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Slashed Rayleigh Distribution

Distribución Slashed Rayleigh

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Abstract

In this article we study a subfamily of the slashed-Weibull family. This subfamily can be seen as an extension of the Rayleigh distribution with more flexibility in terms of the kurtosis of distribution. This special feature makes the extension suitable for fitting atypical observations. It arises as the ratio of two independent random variables, the one in the numerator being a Rayleigh distribution and a power of the uniform distribution in the denominator. We study some probability properties, discuss maximum likelihood estimation and present real data applications indicating that the slashed-Rayleigh distribution can improve the ordinary Rayleigh distribution in fitting real data.

***Key words*:** Kurtosis, Rayleigh Distribution, Slashed-elliptical Distributions, Slashed-Rayleigh Distribution, Slashed-Weibull Distribution, Weibull Distribution.

Resumen

En este artículo estudiamos una subfamilia de la familia slashed-Weibull. Esta subfamilia puede ser vista como una extensión de la distribución Rayleigh con más flexibilidad en cuanto a la kurtosis de la distribución. Esta particularidad hace que la extensión sea adecuada para ajustar observaciones atípicas. Esto surge como la razón de dos variables aleatorias independientes, una en el numerador siendo una distribución Rayleigh y una

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potencia de la distribución uniforme en el denominador. Estudiamos algunas propiedades de probabilidad, discutimos la estimación de máxima verosimilitud y presentamos aplicaciones a datos reales indicando que la distribución slashed-Rayleigh presenta mejor ajuste para datos reales que la distribución Rayleigh.

***Palabras clave*:** curtosis, distribución Rayleigh, distribuciones Slashed-elípticas, distribución Slashed-Rayleigh, distribución Slashed-Weibull, distribución Weibull.

1. Introduction

An important distribution in modeling random phenomena, specially positive ones is the Rayleigh distribution. Other models for positive data are generalized exponential distributions (Gómez, Bolfarine & Gómez 2014). A random variable X follows a Rayleigh distribution where σ is the scale parameter, that we denote $X \sim R(\sigma)$, if its density function is given by

$$f_X(x;\sigma) = \frac{x}{\sigma} e^{-\frac{x^2}{2\sigma}}$$

where $x > 0$ and $\sigma > 0$.

Some properties of this distribution are

$$\begin{aligned} E(X) &= \sqrt{\frac{\pi}{2}\sigma} \\ Var(X) &= \frac{4-\pi}{2}\sigma \\ E(X^r) &= (2\sigma)^{r/2}\Gamma\left(\frac{r+2}{2}\right). \end{aligned}$$

Further details on the Rayleigh distribution can be found in Johnson, Kotz & Balakrishnan (1994). Gómez, Quintana & Torres (2007) and Gómez & Venegas (2008) introduced the class of slash-elliptical distributions. This class of distributions can be regarded as an extension of the class of elliptical distributions studied in Fang, Kotz & Ng (1990).

A random variable T follows a slashed-elliptical distribution with location parameter μ and scale parameter σ, denoted as $T \sim SEl(t;\mu,\sigma,g)$, if it can be represented as

$$T = \sigma \frac{X}{U^{1/q}} + \mu, \tag{1}$$

where $X \sim El(0,1,g)$, $U \sim U(0,1)$ are independent and $q > 0$.

***Note* 1.** Specifically, a random variable X follows an elliptical distribution with location parameter μ and scale parameter σ denoted by $X \sim EL(\mu,\sigma;g)$ if the density function of X is given by

$$f_X(x) = \frac{1}{\sigma} g\left(\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

for a non negative function $g(u), u \geq 0$ (called the density generator), satisfying $\int_0^\infty u^{-\frac{1}{2}} g(u)\, du = 1$.

If $T \sim SEl(0, 1, q)$, then, the density function for T is given by

$$f_T(t; 0, 1, q) = \begin{cases} \frac{q}{2|t|^{q+1}} \int_0^{t^2} v^{\frac{q-1}{2}} g(v)\, dv & if \quad t \neq 0 \\ \\ \frac{q}{1+q} g(0) & if \quad t = 0. \end{cases} \tag{2}$$

In the canonic case, that is, for $q = 1$, density in (2) becomes

$$f_T(t; 0, 1, 1) = \begin{cases} \frac{G(t^2)}{2t^2} & if \quad t \neq 0 \\ \\ \frac{1}{2} g(0) & if \quad t = 0 \end{cases} \tag{3}$$

where $G(x) = \int_0^x g(v)\, dv$.

Arslan (2008) discusses asymmetric versions of the family of slashed-elliptical distributions. Gómez, Olivares-Pacheco & Bolfarine (2009) propose an extension of the Birnbaum-Saunders distribution based on slashed-elliptical distributions.

Olivares-Pacheco, Cornide-Reyes & Monasterio (2010) introduce an extension of the two parameter Weibull distribution to make it more flexible in terms of kurtosis and is called slashed-Weibull distribution. Let $W \sim SW(\alpha, \beta, q)$, then W is distributed as the slashed-Weibull distribution. The density function of W is given by

$$f_W(w; \alpha, \beta, q) = \frac{q\beta}{\alpha^\beta w^{\beta-1}} T_W(w; \alpha, \beta, q) \tag{4}$$

where $\alpha > 0$, $\beta > 0$, $q > 0$ and $T_W(w; \alpha, \beta, q)$ are defined as

$$T_W(w; \alpha, \beta, q) = \int_0^1 u^{\beta+q-1} e^{-(uw/\alpha)^\beta}\, du$$

If $X \sim SW(\alpha, 2, q)$, then X is distributed as the slashed-Rayleigh distribution, denoted as $X \sim SR(\alpha, q)$. The density function of X is given by

$$f_X(x; \alpha, q) = \frac{2q}{\alpha^2} x T_X(x; \alpha, 1) \tag{5}$$

where $x > 0$, $\alpha > 0$, $q > 0$ and $T_X(x; \alpha, q)$ defined as

$$T_X(x; \alpha, q) = \int_0^1 u^{q+1} e^{-(ux/\alpha)^2}\, du$$

Several extensions have been considered in statistical literature for the Rayleigh distribution. Among the most important we consider Vodă (1976); Balakrishnan & Kocherlakota (1985); Surles & Padgett (2001); Kundu & Raqab (2005); Manesh & Khaledi (2008) and Cordeiro, Cristino, Hashimoto & Ortega (2013). The study of the extension proposed in this paper is motivated by works of Gómez et al. (2007); Gómez & Venegas (2008); Olivares-Pacheco et al. (2010) and Olmos, Varela, Gómez & Bolfarine (2012).

We consider an extension based on the ratio between two independent random variables, one in the numerator corresponding to a random variable with Rayleigh distribution and in the denominator a power of a uniform random variable. This extension allows fitting the Rayleigh distribution to real data, being able to accommodate atypical observations (high kurtosis). Another interesting feature is that the resulting model has "closed form" in the sense that it is represented in terms of known functions such as the Gamma function.

The paper is organized as follows. Section 2 is devoted to an extension of the Rayleigh distribution and we derive its density, moments and kurtosis and asymmetry coefficients. In Section 3 we discuss moments and maximum likelihood estimation and present applications to two real data sets. The application illustrates the good performance of the model proposed in real applications. Final conclusions are reported in Section 4.

2. Slashed Rayleigh Distribution

In this section we introduce the new density, its stochastic representation, some properties, and graphical representations.

2.1. Definition

A random variable T follows a slashed-Rayleigh distribution with scale parameter σ kurtosis parameter q, denoted by $T \sim SR(\sigma, q)$, if it can be represented as

$$T = \frac{X}{U^{1/q}}, \tag{6}$$

where $X \sim R(\sigma)$ and $U \sim U(0,1)$ are independent, with $q > 0$.

2.2. Density Function

The following proposition reveals the probability density function (pdf) for a random variable T generated using the stochastic representation given in (6) according to the slashed-Rayleigh.

Proposition 1. *Let $T \sim SR(\sigma, q)$. Then, the pdf of T is given by*

$$f_T(t; \sigma, q) = \frac{q(2\sigma)^{q/2}}{t^{q+1}} \Gamma\left(\frac{q+2}{2}\right) F\left(\frac{t^2}{2\sigma}, \frac{q+2}{2}, 1\right) \tag{7}$$

where $\sigma > 0, q > 0$, $t > 0$ and F is the commulative distribution function of the Gamma distribution.

Proof . Using the stochastic representation given in (6) and multiplying by the Jacobian of the transformation, the pdf associated with T is given by

$$f_T(t;\sigma,q)=\int_0^1 \frac{qtw^{q+1}}{\sigma}e^{-\frac{t^2w^2}{2\sigma}}dw$$

Making the variable transformation $u=\frac{t^2w^2}{2\sigma}$ it follows that

$$f_T(t;\sigma,q)=\frac{q(2\sigma)^{q/2}}{t^{q+1}}\int_0^{t^2/2\sigma} u^{q/2}e^{-u}\,du$$

so that the result follows after identifying a Gamma distribution inside the integral sign. □

In the particular case where $\sigma=q=1$ it follows that the canonic slashed-Rayleigh distribution, denoted as $T\sim SR(1,1)$ is obtained. Then, the density function for T is given by

$$f_T(t)=\sqrt{\pi/2}t^{-2}F\left(t^2/2,3/2,1\right)\ ,\quad t>0 \tag{8}$$

Figure 1 depicts some of the shapes that the slashed-Rayleigh distribution can take for different values of the parameters σ and q.

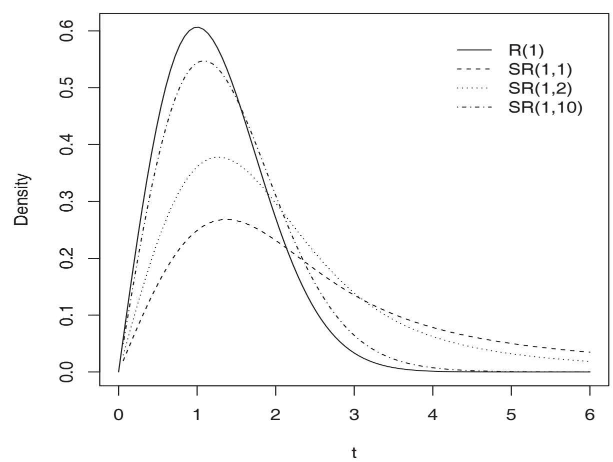

FIGURE 1: Plot of the slashed-Rayleigh density, $SR(\sigma,q)$.

2.3. Properties

In this section some basic properties of the slashed-Rayleigh distribution are considered.

Proposition 2. *Let $T\sim SR(\sigma,q)$, then*

1. $\lim_{q \to \infty} f_T(t; \sigma, q) = \frac{x}{\sigma} e^{-\frac{x^2}{2\sigma}}$

2. $F_T(t; \sigma, q) = P(T < t) = \left(1 - e^{-\frac{k^2}{2\sigma}}\right) - k^{-q}(2\sigma)^{\frac{q}{2}} \Gamma\left(\frac{q+2}{2}\right) F\left(\frac{k^2}{2\sigma}, \frac{q+2}{2}, 1\right)$

***Note* 2.** Property 1 reveals that as $q \to \infty$ the slashed-Rayleigh converges to the ordinary Rayleigh distribution.

2.4. Moments

Proposition 3. *Let $T \sim SR(\sigma, q)$. Then, for $r = 1, 2, \ldots$ and $q > r$, it follows that the r-th moment is given by*

$$\mu_r = E(T^r) = (2\sigma)^{r/2} \Gamma\left(\frac{r+2}{2}\right) \frac{q}{q-r} \tag{9}$$

***Proof*.** Using the stochastic representation for the distribution given in (6), we have that

$$\begin{aligned} \mu_r &= E\left(T^r\right) \\ &= E\left(\left(\frac{X}{U^{\frac{1}{q}}}\right)^r\right) \\ &= E\left(X^r U^{-\frac{r}{q}}\right) \\ &= E\left(X^r\right) E\left(U^{-\frac{r}{q}}\right) \end{aligned}$$

where it follows that $E\left(U^{-\frac{r}{q}}\right) = \frac{q}{q-r}$, $q > r$ and $E\left(X^r\right) = (2\sigma)^{r/2}\Gamma\left(\frac{r+2}{2}\right)$ are the moments for the distribution $R(\sigma)$. □

Proposition 4. *Let $T \sim SR(\sigma, q)$, then it follows that*

$$E(T) = \frac{q}{q-1}\sqrt{\frac{\pi\sigma}{2}} \ , \ q > 1 \quad \text{and} \quad Var(T) = \frac{\sigma q\left[4(q-1)^2 - \pi q(q-2)\right]}{2(q-1)^2(q-2)} \ , \ q > 2 \tag{10}$$

Proposition 5. *Let $T \sim SR(\sigma, q)$. Then, the asymmetry ($\sqrt{\beta_1}$) and kurtosis (β_2) coefficients for $q > 3$ and $q > 4$ are given respectively by*

$$\sqrt{\beta_1} = \frac{\sqrt{4\pi(q-2)}\left[3(q-1)^3(q-2) - 6q(q-1)^2(q-3) + q^2\pi(q-2)(q-3)\right]}{\sqrt{q}(q-3)[4(q-1)^2 - q\pi(q-2)]^{3/2}}$$

$$\begin{aligned} \beta_2 = &\frac{(q-2)[8(q-1)^2(q-2)\{4(q-1)(q-3) - 3\pi q(q-4)\}]}{q(q-3)(q-4)[4(q-1)^2 - q\pi(q-2)]^2} \\ &+ \frac{(q-2)\left[\pi q^2(q-3)(q-4)\{24(q-1)^2 - 3\pi q(q-2)\}\right]}{q(q-3)(q-4)[4(q-1)^2 - q\pi(q-2)]^2} \end{aligned}$$

***Note* 3.** It follows that as $q \to \infty$ the asymmetry and kurtosis coefficients converge to

$$\sqrt{\frac{4\pi(\pi-3)^2}{(4-\pi)^2}}$$

and

$$\frac{32-3\pi^2}{(4-\pi)^2}$$

respectively, which correspond to those for the Rayleigh distribution. Figures 2 and 3 depict plots for the asymmetry and kurtosis coefficients, respectively.

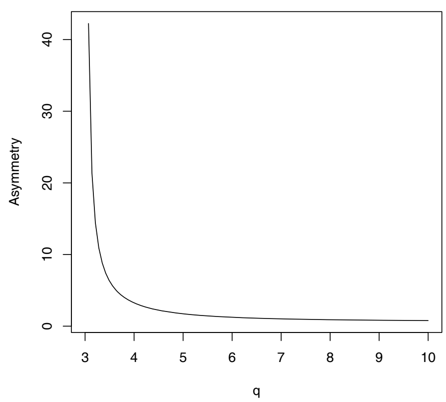

FIGURE 2: Asymmetry coefficient

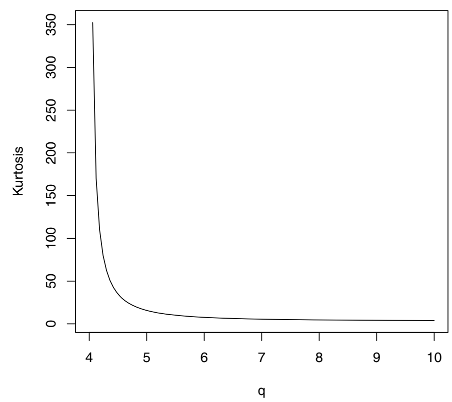

FIGURE 3: Kurtosis coefficient

3. Inference

In this section we discuss moments and maximum likelihood estimation for parameters σ and q for the slashed-Rayleigh distribution.

3.1. Moment Estimators

In the next proposition we present analytical expressions for the moment estimators of the parameters σ and q.

Proposition 6. *Let $T_1, \ldots, T_n$ a random sample for the random variable $T \sim SR(\sigma, q)$. Then, moment estimators for $\boldsymbol{\theta} = (\sigma, q)$, with $q > 2$, are given by*

$$\widehat{\sigma}_M = \tfrac{2\overline{T}^2(\widehat{q}-1)^2}{\pi\widehat{q}^2} \quad \textit{and} \quad \widehat{q}_M = 1 + \left(\tfrac{\pi\overline{T}^2}{\pi\overline{T^2}-4\overline{T}^2}\right)^{1/2} \quad , \textit{ if } \quad \pi\overline{T^2} > 4\overline{T}^2,$$

where $\overline{T}$ is the sample mean, and $\overline{T^2}$ is the sample mean for square of the sample units.

Proof. Using (9), it follows that

$$E(T) = \frac{q}{q-1}\sqrt{\frac{\pi\sigma}{2}} \quad and \quad E(T^2) = \frac{q}{q-2}2\sigma \ \ , if \ \ q > 2 \tag{11}$$

and replacing $E(T)$ by $\overline{T}$ and $E(T^2)$ by $\overline{T^2}$ in (11), we obtain a system of equations for which the solution leads to the moment estimators $(\widehat{\sigma}_M, \widehat{q}_M)$ for (σ, q). □

3.2. Maximum Likelihood Estimation

For a random sample $T_1, \ldots, T_n$ from the distribution $SR(\sigma, q)$, the log likelihood function can be written as

$$l(\sigma, q) = n\log(q) + \frac{nq}{2}\log(2\sigma) + n\log\left(\Gamma\left(\frac{q+2}{2}\right)\right) - (q+1)\sum_{i=1}^{n}\log(t_i) + \sum_{i=1}^{n}\log(F(t_i)) \tag{12}$$

so that the maximum likelihood equations are given by

$$\sum_{i=1}^{n}\frac{F_1(t_i)}{F(t_i)} = -\frac{nq}{\sigma} \tag{13}$$

$$\frac{n}{q} + \frac{n}{2}\log(2\sigma) + \frac{n}{2}\Psi\left(\frac{q+2}{2}\right) + \sum_{i=1}^{n}\frac{F_2(t_i)}{F(t_i)} = \sum_{i=1}^{n}\log(t_i) \tag{14}$$

where $F(t_i) = F\left(\frac{t_i^2}{2\sigma}, \frac{q+1}{2}, 1\right)$, $F_1(t_i) = \frac{\partial}{\partial\sigma}F(t_i)$, $F_2(t_i) = \frac{\partial}{\partial q}F(t_i)$ and Ψ is digamma function. The solution for the equations (13-14) can be obtained by using the function optim available in software R, the specific method is the L-BFGS-B developed by Byrd et al. (1995) which allows box constraint. This uses a limited-memory modification of the quasi-Newton method.

3.3. Observed Information Matrix

In this subsection we consider the observed information matrix for the slashed-Rayleigh distribution.

Let $T \sim SR(\sigma, q)$, so that the observed information matrix is given by

$$\begin{pmatrix} -\frac{nq}{2\sigma^2} + \sum_{i=1}^{n}\frac{\partial}{\partial\sigma}\left(\frac{F_1(t_i)}{F(t_i)}\right) & \frac{n}{q} + \sum_{i=1}^{n}\frac{\partial}{\partial q}\left(\frac{F_1(t_i)}{F(t_i)}\right) \\ \frac{n}{2\sigma} + \sum_{i=1}^{n}\frac{\partial}{\partial\sigma}\left(\frac{F_2(t_i)}{F(t_i)}\right) & -\frac{n}{q^2} + \frac{n}{4}\Psi_1\left(\frac{q+2}{2}\right) + \sum_{i=1}^{n}\frac{\partial}{\partial q}\left(\frac{F_2(t_i)}{F(t_i)}\right) \end{pmatrix}$$

where $F(t_i) = F\left(\frac{t_i^2}{2\sigma}, \frac{q+2}{2}, 1\right)$, $F_1(t_i) = \frac{\partial}{\partial\sigma}F(t_i^2)$, $F_2(t_i) = \frac{\partial}{\partial q}F\left(\frac{t_i^2}{2\sigma}, \frac{q+2}{2}, 1\right)$ and Ψ_1 is the trigamma function.

3.4. Simulation Study

Using the stochastic representation considered in (6), we generate 1000 samples distributed as $SR(1,1)$, $SR(1,2)$, $SR(1,3)$, $SR(1,4)$ and $SR(1,5)$, respectively, for the sample sizes 30, 50 and 100. For each sample we compute the maximum likelihood estimators (MLEs) using the moment estimators as starting values.

Table 1: Maximum likelihood estimators for samples generated with $\sigma = 1$ and several values of the parameter q.

q	$n = 30$ (SD)	$n = 50$ (SD)	$n = 100$ (SD)
1	1.060303 (0.2435404)	1.025251 (0.1672472)	1.0141615 (0.1141615)
2	2.129808 (0.6008310)	2.075064 (0.4092104)	2.0637530 (0.3004835)
3	3.241356 (1.2942610)	3.226699 (0.9393005)	3.1032040 (0.5203248)
4	4.502840 (2.3793270)	4.306673 (1.6915380)	4.1451880 (0.8128965)
5	6.320374 (5.5379420)	5.811159 (3.6906960)	5.1960890 (1.3196100)

For each generated sample, MLEs were computed numerically using a Newton-Raphson type procedure. Empirical means and standard deviations (SD) are reported. Notice that the empirical means become very close to the true values and standard deviations become small, an expected result since MLEs are consistent.

4. Two Illustrative Data Sets

4.1. Illustration 1

Devore (2005), with pedagogic interest, presents a data set corresponding to a sample of 26 units related to contaminant aluminum (ppm) in certain type of plastic material. This data set can also be found in Aubin (1990). Using results in Subsection 3.1, moment estimators were computed and are given by $\widehat{\sigma}_M = 6360.913$ and $\widehat{q}_M = 3.341$. Using these estimates as starting values for the Newton-Raphson procedure, maximum likelihood estimates were computed. Table 2 presents descriptive statistics for the amount of contaminated aluminum in the data set where b_1 and b_2 are the coefficients of asymmetry and kurtosis, respectively. Notice that the data set presents high positive asymmetry and also high kurtosis.

Table 2: Summary statistics for rupture times.

n	$\overline{X}$	s^2	b_1	b_2
26	142.6538	9,644.075	2.028009	8.07527

Table 3 presents parameter estimates for the R, W and SR models, using maximum likelihood (MLE) approach and the corresponding Akaike information criterion (AIC) for model choice. For these data, AIC shows a better fit of the SR model. Standard deviations (SD) were computed using the inverse of the Hessian matrix. We also computed the Kolmogorov-Smirnov statistic (KSS), for which the corresponding values for models R and SR were 0.282 and 0.178 respectively, which also indicates that the best fit is presented by the SR model. Figure 4 depicts the

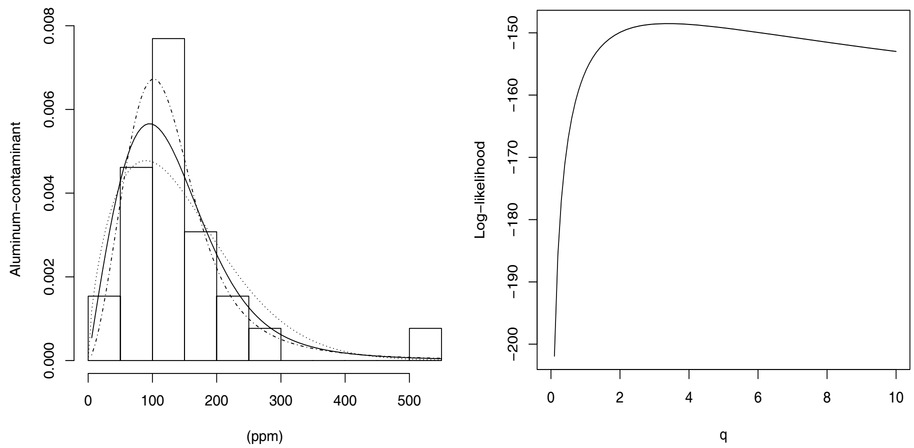

Figure 4: Models fitted by the maximum likelihood method for aluminum-contaminant data set: SR (solid line), R (dashed line), W (doted line) and SW (doted and dashed line). Log-likelihood profile for the SR distribution with $\sigma = 6,360.913$.

Table 3: Parameter estimates and log-likelihood values for R, W and SR models for the aluminum-contaminant life data set.

Parameter	R (SD)	W (SD)	SW (SD)	SR (SD)
θ	-	1.631	2.555	-
		(0.226)	(0.672)	
σ	14,811.630	160.569	102.426	6,360.913
	(2,965.821)	(20.477)	(18.791)	(2,324.110)
q	-	-	2.700	3.386
			(1.140)	(1.615)
LL	-151.5879	-150.3446	-148.039	-148.5184
AIC	305.1758	304.6892	302.078	301.0368

histogram and the fitted densities with parameters replaced by the MLEs. It also depicts likelihood profile for the fitted models for parameter q. Figures 5 and 6 presents qqplots for both models. Results also corroborate the good performance of the SR model.

4.2. Illustration 2

We consider in this application the data set from Devore (2005), corresponding to the lifetime of 50 units of a type of drilling machine.

Using results in Section 3.1, moment estimators were computed, leading to $\widehat{\sigma} = 3460.013$ and $\widehat{q} = 2.619$. These estimates were then used as starting values for the optim algorithm for maximizing the likelihood function, Table 4 presents summary statistics for the lifetime data.

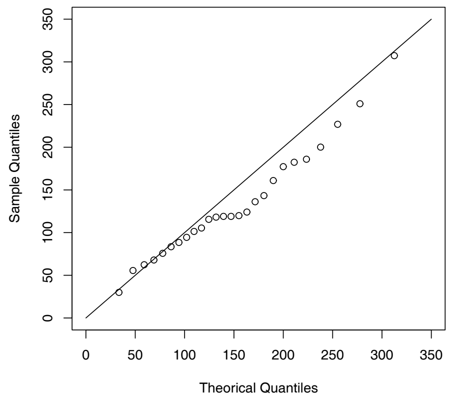

FIGURE 5: qqplot: R model

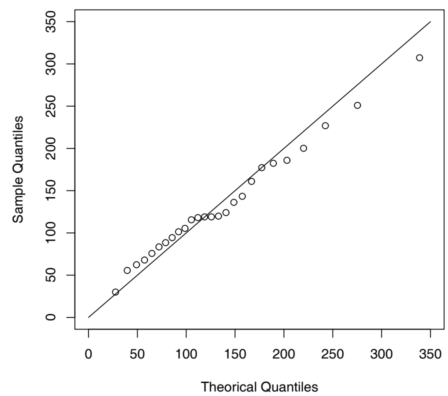

FIGURE 6: qqplot: SR model

TABLE 4: Summary statistics for lifetime data.

n	$\overline{X}$	s^2	b_1	b_2
50	119.26	9,503.013	1.970068	4.376542

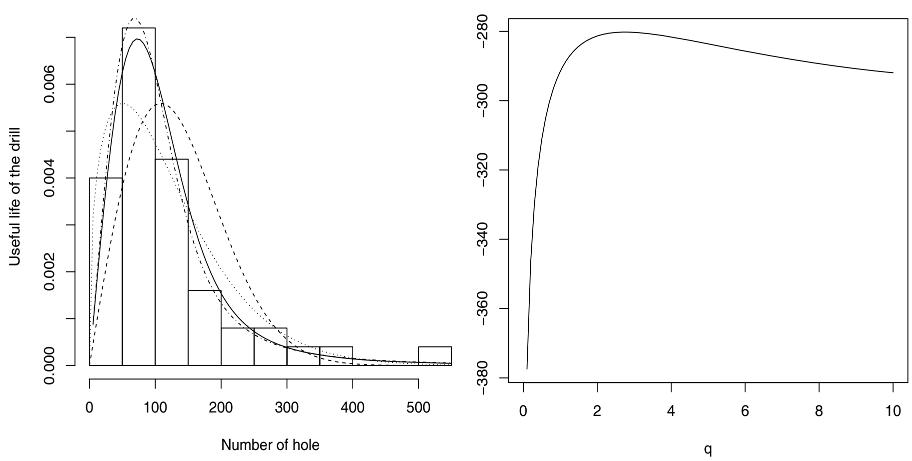

FIGURE 7: Models fitted by maximum likelihood method for the useful life of the drill data set: SR (solid line), R (dashed line), W (dotted line) and SW (dotted and dashed line). Profile log-likelihood for the parameter q of the SR model.

Table 5 depicts parameter estimates for models R, W, SW and SR, using the maximum likelihood (MLE) approach and the corresponding Akaike information criterion (AIC). For these data, AIC shows a better fit of the SR model. Standard deviations (SD) were computed using the inverse of the Hessian matrix. We also computed the Kolmogorov-Smirnov statistic (KSS), so that corresponding values for the models R and SR were 0.347 and 0.214 respectively, which also indicates that the best fit is presented by the SR model. Figure 7 presents the histogram for the data with the fitted densities and the log-likelihood profile for parameter q. Figures 8 and 9 depicts qqplots for both models.

Table 5: Parameter estimates and log-likelihood values for R, W, SW and SR models for the useful life of the drill data set.

Parameters	R (SD)	W (SD)	SW (SD)	SR (SD)
θ	-	1.370	75.058	-
		(0.140)	(11.904)	
σ	11,767.950	131.365	2.049	3,460.012
	(1,614.388)	(14.360)	(0.386)	(1,127.119)
q	-	-	2.144	2.482
			(0.627)	(0.765)
LL	-293.6905	-285.1543	-282.5264	-282.741
AIC	589.381	574.308	571.052	569.483

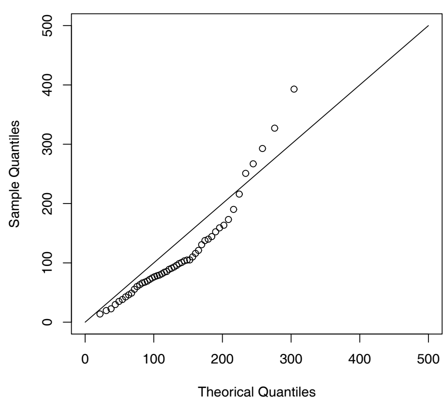

Figure 8: qqplot: R model

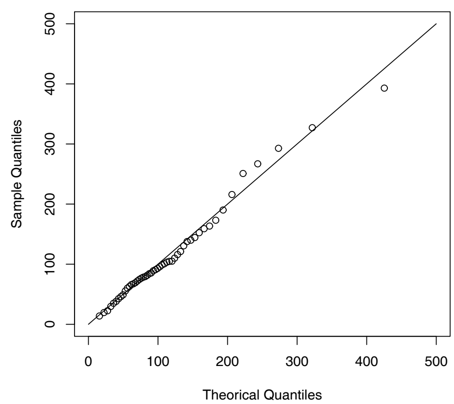

Figure 9: qqplot: SR model

5. Concluding Remarks

In this paper we study a subfamily of the slash-Weibull distribution. This model arises from the ratio between two independent random variables, the Rayleigh distribution in the numerator and the power of uniform random variable in the denominator. Moment estimators for the slashed-Rayleigh distribution are obtained explicitly and can be used as initial values for the computation of the maximum likelihood estimators which requires numerical procedures such as the Newton-Rapshon algorithm. The derivation of the asymmetry and kurtosis coefficients illustrates the fact that the slashed-Rayleigh distribution is able to fit data sets for which the Rayleigh distribution is adequate but with an excess of kurtosis. Applications to real data have demonstrated that the slashed-Rayleigh distribution can present better fit than distributions such as the Rayleigh and Weibull. It also indicated that the slashed-Rayleigh can present better fit than the slashed-Weibull distribution.

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