

## On the instability of nonlinear functional differential equations of fifth order

Cemil Tunç<sup>1,a</sup>

**Abstract.** The author gives sufficient conditions for non-existence of periodic solutions of two higher order nonlinear delay differential systems. Our technical approach is based on the construction of two suitable Lyapunov type functionals. An example is given to illustrate the obtained results. The main results here improve recent results found on the topic in the literature from the case of without delay to the delay case.

**Keywords:** Non-existence of periodic solutions, Lyapunov functional, nonlinear, differential equation, delay, fifth order.

**Resumen.** El autor presenta condiciones suficientes para la existencia de soluciones periódicas de dos ecuaciones diferenciales con retraso de alto orden. El enfoque técnico se fundamenta en la construcción de funcionales de Lyapunov adecuados. Un ejemplo ilustra los resultados obtenidos. El resultado principal presentado en este artículo mejora un resultado reciente en el tema de pasar el caso sin atraso al caso con atraso.

**Palabras claves:** .

Mathematics Subject Classification: 34D05, 34D20.

Recibido: julio de 2017

Aceptado: noviembre de 2017

### 1. Introduction

Qualitative properties of solutions to scalar differential equations and systems of differential equations of fifth order have been studied by many authors; see for example the references of this article such as Ezeilo [1, 2, 3], Ezeilo and Tejumola [4], Li and Duan [7], Li and Yu [8], Sadek [9], Sun and Hou [10], Tejumola [11], Tunç [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], Tunç and Ateş [23], Tunç and Erdoğan [25], Tunç and Karta [26], Tunç and Şevli [24] and their references. However, most of these publications only consider instability of solutions and non-existence of periodic solutions for scalar or system of differential equations of fifth order without delay. In this article, we investigate non-existence of periodic solutions for two systems of differential equations of fifth order with

<sup>1</sup>Department of Mathematics, Faculty of Sciences, Van Yuzuncu Yil University, Van, Turkey

<sup>a</sup>cemtunc@yahoo.com

constant delay. This article is motivated by the references of this paper, [1-3], [5], [7, 8, 9, 10, 11, 13, 12, 14, 15, 16, 17, 19, 18, 20, 21, 22, 23, 25, 26, 24], and that can be found in the literature.

We consider delay differential systems of fifth order

$$\begin{aligned} X^{(5)} + \Phi_1(\ddot{X})X^{(4)} + \Phi_2(\dot{X})\ddot{X} + \Phi_3(X, \dot{X}, \ddot{X})\dot{X} \\ + \Phi_4(\dot{X}) + \Phi_5(X(t-\tau)) = 0 \end{aligned} \quad (1)$$

and

$$\begin{aligned} X^{(5)} + AX^{(4)} + \Psi_2(\ddot{X})\ddot{X} + \Psi_3(\dot{X})\ddot{X} \\ + \Psi_4(\dot{X}) + \Psi_5(X(t-\tau)) = 0, \end{aligned} \quad (2)$$

respectively, where  $X \in \mathbb{R}^n$ ,  $t \in \mathbb{R}^+$ ,  $\mathbb{R}^+ = [0, \infty)$ ,  $\tau \in \mathbb{R}$ ,  $\tau > 0$  with  $t - \tau \geq 0$ ,  $A$  is a constant  $n \times n$ -symmetric matrix,  $\Phi_1, \Phi_2, \Phi_3, \Psi_2$  and  $\Psi_3$  are  $n \times n$ -symmetric continuous matrix functions,  $\Phi_4 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\Phi_5 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\Psi_4 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\Psi_5 : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $\Phi_4(0) = \Phi_5(0) = \Psi_4(0) = \Psi_5(0) = 0$  are continuous functions. The continuity of the matrix functions  $\Phi_1, \Phi_2, \Phi_3, \Psi_2, \Psi_3$  and the vector functions  $\Phi_4, \Phi_5, \Psi_4, \Psi_5$  guarantees the existence of the solutions of Eq. (1) and Eq. (2). The assumptions  $\Phi_4(0) = \Phi_5(0) = \Psi_4(0) = \Psi_5(0) = 0$  imply that both of Eq. (1) and Eq. (2) have the zero solution  $X(t) \equiv 0$ . In addition, we assume that the matrix functions  $\Phi_1, \Phi_2, \Phi_3, \Psi_2, \Psi_3$  and the vector functions  $\Phi_4, \Phi_5, \Psi_4, \Psi_5$  satisfy the Lipschitz condition with respect to their respective arguments. This fact guarantees the uniqueness of solutions of Eq. (1) and Eq. (2).

Let the symbols  $J_{\Phi_1}(W)$ ,  $J_{\Phi_2}(Z)$ ,  $J_{\Phi_4}(Y)$ ,  $J_{\Phi_5}(X)$ ,  $J_{\Psi_2}(Z)$ ,  $J_{\Psi_3}(Y)$ ,  $J_{\Psi_4}(Y)$  and  $J_{\Psi_5}(X)$  represent the Jacobian matrices corresponding to  $\Phi_1, \Phi_2, \Phi_4, \Phi_5, \Psi_2, \Psi_3, \Psi_4$  and  $\Psi_5$ , respectively. Throughout this paper, we assume that these Jacobian matrices exist and are continuous and symmetric.

We consider the equivalent differential systems corresponding to Eq. (1) and Eq. (2), respectively:

$$\begin{aligned} \dot{X} = \dot{Y}, \dot{Y} = Z, \dot{Z} = W, \dot{W} = U, \\ \dot{U} = -\Phi_1(W)U - \Phi_2(Z)W - \Phi_3(X, Y, Z)Z \\ - \Phi_4(Y) - \Phi_5(X) + \int_{t-\tau}^t J_{\Phi_5}(X(s))Y(s)ds \end{aligned} \quad (3)$$

and

$$\begin{aligned} \dot{X} = \dot{Y}, \dot{Y} = Z, \dot{Z} = W, \dot{W} = U, \\ \dot{U} = -AU - \Psi_2(Z)W - \Psi_3(Y)Z - \Psi_4(Y) \\ - \Psi_5(X) + \int_{t-\tau}^t J_{\Psi_5}(X(s))Y(s)ds, \end{aligned} \quad (4)$$

respectively.

We prove here two new results on the non-existence of periodic solutions of Eq. (1) and Eq. (2), respectively. The aim of this work is to improve the results of Tejumola [11, Theorem 3, Theorem 5] when  $n = 1$  and Tunç and Ateş [23] from the cases of the without delay to the cases of with constant delay. These are the novelty and originality of this article, and its contribution to the literature.

It is clear that  $X, Y, Z, W$  and  $U$  represent  $X(t), Y(t), Z(t), W(t)$  and  $U(t)$  respectively.

Let  $X = (x_1, x_2, \dots, x_n)$  and  $Y = (y_1, y_2, \dots, y_n)$ . The symbol  $\langle X, Y \rangle$  represents the usual scalar product  $\sum_{i=1}^n x_i y_i$  for any pair  $X, Y$  in  $\mathfrak{R}^n$ . In addition,  $\lambda_i(A)$ , ( $A = (a_{ij})$ ), ( $i, j = 1, 2, \dots, n$ ), are eigenvalues of  $n \times n$ -symmetric matrix  $A$ , and the matrix  $A = (a_{ij})$  is said to be positive definite if and only if the quadratic form  $X^T A X$  is positive definite, where  $X \in \mathfrak{R}^n$  and  $X^T$  denotes the transpose of  $X$ .

Consider the linear constant coefficient differential equation of fifth order:

$$x^{(5)} + a_1 x^{(4)} + a_2 \ddot{x} + a_3 \ddot{x} + a_4 \dot{x} + a_5 x = 0, \tag{5}$$

where  $a_1, a_2, \dots, a_5$  are some real constants. It can be followed from Tejumola [9] that if either of the hypotheses

(A1)  $a_1 \neq 0$ ,  $\text{sgn } a_1 = \text{sgn } a_5$ ,  $a_3 \text{sgn } a_1 < 0$

and

(A2)  $a_2 < 0$ ,  $a_4 > 0$

holds, then Eq. (5) has no non-trivial periodic solutions of any period. In addition, it should be noted that these odd and even subscripts features run through the generalized criteria obtained for the non-linear equations studied here.

We now consider only equations in which the right member does not depend explicitly on the time:

$$\frac{dx_i}{dt} = X_i(x_1, \dots, x_n), \quad (i = 1, 2, \dots, n),$$

where the functions  $X_i$  are defined and continuous in the region

$$\|x\| < H, \quad (H = \text{constant}).$$

We also require that the functions  $X_i$  have continuous partial derivatives  $\frac{\partial X_i}{\partial x_j}$ , ( $i, j = 1, 2, \dots, n$ ), in the region  $\|x\| < H$ .

**Theorem 1.1.** (Krasovskii [6]) *Let  $\bar{H}_0$  be a closed region, and suppose that for  $\|x\| \leq \bar{H}_0$ , the function  $v(x)$  has a derivative  $\frac{dv}{dt}$  that is positive-definite in the region  $v \geq 0$ . Suppose further that the point  $x = 0$  belongs to the closure of the region  $v > 0$ . Then the null solution  $x = 0$  of the differential equation  $\frac{dx_i}{dt} = X_i(x_1, \dots, x_n)$  is unstable, and there is a trajectory  $x(x_0, t)$  that converges on the point  $x = 0$  for  $t \rightarrow -\infty$ ,*

$$\lim \|x(x_0, t)\| = 0 \text{ for } t \rightarrow -\infty;$$

moreover,  $\|x_0\| < H_0$ .

## 2. Instability

We need the following lemma while proving our instability results.

**Lemma 2.1.** (Horn & Johnson [5]). *Let  $A$  be a real symmetric  $n \times n$ -matrix and*

$$a' \geq \lambda_i(A) \geq a, \quad (i = 1, 2, \dots, n),$$

where  $a'$  and  $a$  are some positive constants.

Then

$$a' \langle X, X \rangle \geq \langle AX, X \rangle \geq a \langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle.$$

### A. Hypotheses

We assume there exist positive constants  $a_1$ ,  $a_4$  and  $a_5$ ,  $a'_5$  and  $a_3 (< 0)$  such that the following conditions hold:

**(C1)**  $\lambda_i(\Phi_1(W)) \geq a_1$ ,  $\lambda_i(\Phi_3(X, Y, Z)) \leq a_3$ ,  $\Phi_4(0) = 0$ ,  $\Phi_4(Y) \neq 0$  when  $Y \neq 0$ ,  $\Phi_5(0) = 0$ ,  $\Phi_5(X) \neq 0$  when  $X \neq 0$ .

**(C2)** The Jacobian matrices  $J_{\Phi_1}(W)$ ,  $J_{\Phi_2}(Z)$ ,  $J_{\Phi_4}(Y)$  and  $J_{\Phi_5}(X)$  exist and are continuous and symmetric such that  $\lambda_i(J_{\Phi_4}(Y)) \geq a_4$  and  $a'_5 \geq \lambda_i(J_{\Phi_5}(X)) \geq a_5$ .

**(C3)**  $\tau < \min(\frac{-2a_3}{a'_5}, \frac{2a_5}{a'_5})$ .

The first instability theorem of this paper is given below.

**Theorem 2.2.** *If hypotheses (C1)–(C3) hold, then Eq. (1) has no non-trivial periodic solution of any period.*

*Remark 2.3.* There is no restriction on matrix function  $\Phi_2$ , except  $\Phi_2$  is an  $n \times n$ -symmetric continuous matrix function.

We have here some equalities that play important role in the sequel.

**Lemma 2.4.** *We assume that  $\Phi_4(0) = 0$ . Then the following hold:*

- 1<sup>0</sup>)  $\frac{d}{dt} \int_0^1 \langle \Phi_1(\sigma W)W, Z \rangle d\sigma = \langle \Phi_1(W)U, Z \rangle + \int_0^1 \langle \Phi_1(\sigma W)W, W \rangle d\sigma.$
- 2<sup>0</sup>)  $\frac{d}{dt} \int_0^1 \langle \sigma \Phi_2(\sigma Z)Z, Z \rangle d\sigma = \langle \Phi_2(Z)W, Z \rangle.$
- 3<sup>0</sup>)  $\frac{d}{dt} \int_0^1 \langle \Phi_4(\sigma Y), Y \rangle d\sigma = \langle \Phi_4(Y), Z \rangle d\sigma.$
- 4<sup>0</sup>)  $\frac{d}{dt} \int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta ds = \|Y\|^2 \tau - \int_{t-\tau}^t \|Y(s)\|^2 ds.$

**Proof.** First, we will give the proof of the equality 1<sup>0</sup>).

It is clear that

$$\begin{aligned} \frac{d}{dt} \int_0^1 \langle \Phi_1(\sigma W)W, Z \rangle d\sigma &= \int_0^1 \langle \Phi_1(\sigma W)W, W \rangle d\sigma + \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle \Phi_1(\sigma W)U, Z \rangle d\sigma \\ &\quad + \int_0^1 \langle \Phi_1(\sigma W)U, Z \rangle d\sigma \\ &= \sigma \langle \Phi_1(\sigma W)U, Z \rangle|_0^1 + \int_0^1 \langle \Phi_1(\sigma W)W, W \rangle d\sigma \\ &= \langle \Phi_1(W)U, Z \rangle + \int_0^1 \langle \Phi_1(\sigma W)W, W \rangle d\sigma. \end{aligned}$$

This result completes the proof of the equality 1<sup>0</sup>).

We now give the proof of 3<sup>0</sup>).

It follows that

$$\begin{aligned} \frac{d}{dt} \int_0^1 \langle \Phi_4(\sigma Y), Y \rangle d\sigma &= \int_0^1 \sigma \langle J_{\Phi_4}(\sigma Y)Z, Y \rangle d\sigma + \int_0^1 \langle \Phi_4(\sigma Y), Z \rangle d\sigma \\ &= \int_0^1 \sigma \langle J_{\Phi_4}(\sigma Y)Y, Z \rangle d\sigma + \int_0^1 \langle \Phi_4(\sigma Y), Z \rangle d\sigma \\ &= \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle \Phi_4(\sigma Y), Z \rangle d\sigma + \int_0^t \langle \Phi_4(\sigma Y), Z \rangle d\sigma \\ &= \sigma \langle \Phi_4(\sigma Y)Z \rangle|_0^1 = \langle \Phi_4(Y), Z \rangle. \end{aligned}$$

The proofs of the equalities 2<sup>0</sup>) and 4<sup>0</sup>) can be completed by following a similar way given in Sadek [9], Tunç ([14], [15]) and Tunç and Ateş [23]. Therefore, we omit the details.

**Proof of Theorem 1.** We define an auxiliary functional  $V_0 = V_0(X, Y, Z, W, U)$

by

$$\begin{aligned}
 V_0 &= \int_0^1 \langle \Phi_1(\sigma W)W, Z \rangle d\sigma + \int_0^1 \langle \sigma \Phi_2(\sigma Z)Z, Z \rangle d\sigma - \frac{1}{2} \langle W, W \rangle \\
 &+ \int_0^1 \langle \Phi_4(\sigma Y), Y \rangle d\sigma + \langle \Phi_5(X), Y \rangle + \langle U, Z \rangle \\
 &- \mu \int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta ds,
 \end{aligned} \tag{6}$$

where  $\mu \in \mathfrak{R}$ ,  $\mu > 0$  which will be chosen later.

Clearly,  $V_0(0, 0, 0, 0, 0) = 0$ .

Since  $\frac{\partial}{\partial \sigma} \Phi_4(\sigma Y) = J_{\Phi_4}(\sigma Y)Y$  and  $\Phi_4(0) = 0$ , then by an integration from  $\sigma_1 = 0$  to  $\sigma_1 = 1$ , we have

$$\Phi_4(Y) = \int_0^1 J_{\Phi_4}(\sigma Y)Y d\sigma_1.$$

From the last equality and hypothesis (C2), it follows that

$$\begin{aligned}
 \int_0^1 \langle \Phi_4(\sigma Y), Y \rangle d\sigma &= \int_0^1 \int_0^1 \langle \sigma_1 J_{\Phi_4}(\sigma_1 \sigma_2 Y)Y, Y \rangle d\sigma_2 d\sigma_1 \\
 &\geq \int_0^1 \int_0^1 \langle \sigma_1 a_4 Y, Y \rangle d\sigma_2 d\sigma_1 \geq \frac{1}{2} a_4 \langle Y, Y \rangle.
 \end{aligned}$$

Then, it is obvious that

$$V_0(0, \varepsilon, 0, 0, 0) \geq \frac{1}{2} a_4 \langle \varepsilon, \varepsilon \rangle = \frac{1}{2} a_4 \|\varepsilon\|^2 > 0$$

for all arbitrary  $\varepsilon \neq 0$ ,  $\varepsilon \in \mathfrak{R}^n$ .

Assume that  $(X, Y, X, W, U)$  is an arbitrary solution of system (3). Hence, from (6) and (3) the time derivative of auxiliary functional  $V_0$  leads that

$$\begin{aligned}
 \frac{d}{dt} V_0 &= \frac{d}{dt} \int_0^1 \langle \Phi_1(\sigma W)W, Z \rangle d\sigma + \langle J_{\Phi_5}(X)Y, Y \rangle + \frac{d}{dt} \int_0^1 \langle \sigma \Phi_2(\sigma Z)Z, Z \rangle d\sigma \\
 &- \langle \Phi_3(X, Y, Z)Z, Z \rangle + \frac{d}{dt} \int_0^1 \langle \Phi_4(\sigma Y), Y \rangle d\sigma - \langle \Phi_1(W)U, Z \rangle \\
 &- \langle \Phi_2(Z)W, Z \rangle - \langle \Phi_4(Y), Z \rangle + \left\langle \int_{t-\tau}^t J_{\Phi_5}(X(s))Y(s) ds, Z \right\rangle \\
 &- \mu \frac{d}{dt} \int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta ds.
 \end{aligned} \tag{7}$$

By Cauchy-Schwarz inequality in  $\mathfrak{R}^n$  and hypothesis (C2), we have

$$\begin{aligned} \langle \int_{t-\tau}^t J_{\Phi_5}(X(s))Y(s)ds, Z \rangle &\geq -\|Z\| \|\int_{t-\tau}^t J_{\Phi_5}(X(s))Y(s)ds\| \\ &\geq -\|Z\| \int_{t-\tau}^t \|J_{\Phi_5}(X(s))\| \|Y(s)\| ds \\ &\geq -a'_5 \|Z\| \int_{t-\tau}^t \|Y(s)\| ds \\ &\geq -\frac{1}{2} a'_5 \tau \|Z\|^2 - \frac{1}{2} a'_5 \int_{t-\tau}^t \|Y(s)\|^2 ds. \end{aligned}$$

Bringing together all the above estimates, in view of Lemma 1, Lemma 2 and (7), by (C1) and (C2) we obtain

$$\begin{aligned} \dot{V}_0 &\geq \langle a_5 Y, Y \rangle - \langle a_3 Z, Z \rangle + \int_0^1 \langle \Phi_1(\sigma W)W, W \rangle d\sigma \\ &\quad - \mu \tau \|Y\|^2 - \frac{1}{2} a'_5 \tau \|Z\|^2 + (\mu - \frac{1}{2} a'_5) \int_{t-\tau}^t \|Y(s)\|^2 ds. \end{aligned}$$

Let  $\mu = \frac{1}{2} a'_5$ . Then, it is clear that

$$\begin{aligned} \dot{V}_0 &\geq \langle (a_5 - 2^{-1} a'_5 \tau) Y, Y \rangle - \langle (a_3 + 2^{-1} a'_5 \tau) Z, Z \rangle \\ &\quad + \int_0^1 \langle \Phi_1(\sigma W)W, W \rangle d\sigma. \end{aligned}$$

Hypothesis (C3) implies that  $a_5 - 2^{-1} a'_5 \tau > 0$  and  $-a_3 - 2^{-1} a'_5 \tau > 0$ . In view of these inequalities and (C1), which guarantee that  $\Phi_1(W)$  is positive-definite, we have  $\dot{V}_0 \geq 0$ .

Thus, the hypotheses of Theorem 1 imply that  $\dot{V}_0(t) \geq 0$  for all  $t \geq 0$ , that is,  $\dot{V}_0$  is positive semi-definite. Finally,  $\dot{V}_0 = 0, (t \geq 0)$ , necessarily implies that  $Y = 0$  for all  $t \geq 0$ , and  $Z = \dot{Y} = 0, W = \dot{Y} = 0, \dot{W} = \ddot{Y} = 0$  for all  $t \geq 0$  so that

$$X = \xi, \quad Y = Z = W = U = 0.$$

From the last estimate and system (3), we have  $\Phi_5(\xi) = 0$ , which necessarily implies that  $\xi = 0$  only since  $\Phi_5(0) = 0$ . Then, it is clear that

$$X = Y = Z = W = U = 0 \quad \text{for all } t \geq 0.$$

Hence, we can conclude that all Krasovskii's properties hold (see, Krasovskii [6] and Tunç and Ateş [23]). Therefore, the Lyapunov functional  $V_0$  satisfies the hypothesis in Krasovskii [6] if the hypotheses of Theorem 1 hold. Thus, the basic properties of the Lyapunov functional  $V_0$ , which were verified above, prove that system (3) have no non-trivial periodic solutions of any period. Since system (3) is equivalent to Eq. (1), this completes the proof of Theorem 1.  $\square$

**Example 2.5.** Let  $n = 2$ . We choose the matrices  $\Phi_1, \Phi_2, \Phi_3$  and vectors  $\Phi_4$  and  $\Phi_5$  as the below:

$$\begin{aligned}\Phi(W) &= \begin{bmatrix} 2 + (1 + w_1^2)^{-1} & 1 \\ 1 & 2 + (1 + w_1^2)^{-1} \end{bmatrix}, \\ \Phi_2(Z) &= \begin{bmatrix} -2 - z_1^2 & 1 \\ 1 & -2 - z_1^2 \end{bmatrix}, \\ \Phi_3(X, Y, Z) &= \begin{bmatrix} -4 - x_1^2 - y_1^2 - z_1^2 & 1 \\ 1 & -4 - x_1^2 - y_1^2 - z_1^2 \end{bmatrix}, \\ \Phi_4(Y) &= \begin{bmatrix} 2y_1 + \arctan y_1 \\ 2y_2 + \arctan y_2 \end{bmatrix}, \Phi_4(0) = 0, \Phi_4(\xi i) \neq 0, \xi \neq 0, \\ \Phi_5(X) &= \begin{bmatrix} 3x_1 + \arctan x_1(t - \tau) \\ 3x_2 + \arctan x_2(t - \tau) \end{bmatrix}, \Phi_5(0) = 0.\end{aligned}$$

It is obvious that

$$J_{\Phi_4}(Y) = \begin{bmatrix} 2 + (1 + y_1^2)^{-1} & 0 \\ 0 & 2 + (1 + y_2^2)^{-1} \end{bmatrix}$$

and

$$J_{\Phi_5}(X) = \begin{bmatrix} 3 + (1 + x_1^2(t - \tau))^{-1} & 0 \\ 0 & 3 + (1 + x_2^2(t - \tau))^{-1} \end{bmatrix}.$$

By some trivial elementary operations, we can obtain the following relations:



$$\begin{aligned} \lambda_1(\Phi_1(W)) &= 1 + \frac{1}{1 + w_1^2}, \\ \lambda_2(\Phi_1(W)) &= 3 + \frac{1}{1 + w_1^2}, \\ \lambda_i(\Phi_1(W)) &\geq 1 = a_1, \\ \lambda_1(\Phi_2(Z)) &= -1 - z_1^2, \\ \lambda_2(\Phi_2(Z)) &= -3 - z_1^2, \\ \lambda_i(\Phi_2(Z)) &< 0, \\ \lambda_1(\Phi_3(Y)) &= -3 - x_1^2 - y_1^2 - z_1^2, \\ \lambda_2(\Phi_3(Y)) &= -5 - x_1^2 - y_1^2 - z_1^2, \\ \lambda_i(\Phi_3(Y)) &\leq -3 = a_3, \\ \lambda_1(J_{\Phi_4}(Y)) &= 2 + \frac{1}{1 + y_1^2}, \\ \lambda_2(J_{\Phi_4}(Y)) &= 2 + \frac{1}{1 + y_2^2}, \\ \lambda_i(J_{\Phi_4}(Y)) &\geq 2 = a_4, \\ \lambda_1(J_{\Phi_5}(X)) &= 3 + \frac{1}{1 + x_1^2(t - \tau)}, \\ \lambda_2(J_{\Phi_5}(X)) &= 3 + \frac{1}{1 + x_2^2(t - \tau)}, \\ a_5 = 3 &\leq \lambda_i(J_{\Phi_5}(X)) \leq 4 = a'_5. \end{aligned}$$

If

$$\tau < \min\left(\frac{-2a_3}{a'_5}, \frac{2a_5}{a'_5}\right) = \min\left(\frac{6}{4}, \frac{6}{4}\right) = \frac{3}{2},$$

then all the hypotheses of Theorem 1 hold. Hence, for the particular choices, the corresponding differential equation has no non-trivial periodic solution of any period.

**B. Hypotheses**

We assume there exist constants  $b_1(> 0)$ ,  $b_2(< 0)$ ,  $b_3(< 0)$ ,  $b_4(> 0)$ ,  $b_5(> 0)$  and  $b'_5(> 0)$  such that the following conditions hold:

- (H1)  $\lambda_i(A) \geq b_1, \lambda_i(\Psi_2(Z)) \leq b_2, \lambda_i(\Psi_3(Y)) \leq b_3, \Psi_4(0) = 0, \Psi_4(Y) \neq 0$  when  $Y \neq 0, \Psi_5(0) = 0, \Psi_5(X) \neq 0$  when  $X \neq 0$ .
- (H2) The Jacobian matrices  $J_{\Psi_2}(Z), J_{\Psi_3}(Y), J_{\Psi_4}(Y)$  and  $J_{\Psi_5}(X)$ , exist and are continuous and symmetric such that  $\lambda_i(J_{\Psi_4}(Y)) \geq b_4$  and  $b'_5 \geq \lambda_i(\Psi_5(X)) \geq b_5$ .
- (H3)  $\tau < \frac{b_4}{2b'_5}$ .

The second instability theorem of this paper is given below.

**Theorem 2.6.** *If hypotheses (H1)–(H3) hold, then Eq. (2) has no non-trivial periodic solution of any period.*

*We have here some equalities that play important role in the proof of Theorem 2.*

**Lemma 2.7.** *We assume that  $\Psi_5(0) = 0$ . Then the following hold:*

$$\begin{aligned} 1^0) \quad & \frac{d}{dt} \int_0^1 \langle \Psi_2(\sigma Z)Z, Y \rangle d\sigma = \langle \Psi_2(Z)W, Y \rangle + \int_0^1 \langle \Psi_2(\sigma Z)Z, Z \rangle d\sigma. \\ 2^0) \quad & \frac{d}{dt} \int_0^1 \langle \sigma \Psi_3(\sigma Y)Y, Y \rangle d\sigma = \langle \Psi_3(Y)Z, Y \rangle. \\ 3^0) \quad & \frac{d}{dt} \int_0^1 \langle \Psi_5(\sigma X), X \rangle d\sigma = \langle \Phi_5(X), Y \rangle. \end{aligned}$$

**Proof.** We now give only the proof of 3<sup>0</sup>).

It is obvious that

$$\begin{aligned} \frac{d}{dt} \int_0^1 \langle \Psi_5(\sigma X), X \rangle d\sigma &= \int_0^1 \sigma \langle J_{\Psi_5}(\sigma X)Y, X \rangle d\sigma + \int_0^1 \langle \Psi_4(\sigma X), Y \rangle d\sigma \\ &= \int_0^1 \sigma \langle J_{\Psi_5}(\sigma X)X, Y \rangle d\sigma + \int_0^1 \langle \Psi_5(\sigma X), Y \rangle d\sigma \\ &= \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle \Psi_5(\sigma X), Y \rangle d\sigma + \int_0^t \langle \Psi_5(\sigma X), Y \rangle d\sigma \\ &= \sigma \langle \Psi_5(\sigma X)Y \rangle \Big|_0^1 = \langle \Psi_5(X), Y \rangle. \end{aligned}$$

The proofs of the equalities 1<sup>0</sup>) and 2<sup>0</sup>) can be easily done by following a similar way given in Sadek [9], Tunç([14],[15]) and Tunç and Ateş [23]. Therefore, we omit the details of the proofs.

**Proof of Theorem 2.** Define an auxiliary functional  $V_1 = V_1(X, Y, Z, W, U)$  by

$$\begin{aligned} V_1 &= - \int_0^1 \langle \Psi_2(\sigma Z)Z, Y \rangle d\sigma - \langle U, Y \rangle - \int_0^1 \langle \sigma \Psi_3(\sigma Y)Y, Y \rangle d\sigma \\ &\quad - \frac{d}{dt} \int_0^1 \langle \Psi_5(\sigma X), X \rangle d\sigma - \langle AY, W \rangle + \frac{1}{2} \langle AZ, Z \rangle \\ &\quad + \langle Z, W \rangle - \lambda \int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta ds. \end{aligned} \tag{8}$$

Then, by hypothesis (H1), we follow that

$$V_1(0, 0, 0, 0, 0) = 0$$

and

$$V_1(0, 0, \varepsilon, \varepsilon, 0) \geq \frac{1}{2}(b_1 + 1)\|\varepsilon\|^2 > 0,$$

where  $\varepsilon \neq 0, \varepsilon \in \mathfrak{R}^n$ .

Differentiating the auxiliary functional  $V_1$  with respect to  $t$  along system (4), from (8) we find

$$\begin{aligned} \dot{V}_1 = & -\frac{d}{dt} \int_0^1 \langle \Psi_2(\sigma Z)Z, Y \rangle d\sigma - \frac{d}{dt} \int_0^1 \langle \sigma \Psi_3(\sigma Y)Y, Y \rangle d\sigma \\ & - \frac{d}{dt} \int_0^1 \langle \Psi_5(\sigma X), X \rangle d\sigma - \langle \Psi_2(Z)W, Y \rangle \\ & + \langle \Psi_3(Y)Z, Y \rangle + \langle \Psi_4(Y), Y \rangle + \langle \Psi_5(X), Y \rangle \\ & + \langle W, W \rangle - \left\langle \int_{t-\tau}^t J_{\Psi_5}(X(s))Y(s)ds, Y \right\rangle \\ & - \lambda\tau \|Y\|^2 + \lambda \int_{t-\tau}^t \|Y(\theta)\|^2 d\theta. \end{aligned}$$

Using the equalities given in Lemma 3 for  $\dot{V}_1$ , we obtain

$$\begin{aligned} \dot{V}_1 = & \langle \Psi_4(Y), Y \rangle - \int_0^1 \langle \Psi_2(\sigma Z)Z, Z \rangle d\sigma + \langle W, W \rangle \\ & - \lambda\tau \|Y\|^2 + \lambda \int_{t-\tau}^t \|Y(\theta)\|^2 d\theta. \end{aligned}$$

Meanwhile, by hypotheses (H1), (H2),  $\frac{\partial}{\partial \sigma} \Psi_4(\sigma Y) = J_{\Psi_4}(\sigma Y)Y$  and  $\Psi_4(0) = 0$ , it is clear that

$$\begin{aligned} \Psi_4(Y) &= \int_0^1 J_{\Psi_4}(\sigma_1 Y)Y d\sigma_1, \\ \langle \Psi_4(Y), Y \rangle &= \left\langle \int_0^1 J_{\Psi_4}(\sigma_1 Y)Y d\sigma_1, Y \right\rangle \geq \frac{1}{2} b_4 \langle Y, Y \rangle \end{aligned}$$

and

$$-\int_0^1 \langle \Psi_2(\sigma Z)Z, Z \rangle d\sigma \geq -\langle b_2 Z, Z \rangle.$$

In addition, by Cauchy-Schwarz inequality in  $\mathfrak{R}^n$  and hypothesis (H2), it is clear that

$$\begin{aligned} -\left\langle \int_{t-\tau}^t J_{\Psi_5}(X(s))Y(s)ds, Y \right\rangle &\geq -\|Y\| \left\| \int_{t-\tau}^t J_{\Psi_5}(X(s))Y(s)ds \right\| \\ &\geq -\|Y\| \int_{t-\tau}^t \|J_{\Psi_5}(X(s))\| \|Y(s)\| ds \\ &\geq -b'_5 \|Y\| \int_{t-\tau}^t \|Y(s)\| ds \\ &\geq -\frac{1}{2} b'_5 \tau \|Y\|^2 - \frac{1}{2} b'_5 \int_{t-\tau}^t \|Y(s)\|^2 ds. \end{aligned}$$

Hence

$$\begin{aligned} \dot{V}_1 &\geq \frac{1}{2} \langle b_4 Y, Y \rangle - \langle b_2 Z, Z \rangle + \langle W, W \rangle \\ &\quad - \frac{1}{2} b'_5 \tau \|Y\|^2 - \frac{1}{2} b'_5 \int_{t-\tau}^t \|Y(s)\|^2 ds \\ &\quad - \lambda \tau \|Y\|^2 + \lambda \int_{t-\tau}^t \|Y(\theta)\|^2 d\theta. \end{aligned}$$

Let  $\lambda = \frac{1}{2} b'_5$ . Then, it is clear that

$$\dot{V}_1 \geq \frac{1}{2} \langle (b_4 - 2b'_5 \tau) Y, Y \rangle - \langle b_2 Z, Z \rangle - \langle W, W \rangle \geq 0$$

by hypotheses (H1) and (H3) of Theorem 2. The rest of the proof is similar to the proof of Theorem 1. Therefore, we omit the details of the proof.  $\square$

## Acknowledgement

The author of this paper would like to thank the main editor and anonymous referees for their valuable comments and suggestions leading to improvement of this paper.

## References

- [1] J. O. C. Ezeilo, *Instability theorems for certain fifth-order differential equations*, Math. Proc. Cambridge Philos. Soc. **84** (1978), no. 2, 343–350.
- [2] ———, *A further instability theorem for a certain fifth-order differential equation*, Math. Proc. Cambridge Philos. Soc. **83** (1979), no. 3.
- [3] ———, *Extension of certain instability theorems for some fourth and fifth order differential equations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **66** (1979), no. 2, 239–242.
- [4] J. O. C. Ezeilo and H. O. Tejumola, *Boundedness and periodicity of solutions of a certain system of third-order non-linear differential equations*, Ann. Mat. Pura Appl. **74** (1966), no. 4, 286–316.
- [5] R. A. Horn and C. R. Johnson, *Topics in matrix analysis*, Cambridge University Press, 1994, Cambridge.
- [6] N. N. Krasovskii, *On conditions of inversion of A. M. Lyapunov's theorems on instability for stationary systems of differential equations*, Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 17–20.

- [7] W.-J. Li and K.-C. Duan, *Instability theorems for some nonlinear differential systems of fifth order*, J. Xinjiang Univ. Natur. Sci. **17** (2000), no. 3, 1–5.
- [8] W.-J. Li and Y. H. Yu, *Instability theorems for some fourth-order and fifth-order differential equations*, J. Xinjiang Univ. Natur. Sci. **7** (1990), no. 2, 7–10.
- [9] A. I. Sadek, *Instability results for certain systems of fourth and fifth order differential equations*, Appl. Math. Comput. **145** (2003), no. 2-3, 541–549.
- [10] W. J. Sun and X. Hou, *New results about instability of some fourth and fifth order nonlinear systems*, J. Xinjiang Univ. Natur. Sci. **16** (1999), no. 4, 14–17.
- [11] H. O. Tejumola, *Integral conditions of existence and non-existence of periodic solutions of some sixth and fifth order ordinary differential equations*, J. Nigerian Math. Soc. **31** (2012), 23–33.
- [12] C. Tunç, *An instability result for a certain non-autonomous vector differential equation of fifth order*, Panamer. Math. J. **3** (15), 51–58.
- [13] ———, *On the instability of solutions of certain nonlinear vector differential equations of fifth order*, Panamer. Math. J. **14** (2004), no. 4, 25–30.
- [14] ———, *Further results on the instability of solutions of certain nonlinear vector differential equations of fifth order*, Appl. Math. Inf. Sci. **2** (2008), no. 1, 51–60.
- [15] ———, *Instability of solutions for certain nonlinear vector differential equations of fourth order*, Nelīniīni Kōliv. **12** (2009), no. 1, 123–132.
- [16] ———, *An instability theorem for a certain fifth-order delay differential equation*, Filomat **25** (2011), no. 3, 145–151.
- [17] ———, *On the instability of solutions of some fifth order nonlinear delay differential equations*, Appl. Math. Inf. Sci. **5** (2011), no. 1, 112–121.
- [18] ———, *Instability for nonlinear differential equations of fifth order subject to delay*, Nonlinear Dyn. Syst. Theory **12** (2012), no. 2, 207–214.
- [19] ———, *Instability of non-linear functional differential equations of fifth order*, ITB J. Sci. **44A** (2012), no. 3, 23–249.
- [20] ———, *Instability of solutions for nonlinear functional differential equations of fifth order with  $n$ -deviating arguments*, Bul. Acad. Stiinte Repub. Mold. Mat. (2012), no. 1, 3–14.
- [21] ———, *Instability of a fifth-order nonlinear vector delay differential equation with multiple deviating arguments*, J. Math., 2013, Art. ID 809748, 5 pp.

- [22] ———, *Instability of a fifth order non-linear vector delay differential equation with multiple deviating arguments*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **76** (2014), no. 1, 155–162.
- [23] C. Tunç and M. Ateş, *Instability of certain nonlinear differential equations of fifth order*, J. Indones. Math. Soc. **22** (2016), no. 1, 83–92.
- [24] C. Tunç and H. Şevli, *On the instability of solutions of certain fifth order nonlinear differential equations*, Mem. Differential Equations Math. Phys. **35** (2005), 147–156.
- [25] C. Tunç and F. Erdogan, *On the instability of solutions of certain non-autonomous vector differential equations of fifth order*, SUT J. Math. **43** (2007), no. 1, 35–48.
- [26] C. Tunç and M. Karta, *A new instability result to nonlinear vector differential equations of fifth order*, Discrete Dyn. Nat. Soc., 2008, Art. ID 971534, 6 pp.