Partial projective representations and the partial Schur multiplier: a survey

Representaciones parciales projectivas y el multiplicador parcial de Schur

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Abstract. We present a short survey on partial projective representations, the partial Schur multiplier and related notions.

Keywords: partial actions, partial representations, partial Schur multiplier, partial factor set.

Resumen. Presentamos un pequeño resumen sobre representaciones parciales proyectivas, el multiplicador parcial de Schur y nociones relacionadas.

Palabras claves: Acciones parciales, representaciones parciales, multiplicador parcial de Schur, conjuntos factores parciales.

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Partial representations of groups were introduced in the theory of C^*-algebras by R. Exel [15, 16], and independently by J. Quigg and I. Reaburn [27] as an important ingredient of a new approach to C^*-algebras generated by partial isometries on a Hilbert space (see the survey [5]).

This concept has a strong relation to partial actions, [16, 6, 18]. Kellendok and Lawson in [22] pointed out several branches of mathematics where partial actions are relevant, for instance $\mathbb{R}$-trees [28, 20], model theory, the profinite topology of groups and their relation [21, 3], graph immersions and inverse semigroups [22], topology and group presentations [**?**, 1].

Amenability was one of the ingredients in [18], where a machinery was developed, based on the interaction between partial actions and partial representations, permitting to study representations of partial crossed products as well as their ideals. This rich interaction was used in [17] to define and investigate the Cuntz-Krieger algebras with infinite number of states, and then in [19] to study their KMS state structure.

We notice that the term *partial action* was used in the litterature in a different sense. In our sense partial actions are defined as follows.

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Definition 0.1. Let G be a group with identity element 1 and $\mathcal{X}$ a set, a partial action α of G on $\mathcal{X}$ is a family of bijections $\alpha = \{\alpha_g \colon \mathcal{X}_{g^{-1}} \to X_g, \}$, where $\mathcal{X}_g \subseteq \mathcal{X}$, for all $g \in G$, and:

- $\mathcal{X}_1 = \mathcal{X}$ and α_1 is the identity of $\mathcal{X}$.
- α_{gh} is an extension of $\alpha_g \circ \alpha_h$, for all $g, h \in G$.

Another key tool that relates partial actions and partial representations are inverse semigroups, the most important is the symmetric inverse semigroup $\mathcal{I}(\mathcal{X})$, which consists of partial bijections of $\mathcal{X}$, that is bijections between subsets of $\mathcal{X}$, where the multiplication is given by composition. For a partial bijection ψ we denote by $\mathrm{dom}(\psi)$ and $\mathrm{ran}(\psi)$ its domain and range, respectively. Given $\psi, \phi \in \mathcal{I}(\mathcal{X})$, then $\psi\phi$ is the composition of partial maps in the largest domain where it makes sense, that is

$$\mathrm{dom}(\psi\phi) = \phi^{-1}(\mathrm{ran}(\phi) \cap \mathrm{dom}(\phi)) \text{ and } \mathrm{ran}(\psi\phi) = \psi(\mathrm{ran}(\phi) \cap \mathrm{dom}(\phi)).$$

With respect to this operation $\mathcal{I}(\mathcal{X})$ becomes an inverse monoid with zero, the zero element being is the map map $\emptyset \to \emptyset$. Thus similar to classical actions a partial action can be seen as a map $\alpha : G \ni g \mapsto \alpha_g \in \mathcal{I}(\mathcal{X})$, and the next holds.

Proposition 0.2. *[16] A map $\alpha : G \ni g \mapsto \alpha_g \in \mathcal{I}(\mathcal{X})$ induces a partial action of G in $\mathcal{X}$, if and only if:*

- $\alpha(1)$*, is the identity of* $\mathcal{X}$*.*
- $\alpha(g)\alpha(h)\alpha(h^{-1}) = \alpha(gh)\alpha(h^{-1})$.

  If the items above hold, then
- $\alpha(g^{-1})\alpha(g)\alpha(h) = \alpha(g^{-1})\alpha(gh)$,

for all $g, h \in G$.

Thus the following is natural.

Definition 0.3. Let G be a group and S a monoid or an algebra over a field K. A partial representation of G is a map $\pi \colon G \to S$ such that

- $\pi(1) = 1_S$.
- $\pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1})$.
- $\pi(g^{-1})\pi(g)\pi(h) = \pi(g^{-1})\pi(gh)$,

for all $g, h \in G$.

Similarly to the case of usual representations, there is an algebra, called the partial group algebra, and denoted $K_{\mathrm{par}}G$, which controls the partial representations of a group G. The algebra $K_{\mathrm{par}}G$, is exactly the semigroup algebra $K\mathcal{S}(G)$, where $\mathcal{S}(G)$ is the Birget-Rhodes expansion of G (see [22, 2]). Taking the canonical partial representation

$$\iota : G \in g \to (\{1, g\}, g) \in K_{\mathrm{par}}G, \tag{1}$$

We have the following properties of $K_{\mathrm{par}}G$.

Proposition 0.4. *Let G be a group. Then*

- *[7] For any partial representation π of G on a K-algebra $\mathcal{A}$, there is a unique K-algebra homomorphism $\tilde{\pi}\colon K_{\mathrm{par}}G \to \mathcal{A}$ such that $\tilde{\pi}\iota = \pi$.*
- *[7, 8] If G is finite, denote by $\mathcal{C}$ a full set of representatives of the conjugacy classes of subgroups of G. Then the partial group algebra of G over K is of the form:*

$$K_{\mathrm{par}}G \cong \bigoplus_{\substack{H\in\mathcal{C} \\ 1\leq m\leq (G:H)}} c_m(H)M_m(KH), \tag{2}$$

where $c_m(H) =$

$$\tfrac{1}{m}(G\colon N_G(H))\left(\binom{(G:H)-1}{m-1} - \sum_{\substack{H<B\leq G \\ (B:H)\mid m}} \frac{m/(B:H)c_{m/(B:H)}(B)}{(G:N_G(B))}\right)$$

and $c_m(H)M_m(KH)$ means the direct sum of $c_m(H)$ copies of $M_m(KH)$.

Algebraic results on partial representations appeared also in [16], [29], [30], [22] and [6], whereas structural results on partial representations and the corresponding partial group algebras for finite groups, such as the isomorphism problem for partial group algebras where obtained in [7], [8], [13] and most recently in [12]. The case of arbitrary groups was considered in [14], where the study of partial representations was reduced to some "purely partial" representations, called *elementary*.

Recall that a projective representation of a group G can be defined as a homomorphism from G to the projective linear group $PGL_n(K)$. The concept of a partial projective representation of a group G over a field K was introduced and studied in [9], as a first step to develop a new cohomological theory based on partial actions. More specifically, let $PM_n(K)$ be the monoid of the projective $n\times n$ matrices over a field K, i.e., $PM_n(K) = M_n(K)/\lambda$, where λ is the congruence given by

$$A\lambda B \Longleftrightarrow A = cB \ \text{ for some } \ c\in K^*.$$

Then a partial projective representation of G is a map $G\to M_n(K)$ such that the map $G\to PM_n(K)$ s a partial homomorphism, where $G\to PM_n(K)$ is obtained as a composition of $G \to M_n(K)$ with the natural homomorphism $M_n(K) \to PM_n(K)$.

Taking into consideration the semigroup $\mathcal{S}(G)$, we have the next:

Proposition 0.5. *[9] A map $\Gamma\colon G \to M_n(K)$ is a partial projective representation exactly when $\Gamma = \tilde{\Gamma}\iota$, for some projective representation $\tilde{\Gamma} : \mathcal{S}(G) \to M_n(K)$, and ι is given by* (1).

Partially defined factor sets appeared naturally as one may notice in the next.

Proposition 0.6. *[9] Given a partial projective representation $\Gamma\colon G \to M_n(K)$, there is a unique partially defined function $\sigma\colon G \times G \to K^*$, such that*

$$\operatorname{dom}\sigma = \{(x, y) \mid \Gamma(x)\Gamma(y) \neq 0\}, \tag{3}$$

and

$$\Gamma(x^{-1})\Gamma(x)\Gamma(y) = \Gamma(x^{-1})\Gamma(xy)\sigma(x, y)$$

and

$$\Gamma(x)\Gamma(y)\Gamma(y^{-1}) = \Gamma(xy)\Gamma(y^{-1})\sigma(x, y),$$

for every $(x, y) \in \operatorname{dom}\sigma$.

Definition 0.7. The function σ associated with a partial projective representation Γ as above is called a factor set of Γ or a partial factor set of G.

For convenience, we set $\sigma(x, y) = 0$ when $(x, y) \notin \operatorname{dom}\sigma$ (making σ totally defined), and keep the notation $\operatorname{dom}\sigma$ for (3).

One may expect that partial factor sets satisfy the "cohomological equality". But this is true only in a restricted form.

Proposition 0.8. *[9] Let $\Gamma\colon G \to M_n(K)$, be a partial projective representation with a partial factor set σ. Then For $x, y, z \in G$ such that $\Gamma(x)\Gamma(y)\Gamma(z) \neq 0$ we have $\sigma(x, y)\sigma(xy, z) = \sigma(x, yz)\sigma(y, z)$.*

However, partial factor sets of G form a commutative inverse semigroup with respect to the pointwise multiplication

$$\sigma\tau(x, y) = \sigma(x, y)\tau(x, y), \text{ for all } x, y \in G,$$

and this semigroup is denoted by $pm(G)$. Equivalence between partial factor sets is defined as in the classical case, that is, $\sigma, \tau \in pm(G)$ are cohomologous or equivalent, if and only if, there exists $\rho\colon G \to K^*$, such that

$$\sigma(x, y) = \tau(x, y)\rho(x)\rho(y)\rho(xy)^{-1},$$

for all $x, y \in G$. This leaded to the definition of the corresponding *partial Schur multiplier* $pM(G)$. It is a generalization of the classical Schur multiplier

$M(G) = H^2(G, \mathcal{C}^*)$, and as in the classical case, a key problem in the theory of partial projective representations of G is the study of the structure of $pM(G)$.

Unlike the usual Schur multiplier $M(G)$, the partial Schur multiplier $pM(G)$ is not a group, but it is a semilattice of abelian groups called *components*. To describe this components one needs the abstract semigroup $\mathcal{T}$ generated by the symbols g, h, t with relations

$$g^2 = h^2 = 1,\ (gh)^3 = 1,\ t^2 = t,\ gt = t,\ tght = thgh,\ tht = 0.$$

Then there is an action of $\mathcal{T}$ on $G \times G$ given by the transformations

$$g\colon (x,y) \mapsto (xy, y^{-1}). \tag{4}$$
$$h\colon (x,y) \mapsto (y^{-1}, x^{-1}). \tag{5}$$
$$t\colon (x,y) \mapsto (x, 1). \tag{6}$$

Using this transformations one gets the following:

Theorem 0.9. *[9] The semigroups $pm(G)$ and $pM(G)$ are semilattices of (abelian) groups*

$$pm(G) = \bigcup_{X \in C(G)} pm_X(G),\ \ pM(G) = \bigcup_{X \in C(G)} pM_X(G),$$

where $C(G)$ is the semilattice of the $\mathcal{T}$-subsets from $G \times G$ with respect to set-theoretic intersection, $pm_X(G)$ is the group of partial factor sets of G with domain X and $pM_X(G)$ consists of cohomology classes in $pm_X(G)$.

Working over algebraically closed fields, the classification of the components was started in [10, 11]. Indeed, as a first step we have an explicit description of partial factor sets.

Theorem 0.10. *[11] Let K be an algebraically closed field and τ be a partial factor set of G with domain X. Then there is a partial factor set σ, cohomologous to τ, such that for all $(a,b) \in X$*

$$\sigma(a,b)\sigma(b^{-1}, a^{-1}) = 1_K, \tag{7}$$

$$\sigma(a,b) = \sigma(b^{-1}a^{-1}, a) = \sigma(b, b^{-1}a^{-1}), \tag{8}$$

$$\sigma(a,1) = 1_K. \tag{9}$$

Conversely, if $\sigma\colon G \times G \to K$ is a partially defined map with $\operatorname{dom}\sigma \in C(G)$ such that (7), (8) and (9) are satisfied for any $(a,b) \in \operatorname{dom}\sigma$, then σ is a partial factor set of G.

For every $X \in C(G)$, the subgroup of $\tilde{pm}_X(G)$ formed by all the maps $\sigma\colon G \times G \to K$ satisfying (7) - (9) is denoted by $\tilde{pm}_X(G)$, then any element in $\tilde{pm}_X(G)$ is determined by its values in the set $\tilde{X} = X \cap \{(x,y) \in G \times G \mid x, y, xy \neq 1\}$. Hence equations (4) - (5) induce an action of the symmetric group S_3 on $\tilde{X}$, the orbits of this action are called *effective* orbits of X. Thus one comes to:

Corollary 0.11. *[11] Let $X \in C(G)$. Then:*

- *Every partial factor set of $pm_X(G)$ is equivalent to some element of $p\tilde{m}_X(G)$.*
- *The kernel $N_X = \{\sigma \in p\tilde{m}_X(G) \mid \sigma \sim 1\}$ of the natural epimorphism $p\tilde{m}_X(G) \to pM_X(G)$ consists of those $\sigma \colon G \times G \to K$ for which there is $\rho : G \times G \to K^*$ satisfying the following conditions:*

$$\rho(1) = 1_K, \quad \rho(a)\rho(a^{-1}) = 1, \text{ for every } a \in G \quad \text{with} \quad (a,1) \in X \quad \text{and}$$

$$\sigma(a,b) = \begin{cases} \rho(a)\rho(b)\rho(ab)^{-1}, & \text{if } (a,b) \in X, \\ 0, & \text{if } (a,b) \notin X. \end{cases}$$

- *Let $s = s(G,X)$ be the cardinality of the set of effective S_3-orbits of X and $\{(a_i,b_i)\}_{1 \le i \le s}$ a full set of representatives of these orbits. Then the map*

$$\phi : (K^*)^s \ni x \mapsto \sigma_x \in p\tilde{m}_X(G),$$

in which $x = (x_i)_{1 \le i \le s}$ and $\sigma_x(a_i,b_i) = x_i$, is an isomorphism of multiplicative groups.

- *For every domain $Y \in C(G)$ such that $Y \supseteq X$, there is a group epimorphism $\psi_X^Y \colon pM_Y(G) \to pM_X(G)$. In particular, $pM_X(G)$ is an epimorphic image of the total component $pM_{G\times G}(G)$.*

In view of last item above, particular attention was payed to $pM_{G\times G}(G)$. This component contains $M(G)$, but in general does not coincide with it, in particular, it is known that $M(G)$ is always a finite group, (see [**?**]) whereas $pM_{G\times G}(G)$ can be infinite, even for groups of small order. For instance we have.

Proposition 0.12. *Let K be the base field of partial projective representations of G, if K is algebraically closed, we have:*

*(i) [11, **?**] If $G = C_n$ is the cyclic group of order n, then*

- *For $1 \le n \le 3$, we have $pM_{C_n\times C_n}(C_n) \cong C_1$.*
- *If $n \ge 4$, then $pM_{C_n\times C_n}(C_n) \cong (K^*)^{T_n}$, where $T_n = \begin{cases} \frac{(n-1)(n-2)}{6} & , \text{ if } 3 \nmid n, \\ \frac{(n-1)(n-2)+4}{6} & , \text{ if } 3 \mid n. \end{cases}$*

(ii) [23] If G is the elementary abelian 2-group of order 2^n, then: $pM_X(G) \cong (K^*)^s$, where s is the number of effective orbits in X.

(iii) [4] When G is the dihedral group $D_{2m} = \langle a,b \mid a^m = b^2 = (ab)^2 = 1\rangle, m \in \mathbb{N}$, then

$$pM_{D_{2m}\times D_{2m}}(D_{2m}) \simeq (K^*)^{d_m - \lfloor \frac{m-1}{2} \rfloor},$$

for some $d_m \in \mathbb{N}$.

(iv) [26] When $G = S_3$ *we have*

$$pM_{S_3 \times S_3}(S_3) \simeq (K^*)^3.$$

(v) [4] In the case of the infinite dihedral group $D_\infty = \langle a, b \mid b^2 = (ab)^2 = 1 \rangle$ *we have*

$$pM_{D_\infty \times D_\infty}(D_\infty) \simeq (K^*)^{(\mathbb{N}\times\mathbb{N})\times(\mathbb{N}\times\mathbb{Z})}.$$

(vi) [4] For Dicyclic groups $G = \mathrm{Dic}_m = \langle a, b \mid a^{2m} = 1,\ b^2 = a^m,\ b^{-1}ab = a^{-1} \rangle$*, one gets*

$$pM_{G\times G}(G) \simeq (K^*)^{dc_m - 2m + 1},$$

where $dc_m = \begin{cases} \frac{(4m-1)(4m-2)+4}{6}, & \text{if } m \equiv 0 \pmod 3, \\ \frac{(4m-1)(4m-2)}{6}, & \text{if } m \not\equiv 0 \pmod 3. \end{cases}$

(vii) [4] Finally, for the integers, we get $pM_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{Z}) \simeq (K^*)^{\mathbb{N}}$.

Notice that if G is a finite cyclic group of order ≤ 5, then all the components of $pM(G)$ are trivial, and in this case we say that $pM(G)$ is *trivial* or that G has *trivial Schur multiplier.* We have the following.

Theorem 0.13. *[24] Let G be a finite group. Then*

- *The partial Schur multiplier of G is trivial if and only if G is a cyclic group of order ≤ 5.*
- *There is an isomorphism $pM_{G\times G}(G) \cong K^*$ if and only if $G \cong C_2 \times C_2$.*

In order to know in which way $M(G)$ is contained in $pM_{G\times G}(G)$, it is helpful to state some conditions for a $\sigma \in p\tilde{m}_{G\times G}(G)$ being a classical factor set. For this we have.

Proposition 0.14. *[24] A partial factor set σ, satisfies $\sigma \in p\tilde{m}_{G\times G}(G) \cap Z^2(G, K^*)$, if and only if, σ verifies the 2-cocycle identity and $\sigma(x, x^{-1}) = 1$, for all $x \in G$.*

It is also important to remark that domains of partial factor sets can be formed by taking unions of domains of partial factor sets corresponding to elementary partial representations [10]. To state this concept we recall that for a finite group G one has a K-algebra homomorphism (see Proposition 0.4 above) $\psi : K_{par}G \to \oplus M_l(KH)$. Let $Pr = Pr_l$ be the projection of $\oplus M_l(KH)$ onto the matrix algebra $M_l(KH)$. Consider also the map $\iota\colon G \ni g \mapsto (\{1, g\}, g) \in K_{\mathrm{par}}G$. A function of the form

$$\Gamma = Pr \circ \psi \circ \iota : G \to M_l(KH)$$

is called an *elementary partial representation* of G and we shall say that the set $D = \{(x, y) \in G \times G \,|\, \Gamma(x)\,\Gamma(y) \neq 0\}$ is an *elementary domain.* A natural question is to wonder which finite groups contain only elementary domains. For this we have the following:

Proposition 0.15. *[25] The finite groups containing only elementary domains are C_1, C_2 and C_3.*

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