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VECTOR VALUED CHEBECHEV SYSTEMS

by

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Abstract. Let I be the unit interval and X be a real Banach space. The space of continuous functions on I with values in X is denoted by $C(I,X)$. The object of this paper is to introduce Chebechev systems in $C(I,X)$ and study the basic properties of such systems, and its relation to interpolation. It is also proved that a subspace that is generated by a weak Chebechev system in $C(I,X)$ is a Chebechev subspace.

Introduction. Let $I = [0,1]$, and $C(I)$ be the space of real valued continuous functions on I. Let $\{u_1,\ldots,u_n\}$ be a set of n-elements in $C(I)$. The functions $u_1,\ldots,u_n$ are said to form a Chebechev system if for every set of points $0 = t_1 < t_2 < \ldots < t_n = 1$,

$$\begin{vmatrix} u_1(t_1) & \ldots & u_1(t_n) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ u_n(t_1) & \ldots & u_n(t_n) \end{vmatrix} > 0$$

For the basic properties of Chebechev systems we refere to [2].

There are two ways in which one can try to generalize the concept of a Chebechev system. The first is to consider real valued continuous functions with domain on a compact set in a finite (or even infinite) dimensional vector space. It turned out, as Michili pointed out in [3], that such generalization is impossible, unless one puts very severe conditions and restrictions on the compact set under consideration.

Another way to generalize Chebechev systems is to consider continuous functions on I but with values in a real Banach space X. It is the object of this paper to consider such generalization. So, we define Chebechev systems for continuous functionss on I with values in a Banach space X. The basic properties of such systems are then discussed, and some results on interpolation are presented. Weak Chebechev systems are also defined.

§1. Notations. Let X be a real Banach space and let $C(I,X)$ denote the space of all continuous functions defined on I with values in X. For $f \in C(I,X)$, we set $\|f\|_\infty = \sup_t \|f(t)\|$. It is known, [1], that $C(I,X)$ is isometrically isomorphic to $C(I)\check{\otimes}X$, the completion of the injective tensor product of $C(I)$ with X.

The element $g\otimes x \in C(I)\check{\otimes}X$ denotes the function $u(t) = g(t)x$ in $C(I,X)$. The dual of X is denoted by X^*. If $F = \{x_1,\ldots,x_n\}$ is a set of n-elements in X, we let $[F]$ to denote the linear span of $x_1,\ldots,x_n$. For $x \in X$ and $x^* \in X^*$, $\langle x,x^*\rangle$ denotes the value of x^* at x.

§2. Chebechev Systems in $C(I,X)$. Let $U = \{u_1,\ldots,u_n\} \in X$ and $T = \{t_1,\ldots,t_n\} \subset I$. Set $E(U,T) = \{u_1(t_1),\ldots,u_1(t_n),u_2(t_1),\ldots,u_2(t_n),\ldots,u_n(t_1),\ldots,u_n(t_n)\}$, and $S(U,T) = [E(U,T)]$. Clearly $S(U,T)$ is a finite dimensional subspace of X. For $x^* \in X^*$, set

$$M(U,T,x^*) = \begin{vmatrix} \langle u_1(t_1),x^*\rangle & \ldots & \langle u_n(t_1),x^*\rangle \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \langle u_1(t_n),x^*\rangle & \ldots & \langle u_n(t_n), {}^*\end{vmatrix}$$

DEFINITION 2.1. Let $U = \{u_1,\ldots,u_n\} \subset C(I,X)$. Then U is said to form a Chebechev system if for every set $T = \{t_1,\ldots, t_n\}$ of *n-distinct* elements in I, there exists at least one $x^* \in X^*$, which does not vanish identically on $S(U,T)$ such that $M(U,T,x^*) \neq 0$.

EXAMPLE 2.2. Let e be a fixed element in X. Put $u_1(t) = te,\ldots,u_n(t) = t^n e$. Then $S(U,T) = [\{e\}]$. Using the Hahn Banach Theorem, let $x^* \in X^*$ be such that $\langle e,x^*\rangle = 1$. Then, if $T = \{t_1,\ldots,t_n\}$, one has

$$M(U,T,x^*) = \begin{vmatrix} t_1 & \ldots & t_1^n \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ t_n & \ldots & t_n^n \end{vmatrix}$$

Now, if $t_i \neq t_j$ for $i \neq j$, then $M(U,T,x^*) \neq 0$, since the set of real functions $g_i(t) = t^i$ is a Chebechev system in $C(I)$, [2].

EXAMPLE 2.3. Let $g_1,\ldots,g_n$ be a Chebechev system in $C(I)$ and $x_1,\ldots,x_n$ be arbitrary in X. Consider the set of elements in $C(I,X)$ defined by

$$u_1 = g_1 \otimes x_1,\ldots,u_n = g_n \otimes x_n .$$

If $T = \{t_1,\ldots,t_n\}$, $t_i \neq t_j$ if $i \neq j$, then by, choosing $x^* \in X^*$ such that $\langle x_i,x^*\rangle = 1$ for $i = 1,\ldots,n$, we have

$$S(U,T,X^*) = \begin{vmatrix} g_1(t_1) & \ldots & g_n(t_1) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ g_1(t_n) & \ldots & g_n(t_n) \end{vmatrix} \neq 0 .$$

Thus $U = \{u_1,\ldots,u_n\}$ is a Chebechev system in $C(I,X)$.

THEOREM 2.4. *If $U = \{u_1,\ldots,u_n\}$ is a Chebechev system in $C(I,X)$, then $u_1,\ldots,u_n$ are linearly independent.*

Proof. If possible assume $u_1,\ldots,u_n$ be linearly dependent. With no loss of generality, we assume $u_1 = \sum_{j=2}^{n} a_j u_j$. Let x^* be any element in X^* and $T = \{t_1,\ldots,t_n\}$, $t_i \neq t_j$ for $i \neq j$. Consider

$$M(U,T,X^*) = \begin{vmatrix} \langle u_1(t_1),x^*\rangle & \ldots & \langle u_n(t_1),x^*\rangle \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \langle u_1(t_n),x^*\rangle & \ldots & \langle u_n(t_n),x^*\rangle \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{j=2}^{n} a_j \langle u_j(t_1),x^*\rangle & \ldots & \langle u_n(t_1),x^*\rangle \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \sum_{j=2}^{n} a_j \langle u_j(t_n),x^*\rangle & \ldots & \langle u_n(t_n),x^*\rangle \end{vmatrix}$$

By performing elementary row operations on $M(U,T,x^*)$, one can get a column of zeros. Hence $M(U,T,x^*) = 0$ for all $x^* \in X^*$. This is a contradiction. Thus $u_1,\ldots,u_n$ must be linearly independent. ▲

Now we prove Zielk'x Theorem, [5], for the vector valued case.

THEOREM 2.5. *Let* $U = \{u_1,\ldots,u_n\} \subset C(I,X)$. *Then the following are equivalent:*

(i) $\{u_1,\ldots,u_n\}$ *is a Chebechev System.*
(ii) *Every element* u *in* $[U]$ *has at most* n-1 *zeros.*

Proof. (i) → (ii). If possible, let there exist $u \in [U]$ and $T = \{t_1,\ldots,t_n\}$, $t_i \neq t_j$ for $i \neq j$, such that $u(t_i) = 0$ for all $i = 1,2,\ldots,n$. If $u = a_1u_1+\ldots+a_nu_n$, it follows that

$$\begin{gathered} a_1\langle u_1(t_1),x^*\rangle+\ldots+a_n\langle u_n(t_1),x^*\rangle = 0 \\ \cdot \\ \cdot \\ \cdot \\ a_1\langle u_1(t_n),x^*\rangle+\ldots+a_n\langle u_n(t_n),x^*\rangle = 0 . \end{gathered}$$

Consequently, the matrix

$$\begin{pmatrix} \langle u_1(t_1), x^*\rangle \ldots \langle u_n(t_1), x^*\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle u_1(t_n), x^*\rangle \ldots \langle u_n(t_n), x^*\rangle \end{pmatrix}$$

is not invertible, for all x^*. Hence $M(U,T,x^*) = 0$ for all x^*, which contradicts the assumption on U. Thus u can have at most n-1-zeros.

(i) → (ii). Let $T = \{t_1,\ldots,t_n\}$, $t_i \neq t_j$ for $i \neq j$. By the assumption, it follows that for all $a = (a_1,\ldots,a_n) \in R^n$

$$a_1u_1(t_j)+\ldots+a_nu_n(t_j) = x_j \ ,$$

$j = 1,\ldots,n$, and $x_j \neq 0$ for at least one j. Assume, with no loss of generality, that $x_n \neq 0$. By the Hahn Banach Theorem, there is at least one $x^* \in X^*$ such that $\langle x_n, x^*\rangle \neq 0$. For such x^*, one has

$$\begin{pmatrix} \langle u_1(t_1), x^*\rangle \ldots \langle u_n(t_1), x^*\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle u_1(t_n), x^*\rangle \ldots \langle u_n(t_n), x^*\rangle \end{pmatrix} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} = \begin{pmatrix} \langle x_1, x^*\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle x_n, x^*\rangle \end{pmatrix},$$

for all $(a_1,\ldots,a_n) \in R^n$. But

$$\begin{pmatrix} \langle x_1, x^*\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle x_n, x^*\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix},$$

unless $(a_i,\ldots,a_n) = (0,\ldots,0) \in R^n$. Hence

$$\begin{vmatrix} \langle u_1(t_1), x^*\rangle \ldots \langle u_n(t_1), x^*\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle u_1(t_1), x^*\rangle \ldots \langle u_n(t_n), x^*\rangle \end{vmatrix} \neq 0 \ .$$

Consequently, $M(T,U,x^*) \neq 0$. This implies that $U = \{u_1,\ldots,u_n\}$ is a Chebechev system. ▲

For $T = t_1,\dots,t_n$, $t_i \neq t_j$ for $i \neq j$, set

$$e_1 = \begin{pmatrix} u_1(t_1) \\ \cdot \\ \cdot \\ \cdot \\ u_1(t_n) \end{pmatrix}, \dots, e_n = \begin{pmatrix} u_n(t_1) \\ \cdot \\ \cdot \\ \cdot \\ u_n(t_n) \end{pmatrix}$$

Let $X(T)$ be the vector space generated by $e_1,\dots,e_n$, where addition and scalar multiplication are defined in the natural way.

Now, we prove an interpolation theorem concerning vector valued Chebechev systems.

THEOREM 2.6. *Let $U = \{u_1,\dots,u_n\}$ be a Chebechev system in $C(I,X)$, and $T = \{t_1,\dots,t_n\}$, $t_i \neq t_j$ for $i \neq j$. If, $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is an arbitrary element in $X(T)$, then there exists a unique $u \in [U]$ such that $u(t_i) = y_i$, $i = 1,2,\dots,n$.*

Proof. First, we claim that $X(T)$ has dimension n. To see that, let $\lambda_1,\dots,\lambda_n$ be real numbers such that

$$\sum_{i=1}^{n} \lambda_i e_i = 0 .$$

Then

$$\begin{gathered} \lambda_1 u_1(t_1)+\dots+\lambda_n u_n(t_1) = 0 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_1 u_1(t_n)+\dots+\lambda_n u_n(t_n) = 0 . \end{gathered}$$

Hence, for every x^* X^*, we have

$$\lambda_1 \langle u_1(t_1),x^*\rangle+\dots+\lambda_n \langle u_n(t_1),x^*\rangle = 0$$

$$\lambda_1 \langle u_1(t_n),x^*\rangle+\dots+\lambda_n \langle u_n(t_n),x^*\rangle = 0 .$$

If $\lambda_i = 0$ for $i = 1,\dots,n$, it follows that the map

$$\begin{pmatrix} \langle u_1(t_1),x^*\rangle \dots \langle u_n(t_1),x^*\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle u_1(t_n),x^*\rangle \dots \langle u_n(t_n),x^*\rangle \end{pmatrix} : R^n \to R^n$$

is not invertible for all $x^* \in X^*$. This contradicts the fact that $U = \{u_1,\ldots,u_n\}$ is a Chebechev system. Hence $e_1,\ldots,e_n$ are independent in $X(T)$, and form a basis.

Now, consider the linear operator $B:R^n \to X(T)$ given by

$$B\begin{pmatrix}\lambda_1\\ \cdot\\ \cdot\\ \cdot\\ \lambda_n\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^{n}\lambda_i u_i(t_1)\\ \cdot\\ \cdot\\ \sum_{i=1}^{n}\lambda_i u_i(t_n)\end{pmatrix}$$

so that

$$B\begin{pmatrix}\lambda_1\\ \cdot\\ \cdot\\ \cdot\\ \lambda_n\end{pmatrix} = \begin{pmatrix}u_1(t_1)\ldots u_n(t_1)\\ \cdot\\ \cdot\\ \cdot\\ u_1(t_n)\ldots u_n(t_n)\end{pmatrix}\begin{pmatrix}\lambda_1\\ \cdot\\ \cdot\\ \cdot\\ \lambda_n\end{pmatrix}$$

Using the same arguement as in the first part of the proof, we get that $B:R^n \to X(T)$ is invertible. Since $\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix} \in X(T)$, then there exists a unique $\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix} \in R^n$ such $B\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix} = \begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}$. Hence, there is a unique $u \in [U]$ such that $u(t_i) = y_i$. This ends the proof of the theorem. ▲

REMARK 2.7. The definition we gave for Chebechev systems is not restrictive. To prove part of Jackson's Theorem, [5], we need to modify our definition. It is an interesting question whether Theorem 2.9 below is still valid for Chebechev systems according to Definition 2.1.

DEFINITION 2.8. A set of elements $U = \{u_1,\ldots,u_n\}$ in $C(I,X)$ is said to form a *weak Chebechev system if for every* $T = \{t_1,\ldots,t_n\}$, $t_i \neq t_j$ for $i \neq j$, there exist n-linear functionals $x_1^*,\ldots,x^*$ in X^*, (not necessarily distinct) such that

$$\begin{vmatrix} \langle u_1(t_1), x_1^*\rangle \ldots \langle u_n(t_1), x_1^*\rangle \\ \vdots \\ \langle u_1(t_n), x_n^*\rangle \ldots \langle u_n(t_n), x_n^*\rangle \end{vmatrix} \neq 0$$

The results that were proved for Chebechev systems in this paper can be also proved for weak-Chebechev systems, using the same techniques.

One basic property of Scalar Chebechev systems is the existence and uniquencess of best approximants in the span of the Chebechev system of every $f \in C(I)$. We now prove the same result for weak Chebechev systems. We assume in the following that X has the so-called approximation property, [1].

THEOREM 2.9. *Let $U = \{u_1,\ldots,u_n\}$ be a weak Chebechev system in C I,X . Then, for every $f \in C(I,X)$, there exists a unique $u \in [U]$ such that $\|f-u\|_\infty = \inf\{\|f-v\|_\infty : v \in [U]\}$. That is, $[U]$ is a Chebechev subspace of $C(I,X)$.*

Proof. Assume if possible that there exist n-linearly independent extremal points (L_i) of the unit ball of $[C(I,X)]^*$, and some $u \in [U]$, $u \neq 0$, such that $L_i(u) = 0$, $i = 1,\ldots,n$.

It is known, [1], that $[C(I,X)]^*$ is $M(I)\hat{\otimes}X^*$, the completion of the projective tensor product of $M(I)$ with X^*, where $M(I)$ is the space of all regular Borel measures on I. Since $M(I)$ and X^* are dual spaces, it follows that the extreme points of the unit ball of $M(I)\hat{\otimes}X^*$ are elements of the form $\mu\hat{\otimes}x^*$, where μ is an extreme element of the unit ball of $M(I)$ and x^* is an extreme element of the unit ball of X^* . But it is well known that the extreme elements of the unit ball of $M(I)$ are the point mass evaluations. Hence L_i is of the form $\delta t_i\hat{\otimes}x_i^*$, $t_i \in (0,1)$, and x_i^* is an extreme point of the unit ball of X^* . Thus, there exist $T = \{t_1,\ldots,t_n\}$, $t_i \neq t_j$ for $i \neq j$, and $x_i^* \in X^*$, such that

$$\langle u(t_i), x^*\rangle = 0, \quad i = 1,\ldots,n \ . \tag{*}$$

If $u = \sum_{i=1}^{n} a_i u_i$, then (*) can be written in the form

$$\begin{pmatrix} \langle u_1(t_1),x_1\rangle \dots \langle u_n(t_1),x_1\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle u_1(t_n),x_n\rangle \dots \langle u_n(t_n),x_n\rangle \end{pmatrix} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

But since it is assumed that $u \neq 0$, this equation violates the fact that $U = \{u_1,\dots,u_n\}$ is a weak Chebechev system. This contradiction implies that there do not exist extreme points L_i of the unit ball of $[C(I,X)]^*$ such that $L_i(u) = 0$ for all $i = 1,\dots,n$, and some $u \in [U]$, $u \neq 0$. Hence by Corollary 2.1 of [4, p.213], $[U]$ is Chebechev. ▲

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