Revista Colombiana de Estadística
July 2015, Volume 38, Issue 2, pp. 385 to 397
DOI: http://dx.doi.org/10.15446/rce.v38n2.51667

A Note on Generalized Exponential Type Estimator for Population Variance in Survey Sampling

Una nota sobre el estimador exponencial generalizado para la varianza poblacional en muestreo de encuestas

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Abstract

Recently a new generalized estimator for population variance using information on the auxiliary variable has been introduced by Asghar, Sanaullah & Hanif (2014). In that paper there was some inaccuracy in the bias and MSE expressions. In this paper, we provide the correct expressions for bias and MSE of the Asghar et al. (2014) estimator, up to the first order of approximation. We also propose a new generalized exponential type estimator for population variance which performs better than the existing estimators. Four data sets are used for numerical comparison of efficiencies.

***Key words*:** Auxiliary Variable, Bias, Efficiency, Mean Square Error, Variance.

Resumen

Recientemente, un nuevo estimador generalizado de varianza de la población utilizando información sobre la variable auxiliar ha sido introducida por Asghar et al. (2014). En ese documento había alguna inexactitud en las expresiones de sesgo y ECM. En este trabajo, proporcionamos las expresiones correctas de sesgo y ECM de Asghar et al. (2014) hasta el primer orden de aproximación. También proponemos un nuevo estimador tipo exponencial generalizado de la varianza de la población que se comporta mejor que los estimadores existentes. Cuatro conjuntos de datos se utilizan para la comparación numérica de la eficiencia.

***Palabras clave*:** error cuadrático medio, sesgo, variable auxiliar, varianza, eficiencia.

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1. Introduction

In applications, the auxiliary information is frequently used to increase precision of the estimators by taking advantage of the correlation between the study variable and the auxiliary variable. Singh, Upadhyaya & Namjoshi (1988) introduced several estimators for population variance using the known population mean and population variance of the auxiliary variable. Isaki (1983) introduced the traditional ratio and regression estimator for population variance. Some related work in this direction is also due to Jhajj, Sharma & Grover (2005), Kadilar & Cingi (2006), Gupta & Shabbir (2008), Bansal, Javed & Khanna (2011), Singh, Chauhan, Swan & Smarandache (2011), Upadhyaya, Singh, Chatterjee & Yadav (2011), Subramani & Kumarapandiyan (2012), Nayak & Sahoo (2012) and Yadav & Kadilar (2013, 2014).

The following notations will be needed to discuss various estimators. Consider a finite population $U = \{1, 2 \ldots, i, \ldots N\}$ of N identifiable units. Let y_i and $x_i (i = 1, 2, \ldots, N)$ be the values on the ith population unit for the study variable y and the auxiliary variable x respectively. Let $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ and $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$ be the population means and $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \overline{Y})^2$ and $S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{X})^2$ be the population variances for y and x respectively. The focus here is on estimating the finite population variance based on a sample of size n from the underlying population by using simple random sampling without replacement (SRSWOR) scheme. Let $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ be the sample means and $s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2$ and $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$ be the corresponding sample variances for y and x respectively.

To obtain the biases and *MSE*s expressions for various estimators, we define the following error terms:
Let $\phi_0 = \frac{s_y^2}{S_y^2} - 1$, $\phi_1 = \frac{\overline{x}}{\overline{X}} - 1$, $\phi_2 = \frac{s_x^2}{S_x^2} - 1$, such that $E(\phi_i) = 0$ for $i = 0,\ 1,\ 2.$ $E(\phi_0^2) = \gamma\delta_{40}^*$, $E(\phi_1^2) = \gamma C_x^2$, $E(\phi_2^2) = \gamma\delta_{04}^*$, $E(\phi_0\phi_1) = \gamma\delta_{21}C_x$, $E(\phi_0\phi_2) = \gamma\delta_{22}^*$, $E(\phi_1\phi_2) = \gamma\delta_{03}C_x$, where $\delta_{40}^* = (\delta_{40} - 1)$, $\delta_{04}^* = (\delta_{04} - 1)$, $\delta_{22}^* = (\delta_{22} - 1)$, and $\gamma = \frac{1}{n}$ as finite population correction factor term is ignored. Also $\delta_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{s/2}}$, where $\mu_{rs} = \frac{1}{N-1}\sum\limits_{i=1}^{N}(y_i - \overline{Y})^r(x_i - \overline{X})^s$.

The variance of the usual variance estimator $\widehat{S}_0^2 = s_y^2$, is given by

$$Var(\widehat{S}_0^2) = \gamma S_y^4 \delta_{40}^*. \tag{1}$$

The traditional ratio estimator $(\widehat{S}_{R1}^2)$ by Isaki (1983), when S_x^2 is known, is given by

$$\widehat{S}_{R1}^2 = s_y^2\left(\frac{S_x^2}{s_x^2}\right) \tag{2}$$

The bias and *MSE* respectively of $\widehat{S}^2_{R1}$, to first order of approximation, are given by

$$Bias(\widehat{S}^2_{R1}) \cong S^2_y \gamma [\delta^*_{04} - \delta^*_{22}] \tag{3}$$

and

$$MSE(\widehat{S}^2_{R1}) \cong S^4_y \gamma [\delta^*_{40} + \delta^*_{04} - 2\delta^*_{22}]. \tag{4}$$

Another form of ratio estimator ($\widehat{S}^2_{R2}$) when $\overline{X}$ is known, is given by

$$\widehat{S}^2_{R2} = s^2_y \left(\frac{\overline{X}}{\overline{x}} \right). \tag{5}$$

The bias and *MSE* respectively of $\widehat{S}^2_{R2}$, to first order of approximation, are given by

$$Bias(\widehat{S}^2_{R2}) \cong S^2_y \gamma \left[C^2_x - \delta_{21} C_x \right] \tag{6}$$

and

$$MSE(\widehat{S}^2_{R2}) \cong S^4_y \gamma \left[\delta^*_{40} + C^2_x - 2\delta_{21} C_x \right]. \tag{7}$$

Yadav & Kadilar (2013) suggested the following exponential type estimator for population variance:

$$\widehat{S}^2_{YK} = S^2_y \exp \left(\frac{S^2_x - s^2_x}{S^2_x + (a-1)s^2_x} \right), \tag{8}$$

where a is a constant.

The bias and minimum *MSE* of $\widehat{S}^2_{YK}$, to first order of approximation, at optimum value of a i.e. $a_{opt} = \frac{\delta^*_{04}}{\delta^*_{22}}$, is given by

$$Bias(\widehat{S}^2_{YK}) \cong S^2_y \gamma \left[\left(\frac{1}{2a^2} - \frac{1}{a^2} + \frac{1}{a} \right) \delta^*_{04} - \frac{1}{a} \delta^*_{22} \right] \tag{9}$$

and

$$MSE(\widehat{S}^2_{YK})_{\min} \cong S^4_y \gamma \left[\delta^*_{40} - \frac{\delta^{*2}_{22}}{\delta^*_{40}} \right] \tag{10}$$

The usual regression estimator $\widehat{S}^2_{Reg1}$ by Isaki (1983), is given by

$$\widehat{S}^2_{Reg1} = s^2_y + b_1 \left(S^2_x - s^2_x \right), \tag{11}$$

where b_1 is the sample regression coefficient. Corresponding population regression coefficient is given by $\beta_1 = \frac{S^2_y \delta^*_{22}}{S^2_x \delta^*_{04}}$.

The MSE of $\widehat{S}^2_{Reg1}$, is given by

$$MSE(\widehat{S}^2_{Reg1}) = S_y^4\gamma\left[\delta^*_{40} - \frac{\delta^{*2}_{22}}{\delta^*_{04}}\right]. \tag{12}$$

Another regression estimator $\widehat{S}^2_{Reg2}$, is given by

$$\widehat{S}^2_{Reg2} = s_y^2 + b_2\left(\overline{X} - \overline{x}\right), \tag{13}$$

where b_2 is the sample regression coefficient. The corresponding population regression coefficient is given by $\beta_2 = \frac{S_y^2\delta_{21}}{\overline{X}C_x}$.

The MSE of $\widehat{S}^2_{Reg2}$, is given by

$$MSE(\widehat{S}^2_{Reg2}) = S_y^4\gamma\left[\delta^*_{40} - \delta^2_{21}\right]. \tag{14}$$

2. Asghar Estimator

Recently Asghar et al. (2014) introduced the following generalized exponential type estimator for population variance. Other work for exponential type estimators for finite population means can be found at Shabbir, Haq & Gupta (2014)

$$\widehat{S}^2_A = \lambda s_y^2 \exp\left\{\frac{c\left(\overline{X} - \overline{x}\right)}{\overline{X} + (d-1)\,\overline{x}}\right\}, \tag{15}$$

where $0 < \lambda \leq 1$, $-\infty < c < \infty$ and $d > 0$.

Asghar et al. (2014) reported the bias and minimum MSE of $\widehat{S}^2_A$, to first order of approximation at optimum values of $\omega(=\frac{c}{d})_{opt} = \delta_{21}\left(C_x\right)^{-1}$ and $\lambda_{opt} = \left(\delta_{40} - \delta^2_{21}\right)^{-1}$, given by

$$Bias(\widehat{S}^2_A) \cong S_y^2\gamma\left[\lambda\left\{1 + \frac{1}{2}\omega^2C_x^2 - \omega\delta_{21}C_x\right\}\right] - S_y^2 \tag{16}$$

and

$$MSE(\widehat{S}^2_A)_{\min} \cong S_y^4\gamma\left[1 - \frac{1}{\delta_{40} - \delta^2_{21}}\right]. \tag{17}$$

However, the bias and minimum MSE of $\widehat{S}^2_A$ expressions reported in (16) and (17) require some revisions and so does their study based on efficiency and numerical comparisons.

The revised bias and MSE of $\widehat{S}^{*2}_A$ (say) instead of $\widehat{S}^2_A$, to first degree of approximation are derived below.

Rewriting (15) in error terms, we have

$$\widehat{S}^{*2}_A \cong \lambda S_y^2\left(1 + \phi_0\right)\exp\left[-\frac{c}{d}\phi_1\left\{1 + \phi_1 - \frac{1}{c}\phi_1\right\}^{-1}\right] \text{ or}$$

$$\widehat{S}^{*2}_A - S_y^2 \cong (\lambda - 1)\,S_y^2 + \lambda S_y^2\left[\phi_0 - \frac{c}{d}\phi_1 + \left(\frac{c^2}{2d^2} + \frac{c}{d} - \frac{c}{d^2}\right)\phi_1^2 - \frac{c}{d}\phi_0\phi_1\right]. \tag{18}$$

From (18), the bias of $\widehat{S}_A^{*2}$, to first degree of approximation, is given by

$$Bias(\widehat{S}_A^{*2}) \cong S_y^2 \left[(\lambda-1)+\lambda\gamma\left\{\left(\frac{c^2}{2d^2}+\frac{c}{d}-\frac{c}{d^2}\right)C_x^2-\frac{c}{d}\delta_{21}C_x\right\}\right]. \quad (19)$$

Also from (18), the *MSE* of $\widehat{S}_A^{*2}$, to first degree of approximation, is given by

$$MSE(\widehat{S}_A^{*2}) \cong S_y^4 E\left[(\lambda-1)+\lambda\left\{\phi_0-\frac{c}{d}\phi_1+\left(\frac{c^2}{2d^2}+\frac{c}{d}-\frac{c}{d^2}\right)\phi_1^2-\frac{c}{d}\phi_0\phi_1\right\}\right]^2 \text{ or}$$

$$\begin{aligned} MSE(\widehat{S}_A^{*2}) \cong S_y^4 \Bigg[1 &+ \lambda^2\left\{1+\gamma\delta_{40}^*+2\left(\frac{c^2}{d^2}+\frac{c}{d}-\frac{c}{d^2}\right)\gamma C_x^2-4\frac{c}{d}\gamma\delta_{21}C_x\right\} \\ &- 2\lambda\left\{1+\left(\frac{c^2}{2d^2}+\frac{c}{d}-\frac{c}{d^2}\right)\gamma C_x^2-\frac{c}{d}\gamma\delta 21 C_x\right\}\Bigg]. \end{aligned} \quad (20)$$

The minimum *MSE* of $\widehat{S}_A^{*2}$ at optimum value of λ i.e.

$$\lambda_{opt}=\frac{1+\left(\frac{c^2}{2d^2}+\frac{c}{d}-\frac{c}{d^2}\right)\gamma C_x^2-\frac{c}{d}\gamma\delta_{21}C_x}{1+\gamma\delta_{40}^*+2\left(\frac{c^2}{d^2}+\frac{c}{d}-\frac{c}{d^2}\right)\gamma C_x^2-4\frac{c}{d}\gamma\delta_{21}C_x} \text{ is given by}$$

$$MSE(\widehat{S}_A^{*2})_{\min} \cong S_y^4\left[1-\frac{\left\{1+\left(\frac{c^2}{2d^2}+\frac{c}{d}-\frac{c}{d^2}\right)\gamma C_x^2-\frac{c}{d}\gamma\delta_{21}C_x\right\}^2}{\left\{1+\gamma\delta_{40}^*+2\left(\frac{c^2}{d^2}+\frac{c}{d}-\frac{c}{d^2}\right)\gamma C_x^2-4\frac{c}{d}\gamma\delta_{21}C_x\right\}}\right]. \quad (21)$$

Note. In expression (20), the principal constant is λ. The roles of other constants c and d are to provide a wider family of estimators. Clearly one can obtain many estimators by choosing the different values of c and d.

If $c=1$ and $d=2$ in (15), (19) and (21), we get the estimator $\widehat{S}_{A1}^2$ (say), with the bias and minimum *MSE*, given by

$$\widehat{S}_{A1}^2=\lambda s_y^2\exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right) \quad (22)$$

$$Bias(\widehat{S}_{A1}^2)=S_y^2\left[(\lambda-1)+\lambda\gamma\left\{\left(\frac{3}{8}C_x^2-\frac{1}{2}\delta_{21}C_x\right)\right\}\right] \quad (23)$$

$$MSE(\widehat{S}_{A1}^2)_{\min}\cong S_y^4\left[1-\frac{\left\{1+\gamma\left(\frac{3}{8}C_x^2-\frac{1}{2}\delta_{21}C_x\right)\right\}^2}{1+\gamma\left(\delta_{40}^*+C_x^2-2\delta_{21}C_x\right)}\right]. \quad (24)$$

Alternatively one can think of a different exponential type estimators as described in the following section.

3. Proposed Estimator

Motivated by Yadav & Kadilar (2014) and Asghar et al. (2014), we propose the following general class of exponential type estimators:

$$\widehat{S}_P^2 = \lambda s_y^2 \exp\left(\frac{c\left(\overline{X} - \overline{x}\right)}{\overline{X} + (d-1)\,\overline{x}}\right) \exp\left(\frac{e\left(S_x^2 - s_x^2\right)}{S_x^2 + (f-1)\,s_x^2}\right), \tag{25}$$

where $-\infty < c < \infty$, $-\infty < e < \infty$, $d > 0$ and $f > 0$. We can generate many more estimators from (25), for $\lambda = 1$, as described below:

(i) For $c = e = 0$ in (25), we get the usual variance estimator $\widehat{S}_0^2 = s_y^2$.

(ii) For $c = 1,\ d = 2$ and $e = 0$ in (25), we get $\widehat{S}_1^2 = s_y^2 \exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)$.

(iii) For $c = d = 1$ and $e = 0$ in (25), we get $\widehat{S}_2^2 = s_y^2 \exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}}\right)$.

(iv) For $c = 0,\ f = 2$ and $e = 1$ in (25), we get $\widehat{S}_3^2 = s_y^2 \exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)$.

(v) For $c = 0$ and $e = f = 1$ in (25), we get $\widehat{S}_4^2 = s_y^2 \exp\left(\frac{S_x^2-s_x^2}{S_x^2}\right)$.

(vi) For $c = 1$ and $e = 0$ in (25), we get $\widehat{S}_5^2 = s_y^2 \exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}+(d-1)\overline{x}}\right)$.

(vii) For $c = 0$ and $e = 1$ in (25), we get $\widehat{S}_6^2 = s_y^2 \exp\left(\frac{S_x^2-s_x^2}{S_x^2+(f-1)s_x^2}\right)$.

(viii) For $d = 2$ and $e = 0$ in (25), we get $\widehat{S}_7^2 = s_y^2 \exp\left(\frac{c\left(\overline{X}-\overline{x}\right)}{\overline{X}+\overline{x}}\right)$.

(ix) For $c = 0$ and $f = 1$ in (25), we get $\widehat{S}_8^2 = s_y^2 \exp\left(\frac{e\left(S_x^2-s_x^2\right)}{S_x^2+s_x^2}\right)$.

(x) For $c = e = 1$ and $d = f = 2$ in (25), we get $\widehat{S}_9^2 = s_y^2 \exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)$ $\exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)$.

(xi) For $c = d = e = f = 1$ in (25), we get $\widehat{S}_{10}^2 = s_y^2 \exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}}\right) \exp\left(\frac{S_x^2-s_x^2}{S_x^2}\right)$.

The bias and *MSE* expressions for the estimators $\widehat{S}_i^2$, $(i = 1, 2, \ldots, 10)$ to first order of approximation, are given by

$$Bias(\widehat{S}_1^2) \cong S_y^2\gamma\left[\frac{3}{8}C_x^2 - \frac{1}{2}\delta_{21}C_x\right] \tag{26}$$

$$Bias(\widehat{S}_2^2) \cong S_y^2\gamma\left[\frac{1}{2}C_x^2 - \delta_{21}C_x\right] \tag{27}$$

$$Bias(\widehat{S}_3^2) \cong S_y^2\gamma\left[\frac{3}{8}\delta_{04}^* - \frac{1}{2}\delta_{22}^*\right] \tag{28}$$

$$Bias(\widehat{S}_4^2) \cong S_y^2\gamma\left[\delta_{04}^* - \delta_{22}^*\right] \tag{29}$$

$$Bias(\widehat{S}_5^2) \cong S_y^2\gamma\left[\left(\frac{1}{2d^2} - \frac{1}{d^2} + \frac{1}{d}\right)C_x^2 - \frac{1}{d}\delta_{21}C_x\right] \tag{30}$$

$$Bias(\widehat{S}_6^2) \cong S_y^2\gamma\left[\left(\frac{1}{2f^2} - \frac{1}{f^2} + \frac{1}{f}\right)\delta_{04}^* - \frac{1}{f}\delta_{22}^*\right] \tag{31}$$

$$Bias(\widehat{S}_7^2) \cong S_y^2\gamma\left[\left(\frac{1}{8}c^2 + \frac{1}{4}c\right)\delta_{04}^* - \frac{1}{2}c\delta_{22}^*\right] \tag{32}$$

$$Bias(\widehat{S}_8^2) \cong S_y^2\gamma\left[\left(\frac{1}{8}e^2 + \frac{1}{4}e\right)C_x^2 - \frac{1}{2}e\delta_{21}C_x\right] \tag{33}$$

$$Bias(\widehat{S}_9^2) \cong S_y^2\gamma\left[\frac{3}{8}\left(C_x^2 + \delta_{04}^*\right) - \frac{1}{2}\left(\delta_{21}C_x + \delta_{22}^*\right) + \frac{1}{4}\delta_{03}C_x\right] \tag{34}$$

$$Bias(\widehat{S}_{10}^2) \cong S_y^2\gamma\left[\left(C_x^2 + \delta_{04}^*\right) - \left(\delta_{21}C_x + \delta_{22}^*\right) + \delta_{03}C_x\right] \tag{35}$$

$$MSE(\widehat{S}_1^2) \cong S_y^4\gamma\left[\delta_{40}^* + \frac{1}{4}C_x^2 - \delta_{21}C_x\right] \tag{36}$$

$$MSE(\widehat{S}_2^2) \cong S_y^4\gamma\left[\delta_{40}^* + C_x^2 - 2\delta_{21}C_x\right] \tag{37}$$

$$MSE(\widehat{S}_3^2) \cong S_y^4\gamma\left[\delta_{40}^* + \frac{1}{4}\delta_{04}^* - \delta_{22}^*\right] \tag{38}$$

$$MSE(\widehat{S}_4^2) \cong S_y^4\gamma\left[\delta_{40}^* + \delta_{04}^* - 2\delta_{22}^*\right] \tag{39}$$

$$MSE(\widehat{S}_5^2)_{\min} \cong S_y^4\gamma\left[\delta_{40}^* - \delta_{21}^2\right] \quad \text{for} \quad d_{opt} = \frac{C_x}{\delta_{21}} \tag{40}$$

$$MSE(\widehat{S}_6^2)_{\min} \cong S_y^4\gamma\left[\delta_{40}^* - \frac{\delta_{22}^{*2}}{\delta_{04}^*}\right] \quad \text{for} \quad f_{opt} = \frac{\delta_{04}^*}{\delta_{22}^*} \tag{41}$$

$$MSE(\widehat{S}_7^2)_{\min} \cong S_y^4\gamma\left[\delta_{40}^* - \delta_{21}^2\right] \quad \text{for} \quad c_{opt} = \frac{2\delta_{21}}{C_x} \tag{42}$$

$$MSE(\widehat{S}_8^2)_{\min} \cong S_y^4\gamma\left[\delta_{40}^* - \frac{\delta_{22}^{*2}}{\delta_{04}^*}\right] \quad \text{for} \quad e_{opt} = \frac{2\delta_{22}^*}{\delta_{04}^*} \tag{43}$$

$$MSE(\widehat{S}_9^2) \cong S_y^4\gamma\left[\delta_{40}^* + \frac{1}{4}\left(C_x^2 + \delta_{04}^*\right) - \left(\delta_{21}C_x + \delta_{22}^*\right) + \frac{1}{2}\delta_{03}C_x\right] \tag{44}$$

$$MSE(\widehat{S}_{10}^2) \cong S_y^4\gamma\left[\delta_{40}^* + \left(C_x^2 + \delta_{04}^*\right) - 2\left(\delta_{21}C_x + \delta_{22}^*\right) + 2\delta_{03}C_x\right] \tag{45}$$

For the general class of estimators given in (25), we can rewrite the proposed estimator in error terms as follows:

$$\widehat{S}_p^2 = S_y^2\,(1+\phi_0)\exp\left(\frac{-c\phi_1}{1+(d-1)(1+\phi_1)}\right)\exp\left(\frac{-e\phi_2}{1+(f-1)(1+\phi_2)}\right) \quad \text{or}$$

$$\begin{aligned}\widehat{S}_p^2 =& S_y^2\big[1+\phi_0-\frac{c}{d}\phi_1-\frac{e}{f}\phi_2-\frac{c}{d}\phi_0\phi_1-\frac{e}{f}\phi_0\phi_2+\alpha_1\phi_1^2+\alpha_2\phi_2^2\\ &+\alpha_3\phi_1\phi_3\big],\end{aligned} \tag{46}$$

where $\alpha_1=\frac{c}{d}-\frac{c}{d^2}+\frac{c^2}{2d^2}$, $\alpha_2=\frac{e}{f}-\frac{e}{f^2}+\frac{e^2}{2f^2}$ and $\alpha_3=\frac{ce}{df}$.

Also from (46), the bias of $\widehat{S}_p^2$, to first order of approximation, is given by

$$Bias(\widehat{S}_p^2)\cong S_y^2\gamma\left[\alpha_1C_x^2+\alpha_2\delta_{04}^*+\alpha_3\delta_{03}C_x-\frac{c}{d}\delta_{21}C_x-\frac{e}{f}\delta_{22}^*\right]. \tag{47}$$

From (46), the *MSE* of $\widehat{S}_p^2$, to first order of approximation, is given by

$$\begin{aligned}MSE(\widehat{S}_p^2)\cong S_y^4\gamma\big[&\delta_{40}^*+\omega_1^2C_x^2+\omega_2^2\delta_{04}^*-2\omega_1\delta_{21}C_x-2\omega_2\delta_{22}^*\\ &+2\omega_1\omega_2\delta_{03}C_x\big],\end{aligned} \tag{48}$$

where $\omega_1=\frac{c}{d}$ and $\omega_2=\frac{e}{f}$.

From (48), the optimum values of ω_1 and ω_2, are given by

$$\omega_{1(opt)}=\frac{\delta_{04}^*\delta_{21}-\delta_{03}\delta_{22}^*}{C_x\left(\delta_{04}^*-\delta_{03}^2\right)} \quad \text{and} \quad \omega_{2(opt)}=\frac{\delta_{22}^*-\delta_{03}\delta_{21}}{\delta_{04}^*-\delta_{03}^2}.$$

Substituting the optimum values of ω_1 and ω_2 in (48), we can get the corresponding minimum *MSE* of $\widehat{S}_p^2$ as, given by

$$MSE(\widehat{S}_p^2)_{\min}\cong S_y^4\gamma\left[\delta_{40}^*-\delta_{21}^2-\frac{\left(\delta_{22}^*-\delta_{21}\delta_{03}\right)^2}{\delta_{04}^*-\delta_{03}^2}\right]. \tag{49}$$

4. Comparison of Estimators

Now we compare the proposed estimator $\widehat{S}_p^2$ with existing estimators.

(i) By (1) and (49)
$MSE(\widehat{S}_p^2)_{\min<MSE(\widehat{S}_0^2)}$ if

$$\delta_{21}^2+\frac{\left(\delta_{22}^*-\delta_{21}\delta_{03}\right)^2}{\delta_{04}^*-\delta_{03}^2}>0.$$

(ii) By (4) and (49)
$MSE(\widehat{S}_p^2)_{\min} < MSE(\widehat{S}_{R1}^2)$ if

$$(\delta_{04}^* - 2\delta_{22}^*) + \delta_{21}^2 + \frac{(\delta_{22}^* - \delta_{21}\delta_{03})^2}{\delta_{04}^* - \delta_{03}^2} > 0.$$

(iii) By (7) and (49)
$MSE(\widehat{S}_p^2)_{\min} < MSE(\widehat{S}_{R2}^2)$ if

$$(C_x - \delta_{21})^2 + \frac{(\delta_{22}^* - \delta_{21}\delta_{03})^2}{\delta_{04}^* - \delta_{03}^2} > 0.$$

(iv) By (24) and (49)
$MSE(\widehat{S}_p^2)_{\min} < MSE(\widehat{S}_{A1}^2)_{\min}$ if

$$1 - \frac{\left\{1 + \gamma\left(\frac{3}{8}C_x^2 - \frac{1}{2}\delta_{21}C_x\right)\right\}^2}{1 + \gamma\left(\delta_{40}^* + C_x^2 - 2\delta_{21}C_x\right)} - \delta_{40}^* + \delta_{21}^2 + \frac{(\delta_{22}^* - \delta_{21}\delta_{03})^2}{\delta_{04}^* - \delta_{03}^2} > 0.$$

(v) By (36) and (49)
$MSE(\widehat{S}_p^2)_{\min} < MSE(\widehat{S}_1^2)$ if

$$\left(\frac{C_x}{2} - \delta_{21}\right)^2 + \frac{(\delta_{22}^* - \delta_{21}\delta_{03})^2}{\delta_{04}^* - \delta_{03}^2} > 0.$$

(vi) By (37) and (49)
$MSE(\widehat{S}_p^2)_{\min} < MSE(\widehat{S}_2^2)$ if

$$(C_x - \delta_{21})^2 + \frac{(\delta_{22}^* - \delta_{21}\delta_{03})^2}{\delta_{04}^* - \delta_{03}^2} > 0.$$

(vii) By (38) and (49)
$MSE(\widehat{S}_p^2)_{\min} < MSE(\widehat{S}_3^2)$ if

$$\left(\frac{\delta_{04}^*}{4} + \delta_{22}^{*2} + \delta_{21}^2\right) + \frac{(\delta_{22}^* - \delta_{21}\delta_{03})^2}{\delta_{04}^* - \delta_{03}^2} > 0.$$

(viii) By (39) and (49)
$MSE(\widehat{S}_p^2)_{\min} < MSE(\widehat{S}_4^2)$ if

$$(\delta_{04}^* - 2\delta_{22}^*) + \delta_{21}^2 + \frac{(\delta_{22}^* - \delta_{21}\delta_{03})^2}{\delta_{04}^* - \delta_{03}^2} > 0.$$

(ix) By [(14), (40), (42) and (49)]
$MSE(\widehat{S}_p^2)_{\min} < [MSE(\widehat{S}_{Reg2}^2), MSE(\widehat{S}_5^2)_{\min},\ MSE(\widehat{S}_7^2)_{\min}]$ if

$$\frac{(\delta_{22}^* - \delta_{21}\delta_{03})^2}{\delta_{04}^* - \delta_{03}^2} > 0.$$

(x) By [(10), (12), (41), (43) and (49)]
$MSE(\widehat{S}_p^2)_{\min} < [MSE(\widehat{S}_{YK}^2)_{\min},\ MSE(\widehat{S}_{Reg1}^2),\ MSE(\widehat{S}_6^2)_{\min}\ MSE(\widehat{S}_8^2)_{\min}]$
if

$$\left(\delta_{22}^*\frac{\delta_{03}}{\sqrt{\delta_{04}^*}} - \delta_{21}\sqrt{\delta_{04}^*}\right)^2 > 0.$$

(xi) By (44) and (49)
$MSE(\widehat{S}_p^2)_{\min} < MSE(\widehat{S}_9^2)$ if

$$\left(\frac{C_x}{2} - \delta_{21}\right)^2 + \left(\frac{\delta_{04}^*}{4} - \delta_{22}^*\right) + \frac{\delta_{03}C_x}{2} + \frac{\left(\delta_{22}^* - \delta_{21}\delta_{03}\right)^2}{\delta_{04}^* - \delta_{03}^2} > 0.$$

(xii) By (45) and (49)
$MSE(\widehat{S}_p^2)_{\min} < MSE(\widehat{S}_{10}^2)$ if

$$\left(C_x - \delta_{21}\right)^2 + \left(\delta_{04}^* - 2\delta_{22}^*\right) + 2\delta_{03}C_x + \frac{\left(\delta_{22}^* - \delta_{21}\delta_{03}\right)^2}{\delta_{04}^* - \delta_{03}^2} > 0.$$

The proposed estimator will perform better than the competing estimators when Conditions (i)-(xii) are satisfied.

5. Numerical Examples

We use the following four data sets for a numerical comparison of estimators.

Population 1: [source: Murthy (1967, p. 226)]
Let y be the output and x be the number of workers in a factory.
$N = 80,\ n = 25,\ \overline{Y} = 5182.638,\ \overline{X} = 285.125,\ C_y = 0.3542,\ C_x = 0.9485,$
$\rho_{yx} = 0.9140,\ \delta_{40} = 2.2383,\ \delta_{21} = 0.5427,\ \delta_{04} = 3.5360,\ \delta_{22} = 2.2943,$
$\delta_{03} = 1.2680.$

Population 2: [source: (Murthy 1967, p. 228)]
Let y be the output and x be the fixed capital.
$N = 80,\ n = 25,\ \overline{Y} = 5182.638,\ \overline{X} = 1126.463,\ C_y = 0.3542,\ C_x = 0.7507,$
$\rho_{yx} = 0.9413,\ \delta_{40} = 2.2384,\ \delta_{21} = 0.4518,\ \delta_{04} = 2.8306,\ \delta_{22} = 2.1930,$
$\delta_{03} = 1.0237.$

Population 3: [source: Cochran (1977, p. 203)]
Let y be the actual weight of peaches on each tree and x be the eye estimate of weight of peaches on each tree.
$N = 10,\ n = 5,\ \overline{Y} = 56.9,\ \overline{X} = 54.2961,\ C_y = 0.1840,\ C_x = 0.1621,$
$\rho_{yx} = 0.9237,\ \delta_{40} = 1.9249,\ \delta_{21} = 0.1873,\ \delta_{04} = 2.5932,\ \delta_{22} = 2.1149,$
$\delta_{03} = 0.4956.$

Population 4: [source: Sarndal, Swensson & Wretman (1992, p. 652-659)]
y =P85-1985 population in thousand,
x =RMT85-Revenues from the 1985 municipal taxation (in millions kronor).
$N = 284,\ n = 35,\ \overline{Y} = 29.36,\ \overline{X} = 245.088,\ C_y = 1.76,\ C_x = 2.43,$
$\rho_{yx} = 0.961,\ \delta_{40} = 88.92,\ \delta_{21} = 7.87,\ \delta_{04} = 88.88,\ \delta_{22} = 78.79,$
$\delta_{03} = 8.77.$

The results based on Populations 1-4 are given in Table 1.

Table 1: The percentage relative efficiency of different estimators with respect to $\widehat{S}_0^2$ for Populations 1-4.

Estimator	Pop.1	Pop.2	Pop.3	Pop.4
$\widehat{S}_0^2$	100	100	100	100
$\widehat{S}_{R1}^2$	104.436	181.318	320.812	434.817
$\widehat{S}_{R2}^2$	111.715	110.215	103.868	158.196
$\widehat{S}_{A1}^2$	136.384	124.423	120.920	331.049
$\widehat{S}_1^2$	130.559	119.063	102.640	125.114
$\widehat{S}_2^2$	111.715	110.215	103.868	158.196
$\widehat{S}_3^2$	214.239	246.178	444.023	273.894
$\widehat{S}_4^2$	104.436	181.318	320.812	434.817
$\widehat{S}_5^2,\ \widehat{S}_7^2,\ \widehat{S}_{Reg2}^2$	131.207	119.736	103.943	338.374
$\widehat{S}_6^2,\ \widehat{S}_8^2,\ \widehat{S}_{Reg1}^2,\ \widehat{S}_{YK}^2$	214.340	268.678	639.150	461.244
$\widehat{S}_9^2$	139.211	179.734	411.659	350.172
$\widehat{S}_{10}^2$	35.776	58.826	223.122	288.273
$\widehat{S}_p^2$	226.001	351.363	806.889	463.471

We use following expression to obtain the percent relative efficiency (PRE) of different estimators with respect to S_0^2.

$$PRE = \frac{Var(S_0^2)}{MSE(S_i^2)} \times 100, \text{ where } i = 0,\ R1,\ R2,\ A1,\ 1-10,\ Reg1,\ Reg2,\ YK, P.$$

6. Conclusion

We have developed a new generalized class of exponential type estimators for population variance which is more efficient than many of the existing estimators. We have also improved the bias and *MSE* expressions of the Asghar et al. (2014) estimator. While we note that the improved Asghar et al. (2014) estimator $\widehat{S}_{A1}^2$ performs better than several estimators. This estimator can be improved further as shown by the performance of $\widehat{S}_p^2$. We also note that the estimators ($\widehat{S}_6^2,\ \widehat{S}_8^2,\ \widehat{S}_{Reg1}^2,\ \widehat{S}_{YK}^2$) and ($\widehat{S}_5^2,\ \widehat{S}_7^2,\ \widehat{S}_{Reg2}^2$) have the same *MSE* for all four populations.

[Received: July 2014 — Accepted: January 2015]

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