Revista Colombiana de Matemáticas
Volumen V (1971) Págs. 19 - 23

A proof of the inverse function theorem

by

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The purpose of this note is to give a proof, hopefully new, of the inverse function theorem as an application of the local existence theorem for differential equations. If Ω is an open subset of a Banach space E and $T:\Omega \to E$ is of class C^1, then T is locally bounded and satisfies a Lipschitz condition at every point.

Theorem. Let Ω be an open subset of the Banach space E and let $T:\Omega \to E$ be a function of class C^1. Assume that $u_o = T'(x_o)$ is an isomorphism from E onto E, and let $y_o = T(x_o)$. Then, there exist neighborhoods $V(x_o)$ and $W(y_o)$ such that the equation $T(x) = y$ has one and only one solution $x \in V$ for every $y \in W$.

Proof. Let $L(E)$ be the space of all continuous linear maps from E into E and let $Is(E)$ be the set of all continuos isomorphisms from E onto E.

(a) There is a ball $B_1(x_o)$ such that $T'(x) \in Is(E)$ for all $x \in B_1$ and a constant M such that $\|(T'(x))^{-1}\| < M$ for all $x \in B_1$. This follows from the fact that $Is(E)$ is an open subset of $L(E)$ and from the continuity of the map $u \to u^{-1}$, $u \in Is(E)$.

(b) There is a ball $B_2(x_o)$ contained in B_1 such that $T(x) = T(x')$ implies $x = x'$ if $x, x' \in B_2$. Given $\in > 0$ there is a ball $B(x_o)$ such that

(1) $$\| T(x) - T(x') - u_o(x - x') \| \leqq \in \| x - x' \|$$

for all $x, x' \in B$. To show (1) we define

(2) $$G(t) = T(tx + (1 - t)\, x') - tu_o(x - x')$$

Then,

(3) $$G(1) - G(0) = \int_0^1 (T'(tx + (1 - t)\, x') - u_o) \cdot (x - x')\, dt$$

Now, (1) follows from (3) and the continuity of T' at u_o. We derive the existence of the ball B_2 from (1). Assume that for every ball $B(x_o) \in \Omega$ there are points $x, x' \in B$ such that $x \neq x'$ and $T(x) = T(x')$. Let $\in_n \to 0$. Then, from (1), we can define sequences $\{x_n\}$ and $\{x'_n\}$ converging to x_o, and such that $x_n \neq x'_n$, $T(x_n) = T(x'_n)$ and

$$\| u_o \cdot (x_n - x'_n \| \leqq \in_n \| x_n - x'_n \|$$

Let

$$w_n = (x_n - x'_n)(\| x_n - x'_n \|)^{-1}$$

Then $\| w_n \| = 1$, $u_o \cdot w_n \to 0$. Since $u_o \in Is(E)$, it follows that $w_n \to 0$. This is a contradiction.

(c) There is a ball $V = B_3(x_o) \in B_2$ such that $\| T(x) - y_o \| \geq \beta > 0$ for some constant β and for all x in the boundary of B_3. Let $a = || u_o^{-1} ||$. Then, there is a ball $B_3(x_o) \subset B_2(x_o)$ of radius r such that

(4) $$T(x) - T(x_o) = u_o \cdot (x - x_o) + 0(x - x_o)$$

and

$$|| 0(x - x_o) || \leq \frac{1}{2a} || x - x_o ||$$

for all $x \in B_3$. Thus, from (4) we obtain,

$$x - x_o = u_o^{-1} (T(x) - T(x_o) - 0(x - x_o))$$

$$|| x - x_o || < a (|| T(x) - T(x_o) || + || 0(x - x_o) ||)$$

Therefore,

$$|| T(x) - T(x_o) || > \frac{r}{2a} = \beta$$

for all x such that $|| x - x_o || = r$.

(d) Let $W = \{ y \mid || y - y_o || < \frac{\beta}{2} \}$. For every $y \in W$ we consider the differential equation

(5) $$\dot{\varphi} = F(\varphi), \quad \varphi(0) = x_o$$

where,

(6) $$F(x) = -(S'(x))^{-1} S(x), \quad S(x) = T(x) - y$$

Let $[0, a]$ be the maximun interval of existence of the solution of (5). Then,

(7) $$S'(\Phi(t)) \cdot \dot{\Phi} = -S(\Phi(t))$$

Therefore,

(8) $$S\Phi(t) = e^{-t} S(x_o) \quad , \quad S(x_o) = T(x_o) - y$$

Let us show that $\Phi(t)$ remains in the ball V defined in (c). Assume that $\| \Phi(t) - x_o \| = r$ for some $t' > 0$. Then $(\Phi(t') = z)$,

$$\| S(z) \| = e^{-t'} \| y - y_o \| = \| T(z) - y \| \geq \| y - y_o \|$$

This would imply that $e^{-t'} \geq 1$, $t' > 0$. This is a contradiction. Therefore $\Phi(t)$ remains in V. Next we show that $\Phi(t)$ exists for all $t > 0$. From

(9) $$\Phi(t) - \Phi(s) = \int_t^s \Phi'(z)\, dz = \int_t^{s'} F(\Phi(z))\, dz$$

and (8) we obtain

(10) $$\| \Phi(t) - \Phi(s) \| \leq M \| y - y_o \| \cdot | e^{-t} - e^{-s} |$$

$\to 0$ when $t, s \to a$. (M is the constant defined in (a)). Therefore $\Phi(t)$ can be extended to a continuous function in the closed interval $[0, a]$. Thus, $\Phi(t)$ exists for all $t \geq 0$ by the local existence theorem for differential equations. Let x^* be the limit of $\Phi(t)$ when

t tends to infinity (this limit exists by (10)) . Then, from (8) , we see that $S(x^*) = 0 = T(x^*) - y$. This concludes the proof .

BIBLIOGRAPHY

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