The Hilbert's Nullstellensatz over skew Poincaré-Birkhoff-Witt extensions

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Title in English

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Título en español

El teorema de los ceros de Hilbert sobre las extensiones de Poincaré-Birkhoff-Witt torcidas

Abstract: In this work we study several versions of the Hilbert's Nullstellensatz. We begin with a commutative review of its geometric interpretation following the study of affine and projective case. Later, we consider its algebraic interpretation. Next, we present several treatments to the non-commutative interpretation. Therefore, we begin with Ore extensions, their properties and obstructions with classical methods. We consider a relationship between the Hilbert's Nullstellensatz and the notion of generic flatness. Subsequently we use the filtration-graduation technique over almost normalizing extensions (also called almost commutative algebras) with the aim of state a theorem that helps us to guarantee conditions such that the Hilbert's Nullstellensatz holds.
Finally, we study skew Poincaré-Birkhoff-Witt extensions together with some of their homological and ring-theoretical properties in order to extend Hilbert's Nullstellensatz to such extensions.

Resumen: En este trabajo estudiaremos algunas versiones del teorema de ceros de Hilbert (Nullstellensatz). Empezaremos con una revisión conmutativa de la interpretación geométrica con el estudio del caso afín y proyectivo. Luego, consideramos su versión algebraica. Después, presentaremos varios desarrollos en el caso no conmutativo. De esta forma, empezamos con las extensiones de Ore, sus propiedades y obstrucciones con los métodos clásicos. Consideraremos una relación entre el teorema de ceros de Hilbert y la noción de plenitud genérica. Posteriormente usaremos la técnica de filtración graduación sobre las extensiones casi normalizadoras (tambien llamadas algebras casi conmutativas) con el objetivo de establecer un teorema que nos ayude a garantizar condiciones para que el teorema de ceros de Hilbert se cumpla.
Por último, estudiaremos las extensiones de Poincaré-Birkhoff-With torcidas junto con algunas de sus propiedades homológicas y de teoría de anillos para poder extender el teorema de ceros de Hilbert sobre estas extensiones.

Keywords: Hilbert's Nullstellensatz, skew PBW extension, Jacobson ring, generic flatness.

Palabras clave: Teorema de ceros de Hilbert, extensión PBW torcida, anillo de Jacobson, plenitud genérica.

Dedicated to

Johana. Thank you for your support all these years.

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Contents

Introduction

One of the most important results for the polynomial ring over a field is the Hilbert's Nullstellensatz, which establishes a fundamental relationship between geometry and algebra. This is one of the three fundamental theorems for such structure proved by Hilbert (the other two are Hilbert's Basis Theorem and Hilbert's Syzygy Theorem). Nullstellensatz establishes a relationship between a radical of a polynomial ideal and the ideal of a variety of the polynomial ideal. Since its formulation, multiples authors have given other versions of the Nullstellensatz such as the algebraic version by Zariski given in 1946 that we will address in Section 1.1. Such version states that if we have a finitely generated k-algebra over a field k and this algebra turns out to be also a field, then this is a finite algebraic extension of k.

Thinking about non-commutative algebras, since these non-commutative structures have a polynomial form, a natural question is whether there exists a Nullstellensatz for these objects. The answer to this question is more or less affirmative. If we think in a geometrical version of Hilbert's Nullstellensatz we can found several difficulties, especially over the notion of the variety of an ideal, i.e, the collection of points in which the polynomials of the ideal vanish. We overlook this problem addressing the algebraic version of the Hilbert's Nullstellensatz. Several authors have established versions of this result for some non-commutative algebras such as Ore extensions [Irv79a], almost normalizing extensions [MR88], while others have given a general versions over algebras imposing some conditions [ASZ99].

The main family of non-commutative rings of interest for us in this work are the skew PBW extensions defined by Gallego and Lezama [GL11] as a generalization of classical PBW extensions introduced by Bell and Goodearl [BG88]. Several ring-theoretical and homological ring properties have been studied in recent years (e.g. [Art15], [LR14], [LAR15], [Rey15], [RS17c],[RS16a], [Rey14], [RS18b],[RS17b], [RS16b], [RS18c], [RS18a]). Since skew PBW extensions have a polynomial form, we ask if there is a version of the Hilbert's Nullstellensatz for these extensions and which are necessary conditions for such theorem to hold.

This document is organized as follows: Chapter 1 is dedicated to present classical Hilbert's Nullstellensatz, beginning in Section 1.1 with the commutative case. Although there is a well-known clasical version together with many interpretations, we take in this work a geometrical (affine and projective case) and an algebraic one. In Section 1.2 we our focus our attention on a non-commutative perspective by making a historical review of necessary conditions present in some formulations of the Hilbert's Nullstellensatz for different types of extensions. Firstly, we focus on Ore extensions. Then, we study the filtration-graduation technique over a special class of non-commutative structures known as almost normalizing extensions. Later, we address some conditions appearing in the literature for an algebra to satisfy Hilbert's Nullstellensatz.

In Chapter 2 we recall skew PBW extensions. Section 2.1 contains the definition of such extensions together with some ring-theoretical and homological properties of them that will be needed later to state Theorem 2.2.3. In Section 2.2 we present the main results of this work: we formulate a theorem that guarantees, on certain conditions, that skew PBW extensions satisfy Nullstellensatz. Finally, in Section 2.3 we classify some examples of skew PBW extensions and determine which versions of the theorems are satisfied.

Lastly, we state some possible future work that could be developed having in mind other versions of the Hilbert's Nullstellensatz. Also, we enunciate several open questions about this topic.

CHAPTER 1

The Hilbert's Nullstellensatz

In commutative algebra, the Hilbert's Nullstellensatz establishes a fundamental relationship between geometry and algebra. Some algebraic approaches have been established in the non-commutative case. If we think in a geometrical version we can found several difficulties; the reason for this is that we can notice some obstacles when we try to see the set of points that vanish a non-commutative polynomial we could have difficulties when we commute some variable even with constants and variables. Due to this reason, we want another interpretation of the Nullstellensatz for the non-commutative case.

In this chapter we will see some interpretations of the Hilbert's Nullstellensatz in the commutative case (geometrical and algebraic) and some approaches algebraic that have been given to enunciate a non-commutative version of the Nullstellensatz.

1.1 Commutative case

The Hilbert's Nullstellensatz is one of the three fundamental theorems about polynomial ring over a field. This result states that over an algebraically closed field, different ideals can give the same variety. The two other theorems are Hilbert's Basis Theorem that asserts that polynomial ring over a field is Noetherian, and Hilbert's Syzygy Theorem that concerns the relations, or syzygies in Hilbert's terminology, between the generators of an ideal, or, more generally, a module.

Several formulations of the theorem have been given throughout history. The most important formulation establishes a relationship between the radical of a polynomial ideal and the ideal of a variety of a polynomial ideal. We can find other statements such as an algebraic formulation that gives conditions to know if a field extension is a finite algebraic extension. Let us begin with the study of its geometric formulation.

1.1.1 Affine case

One important link between algebra and geometry is the study of the polynomial ring $k[x_1, \dots, x_n]$ over a field k. For any polynomials $f_1, \dots, f_s \in k[x_1, \dots, x_n]$ we would like to know which are the collections of points that vanish, while for a set of points we want to find which are the set of polynomials that vanish in those points. These questions are studied from the definitions of a variety and an ideal.

Definition 1.1.1 ([CLD15], Definition 1.1.4)**.** Given a field k and a positive integer n, we define the n-dimensional **affine space** over k to be the set

$$k^n := \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in k\}.$$

For example, if we consider the case $k = \mathbb{R}$, we get the familiar space $\mathbb{R}^n$ that usually use. For $n = 1$ the affine space is named affine line; $n = 2$ is named affine plane.

Definition 1.1.1 is the cornerstone of classical algebraic geometry. On the affine space, we will define affine variety which will be the collections of points that vanish certain polynomials.

Definition 1.1.2 ([CLD15], Definition 1.2.1)**.** Let k be a field, and let $f_1, \ldots, f_s$ be polynomials in $k[x_1, \ldots, x_n]$. Then we set

$$\mathbf{V}(f_1, \ldots, f_s) := \{(a_1, \ldots, a_n) \in k^n \mid f_i(a_1, \ldots, a_n) = 0, \text{ for all } 1 \leq i \leq s\}.$$

We call $\mathbf{V}(f_1, \ldots, f_s)$ the **affine variety** defined by $f_1, \ldots, f_s$.

Thus, an affine variety $\mathbf{V}(f_1, \ldots, f_s) \subset k^n$ is the set of all solutions of the system of equations $f_1(x_1, \ldots, x_n) = \cdots = f_s(x_1, \ldots, x_n) = 0$. Now we can think on the set of all polynomials vanishing on collections of points given in Definition 1.1.2. The set of polynomials will be called **the ideal of a variety**.

Definition 1.1.3 ([CLD15], Definition 1.4.5)**.** Let $V \subset k^n$ be an affine variety. Then we set

$$\mathbf{I}(V) := \{f \in k[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0, \text{ for all } (a_1, \ldots, a_n) \in V\}$$

The Definition 1.1.3 of $\mathbf{I}(V)$ is an ideal as we see in Lemma 1.1.4.

Lemma 1.1.4 ([CLD15], Lemma 1.4.6)**.** *If $V \subset k^n$ is an affine variety, then $\mathbf{I}(V) \subset k[x_1, \ldots, x_n]$ an ideal. We will call $\mathbf{I}(V)$ the **ideal of** V.*

Proof. We can note that $0 \in \mathbf{I}(V)$ since the zero polynomial vanishes on all of k^n, in particular it vanishes on V. We suppose that $f, g \in \mathbf{I}(V)$ and $h \in k[x_1, \ldots, x_n]$. Let $(a_1, \ldots, a_n)$ be an arbitrary point of V. Then we have that $f(a_1, \ldots, a_n)+g(a_1, \ldots, a_n) = 0$ and $h(a_1, \ldots, a_n)f(a_1, \ldots, a_n) = h(a_1, \ldots, a_n)0 = 0$, so $\mathbf{I}(V)$ is an ideal. $\square$

Ideals are algebraic objects, while varieties are geometric objects. We can notice a connection algebra-geometric if we take some polynomials $f_1, \ldots f_s \in k[x_1, \ldots, x_n]$ and find a variety for them $\mathbf{V}(f_1, \ldots f_s)$; next, we calculate the ideal of this variety $\mathbf{I}(\mathbf{V}(f_1, \ldots f_s))$.

$$\begin{array}{ccccc} \text{Polynomials} & & \text{Variety} & & \text{Ideal} \\ f_1, \ldots f_s & \longrightarrow & \mathbf{V}(f_1, \ldots f_s) & \longrightarrow & \mathbf{I}(\mathbf{V}(f_1, \ldots f_s)). \end{array}$$

We can note a relationship between varieties and ideals, and we could think that $\mathbf{I}(\mathbf{V}(f_1, \dots f_s)) = \langle f_1, \dots f_s \rangle$ but this, unfortunately, is not always true. We only can say that $\langle f_1, \dots f_s \rangle \subset \mathbf{I}(\mathbf{V}(f_1, \dots f_s))$. The reason is that if we take $f \in \langle f_1, \dots f_s \rangle$, means that $f = \sum_{i=1}^{s} h_i f_i$ for some polynomials $h_i, \dots, h_s \in k[x_1, \dots, x_s]$. Since $f_1, \dots f_s$ vanish on $\mathbf{V}(f_1, \dots f_s)$, So must $\sum_{i=1}^{s} h_i f_i$. An example that shows that equality need not occur is $\langle x^3, y^2 \rangle \subsetneq \mathbf{I}(\mathbf{V}(x^3, y^2))$. We first compute $\mathbf{I}(\mathbf{V}(x^3, y^2))$. The equations $x^3 = y^2 = 0$ imply that $\mathbf{V}(x^3, y^2) = \{(0,0)\}$, and we can see that the ideal of $\{(0,0)\}$ is $\langle x, y \rangle$ and this is strictly larger than $\langle x^3, y^2 \rangle$.

Over an algebraically closed field we have the relationship between $\mathbf{I}(\mathbf{V}(f_1, \dots f_s))$ and $\langle f_1, \dots f_s \rangle$. One approach for this relationship is the Weak Nullstellensatz that says us what happen if $\mathbf{V}(I) = \emptyset$.

Proposition 1.1.5 ([CLD15],Theorem 4.1.1. (The Weak Nullstellensatz))**.** *Let k be an algebraically closed field and let $I \subset k[x_1, \dots, x_n]$ be an ideal satisfying $\mathbf{V}(I) = \emptyset$. Then $I = k[x_1, \dots, x_n]$.*

We need an algebraically closed field because every nonconstant polynomial has a root in $k[x]$ and we can use this to prove Proposition 1.1.5 using induction over n. Hence, the only way that we could have $\mathbf{V}(I) = \emptyset$ would be to have f be a nonzero constant. In this case, $1/f \in k$. Thus $1 \in I$ which means that $g \in I$, for all $g \in k[x_1, \dots, x_n]$. Hence, $I = k[x_1, \dots, x_n]$.

By the Weak Nullstellensatz, one might think that the correspondence between ideals and varieties is one-to-one provided if only we restricts to algebraically closed fields. Unfortunately if we take, like before, the ideals $\langle x^3, y^2 \rangle$ and $\langle x, y \rangle$ we have that $\mathbf{V}(x^3, y^2) = \mathbf{V}(x, y) = \{(0,0)\}$ over any field define the same variety. These examples illustrate a basic reason why different ideals can define the same variety, a power of a polynomial vanishes on the same set as the original polynomial. The Hilbert Nullstellensatz states that over an algebraically closed field, this is the reason that different ideals can give the same variety.

Proposition 1.1.6 ([CLD15], Theorem 4.1.2. (Hilbert's Nullstellensatz))**.** *Let k be an algebraically closed field. If $f, f_1, \dots, f_s \in k[x_1, \dots, x_n]$ are such that $f \in \mathbf{I}(\mathbf{V}(f_1, \dots, f_s))$, then there exists an integer $m \geq 1$ such that $f^m \in \langle f_1, \dots, f_s \rangle$ (and conversely).*

The idea of the proof is to take a nonzero polynomial f which vanishes at every common zero of the polynomials $f_1, \dots, f_s$ and show that there exists an integer $m \geq 1$ and polynomials $g_1, \dots, g_s$ such that $f^m = \sum_{i=1}^{s} g_i f_i$. For this, we take a special ideal $\tilde{I} := \langle f_1, \dots, f_s, 1 - yf \rangle \subseteq k[x_1 \dots, x_n, y]$ and prove that $\mathbf{V}(\tilde{I}) = \emptyset$ to reduce to the weak Nullstellensatz to this let $(a_1, \dots, a_n, a_{n+1}) \in k^{n+1}$. Either $(a_1, \dots, a_n)$ is a common zero of $f_1, \dots, f_s$, or $(a_1, \dots, a_n)$ is not a common zero of $f_1, \dots, f_s$. In the first case $f(a_1, \dots, a_n) = 0$, since f vanishes at any common zero of $f_1, \dots, f_s$. Thus, $1 - yf$ takes the value $1 - a_{n+1} f(a_1, \dots, a_n) = 1 \neq 0$ at point $(a_1, \dots, a_n, a_{n+1})$. In particular $(a_1, \dots, a_n, a_{n+1}) \notin \mathbf{V}(I)$. In the second case, for some $1 \leq i \leq s$, we must have $f_i(a_1, \dots, a_n) = 0$. Thinking of f_i as a function of $n+1$ variables which does not depend on the last variable, we have $f_i(a_1, \dots, a_n, a_{n+1}) \neq 0$. In particular, we conclude that $(a_1, \dots, a_n, a_{n+1}) \notin \mathbf{V}(I)$. Since $(a_1, \dots, a_n, a_{n+1}) \in k^{n+1}$ was arbitrary, we obtain $\mathbf{V}(\tilde{I}) = \emptyset$. By the weak Nullstellensatz we know that $1 \in \tilde{I}$, and hence $1 = \sum_{i=1}^{s} p_i(x_1, \dots, x_n, y) f_i + q(x_1, \dots, x_n, y)(1 - yf)$, for some polynomials $p_i, q \in$

$k[x_1, \ldots, x_n, y]$. If we take $y = 1/f(x_1, \ldots, x_n)$ we have that $1 = \sum_{i=1}^{s} p_i(x_1, \ldots, x_n, 1/f) f_i$. If we multiply both sides of this equation by a power f^t, where t is chosen sufficiently large to clear denominators, we have $f^t = \sum_{i=1}^{s} g_i f_i$, for some polynomials $g_i \in k[x_1, \ldots, x_n]$. The election of that ideal $\tilde{I}$ is known as "Rabinowitz trick".

To explore the relationship between ideals and varieties, it is natural to formulate Hilbert's Nullstellensatz in terms of ideals.

Definition 1.1.7 ([CLD15], Definition 4.2.4.)**.** Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. The **radical** of I, denoted $\sqrt{I}$, is the set

$$\{f \mid f^m \in I, \text{ for some integer } m \geq 1\}.$$

It is not hard to see that $\mathbf{I}(V)$ is a radical ideal, if we take a element $a \in V$. If $f^m \in \mathbf{I}(V)$, then $(f(a))^m = 0$. But this can happen only if $f(a) = 0$, and since a was arbitrary, we must have $f \in \mathbf{I}(V)$. From Definition 1.1.7 and Theorem 1.1.6 we can state ideal-theoretic form of the Nullstellensatz.

Proposition 1.1.8 ([CLD15]. Theorem 4.2.6. (The Strong Nullstellensatz))**.** *Let k be an algebraically closed field. If I is an ideal in $k[x_1, \ldots, x_n]$, then $\mathbf{I}(\mathbf{V}((I)) = \sqrt{I}$.*

The proof of the strong Nullstellensatz consists of the following: for $\sqrt{I} \subset \mathbf{I}(\mathbf{V}(I))$, taking $f \in \sqrt{I}$ implies that $f^m \in I$ for some m. Hence, f^m vanishes on $\mathbf{V}(I)$, which implies that f vanishes on $\mathbf{V}(I)$. Thus, $f \in \mathbf{I}(\mathbf{V}(I))$. Conversely, we use the Hilbert's Nullstellensatz to guarantee that $\mathbf{I}(\mathbf{V}(I)) \subset \sqrt{I}$.

1.1.2 Proyective case

In some cases, affine case is not enough to verify all points where a polynomial vanished, because, in a certain sense, we missing some "points at infinity". To recover these points, we will add them to reach definition of the projective space $\mathbb{P}^n$. Then we will introduce homogeneous polynomial for this space. Later, we will define projective varieties over $\mathbb{P}^n$ and, such as in Section 1.1.1, we study a projective version of an algebraic-geometry relationship due to the Hilbert's Nullstellensatz.

Before we give a definition of projective space, we consider an equivalence relation $\sim$ on the nonzero points of k^{n+1} by setting $(x'_0, \ldots, x'_m) \sim (x_0, \ldots, x_m)$ if are parallel, i.e. there is a nonzero element $\alpha \in k$ such that $(x'_0, \ldots, x'_m) = \alpha(x_0, \ldots, x_m)$. With the equivalence relation we can define the projective space

Definition 1.1.9 ([CLD15], Definition 8.2.1)**.** The **n-dimensional projective space** over a field k, denoted $\mathbb{P}^n(k)$ or $\mathbb{P}^n$, is the set of equivalence classes of $\sim$ on $k^{n+1} \setminus \{(0, \ldots, 0)\}$. Thus, $\mathbb{P}^n = (k^{n+1} \setminus \{(0, \ldots, 0)\}) / \sim$. Given a $(n+1)$-tuple $(x_0, \ldots, x_n) \in k^{n+1} \setminus \{(0, \ldots, 0)\}$, its equivalence class $p \in \mathbb{P}^n$ will be denoted $(x_0 : \cdots : x_n)$, and we will say that $(x_0 : \cdots : x_n)$ are **homogeneous coordinates** of p. Thus $(x'_0 : \cdots : x'_n) = (x_0 : \cdots : x_n)$ if and only if $(x'_0, \ldots, x'_m) = \alpha(x_0, \ldots, x_m)$, for some $\alpha \in k \setminus \{0\}$.

We want to define varieties in projective case. If we try to replicate affine case, we will have problems. For example in the 2-dimensional projective space $\mathbb{P}^2$ when we take some $f \in k[x_0, \ldots, x_n]$ and we try to construct $\mathbf{V}(x_1^2 - x_2)$, the point $p = (3 : 2 : 4)$ satisfy the equation $x_1^2 - x_2 = 0$. However, we notice that p can be represented by a different homogeneous component, for example $p = (6 : 4 : 8)$; if we substitute these components into our polynomial, we obtain that $4^2 - 8 = 4 \neq 0$. We can overlook this problem using homogeneous polynomials. We recall that a polynomial f is **homogeneous of total degree** d, if every term appearing in f has total degree exactly d. We define varieties over projective space $\mathbb{P}^n$.

Definition 1.1.10 ([CLD15], Defintion 8.2.5). Let k be a field and let $f_1, \ldots, f_s \in k[x_0, \ldots, x_n]$ be homogeneous polynomials. We set

$$\mathbf{V}(f_1, \ldots, f_s) = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid f_i(a_0, \ldots, a_n) = 0 \text{ for all } 1 \leq i \leq s\}.$$

We call $\mathbf{V}(f_1, \ldots, f_s)$ the **projective variety** defined by $f_1, \ldots, f_s$.

We could think that, such as the polynomial $f = x_1^2 - x_2$, we can have problems with different representations of p. Nevertheless, in homogeneous polynomials this problem is completely avoided.

Proposition 1.1.11 ([CLD15], Proposition 8.2.4). *Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial. If f vanishes on any one set of homogeneous coordinates for a point $p \in \mathbb{P}^n$ then f vanishes for all homogeneous coordinates of p. In particular $\mathbf{V}(f)$ is a well-defined subset of $\mathbb{P}^n$.*

Proof. We take $(a_0 : \cdots : a_n) = (\lambda a_0 : \cdots : \lambda a_n)$ homogeneous coordinate for $p \in \mathbb{P}^n$ and we assume that $f(a_0, \ldots, a_n) = 0$. If f is homogeneous of total degree t, we have that every term in f have the form $cx_0^{\alpha_0} \cdots x_n^{\alpha_n}$, with $\alpha_0 + \cdots + \alpha_n = t$. When we substitute $x_i = \lambda a_i$ we have $c(\lambda a_0)^{\alpha_0} \cdots (\lambda a_n)^{\alpha_n} = c(\lambda)^{\alpha_0}(a_0)^{\alpha_0} \cdots (\lambda)^{\alpha_n}(a_n)^{\alpha_n} = c\lambda^t (a_0)^{\alpha_0} \cdots (a_n)^{\alpha_n}$. All terms in f have a common factor λ^t and hence $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^t f(a_0, \ldots, a_n) = 0$. □

Definition 1.1.12 ([CLD15], Definition 8.3.1). An ideal I in $k[x_0, \ldots, x_n]$ is said to be **homogeneous** if for each $f \in I$, the homogeneous components f_i of f are in I as well.

We can note that not all ideals have this property. For instance, let $I = \langle x^2 - y \rangle \subseteq k[x, y]$. The homogeneous components of $f = x^2 - y$ are $f_1 = x^2$ and $f_2 = -y$. Neither of these polynomials is in I since neither is a multiple of $x^2 - y$. Hence, I is not a homogeneous ideal.

One way to create examples of homogeneous ideal is to consider the ideal generated by the defining equations of a projective variety. But there is another way such as a projective variety can gives us a homogeneous ideal.

Proposition 1.1.13 ([CLD15], Proposition 8.2.4). *Let $V \subseteq \mathbb{P}^n$ be a projective variety and let*

$$\mathbf{I}(V) = \{f \in k[x_0, \ldots, x_n] \mid f(a_0, \ldots, a_n) = 0, \text{ for all } (a_0 : \cdots : a_n) \in V\}.$$

If k is an infinite field, then $\mathbf{I}(V)$ is a homogeneous ideal in $k[x_0, \ldots, x_n]$.

We have a relationship between a projective variety and a homogeneous ideal, such as in the affine case. For the Hilbert's Nullstellensatz we define the radical of a homogeneous ideal as is usual:

$$\sqrt{I} := \{f \in k[x_0, \dots, x_n] \mid f^m \in I, \text{ for some } m \geq 1\}.$$

The radical of a homogeneous ideal is always homogeneous. We expect an especially close relationship between projective varieties and homogeneous ideals over an algebraically closed field k, such as in affine case. We could think that the weak and strong Nullstellensatz that we have seen in section 1.1.1 can be extended to projective varieties and homogeneous ideals. Unfortunately this is not possible; in particular the Weak Nullstellensatz fails for certain homogeneous ideals. For example, if we consider the ideal $\langle x_0, x_1, \dots, x_n\rangle \subseteq k[x_0, \dots, x_n]$ with k algebraically closed, then $\mathbf{V}(I) \subseteq \mathbb{P}^n$ is defined by equations $x_0 = \cdots = x_n = 0$ which have not solutions in $\mathbb{P}^n$. It follows that $\mathbf{V}(I) = \emptyset$, yet $I \neq k[x_0, \dots, x_n]$. However, we can give a version projective Nullstellensatz having this problem in mind.

Proposition 1.1.14 ([Lez19a], Theorem 4.2.17; [CLD15], Theorem 8.3.9 (Projective Nullstellensatz))**.** *Let k be an algebraically closed field and $J \subseteq k[x_0 \dots, x_n]$ be a homogeneous ideal. Then,*

(i) $\mathbf{V}(J) = \emptyset \Leftrightarrow \langle x_0, \dots, x_n\rangle \subseteq \sqrt{J}$.

(ii) *If* $\mathbf{V}(J) \neq \emptyset$, *then* $\mathbf{I}(\mathbf{V}(J)) = \sqrt{J}$.

1.1.3 Algebraic formulation

One version of the Hilbert's Nullstellensatz is given in [AM69, Chapters 5 & 7] around the definition of integral dependence and Noetherian rings (a weak version). For this, we remember that A is a k-algebra (with k a field), if it is a vector space equipped with a bilinear product, and it is a finitely generated k-algebra if there exist finitely elements $x_1, \dots, x_n \in A$ such that every element of A can be written as a linear combination of these elements. Let us remember some definitions with the aim of establishing the theory.

Definition 1.1.15 ([Fra03], Definition 29.6)**.** Let L, F be fields, with L a field extension of F. An element $a \in L$ is called an **algebraic** over F, if there exists some non-zero polynomial $g(x)$ with coefficients in F such that $g(a) = 0$. If a is not algebraic over F, then a is **transcendental** over F.

We are talking about elements that vanish a polynomial; we can think of it such as a kind of variety. Considering the case when every element of a field vanishes, we can extend Definition 1.1.15.

Definition 1.1.16 ([Fra03], Definition 31.1)**.** Let L, F be fields, with L a field extension of F. L is called **algebraic**, if every element of L is algebraic over F.

In field theory we remember that a field F is algebraically closed if contains a root for every non-constant polynomial in $F[x]$. Hence, from Definition 1.1.16, we can said that L

is algebraically closed. This definition of an algebraic element in fields can be extended to the context of rings.

Let B be a ring, A a subring of B. An element x of B is said **integral** over A, if x is a root of a monic polynomial with coefficients in A, that is, if x satisfies an equation of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

where $a_i \in A$, for $0 \leq i \leq n-1$. Clearly, every element of A is integral over A.

Remark 1.1.17. Let $f : A \to B$ be a ring homomorphism, so that B is an A-algebra. Then f is said to be **integral**, and B is said to be an **integral A-algebra**, if B is integral over the subring $f(A)$.

Let us remind that an integral domain is a nonzero commutative ring in which product of any two non-zero elements is non-zero. One of the important tools to give an algebraic version of the theorem is the definition of a valuation ring.

Definition 1.1.18 ([AM69], page 65)**.** Let B be an integral domain, K its field of fractions. B is a **valuation ring** of K if, for each $x \in K$ nonzero, either $x \in B$ or $x^{-1} \in B$ (or both).

Valuation rings have many properties like being a local ring (they have a unique maximal ideal), and they are integrally closed over its field of fractions. Other way to characterize valuation rings of a field K is that valuation rings B of K have K as their field of fractions, and their ideals are totally ordered by inclusion. Equivalently, their principal ideals are totally ordered by inclusion.

Proposition 1.1.19 ([AM69], Proposition 5.23)**.** *Let $A \subseteq B$ be integral domains, B finitely generated over A. Let v be a non-zero element of B. Then there exists $u \neq 0$ in A with the following property: any homomorphism f of A into an algebraically closed field Ω such that $f(u) \neq 0$ can be extended to a homomorphism g of B into Ω such that $g(v) \neq 0$.*

The proof of Proposition 1.1.19 works by induction on the number of generators of B over A and reduce to the case where B is generated over A by a single element. If the element is transcendental, we can make a function and extend it if we consider that the field is infinite. If the element is algebraic, we use that v is a polynomial in the element and use v^{-1}, and take equations with the element and v^{-1} and u as the multiplication of the lead coefficients of the equations; hence we extend the function.

If we use Proposition 1.1.19, we can give an algebraic version of the Hilbert's Nullstellensatz. Following [AM69, page 80], we recall that a ring A is said to be **Noetherian**, if it satisfies one of the following three equivalent conditions:

1. Every non-empty set of ideals in A has a maximal element.

2. Every ascending chain of ideals in A is stationary.

3. Every ideal in A is finitely generated.

Noetherian rings are one of the most important class of rings in commutative and non-commutative algebra. Next, we recall some of their properties which are essential for the purpose of this work.

Proposition 1.1.20 ([AM69], Proposition 7.8)**.** *Let $A \subseteq B \subseteq C$ be rings. Suppose that A is Noetherian, that C is finitely generated as an A-algebra and that C is either* (i) *finitely generated as a B-module or* (ii) *integral over B. Then B is finitely generated as an A-algebra.*

Proof. We follow the proof given in [AM69]. Let $x_1, \ldots, x_m$ the elements that generated C as an A-algebra, and let $y_1, \ldots, y_n$ be the elements that generate C as a B-module. Then we can find expressions of the form

$$x_i = \sum_j b_{ij} y_i \text{ with } b_{ij} \in B \tag{1.1.1}$$

$$y_i y_j = \sum_k b_{ijk} y_i \text{ with } b_{ijk} \in B. \tag{1.1.2}$$

In particular $y_i y_i = \sum_k b_{iik} y_i$. Let B_0 be the algebra generated over A by the b_{ij} and the b_{ijk}. Since A in Noetherian, so B_0, and $A \subseteq B_0 \subseteq B$.

Any element of C is a polynomial in the x_i with coefficient in A. Substituting the expression (1.1.1) in the polynomial and making repeated use of expression (1.1.2) we can show that each element of C is a linear combination of the y_i with coefficients in B_0, and hence C is finitely generated as a B_0-module. Since B_0 is Noetherian and B is a submodule of C, it follows that B is finitely generated as B_0-module. Since B_0 is finitely generated as an A-algebra, it follows that B is finitely generated as an A-algebra. □

From valuation rings or Noetherian rings we can state a version of Hilbert's Nullstellensatz that gives us a method to identifying a finite algebraic extension through a finitely generated k-algebra.

Proposition 1.1.21 ([AM69], Proposition 7.9. (Hilbert's Nullstellensatz))**.** *Let k be a field, E a finitely generated k-algebra. If E is a field then it is a finite algebraic extension of k.*

If we use valuation rings, we can follow Proposition 1.1.21 as a corollary of Proposition 1.1.19 taking $B = E$, $A = k$, $v = 1$ and $\Omega =$ algebraic clousure of k, whereas if we use the Noetherian rings, we assume k is infinite and E is a simple transcendental extension $k[x]$. We claim that if $f_1, \ldots, f_n \in E$, then the k-algebra A they generate is smaller than E. To see this, we choose $a \in k$ away from the poles of the rational functions f_i. Then no element of A can have a pole at a, so $1/(x-a) \notin A$, and A is smaller than E.

Proposition 1.1.21 is known as **Zariski's lemma**. In Section 1.2 we will see a generalization of the Proposition 1.1.21 and the Hilbert's Nullstellensatz for some non-commutative rings. We can note that this version of Hilbert's Nullstellensatz does not use varieties of a collection of polynomials, or an ideal of a set of points.

Next, we recall the algebraic version of the weak Hilbert's Nullstellensatz.

Proposition 1.1.22 ([AM69], Corollary 7.10. (Weak Hilbert's Nullstellensatz))**.** *Let k be a field, A a finitely generated k-algebra. Let M be a maximal ideal of A. Then the field A/M is a finite algebraic extension of k. In particular, if k is algebraically closed then $A/M \cong k$.*

We can see that the Hilbert's Nullstellensatz implies the Weak Hilbert's Nullstellensatz. In Proposition 1.1.21, since B is a field, the maximal ideal is generated by 0, and so Proposition 1.1.22 follows from Proposition 1.1.21. For the proof of the Proposition 1.1.22, take $E = A/M$.

The algebraic version of Hilbert's Nullstellensatz allows us to overlook the difficulty of define variety and the ideal of a set of points in a non-commutative structure. We can extend Proposition 1.1.21 to state a general version of Hilbert's Nullstellensatz in the commutative and non-commutative case.

1.2 Non-commutative case

In Section 1.1 we saw that the Hilbert's Nullstellensatz has a geometric and an algebraic version. If we try to extend the Hilbert's Nullstellensatz in its geometrical form, we would have some difficulties. In some cases, a set of points can vanish an expression, but when we commute variables or even a variable and a constant, possibly this set does not vanish the new polynomial. That is the reason we will concentrate on approaches in the algebraic version of the Hilbert's Nullstellensatz. All these methods have been studied throughout the 20th century.

1.2.1 First approach: Ore extensions

One of the most important approaches for the non-commutative case is presented in [Irv79a] which provides us a relationship between Hilbert's Nullstellensatz for Ore extensions defined by [Ore33] and the notion of generic flatness (see Definition 1.2.12). Before that, let us review some facts of commutative ring theory.

Definition 1.2.1 ([Irv79a], page 259)**.** A commutative domain R is called G**-domain**, if its fraction field is a finitely-generated R-algebra, that is if there exists a finite number of non-zero elements $u_1, \ldots, u_n$ such that $R[u_1^{-1}, \ldots, u_n^{-1}] = K$, with K the field of fractions of R.

From Definition 1.2.1, it follows that the fraction field is generated by a single element $u = \prod_{i=1}^{n} u_i$.

Definition 1.2.2 ([Irv79a], page 260)**.** A prime ideal P of a ring A is a G**-ideal**, if the primes which properly contain P intersect in an ideal properly containing P.

We can note that a commutative ring R is a G-domain, if $\{0\}$ is G-ideal or R/P is a G-domain.

Proposition 1.2.3 ([Irv79a], Proposition 1)**.** *Let R be a commutative ring. G-ideals of R are precisely the intersections of the maximal ideals in $R[t]$.*

Over a commutative ring, maximal ideals are G-ideals, and G-ideals are prime ideals. We can enunciate a class of rings in which converse holds.

Definition 1.2.4 ([Irv79a], page 260)**.** A commutative ring R is **Jacobson ring**, if every G-ideal is maximal.

In particular, nilradical in a Jacobson ring coincides with Jacobson radical; this property is the usual definition of a Jacobson rings and it is used for commutative rings. We can find other characterizations of Jacobson rings. For example, a ring R is a Jacobson ring if $J(R/P) = 0$, for every prime ideal P of R. We can observe some examples of Jacobson rings, such as $\mathbb{Z}$ or $k[x]$, with k a field, every field is a Jacobson ring.

More recently, Jacobson rings (also known such as Hilbert rings) can be seen using primitive ideals. Let us enunciate when a ideal is known as primitive.

Definition 1.2.5 ([GW04], page 60)**.** Let R be a ring and I an ideal of R. We say that I is a **left (right) primitive ideal**, if there exists a simple left (right) R-module M such that $\mathrm{Ann}(M) = I$. A right (left) primitive ring is any ring in which 0 is a right (left) primitive ideal, i.e., any ring which has a faithful simple right (left) module.

Definition 1.2.6 ([MR01], page 342)**.** R is a **Jacobson ring**, if every prime ideal in R is an intersection of primitive ideals.

We can note that for commutative rings, primitive ideals are equivalent to maximal ideals (over a commutative ring R, any simple module is isomorphic to R/I for some maximal ideal I, and $\mathrm{Ann}(R/I) = I$); therefore, in Jacobson ring every prime ideal is an intersection of maximal ideals. The most important property in commutative algebra establishes a relation between fields and finitely generated algebras.

Proposition 1.2.7 ([GW10], Proposition 1.7)**.** *Let K be a (not necessarily algebraically closed) field and let A be a finitely generated K-algebra. Then A is Jacobson.*

Proposition 1.2.8 ([Irv79a], Proposition 2)**.** *If R is a Jacobson ring, so is $R[t]$.*

Proposition 1.2.7 can be used to deduce the Hilbert's Nullstellensatz, and might it self be considered a version of the Nullstellensatz. We might think, over Proposition 1.2.8, that in a non-commutative ring introduced by Ore [Ore33], if we start from a Jacobson ring and the extension of this ring results Jacobson ring we could conclude the Nullstellensatz. Unfortunately, at least in the Ore extensions this result does not hold. In [Irv79a], [Irv79b] and [PS77] we can see non-commutative extensions of Jacobson rings which are not necessarily Jacobson. In [NIM14], under suitable conditions, an Ore extension of a Jacobson ring is also Jacobson. This tells us that to extend the result we have to consider other notions to conclude the Nullstellensatz.

Proposition 1.1.21 establishes a way to conclude the algebraic version of the Nullstellensatz; we can extend this proposition to state a non-commutative version of the theorem.

Definition 1.2.9 ([Irv79a], page 261)**.** Let R be a commutative ring and A an R-algebra. We say that A satisfies the **(right) strong Nullstellensatz**, if for every simple right A-module M, the annihilator I intersects R in a G-ideal.

Definition 1.2.10 ([Irv79a], page 262)**.** Let A be an algebra over a field k. Then A satisfies the **Nullstellensatz**, if for any simple right A-module M, the division ring $\mathrm{End}_A(M)$ is algebraic over k.

Definition 1.2.10 extends Proposition 1.1.21. Let K be a finitely-generated algebra over k. Then K is a simple K-module, which equals its own endomorphism ring, and hence we have the algebraic version of the Hilbert's Nullstellensatz.

The use of the word "strong" is justified by Proposition 1.2.11.

Proposition 1.2.11 ([Irv79a], Proposition 3)**.** *Let $A[x]$ be satisfy the strong Nullstellensatz as a $k[x]$-algebra. Then A satisfies the Nullstellensatz*

Proof. Let M be a simple A-module and let ϕ an A-endomorphism of M. We want to see that ϕ is algebraic over k. We can view M as an $A[x]$-module, with x acting as ϕ does. By hypothesis $A[x]$ satisfy the strong Nullstellensatz, i.e. some G-ideal of $k[x]$ annihilates M. But $k[x]$ is not a G-domain, so every G-ideal of $k[x]$ is maximal. Hence, for some non-zero polynomial $p(x)$ in $k[x]$, we have $Mp(x) = \langle 0 \rangle$. This implies that $p(\phi)$ is the zero endomorphism, and ϕ is algebraic over k. □

There are many examples of algebras in which the Nullstellensatz holds, and some of these are finitely-generated PI-algebras, or enveloping algebras of finite-dimensional Lie algebras. In Chapter 2, we are going to develop some examples of skew PBW extensions that satisfy Nullstellensatz.

We have a more general version of the Nullstellensatz, even for the non-commutative case. However, in general, it is not easy to verify. Hence we will develop several tools in order to make such verification easier. For that we define the notion of generic flatness.

Definition 1.2.12 ([Irv79a], page 263)**.** We say that an algebra A over a commutative domain R satisfies **generic flatness**, if for any finitely-generated A-module M, there exists a non-zero element c in R such that $M_c = M \otimes_R R_c$ is free over the localization R_c.

In other words, an algebra A satisfies generic flatness over R, if for every simple A-module M there is $c \neq 0$ in R such that $M[c^{-1}]$ is free over $R[c^{-1}]$ ([Row88, Definition 2.12.32]).

We identify a relationship between generic flatness and Nullstellensatz. This interaction is presented in [Duf73] and justified in [Irv79a].

Proposition 1.2.13 ([Irv79a], Proposition 4)**.** *Let R be a commutative ring and let A be an R-algebra. Suppose for each prime P of R, the R/P-algebra A/PA satisfies generic flatness. Then A satisfies the strong Nullstellensatz.*

Proof. Let M be a simple A-module. We want to see that the annihilator I intersect R in a G-ideal. Its annihilator in R is a prime ideal, which we may set equal to $\{0\}$. So we can assume R is a domain over which A satisfies generic flatness, and we must prove that R is a G-domain.

By hypothesis, there exists an element $c \in R$ such that M_c is free over R_c. Since c is central, it belongs to $\mathrm{End}_A(M)$ and must be invertible, so $M_c = M$. Let $\{m_i \mid i \in I\}$ be a basis for M over R_c, r a non-zero element of R_c, and $i \in I$. By Schur's Lemma, $r^{-1}m_i$ is in M, which means that we can write $r^{-1}m_i = tm_i + \sum_{j\neq i} t_j m_j$, where t_i and m_i are unique elements of R_c. Thus $m_i = rtm_i + \sum_{j\neq i} rt_j m_j$, and $rt = 1$, so $r^{-1} = t$ lies in R_c, and we can conclude that R_c is a field. This fact proves that R is a G-domain. □

Propositions 1.2.11 and 1.2.13 imply the following corollary.

Corollary 1.2.14 ([Irv79a], page 266)**.** *Let A be an algebra over the field k, and assume that $A[x]$ satisfies generic flatness as a $k[x]$-algebra. Then A satisfies the Nullstellensatz.*

Notice that Definition 1.2.12 is a powerful tool to the task of verifying the Nullstellensatz. We remark that, up to this moment, the ring is not ask to be Noetherian. However, when this property holds we can conclude that it is Jacobson.

Proposition 1.2.15 ([Irv79a], Proposition 5)**.** *Let R be a commutative Jacobson ring and A an R-algebra which satisfies the strong Nullstellensatz. Assume that for each maximal ideal M of R, the algebra $A/M[x]$ satisfies the Nullstellensatz over $k = R/M$. Then the Jacobson radical A is nilpotent. In particular if A is Noetherian, then A is a Jacobson ring.*

Proof. Let a be an element in the Jacobson radical of A. We claim that $(1-ax)A[x] = A[x]$. By contradiction, let I be a maximal right ideal of $A[x]$ containing $(1-ax)A[x]$. Denote by N the module $A[x]/I$, which is simple over $A[x]$. By assumption, the annihilator of N intersects R in a maximal ideal M, and the $A[x]$-endomorphism of N induced by x is invertible and algebraic over $k = R/M$. Let ϕ be this endomorphism, and $v \in N$ the image of 1 under the canonical map of $A[x]$ to M. Then $v(1-ax) = v(1-a\phi) = 0$, so $v\phi^{-1} = va$. The element ϕ can be expressed as $p(\phi^{-1})$, for some polynomial $p \in k[\phi]$, since ϕ is algebraic over $k = R/M$. It follows that $vp(\phi^{-1}) = vp(a)$, and so $0 = v(1-a\phi) = v(1-ap(a))$. Since a is in Jacobson ring, the element $1-ap(a)$ must be invertible, which is a contradiction. This fact allows us to conclude that $(1-ax)A[x] = A[x]$. Therefore, for some element $f(x)$, we have $f(x) = 1 + \sum_{i>0} a^i x^i$, so a must be nilpotent, if this not the case we could have a sum that never ends.

□

As we have seen, for verifying the Nullstellensatz it is enough to check the condition of generic flatness. In the particular case of Ore extension we shall prove that they all satisfy generic flatness. For that we recall the definition of these extensions.

Definition 1.2.16 ([GW04], page 34)**.** Let A be a ring, σ a ring endomorphism of A, and δ a σ-derivation on A. We shall write $B = A[x; \sigma, \delta]$ provided

(a) B is a ring, containing A as a subring;

(b) x is an element of B;

(c) B is a free left A-module with basis $\{1, x, x^2, \ldots\}$;

(d) $xa = \sigma(a)x + \delta(a)$ for all $a \in A$.

Such a ring B is called **Ore extension** over A (for some authors **skew polynomial rings** over A).

We recall that for A a ring, $\sigma : A \to A$ be a ring homomorphism, $\delta : A \to A$ is a σ-derivation which means that δ is a homomorphism of abelian groups satisfying $\delta(a_1a_2) = \sigma(a_1)\delta(a_2) + \delta(a_1)a_2$, for every $a_1, a_2 \in A$.

In general, we begin with an endomorphism σ of A, and a right σ-derivation δ on A. When we have σ an automorphism, any element of B can be written either as a polynomial in x with all its coefficients on the left or as a polynomial with all its coefficients on the right. If A is an algebra over a commutative ring R, we want B to be an R-algebra as well. This will be the case if σ is an R-automorphism and δ vanished on R.

One of the most important result for Ore extension is the Hilbert's Basis Theorem which establishes a sufficient condition for the Noetherian property to extend from the coefficient ring to all extension.

Proposition 1.2.17 ([GW04], Theorem 2.6 (Hilbert's basis Theorem))**.** *Let $B = A[x; \sigma, \delta]$, where σ is an automorphism of A. If A is right (left) Noetherian, then so is B.*

We want to extend from the coefficient ring the property of generic flatness to the Ore extension. We will do it by guarantying that the extension is an algebra.

Proposition 1.2.18 ([Irv79a], Theorem 1)**.** *Let R be a commutative domain, and let A be a right Noetherian R-algebra which satisfies right generic flatness. Let σ be an $R-$automorphism of A and δ a σ-derivation such that $\delta(R) = 0$. Then the Ore extension $B = A[x; \sigma, \delta]$ satisfies right generic flatness.*

Proof. We follow the proof presented in [Irv79a]. Due to Lemma 1.2.24 below, it suffices to prove generic flatness for cyclic modules, so let B/I be a cyclic right B-module, where I is a right ideal of B. As A-module B can be written as an infinite direct sum: $B = \bigoplus_{i=0}^{\infty} x^i A$ We define a series of right A-submodules

$$L_{-1} := \langle 0 \rangle,\ L_0 := A, \ldots,\ L_k := \sum_{i=0}^{k} x^i A.$$

Let $I_k := I + L_k$ be the right A-submodule. Then by the isomorphism Theorem and the modular lay, we can assert that $I_k/I_{k-1} \cong L_k + I/L_{k-1} + I \cong L_k + (L_{K-1} + I)/L_{k-1} + I \cong L_k/(L_k \cap (L_{k-1} + I)) \cong L_k/(L_{k-1} + (I \cap L_k))$. Let us consider $I \cap L_k$ more closely. Every element has degree at most k, and it can be written with its coefficients on the left, so that it has the form $\sum_{i=0}^{k} a_i x^i$.

Let $Q_k \subset A$ be the set of leading coefficient $\{a_k\}$ of elements in $I \cap L_k$. By writing coefficients on the left, it follows that $Q_k \subset Q_{k+1}$. This fact is true because $Ix \subset I$. Also, each Q_k is a right ideal of A. Now, if $a \in A$, and $a_l \in Q_k$, then

$$\left(\sum_{i=0}^{k} a_i x_i\right) \sigma^k(a) = a_k a x^k + (\text{lower degree terms}),$$

so that $a_m a \in Q_k$. By the definition of Q_k, we see that $L_{k-1} + (I \cap L_k)$ is equals to $L_{k-1} + Q_k x^k$. Therefore, I_k/I_{k-1} is isomorphic as an R-module to

$$L_k/(L_{k-1} + Q_k x^k) \cong (L_{k-1} + Ax^k)/(L_{k-1} + Q_k x^k).$$

Other application of isomorphism Theorem shows that the above is isomorphic to $Ax^k/(Q_k x^k + (L_{k-1} \cap Ax^k))$. This is because

$$\begin{aligned}(L_{k-1} + Ax^k)/(L_{k-1} + Q_k x^k) &\cong (L_{k-1} + Ax^k + Q_m x^k)/(L_{k-1} + Q_k x^k)\\ &\cong Ax^m/(Ax^m \cap (Q_k x^k + L_{k-1}))\\ &\cong Ax^k/(Q_k x^k + (L_{k-1} \cap Ax^k)),\end{aligned}$$

but $L_{k-1} \cap Ax^k = \langle 0 \rangle$, so we conclude that,

$$I_k/I_{k-1} \cong Ax^k/Q_k x^k \cong A/Q_k \qquad (R\text{-modules}).$$

The ascending chain of right ideal $\{Q_k\}$ must become stationary at some integer n, since A is right Noetherian. For each $1 \leq n$, by assumption of generic flatness on A, there exist f_i in R such that $(A/Q_i)_{f_i}$ is free over R_{f_i}. Let $f = \prod f_i$. Then every R_f-module $(I_k/I_{k-1})_f \cong (A/Q_k)_f$ is free, and so $(B/I)_f$ is free over R_f. □

We can iterate the construction of Ore extension (or skew polynomial ring) with the aim of obtaining a iterated Ore extension (or iterated skew polynomial ring) of the form $A[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$. Notice that σ_i and δ_i must be defined as

$$\sigma_i, \delta_i : A[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}] \to A[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}].$$

But we can impose some conditions because this construction can be difficult if we restrict when δ is an automorphism; the conditions are the following:

$$\begin{aligned}\sigma_i(x_j) &= x_j, \qquad j < i,\\ \delta_i(x_j) &= 0, \qquad j < i,\\ \sigma_i\sigma_1 &= \sigma_1\sigma_i, \qquad 1 \leq i \leq n,\\ \delta_i\delta_1 &= \delta_1\delta_i, \qquad 1 \leq i \leq n,\end{aligned}$$

when the two last relations are understood to be restricted to A.

Proposition 1.2.17 can be extended to an iterated Ore extensions with the aim of determining if this extensions are right (left) Noetherian.

Proposition 1.2.19 ([GW04], Corollary 2.7)**.** *Let $B = A[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ be a iterated Ore extension, where each σ_i is an automorphism of the ring $A[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}]$. If A is right (left) Noetherian, then so is B.*

We saw in Proposition 1.2.18 that, under the condition of the Ore extension A being an R-algebra, the extension satisfy the condition of generic flatness. The next result extends this scenario to iterated Ore extensions.

Corollary 1.2.20 ([Irv79a], page 270)**.** *Let R be a commutative Noetherian domain and A an R-algebra obtained from R by a finite sequence of Ore extensions, each of which preserves the R-algebra structure. Then A satisfies generic flatness.*

Proof. This follows from the Theorem 1.2.18 by induction on the number of extensions required to obtain A, since we have σ an automorphism and an Ore extension of a Noetherian ring is Noetherian by Proposition 1.2.19. □

Proposition 1.2.21 ([Irv79a], Theorem 2)**.** *Let A be a finitely iterated Ore extension of the commutative Noetherian Jacobson ring R (resp., field k), which preserves the algebra structure. Then A satisfies the strong Nullstellensatz (resp., Nullstellensatz), and A is a Jacobson ring.*

Proof. If A is an R-algebra, Corollary 1.2.20 insures that Proposition 1.2.13 holds, so A satisfies the strong Nullstellensatz.

On the other hand, when A is an algebra over the field k, the ring $A[t]$ can be viewed as an iterated ore extension of $k[t]$. The Corollary 1.2.20 now insures that the hypotheses of Proposition 1.2.13 and its Corollary keep, so that a satisfies the Nullstellensatz.

In either case, we can apply Proposition 1.2.15, since $A[t]$ is an iterated Ore extension of R or k. So $A/M[x]$ is an iterated Ore extension of R/M, and satisfies the Nullstellensatz by the first part of this Corollary 1.2.20.

□

Since generic flatness and the strong Nullstellensatz are satisfied by several important families of finitely-generated Noetherian algebras, one might hope that all finitely-generated Noetherian algebras satisfy these properties. However, this is not the case, as we can appreciated in [Irv79a] and [Irv79b].

Remark 1.2.22 ([Irv79a], page 272)**.** Let $R = \mathbb{Z}[\frac{1}{2}, y, y^{-1}]$ and let T be the multiplicatively closed subset generated by the set of elements $\{y + 2n \mid n \in \mathbb{Z}\}$, and let $S = R_T$ be the corresponding localization. The ring S is a commutative, Noetherian, Jacobson, with an automorphism σ defined by $\sigma(y) = y + 2$.

We now construct the associated twisted group ring $A = S[x, x^{-1}; \sigma]$, which consists of polynomials in x and x^{-1} satisfying the relation $x^{-1}yx = y + 2$ and more generally, $x^{-n}yx^n = y + 2n$ and $x^{-n}y^{-1}x^n = (y + 2n)^{-1}$.

The ring A is a finitely-generated, Noetherian, Jacobson $\mathbb{Z}$-algebra which is a primitive ring. Hence A does not satisfy the strong Nullstellensatz or generic flatness.

A is finitely generated by $\frac{1}{2}$, y, y^{-1}, x and x^{-1}. Also A can be viewed as a factor ring of an iterated Ore extension of S by two automorphisms, σ and σ^{-1}. This implies that A is Noetherian, now, due to [GM74, Theorem (1.11)] we can conclude that A is a Jacobson ring. Notice however that A is not a finitely- iterated Ore extension of $\mathbb{Z}$, and that A is an iterated Ore extension of S without S-structure. Then Theorem 1.2.18 and 1.2.21 does not apply.

We prove primitivity by constructing an explicit faithful, simple module. Let V be the $\mathbb{Q}$-vector space with basis $\{v_n \mid n \in \mathbb{Z}\}$. We make V an A-module as follows: $xv_n = v_{n+1}$, $x^{-1}v_n = v_{n-1}$ and $yv_n = (2n+1)v_n$, and $y^{-1}v_n = (2n+1)^{-1}v_n$. This is well-defined, since $2n+1$ can never be zero, and the relation $x^{-1}yx = y+2$ we can see because $x^{-1}yxv_n = x^{-1}yv_{n+1} = x^{-1}(2n+3)v_{n+1} = (2n+3)v_n = (2n+1)v_n + 2v_n = yv_n + 2v_n = (y+2)v_n$.

The module V is faithful, for see this suppose $a \in A$ annihilates V and we must conclude that $a = 0$. We can write $a = \sum_i s_i x^i$, for $s_i \in S$. Then $av_n = \sum_i s_i x^i v_n = \sum_i s_i v_{n+i}$, so each s_i must also annihilates V. But then, multiplying s_i by an element in the set $\{2^l\} \cup T$, with $l \geq 1$, we can assume some $f(y)$ of $\mathbb{Z}[y]$ annihilates V. This mean $0 = f(y)v_n = f(2n+1)v_n$, so f vanishes on the set $\{2n+1 \mid n \in Z\}$. Therefore $f = 0$ and we can conclude that $a = 0$.

To prove simplicity, let $v = \sum_{i=m}^{n} c_i v_i$ be a non-zero vector. Suppose $m \neq n$ and $c_m, c_n \neq 0$. Then $yv = \sum_{i=m}^{n} c_i(2i+1)v_i$, and $yv - (2m+1)v = \sum_{i=m}^{n} 2c_i(i-m)v_i$, So Av contains a vector with fewer non-zero coefficient. Continuing this process, we find that Av contains a vector of the form cv_n, with $c \in \mathbb{Q}$. If we can show that Av contains $\mathbb{Q}v_n$, we will be done, for then the action of x allows us to conclude that $Av = V$. But observe that $(x^{-i}y^{-1}x^i)v_n = (2(n+i)+1)^{-1}v_n$, and as i varies, we obtain multiples of v_n by the inverse of all odd primes. Since A already contains $\frac{1}{2}$, we see that Av does contain $\mathbb{Q}v_n$.

Until now we have seen that an Ore extension which comes from a Jacobson ring is not necessarily a Jacobson ring [NIM14]. We can find examples of finitely generated Jacobson Noetherian Ore extensions (also been algebras) which does not satisfy the Nullstellensatz or generic flatness condition. For this reason we still need to develop new tools for verification purposes of the Hilbert's Nullstellensatz.

1.2.2 Filtration-graduation technique

Another approach to the Hilbert's Nullstellensatz in the non-commutative case is given in [MR88], which gives an extended version of generic flatness to see how the theorem is satisfied in certain non-commutative affine algebras over a field.

In [MR88, page 227] we can note that for certain non-commutative affine algebras R over a field k, if the following properties hold:

(i) (Endomorphism property) For each simple right R-module M, $\mathrm{End}(M)$ is algebraic over k.

(ii) (Radical property) The Jacobson radical of each factor ring of R is nilpotent.

It is said that R **satisfies the Nullstellensatz over** k.

Next, we recall the definition of generic flatness given in [MR88] over a commutative integral domain D (compare with Definition 1.2.12).

Definition 1.2.23 ([MR88], Definition 1)**.** We say that a D-algebra R is **generically flat** over D, if for each finitely generated M_R, there exists $0 \neq d \in D$ such that M_d is free over D_d.

In practice, it is sufficient to check the Definition 1.2.23 for each cyclic module as we will see in the following result.

Lemma 1.2.24 ([MR01], page 349). *Let R be a K-algebra over a integral domain K. If, for each cyclic module N_R there exist $0 \neq f \in K$ such that N_f is free over K_f, then R is generically flat over K.*

Proof. Suppose $M_R = \sum_{i=1}^{n} m_i R$. We are going to use induction on n. When $n = 1$ we are done. We can, therefore, assume the existence of $0 \neq v, w \in K$ such that $(m_1 R)_v$ is free over K_v and $(M/m_1 R)_w$ is free over K_w. Let $f = vw$; the both $(m_1 R)_f$ and $(M/m_1 R)_f$ are free over K_f. Hence the short exact sequence

$$0 \longrightarrow (m_1 R)_f \longrightarrow M_f \longrightarrow (M/m_1 R)_f \longrightarrow 0$$

splits and M_f is free over K_f. $\square$

There is a well-know connection between the Definition 1.2.23 and the Nullstellensatz as we can see in [Duf73], or Proposition 1.2.13 and Corollary 1.2.14.

Lemma 1.2.25 ([MR88], Lemma 2). *Let R be a k-algebra and x be a central indeterminate. If $R[x]$ is generically flat over $k[x]$ then R has the endomorphism property.*

Proof. We follow the proof given in [MR88]. Let M_R be a simple module and assume that $\mathrm{End}(M)$ is algebraic over k. Then there is an embedding $k[x] \hookrightarrow \mathrm{End}(M)$. By hypothesis, M_d is free over $k[x]_d$ for some $d \in k[x]$ non-zero. Thus, if A is proper nonzero ideal of $k[x]_d$ then AM_d is a proper nonzero submodule of M_d. However since $k[x] \hookrightarrow \mathrm{End}(M)$ then $k(x) \hookrightarrow \mathrm{End}(M)$ and also $M_d = M$. It follows that $k[x]_d$ must be a field, which is a contradiction. $\square$

Next we note a link with radical property given for the Lemma.

Lemma 1.2.26 ([MR88], Lemma 3). *If R is a k-algebra, x is a central indeterminate and $R[x]$ has the endomorphism property then R satisfies the Nullstellensatz.*

Throughout this chapter we have tried to explain how a algebraic structure satisfy the Hilbert's Nullstellensatz. In the following we are going to concentrate our attention on the endomorphism property (Nullstellensatz) and on the generic flatness notion.

Corollary 1.2.27 ([MR88], Corollary 4). *If R is a k-algebra such that $R[x, y]$ is generically flat over $k[y]$ then R satisfies the Nullstellensatz.*

With this in mind we now extend Definition 1.2.23.

Definition 1.2.28 ([MR88], Definition 5). A D-algebra R is $(\mathbb{N}, \mathbb{N})$**-generically flat** over D, if $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is generically flat over $D[y_1, \ldots, y_m]$ for all $n, m \in \mathbb{N}$.

We are going to see a sequence of results for the existence of a large class of such algebras.

Lemma 1.2.29 ([MR88], Lemma 6)**.** *Let $R \subseteq S$ be D-algebras and let R be $(\mathbb{N}, \mathbb{N})$-generically flat over D. Suppose that either one of the following conditions holds:*

(i) *S is a finite extension of R (i.e. S is finitely generated as a right R-module);*

(ii) *S is generated over R by an element z such that $zR = Rz$.*

Then S is $(\mathbb{N}, \mathbb{N})$-generically flat over D.

Proof. Let us see the two cases.

(i) The fact that $S[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is a finitely generated $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$-module shows that it is enough to prove that S is generically flat over D. However, any finitely generated right S-module is also finitely generated as a right R-module.

(ii) Once again it is enough to show that S is generically flat over D, so we consider, for lemma 1.2.24, a cyclic S-module M, say $M \cong S/I$, with I a right ideal of S. If one defines, for each n

$$I_n = \{r \in R \mid rz^n \in z^{n-1}R + \cdots + zR + R + I\},$$

then one obtains a chain of R-modules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots \subseteq M = \bigcup M_n$$

in which $M_n = \left(I + \sum_{i=0}^{n-1} Rz^i\right)/I$ and $M_{n+1}/M_n \cong R/I_n$ as a D-module. Now, let $N = R[x]/\sum x^n I_n$ where x is a central indeterminate. N is a cyclic $R[x]$-module and so, by hypothesis, N_d is free over D_d for some $0 \neq d \in D$ (in this point the condition $(\mathbb{N}, \mathbb{N})$-generic flatness is needed). It follows that each D_d-direct summand $(R/I_n)_d$ of N_d is projective and hence M_d splits; $M_d \cong \oplus(R/I_n)_d \cong N_d$. Thus M_d is free.

□

Next, we define graded and filtered rings with the aim of characterizing the generic flatness property and so the Nullstellensatz in several non-commutative structures.

Definition 1.2.30 ([AM69], page 106)**.** A **graded ring** is a ring A with a family $\{A_p\}_{p\geq 0}$ of subgroups of the additive group of A, such that

(i) $A_pA_q \subseteq A_{p+q}$, for all $p, q \geq 0$.

(ii) $A = \bigoplus_{p=0}^{\infty} A_p$.

The family $\{A_p\}_{p\geq 0}$ is called a **grading** of A.

We have that A_0 is a subring of A, and each A_p is an A_0-module.

Definition 1.2.31 ([MR01], Definition 1.6.1)**.** A **filtered ring** is a ring A with a family $\{F_p(A)\}_{p\geq 0}$ of subgroups of the additive group of A such that

(i) $F_p(A)F_q(A) \subseteq F_{p+q}(A)$, for all $p, q \geq 0$.

(ii) $A = \bigcup_{p=0}^{\infty} F_p(A)$.

(iii) $F_p(A) \subset F_q(A)$, for $p < q$.

(iv) $1 \in F_0(A)$.

The family $\{F_p(A)\}_{p\geq 0}$ is called a **filtration** of A.

The notion of filtered and graded ring have several important properties. Some of them will be described throughout this chapter.

Proposition 1.2.32 ([MR01], page 26)**.** *Every graded ring is a filtered ring.*

Proof. Let A be a ring with graduation $\{A_p\}_{p\geq 0}$. A is a filtered ring with filtration $\{F_p(A)\}_{p\geq 0}$, where $F_p(A) := \bigoplus_{n=0}^{p} A_n$. We can note that this definition is a filtration: we have that $F_p(A)F_q(A) = \bigoplus_{n=0}^{p} A_n \bigoplus_{n=0}^{q} A_n \subseteq \bigoplus_{n=0}^{p+q} A_n = F_{p+q}(A)$; $\bigcup_{p=0}^{\infty} = \bigoplus_{p=0}^{\infty} A_p = A$; $F_p(A) = \bigoplus_{n=0}^{p} A_n \subseteq \bigoplus_{n=0}^{q} A_n = F_q(A)$, if $p < q$ and $1 \in F_0(A)$. $\square$

Proposition 1.2.33 ([MR01], Page 26)**.** *If A is a filtered ring, then there is a graded ring* $\mathrm{Gr}(A)$ *associated to A.*

Proof. Given a filtered ring A, first we clarify that we could consider several graduation associated to the ring A. We are going to describe one (different from the trivial). So that, let $\{F_p\}_{p\in\mathbb{Z}}$ the filtration of A. Due to F_p are subgroups of the abelian group R^+, we can consider $\mathrm{Gr}_i(A) = F_i(A)/F_{i-1}(A)$, with $i \in I$. Let us check that we can defined the ring $\mathrm{Gr}(A) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}(A)_i$[1], whose multiplication for $\overline{a} = a + \mathrm{Gr}(A)_{p-1} \in \mathrm{Gr}(A)_p$, with $a \in F_p(A)$ and $\overline{b} = b + \mathrm{Gr}(A)_{q-1} \in \mathrm{Gr}(A)_q$, with $a \in F_q(A)$ is given by $(a + \mathrm{Gr}(A)_{p-1})(b + \mathrm{Gr}(A)_{q-1}) = ab + a\mathrm{Gr}(A)_{q-1} + \mathrm{Gr}(A)_{p-1}b + \mathrm{Gr}(A)_{p-1}\mathrm{Gr}(A)_{q-1} = ab + \mathrm{Gr}(A)_{p+q-1}$. This multiplication is well defined: take $\overline{a}, \overline{a'} \in \mathrm{Gr}(A)_p$ and $\overline{b}, \overline{b'} \in \mathrm{Gr}(A)_q$, such that $a, a' \in F_p(A)$ and $b, b' \in F_q(A)$ then we have that $\overline{a} - \overline{a'} = \overline{0}$ and $\overline{a} - \overline{a'} = \overline{0}$, that is to say $a - a' \in F_{p-1}(A)$ and $b - b' \in F_{q-1}(A)$. By hypothesis we have that $(a - a')(b - b') = ab - ab' - a'b + a'b' \in F_{p+q-2}(A) \subseteq F_{p+q-1}(A)$. Due to $ab - ab' - a'b + a'b' = ab + (-a'b' + a'b') - ab' - a'b + a'b' = (ab - a'b') + (a' - a)b' + a'(b' - b)$ and $(a' - a)b', a'(b' - b) \in F_{p+q-1}$, then $ab - a'b' \in F_{p+q-1}$. Therefore $\overline{ab} = \overline{a'b'}$ in $\mathrm{Gr}(A)_{p+q}$. According to the defined multiplication the first condition $\mathrm{Gr}(A)_p\mathrm{Gr}(A)_q \subseteq \mathrm{Gr}(A)_{p+q}$ holds. Therefore $\mathrm{Gr}(A)$ is a graded ring with graduation $\{\mathrm{Gr}(A)_p\}_{p\in\mathbb{Z}}$. $\square$

Corollary 1.2.34 ([Lez19b], Corollary 2.2.5)**.** *Let A be a graded ring. Then* $\mathrm{Gr}(A) \cong A$.

[1] $\bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}(A)_i \cong \sum_{i\in\mathbb{Z}} \oplus \mathrm{Gr}'(A)_i$, if we consider $\mathrm{Gr}'(A)_i = (\cdots, 0, \cdots, \mathrm{Gr}(A)_i, \cdots, 0, \cdots)$, with $\mathrm{Gr}(A)_i$ in the i-th entry.

Proof. Let A be a graded ring with grading $\{A_p\}_{p\in\mathbb{Z}}$. Due to the Proposition 1.2.32 we have $\{F_p(A)\}_{p\in\mathbb{Z}}$, with $F_p(A) := \bigoplus_{n\leq p} A_p$ a filtration of A. Then the associated graded $\mathrm{Gr}(A)_p = \bigoplus_{n\leq p} A_p / \bigoplus_{n-1\leq p} A_p \cong A_p$ and $\mathrm{Gr}(A) := \bigoplus_{p\in\mathbb{Z}} \mathrm{Gr}(A)_p \cong \bigoplus_{p\in\mathbb{Z}} A_p = A$. □

Example 1.2.35. Several examples of graded and filtered rings are the following:

1. Let $P = K[x_1, \dots, x_n]$ be the polynomial ring with n-variables. Then consider the group P_m which is generated by $\{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \sum_{i=1}^n \alpha_i = m\}$ as K-module. $\{P_m\}_{m\in\mathbb{N}}$ is a positive graduation of P.

2. Let $S = A[x, \sigma]$ be an Ore extension of endomorphism type. It admits a positive graduation according to the degree of the polynomials, so that $S_p = \{ax^p \mid a \in A\}$, for $p \in \mathbb{N}$ is a positive graduation of S.

3. Let $S = A[x; \sigma, \delta]$ be an Ore extension over A. Then A is a filtered ring with positive filtration $\{F_p(A)\}$ defined as $F_p(A) = \{f \in A \mid \deg(f) \leq p\} \cup \{0\}$.

Proposition 1.2.36 ([MR01], Page 27)**.** *Let A be a $\mathbb{N}$-filtered ring with filtration $\{F_i(A)\}_{i\in\mathbb{N}}$. If $\mathrm{Gr}(A)$ is a domain, then A is a domain.*

Proposition 1.2.37 ([MR01], Theorem 6.9)**.** *Let A be a $\mathbb{N}$-filtered ring with filtration $\{F_i(A)\}_{i\in\mathbb{N}}$. If $\mathrm{Gr}(A)$ is right (left) Noetherian, then A is right (left) Noetherian.*

Finally we have that the associated graded ring of an Ore extension with coefficients in a ring A is well determined by an Ore extension of endomorphism type

Proposition 1.2.38 ([MR01], Page 29)**.** *Let $S = A[x; \sigma, \delta]$ be an Ore extension over A. Then $\mathrm{Gr}(S) \cong A[\overline{x}; \sigma]$.*

The following result establishes that the $(\mathbb{N}, \mathbb{N})$-generically flat inherits from the associated graded ring.

Lemma 1.2.39 ([MR88], Lemma 7)**.** *Let S be a filtered D-algebra with $D \subseteq S_0$ and suppose that the associated graded ring $\mathrm{Gr}(S)$ is $(\mathbb{N}, \mathbb{N})$-generically flat over D. Then so too is S.*

Proof. It is sufficient to show that S is generically flat. If M is a finitely generated right S-module, it can be filtered so that $\mathrm{Gr}(M)$ is finitely generated over $\mathrm{Gr}(S)$. Therefore $(\mathrm{Gr}(M))_d$ is free over D_d for some d, and so, arguing as in Lemma 1.2.29 (ii), $M_d \cong (\mathrm{Gr}(M))_d$ and thus is free. □

In [MR88] it was shown that a kind of special extensions are $(\mathbb{N}, \mathbb{N})$-generically flat and so they satisfy the *Nullstellensatz*.

Definition 1.2.40 ([MR88], Definition 8)**.** Let R, S be k-algebras with $R \subset S$. Then S is an **almost normalizing extension** (also **almost commutative algebra**) of R, if the following conditions hold:

(i) S is generated over R by a finite set of elements $\{x_1, \dots, x_n\}$;

(ii) $x_iR + R = Rx_i + R$;

(iii) $x_ix_j - x_jx_i \in \sum_{k=1}^{n} Rx_k + R$.

Lemma 1.2.41 ([MR88], Lemma 9)**.** *If R is $(\mathbb{N},\mathbb{N})$-generically flat over D and S is an almost normalizing extension of R then S is $(\mathbb{N},\mathbb{N})$-generically flat over D.*

Proof. We filter S by "degree" in the x_i's, i.e., we set $S_n = \sum Rw$ where w ranges over all words of length at most n in the x_i. The associated graded ring $\mathrm{Gr}(S)$ is generated by R and the images of the x_i and so is obtainable from R by a finite number of extensions, as covered by Lemma 1.2.29 (ii). Therefore $\mathrm{Gr}(S)$, and hence S, is $(\mathbb{N},\mathbb{N})$-generically flat over D. □

An example of the existence of $(\mathbb{N},\mathbb{N})$-generic flatness is given in Lemma 1.2.42.

Lemma 1.2.42 ([MR88], Lemma 10)**.** *D is $(\mathbb{N},\mathbb{N})$-generically flat over D.*

Proof. Initially, we are going to show that $R = D[x_1,\dots,x_n]$ is generically flat over D. Let W be the semigroup of all words in $x_1,\dots,x_n$ and order W first by total degree and, subject to that, lexicographically. Suppose that $M \cong R/I$ with I a right ideal of R and for each $w \in W$, let

$$I(w) := \{d \in D \mid dw \in \sum_{v<w} Dv + I\}.$$

Note that if v divides w then $I(v) \subseteq I(w)$. One can show that any subset S of W has finite elements which are not divisible within S. We let $S := \{w \in W \mid I(w) \neq 0\}$ and let $w_1,\dots,w_t$ be its nondivisible elements. We choose $0 \neq d \in I(w_1) \cap \cdots \cap I(w_t)$. Then $(D/I(w))_d$ is free over D_d for all $w \in W$. One can now argue, as in Lemma 1.2.29 (ii), that $M_d \cong \bigoplus (D/I(w))_d$. Thus M_d is free. □

Definition 1.2.43 ([MR88], Definition 11)**.** A K-algebra S is called **constructible**, if it can be obtained from K via a finite number of iterations of finite extensions and almost normalizing extensions.

Proposition 1.2.44 ([MR88], Theorem 12)**.** (i) *Any constructible D-algebra is $(\mathbb{N},\mathbb{N})$-generically flat over D.*

(ii) *Any constructible k-algebra satisfies the Nullstellensatz.*

Proof. (i) This combines Lemmas 1.2.29, 1.2.41 and 1.2.42

(ii) This follows from (i) and Corollary 1.2.27

□

Corollary 1.2.45 ([MR88], Corollary 12)**.** *Let S be a constructible K-algebra, M a simple S-module and P a prime ideal of S.*

(i) *If k is the field of fractions of $K/P \cap K$, then $S/P \otimes_K k$ is a constructible k-algebra, and $J(S/P \otimes_K k) = 0$.*

(ii) *If* $P = \mathrm{Ann}_S(M)$ *then* $\mathrm{End}(M)$ *is algebraic, indeed finite dimensional over* k.

Proof. (i) It is clear that $S/P \otimes_K k$ is constructible over k. Moreover it is prime (since it is a localization with respect to an Ore set) and Noetherian, and hence it has no nonzero nil ideals.

(ii) The embedding $K/p \cap K \hookrightarrow \mathrm{End}(M)$ extends to an embedding $k \hookrightarrow \mathrm{End}(M)$. It follows that M is also a simple module over $S/P \otimes k$ and so its endomorphism ring over $S/P \otimes k$ is algebraic over k. However $\mathrm{End}(M_S) \hookrightarrow \mathrm{End}(M_{S/P\otimes k})$ (in fact they are isomorphic). The finite dimension follows.

□

As we saw in Section 1.1, a Jacobson ring is a ring S such that $J(S/P) = 0$, for all prime ideal P. This fact implies that the Jacobson radical of each factor ring of S is nil.

Proposition 1.2.46 ([MR88], Theorem 14.)**.** *Let* S *be a constructible* K*-algebra with* K *a Jacobson ring, and let* M *be a simple right* S*-module. Then*

(i) S *is a Jacobson ring; and*

(ii) $K/\mathrm{Ann}_K M$ *is a field over which* $\mathrm{End}(M_S)$ *is finite dimensional.*

Following [McC82] we have k a commutative field and K commutative ring. Let A be a finitely generated commutative polynomial algebra over k. Then the following statements are equivalent:

(i) the algebra A satisfies the Hilbert Nullstellensatz;

(ii) each prime ideal of A is an intersection of primitive ideals;

(iii) if M is a simple A-module, the division ring $\mathrm{End}_A(M)$ is algebraic over k.

The second statement, as we could see in Definition 1.2.6, is the definition of Jacobson ring. The third statement for Definition 1.2.10 is our definition of Nullstellensatz. Other perspective in [McC82] is the maximal Nullstellensatz that tells us that an algebra A over K satisfies the maximal Nullstellensatz over K, if for all simple left A-modules M, $\mathrm{Ann}_K(M)$ is a maximal ideal of K and $A/\mathrm{Ann}_A M$ satisfies the Nullstellensatz over the field $K/\mathrm{Ann}_K(M)$ (thus when K is a field the maximal Nullstellensatz coincides with the Nullstellensatz like the definition 1.2.10).

In [McC82] we find some examples of almost normalizing extension like B an Ore extension of A, $B = A[x; \sigma, \delta]$, where σ is an automorphism of A, B a skew Laurent extension $B = A[x, x^{-1}; \sigma]$ where σ is an automorphism of A.

1.2.3 Flatness, freeness and projective

In [Irv79a] and [MR88] the definition of generically flat and generically free are used as the same. However, the two notions are not equivalent (recall that every free module is flat but the converse is not true).

From now on we will follow [ASZ99]. We say that an **R-module M is generically flat over a domain R if there is a simple localization R_s such that $M_s = M \otimes_R R_s$ is flat over R_s.** Generically projective and generically free modules are defined similarly.

A technique to verify that the modules over a fixed ring are generically free is to check that all the associated graded modules are generically projective.

Proposition 1.2.47 ([ASZ99], Proposition 3.8)**.** *Let R be a commutative domain and let $A = \bigcup F_n$ be an $\mathbb{N}$-filtered R-algebra. If every finite graded right $\mathrm{Gr}(A)$-module is generically projective over R, then every finite right A-module is generically free over R.*

Proof. We follow the proof presented in [ASZ99]. Let M be a finite right A-module. Since M is finite, there is a finite R-submodule $N \subset M$ such that $M = NA$. Let $L_n = NF_n$. Thus $\mathrm{Gr}(M) := \bigoplus_n L_n/L_{n-1}$ is a finite graded $\mathrm{Gr}(A)$-module. By hypothesis, there is an $0 \neq f \in R$ such that $\mathrm{Gr}(M)_f$ is projective over R_f. Hence every $(L_n/L_{n-1})_f$ is projective. Therefore

$$M_f = \bigcup_n (L_n)_f \approx \bigoplus_n (L_n/L_{n-1})_f,$$

which it is projective over R_f. By Bass's theorem that asserts that an infinitely generated projective module over a commutative domain is free, we can conclude that M is free if it is an infinitely generated R-module, and M is generically free if it is finitely generated. Thus M is generically free in every case. $\square$

We recall the results of the previous subsections in the following lemma, in which, part (i) is explicitly in Lemma 1.2.25 and the part (ii) is a special case of Proposition 1.2.15.

Lemma 1.2.48 ([ASZ99], Lemma 3.9)**.** *Let A be a right Noetherian algebra over a field k.*

(i) *If every simple right $A[t]$-module is generically free over $k[t]$, then A satisfies the Nullstellensatz.*

(ii) *If $A[t]$ satisfies the Nullstellensatz, then A is a Jacobson algebra.*

The next result establishes some conditions in which the modules of an Ore extensions are generically free, provided that all the modules over the coefficient ring are all generically free as well.

Proposition 1.2.49 ([ASZ99], Theorem 3.10)**.** *Let R be a commutative domain. Let A be a right Noetherian R-algebra such that every finite right A-module is generically free over R. Let $A[x; \sigma, \delta]$ be an Ore extension for an R-linear automorphism σ and a R-linear σ-derivation δ. Then every finite right $A[x; \sigma, \delta]$-module is generically free over R.*

Proof. We follow the proof in [ASZ99]. The Ore extension $A[x; \sigma, \delta]$ is an $\mathbb{N}$-filtered R-algebra and its associated graded ring is $A[x; \sigma]$ with $\deg x = 1$. By Proposition 1.2.47 it suffices to prove that every finite graded right $A[x; \sigma]$-module M is generically projective over R. The automorphism σ of A can be extended to an automorphism of $A[x; \sigma]$ by $\sigma(x) = x$. The σ-twisted module M^σ is M as an R-module with right multiplication defined by $m \cdot a = m\sigma(a)$, for all $m \in M$ and $a \in A[x;\sigma]$. Consider the $A[x;\sigma]$-linear map $M^\sigma[-1] \to M$ defined by multiplication by x. The kernel and cokernel of this map are finitely generated graded modules on which x acts trivially, so they are finite graded right A-modules. They are zero except in finitely many degrees, and for large degree, say $n > n_0$, the linear map $M_{n-1} \to M_n$ is bijective. To make M free over R, it suffices to make M_i free for $i \leq n_0$. Since each M_i is a finite A- module, it is generically free by hypothesis, so this can be done. □

In order to state a result that helps us to describe the Hilbert's Nullstellensatz, let us remind that an $\mathbb{N}$-graded R-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ is called **locally finite**, if each homogeneous component A_i is a finite R-module for every i.

Example 1.2.50. Several examples of locally finite $\mathbb{N}$-graded R-algebras are the following.

1. Let $A = k[x]$ with k a field. We can do a $\mathbb{N}$-graduation as follows:

$$A = k \oplus kx \oplus kx^2 \oplus \cdots \oplus kx^n \oplus \cdots ,$$

   when $A_i = kx^i$ and $A_o = k$, we have that A is a k-algebra. We can note that every A_i is finite dimensional k.

2. We extend the previous example with $A = k[x_1 \ldots, x_n]$. We consider a $\mathbb{N}$-graduation given by

$$A = k \oplus kx_1 + \cdots + kx_n \oplus kx_1^2 + \cdots + kx_n^2 \oplus \cdots \oplus kx_1^n + \cdots + kx_n^n \oplus \cdots ,$$

   when $A_i = kx_1^i + \cdots + kx_n^i$ and $A_o = k$. We have that A is a k-algebra and every A_i is finite dimensional.

A Dedekind domain is an integral domain in which every non-zero ideal is uniquely represented as the product of a finite number of prime ideals ([Mat89], page 284). A consequence of this definition is that every principal ideal domain (PID) is a Dedekind domain.

Proposition 1.2.51 ([ASZ99], Theorem 0.4)**.** *Let A be an $\mathbb{N}$-filtered algebra over a field k, whose associated graded ring is locally finite and right Noetherian. Then A is a Jacobson algebra which satisfies the Nullstellensatz.*

Proof. We follow the proof in [ASZ99]. Let B be an algebra over $k[t]$. Since $k[t]$ is a Dedekind domain, every Noetherian right B-module is generically flat over $k[t]$. If B is also graded, then every right Noetherian, locally finite, graded right B-module is generically projective and hence generically free, over $k[t]$. Together with Proposition 1.2.47, these remarks show that if A is a right Noetherian $\mathbb{N}$-filtered k-algebra, then every finite right $A[t]$-module is generically free over $k[t]$. By Proposition 1.2.49 every finite right $A[x][t]$-module is generically free over $k[t]$. Lemma 1.2.48 (i) guarantees that $A[x]$ satisfies the Nullstellensatz and so for Lemma 1.2.48 (ii) we have that A is a Jacobson algebra. □

CHAPTER 2

The Hilbert's Nullstellensatz over skew PBW extensions

In Chapter 1 we appreciated several treatments with the aim of formulating a non-commutative version of the Hilbert's Nullstellensatz. Now, we will see the definition of skew PBW extensions introduced in [GL11] and several of their properties with the goal of formulating the Hilbert's Nullstellensatz for these extensions. In Section 2.3 we are going to see several examples of skew PBW extensions for which the result holds.

2.1 Definition and properties

Skew PBW extensions were defined in [GL11] as a generalization of the PBW (Poincaré-Birkhoff-Witt) extension introduced in [BG88], as an alternative technique for studying a very wide class of non-commutative rings of polynomial type. Let us remind the definition of the classical PBW extensions.

Definition 2.1.1 ([BG88], page 27)**.** Let R and A be rings. It is said that A is a **Poincaré-Birkhoff-Witt (PBW, for short) extension** of R, if the following conditions hold:

(i) $R \subseteq A$;

(ii) There exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}(x^0 := 1)$;

(iii) $x_i r - r x_i \in R$, for each $r \in R$ and $1 \le i \le n$;

(iv) $x_j x_i - x_i x_j \in R + Rx_1 + \cdots + Rx_n$, for any $1 \le i, j \le n$.

In this situation, we write $A = R\langle x_1, \dots, x_n\rangle$.

The following are examples of PBW extensions: the enveloping algebra of any finite-dimensional Lie algebra; any Weyl algebra $A_n(R)$; any differential operator ring $R[x_1, \dots, x_n; \delta_1, \dots, \delta_n]$ formed from commuting derivations $\delta_1, \dots, \delta_n$ on R; the twisted or smash product differential operator ring, among others (see [BG88]).

Definition 2.1.2 ([GL11], Definition 1)**.** We say that A is a **skew PBW extension** of R (also called **σ-PBW extension** of R), which is denoted by $A := \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) $R \subseteq A$;

(ii) There exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $Mon(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}(x^0 := 1)$;

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exist an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$;

(iv) For any elements x_i, x_j with $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Skew PBW extensions have several ring-theoretical and homological properties (see [LSR17], [RS17a], [NR17], [RJ18]). Let us remind some them.

Proposition 2.1.3 ([GL11], Proposition 3)**.** *Let A a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$.*

Proof. For every $1 \leq i \leq n$ and each $r \in R$ we have elements $c_{i,r}, r_i \in R$ such that $x_i r = c_{i,r} x_i + r_i$; since $Mon(A)$ is a R-basis of A $c_{i,r}$ and r_i are unique for r, so we define $\sigma_i, \delta_i : R \to R$ by $\sigma_i(r) = c_{i,r}$, $\delta_i(r) = r_i$. We can check that σ_i is a ring endomorphism and δ_i is a σ_i-derivation of R, for $r, s \in R$ we have that

$$
\begin{aligned}
x_i(r+s) &= \sigma_i(r+s)x_i + \delta_i(r+s) \\
x_i r + x_i s &= \sigma_i(r)x_i + \delta_i(r) + \sigma_i(s)x_i + \delta(s) \\
&= (\sigma_i(r) + \sigma_i(s))x_i + \delta_i(r) + \delta_i(r),
\end{aligned}
$$

so we have that $\sigma_i(r+s) = \sigma_i(r) + \sigma_i(s)$ and $\delta(r+s) = \delta_i(r) + \delta_i(r)$, and

$$
\begin{aligned}
x_i(rs) &= \sigma_i(rs)x_i + \delta_i(rs) \\
(x_i r)s &= (\sigma_i(r)x_i + \delta_i(r))s \\
&= \sigma_i(r)x_i s + \delta_i(r)s \\
&= \sigma_i(r)(\sigma_i(s)x_i + \delta_i(s)) + \delta_i(r)s \\
&= \sigma_i(r)\sigma_i(s)x_i + \sigma_i(r)\delta_i(s) + \delta_i(r)s,
\end{aligned}
$$

we have $\sigma_i(rs) = \sigma_i(r)\sigma_i(s)$ and $\delta_i(rs) = \sigma_i(r)\delta_i(s) + \delta_i(r)s$ (this is de condition of σ_i-derivation), we can note also that $x_i = x_i 1 = \sigma_i(1)x_i + \delta_i(1)$, so $\sigma_i(1) = 1$ and $\delta_i(1) = 0$. Moreover, by the Definition 2.1.2 (iii), $c_{i,r} \neq 0$ for $r \neq 0$. This means that σ_i is injective. □

Definition 2.1.4 ([GL11], Definition 4)**.** Let A be a skew PBW extension.

(a) A is quasi-commutative if the conditions (iii) and (iv) in the Definition 2.1.2 are replaced by the following:

(iii') For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$.

(iv') For any elements x_i, x_j with $1 \leq i, j \leq n$, there exists an element $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.

(b) A is bijective, if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i, j \leq n$.

Skew PBW extensions are filtered rings, as the following proposition shows.

Proposition 2.1.5 ([LR14], Theorem 2.2)**.** *Let A be an arbitrary skew PBW extension of R. Then, A is a filtered ring with filtration given by*

$$F_m := \begin{cases} R, & \textit{if } m = 0 \\ \{f \in A \mid deg(f) \leq m\}, & \textit{if } m \geq 1. \end{cases}$$

and the corresponding graded ring $\mathrm{Gr}(A)$ *is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then* $\mathrm{Gr}(A)$ *is a quasi-commutative bijective skew PBW extension of R.*

Proposition 2.1.5 gives us a information over associated graded of a skew PBW extension (we obtain a quasi-commutative skew PBW extension). Moreover, we can give a characterization of quasi-commutative skew PBW extension.

Proposition 2.1.6 ([LR14], Theorem 2.3)**.** *Let A be a quasi-commutative skew PBW extension of a ring R.*

(i) A is isomorphic to an iterated skew polynomial ring of endomorphism type.

(ii) If A is bijective, then each endomorphism is bijective.

Proposition 2.1.7 ([LR14], Corollary 2.4. Hilbert Basis Theorem)**.** *Let A be a bijective skew PBW extension of R. If R is a left Noetherian ring, then A is also a left Noetherian ring.*

Proof. According to Proposition 2.1.5, $\mathrm{Gr}(A)$ is a quasi-commutative skew PBW extension, and by the hypothesis, $\mathrm{Gr}(A)$ is also bijective. By Proposition 2.1.6, $\mathrm{Gr}(A)$ is isomorphic to an iterated skew polynomial ring $R[z_1; \sigma_1] \cdots [z_n; \sigma_n]$ such that each σ_i is bijective, $1 \leq i \leq n$. This implies that $\mathrm{Gr}(A)$ is a left Noetherian ring, and hence, A is left Noetherian. □

The following are examples of skew PBW extension: classical polynomial rings, skew polynomial rings of derivation type, Weyl algebra, universal enveloping algebra of a finite dimensional Lie algebra, Woronowicz algebra, q-Heisenberg algebra, additive analogue of the Weyl algebra, multiplicative analogue of the Weyl algebra. For some of these examples we know that the Hilbert's Nullstellensatz holds (see [Irv79a] and [MR88]). The idea is to find the conditions to guarantee the result over general skew PBW extensions.

2.2 Nullstellensatz over skew PBW extensions

We identify several conditions that we should expect skew PBW extensions satisfy in order to the Hilbert's Nullstellensatz to be valid over them. In this section, we shall review some preliminaries properties and enunciate the Hilbert's Nullstellensatz for skew PBW extensions.

Recall from Definition 1.2.10 that a k-algebra A satisfies the Nullstellensatz, if for any simple right A-module M, the division ring $\mathrm{End}_A(M)$ is algebraic over k. This will be our standard version of Hilbert's Nullstellensatz that the following result extends for skew PBW extensions.

Proposition 2.2.1. *Let R be a commutative domain. Let B be a right Noetherian R-algebra such that every finite right B-module is generically free over R. Let A be a skew PBW extension of B for an R-linear automorphism σ_i and a R-linear σ_i-derivation δ_i, for $1 \leq i \leq n$ Then every finite right A-module is generically free over R.*

Proof. Let A be a skew PBW extension of B, with B a right Noetherian R-algebra such that every finite right B-module is generically free over R. We have by Proposition 2.1.5 that A is isomorphic to $B[z_1;\theta_1]\cdots[z_n;\theta_n]$, and we have that θ_i is a R-lineal automorphism, by hypothesis every finite right B-module is generically free over R, then Proposition 1.2.49 every finite right $B[z_1;\theta_1]$-module is generically free over R, so we can conclude that every finite right $B[z_1;\theta_1]\cdots[z_1,;\theta_n]$-module is generically free and by the Proposition 1.2.47 we have that every finite right A-module is generically free over R. □

From Proposition 1.2.51 we know that one hypothesis for an algebra $\mathbb{N}$-filtered to satisfy the Nullstellensatz is that its associated graded ring is locally finite.

Proposition 2.2.2 ([LSR17], Proposition 2.10)**.** *Let A be a K-algebra. A is finitely graded if and only if there exists a graded isomorphism of K-algebras*

$$A \cong K\{x_1,\dots,x_n\}/I,$$

where I is a proper homogeneous two-sided ideal of $K\{x_1,\dots,x_n\}$ that denote the free algebra over K. In such case, A is locally finite, i.e., for every $n \in \mathbb{N}$ $\dim_K A_n < \infty$.

Following Proposition 1.2.51 and Section 2.1 we can give a theorem to guarantee when a skew PBW extension satisfies the Hilbert's Nullstellensatz.

Theorem 2.2.3. *Let A be a bijective skew PBW extension of a Noetherian k-algebra R such that A is also a k-algebra and $\mathrm{Gr}(A)$ is locally finite. Then A is a Jacobson algebra which satisfies the Nullstellensatz.*

Proof. By the Proposition 2.1.7, since A is a bijective skew PBW extension of R Noetherian, A is left Noetherian. According to Proposition 2.1.5, $\mathrm{Gr}(A)$ is a quasi-commutative bijective skew PBW extension isomorphic to an iterated skew polynomial ring $R[z_1;\theta_1][z_2;\theta_2]\cdots[z_n;\theta_n]$ such that each θ_i is bijective, $1 \leq i \leq n$ and by the Proposition 2.1.6 $\mathrm{Gr}(A)$ is left Noetherian, and by the hypothesis $\mathrm{Gr}(A)$ is locally finite. So we have A $\mathbb{N}$-filtered algebra over k (Proposition 2.1.5) whose associated graded ring is locally finite and left Noetherian. Then, by Proposition 1.2.51, A is a Jacobson algebra which satisfies the Nullstellensatz. □

Corollary 2.2.4. *Every bijective skew PBW extension which preserves the k-algebra structure whose associated graded ring is finitely graded is a Jacobson algebra which satisfies the Nullstellensatz*

Proof. If we have A skew PBW extension over k whose associated graded ring $\mathrm{Gr}(A)$ is finitely graded, by Proposition 2.2.2. is locally finite and by Proposition 2.1.7 is left Noetherian. Then from the Theorem 2.2.3 A is a Jacobson algebra which satisfies the Nullstellensatz. □

2.3 Examples

We addressed conditions that skew PBW extensions should satisfy in order to the Hilbert's Nullstellensatz to hold. In this section we will exhibit some particular examples in which such hypothesis hold and thus the theorem.

The examples here presented were studied in [LR14], [GL11], [Rey14], [RJ18], [RS16b], [RS17b], [RS17a], [RS18b], [RS18c]. Here we verify that the Hilbert's Nullstellensatz holds for them.

2.3.1 Classical PBW extensions

Example 2.3.1 (Classical polynomial ring)**.** Let $k[x_1, \ldots, x_n]$ be the polynomial ring with k a field. The polynomial ring is an Ore extension with $\sigma_i = i_{k[x_1,\ldots,x_n]}$ and $\delta_i = 0$ for $1 \leq i \leq n$, Therefore, we have an extension over a Noetherian ring k which preserves the algebra structure. Then, polynomial ring satisfies the hypothesis of Proposition 1.2.21, thus the Nullstellensatz hold and we have a Jacobson algebra.

We can note also that $k[x_1, \ldots, x_n]$ is a $\mathbb{N}$-filtered A algebra over k and its associated graded is $k[x_1, \ldots, x_n]$. We know that $k[x_1, \ldots, x_n]$ is Noetherian and locally finite. Then, for the Proposition 1.2.51 $k[x_1, \ldots, x_n]$ is a Jacobson algebra that satisfies the Nullstellensatz.

Finally, the polynomial ring is a skew PBW extension. Since $x_ir - rx_i = 0$ and $x_ix_j - x_jx_i = 0$ for any $r \in k$ and $1 \leq i, j \leq n$. The k-free basis is $\mathrm{Mon}(k[x_1, \ldots, x_n])$. Every skew PBW extension is filtered in this case $\mathbb{N}$ filtered and its associated graded is locally finite. Then for the Theorem 2.2.3 $k[x_1, \ldots, x_n]$ is a Jacobson algebra that satisfies the Nullstellensatz.

Example 2.3.2 (Universal enveloping algebra of a Lie algebra with K field)**.** Let K be a commutative ring (in this case a field) and $\mathcal{G}$ be a finite dimensional Lie algebra over K with basis $\{x_1 \ldots, x_n\}$. The **universal enveloping algebra of** $\mathcal{G}$, $\mathcal{U}(\mathcal{G})$, with $x_ir - rx_i = 0$ and $x_ix_j - x_jx_i = [x_i, x_j]$.

These algebras not necessarily are Ore extensions. Then, we do not conclude the Nullstellensatz using Proposition 1.2.21.

In [Dix77, page 75] it was shown that $\mathcal{U}(\mathcal{G})$ is an algebra $\mathbb{N}$-filtered; in [Li02, page 30] its associated graded is isomorphic to the classical polynomial ring. Therefore, this algebra is Noetherian and locally finite. Then, due to Proposition 1.2.51, $\mathcal{U}(\mathcal{G})$ is a Jacobson algebra that satisfies the Nullstellensatz.

The universal enveloping algebra of $\mathcal{G}$, $\mathcal{U}(\mathcal{G})$, can be seen such as a skew PBW extension. In [LR14, page 1211] the authors shown that there exists a skew PBW extension $A = \sigma(K)\langle x_1, \ldots, x_n\rangle$ such that $\mathcal{U}(\mathcal{G}) \cong A$. Since $x_ir - rx_i = 0$ and $x_ix_j - x_jx_i = [x_i, x_j]$, with $[x_i, x_i] = K + Kx_1 + \cdots + Kx_n$. In this case A is a $\mathbb{N}$-filtered algebra and its associated graded is isomorphic to the classical polynomial ring and is locally finite. Then, due to Theorem 2.2.3 A is a Jacobson algebra that satisfies the Nullstellensatz.

2.3.2 3-dimensional skew polynomial algebras

Following [Ros95, Definition C4.3], 3-dimensional skew polynomial algebras $\mathcal{A}$ are k-algebra generated by the indeterminates x, y, z restricted to relations $yz - \alpha zy = \lambda$, $zx - \beta xz = \mu$, and $xy - \gamma yx = \nu$, such that, he following conditions hold: (i)$\lambda, \mu, \nu \in k + kx + ky + kz$, and $\alpha, \beta, \gamma \in k^*$; (ii) standard monomials, $\{x^iy^jz^k \mid i, j, k \geq 0\}$, are a k-basis of the algebra.

These algebras are skew PBW extensions. In [RS17b] it was shown that 3-dimensional skew polynomial algebras have a PBW basis, and with the relations defined we can note that these algebras satisfy the condition (iii) and (iv) of Definition 2.1.2. Note that, not necessarily, 3 dimensional skew polynomials algebras are PBW extensions, the condition (iv) of the Definition 2.1.1 fail when α, β,$\gamma \neq 1$.

There exists a classification of 3-dimensional skew polynomial algebras provided by [Ros95, Theorem C.4.3.1]. More precisely, up isomorphism, $\mathcal{A}$ is one of the following algebras:

(a) if $|\{\alpha, \beta, \gamma\}| = 3$, then $\mathcal{A}$ is defined by the relations $yz - \alpha zy = 0$, $zx - \beta xz = 0$, and $xy - \gamma yx = 0$.

(b) if $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma = 1$, then $\mathcal{A}$ is one of the following algebra:

  (i) $yz - zy = z$, $zx - \beta xz = y$, $xy - yx = x$;
  (ii) $yz - zy = z$, $zx - \beta xz = b$, $xy - yx = x$;
  (iii) $yz - zy = 0$, $zx - \beta xz = y$, $xy - yx = 0$;
  (iv) $yz - zy = 0$, $zx - \beta xz = B$, $xy - yx = 0$;
  (v) $yz - zy = az$, $zx - \beta xz = 0$, $xy - yx = x$;
  (vi) $yz - zy = z$, $zx - \beta xz = 0$, $xy - yx = 0$,

  where a, b are any elements of k. All nonzero values of b yield isomorphic algebras.

(c) If $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma \neq 1$, then $\mathcal{A}$ is one of the following algebra:

  (i) $yz - \alpha zy = 0$, $zx - \beta xz = y + b$, and $xy - \gamma yx = 0$;
  (ii) $yz - \alpha zy = 0$, $zx - \beta xz = b$, and $xy - \gamma yx = 0$.

In this case, $b \in k$ is an arbitrary element and, like before, any nonzero values of b give isomorphic algebras.

(d) If $\alpha = \beta = \gamma \neq 1$, then $\mathcal{A}$ is the algebra defined by the relations $yz - \alpha zy = a_1 x + b_1$, $zx - \beta xz = a_2 y + b_2$, and $xy - \gamma yx = a_3 z + b_3$. If $a_i = 0$ for $i = 1, 2, 3$, then all nonzero values of b_i give isomorphic algebras.

(e) If $\alpha = \beta = \gamma = 1$, then $\mathcal{A}$ is isomorphic to one of the following algebras:

(i) $yz - zy = x$, $zx - xz = y$, $xy - yx = z$;

(ii) $yz - zy = 0$, $zx - xz = 0$, $xy - yx = z$;

(iii) $yz - zy = 0$, $zx - xz = 0$, $xy - yx = b$;

(iv) $yz - zy = -y$, $zx - xz = x + y$, $xy - yx = 0$;

(v) $yz - zy = az$, $zx - xz = z$, $xy - yx = 0$;

With $a, b \in k$ arbitrary, and all nonzero values of b generate isomorphic algebras.

These algebras are not necessarily iterated Ore extension (see Example 2.3.3). For this reason, The Nullstellensatz do not hold using Proposition 1.2.21.

3-dimensional skew polynomial algebras $\mathcal{A}$ are skew PBW extensions as we note previously. From Proposition 2.1.5 we have that $\mathcal{A}$ is $\mathbb{N}$-filtered. The associated graded of this algebra is $k_q[x, y, z]$ with q defined by an automorphism and is Noetherian and locally finite. Then, due to Proposition 1.2.51 or Theorem 2.2.3 the 3-dimensional skew polynomial algebras are Jacobson algebras in which the Nullstellensatz hold.

Example 2.3.3 (Dispin algebra $\mathcal{U}(osp(1, 2))$)**.** Dispin algebra $\mathcal{U}(osp(1, 2))$, defined in [Ros95, Definition C4.1], is the enveloping algebra of the Lie superalgebra $osp(1, 2)$. It is generated by the indeterminates x, y, z over the commutative ring K (in this case a field) satisfying the relations $yz - zy = z, zx + xz = y, xy - yx = x$.

These algebras not necessarily are Ore extensions. Hence, we can not use Proposition 1.2.21 to conclude the Nullstellensatz.

We can note that $\mathcal{U}(osp(1, 2))$ is a skew PBW extension over K; in [Ros95, page 99] we can see a basis and with the relations satisfy Definition 2.1.2. In [LR14, page 1215] we note that $\mathcal{U}(osp(1, 2)) \cong \sigma(K)\langle x, y, z\rangle$, its associated graded is $k_q[x, y, z]$, this is Noetherian, and its locally finite (in deed is a K-algebra finitely graded). Then, due to Proposition 1.2.51 or Theorem 2.2.3 $\mathcal{U}(osp(1, 2))$ is a Jacobson algebra in which the Nullstellensatz hold.

2.3.3 Other examples

Example 2.3.4 (Multiplicative analogue of the Weyl algebra)**.** The K-algebra $\mathcal{O}_n(\lambda_{ji})$, defined in [Jat84], is generated by the indeterminates $x_1, \dots, x_n$ subject to the relations: $x_j x_i = \lambda_{ji} x_i x_j$ with $1 \leq i < j \leq n$, and $\lambda_{ji} \in K - \{0\}$.

We can note that $\mathcal{O}_n(\lambda_{ji})$ is not an Ore extension over K, but, $\mathcal{O}_n(\lambda_{ji})$ is an Ore extension over $K[x_1]$ (see [Li02, page 29]). This is a finitely iterated Ore extension $(K[x_1][x_2; \sigma_2] \cdots [x_n; \sigma_n]$ with $x_j x_i = \sigma_j(x_i) x_j = \lambda_{ji} x_i x_j)$ and we have that $K[x_1]$ is a

Jacobson ring, and preserves the algebra structure i.e. $\sigma_i(k) = k$, for all $k \in K$ and we have that $\delta_i = 0$. Then, due to Proposition 1.2.21, $\mathcal{O}_n(\lambda_{ji})$ satisfies the Nullstellensatz .

We can note for the relations that $\mathcal{O}_n(\lambda_{ji})$ the condition (iii) of the Definition 2.1.2 and the condition (iv) ($\delta = 0$). In [LR14], we can note that $\mathcal{O}_n(\lambda_{ji}) \cong A = \sigma(K)\langle x_1, \dots, x_n\rangle$. A is $\mathbb{N}$-filtered and its associated graded is $K_q[x_1, \dots, x_n]$ with q defined by the automorphism σ, its associated graded is Noetherian and locally finite. Then, by Proposition 1.2.51 or Theorem 2.2.3 $\mathcal{O}_n(\lambda_{ji})$ satisfies Nullstellensatz.

Example 2.3.5 (Additive analogue of the Weyl algebra)**.** The k-algebra $A_n(q_1, \dots, q_n)$, introduced in [Kur80], is the algebra generated by the indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$ subject to the relations: $x_jx_i = x_ix_j$ and $y_jy_i = y_iy_j$ for $1 \leq i, j \leq n$, $y_ix_j = x_jy_i$ for $i \neq j$ and $y_ix_i = q_ix_iy_i + 1$ for $1 \leq i \leq n$, where $q_i \in k \setminus \{0\}$.

$A_n(q_1, \dots, q_n)$ is not a Ore extension over k. In [Li02], the authors proved that this algebra is an Ore extension over $k[x_1, \dots, x_n]$, i.e. $k[x_1, \dots, x_n][y_1; \sigma_1, \delta_1] \cdots [y_n; \sigma_n, \delta_n]$ with $y_ix_i = \sigma_i(x_i)y_i + \delta_i(x_i) = q_ix_iy_i + 1$. Here, we have, for the Example 2.3.1 that $k[x_1, \dots, x_n]$ is a Jacobson ring and is commutative Noetherian ring. Then, due to Proposition 1.2.21 $A_n(q_1, \dots, q_n)$ satisfies the Nullstellensatz.

From the relations that describe the algebra, in [LR14] the authors shown that $A_n(q_1, \dots, q_n) \cong \sigma(K)\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$. Hence, the additive analogue of a Weyl algebra is a skew PBW extension, then we have that is $\mathbb{N}$-filtered and its associated graded is $k_q[x_1, \dots, x_n, y_1, \dots, y_n]$ with q defined by the automorphism σ_i, its associated graded is Noetherian and locally finite. Then, by Proposition 1.2.51 or Theorem 2.2.3, $A_n(q_1, \dots, q_n)$ satisfies Nullstellensatz.

Example 2.3.6 (Quantum algebra)**.** The K-algebra $\mathcal{U}'(so(3, K))$, developed in [HKP00] and [Ior02] is generated by I_1, I_2, I_3 subject to relations $I_2I_1 - qI_1I_2 = -q^{1/2}I_3$, $I_3I_1 - q^{-1}I_1I_3 = q^{-1/2}I_2$, $I_3I_2 - qI_2I_3 = -q^{1/2}I_1$, where $q \in K - \{0\}$. This algebra is not a Ore extension. Then, we can not conclude Nullstellensatz with Proposition 1.2.21.

In [AL15] and [RS17b], the authors proved that this algebra is a skew PBW extension $\mathcal{U}'(so(3, K)) \cong \sigma(K)\langle I_1, I_2, I_3\rangle$. Then we have that the algebra is $\mathbb{N}$-filtered and its associated graded is $k_q[I_1, I_2, I_3]$ with q defined by the automorphism σ_i, its associated graded is Noetherian and locally finite. Then, due to Proposition 1.2.51 or Theorem 2.2.3, $\mathcal{U}'(so(3, K))$ satisfies Nullstellensatz.

Example 2.3.7 (q-Heisenberg algebra)**.** The K-algebra $H_n(q)$ introduced in [Ber92] is generated by the set of variables $x_1, \dots, x_n$, $y_1, \dots, y_n$, $z_1, \dots, z_n$ subject to the relations:

$$\begin{array}{llll}
x_jx_i = x_ix_j, & y_jy_i = y_iy_j, & z_jz_i = z_iz_j, & 1 \leq i, j \leq n, \\
z_jy_i = y_iz_j, & z_jx_i = x_iz_j, & y_jx_i = x_iy_j & i \neq j, \\
z_iy_i = qy_iz_i, & z_ix_i = q^{-1}x_iz_i + y_i, & y_ix_i = qx_iy_i & 1 \leq i \leq n,
\end{array}$$

with $q \in K \setminus \{0\}$.

In [Li02], the authors proved that this algebra is an Ore extension over $k[x_1, \dots, x_n]$, i.e. $k[x_1, \dots, x_n][y_1; \sigma_1] \cdots [y_n; \sigma_n][z_1; \theta_1, \delta_1] \cdots [y_n; \theta_n, \delta_n]$ with $y_ix_i = \sigma_i(x_i)y_i = qx_iy_i$, $z_iy_i = \theta_i(y_i)z_i = qy_iz_i$, $z_ix_i = \theta_i(x_i)z_i + \delta_i(x_i) = q^{-1}x_iz_i + y_i$, for $1 \leq i \leq n$. So, due to Proposition 1.2.21 we can conclude that $H_n(q)$ satisfies the Nullstellensatz.

In [LR14] we can see that $H_n(q)$ is a skew PBW extension isomorphic to $\sigma(K)\langle x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n\rangle$. Then we have that is $\mathbb{N}$-filtered and its associated graded is $k_q[x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n]$ with q defined by the automorphisms σ_i and θ_i; its associated graded is Noetherian and locally finite. Then, due to Proposition 1.2.51 or Theorem 2.2.3 $H_n(q)$ satisfies Nullstellensatz.

Throughout this section we observe some examples of extensions in which the Nullstellensatz hold using different enunciated that we studied in this work. On the next table, we will see which property can be used to conclude the theorem in these examples. Using the symbol ✓ if we can conclude the theorem and $\star$ if we do not. We note for I-N Proposition 1.2.21, Z-N Proposition 1.2.51 and SPBW-N Theorem 2.2.3.

Algebras	**I-N**	**Z-N**	**SPBW-N**
Classical polynomial ring $k[x_1, \dots, x_n]$	✓	✓	✓
Universal enveloping algebra of a Lie algebra $\mathcal{G}$,$\mathcal{U}(\mathcal{G})$	$\star$	✓	✓
Multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$	✓	✓	✓
Additive analogue of the Weyl algebra $A_n(q_1, \dots, q_n)$	✓	✓	✓
Quantum algebra $\mathcal{U}'(\mathfrak{so}(3, k))$	$\star$	✓	✓
Some 3-dimensional skew polynomial algebras	$\star$	✓	✓
Dispin algebra $\mathcal{U}(osp(1, 2))$	$\star$	✓	✓
q-Heinsenberg algebra $\mathbb{H}_n(q)$	✓	✓	✓

Table 2.1: Skew PBW extension and the Hilbert's Nullstellensatz

Conclusions

Throughout the document, we saw several versions of the Hilbert's Nullstellensatz in the commutative case. We can note that some of these versions can be extended to the non-commutative case.

For skew PBW extensions, we identify several conditions that these extensions must have to guarantee that Hilbert's Nullstellensatz hold. Some of these restrictions are quite powerful and, for some examples, we can not conclude the theorem.

It becomes a difficult task to try to give a geometric version of the Nullstellensatz in the non-commutative case, especially for skew PBW extensions.

Future work

We noticed that several examples of skew PBW extensions satisfy Hilbert's Nullstellensatz for Theorem 2.2.3. Unfortunately, there exist quite examples in which we can not conclude the Nullstellensatz with Theorem 2.2.3; an example of this case is giving in [ASZ99], we find an algebra that satisfies the Nullstellensatz but is not a skew PBW extension. In other cases, to verify that a structure is a $k-$algebra is difficult and it is an essential condition to conclude the theorem. Thinking about that, we want to state another property to identify more examples that could be not skew PBW extensions and a form to guarantee when an algebra preserve the k-algebra structure. In [LG18] the authors defined finitely semi-graded algebras. These structures are more general than skew PBW extensions and therefore, it would be interesting to extend the results of this work to finitely semi-graded algebras.

We overlook the geometrical version of the theorem due to the difficulty to define a notion of variety and ideal such as the commutative case. An interesting question is stated as a purely geometrical case to the Hilbert's Nullstellensatz for non-commutative structures.

We gave an algebraic version of the Nullstellensatz and we can conclude the affine case of the geometrical version with this. Another possible work would be searching in the literature an algebraic version of the projective case and extended to non-commutative structures.

In classical algebraic geometry, we give a topology for the polynomial ring with the prime spectrum of a ring (the set of all prime ideals). This topology is known as Zariski topology. In the non-commutative cases, there are examples in which we do not have prime ideals. We can substitute the prime spectrum with the primitive spectrum in the non-commutative case and defined a topology with this (known as Jacobson topology). Thinking about that, we saw that some examples are Jacobson algebras; we would be interested in giving a topology to skew PBW extensions.

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