Revista Colombiana de Matemáticas
Vol. XIII (1979) págs. 39 - 48

RINGS OF REAL-VALUED FUNCTIONS AND THE FINITE SUBCOVERING PROPERTY*

by

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Abstract. Let C be a ring of (not necessarily bounded) real-valued functions with a common domain X such that C includes all the constant functions and if $f \in C$ then $|f| \in C$. Without resorting to any topological notions and using only algebraic techniques, we prove that X can be extended to a set X' and every $f \in C$ can be extended to a function f' on X', such that the resulting set C' of the extended functions is a ring isomorphic to C, and such that if E' is any bounding subset of C' with the property that for every $z \in X'$ there exists an $f' \in E'$ with

* The first author acknowledges an F.L.I. grant from Iowa State University. The second author acknowledges a grant from C.S.I.R., South Africa.

$f'(z) \neq 0$, then there exists a finite subset of E' with the same property (E' is *bounding* if for every $f' \in G'$ there exist a constant function $c' \in G'$ and $e' \in E'$ such that $c' \geq |f'| + e'$).
When X is a completely regular Hausdorff space and C its ring of continuous real-valued functions (or - bounded continuous real-valued functions) then X' is the Hewitt realcompactification (or -the Stone-Čech compactification, since in this case the requirement that E' must be bounding can be dropped) of X. In effect, our result shows that the Hewitt real-compactification (or -the Stone-Čech compactification) theorem can be formulated purely algebraically requiring neither the continuity of functions nor a topology on X.

In what follows all the functions are real-valued. Also, all the ring-theoretical statements which are made in connection with a set F of functions (with a common domain) refer to the pointwise addition and multiplication of the elements of F ; similarly, statements pertaining to order among the elements of F refer to the pointwise comparison. Moreover, if f is a function then we let $|f|$ stand for the function whose values are the absolute values of f. Let c be a function whose domain is X. Then, as expected, c is called a *constant* function on X if and only if $c(x) = r$ for every $x \in X$. In particular, if $r = 1$ then c is called the *unit function* on X.

Let E be a set of functions with a common domain X and let u be the unit function on X. We say that u *is covered by the elements of* E if and only if for every $z \in X$ we have:

(1) $f(z) \neq 0$ for some $f \in E$.

Lemma 1. *Let* F *be a subset of a ring* C *of functions with a common domain* X *and let* $u \in C$ *where* u *is the unit function on* X. *If* u *is not covered by the elements of* F *then the ideal* J *of* C *generated by* F *is proper.*

Proof. Since u is not covered by the elements of F, from (1) it follows that for some $z \in X$ it is the case that $f(z) = 0$ for every $f \in F$. But then if $g \in J$ we see that $g(z) = 0$. Hence, $u \notin J$ and J is a proper ideal. ∎

From the Lemma we have immediately:

Corollary. *Let* C *be a ring as mentioned in Lemma* 1 *and* E *be a subset of* C. *If* u *is covered by no finite number of elements of* E *then the ideal* I *of* C *generated by* E *is proper.*

Let C be a set of functions with a common domain X. A set E of functions on X is called a *bounding* set (or a *bounding* subset in case $E \subseteq C$) of C if and only if for every $f \in C$ there exists a constant function c on X such that

(2) $\quad c \geq |f| + e \quad$ for some $\quad e \in E$.

Lemma 2. *Let* C *be a ring of functions with a common domain* X *such that every constant function on* X *is an element of* C *and if* $f \in C$ *then* $|f| \in C$. *Let* M *be both a maximal ideal and bounding subset of* C . *Then* M *is a real ideal of* C *(i.e.,* C/M *is isomorphic to the reals).*

Proof. Let us observe that C is a lattice where $f \vee g$ and $f \wedge g$ are equal, respectively, to $\frac{1}{2}(f+g + |f-g|)$ and $\frac{1}{2}(f+g - |f-g|)$. Thus, by [1], p.66, the maximal ideal M is an absolutely convex ideal of C and C/M is a totally ordered field. However, M is bounding subset of C and therefore for every $f \in C$, there exists a constant c such that, in view of (2) and using an obvious notation, we have $c+M \geq |f|+M$. Hence, C/M is also Archimedean. Moreover, since every constant function is an element of C we see that C/M has a subfield isomorphic to the reals. But then, as such, C/M itself is isomorphic to the reals and M is a real ideal of C, as desired. ∎

Based on the above, we have:

Theorem. *Let* C *be a ring of (real-valued) func* *tions with a common domain* X *such that every constant function on* X *is an element of* C *and if* $f \in C$ *then* $|f| \in C$ *and where* u *is the function on* X. *Then,* X *can be extended to a set* X' *and every element* f *of* C *can be extended to a function* f' *with* X' *as its domain such that:*

(i) *the resulting set* C' *of the extended functions is a ring.*

(ii) *the correspondence* $f \mapsto f'$ *is a ring isomorphism from* C *onto* C'.

(iii) *if* u' *is covered by the elements of a bounding subset* E' *of* C' *then* u' *is already covered by finitely many elements of* E'.

<u>Proof</u>. Since C contains all the constant functions on X, then the set M_x given by :

(3) $$M_x = \{ f \mid f \in C \quad \text{and} \quad f(x) = 0 \}$$

is a real ideal of C for every $x \in X$. Let

(4) $$\{ M_y \mid y \in Y \}$$

be the set of all the real ideals not of the form (3), and consider

(5) $$X' = X \cup Y$$

Clearly, X' is an extension of X, and

(6) $$\{ M_z \mid z \in X' \}$$

is the set of all the real ideals of C.

To every $f \in C$ let us make correspond a function f' on X defined as:

(7) $f' = f$ on X, and $f'(y) = f(\mathrm{Mod}\ M_y)$ on Y.

Obviously, f' is an extension of f. Moreover, $M_y = \{f \mid f \in C \text{ and } f'(y) = 0\}$ for every $y \in Y$, which by (3) implies

(8) $M_z = \{f \mid f \in C \text{ and } f'(z) = 0\}$ for every $z \in X'$.

Again, from definition (7) it readily follows that:

(9) $f' + g' = (f+g)'$ and $f' g' = (fg)'$

and since C is a ring, we see that C' given by:

(10) $$C' = \{f' \mid f \in C\}$$

is also a ring. Hence (i) is established.

Clearly, if $f' = g'$ then $f = g$ which by (9) implies that the correspondence $f \mapsto f'$ is a ring isomorphism from C onto C'. Hence (ii) is also established.

Next, let the unit function u' be covered by the elements of a bounding set E' of C', i.e., as in (1), for every $x \in X$ we have:

(11) $f'(z) \neq 0$ for some $f' \in E'$.

To prove (iii) we must show that u' is already covered by some finitely many elements of E'. Let us assume to the contrary that u' is covered by no finite number of elements of E'. Thus, by the Corollary, the ideal I' of C' generated by E' is proper. Also, since E' is a bounding subset of C',

(12) $E' \subseteq I'$ and I' is a bounding subset of C'.

Since u' is the extension of u and $f \mapsto f'$ is a ring isomorphism, in view of (12), we see that the subset I of C defined by:

(13) $I = \{ f \mid f' \in I' \}$

is both a proper ideal and a bounding subset of C. As a proper ideal, I is contained in a maximal ideal M of C. But then, since I is a bounding subset of C we see that M is both a maximal ideal and a bounding subset of C. Consequently, from Lemma 2 it follows that M is a real ideal of C and, in view of (6), we have:

(14) $\quad I \subseteq M = M_z \quad$ for some $\quad z \in X'$.

Now, from (12), (13), (14), (8) it follows that

$$f'(z) = 0 \qquad \text{for every} \qquad f' \in E'$$

which contradicts (11). Hence our assumption is false and (iii) is established. ∎

<u>Remark.</u> We observe that X (as well as X') can be topologized with subbasic open sets of the form $\{x \mid f(x) \neq 0\}$ for some $f \in C$ (as well as for some $f' \in C'$), which in fact form a base. It can be readily verified that with respect to this topology all the elements of C (as well as of C') become continuous functions (where reals are topologized as usual). If the elements of C separate points and closed subset of X then X becomes completely regular Hausdorff and X' becomes realcompact (since every real ideal in C' is fixed). Moreover, if C is the ring of all continuous functions on X then X is C-embedded in X' and hence X' is the Hewitt realcompactification of X.

If in the above, C was the ring of all bounded continuos functions on X then X is C^*-embedded in X' and hence X' is the Stone-Čech compactification of X (since in this case the requirement that E' must be a bounding subset of C' can be dropped).

On the other hand, if X had a completely regular topology to start with, this topology coincides with the one defined above for any ring C of continuos functions separating points and closed subsets. Therefore, the above Remark shows that the topological structures involved in the Hewitt realcompactification or the Stone-Čech compactification can be fully recovered from the underlying algebraic structures.

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The authors thank the referee for some improvements in the text.

(Recibido en septiembre de 1978)