0	53
B♭	Major	0	53
C	Major	0	53
F	Major	0	53
0	0	0	0
C	Major	0	53
C	Major	0	53
[4]F♯	Diminished	0	63
G	Major	0	53
D	Major	0	63
G	Minor	0	53
D	Major	7m,-3	43
G	Minor	0	63
[5]A	Major	0	53
0	0	0	0
E	Major	0	63
A	Minor	0	53
B♭	Major	7Ma	2
G	Minor	0	53
G	Major	7m	2
E	Major	0	53
E	Major	0	53
B	Diminished	0	63
B♭	Major	0	63
[6]A♭	Major	0	63
C	Minor	0	53
[7]C	Minor	-3	53
A	Major	7m,-5	7
[8]A	Major	0	53
D	Minor	0	53
D	Minor	0	53
A	Minor	0	53
C	Major	7m	43
F	Major	0	53
F	Major	0	53
B♭	Major	0	53
F	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
0	0	0	0
C	Major	0	53
G	Minor	0	53
G	Minor	0	53
G	Minor	0	53
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
A	Minor	0	53
A	Minor	0	53
E	Minor	0	53
A	Minor	0	53

[1]Bar 4: The chord is analyzed taking into account the notes of the next beat of the bar.
[2]Bar 8: The note E is omitted in the analysis.
[3]Bar 15: The root of the chord arrives late.
[4]Bar 23: The root of the chord arrives late.
[5]Bar 26: The third of the chord arrives late.
[6]Bar 32: The root of the chord arrives late.
[7]Bar 32: The note D is omitted in the analysis.
[8]Bar 32: The fifth of the chord arrives late.

N°6: ***Disprezzata Regina, Regina*** (pp. 65-68, 44 bars)			
Root	**Triad**	**Alteration**	**Inversion**
A	Minor	0	53
A	Minor	0	53
A	Minor	0	53
A	Minor	0	53
A	Minor	0	53
[1]E	Major	0	63
A	Minor	0	53
0	0	0	0
A	Minor	0	53
0	0	0	0
G	Major	7m,-5	7
E	Minor	0	63
0	0	0	0
[2]D	Minor	0	63
E	Minor	0	53
0	0	0	0
C	Major	0	53
0	0	0	0
F	Major	0	53
0	0	0	0
G	Major	0	53
0	0	0	0
C	Major	0	53
0	0	0	0
G	Minor	0	63
[3]G	Minor	-5	63
[4]F♯	Diminished	0	63
G	Major	0	53
D	Major	0	63
G	Major	0	53
G	Major	0	53
C	Major	0	53
[5]G	Major	0	63
A	Minor	0	53

F	Major	0	53
0	0	0	0
D	Minor	0	63
[6]C♯	Diminished	0	63
D	Major	0	53
D	Major	0	53
B	Minor	0	63
C	Major	7Ma,-5	7
A	Minor	0	63
B	Minor	0	53
B	Minor	0	53
D	Minor	0	64
[7]A	Minor	-3	53
A	Minor	0	53
G	Major	0	53
D	Major	0	53
B♭	Major	0	63
B♭	Major	0	63
E♭	Major	0	53
C	Minor	0	53
C	Minor	0	53
D	Major	0	53
0	0	0	0
G	Major	0	53
0	0	0	0
G	Major	0	53
C	Major	0	53
F	Major	0	53
E	Major	0	53
0	0	0	0
E	Major	0	63
E	Major	0	63
E	Major	7m	65
A	Minor	0	53
D	Major	7m	7
B	Minor	0	63
D	Major	0	53
E	Major	0	53
C	Major	0	53
C	Major	7m,-5	7
F	Major	0	53
G	Major	0	53
C	Major	0	53

[1]Bar 6: The third of the chord arrives late. Sustained note in A.
[2]Bar 10: The root of the chord arrives late.
[3]Bar 16: The note C is omitted in the analysis.
[4]Bar 16: The root of the chord arrives late.
[5]Bar 21: The third of the chord arrives late.
[6]Bar 24: The root of the chord arrives late.
[7]Bar 28: The note D is omitted in the analysis.

N°7: ***Intanto il frequente cader*** (pp. 69-71, 28 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B	Major	0	53
B	Major	0	53
D	Major	0	53
[1]A	Major	7m,-3	7
D	Major	-5	53
0	0	0	0
E	Major	0	63
A	Minor	0	53
A	Minor	0	53
A	Major	0	63
A	Major	7m	65
D	Major	0	53
D	Major	7m	2
0	0	0	0
B	Minor	0	53
0	0	0	0
[2]F♯	Diminished	0	63
F♯	Diminished	0	63
A	Minor	0	53
E	Minor	0	64
B	Major	0	53
E	Minor	0	53
0	0	0	0
C	Major	0	53
0	0	0	0
[3]C	Major	0	53
F	Major	0	53
B♭	Major	0	53
0	0	0	0
B♭	Major	0	53
A	Minor	0	53
0	0	0	0
C	Major	0	53
C	Major	0	53
G	Major	0	63
A	Minor	7m,-5	7
G	Major	-5	53
F	Major	0	53
C	Major	0	63
D	Major	-5	53
[4]D	Major	7m	2
B	Minor	-5	53
[5]A	Minor	-5	53
G	Major	0	53
C	Major	0	53
0	0	0	0
C	Major	0	53
[6]G	Major	0	63
G	Major	0	63
B	Diminished	0	53
C	Major	-5	53
0	0	0	0
B♭	Major	0	53
E♭	Major	7Ma,-5	7
C	Minor	0	63
C	Minor	0	63
E♭	Major	0	53
B♭	Major	0	63
D	Minor	0	53
D	Minor	0	53
B♭	Major	7Ma,-5	7
G	Minor	0	63
B♭	Major	7Ma,-5	7
G	Minor	0	63
A	Minor	7m,-5	7
A	Minor	7m,-5	7
F	Major	0	63
G	Minor	7m,-5	7
E	Diminished	0	63
G	Minor	7m,-5	7
E	Diminished	0	63
E	Diminished	0	63
G	Minor	0	53
D	Minor	0	64
[7]A	Major	0	53
D	Minor	-5	53

[1]Bar 3: The note D is omitted in the analysis.
[2]Bar 9: The root of the chord arrives late.
[3]Bar 11: This chord is not in a beat of the bar.
[4]Bar 15: The third of the chord is made by the voice previously.
[5]Bar 16: The note D is omitted in the analysis.
[6]Bar 18: The third of the chord arrives late.
[7]Bar 28: The third of the chord arrives late.

Nº8: ***Se non ci fosse ne l'honor (+ Ecco la sconsolata donna assunta)*** (pp.81-83, 29 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	53
D	Major	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	53
A	Minor	0	63
G	Major	0	63
D	Major	0	53
D	Major	0	53
D	Major	7m	7
D	Major	7m,-5	63
G	Major	0	53
C	Major	0	53
A	Minor	0	53
A	Minor	0	53
E	Minor	0	53
B	Minor	0	63
B	Minor	0	63
D	Major	0	53
E	Minor	0	53
A	Minor	7m	7
F♯	Diminished	7m	65
*B	Major	0	53
E	Minor	0	53
C	Minor	0	53
C	Minor	0	53
C	Minor	0	53
G	Minor	0	53
C	Minor	0	53
C	Minor	7m,-5	2
F	Minor	0	63
G	Minor	0	53
C	Minor	0	53
0	0	0	0
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
A	Minor	0	53
A	Minor	0	53
A	Minor	7m	2
F	Major	0	53
F	Major	7Ma	2
D	Minor	0	53
0	0	0	0
C	Major	0	53
C	Major	7Ma	2
A	Minor	0	53

*Bar 13: The third of the chord arrives late.

Nº9: ***Tu mi vai promettendo balsamo*** (pp. 87-88, 14 bars)			
Root	**Triad**	**Alteration**	**Inversion**
A	Minor	0	53
A	Minor	0	53
E	Minor	0	53
E	Minor	0	53
B	Minor	0	53
B	Minor	0	53
A	Minor	0	53
G	Major	0	63
B	Diminished	0	63
E	Minor	0	53
D	Major	0	53
[1]A	Major	7m	65
D	Major	-5	53
B	Minor	0	53
G	Major	0	63
A	Minor	0	53
A	Minor	0	53
G	Major	0	53
E	Major	0	63
A	Minor	0	53
A	Minor	7m	7
[2]B	Major	0	53
E	Minor	0	53

[1]Bar 8: The third of the chord arrives late.
[2]Bar 13: The third of the chord arrives late.

Nº10: ***Le porpore regali e le grandezze*** (pp. 96-97, 24 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
A	Minor	0	53
A	Minor	0	53
B♭	Major	7Ma,-5	53
G	Minor	0	53
E	Diminished	0	63
F	Major	0	53
0	0	0	0
C	Major	0	63
D	Minor	0	53
C	Major	0	53
C	Major	7Ma	2
A	Major	0	53
D	Minor	0	53
E	Major	0	53
A	Minor	0	53
A	Minor	0	53
F	Major	0	53
0	0	0	0
F	Major	0	53
[1]C	Major	7m	65
F	Major	0	53
[2]C	Major	0	63
D	Minor	0	53
[3]A	Major	0	63
D	Minor	0	53
A	Minor	0	53
A	Minor	0	53
A	Minor	0	53

A	Minor	0	53
B♭	Major	0	53
B♭	Diminished	0	53
A	Major	0	53
D	Minor	0	63
G	Minor	7m	7
[4]A	Major	0	53
D	Minor	0	53

[1]Bar 14: The third of the chord arrives late.
[2]Bar 16: The third of the chord arrives late.
[3]Bar 17: The third of the chord arrives late.
[4]Bar 23: The third of the chord arrives late.

N°11: ***Son risoluto al fine*** (pp. 100-101, 19 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	53
G	Major	0	53
C	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
D	Major	0	53
D	Major	0	53
E	Minor	0	53
D	Major	0	63
G	Major	0	53
C	Major	7Ma	7
*D	Major	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
F	Major	0	53
0	0	0	0
F	Major	0	53
C	Major	0	53
A	Minor	0	63
G	Major	0	53
G	Major	0	53
D	Major	0	53
D	Major	0	53
C♯	Diminished	0	53
D	Major	0	53
0	0	0	0
D	Major	0	53
G	Major	0	53
C	Major	0	53
A	Minor	0	53
D	Minor	7m	65
G	Major	0	53
C	Major	-5	53

*Bar 8: The third of the chord arrives late.

N°12: ***Solitudine amata, eremo*** (pp. 147-150, 48 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Minor	0	53
D	Minor	0	53
A	Major	0	63
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
G	Minor	0	63
A	Major	0	53
A	Major	0	53
G	Minor	0	63
G	Major	0	63
C	Major	0	53
A	Major	0	63
D	Minor	0	53
G	Minor	0	63
A	Minor	0	53
F	Major	0	53
F	Major	0	53
F	Major	-5	53
B	Diminished	0	63
C	Major	0	53
A	Minor	0	53
F♯	Diminished	0	63
G	Major	0	53
D	Major	0	63
[1]D	Major	0	63
G	Major	0	53
A	Major	0	53
A	Major	0	53
G	Major	0	53
A	Major	7m,-5	7
D	Minor	0	64
[2]A	Major	0	53
D	Minor	0	53
G	Minor	0	53
G	Minor	0	53
G	Minor	0	53
F	Major	0	53
E♭	Major	0	53
E♭	Major	0	53
[3]D	Minor	0	53
[4]F	Major	0	53
B♭	Major	0	53
D	Minor	0	53
G	Minor	0	53
E♭	Major	0	53
D	Minor	0	53
F	Major	0	53
F	Major	0	53
D	Minor	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
G	Major	0	53
C	Major	0	53
D	Minor	0	53
G	Major	0	53
C	Major	0	53
G	Major	0	53
D	Minor	0	53
G	Major	0	63
C	Major	0	53
C	Major	0	53
C	Major	7Ma	2
F	Major	0	63

B♭	Major	0	53
[5]C	Major	0	53
F	Major	0	63

[1]Bar 23: The third of the chord has been taken as D♯.
[2]Bar 25: The third of the chord arrives late.
[3]Bar 30: The fifth of the chord is made by the voice.
[4]Bar 31: The third of the chord arrives late.
[5]Bar 48: The third of the chord arrives late.

N°13: ***Il comando tiranno*** (pp. 155-156, 26 bars)			
Root	**Triad**	**Alteration**	**Inversion**
E	Minor	0	53
B	Major	0	63
E	Minor	0	53
B	Major	0	53
B	Major	0	53
B	Major	7m	2
E	Major	0	63
B	Major	0	53
E	Major	0	53
C	Major	0	53
C	Major	0	53
A	Minor	0	53
F	Major	0	53
F	Major	0	53
C	Major	0	53
F	Major	0	63
G	Major	7m,-5	7
E	Minor	0	63
F	Major	0	53
G	Major	0	53
C	Major	-5	53
E♭	Major	0	53
0	0	0	0
E♭	Major	0	53
A♭	Major	0	53
F	Minor	0	53
C	Minor	0	53
*F	Minor	0	63
G	Major	0	53
0	0	0	0
D	Major	0	53
D	Major	0	63
D	Major	0	63
G	Major	0	53
G	Major	7m	2
E	Major	0	53
E	Major	0	63
E	Major	0	63
A	Minor	0	53
B	Major	-3	53
E	Major	0	53

*Bar 18: The root of the chord arrives late.

N°14: ***Nerone Non più, non più*** (pp. 158-159, 22 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
D	Major	0	63
G	Major	0	53
G	Major	0	53
D	Major	0	53
C	Major	0	53
D	Major	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	64
[1]G	Major	0	63
C	Major	0	53
0	0	0	0
D	Minor	0	63
G	Major	0	53
C	Major	0	53
0	0	0	0
A	Minor	0	53
A	Minor	0	53
D	Minor	0	53
A	Major	0	63
D	Minor	0	53
A	Minor	0	53
E	Major	0	63
A	Minor	0	53
A	Minor	0	63
E	Major	0	53
E	Major	0	53
E	Major	0	63
A	Minor	0	53
E	Major	-5	53
A	Minor	0	53
F♯	Diminished	0	63
G	Major	0	53
[2]D	Major	0	63
G	Major	0	53
G	Major	0	53
E	Minor	0	53
C♯	Diminished	0	63
[3]C♯	Diminished	0	63
D	Major	0	53

[1]Bar 6: The third of the chord arrives late.
[2]Bar 19: The third of the chord arrives late.
[3]Bar 22: The root of the chord arrives late.

N°15: ***Nutrice, nutrice*** (pp. 211-213, 26 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	53
F	Major	0	53
0	0	0	0
F	Major	0	53
F	Major	0	53
C	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	53
0	0	0	0
D	Minor	0	53
0	0	0	0
D	Minor	0	53
D	Minor	0	53
0	0	0	0
A	Major	0	63
D	Minor	0	53
0	0	0	0
C♯	Diminished	0	63
D	Minor	0	63
D	Minor	0	63
A	Major	0	53
0	0	0	0
E	Diminished	0	63
G	Minor	0	53

A	Major	0	53
A	Major	0	53
D	Major	0	63
G	Minor	0	53
A	Major	0	53
D	Major	0	53
0	0	0	0
D	Major	0	53
G	Major	0	53
A	Minor	0	63
D	Major	0	53
C	Major	0	63
D	Minor	0	63
E	Major	0	53
A	Minor	0	53
A	Minor	0	53
G	Major	0	53
F	Major	0	53
D	Minor	0	63
C	Major	0	53
G	Major	0	53
F	Major	7Ma	7
*G	Major	0	53
C	Major	0	53

*Bar 26: The third of the chord arrives late.

N°16: ***Andiam, andiam, a Ottavia homai*** (pp. 217-218, 18 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	53
G	Major	0	63
F♯	Diminished	0	63
G	Major	0	53
G	Major	0	53
D	Major	0	63
G	Major	0	53
C♯	Diminished	0	63
C♯	Diminished	0	63
D	Major	-5	63
D	Major	0	63
G	Major	0	63
A	Major	-5	53
0	0	0	0
A	Major	0	53
A	Major	0	53
A	Major	0	53
A	Major	0	53
A	Major	0	53
A	Major	0	53
A	Major	0	53
A	Major	0	63
D	Major	0	53
D	Major	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	53
G	Major	0	63
C	Major	0	53
*D	Major	0	53
G	Major	0	53

*Bar 18: The third of the chord arrives late.

N°17: ***Ma cheveggio, infelice?*** (pp. 242-246, 53 bars)			
Root	**Triad**	**Alteration**	**Inversion**
0	0	0	0
D	Minor	0	53
0	0	0	0
D	Minor	0	53
G	Minor	0	53
A♭	Major	0	53
D	Major	0	63
G	Major	0	53
E	Minor	0	63
A	Major	0	53
D	Major	-5	53
0	0	0	0
C	Minor	0	53
C	Minor	0	53
C	Minor	0	53
C	Minor	0	53
B	Diminished	0	53
C	Minor	0	53
F	Minor	7m	7
[1]G	Major	0	63
C	Major	0	53
0	0	0	0
A	Minor	0	53
G♯	Diminished	-5	53
G♯	Diminished	7°,-5	7
E	Major	0	63
A	Minor	0	53
D	Minor	0	63
0	0	0	0
[2]E	Minor	0	53
0	0	0	0
A	Minor	7m,-5	7
F	Major	0	63
F	Major	0	63
E	Major	0	63
E	Major	0	63
E	Major	0	63
A	Minor	0	53
D	Major	0	63
D	Major	0	63
G	Major	0	53
E	Minor	0	53
D	Minor	0	63
D	Minor	0	53
[3]E	Major	0	53
A	Minor	0	53
0	0	0	0
D	Major	0	53
0	0	0	0
D	Major	0	63
0	0	0	0
G	Major	0	53
0	0	0	0
G	Major	0	63
C	Major	0	53
F	Major	0	53
0	0	0	0
B♭	Major	0	53
0	0	0	0
E♭	Major	0	53
C	Minor	0	53
G	Minor	0	53
F	Minor	7m	7
[4]G	Major	0	53
C	Major	0	53

0	0	0	0
A	Minor	0	53
A	Minor	0	53
E	Minor	7m	7
A	Major	0	53
A	Minor	0	53
A	Minor	0	53
F	Major	0	53
0	0	0	0
C	Major	0	53
G	Major	0	53
C	Major	0	53
0	0	0	0
G	Major	0	53
G	Major	0	53
D	Major	0	63
D	Major	0	63
G	Major	0	53
0	0	0	0
G	Major	0	53
C	Major	0	53
0	0	0	0
D	Minor	0	53
A	Major	0	53
0	0	0	0
[5]G	Minor	7m	7
A	Major	0	53

[1]Bar 13: The third of the chord arrives late.
[2]Bar 19: The third of the chord arrives late.
[3]Bar 29: The third of the chord arrives late.
[4]Bar 39: The third of the chord arrives late.
[5]Bar 53: This chord is not in a beat of the bar, however this has been included in order to take into account the final cadence of the *recitative*.

Appendix F. Acis and Galatea

N°1: ***Stay, shepherd, stay!*** (p. 28, 6 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B♭	Major	0	53
C	Major	0	2
E	Diminished	0	53
F	Major	0	53
A	Major	0	63
D	Minor	0	53
G	Minor	0	53
A	Major	0	53
D	Minor	0	53

N°2: ***Lo! here my love!*** (p. 32, 5 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	63
C	Minor	0	63
F♯	Diminished	7°	7
G	Minor	0	53
G	Major	0	2
C	Minor	0	63
D	Major	0	53
G	Minor	0	53

N°3: ***Oh! didst thou*** (p. 35, 4 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
C♯	Diminished	7°	7
D	Minor	0	53
D	Minor	0	63
E	Major	0	53
A	Minor	0	53

N°4: ***Whither, fairest, art thou*** (p. 70, 14 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
F	Major	0	53
B	Diminished	0	63
C	Major	0	53
E	Major	0	63
E	Major	7m	65
F	Diminished	0	63
A	Major	0	63
A	Major	0	2
D	Major	0	63
*–	–	–	–
G	Minor	0	63
A	Major	0	53
D	Major	0	63
F♯	Diminished	7°	7
G	Minor	0	53
B♭	Major	0	63
E	Major	0	2
D♯	Diminished	7°	7
E	Major	0	53
A	Minor	0	53

*Bars 8-14: These 7 bars do not have been consider in the analysis because the recitative style clearly change in this part.

N°5: ***His hideous love*** (p.78, 8 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	63
C	Major	0	53
B	Diminished	0	63
E	Major	0	63
A	Minor	0	53
D	Major	0	63
G	Major	0	53
C	Major	0	53
D	Major	0	53
G	Major	0	53

N°6: ***Cease, oh cease*** (p. 89, 7 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B	Major	0	63
B	Major	0	65
E	Minor	0	53
E	Major	0	2
A	Minor	0	63
F♯	Diminished	7°	7
G	Minor	0	53
C	Minor	0	53
D	Major	0	53
G	Minor	0	53

N°7: ***Tis done*** (p. 111, 5 bars)			
Root	**Triad**	**Alteration**	**Inversion**
F	Major	0	53
F	Major	0	53
F♯	Diminished	7°	7
G	Major	0	53
G	Major	0	53
C	Minor	0	63
D	Major	0	7
G	Minor	0	53

Appendix G. The Marriage of Hercules and Hebe

N^{o}1: ***Odimi, Alcide! Ah Genitor*** (pp. 18-21, 55 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	53
D	Major	0	53
D	Major	0	53
D	Major	0	53
D	Major	0	53
D	Major	0	53
D	Major	7m	7
D	Major	7m	7
[1]A	Minor	0	64
[1]A	Minor	0	63
D	Major	7m	7
G	Major	0	63
[2]G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	7m	65
G	Major	7m	65
C	Major	0	63
C	Major	0	63
C	Major	0	63
C	Major	7m	65
C	Major	0	63
C	Major	7m	65
F	Major	0	53
F	Major	0	53
G	Major	7m	2
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
D	Major	-5	63
D♯	Diminished	7^{o}	7
B	Major	7m	65
E	Minor	0	53
E	Minor	0	53
E	Major	7m	2
E	Major	7m	2
A	Major	0	63
A	Major	0	63
A	Major	0	63
A♯	Diminished	7^{o}	7
E	Diminished	0	64
A♯	Diminished	7^{o}	7
A♯	Diminished	7^{o}	7
A♯	Diminished	7^{o}	7
[3]F♯	Major	0	53
F♯	Major	7m	7
B	Minor	0	63
B	Minor	0	63
B	Minor	0	63
E	Major	0	53
[4]E	Major	7m	53
A	Major	0	53
A	Major	0	53
A	Major	0	53
[5]E	Major	7m,-3	2
B	Minor	0	63
0	0	0	0
[6]E	Major	0	53
A	Major	0	53
A	Major	0	53
A	Major	7m	7
A	Major	7m	7
A	Major	-5	53
A	Major	7m	7
A	Major	7m	7
A	Major	7m,-5,9	7
[7]A	Major	7m	7
D	Major	0	63
D	Major	0	63
D	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	0	53
G	Major	0	53
E	Major	7m	65
E	Major	0	63
E	Major	7m	65
A	Minor	0	53
A	Minor	0	53
A	Minor	0	53
A	Diminished	7^{o}	7
0	0	0	0
[8]B	Major	7m	7
C	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	7m	7
[9]C	Major	7m	7
F	Major	0	63
F	Major	0	63
F	Major	0	63
[10]B♭	Major	7Ma,-5	2
B♭	Major	0	53
B♭	Major	0	53
B♭	Major	7m,-5	7
E♭	Major	0	53

E♭	Major	0	53
0	0	0	0
F	Major	0	53
B♭	Major	0	53
B♭	Major	0	53
[11]C	Major	7m	65
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
G	Major	7m	2
G	Major	7m	2
C	Major	0	63
C	Major	7m	65
F	Major	0	53
F	Major	0	53
0	0	0	0
[12]G	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
D	Major	0	63
D	Major	0	53
D	Major	7m	65
G	Major	0	53
G	Major	0	53
C	Major	0	63
C	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	63
[13]F	Major	0	63
F	Major	0	63
A	Minor	-5	63
F	Major	7m	65
B♭	Major	0	63
B♭	Major	0	63
A	Major	0	63
A	Major	0	63
C♯	Diminished	7°	7
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
D	Diminished	7°	7
E	Major	7m	2
0	0	0	0
E	Major	7m	7
A	Major	0	63
A	Major	0	63
A	Major	0	63
A	Major	0	63
A	Major	7m	65
D	Major	0	53
D	Major	7m	7
G	Major	0	53
A	Minor	7m,-5	2
G	Major	0	53
G	Diminished	7°	7
0	0	0	0
[14]A	Major	7m	7
D	Major	0	53

[1]Bar 3: Sustained note in D.
[2]Bar 4: The root note is done by the voice.
[3]Bar 16: The F♮ is omitted.
[4]Bar 18: The seventh of the chord arrives late.
[5]Bar 19: Sustained note in A.
[6]Bar 20: The third of the chord arrives late.
[7]Bar 24: The fifth of the chord is done by the voice.
[8]Bar 29: The seventh of the chord arrives late.
[9]Bar 32: The third of the chord arrives late.
[10]Bar 33: The third of the chord is done by the voice.
[11]Bar 37: The seventh of the chord arrives late.
[12]Bar 41: The third of the chord arrives late.
[13]Bar 45: The root note is done by the voice.
[14]Bar 54: The seventh of the chord arrives late.

N°2: ***Fermati, Acide!*** (pp. 32-35, 48 bars)			
Root	**Triad**	**Alteration**	**Inversion**
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
B♭	Major	0	63
C	Major	7m	2
[1]C	Major	7m	2
C	Major	7m	2
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	0	63
C	Major	0	63
C	Major	-5	63
C	Major	0	63
C	Major	7m	65
[2]C	Major	0	63
C	Major	7m	65
C	Major	7m	65
F	Major	0	53
F	Major	0	53
F	Major	0	53
D	Minor	0	63
G	Major	7m	2
G	Major	7m	2
C	Major	0	63
F	Major	0	53
0	0	0	0
[3]G	Major	0	53
D	Major	0	63
D	Major	0	63
D	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63

G	Major	0	63
B	Diminished	0	53
[4]G	Major	7m	63
C	Major	0	53
C	Major	-5	53
C	Major	0	53
A	Minor	0	63
D	Major	7m	2
0	0	0	0
[5]D	Major	0	53
G	Major	0	53
[6]D	Major	0	64
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
E	Major	7m	7
E	Major	0	53
A	Minor	0	63
[7]A	Minor	0	63
A	Minor	0	63
A	Major	7m	65
D	Major	0	53
D	Major	0	53
[8]D	Major	0	7
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	7m,-5	7
D	Major	0	53
D	Major	0	53
D	Major	7m	2
[9]D	Major	7m	2
D	Major	0	2
[10]D	Major	7m	2
G	Major	0	63
[11]G	Major	7m	65
C	Major	0	53
C	Major	0	53
D	Major	7m	2
D	Major	7m	2
0	0	0	0
[12]D	Major	0	53
A	Major	0	63
A	Major	0	63
A	Major	0	63
[13]A	Major	7m	65
C♯	Diminished	0	53
A	Major	7m	65
A	Major	7m,-5	65
A	Major	7m	65
D	Major	0	63
D	Major	-5	63
D	Major	0	63
D	Major	0	63
D	Major	7m	65
D	Major	7m	65
G	Major	0	53
[14]G	Major	7m	2
C	Major	0	63
C	Major	7m	65
F	Major	0	53
0	0	0	0
G	Major	0	53
C	Major	0	53
C	Major	0	53
C	Major	0	53
[15]C	Major	7m	7
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
G	Major	7m	2
G	Major	7m	2
G	Major	7m	2
C	Major	0	63
[16]C	Major	0	63
C	Major	0	63
C	Major	0	63
D	Major	7m	2
D	Major	7m,-5	2
D	Major	7m	2
D	Major	7m	2
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
B	Minor	7m,-5	63
B	Major	0	63
B	Major	0	63
B	Major	0	63
B	Major	7m	65
E	Major	0	53
E	Major	0	53
E	Major	7m	7
[17]E	Major	7m	53
A	Major	0	53
0	0	0	0
E	Major	0	64
[18]B	Major	0	53
E	Major	0	53

[1]Bar 4: The fifth of the chord is made by the voice.
[2]Bar 9: The fifth of the chord is made by the voice.
[3]Bar 12: The root of the chord arrives late.
[4]Bar 16: The seventh of the chord arrives late.
[5]Bar 18: The third of the chord arrives late.
[6]Bar 19: Sustained note in G.
[7]Bar 21: The root of the chord is made by the voice, and it appears previously.
[8]Bar 23: The fifth of the chord is made by the voice.
[9]Bar 26: The fifth of the chord is made by the voice, and it appears previously.
[10]Bar 26: The third of the chord is made by the voice, and it arrives late.
[11]Bar 27: The seventh of the chord arrives late.
[12]Bar 28: The third of the chord arrives late.
[13]Bar 30: The root of the chord is made by the voice.
[14]Bar 34: The third, the fifth and the seventh of the chord arrive late.
[15]Bar 37: The seventh of the chord arrives late.
[16]Bar 40: The fifth of the chord is made by the voice, and it arrives late.
[17]Bar 47: The seventh of the chord arrives late.
[18]Bar 48: This chord is not in a beat of the bar, however this has been included in order to take into account the cadential six-four.

N^o^3: ***Giuno, gl'inganni tuoi*** (p. 42, 13 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	53
[1]D	Major	0	53
D	Major	0	53
D	Major	-5	53

D	Major	0	53
G	Major	0	63
G	Major	0	63
E	Major	7m,-5	7
E	Major	7m	7
E	Major	7m	7
A	Minor	0	63
A	Minor	0	63
C	Major	7m	7
C	Major	0	53
C	Major	7m	7
C	Major	7m	7
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
D	Major	7m,-5	2
D	Major	7m	7
D	Major	7m	7
D	Major	7m,-5	7
[2]D	Major	7m	7
[3]D	Major	7m	7
G	Minor	0	63
G	Minor	0	63
G	Minor	0	63
[4]B♭	Major	0	53
C	Minor	0	53
A	Diminished	0	63
0	0	0	0
[5]D	Major	0	53
G	Minor	0	53

[1] Bar 1: The fifth of the chord is made by the voice, and it arrives late.
[2] Bar 11: The fifth of the chord is made by the voice.
[3] Bar 11: The seventh of the chord arrives late.
[4] Bar 12: The fifth of the chord arrives late.
[5] Bar 13: The third of the chord arrives late.

N^{o}4: ***Dove in queste*** (pp. 50-53, 46 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	63
C	Major	-5	63
C	Major	0	63
C	Major	0	63
C	Major	7m	65
E	Diminished	0	53
A	Minor	0	64
E	Diminished	7^o	7
E	Diminished	0	53
E	Diminished	7^o	7
C	Major	7m	65
F	Major	0	53
F	Major	0	53
F	Major	7m	2
0	0	0	0
[1]F	Major	0	53
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	7m	65
B♭	Major	7m,-5	65
B♭	Major	7m	65
E♭	Major	0	53
E♭	Major	0	53
0	0	0	0
[2]D	Major	0	53
A	Major	0	63
A	Major	7m	65
F	Diminished	0	64
A	Major	0	63
A	Major	7m	65
D	Major	0	53
D	Major	0	53
D	Major	0	53
D	Major	0	53
D	Major	0	53
D	Major	0	53
G	Minor	0	53
G	Minor	0	53
G	Minor	0	53
G	Minor	-5	53
G	Major	7m	7
C	Major	0	63
C	Major	0	63
C	Major	0	63
E	Minor	-5	53
C	Major	0	63
E	Diminished	0	53
C	Major	7m	65
[3]C	Major	7m	65
F	Major	0	53
F	Major	0	53
B♭	Major	0	53
B♭	Major	0	53
G	Minor	0	63
0	0	0	0
F	Major	0	64
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	7m,-5	7
F	Major	7m,-5	7
B♭	Major	0	63
B♭	Major	-5	63
B♭	Major	0	63
C	Major	7m	65
E	Diminished	0	53
C	Major	7m	65
[4]C	Major	7m	65
C	Major	7m	65
C	Major	7m,-5	65
C	Major	7m	65

[5]C	Major	7m	65
F	Major	0	53
F	Major	0	53
F	Major	0	53
G	Major	7m	2
0	0	0	0
[6]G	Major	0	53
C	Major	0	53
C	Major	-5	53
C	Major	0	53
D	Major	7m	2
D	Major	7m	2
D	Major	7m	2
D	Major	7m	2
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
E	Major	7m	65
E	Major	0	63
E	Major	7m	65
A	Minor	0	63
C	Major	-5	53
A	Minor	0	63
A	Minor	-5	63
C	Major	7m	7
C	Major	7m	7
C	Major	7m,-5	7
C	Major	7m	7
F	Major	0	53
F	Major	0	53
F	Major	0	53
G	Minor	7m,-5	2
D	Major	7m	7
G	Major	0	53
G	Major	0	53
G	Major	0	53
A	Minor	7m,-5	2
E	Major	7m	7
[7]E♯	Diminished	0	63
E	Major	7m	53
[8]E	Major	7m	53
A	Major	0	53
A	Major	0	53
A	Major	0	53
A	Major	0	53
A	Major	7m	7
A	Major	7m	7
A	Major	7m	7
D	Major	0	64
A	Major	7m	7
A	Major	7m	7
D	Major	0	63
D	Major	0	63
D	Major	0	63
G	Major	0	53
E♯	Diminished	7º	7
0	0	0	0
[9]F♯	Major	0	53
B	Minor	0	53

[1]Bar 7: The third of the chord arrives late.
[2]Bar 11: The third of the chord arrives late.
[3]Bar 18: The seventh of the chord arrives late.
[4]Bar 25: The fifth of the chord is made by the voice.
[5]Bar 27: The seventh of the chord arrives late.
[6]Bar 29: The third of the chord arrives late.
[7]Bar 41: Sustained note in E.
[8]Bar 41: The seventh of the chord arrives late.
[9]Bar 46: The third of the chord arrives late.

Nº5: ***Poichè la legge*** (pp. 50-53, 46 bars)			
Root	**Triad**	**Alteration**	**Inversion**
A	Major	0	63
C♯	Minor	-5	53
A	Major	0	63
D	Major	0	63
D	Major	0	63
D	Major	0	63
D	Major	-5	63
D	Major	0	63
D	Major	0	63
D	Major	7m	65
F♯	Minor	-5	53
F♯	Diminished	7º	7
D	Major	7m	65
G	Major	0	63
G	Major	0	63
G	Major	7m	65
G	Major	7m	65
C	Major	0	53
0	0	0	0
G	Major	0	64
[1]D	Major	0	53
G	Major	0	53

[1]Bar 6: This chord is not in a beat of the bar, however this has been included in order to take into account the cadential six-four.

Nº6: ***Avversi fatirei*** (pp. 73-77, 67 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B♭	Major	0	53
B♭	Major	0	53
B♭	Major	7m,-5	7
B♭	Major	7m	7
B♭	Major	7m	7
[1]B♭	Major	7m	7
E♭	Major	0	53
E♭	Major	0	53
E♭	Major	0	53
C	Minor	-5	63
F	Major	7m	2
E	Diminished	7m	7
0	0	0	0
[2]F	Major	0	53
C	Major	0	63
[3]E	Minor	-5	53
C	Major	0	63
C	Major	-5	63
E	Diminished	7º	7
C	Major	7m	65
F	Major	0	53
F	Major	0	53
F	Major	0	53
[4]F	Major	0	53
F	Major	0	53
G	Major	7m	43

[5]G	Major	7m	43
C	Major	0	53
C	Major	0	53
[6]B	Diminished	0	64
C	Major	0	53
C	Major	-5	53
C	Major	7m	7
F	Major	0	53
F	Major	0	53
F	Major	0	53
D	Minor	0	63
D	Minor	0	63
D	Minor	0	63
B	Diminished	0	64
E	Major	0	53
E	Major	0	53
E	Major	0	53
[7]E	Major	7m,-3	7
E	Major	0	53
E	Major	0	53
E	Major	7m	7
E	Major	0	53
A	Minor	0	53
B	Minor	7m,-5	2
D	Major	0	63
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
A	Major	7m	2
0	0	0	0
[8]A	Major	0	53
D	Major	0	53
[9]C	Major	0	63
D	Major	0	53
D	Major	0	53
D	Major	0	53
D	Major	7m	7
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
E	Major	7m	7
E	Major	7m	7
[10]B	Diminished	0	64
[11]E	Major	7m	7
A	Minor	0	63
A	Minor	0	63
A	Minor	0	63
[12]A	Minor	0	63
B	Major	0	63
E	Major	0	53
E	Major	0	53
E	Major	0	53
E	Major	0	53
A	Major	0	63
A	Major	0	63
A	Major	0	63

A	Major	0	63
A	Major	0	63
F♯	Major	7m	7
[13]F♯	Major	7m	7
[14]F♯	Major	7m	7
B	Major	0	53
B	Major	0	53
C♯	Major	7m	2
0	0	0	0
[15]C♯	Major	7m	7
F♯	Minor	0	53
F♯	Minor	0	53
F♯	Minor	0	53
F♯	Diminished	7^{o}	7
0	0	0	0
[16]F♯	Diminished	9	64
A	Major	0	53
A	Major	7m	7
A	Major	7m,-5	7
D	Major	0	63
D	Major	-5	63
D	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	7m	65
C	Major	0	53
C	Major	-5	53
C	Major	0	53
C	Major	7m	7
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	7m	7
B♭	Major	0	63
[17]B♭	Major	0	63
D	Diminished	7^{o}	7
D	Diminished	7^{o}	7
B♭	Major	0	63
[18]B♭	Major	0	63
B♭	Major	7m	65
B♭	Major	7m	65
E♭	Major	0	63
E♭	Major	0	63
G	Diminished	7^{o}	7
A♭	Major	0	63
C	Minor	7m	7
A♭	Major	0	63
C	Minor	7m,-5,9	7
A♭	Major	0	63
F	Major	7m	7
[19]F	Major	7m	7
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	7m	7
E♭	Major	0	53
E♭	Major	0	53

F	Major	7m	2
0	0	0	0
[20]F	Major	0	53
D	Major	0	63
D	Major	-5	63
D	Major	0	63
D	Major	7m	65
D	Major	7m	65
[21]D	Major	7m	65
G	Minor	0	53
G	Minor	0	53
G	Minor	0	53
E	Diminished	0	63
A	Major	7m	2
D	Major	0	63
[22]D	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
G	Major	0	63
C	Major	0	63
C	Major	0	63
E	Diminished	7^{o}	7
E	Diminished	0	53
E	Diminished	7^{o}	7
F	Minor	0	53
[23]F	Minor	0	53
F	Major	7m	2
F	Major	7m	2
F	Major	7m	2
F	Major	7m	2
B♭	Major	0	63
B♭	Major	-5	63
B♭	Major	0	63
C♯	Diminished	0	53
D	Minor	0	53
D	Minor	-5	53
D	Minor	0	53
D	Minor	-5	53
D	Minor	0	53
D	Minor	0	53
C	Major	0	63
C	Major	-5	63
C	Major	0	63
E	Diminished	7^{o}	7
E	Diminished	0	53
F	Major	0	53
[24]F	Major	7m	7
B♭	Major	0	53
B♭	Major	0	53
B♭	Diminished	7^{o}	7
0	0	0	0
[25]C	Major	0	53
F	Major	0	53

[1]Bar 3: The seventh of the chord arrives late.
[2]Bar 5: The third of the chord arrives late.
[3]Bar 6: The third of the chord is made by the voice.
[4]Bar 8: The third and the fifth of the chord arrive late.
[5]Bar 9: The root of the chord arrives late.
[6]Bar 9: Sustained note in C.
[7]Bar 13: The voice is used to infer the harmony.
[8]Bar 20: The third of the chord arrives late.
[9]Bar 20: Sustained note in D.
[10]Bar 24: Sustained note in E.
[11]Bar 24: The seventh of the chord arrives late.
[12]Bar 25: The third of the chord arrives late.
[13]Bar 31: The fifth of the chord is made by the voice.
[14]Bar 31: The seventh of the chord arrives late.
[15]Bar 32: The third of the chord arrives late.
[16]Bar 34: The G♯ is omitted in the harmony analysis.
[17]Bar 43: The root of the chord is made by the voice.
[18]Bar 44: The fifth of the chord arrives late.
[19]Bar 48: The seventh of the chord arrives late.
[20]Bar 50: The third of the chord arrives late.
[21]Bar 53: The fifth of the chord arrives late.
[22]Bar 55: The root and the fifth of the chord arrive late.
[23]Bar 58: The third of the chord arrives late.
[24]Bar 65: The seventh of the chord arrives late.
[25]Bar 67: The third of the chord arrives late.

Nº7: ***Ah, madre, e che facesti?*** (pp. 90-92, 42 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	63
C	Major	0	63
C	Major	0	63
C	Major	7m	65
E	Diminished	0	53
C	Major	7m	65
[1]C	Major	7m	65
C	Major	7m	65
A	Minor	0	64
C	Major	7m	65
[2]C	Major	7m	65
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	63
E♭	Major	0	63
E♭	Major	0	63
A♭	Major	0	63
A♭	Major	0	63
G	Major	0	63
B	Diminished	7^{o}	7
B	Diminished	7^{o}	7
B	Diminished	7^{o}	7
C	Minor	0	63
[3]C	Minor	0	63
C	Minor	0	63
C	Major	7m	2
C	Major	7m,-5	2
C	Major	7m	2
F	Major	0	63
[4]F	Major	0	63
F	Major	0	63
A	Minor	-5	53
F	Major	0	63
F	Major	0	63
D	Major	7m,-5	7
D	Major	0	53
D	Major	7m	7
G	Minor	0	53
G	Minor	0	53
A	Major	0	63
A	Major	-5	63
A	Major	0	63
A	Major	0	63

C♯	Diminished	7^{o}	7
[5]C♯	Diminished	7^{o}	7
A	Major	7m	65
D	Major	0	53
D	Major	0	53
[6]D	Major	0	53
E	Major	7m	2
E	Major	7m	2
E	Major	7m	2
A	Major	0	63
A	Major	0	63
A	Major	0	63
D	Major	0	53
E	Major	0	2
0	0	0	0
[7]E	Major	0	53
B	Major	0	63
[8]D♯	Minor	-5	53
B	Major	0	63
B	Major	7m	65
B	Major	7m	65
E	Major	0	53
E	Major	0	53
E	Major	0	53
E	Major	0	53
E	Major	0	53
E	Major	0	53
C♯	Major	0	63
[9]C♯	Major	0	63
C♯	Major	0	63
C♯	Major	0	63
C♯	Major	7m	65
F♯	Minor	0	53
F♯	Minor	0	53
F♯	Minor	0	53
F♯	Major	7m	2
F♯	Major	7m	2
F♯	Major	7m	2
F♯	Major	7m	2
E	Diminished	7^{o}	7
F♯	Major	7m	2
B	Minor	0	63
B	Minor	0	63
E	Major	7m	2
[10]E	Major	7m	2
E	Major	7m	2
[11]E	Major	7m	2
[12]A	Major	7m	2
[13]D	Major	2	65
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
0	0	0	0
D	Major	0	64
[14]A	Major	0	53
D	Major	0	53
D	Major	0	53
D	Major	0	53
D	Major	7m	7
[15]D	Major	7m	7
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	7m	7
C	Major	0	63
C	Major	7m	65
F	Major	0	53
D	Minor	0	63
G	Major	0	2
0	0	0	0
[16]G	Major	0	53
C	Major	0	53

[1]Bar 3: The fifth of the chord arrives late.
[2]Bar 5: The seventh of the chord arrives late.
[3]Bar 11: The fifth of the chord is made by the voice.
[4]Bar 14: The root of the chord is made by the voice.
[5]Bar 18: The third of the chord is made by the voice, and it arrives late.
[6]Bar 19: The fifth of the chord is made by the voice.
[7]Bar 22: The third of the chord arrives late.
[8]Bar 23: The chord is inferred from the neighboring notes.
[9]Bar 27: The fifth of the chord is made by the voice, and it arrives late.
[10]Bar 33: The third of the chord arrives late.
[11]Bar 34: The root of the chord arrives late.
[12]Bar 35: The seventh of the chord arrives late.
[13]Bar 35: The seventh of the chord arrives late.
[14]Bar 36: This chord is not in a beat of the bar, however this has been included in order to take into account the cadential six-four.
[15]Bar 38: The seventh of the chord arrives late.
[16]Bar 42: The third of the chord arrives late.

N^{o}8: ***Vedi che a noi qui volge*** (pp. 100-101, 12 bars)

Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	53
C	Major	-5	53
C	Major	7m	7
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
F	Major	0	63
B♭	Major	0	63
B♭	Major	-5	63
B♭	Major	7m	65
E♭	Major	0	53
0	0	0	0
[1]F	Major	7m,-5	7
D	Major	0	63
[2]D	Major	7m	65
G	Minor	0	53
[3]G	Minor	0	53
G	Minor	0	53
G	Major	0	63
[4]B	Minor	-5	63
B	Diminished	0	63
[5]G	Major	7m	43
[6]G	Major	7m	43
G	Major	0	64
C	Minor	0	53
C	Minor	0	53
D	Major	7m	2
0	0	0	0
D	Major	7m	7
G	Minor	0	53

[1]Bar 6: The third of the chord arrives late.
[2]Bar 6: The root of the chord arrives late.
[3]Bar 7: The third and the fifth of the chord arrive late.
[4]Bar 8: The root and the third of the chord arrive late.
[5]Bar 8: The root and the third of the chord arrive late.
[6]Bar 9: The root of the chord is made by the voice.

N^{o}9: ***Vana l'opra non fù*** (pp. 105-108, 48 bars)			
Root	**Triad**	**Alteration**	**Inversion**
E♭	Major	0	53
E♭	Major	-5	53
E♭	Major	0	53
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
[1]F	Major	0	53
B♭	Major	0	53
D	Diminished	7^{o}	7
[2]D	Diminished	7^{o}	7
D	Diminished	7^{o}	7
[3]D	Diminished	0	53
D	Diminished	7^{o}	7
B♭	Major	7m	65
E♭	Major	7m	2
E♭	Major	7m	2
E♭	Major	7m	2
E♭	Major	7m	2
G	Diminished	0	64
E♭	Major	7m	2
E♭	Major	7m,-5	2
E♭	Major	7m	2
A♭	Major	0	63
[4]A♭	Major	0	63
A♭	Major	0	63
A♭	Major	0	63
A♭	Major	0	63
F	Major	0	53
F	Major	0	53
[5]F	Major	7m	7
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	0	63
D	Minor	7m	7
D	Major	0	63
D	Major	0	63
F♯	Diminished	7^{o}	7
D	Major	7m	65
G	Minor	0	53
G	Minor	0	53
G	Minor	0	53
A	Major	7m	2
0	0	0	0
D	Minor	0	64
[6]A	Major	0	53
E	Major	0	63
A	Minor	0	63
A	Minor	0	63
A	Minor	0	63
D	Major	0	53
[7]D	Major	7m	7
G	Major	0	53
G	Major	0	53
G	Major	7m	7
G	Major	7m	7
C	Major	0	53
C	Major	0	53
C	Major	0	53
[8]C	Major	0	53
[9]C	Major	0	7
C	Major	0	53
C	Major	7m	7
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
G	Major	7m	2
G	Major	7m	2
G	Major	7m	2
G	Major	7m	2
C	Major	0	63
C	Major	-5	63
C	Major	0	63
E	Major	0	53
E	Major	0	53
E	Major	0	53
E	Major	0	53
E	Major	7m,-3	7
E	Major	7m	7
E	Major	7m	7
E	Major	0	53
A	Minor	0	63
[10]A	Minor	0	63
A	Minor	0	63
[11]A	Minor	0	63
D	Minor	0	53
D	Minor	0	53
0	0	0	0
[12]E	Major	0	53
B	Major	0	63
[13]B	Major	0	63
B	Major	0	63
E	Major	7m	2
E	Major	7m	2
E	Major	7m	2
D	Major	7^{o}	7
A	Major	0	63
[14]A	Major	0	63
F♯	Major	7m,-5	7
F♯	Major	7m,-5	7
F♯	Major	7m	7
[15]F♯	Major	7m	7
B	Minor	0	63
B	Minor	0	63
D	Major	7m	65
D	Major	0	63
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	63
F	Major	0	53
0	0	0	0
[16]G	Major	0	53
D	Major	0	63
D	Major	0	63
D	Major	0	63
D	Major	-5	63
D	Major	0	63

D	Major	7m	65
D	Major	0	63
D	Major	7m	65
D	Major	7m	65
G	Major	0	63
G	Major	-5	63
G	Major	0	63
G	Major	0	63
D♯	Diminished	7^{o}	7
A	Diminished	0	64
D♯	Diminished	7^{o}	7
E	Minor	0	53
E	Minor	0	53
E	Minor	0	53
E	Minor	0	53
F♯	Major	7m	2
0	0	0	0
[17]F♯	Major	0	53
B	Minor	0	53

[1]Bar 4: The third of the chord arrives late.
[2]Bar 5: The third and the fifth of the chord arrive late.
[3]Bar 6: The third and the fifth of the chord arrive late.
[4]Bar 10: The root of the chord is made by the voice, and it appears previously.
[5]Bar 13: The E has been taken as E♭.
[6]Bar 17: This chord is not in a beat of the bar, however this has been included in order to take into account the cadential six-four.
[7]Bar 19: The fifth of the chord is made by the voice.
[8]Bar 21: The chord is inferred from the neighboring notes.
[9]Bar 22: The fifth of the chord is made by the voice.
[10]Bar 29: The root of the chord is made by the voice, and it appears late.
[11]Bar 30: The fifth of the chord arrives late.
[12]Bar 31: The third of the chord arrives late.
[13]Bar 32: The fifth of the chord is made by the voice, and it appears previously.
[14]Bar 34: The root of the chord is made by the voice, and it appears late.
[15]Bar 35: The seventh of the chord arrives late.
[16]Bar 39: The third of the chord arrives late.
[17]Bar 48: The third of the chord arrives late.

Nº10: ***Ahimè! La Dea nemica*** (pp. 113-116, 52 bars)

Root	**Triad**	**Alteration**	**Inversion**
B♭	Major	0	53
B♭	Major	0	53
B♭	Major	7m,-5	7
B♭	Major	7m,-5	7
[1]E♭	Major	0	64
B♭	Major	7m	7
[2]B♭	Major	7m	7
E♭	Major	0	63
E♭	Major	-5	63
E♭	Major	0	63
E♭	Major	0	63
E♭	Major	0	63
E♭	Major	0	63
E♭	Major	0	63
E♭	Major	0	63
E♭	Major	0	63
E♭	Major	0	63
E♭	Major	0	63
A♭	Major	0	63
[3]A♭	Major	0	63
A♭	Major	0	63
F	Minor	0	53
F	Minor	0	53
F	Minor	0	53
F	Minor	0	53
[4]F	Major	7m	7
[5]F	Major	7m	7
F	Major	7m	7
[6]F	Major	7m	7
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	0	63
C	Minor	0	63
C	Minor	7m	65
F	Major	7m	2
0	0	0	0
[7]F	Major	0	53
B♭	Major	0	53
[8]B♭	Major	0	53
B♭	Major	0	53
B♭	Major	-5	53
B♭	Major	0	53
E♭	Major	0	64
B♭	Major	0	53
B♭	Major	7m	7
[9]B♭	Major	7m	7
[10]B♭	Major	7m	7
B♭	Major	7m	7
E♭	Major	0	53
[11]D	Diminished	0	64
E♭	Major	0	53
E♭	Major	0	53
E♭	Major	0	53
C	Major	0	53
C	Major	7m	7
[12]C	Major	7m	7
C	Major	-5	53
C	Major	0	53
F	Minor	0	53
0	0	0	0
[13]G	Major	0	53
D	Major	0	63
D	Major	7m	65
G	Minor	0	53
[14]G	Minor	-3	53
G	Minor	0	53
G	Minor	0	53
A	Major	0	63
A	Major	0	63
C♯	Diminished	7^{o}	7
C♯	Diminished	7^{o}	7
C♯	Diminished	7^{o}	7
A	Major	7m	65
D	Minor	0	53
D	Minor	0	53
D	Major	7m	2
[15]D	Major	7m	2
D	Major	7m	2
[16]C	Major	7m,-3	7
D	Major	7m	2
G	Minor	0	63
G	Minor	-5	63
G	Minor	0	63
G	Major	7m	7
C	Major	0	63

C	Major	0	63
C	Major	0	63
F	Major	0	53
D	Minor	0	63
0	0	0	0
A	Minor	0	64
[17]E	Major	0	53
B	Major	0	63
B	Major	0	63
B	Major	0	63
B	Major	7m	65
[18]B	Major	7m	65
B	Major	7m	65
E	Major	0	53
E	Major	0	53
E	Major	0	53
E	Major	0	53
E	Major	0	53
C♯	Major	0	63
F♯	Minor	0	53
F♯	Minor	0	53
F♯	Minor	0	53
F♯	Minor	0	53
0	0	0	0
[19]G♯	Major	0	53
C♯	Minor	0	53
C♯	Minor	0	53
C♯	Minor	0	53
E	Major	0	63
E	Major	0	63
E	Major	0	63
E	Major	0	63
E	Major	0	63
E	Major	7m	65
E	Major	7m	65
A	Major	0	63
F♯	Minor	0	64
D	Major	-5	53
D	Major	0	53
[20]D	Major	7m	7
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
C	Minor	0	63
[21]C	Minor	0	63
C	Minor	0	63
D	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Minor	0	53
C	Minor	0	53
C	Minor	0	53
C	Minor	0	53
0	0	0	0
[22]D	Major	0	53
G	Minor	0	53

[1]Bar 3: The root and the fifth of the chord arrive late.
[2]Bar 4: The seventh of the chord arrives late.
[3]Bar 8: The root of the chord is made by the voice, and it appears previously.
[4]Bar 10: The fifth of the chord is made by the voice.
[5]Bar 11: The third of the chord is made by the voice, and it appears previously.
[6]Bar 11: The seventh of the chord arrives late.
[7]Bar 14: The third of the chord arrives late.
[8]Bar 15: The third of the chord arrives late, and the fifth is made by the voice.
[9]Bar 17: The seventh of the chord arrives late.
[10]Bar 17: The seventh of the chord is made by the voice.
[11]Bar 18: Sustained note in E♭.
[12]Bar 21: The fifth of the chord is made by the voice.
[13]Bar 22: The third of the chord arrives late.
[14]Bar 24: The fifth of the chord is made by the voice, and it appears late.
[15]Bar 28: The fifth of the chord is made by the voice, and it appears late.
[16]Bar 29: The seventh of the chord is made by the voice.
[17]Bar 33: This chord is not in a beat of the bar, however this has been included in order to take into account the cadential six-four.
[18]Bar 35: The seventh of the chord is made by the voice, and it appears previously.
[19]Bar 40: The third of the chord arrives late.
[20]Bar 46: The fifth of the chord is made by the voice.
[21]Bar 48: The root of the chord arrives late.
[22]Bar 52: The third of the chord arrives late.

Nº11: ***Consorte, a tempo giungi*** (pp. 124-126, 33 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	53
G	Major	0	53
[1]F♯	Diminished	0	64
G	Major	0	53
G	Major	0	53
C	Major	0	64
G	Major	0	53
G	Major	7m	7
C	Major	0	64
G	Major	7m	7
C	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
[2]F	Major	7m	7
B♭	Major	0	63
B♭	Major	0	63
C	Minor	7m	65
C	Minor	0	63
F	Major	7m	2
0	0	0	0
[3]F	Major	0	53
C	Major	0	63
F	Major	0	53
F	Major	0	53
F	Major	0	53
B♭	Major	0	63
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
[4]F♯	Diminished	0	64
G	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	63
C	Major	0	63
C	Major	7m	65
C	Major	7m	65
C	Major	7m	65
F	Major	0	53
F	Major	0	53
F	Major	0	53

F	Major	0	53
F	Major	0	53
F	Major	0	53
F	Major	0	53
D	Minor	0	63
G	Major	7m	2
0	0	0	0
[5]G	Major	0	53
C	Major	0	53
C	Major	0	53
[6]G	Major	7m,-5	2
C	Major	0	53
C	Major	0	53
A	Major	0	63
A	Major	0	63
C♯	Diminished	7^{o}	7
[7]C♯	Minor	-5	53
C♯	Diminished	7^{o}	7
[8]C♯	Diminished	7^{o}	7
C♯	Diminished	7^{o}	7
A	Major	7m	65
D	Major	0	63
[9]D	Major	0	63
D	Major	7m	65
G	Major	7m	2
G	Major	0	53
E	Minor	0	63
A	Major	7m	2
0	0	0	0
[10]A	Major	0	53
D	Major	0	53
D	Major	0	53
G	Minor	0	63
[11]G	Minor	0	63
G	Minor	0	63
C	Minor	0	63
C	Minor	0	63
C	Minor	0	63
[12]G	Major	0	53
F	Major	0	53
F	Major	7m	7
[13]B♭	Minor	0	53
[13]B♭	Minor	0	53
[13]B♭	Minor	0	53
0	0	0	0
[14]C	Major	0	53
F	Major	0	53

[1]Bar 1: Sustained note in G.
[2]Bar 6: The seventh of the chord arrives late.
[3]Bar 8: The third of the chord arrives late.
[4]Bar 14: Sustained note in G.
[5]Bar 22: The third of the chord arrives late.
[6]Bar 23: Sustained note in C.
[7]Bar 25: The third of the chord is made by the voice, and it appears previously and late.
[8]Bar 25: The third of the chord is made by the voice.
[9]Bar 26: The fifth of the chord arrives late.
[10]Bar 28: The third of the chord arrives late.
[11]Bar 29: The root of the chord is made by the voice, and it appears previously and late.
[12]Bar 31: Sustained note in E♭, the third of the chord is made by the voice.
[13]Bar 32: One of the B is taken as B♭.
[14]Bar 33: The third of the chord arrives late.

N^{o}12: ***Ora il nostro contento*** (p. 134, 12 bars)			
Root	**Triad**	**Alteration**	**Inversion**
F	Major	0	53
F	Major	0	53
F	Major	0	53
B♭	Major	0	63

B♭	Major	0	63
B♭	Major	0	63
B♭	Major	7m	65
B♭	Major	0	63
B♭	Major	7m	65
[1]D	Diminished	0	53
B♭	Major	7m	65
B♭	Major	-5	63
B♭	Major	7m	65
E♭	Major	0	63
E♭	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
[2]F	Major	0	53
A	Major	0	63
A	Major	0	63
A	Major	7m	65
D	Major	0	53
D	Major	0	53
D	Major	0	53
[3]D	Major	7m	7
G	Major	0	53
G	Major	0	53
G	Major	0	53
0	0	0	0
[4]A	Major	0	53
D	Major	0	53

[1]Bar 5: The fifth of the chord is made by the voice.
[2]Bar 8: The third of the chord is made by the voice, and it appears late.
[3]Bar 10: The seventh of the chord arrives late.
[4]Bar 12: The third of the chord arrives late.

Appendix H. Mithridates, King of Pontus

N°1: ***Vieni, signor (+ Se a me s'unisce Arbate)*** (pp. 16-20, 104 bars)

Root	Triad	Alteration	Inversion
D	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	0	53
E	Minor	0	63
B	Major	7m	65
B	Major	7m	65
E	Minor	0	53
E	Minor	0	53
F♯	Major	7m	2
F♯	Major	0	53
C♯	Major	0	63
C♯	Major	7m	65
F♯	Minor	0	53
B	Major	0	63
E	Major	0	53
E	Major	0	53
B	Minor	0	63
C♯	Major	-5	53
C♯	Major	7m,-5	7
F♯	Minor	0	53
A	Major	7m	43
D	Major	0	53
B	Minor	0	63
F♯	Major	7m	2
B	Minor	0	63
D	Major	7m	65
D	Major	0	63
G	Major	0	63
G	Major	7m	65
C	Major	0	53
C	Major	7m	7
A	Major	0	63
A	Major	0	63
A	Major	7m	65
D	Minor	0	53
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
B♭	Major	0	53
D	Major	0	63
G	Minor	0	53
C	Major	0	63
F	Major	0	53
F	Major	0	53
D	Minor	0	63
C♯	Diminished	0	63
D	Minor	0	53
D	Minor	0	53
E	Major	7m	2
A	Minor	0	63
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
A	Minor	0	53
B	Major	0	53
F♯	Major	0	63
B	Minor	0	63
B	Minor	0	63
D	Major	0	63
G	Major	0	63
A	Major	0	63
A	Major	7m	65
D	Major	0	53
D	Major	0	53
E	Major	7m	2
E	Major	0	53
A	Major	0	63
B	Major	7m	65
E	Major	0	53
A	Major	0	53
A	Major	-5	53
C♯	Major	0	63
F♯	Minor	0	53
F♯	Minor	0	53
D	Major	0	63
C♯	Diminished	0	63
C♯	Diminished	0	63
D	Major	0	53
D	Major	0	53
G	Major	0	53
E	Minor	0	63
F♯	Major	0	53
F♯	Major	0	53
B	Minor	0	63
B	Minor	0	63
B	Major	7m	65
E	Minor	0	53
C	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
D	Minor	0	63
D	Minor	0	63
C	Major	0	63
C♯	Diminished	0	63

D	Minor	0	53
D	Major	7m	65
G	Minor	0	53
C	Major	0	63
F	Major	0	53
A	Major	7m	65
D	Minor	0	53
D	Minor	0	53
D	Minor	0	53
E	Major	0	53
A	Minor	0	53
D	Major	0	63
D	Major	0	63
G	Major	0	53
C	Major	0	63
0	0	0	0
D	Major	0	53
G	Major	0	53
0	0	0	0

N°2: ***Principe, che facemmo!*** (p.56, 5 bars)

Root	Triad	Alteration	Inversion
C	Major	0	63
C	Major	7m	65
A	Major	0	63
A	Major	7m	65
D	Minor	0	53
G	Minor	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
*D	Minor	0	53

*p. 38 of [86]: The orchestra play this chord.

N°3: ***Un tale addio*** (pp. 62-63, 25 bars)

Root	Triad	Alteration	Inversion
B♭	Major	0	63
C	Major	0	63
F	Major	0	53
D	Minor	0	63
A	Major	0	63
D	Minor	0	53
D	Minor	0	53
E	Major	7m	2
A	Minor	0	63
A	Minor	0	63
B	Major	7m	65
E	Minor	0	53
A	Major	7m	65
D	Major	0	53
G	Major	0	53
C	Major	0	53
0	0	0	0
D	Major	0	53
A	Major	0	63
A	Major	7m	65
D	Major	0	53
F♯	Major	7m	2
B	Minor	0	63
B	Minor	0	63
C♯	Major	0	63
F♯	Minor	0	53
F♯	Minor	0	53

A	Major	7m	43
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	0	53
A	Major	0	53

N°4: ***Eccovi in un momento*** (p. 68, 22 bars)

Root	Triad	Alteration	Inversion
F♯	Major	0	63
F♯	Major	7m	65
B	Minor	0	53
C♯	Major	0	63
F♯	Minor	0	53
F♯	Minor	0	53
C♯	Diminished	0	63
D	Major	0	53
D	Major	0	53
G	Major	0	53
E	Minor	0	63
B	Major	0	63
E	Minor	0	53
E	Major	7m	2
A	Minor	0	63
C	Major	0	63
F	Major	0	63
F	Major	7m	65
B♭	Major	0	53
G	Minor	0	63
D	Major	0	53
G	Minor	0	53
0	0	0	0
A	Major	0	53
D	Minor	0	53

N°5: ***Tu mi rivedi, Arbate (+ Su la temuta destra)*** (pp. 89-91, 69 bars)

Root	Triad	Alteration	Inversion
E	Major	0	63
E	Major	7m	65
A	Major	0	53
D	Major	0	63
G	Major	0	53
G	Major	0	53
E	Minor	0	63
E	Minor	0	63
F♯	Major	0	53
F♯	Major	0	53
B	Minor	0	63
B	Minor	0	63
C♯	Major	0	53
F♯	Minor	0	53
G♯	Major	7m	2
0	0	0	0
G♯	Major	0	53
C♯	Major	0	53
0	0	0	0
A	Major	0	63
A	Major	7m	65
D	Major	0	53
G	Major	0	53
0	0	0	0

A	Major	0	53
D	Major	0	63
D	Major	7m	65
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
F♯	Major	0	53
B	Minor	0	53
E	Minor	0	53
F♯	Major	7m	2
0	0	0	0
F♯	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	7m	7
E	Major	0	63
A	Minor	0	63
C	Major	0	63
F	Major	0	53
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
C	Minor	0	63
B	Diminished	0	63
C	Minor	0	63
C	Minor	0	63
D	Major	7m	65
G	Minor	0	53
0	0	0	0
A	Major	0	53
D	Minor	0	53
D	Minor	0	53
E	Major	0	63
A	Minor	0	53
C	Major	0	63
F	Major	0	53
B♭	Major	0	53
C	Major	7m	2
0	0	0	0
C	Major	0	53
F	Major	0	53
0	0	0	0

Nº6: ***Teme Ismene a ragion*** (pp. 99-101, 55 bars)			
Root	**Triad**	**Alteration**	**Inversion**
A	Major	0	63
D	Minor	0	63
D	Minor	0	63
E	Major	7m	65
A	Minor	0	53
B	Major	0	63
E	Minor	0	53
E	Minor	0	53
F♯	Major	7m	2
F♯	Major	0	53
C♯	Major	0	63
F♯	Minor	0	53
F♯	Minor	0	53
G♯	Major	7m	2
C♯	Minor	0	63
C♯	Minor	0	63
D♯	Major	0	53
B	Major	0	63
B	Major	7m	65
E	Major	0	53

E	Major	0	53
E	Major	7m	7
C♯	Major	0	63
F♯	Minor	0	53
F♯	Minor	-5	53
D	Major	0	63
D	Major	0	63
*D	Major	7m	65
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
F♯	Major	7m	2
0	0	0	0
F♯	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	53
C	Major	0	63
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	0	63
D	Major	0	63
G	Minor	0	63
A	Major	0	53
D	Minor	0	53
E	Major	7m	2
0	0	0	0
E	Major	0	53
B	Major	0	63
E	Minor	0	53
E	Minor	0	53
F♯	Major	7m	2
B	Minor	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	53
D	Major	7m	2
0	0	0	0
D	Major	0	53
G	Major	0	53

*Bar 22: The root of the chord is made by the voice.

Nº7: ***Questo è l'amor*** (pp. 115-116, 42 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
C	Major	0	63
B	Diminished	0	63
C	Major	-5	53
C	Major	0	53
*D	Minor	0	63
E	Major	0	53
E	Major	0	53
A	Minor	0	63
A	Minor	0	63
C	Major	0	63
F	Major	0	63
B♭	Major	0	53

C	Major	7m	2
0	0	0	0
C	Major	0	53
F	Major	0	63
F	Major	0	63
B♭	Major	0	63
D	Major	7m	2
G	Minor	0	63
G	Minor	0	63
0	0	0	0
A	Major	0	53
B♭	Major	0	63
B♭	Major	0	63
E♭	Major	0	53
A♭	Major	0	53
F	Minor	0	63
C	Major	7m	65
F	Minor	0	53
F	Minor	0	53
G	Major	7m	2
C	Minor	0	63
A	Diminished	7Ma	43
D	Major	0	53
G	Minor	0	63
A	Major	0	63
A	Major	7m	65
D	Minor	0	53
G	Minor	0	53
0	0	0	0
A	Major	0	53
D	Minor	0	53
0	0	0	0

*Bar 10: The root of the chord arrives late.

N^{o}8: ***Perfido, ascolta (+ Eccomi a'cenni tuoi + Respiro, oh Dei!)*** (pp. 121-124, 77 bars)

Root	Triad	Alteration	Inversion
G	Major	0	63
C	Major	0	63
E	Major	0	63
A	Minor	0	63
B	Major	0	63
E	Minor	0	53
D	Major	0	63
G	Major	0	53
C	Major	0	63
C	Major	0	63
F	Major	7Ma,-5	7
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	53
D	Major	0	63
G	Major	7m	2
G	Major	7m	2
C♯	Diminished	0	63
D	Minor	0	53
D	Major	0	63
G	Major	0	63
C	Major	0	53
0	0	0	0
D	Major	0	53
A	Major	0	63
D	Major	0	53
D	Major	0	53
E	Major	0	63
A	Major	0	53
A	Major	-5	53
E♯	Diminished	7^{o}	7
F♯	Minor	0	53
B	Major	0	63
E	Major	0	53
A	Major	0	53
A	Major	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
E	Major	7m	7
E	Major	0	53
A	Major	0	63
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	0	53
A	Major	0	63
A	Major	0	63
A	Major	7m	65
F♯	Major	0	63
B	Minor	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
C	Minor	0	63
F	Major	7m	2
0	0	0	0
F	Major	0	53
C	Major	0	63
F	Major	0	53
D	Diminished	7^{o}	7
E	Major	7m	2
A	Minor	0	63
C	Diminished	7^{o}	7
0	0	0	0
B	Major	0	53
E	Minor	0	53
G	Major	0	63
C	Major	0	63
F	Major	0	53
D	Minor	0	63
C♯	Diminished	0	63
D	Major	0	53
D	Major	0	53
D	Major	7m	65
D	Major	7m	65
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53

N^{o}9: ***Che dirò? Che ascoltai? (+ Alla tua fede il padre + Oh giorno di dolore!)*** (pp. 130-133, 80 bars)

Root	Triad	Alteration	Inversion
B♭	Major	0	63
B♭	Major	7m	65
G	Major	0	63

G	Major	7m	65
C	Minor	0	53
D	Major	0	63
G	Minor	0	53
C	Major	0	63
C	Major	7m	65
A	Major	0	63
D	Minor	0	63
D	Minor	0	63
E	Major	0	53
E	Major	0	53
B	Diminished	0	63
C	Major	0	53
C	Major	0	53
D	Major	0	63
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
F♯	Major	7m	2
0	0	0	0
F♯	Major	0	53
C♯	Major	0	63
F♯	Minor	0	53
B	Major	0	63
E	Major	0	53
E	Major	0	53
A	Major	0	63
D	Major	0	53
D	Major	0	53
D	Major	7m,-5	7
G	Major	0	53
C	Major	0	53
D	Major	7m	2
0	0	0	0
D	Major	0	53
G	Major	0	53
D♯	Diminished	7^{o}	7
E	Minor	0	53
E	Major	7m	2
A	Minor	0	63
C	Major	7m	7
A	Major	0	63
D	Minor	0	53
D	Minor	-5	53
[1] D	Minor	-5	53
D	Major	7m	65
G	Minor	0	53
G	Minor	0	53
D	Diminished	0	53
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
B♭	Major	0	63
B♭	Major	0	63
G	Major	0	63
C	Minor	0	63
C	Major	7m	65
F	Minor	0	53
B♭	Major	0	63
E♭	Major	0	53
F	Major	7m	2
B♭	Major	0	63
D	Major	0	63
G	Minor	0	63
C	Major	0	63
F	Major	0	63
F	Major	0	63
D	Minor	0	64
C♯	Diminished	0	63
D	Minor	0	63
E	Major	7m	2
A	Minor	0	63
D	Major	0	63
G	Major	0	53
B	Major	0	63
E	Minor	0	53
F♯	Major	7m	2
0	0	0	0
F♯	Major	0	53
[2] G	Major	0	53

[1] Bar 50: The G note has been omitted in the chord analysis.
[2] p. 91 of [86]: The orchestra play this chord.

N^{o}10: ***Qui, dove la vendetta (+ Sedete, o Prenci + Signor, son io + Inclita Ismene)*** (pp. 161-166)			
Root	**Triad**	**Alteration**	**Inversion**
A	Major	0	63
D	Major	0	63
G	Major	0	53
G	Major	0	53
C	Major	0	63
E	Major	7m	2
A	Minor	0	63
B	Major	0	63
E	Minor	0	53
F♯	Major	0	53
B	Minor	0	53
C♯	Major	7m	2
0	0	0	0
C♯	Major	0	53
F♯	Minor	0	53
C♯	Diminished	0	63
D	Major	0	53
D	Major	0	53
G	Major	0	53
C	Major	0	63
E	Major	0	63
A	Minor	0	53
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	53
C	Major	-5	53
C	Major	7m,-5	7
F	Major	0	53
B♭	Major	0	63
B♭	Major	0	63
E♭	Major	0	53
0	0	0	0
F	Major	0	53
C	Major	0	63
C	Major	0	63

F	Major	0	53
0	0	0	0
G	Major	0	53
D	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	0	53
E	Minor	7m,-5	7
C♯	Diminished	0	63
D	Major	0	53
E	Major	7m	2
A	Major	0	63
C♯	Major	0	63
F♯	Minor	0	63
G♯	Major	7m	2
C♯	Minor	0	63
F♯	Minor	0	53
G♯	Major	7m	2
0	0	0	0
G♯	Major	0	53
A	Major	0	53
A	Major	7m	7
F♯	Major	0	63
B	Minor	0	53
B	Major	7m	65
E	Minor	0	53
C	Major	0	63
B	Diminished	0	63
C	Major	0	53
C	Major	0	53
C	Major	-5	53
C	Major	7m,-5	7
F	Major	0	53
A	Major	0	63
D	Minor	0	53
D	Minor	-5	53
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
B♭	Major	0	53
B♭	Major	7m	7
G	Major	0	63
C	Minor	0	53
F	Minor	0	53
0	0	0	0
G	Major	0	53
F♯	Diminished	7°	7
D	Major	7m	65
G	Minor	0	53
G	Minor	0	63
A	Major	0	53
D	Minor	0	63
D	Minor	0	63
E	Major	0	53
A	Minor	0	53
B	Major	0	63
E	Minor	0	53
E	Minor	0	53

F♯	Major	7m	2
B	Minor	0	63
C♯	Major	7m	65
F♯	Minor	0	53
B	Major	0	63
E	Major	0	53
G♯	Major	7m	2
C♯	Minor	0	63
C♯	Minor	0	63
E	Major	0	63
A	Major	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
E	Major	0	53
E	Major	7m	7
A	Major	-5	53
A	Major	0	53
D	Major	0	53
B	Minor	0	63
E	Major	7m	2
0	0	0	0
E	Major	0	53
A	Major	0	53
0	0	0	0

N°11: ***Ah giacché son tradito*** (p. 174, 12 bars)			
Root	**Triad**	**Alteration**	**Inversion**
A	Major	0	63
A	Major	7m	65
D	Major	0	53
D	Major	7m	7
G	Major	0	63
B	Major	0	63
E	Minor	0	63
E	Minor	0	63
G	Major	7m	65
C	Major	0	53
D	Major	7m	2
0	0	0	0
D	Major	0	53
G	Major	0	53
0	0	0	0

N°12: ***E credarai, Signor*** (pp. 179-182, 90 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
C	Major	7m	65
F	Major	0	53
F	Major	7m	7
B♭	Major	-5	53
D	Major	0	63
D	Major	7m	65
G	Minor	0	53
G	Minor	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
B♭	Major	0	53
B♭	Major	0	53
B♭	Major	7m	7
E♭	Major	-5	53
E♭	Major	0	53

E♭	Major	7m	7
C	Major	0	63
C	Major	7m	65
F	Minor	0	53
F	Minor	0	53
G	Major	7m	2
C	Minor	0	63
D	Major	0	63
G	Minor	0	53
C	Major	7m	65
F	Major	0	53
D	Minor	0	63
C♯	Diminished	0	63
D	Minor	0	53
G	Minor	0	53
0	0	0	0
A	Major	0	53
E	Major	0	63
A	Minor	0	63
A	Minor	0	63
D	Major	0	63
G	Major	0	53
B	Major	7m	65
E	Minor	0	53
0	0	0	0
F♯	Major	0	53
C♯	Major	0	63
F♯	Major	0	53
D	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	0	53
G	Major	7m	7
E	Major	0	63
E	Major	7m	65
A	Minor	0	53
A	Minor	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
C	Major	0	53
A	Major	0	53
D	Minor	0	63
D	Major	7m	65
G	Minor	0	53
B♭	Major	0	63
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
C	Minor	0	63
G	Major	7m	2
C	Minor	0	63
E♭	Major	0	63
F	Major	7m	7
F	Major	7m	7
B♭	Major	-5	53
D	Major	0	63
D	Major	7m	65
G	Minor	0	53
C	Minor	0	53
0	0	0	0

D	Major	0	53
A	Major	0	63
D	Minor	0	53
E	Major	7m	2
A	Minor	0	63
A	Minor	0	63
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
F♯	Major	7m	2
B	Minor	0	63
B	Minor	0	63
C♯	Major	7m	65
F♯	Minor	0	53
F♯	Minor	-5	53
D	Major	0	63
D	Major	7m	65
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
E	Minor	0	53
B	Diminished	0	63
C	Major	0	53
D	Major	0	63
G	Major	0	53
C	Major	0	63
E	Major	7m	2
E	Major	0	63
A	Minor	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
E	Minor	0	53

Nº13: ***Sifare, per pietà*** (pp. 188-189, 29 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	63
C	Major	0	63
[1]F	Major	0	53
D	Minor	0	63
E	Major	0	53
E	Major	0	53
A	Minor	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
D	Major	7m	65
G	Major	0	53
E	Minor	0	63
B	Major	7m	43
E	Minor	0	53
F♯	Major	7m	2
B	Major	0	63
B	Major	0	63
C♯	Major	0	63
F♯	Minor	0	53
G♯	Major	7m	2
C♯	Minor	0	63
E	Major	0	63
A	Major	0	63
D	Major	0	53

G	Major	0	53
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
[2]D	Major	0	53

[1]Bar 5: The E note has been omitted in the chord analysis.
[2]p. 189: This chord is played together with the orchestra.

Nº14: ***Pera omai chi m'oltraggia*** (pp. 205-206, 41 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
C	Minor	0	63
0	0	0	0
F	Major	0	53
B♭	Major	0	53
B♭	Major	0	53
G	Minor	0	63
D	Major	0	63
G	Minor	0	53
B♭	Major	7m	65
E♭	Major	0	53
A♭	Major	0	63
C	Major	7m	65
F	Minor	0	53
G	Major	0	53
C	Minor	0	63
C	Minor	0	63
D	Major	0	53
G	Minor	0	63
G	Minor	0	63
A	Major	0	53
D	Minor	0	53
D	Diminished	7º	7
0	0	0	0
E	Major	0	53
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
F♯	Major	7m	2
B	Minor	0	63
B	Major	0	63
E	Minor	0	53
A	Minor	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
E	Minor	0	53

Nº15: ***Re crudel, Re spietato (+ Mio Re, t'affretta)*** (pp. 211-213, 59 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B	Major	0	63
B	Major	7m	65
E	Major	0	53
C♯	Major	7m	65
F♯	Minor	0	53
D	Major	0	63
D	Major	7m	65
B	Major	0	63
E	Minor	0	53
E	Minor	0	53
C♯	Diminished	0	63
F♯	Major	7m	2
0	0	0	0
F♯	Major	0	53
B	Minor	0	63
D	Major	0	63
G	Major	0	63
G	Major	7m	65
C	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
D	Major	0	63
G	Minor	0	63
B♭	Major	7m	43
E♭	Major	0	53
C	Minor	0	63
G	Major	7m	2
C	Minor	0	63
D	Major	7m	2
G	Minor	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
C	Major	0	63
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
E	Major	0	63
A	Major	0	53
D	Major	0	63
G	Major	7Ma,-5	7
E	Minor	0	63
F♯	Major	0	53
B	Minor	0	63
C♯	Major	0	63
F♯	Minor	0	53
A	Major	0	63
D	Major	0	53
D	Major	7m,-5	7
G	Major	0	53
C	Major	0	63
C	Major	0	63
F	Major	7Ma,-5	7
D	Minor	0	63
E	Major	0	53
A	Minor	0	53
C♯	Diminished	7º	7
A	Major	7m	65
D	Minor	0	53
D	Minor	0	53
E	Major	7m	2
0	0	0	0

E	Major	0	53
A	Minor	0	63
0	0	0	0

N°16: ***Lagrime intempestive*** (p. 219, 8 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	53
F♯	Diminished	7°	7
D	Major	7m	65
G	Minor	0	53
A	Major	7m	2
D	Minor	0	63
C	Major	0	63
C	Major	7m	65
F	Minor	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
*C	Minor	0	53

*p. 219: This chord is played together with the orchestra.

N°17: ***Che fai, Regina? (+ Che mi val questa vita)*** (pp. 228-229, 39 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
D	Major	7m	65
B	Major	0	63
E	Minor	0	53
E	Minor	-5	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	0	63
E♭	Major	7Ma,-5	7
C	Minor	0	63
D	Major	0	53
D	Major	0	53
G	Minor	0	63
G	Minor	0	63
F	Major	0	63
B♭	Major	0	53
E♭	Major	0	63
A♭	Major	0	63
A♭	Major	0	63
C	Major	7m	65
F	Minor	0	53
G	Major	7m	2
C	Minor	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	53
C	Major	7m	2
0	0	0	0
C	Major	0	53
F	Major	0	53
0	0	0	0
B♭	Major	0	63
B♭	Major	0	63
C	Major	7m	65
F	Major	0	53
D	Minor	0	63

C♯	Diminished	0	63
D	Minor	0	53
E	Major	7m	2
A	Minor	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	53

N°18: ***Sorte crudel, stelle (+ Teco i patti, Farnace)*** (pp. 238-240, 56 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
A	Major	0	63
A	Major	7m	65
D	Minor	0	53
D	Minor	0	63
E	Major	0	53
E	Major	0	53
A	Minor	0	63
C	Major	0	63
F	Major	0	53
F	Major	0	53
C	Minor	0	63
D	Major	0	53
D	Major	0	53
G	Minor	0	63
C	Major	0	63
F	Major	0	53
F	Major	0	53
G	Major	7m	2
C	Major	0	63
E	Major	0	63
A	Minor	0	63
A	Minor	0	63
B	Major	0	63
E	Minor	0	53
C♯	Diminished	0	63
F♯	Major	7m	2
B	Minor	0	63
D	Major	0	63
G	Major	0	63
A	Major	7m	65
D	Major	0	53
E	Major	0	63
A	Major	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
E	Major	0	53
A	Major	0	63
D	Major	0	63
G	Major	0	53
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53
0	0	0	0

N^{o}19: ***Figlio, amico, non più (+ Ah vieni, o dolce + Reo non si chiami)*** (pp. 255-258, 74 bars)

Root	Triad	Alteration	Inversion
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
D	Minor	0	63
C♯	Diminished	0	63
A	Major	7m	65
D	Minor	0	53
E	Major	7m	65
A	Minor	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
F♯	Major	0	63
F♯	Major	7m	65
B	Minor	0	63
C♯	Major	7m	65
F♯	Minor	0	53
B	Major	7m	65
E	Major	0	53
E	Major	7m,-5	7
A	Major	0	53
A	Major	0	53
*A	Major	0	53
A	Major	7m,-5	7
D	Major	0	53
E	Major	0	63
E	Major	0	63
E	Major	7m	65
A	Major	0	53
A	Major	0	53
F♯	Minor	0	63
C♯	Major	0	63
C♯	Major	7m	65
F♯	Minor	0	53
D	Major	0	63
G	Major	0	53
G	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
0	0	0	0
G	Major	0	53
C	Major	0	53
E	Major	7m	2
A	Minor	0	63
A	Minor	0	63
C	Diminished	7^{o}	7
0	0	0	0
B	Major	0	53
C	Major	0	53
C	Major	7m	7
F	Major	0	53
F	Major	7m	7
B♭	Major	-5	53
B♭	Major	0	53
D	Major	0	63
G	Major	0	53
0	0	0	0
A	Major	0	53
E	Major	0	63
A	Minor	0	63
B	Major	0	63
E	Minor	0	53
F♯	Major	7m	2
B	Minor	0	63
B	Minor	0	63
E	Minor	0	53
F♯	Major	7m	2
0	0	0	0
F♯	Major	0	53
C♯	Major	0	63
F♯	Minor	0	53
B	Major	0	63
E	Major	0	53
A	Major	0	53
D	Major	0	53
0	0	0	0
E	Major	0	53
A	Major	0	53
0	0	0	0

*Bar 24: The fifth of the chord is made by the voice.

Appendix I. Apollo and Hyacinthus

Nº1: *Amice! jam parata sunt omnia* (pp. 7-9, 57 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
E	Major	0	63
E	Major	0	63
A	Major	0	63
D	Major	0	53
G	Major	0	53
G	Major	0	63
C	Major	0	53
C	Major	0	63
F	Major	0	53
A	Major	0	63
D	Minor	0	53
G	Major	0	63
C	Major	0	53
F	Major	0	53
E	Major	0	53
E	Major	0	63
A	Minor	0	63
B	Major	0	63
E	Minor	0	53
D	Major	0	63
G	Major	0	53
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
A	Major	0	63
F♯	Major	7m	65
B	Minor	0	53
E♯	Diminished	0	64
0	0	0	0
C♯	Major	0	53
F♯	Major	0	63
B	Minor	0	63
A	Major	7m	65
D	Major	0	53
D	Major	7m	7
B	Major	0	63
B	Major	7m	7
E	Minor	0	63
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	53
D	Major	7m	65
G	Major	0	53
G	Major	0	53
E	Major	0	63
A	Minor	0	53
A	Minor	0	53
A	Major	7m	2
D	Minor	0	63
G	Minor	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
E	Major	0	63
A	Minor	0	53
A	Major	7m	65
D	Minor	0	53
D	Major	0	63
B	Major	7m	65
E	Minor	0	53
E	Minor	0	53
A	Major	0	2
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53

Nº2: *Heu me! Periimus!* (pp. 18-19, 32 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
G	Major	0	53
B	Major	0	63
E	Minor	0	53
E	Minor	0	53
E	Minor	0	63
A	Minor	0	53
B	Major	0	53
B	Major	7m	2
E	Minor	0	63
G	Major	7m	7
A	Minor	0	53
A	Major	7m	2
D	Minor	0	63
F♯	Diminished	7^{o}	7
G	Minor	0	53
A	Major	0	63
A	Major	0	63
D	Minor	0	53
D	Minor	0	63
F♯	Diminished	7^{o}	7
G	Major	7m	2
C	Minor	0	63
E	Diminished	7^{o}	7
C	Minor	7m	65
F	Minor	0	53
F	Major	7m	2
B♭	Major	0	63
C	Major	0	63
F	Major	0	53
D	Major	0	63

D	Major	0	63
G	Major	0	53
C	Major	0	63
F	Major	0	63
F	Major	0	63
B♭	Major	0	63
C	Major	0	63
F	Major	0	53
D	Major	7m	65
G	Minor	0	53
0	0	0	0
A	Major	0	53
D	Major	0	53
0	0	0	0

N°3: ***Ah nate! Vera loqueris*** (pp. 26-28, 62 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	63
G	Major	7m	2
C	Minor	0	63
C	Major	0	53
A	Major	0	63
D	Minor	0	53
E	Major	7m	65
A	Minor	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
F♯	Major	0	63
F♯	Major	7m	65
B	Minor	0	53
B	Minor	0	53
C♯	Major	7m	65
C♯	Major	7m	65
F♯	Minor	0	53
F♯	Major	7m	2
B	Minor	0	63
E	Minor	0	53
F♯	Major	7m	2
0	0	0	0
F♯	Major	0	53
G	Major	0	53
G	Major	0	53
G	Major	0	63
C	Major	0	63
D	Major	0	63
G	Major	7m	2
C	Major	0	63
D	Major	7m,-5	7
D	Major	7m	7
G	Major	0	53
A	Major	0	63
A	Major	0	63
D	Major	0	53
D	Major	7m	2
G	Major	0	63
C	Minor	0	63
D	Major	0	53
D	Major	0	63
D	Major	0	53
A	Major	7m	2
0	0	0	0

A	Major	0	53
E	Major	0	63
E	Major	7m	65
A	Major	0	53
A	Major	7m	2
0	0	0	0
D	Major	0	63
C♯	Major	0	63
C♯	Major	7m	2
F♯	Minor	0	63
F♯	Major	7m	2
F♯	Major	0	2
B	Minor	0	63
G	Major	0	63
G	Major	7m	65
C	Major	0	53
G	Major	0	53
C	Major	0	53
D	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	0	53
B	Diminished	7°	7
C	Minor	0	53
A	Major	0	63
D	Major	0	63
E	Major	0	63
E	Major	0	63
A	Major	0	63
B	Major	0	63
E	Major	0	53
A	Major	0	63
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	0	53
A	Major	0	53

N°4: ***Amare num quid filia*** (pp. 33-34, 37 bars)			
Root	**Triad**	**Alteration**	**Inversion**
F	Major	0	53
F	Major	0	53
F	Major	0	63
B♭	Major	0	53
C	Major	0	63
F	Major	0	53
D	Major	0	63
E	Major	0	63
A	Minor	0	53
A	Major	7m	2
D	Major	0	63
G	Major	0	53
0	0	0	0
A	Major	0	53
E	Major	0	63
A	Major	7m	2
D	Major	0	63
C♯	Major	0	63
F♯	Major	7m	2
B	Minor	0	63
*G	Major	7m,-5	7
F♯	Major	0	53
F♯	Major	7m	2
F♯	Major	0	63

B	Minor	0	53
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
E	Major	0	63
A	Major	0	53
A	Major	0	53
C♯	Diminished	7^{o}	7
D	Minor	0	53
D	Major	0	63
G	Major	0	63
G	Major	0	63
C	Major	0	63
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53

*Bar 19: The third of the chord is made by the voice.

N^{o}5: *Rex! de salute filii est actum* (pp. 41-46, 117 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B♭	Major	0	53
B	Diminished	7^{o}	7
C	Minor	0	53
C	Minor	0	53
E	Diminished	7^{o}	7
F	Minor	0	53
B♭	Minor	0	63
C	Major	0	53
C	Major	0	63
F	Minor	0	63
F	Major	7m	2
B♭	Minor	0	63
C	Major	0	53
E	Diminished	7^{o}	7
C	Major	7m	65
C	Major	0	63
C	Major	7m	65
F	Minor	0	53
D	Major	0	63
G	Minor	0	53
G	Major	0	63
C	Minor	0	53
C	Major	7m	2
F	Minor	0	63
F	Major	0	63
B♭	Minor	0	53
C	Major	7m	2
0	0	0	0
C	Major	0	53
G	Major	0	63
C	Minor	0	53
B♭	Diminished	7^{o}	7
E	Diminished	7^{o}	7
C	Major	7m	65
F	Minor	0	53
F	Major	7m	2
F	Major	0	63
B♭	Minor	0	53
E♭	Minor	0	63
F	Major	0	53
G	Major	0	63
G	Major	0	63
A	Major	0	63
D	Minor	0	53
G	Minor	0	53
0	0	0	0
A	Major	0	53
B	Major	0	63
E	Minor	0	53
E	Major	7m	2
A	Minor	0	63
G	Major	0	63
C	Major	0	53
A	Major	0	63
A	Major	0	63
D	Major	0	53
D	Major	0	63
E	Major	0	63
E	Major	7m	65
A	Minor	0	53
A	Major	0	63
D	Minor	0	53
G	Major	0	63
C	Major	0	53
F	Major	0	53
F	Major	0	63
B♭	Major	0	53
G	Major	0	63
C	Minor	0	53
*F	Minor	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
G	Minor	0	53
G	Major	7m	2
G	Major	7m	2
C	Minor	0	63
F	Minor	0	63
G	Major	0	53
C	Major	0	63
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53
G	Major	0	63
C	Major	0	63
C	Major	0	63
F	Major	0	53
F	Major	0	63
B♭	Major	0	53
C	Major	7m	2
0	0	0	0
C	Major	0	53
G	Major	0	63
C	Major	0	53
D	Major	7m	2
0	0	0	0

D	Major	0	53
G	Major	0	53
C	Major	0	63
C	Major	0	63
D	Major	0	63
G	Major	0	53
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
E	Major	7m	65
A	Major	0	53
B	Major	0	63
E	Major	0	53
E	Major	0	63
A	Minor	0	53
D	Minor	0	63
E	Major	0	53
C	Major	0	53
C	Major	0	53
A	Major	0	63
D	Minor	0	53
D	Major	7m	2
G	Minor	0	63
C	Major	7m	2
C	Major	7m,-5	2
F	Major	0	63
F	Major	0	63
G	Major	0	63
C	Major	0	53
D	Major	7m	2
0	0	0	0
D	Major	0	53
E	Major	0	63
A	Major	0	63
B	Major	0	63
E	Major	0	53
E	Major	0	63
A	Major	0	53
A	Major	0	63
D	Major	0	53
F♯	Diminished	7^o	7
G	Minor	0	53
G	Major	0	63
C	Major	0	53
D	Major	7m	2
0	0	0	0
D	Major	0	53
A	Major	0	63
D	Major	0	53
D	Minor	0	53
G	Minor	0	63
A	Major	0	53
0	0	0	0

*Bar 52: The D has been omitted in the chord analysis.

D	Major	0	53
G	Minor	0	53
G	Major	7m	2
C	Minor	0	63
C	Major	7m	2
F	Minor	0	63
F	Major	0	63
G	Major	0	63
C	Major	0	53
A	Major	0	63
D	Major	0	53
D	Major	7m	2
G	Major	0	63
C	Minor	0	63
D	Major	0	53
G	Major	0	63
C	Major	0	53
C	Major	7m	2
F	Major	0	63
B♭	Major	0	53
C	Major	7m	2
[1]0	0	0	0
F	Major	0	53
G	Major	7m	2
C	Minor	0	63
E	Diminished	7^o	7
C	Major	7m	65
F	Minor	0	53
F	Major	7m	2
B♭	Minor	0	63
B♭	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	53
B♭	Major	-5	53
G	Minor	0	53
E	Diminished	0	63
F	Major	0	53
A♭	Diminished	7^o	7
A♭	Diminished	7^o	7
E♭	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Minor	0	53
[2]A♭	Major	0	63
D	Major	0	63
G	Minor	0	63
G	Minor	0	63
C	Minor	0	63
D	Major	0	53
0	0	0	0

[1]Bar 27: After the rest there is a scale that have not been included in the analysis.
[2]Bar 42: The chord is inferred from the neighboring notes.

N°6: ***Heu! Numen! ecce!*** (pp. 49-51, 47 bars)			
Root	**Triad**	**Alteration**	**Inversion**
A	Major	0	63
D	Major	0	53
G	Major	0	63
A	Major	0	63
D	Major	7m	2
G	Minor	0	63
C	Minor	0	63

N°7: ***Quocumque me converto*** (pp. 79-81, 60 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Minor	0	63
E	Diminished	7^o	7
C	Major	7m	65
F	Minor	0	53
F	Major	7m	2
B♭	Minor	0	63
C	Major	0	63

C	Major	0	63
F	Major	0	53
D	Major	7m	65
G	Minor	0	53
G	Minor	0	53
G	Major	7m	2
C	Major	0	63
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
G	Major	0	53
C	Major	0	63
C	Major	0	63
B	Major	0	63
E	Minor	0	53
C	Major	0	53
C	Major	0	53
D	Minor	7m,-5	2
C	Major	0	53
D	Major	0	63
G	Major	0	53
G	Major	0	63
C	Major	0	63
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
A♭	Major	7m,-5	53
G	Major	0	53
C	Minor	0	63
D	Major	0	63
G	Minor	0	53
C	Minor	0	53
0	0	0	0
D	Major	0	53
E♭	Major	0	53
E	Diminished	7°	7
C	Major	7m	65
E	Diminished	7°,-5	7
C	Major	0	63
F	Minor	0	53
F	Major	7m	2
B♭	Minor	0	63
B♭	Major	0	63
B♭	Major	0	63
C	Major	0	63
F	Major	0	53
F♯	Diminished	7°	7
G	Minor	0	53
G	Major	7m	2
C	Minor	0	63
C	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	0	53
A	Major	0	63
D	Minor	0	53
D	Minor	0	53
D♯	Diminished	7°	7
E	Minor	0	53
E	Minor	0	63
A	Minor	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
E	Minor	0	53

N°8: ***Rex! me redire*** (p. 91, 6 bars)			
Root	**Triad**	**Alteration**	**Inversion**
A	Major	0	53
A	Major	0	63
D	Major	0	53
D	Major	0	63
G	Major	0	53

N°9: ***Funus et flore aemulo*** (p. 92, 2 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	0	53

N°10: ***Quid video?*** (pp. 92-95, 67 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B	Major	0	63
C♯	Major	0	63
F♯	Major	0	53
F♯	Major	0	53
B	Minor	0	63
A	Major	0	63
D	Minor	0	63
G	Major	7m	2
E	Major	7m	7
C♯	Major	0	63
C♯	Major	0	63
F♯	Minor	0	53
G♯	Major	7m	2
0	0	0	0
G♯	Major	0	53
C♯	Minor	0	63
B	Major	0	63
B	Major	0	63
E	Minor	0	53
E	Major	7m	2
A	Minor	0	63
D	Major	0	63
G	Minor	0	53
E♭	Major	0	63
A	Major	7m	2
0	0	0	0
A	Major	0	53
B♭	Major	0	53
B♭	Major	0	53
B♭	Major	0	63
E♭	Major	0	63
B♭	Major	7m	2
0	0	0	0
B♭	Major	0	53

G	Major	0	63
C	Minor	0	53
D	Major	0	63
G	Minor	0	53
G	Minor	0	53
F	Minor	0	63
G	Major	0	53
G	Major	0	63
G	Major	0	63
A	Major	0	63
D	Major	0	53
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	0	53
A	Major	0	53
*G♯	Diminished	0	64
A	Major	0	53
D	Major	0	63
D	Major	0	63
D♯	Diminished	7^o	7
E	Minor	0	53
E	Major	0	63
A	Minor	0	53
A	Major	7m	2
D	Minor	0	63
C	Major	0	63
F	Major	0	53
G	Major	0	63
C	Major	0	53
D	Major	7m	2
0	0	0	0
D	Major	0	53
A	Major	0	63
D	Major	0	63
E	Major	0	63
E	Major	0	63
A	Major	0	53
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53

*Bar 46: Sustained note in A.

Appendix J. The marriage of Figaro

N°1: *Sag, was hast du denn da zu mes-sen* (p. 30, 19 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
C	Major	7m	65
A	Major	0	63
0	0	0	0
A	Major	0	53
D	Major	0	63
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
0	0	0	0

N°2: *Wohlan denn, so hör* (pp. 38-40, 40 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
A	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
0	0	0	0
C	Major	7m	7
*F	Major	0	53

Bar 40: *The recitative has been consider ended at this point. After this point a different recitative style follows.

N°3: *Sie zögerten so lang* (pp. 47-48, 22 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B♭	Major	0	63
B♭	Major	7m	65
G	Major	0	63
C	Major	0	63
E	Major	0	63
C♯	Major	0	63
C♯	Major	7m	65
F♯	Minor	0	53
E	Major	0	63
A	Major	0	53
A	Major	7m	65
D	Major	0	53
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53
0	0	0	0

N°4: *Noch ist nicht alles verloren* (p. 58, 14 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
E	Major	0	63
A	Major	0	53
B	Major	0	63
B	Major	7m	65
E	Major	0	53
A	Major	0	53
0	0	0	0
B	Major	7m	7
E	Major	0	53

N°5: *Fahr hin, du alte* (pp. 66-68, 42 bars)			
Root	**Triad**	**Alteration**	**Inversion**
A	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
C	Major	7m	65
A	Major	0	63
D	Minor	0	53
E	Major	0	63
E	Major	7m	65
A	Major	7m	2
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
A	Diminished	0	63
B♭	Major	0	53
0	0	0	0

N°6: *Ich bin verloren! Keine Angst* (pp. 76-80, 83 bars)			
Root	**Triad**	**Alteration**	**Inversion**

0	0	0	0
F	Major	0	63
0	0	0	0
A	Diminished	0	53
D	Major	0	63
D	Major	7m	65
G	Minor	0	53
G	Major	7m	2
C	Minor	0	63
B♭	Major	0	63
0	0	0	0
E♭	Major	0	53
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
D	Major	0	63
G	Minor	0	63
A	Major	0	63
A	Major	7m	65
D	Minor	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B	Diminished	0	63
C	Major	0	63
D	Major	7m	65
G	Major	0	53
0	0	0	0
E	Minor	7m	7
A	Major	0	63
D	Major	0	63
B	Major	0	63
B	Major	7m	65
E	Major	0	53
A	Major	0	63
E♯	Diminished	7^{o}	7
F♯	Minor	0	53
E	Major	0	63
A	Major	0	53
A	Major	7m	2
D	Major	0	63
D	Major	7m	65
G	Major	7m	2
C	Major	0	63
C	Major	7m	65
A	Major	0	63
D	Minor	0	63
F	Major	0	63
F	Major	7m	65
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53

D	Major	0	63
D	Major	7m	65
G	Minor	0	53
A	Major	0	63
A	Major	0	65
D	Minor	0	53
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	7m	7
D	Major	0	53

N^{o}8: ***Was soll denn die Komödie?*** (pp. 100-101, 35 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	63
C	Major	0	63
F	Major	0	63
F	Major	7m	2
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
A♭	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Minor	0	53
G	Major	0	63
C	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
C	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	7m	65
C	Major	0	53
0	0	0	0
D	Major	0	53
G	Major	0	53

N^{o}7: ***Basilio, geschwind zu Figaro*** (pp. 95-96, 24 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B♭	Major	0	63
B♭	Major	7m	65
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63

N^{o}9: ***Er lE♭e! Er le be!*** (pp. 104-105, 28 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
A	Major	7m	65
D	Minor	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
0	0	0	0
D	Major	0	53

G	Major	0	53

N^o^10: *Komm nun, liE♭e Susanna* (pp. 121-125, 80 bars)			
Root	**Triad**	**Alteration**	**Inversion**
F	Major	0	63
B♭	Major	0	63
D	Major	0	63
F♯	Diminished	7°	7
G	Minor	0	53
G	Major	7m	2
C	Major	0	63
C	Major	7m	65
F	Major	0	53
0	0	0	0
*–	–	–	–
F	Major	0	53
B♭	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
A	Major	0	63
A	Major	7m	65
D	Major	0	53
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
F	Major	7m	2
B♭	Major	0	63
B♭	Major	7m	65
G	Major	0	63
G	Major	7m	65
C	Major	0	53
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
0	0	0	0
B♭	Major	0	53
F♯	Diminished	0	63
G	Minor	-5	53
D	Major	0	63
G	Major	0	53
G	Major	7m	65
C	Major	0	53
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53

*Bars 15-21: These 7 bars do not have been consider in the analysis because the recitative style clearly change in this part.

N^o^11: *Ach, wie peinlich, Susanna* (pp. 126-127, 29 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	63
C	Major	0	63
0	0	0	0
C	Major	-5	53
G♯	Diminished	0	63
A	Minor	-5	53
E	Major	0	63
A	Minor	0	53
A	Major	7m	2
D	Major	0	63
G	Major	0	53
G	Major	7m	2
C	Minor	0	63
E♭	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	63
0	0	0	0
F	Major	7m	7
B♭	Major	0	53

N^o^12: *Bravo, welch schöne Stimme* (pp. 133-134, 25 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
C	Major	0	63
C	Major	7m	65
A	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	7m	2
C	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53

N^o^13: *Ende nundiese Possen!* (pp. 144-147, 87 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
D	Major	0	63
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
D	Major	0	63
G	Major	0	53
G	Major	7m	2
C	Major	0	63
C	Major	7m	65
F	Major	0	53
F	Major	7m	7
D	Major	7m	65
G	Minor	0	53
G	Major	7m	2
C	Minor	0	63
E	Diminished	7°	7
F	Minor	0	53
E♭	Major	0	63

E♭	Major	7m	65
A♭	Major	0	53
B♭	Major	0	63
G	Major	0	63
C	Minor	0	63
D	Major	0	63
D	Major	7m	65
G	Minor	0	53
0	0	0	0
A	Major	7m	7
D	Minor	0	53
B♭	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
F	Major	7m	2
B♭	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	0	2
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	7m	7
C	Major	0	53

N^{o}14: ***Also wollen Sie nicht öffnen?*** (p. 157, 21 bars)

Root	Triad	Alteration	Inversion
C	Major	0	63
C	Major	0	65
A	Major	0	63
A	Major	0	63
A	Major	7m	2
D	Major	0	63
G	Major	0	53
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
E	Major	7m	2
A	Minor	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	53

N^{o}15: ***O sieh den kleinen*** (pp. 162-163, 36 bars)

Root	Triad	Alteration	Inversion
G	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53

D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
C	Major	7m	65
A	Major	0	63
D	Major	0	63
G	Major	7m	2
C	Major	0	63
C	Major	7m	65
F	Major	0	53
F	Major	7m	2
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
B♭	Major	0	53

N^{o}16: ***Sonderbare Verwirrung!*** (pp. 248-250, 48 bars)

Root	Triad	Alteration	Inversion
D	Major	0	63
G	Major	0	53
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Minor	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Minor	0	53
F	Major	7m	2
E♭	Major	0	63
D	Major	0	63
B	Major	0	63
E	Major	7m	2
A	Minor	0	63
D	Minor	0	63
0	0	0	0
E	Major	0	53
A	Minor	0	53

N^{o}17: ***Und warum warst du mit mir heute*** (p.257, 18 bars)

Root	Triad	Alteration	Inversion
A	Major	0	63
A	Major	7m	2
D	Major	0	63
D	Major	7m	65
G	Major	0	53
0	0	0	0
B	Minor	-5	53
F	Major	0	63
E	Major	0	63

E	Major	7m	65
A	Minor	0	53
B	Major	7m	2
0	0	0	0
B	Major	7m	7
C	Major	0	53

N°18: ***Der Prozeß ist entschieden*** (pp. 271-272, 38 bars)

Root	Triad	Alteration	Inversion
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
E♭	Major	7m	2
A♭	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Minor	0	53
F	Major	7m	2
D	Major	7m	7
G	Minor	0	63
A	Major	0	63
D	Major	7m	2
D	Major	7m	65
G	Minor	0	53
F	Major	0	63
B♭	Major	0	53
0	0	0	0
C	Major	7m	7
F	Major	0	53

N°19: ***Sehn Sie, mein liEber*** (pp. 295-296, 32 bars)

Root	Triad	Alteration	Inversion
F	Major	0	63
B♭	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Minor	0	53
0	0	0	0
G	Major	7m	2
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
D	Major	0	63
G	Minor	0	53
F	Major	0	63
B♭	Major	0	53
0	0	0	0
*–	–	–	–
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
A♭	Major	0	63
C	Major	0	63
C	Major	7m	65

F	Minor	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	53

*Bars 17-19: These 3 bars do not have been consider in the analysis because the recitative style clearly change in this part.

N°20: ***Ganz gewiß, gnäd'ger Herr*** (pp. 306-307, 25 bars)

Root	Triad	Alteration	Inversion
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	7m	7
B♭	Major	0	53
0	0	0	0
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
D	Major	0	63
G	Minor	0	53
G	Major	7m	2
C	Minor	0	63
F	Major	7m	2
B♭	Major	0	53

N°21: ***Der Brief ist gefaltet, wie*** (p. 311, 11 bars)

Root	Triad	Alteration	Inversion
C	Major	0	63
F	Major	0	53
G	Major	7m	2
C	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
0	0	0	0
D	Major	0	53
G	Major	0	53
0	0	0	0

N°22: ***Dies sind, gnädige Frau*** (pp. 315-318, 77 bars)

Root	Triad	Alteration	Inversion
A	Major	0	53
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
E♭	Major	0	53
E♭	Major	7m	2
A♭	Major	0	63

C	Major	0	63
C	Major	7m	65
F	Minor	0	53
F	Major	7m	2
B♭	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	7m	65
G	Major	0	63
C	Major	0	63
C	Major	7m	65
A	Major	0	63
A	Major	7m	65
D	Major	0	53
G	Major	0	63
B	Major	0	63
B	Major	7m	65
E	Minor	0	53
D	Major	0	63
G	Major	0	53
A	Major	7m	65
D	Major	0	53
D	Major	0	63
D	Major	7m	65
G	Major	0	53

0	0	0	0
A	Major	0	53
D	Major	0	53
A	Major	7m	43
D	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	7m	2
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Minor	0	53
G	Diminished	7^{o}	7
0	0	0	0
A	Major	7m	7
D	Major	0	53
B♭	Major	0	53
C	Major	0	63
C	Major	7m	65
A	Major	0	63
A	Major	7m	2
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
0	0	0	0
D	Major	0	53
G	Major	0	53

N°23: ***Barbarina, was suchst du?*** (pp. 341-343, 61 bars)			
Root	**Triad**	**Alteration**	**Inversion**
0	0	0	0
C	Major	0	63
C	Major	7m	65
F	Minor	0	53
F	Major	7m	2
B♭	Minor	0	63
B♭	Major	0	63
B♭	Major	7m	65
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
F	Major	7m	2
B♭	Major	0	63
B♭	Major	0	63
E♭	Major	0	53
0	0	0	0
G	Minor	-5	53
D♭	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
D	Major	0	63
D	Major	7m	65
G	Major	0	53
A	Major	7m	2

N°24: ***Im Pavillon zur Linken*** (pp. 349-351, 44 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
D	Major	7m	65
G	Major	0	53
G	Major	7m	2
C	Major	0	63
E	Major	0	63
E	Major	7m	65
A	Minor	0	53
A	Major	7m	2
D	Major	0	63
0	0	0	0
G	Major	7m	7
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
B♭	Major	7m	65
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	0	63
G	Major	7m	65
C	Major	0	53
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
D	Major	7m	65

G	Major	0	53
G	Major	7m	2
C	Major	0	63
C	Major	7m	65
A	Major	0	63
D	Minor	0	63
F	Major	0	63
B♭	Major	0	53
C	Major	7m	2
0	0	0	0
C	Major	7m	7
F	Major	0	53

N^{o}25: ***Ales ist richtig*** (p. 362, 2 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
C	Major	7m	65
F	Major	0	53
0	0	0	0

N^{o}26: ***Frau Gräfin, Marcellina sagt*** (p. 372, 21 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Minor	0	53
G	Major	7m	2
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	0	53
C	Major	0	63
F	Major	0	53
0	0	0	0
G	Major	7m	7
C	Major	0	53
0	0	0	0

N^{o}27: ***Schändliche, in solcher Weise*** (p. 378, 12 bars)			
Root	**Triad**	**Alteration**	**Inversion**
F	Major	0	63
B♭	Major	0	63
0	0	0	0
D	Major	0	63
G	Minor	0	53
G	Major	7m	2
D	Minor	0	63
E	Major	0	63
E	Major	7m	65
A	Minor	0	53
D	Minor	0	63
0	0	0	0
E	Major	0	53
A	Major	0	53

Appendix K. Cinderella

N°1: *Date for mezzo* (pp. 125-126, 42 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
E	Diminished	0	53
A	Major	0	63
D	Major	0	63
G	Major	0	63
G	Major	0	7
B	Diminished	0	53
C	Major	0	63
F	Major	0	53
B	Diminished	0	64
0	0	0	0
G	Major	7m	7
C	Major	-5	53
C	Major	0	63
F	Major	0	63
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
0	0	0	0
G	Major	7m	7
C	Major	-5	53
C	Major	0	63
F	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	7m	7
C	Major	-5	53

N°2: *Sappiate che fra poco* (pp. 150-151, 40 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
F♯	Diminished	0	53
G	Major	0	63
C	Major	0	63
E	Diminished	0	53
F	Major	0	63
F	Major	7m	65
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
E	Diminished	0	53
A	Major	0	63
C♯	Diminished	0	53
D	Major	0	63
D	Major	7m	65
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	7m	7
C	Major	-5	53

N°3: *Non so che dir* (pp. 182-183, 35 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
A	Major	0	63
D	Major	0	53
E	Major	0	63
A	Major	0	63
D	Major	0	53
0	0	0	0
E	Major	7m	7
A	Major	-5	53

N°4: *Allegrissimamente* (pp.226-228, 58 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
G	Major	0	63
A	Major	7m	2
0	0	0	0
A	Major	7m,-5	7
D	Major	0	53
D	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
A	Diminished	0	64
0	0	0	0
F	Major	7m	7
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63

G	Major	0	63
E	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	63
D	Major	7m	2
0	0	0	0
D	Major	0	53
G	Major	0	53

N°5: ***Grazie, vezzi*** (pp. 301-302, 33 bars)

Root	**Triad**	**Alteration**	**Inversion**
A	Major	0	63
D	Major	0	63
G	Major	0	53
C	Major	0	53
E	Diminished	0	53
A	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	7m	7
B♭	Major	-5	53

N°6: ***Ma bravo*** (pp. 314-318, 89 bars)

Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	53
A	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	53
F	Major	0	53
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	7m	7
D	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
C	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
A♭	Major	0	63
0	0	0	0
B♭	Major	7m	7
B	Major	0	63
E	Major	0	53

A	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	63
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
F♯	Diminished	0	64
0	0	0	0
D	Major	0	53
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	7m	7
A	Major	-5	53

N°7: ***Mi parche*** (pp. 497-501, 85 bars)

Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
E	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
C	Major	7m	43
F	Major	0	53
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E	Diminished	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	63
A♭	Major	0	63
D	Diminished	0	64
0	0	0	0
B♭	Major	7m	7
E♭	Major	-5	53
C	Major	0	63
F	Major	0	53
F	Major	0	63
B♭	Major	0	63
E♭	Major	0	64

F	Major	7m	2
0	0	0	0
F	Major	7m	7
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
F	Major	0	63
B♭	Major	0	63
G	Major	0	63
E	Major	0	63
A	Major	0	63
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	7m	7
A	Major	-5	53

N^{o}8: *Ti sognianco* (pp. 524-526, 60 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
G	Major	0	63
A	Major	7m	2
0	0	0	0
A	Major	0	53
B♭	Major	0	63
E♭	Major	0	63
A♭	Major	0	63
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
A♭	Major	0	53
F	Major	0	63
D	Major	0	63
E	Major	7m	2
0	0	0	0
E	Major	7m	7
A	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	53
D	Major	7m	2
0	0	0	0
D	Major	0	53
G	Major	0	63
C	Major	0	63

D	Major	7m	2
0	0	0	0
D	Major	7m	7
E♭	Major	0	53
E♭	Major	0	63
A♭	Major	0	53

N^{o}9: *La notte* (pp. 563-565, 51 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	63
F	Major	0	53
F	Major	0	63
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	63
B♭	Major	0	63
G	Major	0	63
E	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	63
E	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	7m,-5	7
D	Major	0	53

N^{o}10: *Mi seconda il* (p. 592, 15 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
G	Major	0	63
C	Major	0	53
A	Major	0	63
D	Major	0	53
E	Major	7m	2
E	Major	0	53
A	Major	-5	53

N^{o}11: *Quanto sei* (pp. 596-597, 38 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	63
B♭	Major	0	63

G	Major	0	63
C	Major	0	63
F	Major	0	63
B♭	Major	0	63
B♭	Major	7m,-5	7
A	Major	0	53

N$^{\text{o}}$12: ***Oh fa mal tempo*** (pp. 598-599, 13 bars)

Root	Triad	Alteration	Inversion
B♭	Major	0	63
E♭	Major	0	63
F	Major	0	53
F	Major	7m	2
B♭	Major	0	63
C	Major	7m	43
F	Major	0	53
D	Major	0	63
B	Major	0	63
E	Major	0	53
A	Major	0	63
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	7m	7
A	Major	-5	53

N$^{\text{o}}$13: ***Scusate amico*** (pp. 621-622, 26 bars)

Root	Triad	Alteration	Inversion
D	Major	0	63
G	Major	0	53
C	Major	0	63
D	Major	0	53
D	Major	0	63
G	Major	0	63
C	Major	0	53
C	Major	7m	43
F	Major	0	53
0	0	0	0
G	Major	0	53
A♭	Major	0	53
B♭	Major	0	63
E♭	Major	0	63
F	Major	0	64
0	0	0	0
F	Major	0	53
D	Major	0	63
G	Major	0	63
B♭	Major	7m	7
E♭	Major	0	63
F	Major	7m	43
0	0	0	0
F	Major	7m	7
B♭	Major	-5	53

N$^{\text{o}}$14: ***Dunque noi siam*** (pp. 701-702, 39 bars)

Root	Triad	Alteration	Inversion
D	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
C	Major	0	63
F	Major	0	53
F	Major	0	63
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
0	0	0	0
G	Major	0	53
C	Major	-5	53

N$^{\text{o}}$15: ***La pillola*** (p. 718, 19 bars)

Root	Triad	Alteration	Inversion
G	Major	0	63
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	63
A	Major	0	63
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	7m	7
A	Major	-5	53

Appendix L. The Barber of Seville

N°1: *Gente indiscreta!* (pp. 52-53, 36 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	53
C	Major	0	63
F	Major	0	63
B♭	Major	0	53
E♭	Major	0	63
A♭	Major	0	63
D♭	Major	0	53
D♭	Major	7m	7
B♭	Major	0	63
E♭	Major	0	63
C	Major	0	63
D	Major	0	63
G	Major	0	53
C	Major	0	63
0	0	0	0
D	Major	0	53
G	Major	0	53

N°2: *Ah ah! che bella vita* (pp. 74-81, 185 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	53
C	Major	7m	7
F	Major	0	53
B♭	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	53
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
E	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	53
0	0	0	0
A	Major	0	53
D	Major	0	53
D	Major	0	53
B	Major	0	63
E	Major	0	53
A	Major	0	53
D	Major	0	63
G	Major	0	53
C	Major	0	63
0	0	0	0
D	Major	0	53
G	Major	0	53
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
B♭	Major	0	53
B♭	Major	0	63
E♭	Major	0	63
A♭	Major	0	53
B♭	Major	7m	2
0	0	0	0
B♭	Major	0	53
E♭	Major	0	53
E♭	Major	0	53
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
A♭	Major	0	63
A♭	Major	0	53
B♭	Major	7m	2
0	0	0	0
B♭	Major	0	53
F	Major	0	63
D	Major	0	63
G	Major	0	53
C	Major	0	63
C	Major	7m	65
F	Major	0	53
D	Major	0	63
*D	Major	7m	65
G	Major	0	53
B	Major	0	63
E	Major	0	53
A	Major	0	63
B	Major	0	63
E	Major	0	53
A	Major	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
E	Major	0	53

Bar 167: *The bass has been consider as F♯ instead of F.

N°3: *Oh cielo!* (p. 83, 33 bars)			
Root	**Triad**	**Alteration**	**Inversion**
0	0	0	0
G	Major	0	53
C	Major	0	63
F	Major	0	63
B♭	Major	0	53
0	0	0	0
B♭	Major	0	63
E♭	Major	0	63
E♭	Major	7m	65
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
D	Major	0	53

N°4: *Evviva il mio padrone!* (p. 111, 14 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B	Major	0	63
E	Major	0	53
A	Major	0	63
B	Major	7m	2
0	0	0	0
B	Major	0	53
E	Major	0	53

N°5: *Sì, sì, la vincerò* (pp. 123-127, 123 bars)			
Root	**Triad**	**Alteration**	**Inversion**
E	Major	0	63
A	Major	0	63
D	Major	0	53
G	Major	0	63
C	Major	0	63
F	Major	0	53
0	0	0	0
G	Major	0	53
C	Major	0	53
C	Major	0	53
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
A♭	Major	0	63
0	0	0	0
G	Minor	0	63
F	Major	0	63
B♭	Major	0	63
E♭	Major	0	53
A♭	Major	0	63
A♭	Major	0	63
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
C	Major	7m	7
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	53
F	Major	0	53
B♭	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
C	Major	7m	2
0	0	0	0
C	Major	0	53
F	Major	0	53
F	Major	0	53
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
C	Major	7m	65
A	Major	0	63
D	Major	0	53
B	Major	0	63
B	Major	7m	65
E	Major	0	53
A	Major	0	63
D	Major	0	53
G	Major	0	53
0	0	0	0
A	Major	0	53
E	Major	0	63
A	Major	0	63
D	Major	0	53
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53

N°6: *Ah! che ne dite? - Ma bravi!* (pp. 139-141, 80 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
G	Major	0	63
C	Major	0	53
C	Major	0	63
F	Major	0	53
B♭	Major	0	53
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
B♭	Major	0	53
B♭	Major	0	53
B♭	Major	7m	7
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
C	Major	0	63
F	Major	0	53
B♭	Major	0	63

B♭	Major	7m	65
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
C	Major	0	63
A	Major	0	63
D	Major	0	53
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
A	Major	0	63
D	Major	0	53
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53

N°7: *Ora mi sento meglio* (pp. 153-154, 44 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	53
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
A♭	Major	0	63
B♭	Major	7m	65
E♭	Major	0	53
A♭	Major	0	63
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
F	Major	7m	2
0	0	0	0
F	Major	0	53
B♭	Major	0	53

N°8: *Brontola quanto vuoi* (p.175, 22 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B♭	Major	0	53
B♭	Major	7m	7
G	Major	0	63
G	Major	7m	65
C	Major	0	53
F	Major	0	53
0	0	0	0
G	Major	0	53
C	Major	0	53
A	Major	0	63
D	Major	0	53
G	Major	0	53

C	Major	0	63
F	Major	0	53
0	0	0	0
G	Major	0	53
C	Major	0	53

N°9: *Ma vedi il mio destino* (p. 281, 19 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B	Major	0	63
E	Major	0	53
A	Major	0	63
D	Major	0	63
G	Major	0	53
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
0	0	0	0
C	Major	0	53
F	Major	0	53

N°10: *Insomma mio signore* (pp. 290-293, 91 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B♭	Major	0	63
E♭	Major	0	53
C	Major	0	63
F	Major	0	53
0	0	0	0
G	Major	0	53
D	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63
D	Major	7m	65
G	Major	0	53
C	Major	0	63
F	Major	0	53
B♭	Major	0	53
G	Major	0	63
C	Major	0	63
C	Major	7m	65
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
B	Major	0	63
E	Major	0	63
A	Major	0	53
B	Major	7m	2
0	0	0	0
B	Major	0	53
E	Major	0	53
E	Major	0	53
A	Major	0	63
D	Major	0	63
B	Major	0	63
E	Major	0	53
A	Major	0	63
B	Major	0	63
E	Major	0	53

A	Major	0	63
D	Major	0	63
0	0	0	0
E	Major	0	53
A	Major	0	53

B♭	Major	0	63
G	Major	0	63
C	Major	0	53
A	Major	0	63
D	Major	0	53
G	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53

N°11: ***Bella voce!*** (p. 308, 13 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
G	Major	0	63
C	Major	0	63
D	Major	7m	2
0	0	0	0
D	Major	0	53
G	Major	0	53

N°12: ***Bravo, signor barbiere*** (pp. 310-313, 90 bars)			
Root	**Triad**	**Alteration**	**Inversion**
G	Major	0	53
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	53
A♭	Major	0	63
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	53
B	Major	0	63
E	Major	0	53
F♯	Major	7m	2
0	0	0	0
F♯	Major	0	53
B	Major	0	53
E	Major	0	53
A	Major	0	63
D	Major	0	63
G	Major	0	53
C	Major	0	53
A	Major	0	63
D	Major	0	63
G	Major	0	53
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	53

N°13: ***Ah! disgraziato me! - che vecchio sospettoso!*** (p. 359, 31 bars)			
Root	**Triad**	**Alteration**	**Inversion**
E♭	Major	0	53
C	Major	0	63
F	Major	0	53

N°14: ***Dunque voi don Alonso*** (pp. 368-371, 96 bars)			
Root	**Triad**	**Alteration**	**Inversion**
D	Major	0	63
G	Major	0	53
G	Major	7m	65
C	Major	0	53
A	Major	0	63
D	Major	0	53
G	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63
B	Major	0	63
E	Major	0	53
A	Major	0	63
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	0	53
B	Major	0	63
E	Major	0	53
A	Major	0	63
D	Major	0	53
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	63
G	Major	0	53
C	Major	0	53
F	Major	0	63
B♭	Major	0	53
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	53
G	Major	0	63
C	Major	0	63
A	Major	0	63
D	Major	0	53
G	Major	0	63
C	Major	0	53
0	0	0	0

A	Major	0	63
0	0	0	0
D	Major	0	53
D	Major	0	63
B	Major	0	63
E	Major	0	53
A	Major	0	63
D	Major	0	53
G	Major	0	63
C	Major	0	63
F	Major	0	53
0	0	0	0
G	Major	0	53
C	Major	0	53

A	Major	0	63
D	Major	0	53
E	Major	7m	2
0	0	0	0
E	Major	0	53
A	Major	0	53

N^{o}15: ***Alfine eccoci qua*** (pp. 380-381, 39 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
F	Major	0	53
B♭	Major	0	53
E♭	Major	0	53
A♭	Major	0	63
A♭	Major	7m	65
F	Major	0	63
B♭	Major	0	63
C	Major	0	63
F	Major	0	53
D	Major	0	63
G	Major	0	53
C	Major	0	63
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
C	Major	0	53

N^{o}16: ***Ah, disgraziati noi!*** (pp. 398-400, 65 bars)			
Root	**Triad**	**Alteration**	**Inversion**
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
0	0	0	0
C	Major	0	53
F	Major	0	53
F	Major	0	53
B♭	Major	0	63
G	Major	0	63
C	Major	0	63
C	Major	0	63
F	Major	0	53
B♭	Major	0	63
E♭	Major	0	63
A♭	Major	0	63
B♭	Major	0	63
C	Major	0	63
A	Major	0	63
B	Major	0	63
E	Major	0	53

N^{o}17: ***Insomma, io ho tutti i torti!*** (pp. 420-421, 41 bars)			
Root	**Triad**	**Alteration**	**Inversion**
B♭	Major	0	53
G	Major	0	63
C	Major	0	53
F	Major	0	53
G	Major	7m	2
0	0	0	0
G	Major	0	53
D	Major	0	63
B	Major	0	63
E	Major	0	53
A	Major	0	53
0	0	0	0
B	Major	0	53
E	Major	0	53
A	Major	0	63
D	Major	0	63
G	Major	0	53
A	Major	7m	2
0	0	0	0
A	Major	0	53
D	Major	0	53

Products of the Thesis

The following products are associated with this thesis:

Audiovisual and radio program

- Documentary film "*Expedición marimba*" (52 minutes), UNIMEDIOS, *Universidad Nacional de Colombia*. Link: `https://www.youtube.com/watch?v=DQtetNlfDrA`. Documentary film Director: Luis Eduardo Martínez. This documentary was nominated in the India Catalina awards of 2018 in the category "*Mejor Nuevo Creador*". This documentary was also selected in the "*Muestra Internacional Documental de Bogotá 2018*" (MIDBO 2018).
- Radio program "*Música, física y marimba de chonta en el pacífico colombiano*". "Desde la botica", "*UN radio*", *Universidad Nacional de Colombia*. Date: October 11, 2018.

Prices and Scholarships

- First price in the contest Three Minute Thesis. *Universidad Nacional de Colombia*. Title: "*La consonancia como base para entender las afinaciones y prácticas de la música de marimba de chonta del Pacífico Colombiano: Un patrimonio de la humanidad que está desapareciendo*".
- Scholarship given by the *Centro Latinoamericano de Formación Interdisciplinaria* (CELFI) to participate in the "*Taller sobre aplicaciones matemáticas y computacionales a la música*" carried out between November 14 and 18 of 2016 in the University of Buenos Aires, Argentina.

Publications and publications in process

- Article accepted for publication in an international indexed journal (based on the research included in the Chapter 2 of this thesis):

  - J. Useche, R. Hurtado, and F. Demmer. Interplay between musical practices and tuning in the marimba de chonta music. *Journal of New Music Research*. Accepted date: 9 September 2019. doi: `http://dx.doi.org/10.1080/09298215.2019.1667399`.

- Article published in an international indexed journal (based on the research included in the Chapter 3 of this thesis):

  - J. Useche and R. Hurtado. Melodies as Maximally Disordered Systems under Macroscopic Constraints with Musical Meaning. *Entropy*, 21(5):532, 2019. doi: `https://doi.org/10.3390/e21050532`.

- Participation as author in one of the chapters of a book based on topics of music, mathematics and computation. The chapter refers to probability and statistics in music. Publisher: *Universidad Nacional de Quilmes*. Editors: Pablo Amster and Bruno Mesz. In process of publication.

Press releases

- "*La física y los sonidos de la marimba de chonta*". "*El espectador*". Date: January 13, 2016.

- "*Sonidos de la marimba de chonta, al ritmo de la física*". "*UN Periódico*". Date: September, 2015.

- "*Las voces de la marimba de chonta se niegan a desaparecer*". Author: Nibeth Adriana Duarte. This article won the contest "*Distintas maneras de narrar las Músicas de marimba y los cantos tradicionales del Pacífico Sur*" promoted by the Ministry of Culture of Colombia. Date: November, 2015.

Participation in events and oral presentations

- Participation in the Latin American Conference 2.0 on Complex Networks (LANET 2019) carried out in Cartagena, Colombia from August 5 to August 9, 2019. Title of the oral contribution "Networks of melody: The complex use of tonal consonance in music".

- Participation in the XX Conference on Nonequilibrium Statistical Mechanics and Nonlinear Physics (MEDYFINOL 2018) carried out in Santiago, Chile from December 3 to December 7, 2018. Oral contribution presented by Professor Rafael hurtado. Title: "From microscopic to macroscopic rules in music and their connection with tonal consonance".

- Participation in the XIV Latin American Workshop on Non Linear Phenomena (LAWNP), held from September 21 to 25, 2015, in Cartagena de Indias, Colombia. Title of the poster: "Link weight distribution in networks of musical melodic lines".

- Participation in the 1st Interdisciplinary Symposium on Social Network Analysis (INTERACT), held in the *Universidad de los Andes*, Bogotá, on June 18 and 19, 2018. Title of the poster: "Las notas amigas y las enemigas: teoremas de clusterización aplicados a relaciones signadas en música".

- Invited participant to the "Workshop Science, Art and Cognition" carried out from December 10 to 15 of 2017 at the *Centro Internacional de Ciencias*, in the city of Cuernavaca, Mexico. Title of the oral presentation: "The psychoacoustics of musical intervals and the emergence of macroscopic quantities in melody".

- Participation as invited speaker in the conference: "*Conexiones matemáticas: Matemáticas y música*" held in March 14, 2018, in the Konrad Lorentz University, Bogotá, Colombia.

- Oral presentation in the "*Taller sobre aplicaciones matemáticas y computacionales a la música*" carried out between November 14 and 18 of 2016 in the University of Buenos Aires, Argentina. Title: "*Las afinaciones y prácticas de la música de marimba de chonta de la costa pacífica de Colombia preservan la consonancia tonal*".

- Oral presentation carried out in the sound engineering program of the San Buenaventura Unviersity held in May 2, 2016, in Bogotá, Colombia. Title: "*Física musical*".

- Oral presentation carried out in the *Claustro de San Agustín* of the city of Tunja, Colombia. Title: "*Del timbre de la marimba de chonta a la complejidad de las obras clásicas*". Date: April 6, 2017. This presentation was promoted by the Bank of the Republic of Colombia.

Musical instruments donated

- Traditional marimba of 24 bars made by the maker Francisco Torres in Guapi, Colombia. This marimba stay in the "*Conservatorio de Música de la Universidad Nacional de Colombia*" for musical, pedagogical, and research purposes.

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