Analysis of the Forecasting Performance of the Threshold Autoregressive Model

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Título

Análisis de la Capacidad de Pronóstico del Modelo Autorregresivo de Umbrales

Abstract

In this investigation, we analyze the forecasting performance of the threshold autoregressive (TAR) model. To this aim, we find the Bayesian predictive distribution from this model, and then, we conduct an out-of-sample forecasting exercise, where we compare forecasts from the TAR model with those from a linear model and nonlinear smooth transition autoregressive, self-exciting threshold autoregressive and Markov-switching autoregressive models. For this empirical forecast evaluation, we: i) use the U.S. and Colombian GDP, unemployment rate, industrial production index and inflation time series, which lead us to estimate and forecast forty models; and, ii) use evaluation criteria and statistical tests that are mostly employed in literature. We also compare the in-sample properties of the estimated models. For the overall comparison, we find a satisfactory performance of the TAR model in forecasting the chosen economic time series, and a shape changing characteristic in the Bayesian predictive distributions of this model that may capture the cycles in the economic time series. This gives important signals about the forecasting ability of the TAR model in the economic field.

Resumen

En esta investigación, se analiza la capacidad de pronóstico del modelo Autorregresivo de Umbrales (TAR). Para esta finalidad, se encuentra la distribución predictiva Bayesiana, y luego, se conduce un ejercicio de pronóstico fuera de la muestra, donde se comparan los pronósticos del modelo TAR con auqellos de un modelo lineal y de los modelos no lineales Autorregresivo de Transición Suave, Autorregresivo de Umbrales Auto-Excitado y Autorregresivo de Cambio de Régimen. Para esta evaluación de pronósticos empírica, i) se utilizan las series del PIB, el desempleo, el índice de producción industrial y la inflación de Estados Unidos y Colombia, lo cual lleva a estimar y pronosticar cuarenta modelos; y, ii) se utilizan criterios y test estadísticos los cuales on ampliamente aplicados en la literatura. De igual manera, se comparan las propiedades dentro de la muestra de los modelos estimados. Para todo el ejercicio de comparación, se encuentra un comportamiento satisfactorio del modelo TAR para pronosticar las distintas series económicas, y se encuentra una característica de cambio de forma en la distribución predictiva del modelo TAR que puede capturar los ciclos presentados en las series económicas. Esto arroja importantes indicios sobre la capacidad de pronóstico del modelo TAR en el campo económico.

Keywords: Bayesian predictive distributions; Forecasts comparison; Threshold autoregressive model; Linear model; Nonlinear model.

Palabras clave: Distribuciones predictivas Bayesianas; Comparación de pronósticos; Modelo autorregresivo de umbrales; modelo lineal; modelo no lineal.

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Introduction

Forecasting is one of the main objectives of the time series analysis. Different time series have been used for this purpose, like the economic and financial time series that have been largely analyzed in the forecasting literature (Granger and Newbold, 1973, 1986; Tsay, 2000; Gooijer and Hyndman, 2006). For the past few decades, it has become relevant in this field, the study of nonlinear models and their forecasting performance (van Dijk and Franses, 2003; Meyn and R., 2009), due to the capacity of these models to describe common behaviors of the economic time series such as business cycle, volatility and uncertainty, among others (Tiao and Tsay, 1994; Franses and van Dijk, 2000; Clements et al., 2003; van Dijk and Franses, 2003). This has fostered fairly large studies in which the forecasting performance of nonlinear and linear models are compared using current macroeconomic time series (Teräsvirta, 2006), intensifying the macroeconomic forecasts studies (Hansson et al., 2005).

However, the literature review shows that the forecasting performance of the threshold autoregressive (TAR) model, in the economic field, has not been studied up to now. That makes the aim of this thesis about analyzing the forecasting performance of this model relevant. The TAR model was initially introduced by Tong (1978) throughout the self-exciting threshold autoregressive (SETAR) model. In recent years, it has been studied by Nieto (2005) and Nieto et al. (2013), who proposed a Bayesian methodology to fit a TAR model with an exogenous threshold variable to an observed time series, and Nieto (2008) and Vargas (2012) who addressed the forecasting stage of this model.

Regarding the literature focused on the forecasting performance of time series models, when using economic and financial time series, we highlight the studies of Teräsvirta and Anderson (1992), who analyze the performance of the smooth transition autoregressive (STAR) and autoregressive models to forecast the industrial production index throughout the period 1960-1986. Cao and Tsay (1992) compare the SETAR with the generalized autoregressive conditional heteroskedasticity (GARCH), exponential GARCH and autoregressive moving average (ARMA) models, using the volatility of stock returns of the NYSE and AMEX from 1928 to 1989. Tiao and Tsay (1994) compare the out-of-sample forecasts from SETAR and AR models, using the United States (U.S.) Real Gross National Product from 1947 to 1990 period. Later, Montgomery et al. (1998) compare the forecasting performance of the SETAR, Markov switching autoregressive (MSAR), autoregressive integrated moving average (ARIMA) and vector ARMA (VARMA) models, using the U.S. unemployment rate from 1948 to 1993. Clements and Krolzig (1998) evaluate the performance of the MSAR, SETAR and AR models in forecasting the U.S. GNP from 1947 to 1996, and Clements and Smith (1999) compare the forecasting performance of the SETAR and AR models, using the exchange rate and GNP in several countries.

In recent years, Clements and Smith (2000) evaluate the forecasting performance of the SETAR, VAR and AR models, using the U.S. GNP and unemployment rate from 1948 to 1993. van Dijk et al. (2002) compare forecasts from the STAR and AR models, using the U.S. unemployment rate from 1968 to 1999. Bradley and Jansen (2004) analyze the forecasts of the STAR and multiple-regime STAR models, using the S&P500 index and the U.S. industrial production from 1935 to 1997. Franses and van Dijk (2005) examine the forecasting performance of the STAR, AR and seasonal ARIMA models, using the industrial production series of 18 OECD countries from 1960 to 2002. Later, Deschamps (2008) compares the forecasts from STAR and MSAR models, using the U.S. unemployment rate from 1960 to 2004. Guidolin et al. (2009) evaluate the predictive performance of the SETAR, STAR, MSAR and GARCH models, using the stock and bond returns in 7 developed countries from 1979 to 2007. Geweke and Amisano (2010) evaluate the out-of-sample predictive distributions of the stochastic volatility, Markov normal mixture and GARCH models, using the S&P500 index stock market over the 1972-2005 period.

Thus, based on this literature review, the forecasting performance of the TAR model, using the Bayesian predictive distributions, will be addressed in the following way: forecasts from the TAR model will be compared with forecasts from a linear autoregressive model and nonlinear STAR, SETAR and MSAR models. For this empirical comparison, we will use the Gross Domestic Product, the unemployment rate, the industrial production index and the inflation rate from Colombia and the United States. Models will be evaluated in terms of the properties of unbiased and uncorrelated errors, relative mean square errors, forecast accuracy and encompassing properties. These evaluation criteria are mostly used in the literature.

The outline of this thesis is as it follows. Chapter 1 is devoted to the estimation and forecasting procedure of the TAR, SETAR, STAR, MSAR and lineal models. This Chapter also presents one of the main contributions of this thesis: it introduces a new computation of the Bayesian predictive distribution of the TAR model, which was developed in this study. Chapter 2 briefly describes the evaluation criteria that will be used to evaluate and compare the forecast performance of the TAR model with that of the competing models. Then, in Chapter 3 are presented the U.S. and Colombian economic time series, that were selected for the forecasting evaluation. Additionally, based on the literature review and the economic theory, we define the threshold variable for each macroeconomic variable, in order to estimate the TAR model. Chapter 4 presents the other main contribution of this thesis: the analysis of the forecasting performance of the TAR model against the competing models. For each macroeconomic time series, we first describe the data. Second, we present the estimation for each considered model and analyze their in-sample properties. Third, we present the outputs of the different criteria and statistical tests that were used for the forecast evaluation. Finally, we draw conclusions about the forecasting performance of the TAR model.

Chapter 1

Forecasting Models

This Chapter briefly presents the specification, estimation and predictive procedure of the threshold autoregressive model and the competing models that we have selected for the forecasting comparison analysis. As we mentioned before, these competing models have been widely used and studied in the literature related to the forecasting performance of several time series models using different economic time series.

1.1. Threshold autoregressive model

Nieto (2005) develops a Bayesian methodology to analyze a bivariate threshold autoregressive (TAR) model with exogenous threshold variable and in presence of missing data. This model is expressed through a dynamical system consisting of an input stochastic process $\{Z_t\}$ that represents the threshold process, and an output stochastic process $\{X_t\}$ that is known as the process of interest. Thus, the TAR model is described as:

$$X_t = a_0^{(j)} + \sum_{i=1}^{k_j} a_i^{(j)} X_{t-i} + h^{(j)} \varepsilon_t, \tag{1.1}$$

if Z_t belongs to the real interval $B_j = (r_{j-1},\ r_j]$ for some j; $j = 1, \ldots, l$, where $r_0 = -\infty, r_l = \infty$ and l is a positive integer number. The real numbers $r_j; j = 1, \ldots, l-1$, are known as the threshold values of the process $\{Z_t\}$ and they indicate the number of l regimes for the process $\{Z_t\}$. The coefficients $a_i^{(j)}$ and $h^{(j)}$ are real numbers with $j = 1, \ldots, l; i = 0, 1, \ldots, k_j$. The nonnegative integer numbers $k_1, \ldots, k_l$ denote the autoregressive orders of the process $\{X_t\}$ in each regime. $\{\varepsilon_t\}$ is a Gaussian zero-mean white noise process with variance 1, and it is mutually independent of process $\{Z_t\}$. This model is denoted TAR$(l; k_1, k_2, \ldots, k_l)$, where the structural parameters are $l, r_1, r_2, \ldots, r_{l-1}; k_1, k_2, \ldots, k_l$, and the nonstructural parameters are $a_i^{(j)}$ and $h^{(j)}$.

The TAR model has the faculty to describe a nonlinear relationship between variables X and Z, where the dynamical response of X depends on the location of Z in its sample space. Besides, by using this model, it is possible to explain certain types of heteroscedasticity in $\{X_t\}$ given that a typical path from it may show burst of large values (Nieto, 2005; Nieto and Moreno, 2016).

It is assumed, according to Nieto (2005, 2008) that:

I) $\{Z_t\}$ is exogenous in the sense that there is no feedback of $\{X_t\}$ towards it.

II) $\{Z_t\}$ is a homogeneous Markov chain of order p, $p \geq 1$, with initial distribution $F_0(z, \boldsymbol{\theta}_z)$ and kernel distribution $F_p\left(z_t|z_{t-1}, \ldots,\ z_{t-p}, \boldsymbol{\theta}_z\right)$, where $\boldsymbol{\theta}_z$ is a parameter vector defined in an appropriate numerical space.

III) Those distributions have densities in the Lebesgue measure sense, were $f_0\left(z, \boldsymbol{\theta}_z\right)$ and $f_p\left(z_t|z_{t-1}, \ldots, z_{t-p}, \boldsymbol{\theta}_z\right)$ are the initial and kernel densities, respectively.

IV) The p dimensional Markov chain $\{\boldsymbol{Z}_t\}$ where $\{\boldsymbol{Z}_t\} = (Z_t, Z_{t-1}, \ldots,\ Z_{t-p+1})'$ for all $t > p-1$, has an invariant distribution $f_p\left(z, \boldsymbol{\theta}_{\boldsymbol{z}}\right)$. It is important to remark that a stationary distribution implies that the paths from Z_t are long term stable.

With the assumptions from II) to IV) it is described the dynamic stochastic behavior of $\{Z_t\}$.

One of the main characteristics of the TAR model is its likelihood function. To define it, let $\mathbf{y} = (\mathbf{x}, \mathbf{z})$ were $\mathbf{x}$ and $\mathbf{z}$ are the observed data for $\{X_t\}$ and $\{Z_t\}$ respectively, in the length period $t = 1, 2, \ldots, T$. Additionally, let $\boldsymbol{\theta}_z$ be the vector of parameters of the process $\{Z_t\}$ and $\boldsymbol{\theta}_x$ be the vector of all the nonstructural parameters, that is $\boldsymbol{\theta}_x = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_l, \mathbf{h})$ where $\mathbf{h} = (h^{(1)}, \ldots, h^{(l)})$ and $\boldsymbol{\theta}_j = (a_0^{(j)}, a_1^{(j)}, \ldots, a_{k_l}^{(j)})$ for $j = 1, \ldots, l$. Conditional on $l, r_1, r_2, \ldots, r_{l-1}, k_1, k_2, \ldots, k_l$ and $\mathbf{x}_k = (x_1, x_2, \ldots, x_k)$ where $k = \max\{k_1, \ldots, k_l\}$, the likelihood function is given by the following density function (Nieto, 2005):

$$f\left(\mathbf{y}|\boldsymbol{\theta}_x, \boldsymbol{\theta}_z\right) = f\left(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}_x, \boldsymbol{\theta}_z\right) f\left(\mathbf{z}|\boldsymbol{\theta}_x, \boldsymbol{\theta}_z\right), \tag{1.2}$$

where

$$f\left(\mathbf{z}|\boldsymbol{\theta}_x, \boldsymbol{\theta}_z\right) = f\left(\mathbf{z}_p|\boldsymbol{\theta}_z\right) f\left(z_{p+1}|\mathbf{z}_p; \boldsymbol{\theta}_z\right) \cdots f\left(z_T|\mathbf{z}_{T-1}; \boldsymbol{\theta}_z\right),$$

with $\mathbf{z}_p = (z_1, \ldots, z_p)$ and

$$f\left(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}_x, \boldsymbol{\theta}_z\right) = f\left(x_{k+1}|\mathbf{x}_k, \mathbf{z}, \boldsymbol{\theta}_x, \boldsymbol{\theta}_z\right) \cdots f\left(x_T|x_{T-1}, \ldots, x_1; \mathbf{z}, \boldsymbol{\theta}_x, \boldsymbol{\theta}_z\right).$$

Since $\{\varepsilon_t\}$ is Gaussian, we have

$$f\left(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}_x, \boldsymbol{\theta}_z\right) = (2\pi)^{\frac{-(T-k)}{2}} \left[\prod_{t=k+1}^{T} \left\{h^{(j_t)}\right\}^{-1}\right] \exp\left(-\frac{1}{2}\sum_{t=k+1}^{T} e_t^2\right),$$

where

$$e_t = \frac{x_t - a_0^{(j_t)} - \sum_{i=1}^{k_{j_t}} a_i^{(j_t)} x_{t-i}}{h^{(j_t)}},$$

and the sequence $\{j_t\}$ is the observed time series for the stochastic process $\{J_t\}$. $\{J_t\}$ is a sequence of indicator variables such that $J_t = j$ if $Z_t \in B_j$ for some $j = 1, \ldots, l$. We assume that there is no relation between $\boldsymbol{\theta}_x$ and $\boldsymbol{\theta}_z$ and that the marginal likelihood function of $\mathbf{x}$ does not depend on $\boldsymbol{\theta}_z$.

1.1.1. Model estimation

The identification, estimation and validation of the TAR model is based on the Markov chain Monte Carlo (MCMC) methods and the Bayesian methodology proposed by Nieto (2005), when dealing with complete data according to Hoyos (2006)[1]. Thus, to identify the TAR model, we follow the next steps.

Step 1. Select a maximum number of regimes l_0, and then, the proper thresholds for each $l = 2, \ldots, l_0$, using the minimization of the NAIC criterion[2]. Intermediate draws of the nonstructural parameters are generated for all possible combinations of autoregressive orders.

Step 2. Identify l using again intermediate draws of nonstructural parameters and autoregressive orders.

Step 3. Conditional on l, identify the autoregressive orders $k_1, \ldots, k_l$.

Now, according to Nieto (2005), in the estimation stage it is assumed that the structural parameters $l, r_1, r_2, \ldots, r_{l-1}; k_1, k_2, \ldots, k_l$ are known (identified), so conditional on them, we estimate the nonstructural parameters of the TAR model using Gibbs sampling. For complete time series, we must calculate the conditional density $p\left(\boldsymbol{\theta}|x, \boldsymbol{z}\right)$, where $\boldsymbol{\theta}$ is the vector of parameters of the process $\{X_t\}$ and $\{Z_t\}$. This conditional density is obtained by computing the full conditional densities for the unknown parameters $a_i^{(j)}$ and $h^{(j)}\,(j = 1, \ldots, l; i = 0, 1, \ldots, k_j)$ and the parameters of the distribution of $\{Z_t\}$.

In that sense, let $\boldsymbol{\theta} = (\boldsymbol{\theta}_x, \boldsymbol{\theta}_z)$ be the vector of total unknown parameters in the TAR model, with $\boldsymbol{\theta}_z$ the vector of parameters of the process $\{Z_t\}$ and $\boldsymbol{\theta}_x = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_l, \mathbf{h})$ the vector of all the nonstructural parameters, where $\mathbf{h} = (h^{(1)}, \ldots, h^{(l)})$ and $\boldsymbol{\theta}_j = (a_0^{(j)}, a_1^{(j)}, \ldots, a_{k_l}^{(j)})$, for $j = 1, \ldots, l$.

Thus, following Hoyos (2006), it must be computed the full conditional densities: i) $p\left(\boldsymbol{\theta}_j|\boldsymbol{\theta}_i; i \neq j; \mathbf{h}, \boldsymbol{\theta}_z, \boldsymbol{x}, \boldsymbol{z}\right)$ for $j = 1, \ldots l$, ii) $p\left(h^{(j)}|h^{(i)}; i \neq j; \boldsymbol{\theta}_z, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_l, \boldsymbol{x}, \boldsymbol{z}\right)$ for $j = 1, \ldots, l$, and iii) $p\left(\boldsymbol{\theta}_z|\boldsymbol{\theta}_x, \boldsymbol{x}, \boldsymbol{z}\right)$. It is assumed *a priori* that the parameters among regimes are independent, $\boldsymbol{\theta}_j$ and $h^{(j)}$ are independent, and $\boldsymbol{\theta}_z$ and $\boldsymbol{\theta}_x$ are also independent (Nieto, 2005). From those full conditional distributions, we extract draws for running the Gibbs sampling.

Finally, for model checking, it is used the standardized pseudo residuals proposed by Nieto (2005)[3], who also suggested to use the CUSUM and CUSUMSQ charts for checking heteroscedasticity in $\{\varepsilon_t\}$ and model specification. Additionally, it is used, following Tsay

[1] See Nieto (2005) and Hoyos (2006) for more details.

[2] The NAIC criterion of Tong (1990) is a normalized AIC criterion, where the AIC criterion is divided by the effective number of observations. This criterion is defined as $NAIC = \frac{\sum_{j=i}^{l} AIC_j}{\sum_{j=i}^{l} n_j}$, where AIC_j and n_j are respectively the AIC criterion and the number of observations in the jth regimen, and l is the number of regimes.

[3] For each $t = 1, \ldots, T$, let

$$\hat{e}_t = \frac{\left(X_t - X_{t|t-1}\right)}{h^{(j)}},$$

if $Z_t \in B_j$ for some $j\,(j;\ j = 1, \ldots, l)$, where $X_{t|t-1} = a_0^{(j)} + \sum_{i=1}^{k_j} a_i^{(j)} X_{t-i|t-1}$ is the one-step ahead predictor of X_t. The process $\{\hat{e}_t\}$ is called by Nieto (2005) standardized pseudo residuals.

(1998), the partial autocorrelation function for checking no serial correlations in the residuals.

1.1.2. Forecasting procedure

Predictive function

Nieto (2008) develops a forecasting procedure for a TAR model based on the Bayesian analysis and the quadratic loss function criterion, with which the best prediction for the variable X_{T+h} is obtained by means of the conditional expectation $E\left(X_{T+h}|\boldsymbol{x}_T, \boldsymbol{z}_T\right)$, where $\boldsymbol{x}_T = (x_1, \ldots,\ x_T)$ and $\boldsymbol{z}_T = (z_1, \ldots,\ z_T)$ are the observations of the respectively variables X_t and Z_t, respectively for $t = 1, \ldots, T$, with T being the length of the sample period and $h \geq 1$ the forecast horizon.

However, Nieto's (2008) predictive distributions do not contemplate the uncertainty in the nonstructural parameters. Thus, in this thesis we find the Bayesian predictive distribution for the TAR model using the joint conditional predictive distribution. This methodology, which is commonly used in the literature[4], is a different proposal than that of Vargas (2012), reducing the complexity in the definition of this author, and making easier the computation of the Bayesian predictive distribution for the TAR model. In that sense, the joint predictive distribution for the TAR model, proposed in this thesis, is a different and a new contribution to the literature. Under this finding, we are going to evaluate the forecasting performance of the TAR model using economic time series.

Bayesian Predictive function

The Bayesian predictor of X_{T+h}, under the quadratic loss function criterion, is the conditional expectation $E\left(X_{t+h}|\boldsymbol{x}_T, \boldsymbol{z}_T\right), h \geq 1$. However, the analytical expression of this conditional expectation is not always easy to obtain for nonlinear models. In that sense, we focus on the joint conditional predictive distribution $p\left(x_{T+1}, ..., x_{T+h}, z_{T+1}, ..., z_{T+h}|\boldsymbol{x}_T, \boldsymbol{z}_T\right)$ from which the marginal predictive distributions $p\left(x_{T+h}|\boldsymbol{x}_T, \boldsymbol{z}_T\right)$ can be obtained. We obtain the following Proposition.

Proposition 1.1. *Under the assumptions specified in equation (1.1) and assuming that i) Z_{T+i} and X_{T+j} are independent for all $i > j \geq 0$, conditional on $\boldsymbol{x}_{T+j-1}$ and $\boldsymbol{z}_{T+j}$, with the convention that the conditioning set is only $\boldsymbol{z}_T$ when $j = 0$, and ii) the set $\{Z_{T+h-1}, \ldots,\ Z_{T+1}\}$ is independent of $\boldsymbol{x}_T$ conditional on z_{T+h} and $\boldsymbol{z}_T$, then, for each $h \geq 1$, the joint predictive density of $X_{T+1}, \ldots, X_{T+h}$ and $Z_{T+1}, \ldots, Z_{T+h}$ given $\boldsymbol{x}_T$ and $\boldsymbol{z}_T$, is*

$$p\left(x_{T+1}, \ldots, x_{T+h}, z_{T+1}, \ldots, z_{T+h}|\boldsymbol{x}_T, \boldsymbol{z}_T\right) = \int p\left(x_{T+1}, \ldots, x_{T+h}, z_{T+1}, \ldots, z_{T+h}|\boldsymbol{x}_T, \boldsymbol{z}_T, \boldsymbol{\theta}_x,\right) p\left(\boldsymbol{\theta}_x|\boldsymbol{x}_T, \boldsymbol{z}_T\right) d\boldsymbol{\theta}_x, \quad (1.3)$$

[4] Among this forecasting literature, we mention the study of Geweke and Terui (1993), who analyze the forecast procedure for a SETAR model, and obtain the Bayesian h-step ahead forecast based on the joint predictive density function. Calderón (2014) also uses this approach, in order to find the predictive distributions of the Multivariate TAR model.

where $p(\boldsymbol{\theta}_x|\boldsymbol{x}_T,\boldsymbol{z}_T)$ *is the posterior distribution of all the nonstructural parameters of the TAR model, which is obtained following Hoyos (2006), and*

$$\begin{aligned} &p(x_{T+1},\dots,x_{T+h},z_{T+1},\dots,z_{T+h}|\boldsymbol{x}_T,\boldsymbol{z}_T,\boldsymbol{\theta}_x) \\ &=\prod_{i=1}^{h} p(x_{T+i}|\boldsymbol{x}_{T+i-1},\boldsymbol{z}_{T+i},\boldsymbol{\theta}_x)\, p(z_{T+i}|\boldsymbol{z}_{T+i-1}), \end{aligned} \tag{1.4}$$

with $\boldsymbol{x}_{T+i-1}=(\boldsymbol{x}_T,x_{T+1},\dots,x_{T+i-1})$ *and* $\boldsymbol{z}_{T+i-1}=(\boldsymbol{z}_T,z_{T+1},\dots,z_{T+i-1})$.

Proof. Notice that the Bayesian predictive distribution in equation (1.3) is based on the definition in equation (A.2) in the Appendix A. Therefore,

$$\begin{aligned} &p(x_{T+1},\dots,x_{T+h},z_{T+1},\dots,z_{T+h}|\boldsymbol{x}_T,\boldsymbol{z}_T) \\ &=\int p(x_{T+1},\dots,x_{T+h},z_{T+1},\dots,z_{T+h},\boldsymbol{\theta}_x|\boldsymbol{x}_T,\boldsymbol{z}_T)\,d\boldsymbol{\theta}_x \\ &=\int p(x_{T+1},\dots,x_{T+h},z_{T+1},\dots,z_{T+h}|\boldsymbol{x}_T,\boldsymbol{z}_T,\boldsymbol{\theta}_x)\,p(\boldsymbol{\theta}_x|\boldsymbol{x}_T,\boldsymbol{z}_T)\,d\boldsymbol{\theta}_x. \end{aligned}$$

Now, we have that:

$$\begin{aligned} &p(x_{T+1},\dots,x_{T+h},z_{T+1},\dots,z_{T+h}|\boldsymbol{x}_T,\boldsymbol{z}_T,\boldsymbol{\theta}_x) \\ &=\prod_{i=1}^{h} p(x_{T+i}|\boldsymbol{x}_{T+i-1},\boldsymbol{z}_{T+i},\boldsymbol{\theta}_x)\, p(z_{T+i}|\boldsymbol{x}_{T+i-1},\boldsymbol{z}_{T+i-1},\boldsymbol{\theta}_x), \end{aligned}$$

which is defined under the assumptions of the TAR model presented in Chapter 1 and Proposition 1.1. Additionally, under condition ii) in Proposition 1.1, we have that

$$p(z_{T+i}|\boldsymbol{x}_{T+i-1},\boldsymbol{z}_{T+i-1},\boldsymbol{\theta}_x)=p(z_{T+i}|\boldsymbol{z}_{T+i-1}),$$

which give us the expression in equation (1.3). ■

On the other hand, in order to forecast the threshold variable $\{Z_{T+h}\}$, Nieto (2008) finds that:

$$\begin{aligned} p(z_{T+h}|\boldsymbol{z}_T)=&\int\cdots\int p(z_{T+h}|z_{T+h-1},\dots,z_{T+1},\boldsymbol{z}_T) \\ &\times p(z_{T+h-1}|z_{T+h-2},\dots,z_{T+1},\boldsymbol{z}_T)\times\cdots\times p(z_{T+1}|\boldsymbol{z}_T)\,dz_{T+1}\cdots z_{T+h-1}. \end{aligned} \tag{1.5}$$

From the above Proposition we observe, as Nieto (2008), that the densities in equation (1.4) satisfy that: i) $p(z_{T+i}|\boldsymbol{z}_{T+i-1})$ is the kernel density of the Markov chain $\{Z_t\}$, and ii) $p(x_{T+i}|\boldsymbol{x}_{T+i-1},\boldsymbol{z}_{T+i},\boldsymbol{\theta}_x)$ is a normal distribution with mean $a_0^{(j)}+\sum_{m=1}^{k_j} a_m^{(j)} x_{T+i-m}$ and variance $\left[h^{(j)}\right]^2$ if $z_{T+i}\in B_j$ for some $j=1,\dots,l$ and for $i=1,\dots,h$.

To draw samples of the distribution in equation (1.3), we define the following steps for the *ith* iteration:

Step 1. Extract a random draw $\boldsymbol{\theta}_x^{(i)}$ from $p(\boldsymbol{\theta}_x|\boldsymbol{x}_T,\boldsymbol{z}_T)$.

Step 2. Extract a random draw $z_{T+1}^{(i)}$ from $p(z_{T+1}|\boldsymbol{z}_T)$.

Step 3. Extract a random draw $x_{T+1}^{(i)}$ from $p\left(x_{T+1}\middle| z_{T+1}^{(i)}, \boldsymbol{x}_T, \boldsymbol{z}_T, \boldsymbol{\theta}_x^{(i)}\right)$.

Step 4. Extract a random draw $z_{T+2}^{(i)}$ from $p\left(z_{T+2}\middle| z_{T+1}^{(i)}, \boldsymbol{z}_T\right)$.

Step 5. Extract a random draw $x_{T+2}^{(i)}$ from $p\left(x_{T+2}\middle| z_{T+1}^{(i)}, z_{T+2}^{(i)}, x_{T+1}^{(i)}, \boldsymbol{x}_T, \boldsymbol{z}_T, \boldsymbol{\theta}_x^{(i)}\right)$.

Step 6. Repeat until it is extracted a random draw $z_{T+h}^{(i)}$ from $p\left(z_{T+h}\middle| z_{T+1}^{(i)}, \ldots, z_{T+h-1}^{(i)}, \mathbf{z}_T\right)$, and then, a random draw $x_{T+h}^{(i)}$ from $p\left(x_{T+h}\middle| z_{T+1}^{(i)}, \ldots, z_{T+h}^{(i)}, x_{T+1}^{(i)}, \ldots, x_{T+h-1}^{(i)}, \boldsymbol{x}_T, \boldsymbol{z}_T, \boldsymbol{\theta}_x^{(i)}\right)$.

Hence, we extract random draws recursively until we have the sets $\left\{x_{T+h}^{(i)}\right\}(h \geq 1; i = 1, \ldots, N)$ and $\left\{z_{T+h}^{(i)}\right\}(h \geq 1; i = 1, \ldots, N)$, N large enough. From those sets we can compute for X_{T+h} and Z_{T+h}: i) the mean of the predictive distribution, that is a numerical approximation to the point forecast, by averaging each of the $x_{T+l}\,(l = 1, \ldots, h)$ and $z_{T+l}\,(l = 1, \ldots, h)$ over the N replications, ii) the variance of the predictive distribution, that give us an approximation to the uncertainty of the forecast and, iii) the credible intervals for the point forecast.

As we can see, in this forecasting procedure using the Bayesian methodology, we incorporate the uncertainty of the unknown parameters in the predictive distributions.

1.2. Self-exciting threshold autoregressive model

The self-exciting threshold autoregressive (SETAR) model was introduced by Tong (1978) and Tong and Lim (1980), in which the threshold variable is the lagged variable X_{t-d} for some positive integer d. This model has been extensively analyzed, with the assumption that the number of regimes and the autoregressive orders are known. Hence, a stochastic process $\{X_t\}$ is a SETAR process if it follows the equation:

$$X_t = \Phi_0^{(j)} + \sum_{i=1}^{p_j} \Phi_i^{(j)} X_{t-i} + \varepsilon_t^{(j)}, \qquad \text{if } r_{j-1} < X_{t-d} \leq r_j, \tag{1.6}$$

where $j = 1, \ldots, k$ are the regimes, with k a positive integer, and the positive integer d is the delay parameter. The real numbers $-\infty = r_0 < r_1 < \ldots < r_k = \infty$ are the thresholds, $\Phi_i^{(j)}$ with $i = 1, \ldots, p_j; j = 1, \ldots, k$, are the coefficients and for each j, $\left\{\varepsilon_t^{(j)}\right\}$ is a sequence of independent and identically distributed Gaussian random variables with mean 0 and variance σ_j^2 (Tiao and Tsay, 1994; Tsay, 2005). The autoregressive orders of the time series in each regime are denoted by p_j.

We can also express the SETAR model in equation (1.6) as:

$$X_t = \sum_{j=1}^{l} \left(\Phi_0^{(j)} + \Phi_1^{(j)} X_{t-1} + \ldots + \Phi_{p_j}^{(j)} X_{t-p_j} + \varepsilon_t^{(j)}\right) I\left(r_{j-1} < X_{t-d} \leq r_j\right),$$

The SETAR model is a piecewise linear autoregressive model, but liable to move between regimes when the process crosses a threshold (Clements et al., 2003). As this model can produce limit cycles, time irreversibility and asymmetric behavior of a time series (Tsay, 1989; Tiao and Tsay, 1994), has been applied to several economic and financial time series (Tiao and Tsay, 1994; Montgomery et al., 1998; Clements et al., 2003, amog others).

1.2.1. Model estimation

We estimate the SETAR model based on the approach of Tsay (1989), which consists in the following steps.

Step 1. Specify a linear autoregressive model for the time series. Select a tentative autoregressive order p by means of the partial autocorrelation function of X_t or some information criteria, and the set of possible values for the delay parameter d.

Step 2. Fit arranged autoregressions for a given p and every element d, and evaluate the null hypothesis of linearity using the nonlinearity test $\hat{F}(p,d)$ proposed by Tsay (1989)[5]. Select the delay parameter d based on the minimum p-value of the F statistics.

Step 3. For a given p and d, locate the possible threshold values by using the scatterplots of the standardized residuals versus X_{t-d}, and the scatterplot of the t ratios of recursive estimates of an AR coefficient versus X_{t-d}.

Step 4. Redefine the autoregressive order and threshold values in each regime. We use the NAIC criterion of Tong (1990).

Step 5. Finally, estimate the model by means of linear autoregression techniques and check the model.

1.2.2. Forecasting procedure

The optimal one-step ahead forecast from the SETAR model is:

$$X_{T+1|T} = E\left(X_{T+1}|X_1, X_2, \ldots,\ X_T\right) = \Phi_0^{(j)} + \sum_{i=1}^{p_j} \Phi_i^{(j)} X_{T+1-i}, \qquad \text{if } r_{j-1} < X_{T+1-d} \leq r_j.$$

Nevertheless, when the forecast horizon h is greater than one period, an analytic expression for X_{T+h} is not available, so it is necessary to use simulation techniques such as Monte

[5] According to Cao and Tsay (1992, p. S170): *"To detect the threshold nonlinearity, Tsay (1989) proposed another F-test based on the arranged autoregression. The test consists of two steps. First, for a prespecified AR order p and a threshold lag d, fit recursively an arranged autoregression of order p of the series Y_t. Assuming that the recursion begins with the first b observations, calculate the standardized predictive residual $\hat{e}_t$ for $t > b$. Secondly, regress the predictive residual $\hat{e}_t$ on $(1, Y_{t-1,}, Y_{t-2}, \ldots, Y_{t-p})$, save the corresponding residuals $\hat{\varepsilon}_t$, and form the F-statistic*

$$\hat{F}(p,d) = \frac{\left(\sum \hat{e}_t^2 - \sum \hat{\varepsilon}_t^2\right)/(p+1)}{\sum \hat{\varepsilon}_t^2/(T-d-b-p-h)}$$

where $h = \max(1,\ p+1-d)$ and the summation is summing over t from $b+1$ to $T-d-h+1$. Under the null hypothesis that Y_t is an AR(p) process, the $\hat{F}(p,d)$ statistic is asymptotically an F distribution with degrees of freedom $p+1$ and $T-d-b-p-h$".

Carlo or Bootstrap to compute the conditional expectation $E\left(X_{T+h}|X_1,\ldots,\ X_{T+h-1}\right)$. In this study, we use the Monte Carlo method[6] to obtain the h-step ahead forecast. With this method, the h-step ahead forecast is simulated k times, with k some large number, and the value of the forecast is obtained by averaging these k repetitions (Franses and van Dijk, 2000).

Finally, Tong (1983) defines the sample MSE of the h-step ahead forecasts as:

$$var\left(h\right)=\frac{1}{\ell}\sum_{T_0=T+0}^{T+\ell-1}\left(X_{T_0+h}-X_{T_0+h|T_0}\right)^2, \qquad h=1,2,\ldots\ ,$$

where ℓ is the number of h-step ahead forecasts available in the forecasting subsample and T_0 is the forecast origin.

1.3. Smooth transition autoregressive model

The smooth transition autoregressive (STAR) model was first suggested by Chan and Tong (1986) and then developed by Teräsvirta and Anderson (1992) and Teräsvirta (1994). A stochastic process X_t follows a STAR model if it satisfies (Teräsvirta and Anderson, 1992):

$$X_t=\Phi_{10}+\sum_{i=1}^{p_1}\Phi_{1i}X_{t-i}+F\left(X_{t-d}\right)\left(\Phi_{20}+\sum_{i=1}^{p_2}\Phi_{2i}X_{t-i}\right)+\varepsilon_t, \tag{1.7}$$

where d is the delay parameter, $p_j; j=1,2$ is the autoregressive order of the jth regime, $\Phi_{ji}; j=1,2, i=1,2,\ldots,p_j$ are the coefficients and ε_t are iid sequences with mean 0 and variance σ^2. F is a transition function which is bounded between 0 and 1.

We consider two transition function: the logistic function,

$$F\left(X_{t-d}\right)=\left(1+\exp\left\{-\gamma\left(X_{t-d}-c\right)\right\}\right)^{-1}, \qquad \gamma>0, \tag{1.8}$$

and the exponential function,

$$F\left(X_{t-d}\right)=1-\exp\left\{-\gamma\left(X_{t-d}-c\right)^2\right\}, \qquad \gamma>0, \tag{1.9}$$

where γ and c are respectively the scale and location parameters. In that way, γ determines the smoothness of the change from one regimen to the other, c is interpreted as the threshold variable between both regimes, and X_{t-d} is assumed to be the transition variable (van Dijk et al., 2002).

The model in equation (1.7) with the transition function in equation (1.8) is the Logistic STAR (LSTAR), and with the transition function in equation (1.9) is the Exponential STAR (ESTAR). The LSTAR model can describe phases of contraction and expansion of an economy, where the transition from one phase to the other may be smooth, and the ESTAR model can explain an economy which moves from a high or low growth to a more normal growth, thus, the STAR family models are commonly used for modelling non linearities in business cycles. Because there is not a strong economic theory to choose between LSTAR or

[6] We use the Monte Carlo method taking into account that this method has been extensively used in the literature (Tiao and Tsay, 1994; Clements and Krolzig, 1998; Clements and Smith, 1999, among others).

ESTAR models, the choice of those models is based on the data (Teräsvirta and Anderson, 1992).

One characteristic of the LSTAR model is when γ becomes very large, given that the logistic function $F(X_{t-d})$ approaches the indicator function $I(X_{t-d} > c) = 1$ if $X_{t-d} > c$ and zero otherwise, and consequently, the change of $F(X_{t-d})$ from 0 to 1 becomes instantaneous. If that happens, the LSTAR model becomes a SETAR model with two regimes. When $\gamma \to 0$, the logistic function approximates a constant equal to 0.5, and, when $\gamma = 0$, the LSTAR model becomes a linear AR model with parameters $\Phi_j = (\Phi_{1j} + \Phi_{2j})/2,\ j = 0, 1, \ldots, p_j$ (van Dijk et al., 2002).

1.3.1. Model estimation

For the forecasting comparison analysis, we fit the Teräsvirta's (1994) traditional STAR model with two regimes, given that this model "(...) allows the business cycle indicator to alternate between (...) two different phases of the business cycle" (Teräsvirta and Anderson, 1992, p. S120). Hence, the specification of the STAR models we use is based on the approach of Teräsvirta and Anderson (1992) and Teräsvirta (1994), which consists on the following steps.

Step 1. Specify a linear autoregressive model for $\{X_t\}$.

Step 2. Perform the nonlinearity test for different values of d, and if it is rejected the null hypothesis of linearity, select the d with the minimum p-value of the F statistics[7].

Step 3. Choose between the LSTAR and the ESTAR models through a sequence of tests of nested hypothesis[8].

[7] According to Teräsvirta and Anderson (1992) and Teräsvirta (1994), the test of linearity against the STAR model, where d is assumed known, is $H_0 : \beta_{2j} = \beta_{3j} = \beta_{4j} = 0, j = 1, \ldots, p$, against the alternative that H_0 is not valid in the artificial regression $x_t = \beta_0 + \sum_{j=1}^{p} \beta_1 x_{t-j} + \sum_{j=1}^{p} \beta_2 x_{t-j} x_{t-d} + \sum_{j=1}^{p} \beta_3 x_{t-j} x_{t-d}^2 + \sum_{j=1}^{p} \beta_4 x_{t-j} x_{t-d}^3 + v_t$, where v_t is an *iid* sequence with zero mean and variance σ_v^2, and the coefficients $\beta_{ij}; i = 1, \ldots, 4, j = 1, \ldots, p$, are functions of the parameters $\Phi_{ji}; j = 1, 2; i = 1, 2, \ldots, p; \gamma, c$. The LM statistic based on the above artificial regression is calculated according to van Dijk et al. (2002).

[8] Teräsvirta and Anderson (1992) and Teräsvirta (1994) propose the selection of the models through a sequence of F tests within the artificial regression use in the nonlinearity test: i) $H_{01} : \beta_{4j} = 0$ *vs.* $H_{11} : \beta_{4j} \neq 0; j = 1, \ldots, p$, ii) $H_{02} : \beta_{3j} = 0 \,|\beta_{4j} = 0$ *vs.* $H_{12} : \beta_{3j} \neq 0| \,\beta_{4j} = 0; j = 1, \ldots, p$, and, iii) $H_{03} : \beta_{2j} = 0 \,|\beta_{3j} = \beta_{4j} = 0$ *vs.* $H_{13} : \beta_{2j} \neq 0| \,\beta_{3j} = \beta_{4j} = 0; j = 1, \ldots, p$, where $\beta_{ij}; i = 1, \ldots, 4, j = 1, \ldots, p$, are functions of the parameters $\Phi_{ji}; j = 1, 2; i = 1, 2, \ldots, p; \gamma, c$. Therefore: i) If H_{01} is not rejected and H_{02} is rejected, the best model is an ESTAR; ii) If H_{01} is rejected or H_{02} is not rejected, there is evidence in favor of a LSTAR model. Rejecting H_{02} is not too informative to choose between both models; iii) If H_{01} and H_{02} are not rejected and H_{03} is rejected, the model to choose is the LSTAR. However, not rejecting H_{03} and rejecting H_{02} often suggests an ESTAR model. Additionally, Teräsvirta (1994, p. 212) indicates that *"If the model is a LSTAR model, then typically H_{01} and H_{03} are rejected more strongly than H_{02}. For an ESTAR model, the situation may be expected to be the opposite. I propose the following decision rule. After rejecting the general null hypothesis, carry out the three F tests. If the p-value of F_3 (the test of H_{02}) is the smallest of the three, select an ESTAR model; if not, choose a LSTAR model".*

Step 4. Finally, once it has been specified d and the transition function $F(X_{t-d})$, estimate the parameters in the model by nonlinear least squares and check the model.

1.3.2. Forecasting procedure

The optimal one-step ahead forecast of X_{T+1} made at time T is obtained by the conditional expectation:

$$\begin{aligned} X_{T+1|T} &= E\left(X_{T+1}|X_1,\ldots,X_T\right) \\ &= \Phi_{10}+\sum_{i=1}^{p_1}\Phi_{1i}X_{T+1-i}+F\left(X_{T+1-d}\right)\left(\Phi_{20}+\sum_{i=1}^{p_2}\Phi_{2i}X_{T+1-i}\right), \end{aligned}$$

where $F(X_{t-d})$ is the transition function.

As it was mentioned before, for forecast horizons greater than one, $h>1$, an analytic expression for X_{T+h} is not available, thus we use the Monte Carlo method to approximate the conditional expectation.

The MSE of the h-step ahead forecasts are:

$$var\left(h\right)=\frac{1}{\ell}\sum_{T_0=T+0}^{T+\ell-1}\left(X_{T_0+h}-X_{T_0+h|T_0}\right)^2, h=1,2,\ldots,$$

where ℓ is the number of h-step ahead forecasts available in the forecasting subsample and T_0 is the forecast origin (Deschamps, 2008).

1.4. Markov-switching autoregressive model

The Markov-switching autoregressive (MSAR) model that we consider is based on the model of Hamilton (1989, 1994), which is widely used in the literature and is given by:

$$X_t=\Phi_{s_t,0}+\sum_{i=1}^{p}\Phi_{s_t,i}X_{t-i}+\varepsilon_t\ , \tag{1.10}$$

where ε_t is a sequence of iid normal random variables with zero-mean and variance σ^2 and, the regimen variable s_t follows an m-state Markov chain, with s_t independent of ε_t for all t. Thus, both the intercept and the autoregressive parameters depend upon an unobservable $s_t \in \{1,2,\ldots,m\}$ with m an integer (Clements and Krolzig, 1998).

A more general specification of the MSAR model allows the intercept, the autoregressive parameters and the variance of the model to switch. An example of this case is presented by Tsay (2005), where the transition between states is governed by a hidden two-state Markov chain, and the time series X_t follows a MSAR model if it satisfies:

$$X_t=\begin{cases}\Phi_{10}+\sum_{i=1}^{p}\Phi_{1i}X_{t-i}+\varepsilon_{1t}, & \text{if } s_t=1\\ \Phi_{20}+\sum_{i=1}^{p}\Phi_{2i}X_{t-i}+\varepsilon_{2t}, & \text{if } s_t=2,\end{cases} \tag{1.11}$$

where $\varepsilon_{it}, i = 1, 2$ are *iid* random variables with zero-mean and finite variance and are independent of each other. s_t assumes values in $\{1, 2\}$ and is the unobserved first-order Markov chain with transition probabilities $P(s_t = i|s_{t-1} = j) = p_{ij}$, with $i, j = 1, 2$, where p_{ij} is the probability of moving from state j to state i and $\sum_{i=1}^{2} p_{ij} = 1$. The unconditional probability that the two-states process will be in regime 1 at any given date is given by $P(S_t = 1) = \frac{1-p_{22}}{2-p_{11}-p_{22}}$, and the unconditional probability that the process will in regime 2 is only 1 minus this value (Hamilton, 1994).

1.4.1. Model estimation

According to Teräsvirta (2006), the estimation of a MSAR model is more complicated than the estimation of another model such as those mentioned above, because this model has two unobservable processes: the Markov chain that indicates the regime and the error process. Therefore, we set the number of regimes in two, taking into account that this number of regimes is the most used in the literature, related to different economic time series, and following the suggestions of (Teräsvirta, 2006, p. 431), who argues that *"(...) testing linearity against the MS-AR alternative is computationally demanding. (...) Most practitioners fix the number of regimes in advance, and the most common choice appears to be two regimes"*.

Once it is defined the number of regimes, we estimate the MSAR model under Hamilton's (1994) approach, which is by means of maximum likelihood estimation. Thus, through this estimation technique, the intercepts, autoregressive parameters and transition probabilities governing the Markov chain of the unobserved regimes are estimated (Clements and Krolzig, 1998). Finally, the model is checked.

1.4.2. Forecasting procedure

According to Teräsvirta et al. (2010), forecasts from the MSAR model can be obtained analytically by a sequence of linear operations. Thus, the 1-step ahead forecast, given the information and including the forecast of $p_{j,T+1}$, is:

$$X_{T+1} = E\left(X_{T+1}|X_1, \ldots,\ X_T\right) = \sum_{j=1}^{m} p_{j,T+1|T}\left(\Phi_{j0} + \Phi_{j1}X_T + \ldots + \Phi_{jp}X_{T+1-p}\right),$$

where $p_{j,T+1|T}$ is a forecast of $p_{j,T+1} = P(s_{T+1} = j|X_T)$ from $\mathbf{p}'_{T+1|T} = \boldsymbol{a}'_T\mathbf{P}$, where $\boldsymbol{a}_T = (p_{1,T}, \ldots, p_{m,T})'$ with $p_{j,T} = P\left(s_T = j|X_1, \ldots, X_T\right); j = 1, \ldots, m$, m is the number of states or regimes, and $\mathbf{P} = [p_{ij}]$ is the matrix of transition probabilities (Hamilton, 1993; Teräsvirta et al., 2010).

The h-step ahead forecast for $h > 1$ can be obtained as:

$$X_{T+h} = E\left(X_{T+h}|X_1, \ldots,\ X_T\right) = \sum_{j=1}^{m} p_{j,T+h|T}\left(\Phi_{j0} + \Phi_{j1}X_{T+h-1} + \ldots + \Phi_{jp}X_{T+h-p}\right),$$

where the forecast $p_{j,T+h|T}$ is calculated from $\mathbf{p}'_{T+h|T} = \boldsymbol{a}'_T\mathbf{P}^h$ with

$$\mathbf{p}_{T+h|T} = \left(p_{1,T+h|T}, \ldots, p_{m,T+h|T}\right)'$$

and $p_{j,T+h} = P\left(s_{T+h} = j|X_1, \ldots, X_T\right); j = 1, \ldots, m$ (Hamilton, 1993; Teräsvirta et al., 2010). The row i, column j element of $\mathbf{P}^h$ defines the probability that the process will be in state i at date $t+h$ given that it is currently in state j (Hamilton, 1993, 1994).

The MSE of the h-step ahead forecasts are:

$$var\left(h\right) = \frac{1}{\ell} \sum_{T_0=T+0}^{T+\ell-1} \left(X_{T_0+h} - X_{T_0+h|T_0}\right)^2, h = 1, 2, \ldots,$$

Given that forecasts from the MSAR model can be calculated analytically, the forecast region that we have to construct is through the interval symmetric about the mean, which is usually constructed using the mean plus or minus a multiple of the standard deviation, that is (Hyndman, 1995):

$$R_\alpha = \left(X_{T+h|T} - \omega, X_{T+h|T} + \omega\right),$$

where ω is chosen such that $P\left(X_{T+h|T} \in R_\alpha|X_1, \ldots, X_T\right) = 1 - alpha$.

1.5. Autoregressive model

A stochastic process X_t follows and AR(p) model if it satisfies:

$$X_t = \phi_1 X_{t-1} + \ldots + \Phi_p X_{t-p} + \varepsilon_t \tag{1.12}$$

where $\{\varepsilon_t\}$ is a white noise series, $\Phi_1, \ldots, \Phi_p$ are the autoregressive parameters and p is a non-negative integer number that indicates the order of the autoregressive component.

Now, $\Phi\left(B\right) X_t = \varepsilon_t$ will define a stationary process if the characteristic equation $\Phi\left(B\right) = 0$ has all their roots outside the unit circle. The AR(p) model in equation (1.12) can be written in the equivalent form

$$\left(1 - \Phi_1 B^1 - \Phi_2 B^2 - \ldots - \Phi_p B^p\right) X_t = \varepsilon_t,$$

or

$$\Phi\left(B\right) X_t = \varepsilon_t,$$

with $\Phi\left(B\right) X_t = \left(1 - \Phi_1 B^1 - \Phi_2 B^2 - \ldots - \Phi_p B^p\right) X_t$, where B is the backshift operator such that $B^i X_t = X_{t-i}$.

1.5.1. Model estimation

The estimation of the linear model is based on the procedure of Box et al. (2016) consisting of model identification, estimation and diagnostic checking.

Step 1. Use the autocorrelation function and partial autocorrelation function to identify the degree of differencing d and the possible order p. Then, use some information criteria to determine the order p of the model that minimizes the information criteria value.

Step 2. Estimate the parameters of the model under the least squares criterion.

Step 3. Check for possible model inadequacy. When the model is adequate, the residual series should behave as a white noise.

1.5.2. Forecasting procedure

The h-step ahead forecast from the AR(p) model in equation (1.12), based on the minimum squared error loss function, is the conditional expectation $E\left(X_{T+h}|F_T\right)$ with F_T the information available at time T, which can be calculated by means of:

$$X_{T+h|T} = E\left(X_{T+h}|F_T\right) = \phi_1 X_{T+h-1} + \ldots + \Phi_p X_{T+h-p}.$$

This forecast can be computed recursively. The associated h-step ahead forecast error is $e_{T+h|T} = X_{T+h} - X_{T+h|T}$.

Chapter 2

Evaluation criteria

In this Chapter, we explain the statistics that are commonly used in the literature to evaluate and compare the forecasting performance of a TAR model with the forecasting performance of the competing models.

To this aim, we first evaluate the individual properties for each model and each horizon, of unbiased forecasts with uncorrelated forecasts errors, which are based on Schuh's (2001) suggestions about considering these properties as the basic principles of the economic forecasting, that are used to evaluate the performance of forecasts. Then, we compare forecasts from the TAR model with those from the other models, based on the relative mean square error, the comparison tests of Diebold and Mariano (1995) and Harvey et al. (1997), and the forecast encompassing tests of Chong and Hendry (1986), Ericsson (1992) and Harvey et al. (1998). We also evaluate the shape of the predictive distributions in order to find if they handle the economic cycles.

2.1. Unbiased forecasts

Forecasts should be unbiased. Following Clements (2005), bias is tested by whether the sample mean of the forecast errors $e_{t_0+h|t_0} = y_{t_0+h} - y_{t_0+h|t_0}$, where h is the forecast horizon, $y_{t_0+h|t_0}$ is the forecast in period $t_0 + h$ made at time t_0, and y_{t_0+h} is the actual value, is significantly different from zero.

This principle is evaluated by means of the test of Holden and Peel (1989), who evaluate $\beta_0 = 0$ in the following regression:

$$y_{t_0+h} - y_{t_0+h|t_0} = \beta_0 + \omega_{t_0+h},$$

where β_0 is a constant, ω_{t_0+h} is an error term that is assumed to be from a series of independent and identically distributed normal random variables with zero mean, and under the null hypothesis is equal to the forecast error. Then, it is compared the t-statistic of the null hypothesis of unbiasedness, that is $\beta_0 = 0$ to the Student's t distribution.

2.2. Uncorrelated forecasts

1-step ahead forecast errors should not be correlated with past errors (Schuh, 2001; Melo and Núñez, 2004). Hence, correlation is evaluated by means of the Ljung-Box Q statistic $Q = n\,(n+2)\sum_{j=1}^{k}\frac{\hat{\rho}_j^2}{n-j}$, where n is the sample size (number of observations) and $\hat{\rho}_j$ is the estimate of the autocorrelation coefficient at lag j and k is the number of lags being tested. Under the null hypothesis of no serial correlation, Q is asymptotically distributed as a $\chi^2_{1-\alpha,k}$, where $1-\alpha$ is the quantile of the chi-squared distribution with k degrees of freedom.

2.3. Relative mean square error

In applications, it is often chosen one of the following three measures, which are usually used to evaluate the performance of the point forecasts: the mean square error (MSE), the mean absolute deviation (MAD) and the mean absolute percentage error (MAPE) (Tsay, 2005). Following the literature review, we use the MSE, which is defined as:

$$MSE\,(h) = \frac{1}{\ell}\sum_{j=t_0+0}^{t_0+\ell-1}\left(e_{j+h|j}\right)^2, \tag{2.1}$$

where ℓ is the number of h-step ahead forecasts available in the forecasting subsample, t_0 is the forecast origin, and

$$e_{j+h|j} = y_{j+h} - y_{j+h|j}, \tag{2.2}$$

with $y_{j+h|j}$ the forecast of y_{j+h} made a time j, and y_{j+h} the actual value, is the forecast error.

In the literature, it is common to compare the relative MSE, which is defined as the MSE given by a model divided by the MSE of the benchmark model. If the relative MSE is greater than 1, then the MSE of the model is greater than the MSE of the benchmark model. But, if the relative MSE is less than 1, this model has a smaller MSE than the benchmark model. Following the literature review, we use this measure for the out-of-sample forecasting comparison.

2.4. Theil's U statistic

Theil (1966) proposes the following measure of forecast performance:

$$U = \frac{\sqrt{\frac{1}{\ell}\sum_{j=t_0+0}^{t_0+\ell-1}\left(e_{j+h|j}\right)^2}}{\sqrt{\frac{1}{\ell}\sum_{j=t_0+0}^{t_0+\ell-1}\left(y_{j+h} - y_{j-1+h}\right)^2}}, \tag{2.3}$$

where $e_{j+h|j}$ and y_{j+h} are those defined in equation (2.2).

Theil's U statistic compares the root mean square error (RMSE), that is $\sqrt{MSE}$, of the forecasts, with the RMSE of the "naïve model", where the latter is the model with no changes, that is to say, the forecast is the last observed value (Granger and Newbold, 1986). Thus, if $U = 0$, we have the perfect forecast. If $U < 1$, forecasts from the evaluated model are more accurate than those from the naïve model, but if $U \geq 1$, then forecasts from the evaluated model are as good as or worse than those from the naïve model.

2.5. Diebold-Mariano test

The test of Diebold and Mariano (1995) is widely used in the literature to compare the point forecast from two competing models. They propose a test to evaluate the equality of the MSE from two competing forecasts, as a measure of forecast accuracy. The null hypothesis indicates equal MSE for the two forecasts (equal predictive accuracy), against the alternative that one model has a smaller MSE (better predictive accuracy) than the other model[9].

Thus, having generated ℓ h-step ahead forecasts from two different models m_1 and m_2, we have two sets of forecast errors $\left\{e^{(m_i)}_{j+h|j}\right\}$ $(j = t_0 + 0, ..., t_0 + \ell - 1; i = 1, 2)$. The hypothesis of equal forecast accuracy can be represented as the mean of the difference between the MSE of the considered models:

$$\bar{d} = \frac{1}{\ell} \sum_{j=t_0+0}^{t_0+\ell-1} d_j, \tag{2.4}$$

where $d_j = \left[e^{(m_1)}_{j+h|j}\right]^2 - \left[e^{(m_2)}_{j+h|j}\right]^2$. $\bar{d}$ has an approximate asymptotic variance of (Harvey, 1997):

$$V\left(\bar{d}\right) \approx \frac{1}{\ell}\left[\gamma_0 + 2\sum_{k=1}^{h-1}\gamma_k\right], \tag{2.5}$$

where γ_k is the kth autocovariance of $\{d_j\}$, which can be estimated by (Harvey, 1997):

$$\hat{\gamma}_k = \frac{1}{\ell}\sum_{j=k+1}^{\ell}\left(d_j - \bar{d}\right)\left(d_{j-k} - \bar{d}\right). \tag{2.6}$$

Then, the Diebold and Mariano (1995) statistic is:

$$DM = \frac{\bar{d}}{\sqrt{\hat{V}\left(\bar{d}\right)}}, \tag{2.7}$$

where $\hat{V}\left(\bar{d}\right)$ is obtained by substituting equation (2.6) in equation (2.5), and under the null hypothesis, this test has an asymptotic standard normal distribution.

Given that the alternative hypothesis indicates that forecasts from one model (let us say, m_1) are better than those from the other model (let us say, m_2), we are going to evaluate separately, the null hypothesis of equal predictive accuracy against both the alternative

[9] Moreover, positive (negative) values for the DM test show that one model has a bigger (smaller) MSE than the other model (Bradley and Jansen, 2004).

hypothesis when forecasts from m_2 are better than those from m_1, and the other alternative hypothesis when forecast from m_1 are better than those from m_2.

Although Diebold and Mariano (1995) show that the performance of their test statistic is good for small samples, autocorrelated forecasts errors and forecast errors with non-normal distributions, this test statistic could be over sized for small number of forecasts and more than one-step ahead forecasts. Then, Harvey et al. (1997) propose a modified DM test (MDM), which improves the finite sample performance of the test, by means of:

$$MDM = \left[\frac{\ell + a - 2h + \ell^{-1} h \left(h - 1 \right)}{\ell} \right] DM, \tag{2.8}$$

where DM is the Diebold and Mariano (1995) test statistic in equation (2.7). This test statistic performs much better at all forecast horizons and when the forecast errors are autocorrelated or have non-normal distribution (Harris and Sollis, 2003). In this case, we also evaluate both alternative hypotheses, when forecasts from one model are better than those from the other model and vice versa.

2.6. Forecast encompassing test

Forecast encompassing is when forecasts from one model use all the relevant information readily available and no other model or information should improve those forecasts. In other words, forecasts from a model contains useful information that is missing in forecasts from another model.

One of the most common test statistic to evaluate forecast encompassing is the one proposed by Chong and Hendry (1986) (CH), who develop a test by regressing the h-step ahead forecast error from the evaluated model m_1, $\left(e^{(m_1)}_{j+h|j} \right)$, on the h-step ahead forecast from the competing model m_2, $\left(\hat{y}^{(m_2)}_{j+h|j} \right)$:

$$e^{(m_1)}_{j+h|j} = \alpha \hat{y}^{(m_2)}_{j+h|j} + \varepsilon_{j+h}, \tag{2.9}$$

for $j = t_0 + 0, ..., t_0 + \ell - 1$ with t_0 the forecast origin. The null hypothesis $\alpha = 0$ indicates that forecasts from m_1 encompass those from m_2. It is used a t-statistic for testing $\alpha = 0$, and under the null hypothesis the t-statistic has an asymptotic normal distribution (Harris and Sollis, 2003). According with Clements and Hendry (2002, p. 275) *"the rationale is that, optimally, the error to be made by forecaster 1 should be uncorrelated with all information available when the forecast is made, while correlation of that error with forecast 2 implies that information in the latter is of some value in anticipating the former"*.

Later, Ericsson (1992) (ER) tests the null hypothesis of forecast encompassing by regressing the forecast error from m_1 on the difference in the forecast errors from m_1 and m_2:

$$e^{(m_1)}_{j+h|j} = \alpha \left(e^{(m_1)}_{j+h|j} - e^{(m_2)}_{j+h|j} \right) + \varepsilon_{j+h}, \tag{2.10}$$

for $j = t_0 + 0, ..., t_0 + \ell - 1$. The null hypothesis $\alpha = 0$ indicates that forecasts from m_1 encompass those from m_2. It is used a t-statistic to test $\alpha = 0$, and under the null hypothesis, the t-statistic has an asymptotic normal distribution (Harris and Sollis, 2003).

Harvey et al. (1998) (HLN) propose the following test:

$$HLN = \frac{\bar{c}}{\sqrt{var(\bar{c})}}, \tag{2.11}$$

where $\bar{c} = \ell^{-1} \sum_{j=t_0+0}^{t_0+\ell-1} c_j$ and $c_j = e^{(m_1)}_{j+h|j} \left(e^{(m_1)}_{j+h|j} - e^{(m_2)}_{j+h|j} \right)$. Under the null hypothesis of forecast encompassing, that test statistic has an asymptotic standard normal distribution (Harris and Sollis, 2003).

In addition, Harvey et al. (1998, p. 254) argue that *"given a record of past forecast errors, it is natural to test for forecast encompassing through a simple least squares regression approach. For one-step-ahead prediction, it may be reasonable to assume, at least as a reference case, that the forecast errors are not autocorrelated so that the regression based test is very straight forward to implement. If forecasts are for longer horizons, however, errors from optimal forecasts will be autocorrelated, for which some allowance is necessary in the development of valid tests"*.

2.7. Graphical analysis

By using graphical analyses, we compare the TAR model with the SETAR and STAR models based on their ability to describe the predictive distributions of the economic time series, in terms of their capacity to handle economic cycles. We make this comparison exercise based on Wong and Li (2000), who study the annual record of the numbers of Canadian lynx trapped in the Mackenzie River district of north-west Canada, from 1821 to 1934. They find that the shapes of the predictive distributions of the mixture autoregressive model change over time, being unimodal when the time series is ascending to a peak, and bimodal when the series is descending to a trough. They explain this behavior, saying that the values of the troughs are more variable than the values of the peaks, which have either large or small values.

In that sense, The MSAR and AR models are not considered in this analysis, because forecasts from these models are computed analytically, where it is assumed Gaussian innovation terms, and under this assumption, both the marginal and the conditional distributions of the time series are Gaussian. Therefore, the predictive distributions of the MSAR and AR models are unimodal, making not possible identifying shape changing characteristics in their predictive distributions, which is the objective of this analysis.

Chapter 3

Selection of the data

This Chapter is devoted to present a description of the different macroeconomic variables used in the forecasting evaluation. We use the following United States (U.S.) and Colombian macroeconomic time series that are commonly used in literature: Gross Domestic Product (GDP), unemployment rate, industrial production index and inflation.

In addition, to estimate the TAR model, we select the variables that will be the threshold process. The selection is based on the economic theory and a literature review about the use of economic indicators as proxies for the output growth, industrial production, unemployment and inflation.

3.1. Unemployment rate

To fit a TAR model for the unemployment rate, from a macroeconomic point of view, we based on the analysis of the aggregate demand - aggregate supply (AD - AS) macroeconomic model[10], which explains the effects of output on the price level and vice versa, throughout the behavior of the labor market and the relation between goods and financial markets (Blanchard and Johnson, 2013).

That macroeconomic model, from the aggregate supply side, postulates that those effects are captured by the following underlying impacts: An increase in output leads to a decrease in unemployment, which leads to an increase in nominal wages, which leads in turn to an increase in the price level, which leads to a decrease in the demand for output (Blanchard and Johnson, 2013), starting over the cycle from the aggregate demand side.

Given that in this general model the output behavior affects the unemployment rate[11], we select as the threshold process the GDP, because it measures the output of an economy through the value of all final goods and services produced by labor in a period in an economy. Furthermore, GDP captures expansions and contractions of an economy, that is, the business or economic cycles, which has been associated with the unemployment

[10] AD-AS is based on Keynes's (1936) general theory of employment, interest and money, although it incorporates other theories that explain labor, goods and financial markets.

[11] Okun's law argues that a strong enough growth can decrease the unemployment rate, but it must be done with caution because the economy can overheat and leads in turn to a pressure on inflation (Blanchard and Johnson, 2013).

rate in the literature. Montgomery et al. (1998), Rothman (1998), van Dijk et al. (2002), Deschamps (2008) among others, argue that the U.S. unemployment rate tends to move countercyclically with the U.S. business cycle. Another remarkable characteristic of the unemployment rate is that it rises quickly but decays slowly, which suggests that the time series could be nonlinear (Tsay, 2005).

Regarding the Colombian case, we highlight the studies of Guzmán et al. (2003) and Vivas (2011), who describe that the unemployment rate also tends to move countercyclically with Colombian business cycles.

Therefore, we fit a TAR model for the Colombian and U.S. unemployment rate with the GDP as the threshold process. In the Colombian case, we use the ISE index as proxy of the GDP, given that the unemployment rate time series is in monthly data and the ISE index, which is calculated by the Colombian National Administrative Department of Statistics (referred to as DANE), is a monthly estimate of the Colombian real economic activity.

3.2. Gross domestic product

There is an extensive literature in forecasting the U.S. Gross Domestic Product (GDP). Among the different several variables used to predict GDP, the term spread, defined as the difference between long-term interest rates and short-term interest rates on bonds of equal credit quality (such as government debt), has been used extensively in the literature[12]. Authors as Laurent (1988), Harvey (1989), Estrella and Hardouvelis (1991), Stock and Watson (2003), Elger et al. (2006), Rossi and Sekhposyan (2010) among others, use this variable to forecast output growth[13].

Stock and Watson (2003, p. 173), based on Bernanke and Blinder (1992), say that *"the standard economic explanation for why the term spread has predictive content for output is that the spread is an indicator of an effective monetary policy: monetary tightening results in short-term interest rates that are high, relative to long-term interest rates, and these high short rates in turn produce an economic slowdown"*.

In the case of Colombia, the term spread has been also used. Arango et al. (2004) estimate a logit model to estimate the probability of change between economic expansion and recession conditioned on the spread and the inflation differential. Arango and Florez (2004) use the same methodology from Harvey (1988, 1997), who uses the spread to predict the U.S. consumption growth as proxy of the expected economy growth. Garzón and Tobos (2014) find, by estimating a VAR from 2000 to 2013 and the Granger causality test, that the active interest rate and the Colombian peso market exchange rate may help to improve the forecasts of the Colombian GDP.

[12] Different authors mention different variables as leading indicators of the economic activity and the inflation, such as interest rates, term spreads, unemployment, consumer price and producer price index, stock returns, dividend yields, exchange rates, money supply, among others.

[13] However, the term spread has lost its ability to predict economic activity in the past few decades (Stock and Watson, 1999, 2003; Estrella and Trubin, 2006; Wheelock and Wohar, 2009; Kuosmanen and Vataja, 2014). Despite of that, this variable has been used in the U.S. to predict recessions in the economic activity, and also, it is considered as the best indicator of economic activity and a useful tool for forecasting (Estrella and Trubin, 2006; Wheelock and Wohar, 2009; Kuosmanen and Vataja, 2014).

Therefore, based on the above review, we select the term spread as the threshold process to estimate the TAR model for both the U.S. and Colombian GDP. The term spread for the U.S. is based on Stock and Watson (2003) and Rossi and Sekhposyan (2010, 2014), which is defined as the difference between the 10-Year Treasury Constant Maturity rate (as the long-term government bond rate) and the effective federal funds rate (as the overnight rate). For the Colombian case, we use the Treasury bonds known as TES, as the long-term government bond rate, and the inter-bank interest rate as the overnight rate.

3.3. Industrial production index

Throughout the literature review, the industrial production index has been voluminously used as proxy of the economic activity. Stock and Watson (1989) use the U.S. industrial production as the output measure, from 1959 to 1985, and find, as a main result, that money has statistically significant marginal predictive value for the U.S. industrial production. Bernanke (1990) finds a satisfactory performance of the paper-bill spread (the difference between the commercial paper rate and the Treasury bill rate) as predictor of industrial production and unemployment rate, over the 1961-1986 period. Friedman and Kuttner (1992) discuss the Stock and Watson's (1989) study, extending the data of industrial production until 1990, and find that paper-bill spread contains significant predictive value for industrial production, while money does not have this ability.

Later, Thoma and Gray (1998) find that the federal funds rate, the paper-bill spread and the M2 are not useful in forecasting the industrial production, when they are used as a measure of the real economic activity. Based on Friedman and Kuttner (1992), Black et al. (2000) restate that the paper-bill spread provides information content for industrial production or real personal income when using data, over the 1959–1997 period. As we already mentioned, Stock and Watson (2003) and Rossi and Sekhposyan (2010) forecast the U.S. industrial production growth as a proxy of the output growth and use the term spread to forecast this variable.

For the Colombian case, Arango et al. (2004) and Arango and Florez (2004) use this index as proxy of the economic activity.

Based on this literature review, the term spread was considered to help to improve the forecast of the economic activity. Therefore, we select the term spread defined in the above Section, as the threshold process for estimating the TAR model for the U.S. and Colombian industrial production index.

3.4. Inflation

As in the GDP case, there is a widespread literature in forecasting the inflation, and specifically, in state that the spread between the yields on long and short government securities helps to forecast inflation. There are different hypotheses that relate the term structure and inflation. For example, Stock and Watson (2003, p. 795) say that *"According to the expectations hypothesis of the term structure of interest rates, the forward rate (and the term spread) should embody market expectations of future inflation and the future real rate"*. Kozicki (1998) mentions that, when the term spread is an indicator of monetary policy, decreases of this term spread (short-term interest rates are higher than long-term interest rates), when responding to contractionary restrictive monetary policies, predict that real activity will slow and inflation will decrease.

For the U.S. case, we highlight the study of Fama (1990) who forecasts the U.S. inflation, from 1952 to 1988, and finds that the spread between five-year bonds and one-year bonds helps to forecast changes in the one-year inflation rate. Mishkin (1990) finds that spreads with long bond rates contain information about future inflation[14]. Jorion and Mishkin (1991) and Mishkin (1991) find similar results using data on ten OECD countries. Estrella and Mishkin (1997) argue that the term spread is an indicator of the state of monetary policy and helps to forecast inflation at moderate to long horizons. Later, Stock and Watson (2003) study the predictive content of the term spread for inflation, from 1959 to 1999, and find that the term spread helps to forecast the U.S. inflation in the first period. Manzan and Zerom (2013) evaluate the U.S. inflation, from 1959 to 2007, and find that the term spread among other variables helps to forecasts after 1984.

Regarding the Colombian case, in the literature review was not identified studies that uses the term spread for predicting inflation[15].

Nonetheless, we use the term spread as the threshold variable to estimate the TAR model for Colombia and the U.S., supporting that selection in the above arguments.

[14] Nevertheless, he recommends using long bond rates, given that they help to predict inflation.

[15] Avella (2001) uses a small-scale macroeconomic model to analyze the influence in short-term of droughts on Colombian inflation, over the 1990-2001 period. Nuñez (2005) finds that inflation in Colombia, from 1998 to 2003, had been principally affected by supply shocks that had an impact on food prices. Melo et al. (2016) compute forecasts for Colombian inflation over the 2002-2011 period. They support their methodology on the inflation estimates of the Central Bank of Colombia, which use some indicators that affect inflation, such as price of certain foods, improving the predictive performance of their technique.

Chapter 4

Out-of-sample forecast evaluation

This Chapter is devoted to empirically evaluate the predictive performance of the TAR model. We compare the out-of-sample forecasts from the TAR models with those from the competing linear and nonlinear models mentioned in Chapter 1, using the macroeconomic variables mentioned in the previous Chapter.

Consequently, we estimate a TAR model with the procedure of Section 1.1, and we compute its forecasts as it was mentioned in Proposition 1.1. The estimation and forecasting procedures for the competing models are based on Chapter 1. It is highlighted that it has been found in the literature that a satisfactory in-sample fit it is not a guarantee of out-of-sample good forecasts performance, even for linear models, when using economic time series (Clements and Hendry, 1996, 1998). In fact, there is a considerable literature that finds that although nonlinear models fit better than linear models within-sample, their forecasting performance is often no clearly better than that of linear alternatives (Clements and Krolzig, 1998; Clements and Smith, 2000; Clements and Hendry, 2002; Franses and van Dijk, 2005, among others).

In that sense, it becomes more relevant to check the out of sample forecasts performance of each model, than to assess which model could describe better a time series within the estimation sample. This gives the possibility to analyze forecasts from models with not completely satisfactory in-sample properties, but that do not provide enough grounds for questioning the adequacy of the fitted models. Thus, we present a summary of the in-sample properties in terms of the good, regular and bad adequacy of each model, but principally, we focus on the out-of-sample forecasts performance of all models.

After the model estimations, we present the results of the forecasting performance for the TAR and competing models using the evaluation criteria mentioned on the previous Chapter[16]. The out-of-sample forecasts comparison is based on a sequence of rolling forecasts. In this procedure all the data set $X = (x_1, x_2, \ldots, x_T)$ with T the size of the total sample, is divided in two subsamples. The first subsample of the data, called the estimation subsample $X^a = (x_1, x_2, \ldots, x_{t_0})$, is used to estimate the model, and the second subsample, the forecasting subsample $X^b = (x_{t_0+1}, x_{t_0+2}, \ldots, x_T)$, is used to assess the forecasting performance of the model, where t_0 is the forecast origin. Then, a new data point is moved

[16] We use the RATS package (V. 7.1) for doing the above tasks.

from the forecasting subsample into the estimation subsample, and another sequence of forecasts are computed again. This procedure is repeated throughout the sample, rolling forward the forecast origin one-step ahead (Tsay, 2005)[17].

According to this design, it is hold back 30% of the total sample for the forecasting subsample. This is based on Granger (1993), who prefers to hold back a substantial amount of post-sample data, at least 20% of the data. Once the size of the subsamples are established, models are estimated at once, although the new data incorporated into the estimation subsample is used in generating the forecasts as mentioned above[18]. For the TAR, SETAR and STAR models, multi-step ahead forecasts are obtained via simulation of 2000 realizations at each step, where the mean of those realizations (which is considered as an estimate of the mean of the predictive distributions) is treated as a point forecast[19].

We remark here that the forecasting performance of the TAR model, compared with several models, has not been studied before in the literature, in the knowledge of the present author. Therefore, this study could give us important signals about the forecasting ability of a TAR model in the economic field.

4.1. Empirical results for the United States economic time series

4.1.1. Unemployment rate

Description of the data

We use the change in the seasonally adjusted U.S. quarterly unemployment rate, $X_t = u_t - u_{t-1}$, retrieved from the Bureau of Labor Statistics of the U.S. Department of Labor, over the 1948:02-2016:03 period (274 observations). This data set allows us to study the behavior of the series from post second world war to the currently available data, so we can count with a considerable set of information for the estimation and forecasting procedure. This period also allows us to contrast the U.S. unemployment with the U.S. business cycle and output growth. For the U.S. output growth, we use the growth rate of the seasonally adjusted quarterly real GDP (the first difference of the logarithm of the series), in billions of chained 2009 Dollars[20], retrieved from the U.S. Bureau of Economic Analysis over the same period.

[17] This procedure has been commonly used in the literature focused on evaluate different models and their forecast performance for several economic time series, such as Cao and Tsay (1992); Clements and Krolzig (1998); Montgomery et al. (1998); Stock and Watson (1999); van Dijk et al. (2002); Clements et al. (2003); Deschamps (2008); Kolly (2014), among others.

[18] Some studies that use this technique are those of Cao and Tsay (1992); Clements and Krolzig (1998); Koop and Potter (1999); van Dijk et al. (2002); Deschamps (2008) and Teräsvirta et al. (2010).

[19] We ran different experiments with 5000, 10000 and 20000 realizations and found that the forecasts are similar.

[20] This is a measure used to express real prices. Real prices are referred to prices that have been adjusted to remove the effect of changes in the purchasing power of the dollar. This measure, introduced by the U.S. Department of Commerce in 1996, is based on the average weights of goods and services in successive pairs of years. The “chained” word is because the second year in each pair (and its weights) becomes the first year of the next pair.

Figure 4.1 shows the U.S. unemployment rate and the growth rate of the U.S. quarterly real GDP. To compare, the shading areas denote the business cycle contractions from peak to trough, based on the National Bureau of Economic Research (NBER)[21]. The unemployment rate presents a countercyclically behavior with the U.S. business cycle, since the unemployment rate increases during contractions and decays during expansions periods. Tsay (2005) suggests that those characteristics indicate a nonlinear dynamic structure of the series. Regarding the growth rate of the real GDP, we can see more positive than negative growths during the analyzed period, where negative growths are in accordance with upwards of unemployment rate.

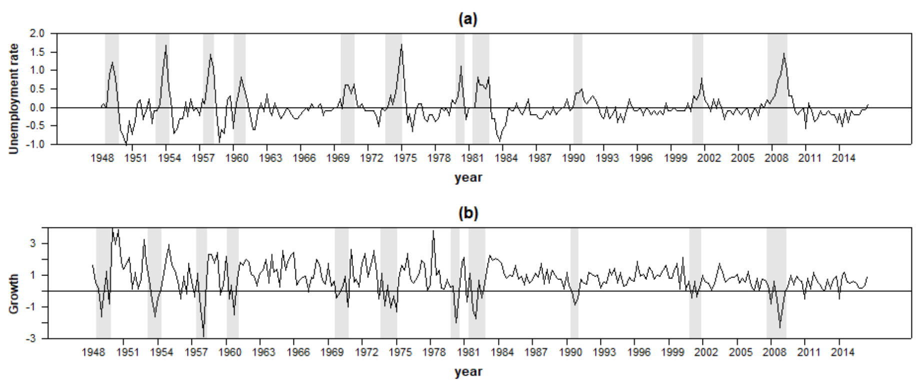


Figure 4.1.: (a) Time plot of the change in the U.S. quarterly unemployment rate and (b) time plot of the growth rate of U.S. quarterly real GDP.

We use as the training subsample, the data from 1948:02 to 1995:04 (191 observations). The remaining observations are reserved for the out-of-sample forecasting evaluation. Then, a sequence of 1 to 8-step ahead forecasts are generated from the forecast origin 1995:04[22]. Afterwards, the forecast origin is moved one period ahead and forecasts are generated again. This procedure is repeated until we compute 82 1-step ahead forecasts, 81 2-step ahead forecasts, and so on until 75 8-step ahead forecast. We use this methodology for the rest of the time series.

Estimation of the TAR model

Based on the estimation procedure in Appendix B.1, the fitted TAR model for the change in the U.S. unemployment is given by:

[21] Information available at the National Bureau of Economic Research web page: `http://www.nber.org/cycles.html`.

[22] The selection of the forecast horizon for quarterly time series is based on the literature review.

$$X_t = \begin{cases} 0.32 + 0.51X_{t-1} + 0.10X_{t-2} - 0.21X_{t-3} - 0.24X_{t-4} & \\ +0.10X_{t-5} + 0.18X_{t-6} - 0.17X_{t-7} + 0.95X_{t-8} & \\ +0.76X_{t-9} - 0.16X_{t-10} - 0.36X_{t-11} + 0.12\varepsilon_t, & \text{if } Z_t \leq 0.19 \\ 0.03 + 0.5X_{t-1} - 0.04X_{t-2} - 0.07X_{t-3} + 0.03\varepsilon_t, & \text{if } 0.19 < Z_t \leq 1.00 \\ -0.25 + 0.07\varepsilon_t, & \text{if } Z_t > 1.00 \end{cases}$$

This model could represent periods in the economy of i) contraction, where growth rates of the GDP less than 0.19% has the greatest increases in the unemployment rate; ii) stabilization, where the regime with growth rates of the GDP between 0.19 and 1.00%, shows low increases and decreases in the unemployment rate; and iii) expansion, where growth rates of the GDP greater than 1.00%, exhibits the greatest decreases in the unemployment rate.

When we check the residuals, in Figure 4.2 we observe that the standardized residuals and squared standardized residuals signal that the noise process is white, and the Ljung-Box statistics for checking "whiteness" are, respectively, $Q(8) = 5.891(0.659)$ and $Q(8) = 5.373(0.717)$ with the number in parenthesis denoting the p-value. Figure 4.3 reports that the CUSUM (with 5% percent of significance) and CUSUMSQ (with 1% percent of significance) behave well, which indicates that there is no statistical evidence for model misspecification or heteroscedasticity in $\{\varepsilon_t\}$[23].

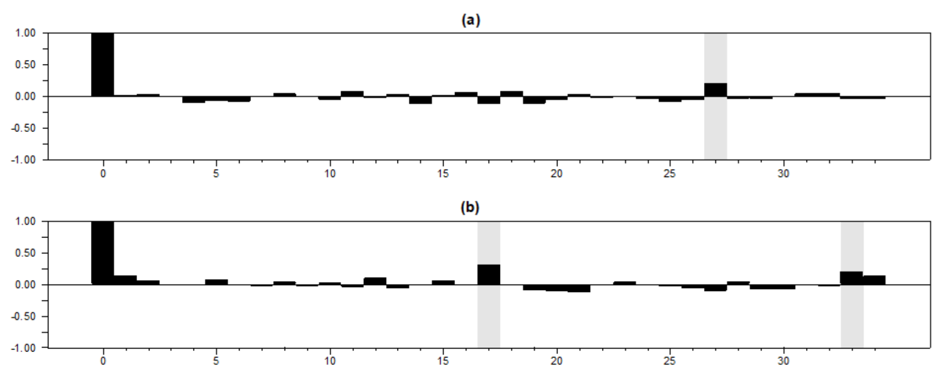


Figure 4.2.: Partial autocorrelation function of the fitted TAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. unemployment rate case.

[23] The significance bands established for the CUSUM and CUSUMSQ are used hereafter, for checking the residuals in the other models and time series.

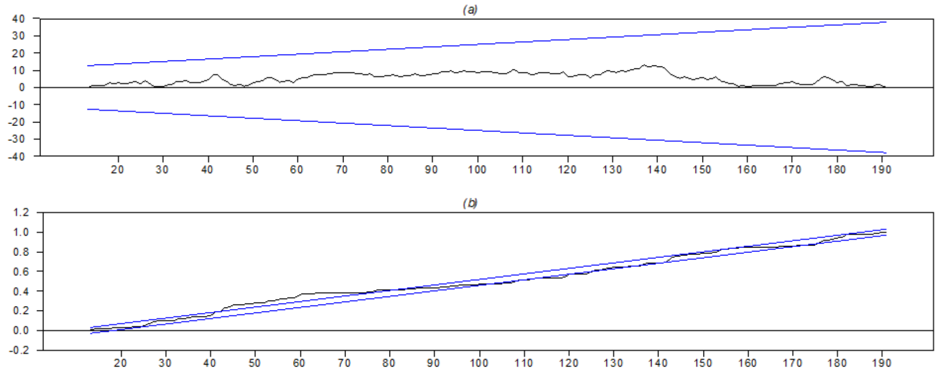


Figure 4.3.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted TAR model. U.S. unemployment rate case.

Estimation of the SETAR model

Based on the estimation procedure in Appendix B.2, the estimated SETAR model for the change in the U.S. unemployment rate is given by:

$$X_t = \begin{cases} 0.21X_{t-1} + \varepsilon_t, & \text{if } X_{t-6} \leq -0.2 \\ 0.67X_{t-1} - 0.21X_{t-3} + \varepsilon_t, & \text{if } X_{t-6} > -0.2 \end{cases}$$

This model could represent periods in the economy of i) stability, where the first regime contains minor variations in the unemployment rate, and ii) instability, where the second regime shows sharp movements in the unemployment rate.

Figure 4.4 shows that the standardized and squared standardized residuals of the model signal that some nonlinear structure in the data is not explained by the model. Furthermore, the Ljung-Box statistics are, respectively, $Q\,(8) = 11.777(0.161)$ and $Q\,(8) = 25.849(0.001)$. Figure 4.5 presents the CUSUM and CUSUMSQ, indicating that there is no statistical evidence for model misspecification but some heteroscedasticity in $\{\varepsilon_t\}$.

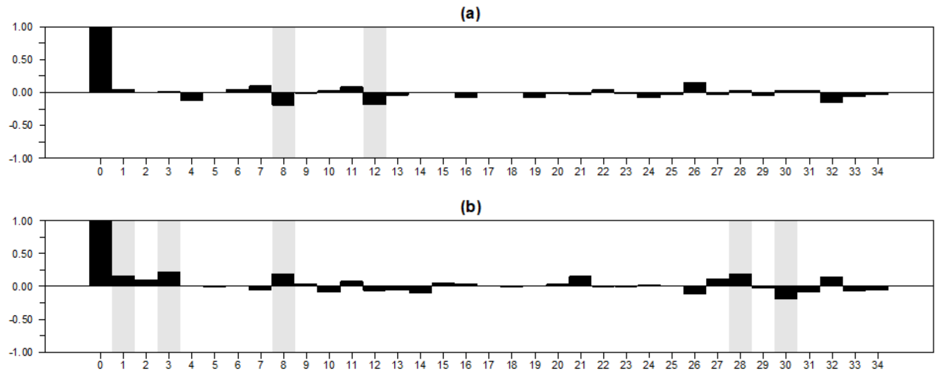


Figure 4.4.: Partial autocorrelation function of the fitted SETAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. unemployment rate case.

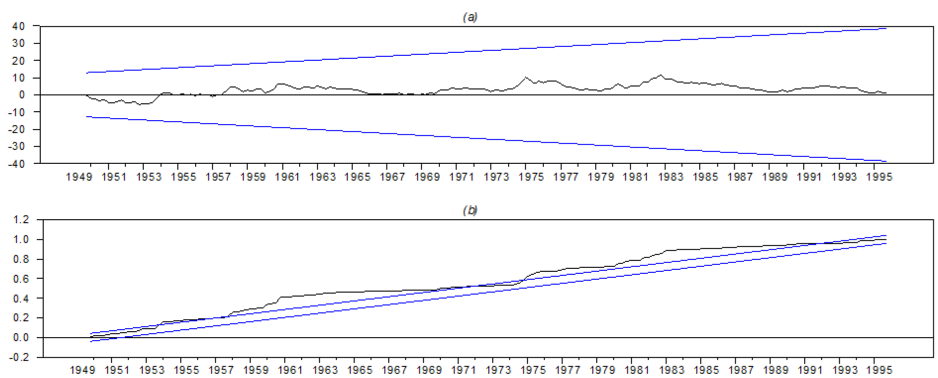


Figure 4.5.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted SETAR model. U.S. unemployment rate case.

Estimation of the STAR model

Based on the results in Appendix B.3, the estimated STAR model for the change in the U.S. unemployment rate is given by:

$$X_t = -0.09 + 0.43X_{t-1} - 0.20X_{t-4} + F\left(X_{t-6}\right)\left(4.24 - 0.56X_{t-1} - 3.37X_{t-2}\right) + \varepsilon_t,$$

where

$$F\left(X_{t-6}\right) = \left(1 + \exp\left\{2.34 \times -1.39\left(X_{t-6} - 1.26\right)\right\}\right)^{-1}.$$

When we check the residuals, Figure 4.6 shows that the standardized and squared standardized residuals of the model signal that the noise process is white, and the Ljung-Box statistics are, respectively, $Q\left(8\right) = 12.736(0.121)$ and $Q\left(8\right) = 16.787(0.032)$. Figure 4.7

reports the CUSUM and CUSUMSQ, indicating that there is no statistical evidence for model misspecification but some heteroscedasticity in $\{\varepsilon_t\}$.

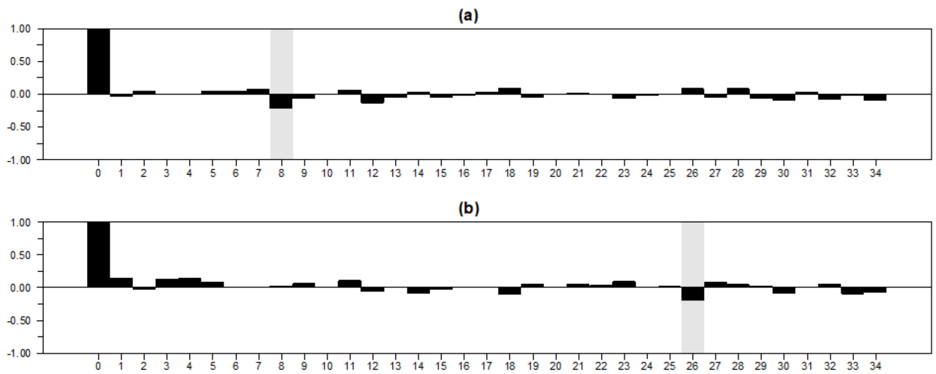


Figure 4.6.: Partial autocorrelation function of the fitted STAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. unemployment rate case.

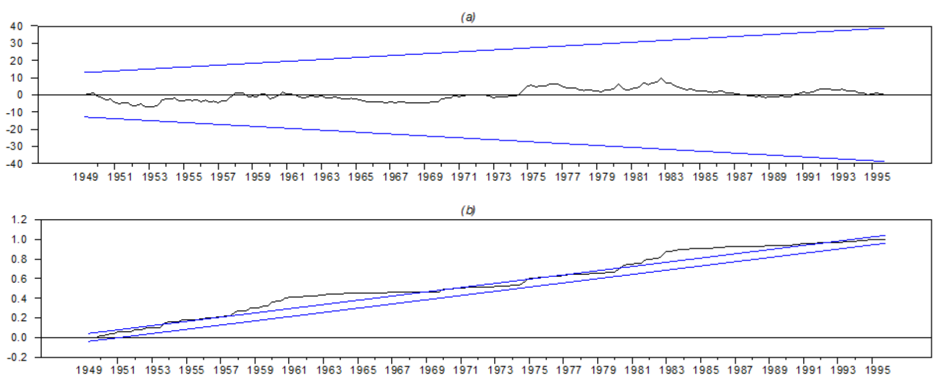


Figure 4.7.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted STAR model. U.S. unemployment rate case.

Estimation of the MSAR model

Following Appendix B.4, the estimated MSAR model for the change in the U.S. unemployment rate is given by:

$$X_t = \begin{cases} -0.07 + 0.28X_{t-1} + 0.09X_{t-2} - 0.12X_{t-4} + \varepsilon_{1t}, & \text{if } s_t = 1 \\ 0.09 + 0.79X_{t-1} - 0.28X_{t-2} - 0.24X_{t-4} + \varepsilon_{2t}, & \text{if } s_t = 2. \end{cases}$$

The conditional mean of X_t for regime 1 is -0.11 and for regime 2 is 0.18. Hence, the first state represents the expansionary periods in the U.S. economy, and the second state represents the contractions. The sample variances of ε_{1t} and ε_{2t} are 0.03 and 0.17, respectively[24].

When we check this model, Figure 4.8 shows that the standardized and squared standardized residuals of the model slightly signal that some nonlinear structure in the data is not explained by the model, and the Ljung-Box statistics are, respectively, $Q(8) = 57.784(0.000)$ and $Q(8) = 2.919(0.939)$. Figure 4.9 reports the CUSUM and CUSUMSQ, which indicate that there is statistical evidence for model misspecification and heteroscedasticity in $\{\varepsilon_t\}$.

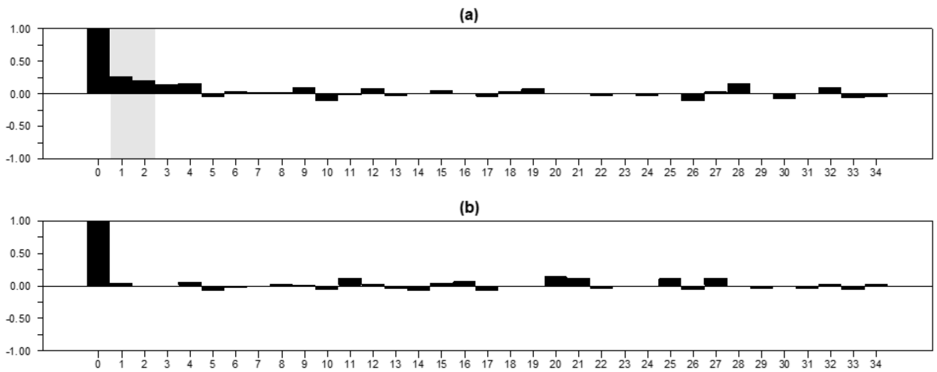


Figure 4.8.: Partial autocorrelation function of the fitted MSAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. unemployment rate case.

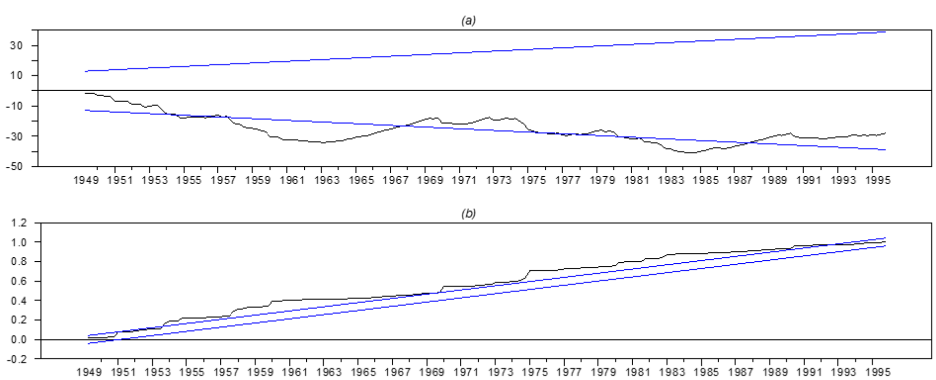


Figure 4.9.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted MSAR model. U.S. unemployment rate case.

[24] The state transition probability $p(i,j)$ gives the probability of moving to state i from j. Thus, the probabilities of moving from one state to the other are $p(s_t = 2|s_{t-1} = 1) = 0.07(0.03)$ and $p(s_t = 1|s_{t-1} = 2) = 0.09(0.05)$, where the number in parenthesis is the standard error. Additionally, the unconditional probability that the process is in regime 1 is 0.56 and that it is in regime 2 is 0.44. The coefficients also allow us to identify that the probability that an expansion is followed by another expansion period is $p(1,1) = 0.93$, and that the probability that a contraction is followed by another contraction period is $p(2,2) = 0.91$.

Estimation of the AR model

We estimate an AR(3) model for the change in the US unemployment rate[25], which is given by:

$$\left(1 - 0.730B + 0.118B^2 + 0.144B^3\right) X_t = a_t, \qquad \hat{\sigma}_a^2 = 0.10.$$

The standard errors of the coefficients are 0.07, 0.09, and 0.07, respectively. When we check the residuals, Figure 4.10 shows that the standardized and the squared standardized residuals slightly signal that some linear structure in the data is not explained by the model, and the Ljung-Box statistics are, respectively, $Q\,(8) = 19.011(0.015)$ and $Q\,(8) = 23.570(0.003)$. Figure 4.11 shows the CUSUM that indicates there is no statistical evidence for model misspecification, and the CUSUMSQ that shows statistical evidence for heteroscedasticity in $\{\varepsilon_t\}$.

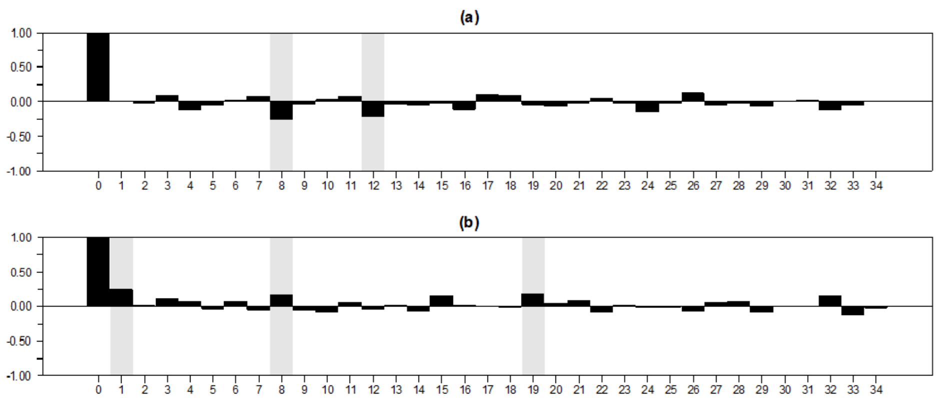


Figure 4.10.: Partial autocorrelation function of the fitted AR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. unemployment rate case.

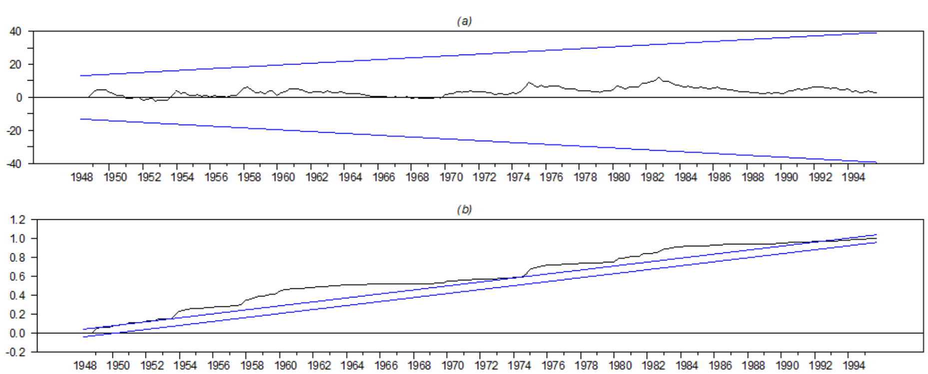


Figure 4.11.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted AR model. U.S. unemployment rate case.

[25] As reported by the Augmented Dickey-Fuller (ADF) ($p-$value $= 0.000$), Phillips Perron (PP) ($p-$value $= 0.000$) and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) (Test statistic $= 0.055$ and Critical value $= 0.463$, the change in the unemployment rate is stationary at the 5% significance level.

Table 4.1 shows a summary of the model adequacy of the analyzed models. Globally, it is observed that the TAR and STAR models exhibit a reasonable in-sample fit.

Model adequacy	AR	TAR	SETAR	STAR	MSAR
White noise	2	1	2	1	2
Model specification	1	1	1	1	2
Homoscedasticity	3	1	2	2	3

Cells with number 1 and highlighted green represent good adequacy, 2 and highlighted orange represent regular adequacy, and 3 and highlighted red represent bad adequacy.

Table 4.1.: Model adequacy. U.S. unemployment rate case.

Forecasting evaluation

As it was mentioned in Chapter 2, firstly we are going to evaluate the individual properties for each model and each horizon. Table 4.2 shows Holden and Peel's (1989) (HP) unbiased test and Ljung-Box's Q correlation test, which were described in Section 2.1 and 2.2, respectively. It is observed that, at the 10% significance level, forecasts errors of all models are unbiased, and only the SETAR and STAR exhibit forecasts errors that, globally, are not autocorrelated at horizons greater than 1 period.

(a)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.921	0.252	0.747	0.540	0.811
2	0.856	0.229	0.476	0.537	0.798
3	0.825	0.203	0.680	0.429	0.742
4	0.872	0.260	0.481	0.506	0.790
5	0.876	0.298	0.224	0.758	0.741
6	0.909	0.275	0.836	0.530	0.772
7	0.965	0.199	0.700	0.317	0.811
8	0.995	0.223	0.766	0.413	0.814

(b)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.000	0.000	0.000	0.000	0.000
2	0.658	0.000	0.210	0.376	0.000
3	0.000	0.000	0.553	0.036	0.000
4	0.000	0.000	0.221	0.196	0.000
5	0.000	0.000	0.721	0.485	0.000
6	0.001	0.000	0.821	0.790	0.000
7	0.527	0.000	0.891	0.603	0.000
8	0.000	0.000	0.033	0.158	0.000

Cells highlighted green have a p–value less than 0.1.

Table 4.2.: p–values of the (a) unbiased test and (b) correlation test for the first 4 lags. U.S. unemployment rate case.

Secondly, we compare, at the 10% significance level, forecasts from the TAR model with those from the other considered models. Table 4.3 shows the relative MSE of forecasts from the estimated models, using the lineal model as the benchmark model. For the overall comparison, the TAR model and the linear model are very close in MSE. We observe that the MSAR model has the smallest MSE among all the estimated models, except the 1 period horizon, where the MSE of the SETAR model is much better.

Horizon	MSE	AR	TAR	SETAR	STAR	MSAR
1	0.053	1.000	1.127	0.943	0.956	1.117
2	0.079	1.000	1.065	1.243	1.981	0.764
3	0.101	1.000	0.937	1.438	1.933	0.602
4	0.106	1.000	0.893	1.800	2.068	0.575
5	0.103	1.000	0.950	1.848	1.932	0.600
6	0.101	1.000	0.947	1.261	1.700	0.613
7	0.100	1.000	0.946	1.481	1.582	0.627
8	0.101	1.000	1.010	1.723	1.771	0.631

(1) The column marked by MSE shows the MSE of forecasts from the AR model. (2) Cells highlighted green are the smallest values for each forecasts horizon.

Table 4.3.: Relative MSE of forecasts. U.S. unemployment rate case.

Table 4.4 shows the results of the DM test of Diebold and Mariano (1995) and the modified DM test (MDM) of Harvey et al. (1997), where both of them evaluate the null hypothesis that indicates equal predictive accuracy (equal MSE) for the two evaluated forecasts. Hereafter, the benchmark model is the TAR, so we can evaluate its performance with respect to the performance of the other models. We analyze both directions of the test, thus, tables (a) and (c) in Table 4.4 shows the p-value of the DM and MDM tests with the null of equal accuracy versus the alternative that says that forecasts from the competing models are more accurate than the TAR model (the competing model has a smaller MSE); while tables (b) and (d) shows the p-value for the null of equal accuracy versus the alternative that says the TAR model is more accurate than the competing models (the TAR model has a smaller MSE). In general, we observe that forecasts from the TAR model are more accurate than those from the SETAR and STAR models, and there is not a significant difference with the MSAR and AR models, using the DM and MDM tests.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.328	0.286	0.266	0.477
2	0.310	0.752	0.999	0.159
3	0.827	1.000	0.998	0.148
4	0.990	1.000	1.000	0.146
5	0.864	1.000	1.000	0.126
6	0.827	0.984	1.000	0.148
7	0.828	0.974	1.000	0.164
8	0.424	0.993	0.999	0.127

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.672	0.714	0.734	0.523
2	0.690	0.248	0.001	0.841
3	0.173	0.000	0.002	0.852
4	0.010	0.000	0.000	0.854
5	0.136	0.000	0.000	0.874
6	0.173	0.016	0.000	0.852
7	0.172	0.026	0.000	0.836
8	0.576	0.007	0.001	0.873

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.354	0.314	0.289	0.481
2	0.335	0.731	0.996	0.193
3	0.766	0.999	0.993	0.186
4	0.971	1.000	1.000	0.182
5	0.807	1.000	1.000	0.165
6	0.776	0.998	1.000	0.184
7	0.783	0.981	1.000	0.200
8	0.428	0.998	1.000	0.160

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.646	0.686	0.711	0.519
2	0.665	0.269	0.004	0.807
3	0.234	0.001	0.007	0.814
4	0.029	0.000	0.000	0.818
5	0.193	0.000	0.000	0.835
6	0.224	0.002	0.000	0.816
7	0.217	0.019	0.000	0.800
8	0.572	0.002	0.000	0.840

Cells highlighted green have a p–value less than 0.1.

Table 4.4.: DM test when (a) H_1 : Forecasts from competing model (F2) are better than forecasts from TAR model (F1) and (b) H_1 : F1 are better than F2; MDM test when (c) H_1 : F2 are better than F1 and (d) H_1 : F1 are better than F2. U.S. unemployment rate case.

When checking the encompassing tests, we also evaluate both directions of these tests as it was done by Clements et al. (2003) and Bradley and Jansen (2004), that is, one model encompasses the other and vice versa. The analysis is as follows: (i) if the TAR and the competing models encompass each other, then no one is better than the other; but (ii) if the TAR model encompasses (does not encompass) the competing model and it is not encompassed (is encompassed) by the latter, then the TAR model is better (is not better) than the competing model.

Table 4.5 shows the forecast encompassing tests of Chong and Hendry (1986) (CH), Ericsson (1992) (ER) and Harvey et al. (1998) (HLN). All of these tests evaluate the null hypothesis of forecast encompassing. Table 4.5 suggests in general, at the 10% significance level, that under the CH criteria, the TAR model could be encompassed by the competing models for the 4 and 5-step ahead forecast, in general. However, the ER and HLN tests shows that the TAR model encompasses the AR, SETAR and STAR models, while those competing models do not encompass the TAR model. Therefore, forecasts from the TAR model contain all the relevant information with respect to the forecasts from these three models. The MSAR model is the only one that encompasses the TAR model.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.035	0.016	0.017	0.044
2	0.111	0.267	0.385	0.000
3	0.086	0.366	0.621	0.003
4	0.068	0.043	0.069	0.002
5	0.273	0.081	0.081	0.001
6	0.138	0.203	0.234	0.000
7	0.188	0.531	0.844	0.002
8	0.735	0.918	0.524	0.004

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.064	0.153	0.033	0.004
2	0.007	0.163	0.041	0.000
3	0.440	0.421	0.255	0.352
4	0.363	0.903	0.793	0.202
5	0.436	0.891	0.789	0.341
6	0.155	0.366	0.639	0.064
7	0.023	0.193	0.349	0.029
8	0.385	0.571	0.765	0.770

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.177	0.076	0.058	0.438
2	0.136	0.248	0.299	0.017
3	0.812	0.259	0.972	0.006
4	0.297	0.047	0.073	0.004
5	0.748	0.101	0.078	0.001
6	0.805	0.158	0.217	0.007
7	0.693	0.516	0.969	0.011
8	0.168	0.684	0.289	0.001

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.729	0.717	0.730	0.509
2	0.615	0.013	0.000	0.958
3	0.019	0.000	0.000	0.565
4	0.000	0.000	0.000	0.548
5	0.013	0.000	0.000	0.458
6	0.012	0.000	0.000	0.653
7	0.007	0.000	0.000	0.689
8	0.316	0.000	0.000	0.362

(e)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.160	0.142	0.124	0.234
2	0.072	0.143	0.826	0.053
3	0.404	0.150	0.514	0.086
4	0.842	0.982	0.954	0.086
5	0.376	0.969	0.968	0.074
6	0.399	0.091	0.905	0.078
7	0.343	0.265	0.515	0.091
8	0.076	0.340	0.130	0.081

(f)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.353	0.341	0.352	0.244
2	0.310	0.005	0.000	0.521
3	0.039	0.000	0.000	0.685
4	0.001	0.000	0.000	0.690
5	0.011	0.000	0.000	0.722
6	0.048	0.000	0.000	0.654
7	0.057	0.000	0.000	0.638
8	0.202	0.000	0.000	0.750

Cells highlighted green have a p–value less than 0.1.

Table 4.5.: CH test when (a) H_0 : F1 encompasses F2 and (b) H_0: F2 encompasses F1; ER test when (c) H_0 : F1 encompasses F2 and (d) H_0: F2 encompasses F1; HLN test when (e) H_0 : F1 encompasses F2 and (f) H_0: F2 encompasses F1. U.S. unemployment rate case.

Table 4.6 reports the Theil's U statistic. In general, we find that all models have the same predictive performance of the naïve model, where the MSAR model has the smallest values of the test, and the TAR model has, generally, the second smallest values.

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.998	1.060	0.969	0.976	1.055
2	1.206	1.245	1.345	1.698	1.055
3	1.361	1.317	1.631	1.892	1.055
4	1.392	1.315	1.868	2.002	1.055
5	1.368	1.334	1.860	1.902	1.060
6	1.353	1.316	1.519	1.764	1.059
7	1.336	1.299	1.626	1.680	1.058
8	1.332	1.339	1.749	1.773	1.059

Cells highlighted green represent the lowest value for each forecast horizon.

Table 4.6.: Theil's U statistic. U.S. unemployment rate case.

Now, we present a summary of the above forecasts comparison evaluation. Regarding the individual properties, all models have unbiased forecasts, but only the SETAR and STAR present uncorrelated forecasts errors. Additionally, Table 4.7 indicates how many models are outperformed by the TAR model, in terms of their forecasting performance, at each forecast horizon and for each evaluation criteria. It is observed that, under the relative MSE criteria, the TAR model outperforms more than 2 models, although the MSAR model has the smallest MSE. The DM and MDM statistics suggest that forecasts from the TAR model are more accurate than those from the SETAR and STAR models, while differences in MSE of forecast between the TAR, MSAR and AR models are not statistically significant.

Also, it is observed that the TAR model encompasses the AR, SETAR and STAR models. That is, given that forecasts from the TAR model are available, the competing models provide no further useful incremental information for prediction (Clements and Hendry, 2002). Finally, forecasts from the TAR model have, after forecasts from the MSAR, the smallest Theil's U statistic at horizons greater than 2 periods ahead. Thus, we can say that forecasts from the TAR model have a good performance for predicting the U.S. unemployment rate, given that this model appears to be marginally preferred to the competing models.

Horizon	Relative MSE	DM - MDM	Encompassing tests	U-Theil
1	0	0	0	0
2	2 (SETAR, STAR)	1 (STAR)	2 (STAR, AR)	2 (SETAR, STAR)
3	3 (SETAR, STAR, AR)	2 (SETAR, STAR)	3 (SETAR, STAR, AR)	2 (SETAR, STAR)
4	3 (SETAR, STAR, AR)	3 (SETAR, STAR, AR)	3 (SETAR, STAR, AR)	3 (SETAR, STAR, AR)
5	3 (SETAR, STAR, AR)	2 (SETAR, STAR)	3 (SETAR, STAR, AR)	3 (SETAR, STAR, AR)
6	3 (SETAR, STAR, AR)	2 (SETAR, STAR)	3 (SETAR, STAR, AR)	3 (SETAR, STAR, AR)
7	3 (SETAR, STAR, AR)	2 (SETAR, STAR)	3 (SETAR, STAR, AR)	3 (SETAR, STAR, AR)
8	3 (SETAR, STAR, AR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)

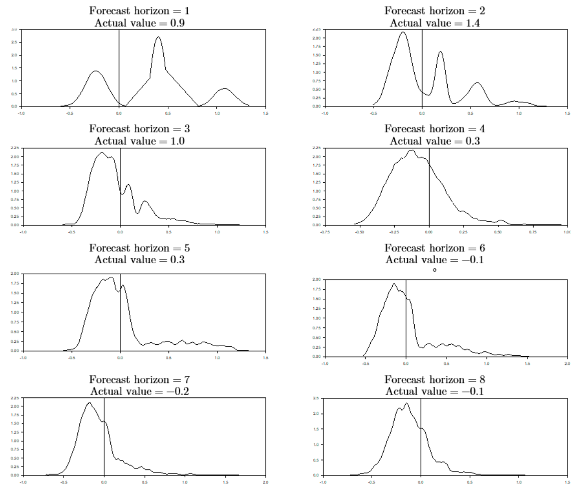
(1) Cells highlighted green report that the TAR model outperforms 3 or 4 models, highlighted orange report that the TAR model outperforms 1 or 2 models, and highlighted red report that the TAR model does not outperform any model. (2) Models in parenthesis are outperformed by the TAR model.

Table 4.7.: Summary of the forecasting performance of the TAR model. U.S. unemployment rate case.

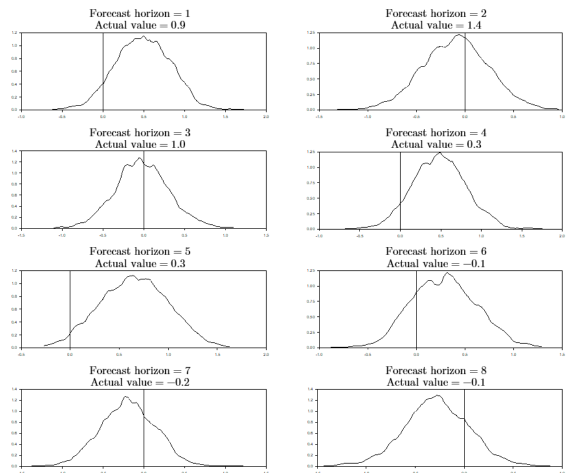
Finally, we compare the TAR model with the SETAR and STAR models based on their ability to describe the predictive distributions of the time series[26], in terms of the capacity to handle cycles. As it was mentioned in Chapter 2, we make this comparison exercise based on Wong and Li (2000), who suggest that the predictive distribution is unimodal when the series is ascending to a peak and bimodal when the series is descending to a trough.

Figure 4.12 shows the predictive distributions for horizons from 1 to 8, when the forecast origin is 2008:03. Based on the business cycle contractions from peak to trough of the NBER, there is a business contraction from 2007:04 to 2009:02. According to this, the predictive distributions of the TAR model tend to be bimodal when the time series is in a trough between 2008:04 and 2009:02 (positive increments of the unemployment rate) and become more unimodal when the time series is regaining an expansionist trend (negative increments of the unemployment rate). The predictive distributions of the SETAR and STAR models do not tend to be bimodal when the time series is in the peak. This suggests that the predictive distributions of the TAR model seem to handle cycles of the times series reasonably well.

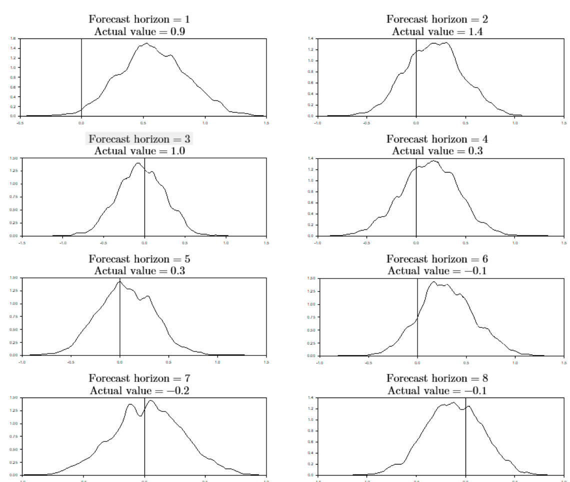
[26] Hereafter, for all models and all the economic time series, we generate the 1 to 8-step predictive distributions for the U.S. and Colombian cases, using the Monte Carlo approach with 2000 replications.



(a)



(b)



(c)

Figure 4.12.: 1 to 8-step predictive distributions of the change in the unemployment rate series, for the (a) TAR, (b) SETAR and (c) STAR models. U.S. unemployment rate case.

4.1.2. Gross domestic product

Description of the data

We use the annual growth rate of the seasonally adjusted U.S. quarterly real output, in billions of chained 2009 Dollars, from 1955:03 to 2016:03 (245 observations), measured by the U.S. Bureau of Economic Analysis. As the threshold value, we use the term spread defined in Section 3.2 over the same period, that is, the difference between the 10-Year Treasury Constant Maturity rate as the long-term government bond rate, and the effective federal funds rate as the overnight rate.

For the forecast comparison, we denote $X_t = [\log(GDP_t) - \log(GDP_{t-4})] * 100$ as the annual growth rate of the real GDP, and $Z_t = (gov_t - overnight_t)$ the spread term, where gov_t is the 10-Year Treasury Constant Maturity rate, and $overnight_t$ is the effective federal funds at time t. Both series are plotted in Figure 4.13 that shows a similar behavior of these series during contractions periods. The shading areas denote the business cycle contractions from peak to trough.

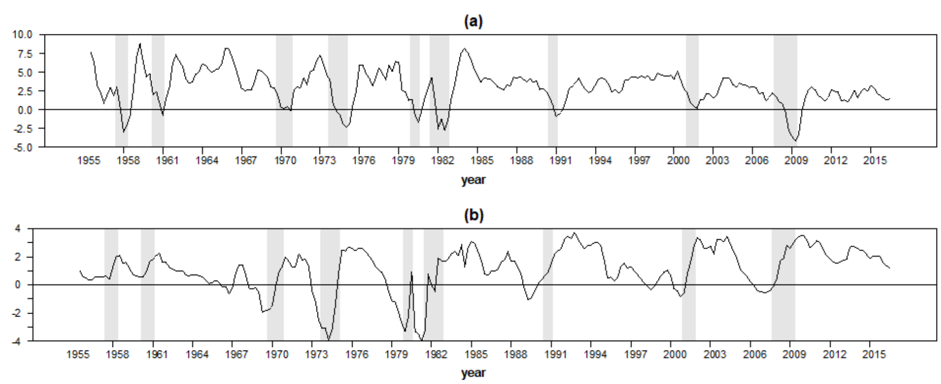


Figure 4.13.: (a) Time plot of the annual growth rate of U.S. real GDP and (b) time plot of U.S. term spread.

We use as the training subsample, data from 1955:03 to 1998:02 (172 observations). The remaining observations are reserved for the out-of-sample forecasting evaluation. A sequence of 1 to 8-step ahead forecasts are generated until we compute 73 1-step ahead forecasts, down to 66 8-step ahead forecast.

Estimation of models

Based on results in Appendix C, Table 4.8 shows that globally, the TAR model presents the best reasonable in-sample fit.

Model adequacy	AR	TAR	SETAR	STAR	MSAR
White noise	3	1	2	3	2
Model specification	1	1	1	1	3
Homoscedasticity	3	2	3	3	3

Cells with number 1 and highlighted green represent good adequacy, 2 and highlighted orange represent regular adequacy, and 3 and highlighted red represent bad adequacy.

Table 4.8.: Model adequacy. U.S. GDP case.

Forecasting evaluation

Table 4.9 shows that at the 10% significance level, only forecasts errors of the TAR model are unbiased, and only the SETAR and STAR exhibit forecasts errors that, globally, are not autocorrelated at horizons greater than 4 periods.

(a)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.001	0.601	0.016	0.019	0.016
2	0.000	0.689	0.072	0.001	0.013
3	0.000	0.432	0.107	0.004	0.006
4	0.000	0.373	0.091	0.016	0.006
5	0.000	0.302	0.042	0.000	0.006
6	0.000	0.202	0.034	0.000	0.004
7	0.000	0.180	0.015	0.000	0.003
8	0.000	0.163	0.000	0.000	0.003

(b)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.406	0.000
3	0.000	0.000	0.036	0.193	0.000
4	0.000	0.000	0.099	0.021	0.000
5	0.000	0.000	0.272	0.189	0.000
6	0.000	0.000	0.509	0.304	0.000
7	0.000	0.000	0.314	0.727	0.000
8	0.000	0.000	0.581	0.776	0.000

Cells highlighted green have a p–value less than 0.1.

Table 4.9.: p–values of the (a) unbiased test and (b) correlation test for the first 4 lags. U.S. GDP case.

Now, we present the forecast comparison at the 10% significance level. Table 4.10 shows, for the overall comparison, that the TAR model and the linear model are very close in MSE of forecasts, but the MSAR model has the smallest MSE among all the estimated models, except for the 1-step ahead forecast, where the MSE of the TAR model is much better.

Horizon	MSE	AR	TAR	SETAR	STAR	MSAR
1	0.546	1.000	0.957	1.654	1.124	0.995
2	1.579	1.000	0.976	2.034	2.700	0.348
3	2.917	1.000	1.051	1.845	2.305	0.184
4	4.399	1.000	1.079	1.928	1.971	0.124
5	5.059	1.000	1.081	1.841	1.478	0.109
6	5.265	1.000	1.048	1.630	1.442	0.106
7	5.233	1.000	1.063	1.511	1.730	0.107
8	5.062	1.000	1.105	1.913	1.798	0.112

(1) The column marked by MSE shows the MSE of forecasts from the AR model. (2) Cells highlighted green are the smallest values for each forecasts horizon.

Table 4.10.: Relative MSE of forecasts. U.S. GDP case.

Henceforth, we define the TAR model as the benchmark model. Table 4.11, that shows the results of the DM and MDM tests, suggests that forecasts from the TAR model are more accurate than those from the SETAR and STAR models, and there is no significant difference with those from the AR model. However, forecasts from the MSAR model are more accurate than those from the TAR model.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.642	0.943	0.792	0.629
2	0.580	0.987	1.000	0.004
3	0.356	0.985	0.990	0.003
4	0.293	0.994	0.955	0.003
5	0.307	0.991	0.958	0.002
6	0.383	0.979	0.980	0.003
7	0.347	0.984	0.998	0.002
8	0.241	0.998	0.999	0.003

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.358	0.057	0.208	0.371
2	0.420	0.013	0.000	0.996
3	0.644	0.015	0.011	0.997
4	0.707	0.006	0.045	0.997
5	0.693	0.009	0.042	0.998
6	0.617	0.021	0.020	0.998
7	0.653	0.016	0.002	0.998
8	0.759	0.002	0.001	0.997

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.643	0.900	0.743	0.630
2	0.579	0.967	0.997	0.014
3	0.354	0.965	0.993	0.010
4	0.301	0.987	0.961	0.009
5	0.337	0.986	0.955	0.007
6	0.402	0.999	0.965	0.008
7	0.373	1.000	0.994	0.007
8	0.282	1.000	0.996	0.009

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.357	0.100	0.257	0.370
2	0.421	0.033	0.003	0.986
3	0.646	0.035	0.007	0.990
4	0.699	0.013	0.039	0.991
5	0.663	0.014	0.045	0.993
6	0.598	0.001	0.035	0.992
7	0.627	0.000	0.006	0.993
8	0.718	0.000	0.004	0.991

Cells highlighted green have a p–value less than 0.1.

Table 4.11.: DM test when (a) H_1 : Forecasts from competing model (F2) are better than forecasts from TAR model (F1) and (b) H_1 : F1 are better than F2; MDM test when (c) H_1 : F2 are better than F1 and (d) H_1 : F1 are better than F2. U.S. GDP case.

Table 4.12 shows, under the CH test, that the TAR model could encompass the linear model. However, under the ER and HLN tests, the TAR, SETAR, STAR and AR models do not encompass each other, so forecasts from all models contain all the same useful information for prediction. Only the MSAR model do encompasses the TAR model.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.940	0.783	0.737	0.974
2	0.988	0.676	0.766	0.123
3	0.732	0.814	0.847	0.097
4	0.622	0.903	0.708	0.067
5	0.517	0.965	0.724	0.057
6	0.422	0.819	0.603	0.058
7	0.385	0.745	0.503	0.079
8	0.357	0.412	0.445	0.075

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.129	0.911	0.895	0.370
2	0.074	0.714	0.500	0.270
3	0.020	0.638	0.159	0.053
4	0.012	0.562	0.165	0.025
5	0.005	0.332	0.014	0.013
6	0.003	0.384	0.015	0.060
7	0.001	0.134	0.002	0.004
8	0.001	0.018	0.000	0.005

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.002	0.003	0.000	0.005
2	0.000	0.019	0.033	0.000
3	0.000	0.001	0.001	0.000
4	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.000
7	0.000	0.000	0.009	0.000
8	0.000	0.007	0.012	0.000

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.000	0.000	0.000	0.001
2	0.000	0.000	0.000	0.446
3	0.000	0.000	0.000	0.727
4	0.000	0.000	0.000	0.358
5	0.000	0.000	0.000	0.337
6	0.000	0.000	0.000	0.826
7	0.001	0.000	0.000	0.357
8	0.001	0.000	0.000	0.431

(e)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.001	0.039	0.010	0.001
2	0.016	0.068	0.090	0.000
3	0.017	0.029	0.043	0.000
4	0.017	0.022	0.020	0.002
5	0.017	0.012	0.014	0.001
6	0.018	0.001	0.022	0.001
7	0.012	0.001	0.033	0.001
8	0.009	0.020	0.041	0.001

(f)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.003	0.036	0.064	0.005
2	0.000	0.005	0.001	0.213
3	0.000	0.003	0.009	0.631
4	0.000	0.005	0.019	0.792
5	0.001	0.002	0.002	0.797
6	0.001	0.000	0.002	0.584
7	0.001	0.000	0.000	0.787
8	0.001	0.000	0.000	0.757

Cells highlighted green have a p–value less than 0.1.

Table 4.12.: CH test when (a) H_0 : F1 encompasses F2 and (b) H_0: F2 encompasses F1; ER test when (c) H_0 : F1 encompasses F2 and (d) H_0: F2 encompasses F1; HLN test when (e) H_0 : F1 encompasses F2 and (f) H_0: F2 encompasses F1. U.S. GDP case.

Table 4.13 shows that the MSAR model is the only one, among all the evaluated models, that has a better predictive performance than the naïve model, and the TAR model has the third-best values of the Theil's U statistic.

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.905	0.885	1.164	0.959	0.903
2	1.528	1.510	2.180	2.511	0.901
3	2.079	2.131	2.823	3.156	0.892
4	2.535	2.634	3.521	3.560	0.892
5	2.535	2.808	3.664	3.282	0.892
6	2.735	2.800	3.491	3.284	0.890
7	2.706	2.791	3.327	3.560	0.887
8	2.649	2.785	3.664	3.552	0.888

Cells highlighted green represent the lowest value for each forecast horizon.

Table 4.13.: Theil's U statistic. U.S. GDP case.

As a summary of the above forecasts comparison evaluation, it is observed that only the TAR model has unbiased forecasts, but the SETAR and STAR present uncorrelated forecasts errors for some forecast periods. Additionally, Table 4.14 let us observe a satisfactory performance of the TAR model. According to the Relative MSE of forecast and the DM and MDM tests, the TAR model has better MSE of forecasts than the SETAR and STAR models. Regarding the encompassing tests, only the MSAR model encompasses the TAR model, while the other models contain all the same useful information for prediction than the TAR model. Additionally, forecasts from the MSAR model are more accurate than those from the naïve model.

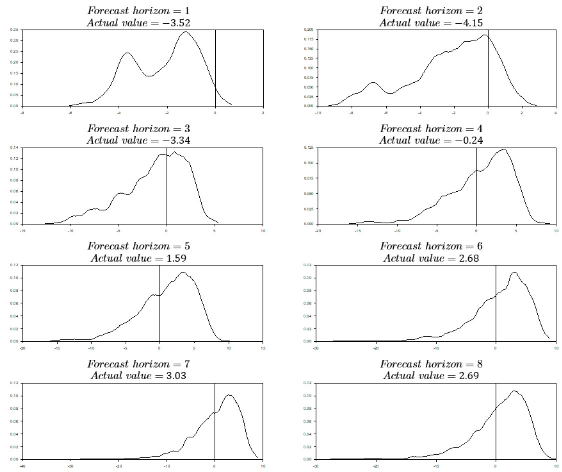
Therefore, from this out-of-sample forecasts comparison, we can conclude that the TAR model could forecasts the growth rate of the real GDP with a good performance, given that of the alternatives, the TAR model appears to be marginally preferred to the competing modes, except the MSAR model that seems to be more competitive according with these tests.

Horizon	Relative MSE	DM - MDM	Encompassing tests	U-Theil
1	4 (AR, SETAR, STAR, MSAR)	1 (SETAR)	2 (AR, STAR)	4 (AR, SETAR, STAR, MSAR)
2	3 (AR, SETAR, STAR)	2 (SETAR, STAR)	0	3 (AR, SETAR, STAR)
3	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)
4	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)
5	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)
6	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)
7	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)
8	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)

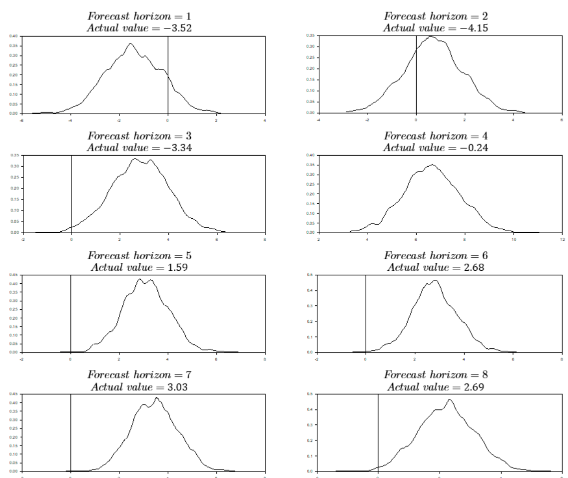
(1) Cells highlighted green report that the TAR model outperforms 3 or 4 models, highlighted orange report that the TAR model outperforms 1 or 2 models, and highlighted red report that the TAR model does not outperform any model. (2) Models in parenthesis are outperformed by the TAR model.

Table 4.14.: Summary of the forecasting performance of the TAR model. U.S. GDP case.

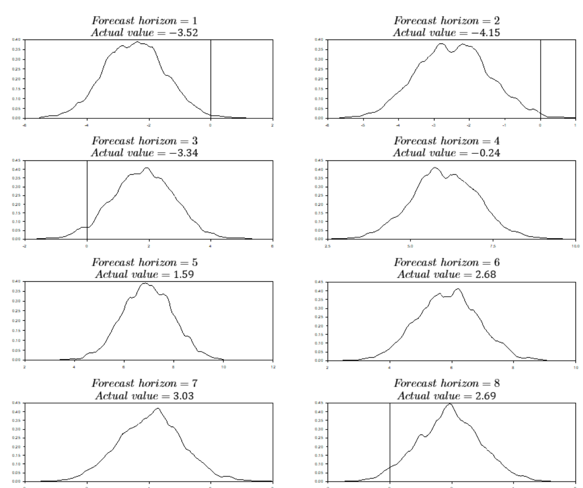
Finally, we compare the TAR model with the SETAR and STAR models based on their ability to describe the predictive distribution of the growth rate of the real GDP, in terms of the capacity to handle cycles. Figure 4.14 shows the predictive distributions for horizons 1 to 8, when the forecast origin is 2008:04. Based on the business contraction from 2007:04 to 2009:02, which is determined by the NBER, the predictive distributions of the TAR model tend to be bimodal when the time series is in a through (negative growths of the real GDP), and become more unimodal when the time series is regaining an expansionist trend (positive growths of the real GDP). The predictive distributions of the SETAR and STAR models slightly capture this behavior. This pattern of the predictive distributions is also observed in other parts of the time series. This suggests that the predictive distributions of the TAR model seem to handle cycles of the times series reasonably well.



(a)



(b)



(c)

Figure 4.14.: 1 to 8-step predictive distributions of the annual growth rate of the U.S. real output, for the (a) TAR, (b) SETAR and (c) STAR models. U.S. GDP case.

4.1.3. Industrial production index

Description of the data

We analyze the annual growth rate of the seasonally adjusted U.S. quarterly industrial production index (indpro), from 1960:01 to 2016:03 (227 observations), which was retrieved from the Federal Reserve Bank of St. Louis (FRED). As the threshold value, we use the term spread defined in Section 3.2 over the same period.

For the forecast comparison, we denote $X_t = \left[\log(Indpro_t) - \log(Indpro_{t-4})\right] * 100$ as the annual growth rate of the industrial production index (indpro), and Z_t, the spread term defined as the difference between the 10-Year Treasury Constant Maturity rate and the effective federal funds. Both series are plotted in Figure 4.15 that shows a similar behavior during contractions periods of these series. The shading areas denote the business cycle contractions from peak to trough.

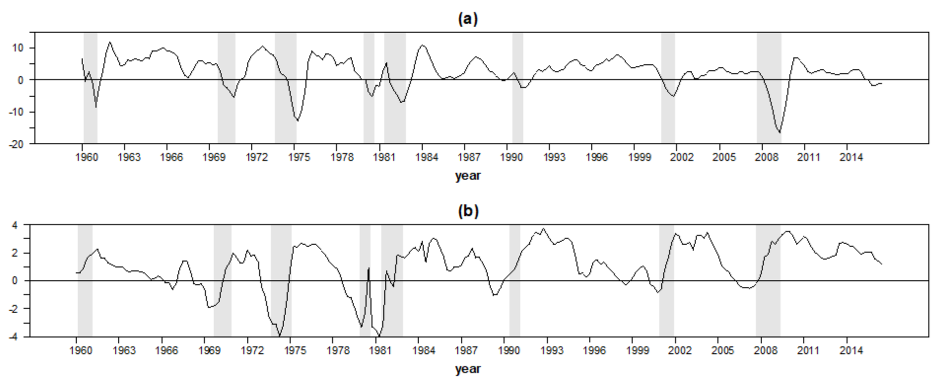


Figure 4.15.: (a) Time plot of the annual growth rate of U.S. quarterly industrial production index and (b) time plot of U.S. term spread.

We use as the training subsample, the data from 1960:01 to 1999:04 (160 observations). The remaining observations are reserved for the out-of-sample forecasting evaluation. By using the procedure mentioned at the beginning of this Chapter, a sequence of 1 to 8-step ahead forecasts are generated until we compute 67 1-step ahead forecasts down to 60 8-step ahead forecast.

Estimation of models

Based on the estimation procedure in Appendix D, Table 4.15 shows that, in general, the TAR shows the best reasonable in-sample fit.

Model adequacy	AR	TAR	SETAR	STAR	MSAR
White noise	2	2	3	3	3
Model specification	2	1	1	1	3
Homoscedasticity	2	2	2	2	2

Cells with number 1 and highlighted green represent good adequacy, 2 and highlighted orange represent regular adequacy, and 3 and highlighted red represent bad adequacy.

Table 4.15.: Model adequacy. U.S. indpro case.

Forecasting evaluation

Table 4.16 shows the HP unbiased test and Ljung-Box's Q correlation test. At the 10% significance level, only forecasts errors of the AR model are unbiased, and only the SETAR and STAR exhibit uncorrelated forecasts errors at horizons greater than 3 and 5 periods, respectively.

(a)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.840	0.000	0.000	0.000	0.001
2	0.853	0.000	0.000	0.000	0.001
3	0.822	0.000	0.000	0.000	0.001
4	0.771	0.000	0.000	0.000	0.002
5	0.769	0.000	0.000	0.000	0.002
6	0.760	0.000	0.000	0.000	0.004
7	0.823	0.000	0.000	0.000	0.006
8	0.912	0.000	0.000	0.000	0.009

(b)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.065	0.059	0.000
4	0.000	0.000	0.166	0.068	0.000
5	0.000	0.000	0.178	0.081	0.000
6	0.000	0.000	0.135	0.121	0.000
7	0.000	0.006	0.405	0.668	0.000
8	0.000	0.002	0.727	0.528	0.000

Cells highlighted green have a p–value less than 0.1.

Table 4.16.: p–values of the (a) unbiased test and (b) correlation test for the first 4 lags. U.S. indpro case.

Now, we present the forecast comparison, also at the 10% significance level. Table 4.17 shows globally, that the TAR model has MSE smaller than those for the linear and SETAR models, but the MSAR model has the smallest MSE among all the estimated models, except at horizons of 1 period, where the MSE of the STAR model is much better.

Horizon	MSE	AR	TAR	SETAR	STAR	MSAR
1	3.352	1.000	0.739	0.559	0.434	0.508
2	15.140	1.000	0.701	0.925	0.846	0.114
4	50.300	1.000	0.685	0.944	0.764	0.035
5	68.231	1.000	0.562	0.865	0.632	0.026
6	81.172	1.000	0.494	0.615	0.493	0.022
7	87.856	1.000	0.455	0.529	0.437	0.020
8	87.565	1.000	0.456	0.561	0.412	0.020

(1) The column marked by MSE shows the MSE of forecasts from the AR model. (2) Cells highlighted green are the smallest values for each forecasts horizon.

Table 4.17.: Relative MSE of forecasts. U.S. indpro case.

Henceforth, we define the TAR model as the benchmark model. Table 4.18 indicates that forecasts from the TAR model are more accurate than those from the SETAR at horizons greater than 1, and there is no significant difference with forecasts form the STAR and AR models. However, forecasts from the MSAR model are more accurate than those from the TAR model.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.755	0.004	0.028	0.006
2	0.765	0.992	0.914	0.017
3	0.781	0.988	0.879	0.016
4	0.777	0.970	0.765	0.016
5	0.862	0.981	0.789	0.015
6	0.893	0.920	0.494	0.014
7	0.906	0.916	0.425	0.015
8	0.898	0.983	0.316	0.022

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.245	0.996	0.972	0.994
2	0.235	0.008	0.086	0.983
3	0.219	0.012	0.121	0.984
4	0.223	0.030	0.235	0.984
5	0.138	0.019	0.211	0.985
6	0.107	0.080	0.506	0.987
7	0.094	0.084	0.575	0.985
8	0.102	0.017	0.684	0.978

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.748	0.004	0.032	0.015
2	0.761	1.000	0.893	0.042
3	0.780	0.999	0.863	0.040
4	0.778	0.977	0.745	0.041
5	0.838	0.979	0.772	0.042
6	0.851	0.926	0.495	0.037
7	0.856	0.903	0.436	0.039
8	0.844	0.991	0.338	0.050

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.252	0.996	0.968	0.985
2	0.239	0.000	0.107	0.958
3	0.220	0.001	0.137	0.960
4	0.222	0.023	0.255	0.959
5	0.162	0.021	0.228	0.958
6	0.149	0.074	0.505	0.963
7	0.144	0.097	0.564	0.961
8	0.156	0.009	0.662	0.950

Cells highlighted green have a p–value less than 0.1.

Table 4.18.: DM test when (a) H_1 : Forecasts from competing model (F2) are better than forecasts from TAR model (F1) and (b) H_1 : F1 are better than F2; MDM test when (c) H_1 : F2 are better than F1 and (d) H_1 : F1 are better than F2. U.S. indpro case.

Table 4.19 shows, under the CH test, that the TAR model could encompass the linear model for the first 4-step ahead forecasts, and the STAR model at horizon greater than 3. However, under the ER and HLN tests and for the overall comparison, the TAR could encompass the linear model, while the TAR, SETAR and STAR do not encompass each other, and the MSAR model could encompass the benchmark model.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.752	0.824	0.411	0.779
2	0.930	0.165	0.907	0.124
3	0.573	0.044	0.594	0.021
4	0.253	0.032	0.503	0.004
5	0.013	0.106	0.367	0.001
6	0.002	0.142	0.222	0.003
7	0.003	0.070	0.128	0.007
8	0.006	0.036	0.090	0.017

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.015	0.613	0.391	0.622
2	0.005	0.461	0.695	0.061
3	0.028	0.015	0.221	0.001
4	0.049	0.001	0.052	0.002
5	0.060	0.002	0.034	0.029
6	0.070	0.023	0.014	0.024
7	0.105	0.022	0.004	0.034
8	0.146	0.017	0.005	0.041

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.179	0.000	0.000	0.000
2	0.246	0.016	0.000	0.000
3	0.247	0.012	0.003	0.000
4	0.227	0.029	0.000	0.000
5	0.427	0.174	0.001	0.000
6	0.610	0.025	0.000	0.090
7	0.688	0.013	0.004	0.000
8	0.595	0.038	0.005	0.000

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.004	0.495	0.493	0.243
2	0.004	0.000	0.000	0.044
3	0.002	0.000	0.000	0.008
4	0.002	0.000	0.000	0.011
5	0.000	0.000	0.000	0.090
6	0.000	0.000	0.000	0.090
7	0.000	0.000	0.013	0.094
8	0.000	0.000	0.062	0.091

(e)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.064	0.000	0.004	0.000
2	0.086	0.014	0.014	0.008
3	0.077	0.012	0.028	0.012
4	0.074	0.021	0.020	0.015
5	0.162	0.116	0.022	0.013
6	0.270	0.046	0.006	0.012
7	0.321	0.055	0.011	0.013
8	0.258	0.054	0.009	0.020

(f)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.036	0.754	0.290	0.873
2	0.053	0.000	0.000	0.903
3	0.060	0.000	0.000	0.935
4	0.061	0.000	0.000	0.948
5	0.051	0.000	0.000	0.901
6	0.048	0.000	0.004	0.910
7	0.045	0.000	0.028	0.897
8	0.048	0.000	0.066	0.888

Cells highlighted green have a p–value less than 0.1.

Table 4.19.: CH test when (a) H_0 : F1 encompasses F2 and (b) H_0: F2 encompasses F1; ER test when (c) H_0 : F1 encompasses F2 and (d) H_0: F2 encompasses F1; HLN test when (e) H_0 : F1 encompasses F2 and (f) H_0: F2 encompasses F1. U.S. indpro case.

Table 4.20 shows that the MSAR model has the smallest value of the test and is the only model that has a better performance than the naïve model. The TAR model has the second-best performance at horizons over the 2 and 5 periods.

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.898	0.772	0.671	0.591	0.640
2	1.894	1.586	1.822	1.742	0.639
3	2.772	2.273	2.676	2.499	0.638
4	3.406	2.818	3.310	2.978	0.637
5	3.963	2.972	3.686	3.151	0.636
6	4.332	3.046	3.396	3.042	0.636
7	4.523	3.050	3.291	2.989	0.633
8	4.494	3.035	3.367	2.884	0.631

Cells highlighted green represent the lowest value for each forecast horizon.

Table 4.20.: Theil's U statistic. U.S. indpro case.

As a summary, it is observed that only the AR model has unbiased forecasts, and only the SETAR and STAR models present uncorrelated forecasts errors for some forecast periods. Additionally, Table 4.21 indicates a satisfactory performance of the TAR model, in general. According to the Relative MSE of forecast, the TAR model has better MSE of forecasts than the AR, SETAR and STAR models. The DM and MDM tests suggest that forecasts from the TAR model are more accurate than those form the lineal model. Regarding the encompassing tests, only the MSAR model encompasses the TAR model under the HLN test, and only the AR model is encompassed by the TAR model. Additionally, forecasts from the MSAR model are more accurate than those from the naïve model. However, the TAR model presents a Theil's U statistic smaller than the SETAR and STAR models.

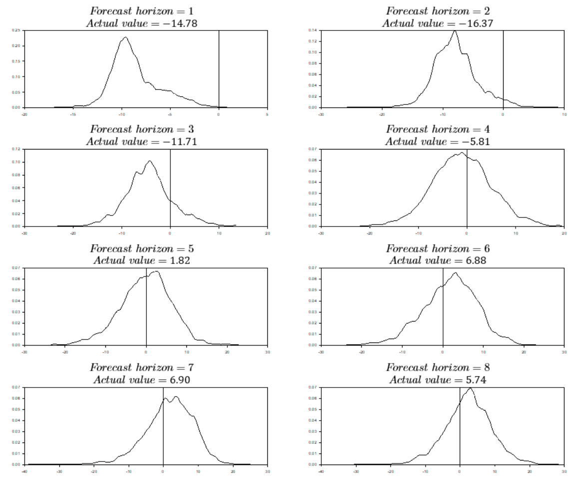
Therefore, from this out-of-sample forecasts comparison, we can conclude that the TAR model could forecasts the growth rate of the industrial production index well, given that of the alternatives, the TAR model appears to be marginally preferred to the competing modes, except the MSAR model that seems to be more competitive according with these tests.

Horizon	Relative MSE	DM - MDM	Encompassing tests	U-Theil
1	1 (AR)	0	1 (AR)	1 (AR)
2	3 (AR, SETAR, STAR)	1 (SETAR)	1 (AR)	3 (AR, SETAR, STAR)
3	3 (AR, SETAR, STAR)	1 (SETAR)	1 (AR)	3 (AR, SETAR, STAR)
4	3 (AR, SETAR, STAR)	1 (SETAR)	1 (AR)	3 (AR, SETAR, STAR)
5	3 (AR, SETAR, STAR)	1 (SETAR)	1 (AR)	3 (AR, SETAR, STAR)
6	2 (AR, SETAR)	1 (SETAR)	1 (AR)	2 (AR, SETAR)
7	2 (AR, SETAR)	1 (SETAR)	1 (AR)	2 (AR, SETAR)
8	2 (AR, SETAR)	1 (SETAR)	1 (AR)	2 (AR, SETAR)

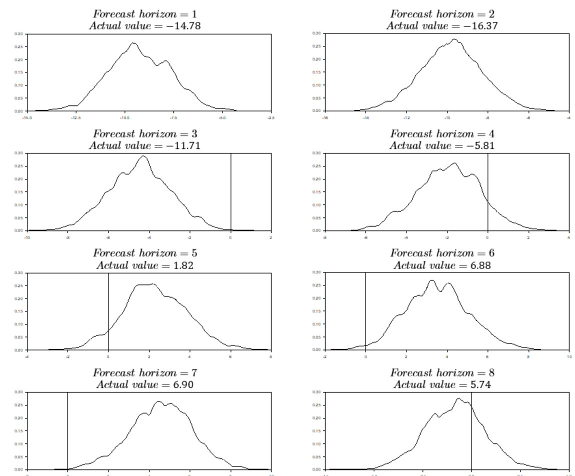
(1) Cells highlighted green report that the TAR model outperforms 3 or 4 models, highlighted orange report that the TAR model outperforms 1 or 2 models, and highlighted red report that the TAR model does not outperform any model. (2) Models in parenthesis are outperformed by the TAR model.

Table 4.21.: Summary of the forecasting performance of the TAR model. U.S. indpro case.

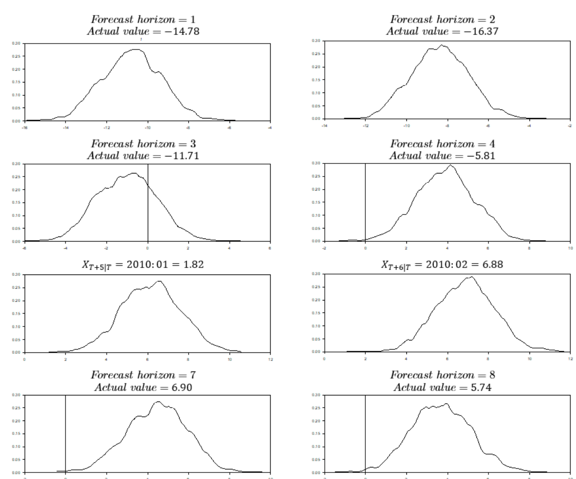
Finally, we compare the TAR model with the SETAR and STAR models based on their ability to describe the predictive distribution of the time series, in terms of the capacity to handle cycles. Figure 4.16 shows the predictive distributions for horizons 1 to 8, when the forecast origin is 2008:04. Based on the business contraction from 2007:04 to 2009:02, the predictive distributions of the TAR model tend to be bimodal when the time series is in a through and tend to become unimodal when the time series is begining an expansionist trend. The predictive distributions of the SETAR and STAR models slightly tend capture this behavior. This pattern of the predictive distributions is also observed in other parts of the time series. This suggests that the predictive distributions of the TAR model seem to handle cycles of the times series reasonably.



(a)



(b)



(c)

Figure 4.16.: 1 to 8-step predictive distributions of the change in the unemployment rate series, for the (a) TAR, (b) SETAR and (c) STAR models. U.S. indpro case.

4.1.4. Inflation

Description of the data

We analyze, as a proxy of the inflation, the seasonally adjusted U.S. quarterly Consumer Price Index (CPI), from 1954:04 to 2016:03 (248 observations), which was retrieved from the Federal Reserve Bank of St. Louis (FRED). As the threshold value, we use the term spread defined in Section 3.2 over the same period[27].

For the forecast comparison, we denote $X_t = [\log(CPI_t) - \log(CPI_{t-1})]*100$ as the growth rate of the CPI, and Z_t, the spread term defined as the difference between the 10-Year Treasury Constant Maturity rate and the effective federal funds. Both series are plotted in Figure 4.17, where the shading areas denote the business cycle contractions from peak to trough.

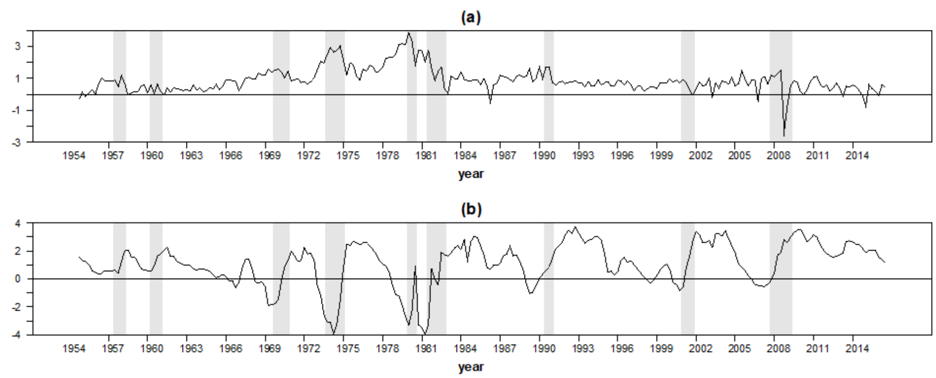


Figure 4.17.: (a) Time plot of the growth rate of U.S. quarterly CPI and (b) time plot of U.S. term spread.

We use as the training subsample, data from 1954:04 to 1998:01 (174 observations). The remaining observations are reserved for the out-of-sample forecasting evaluation. By using the procedure mentioned at the begining of this Chapter, a sequence of 1 to 8-step ahead forecasts are generated until we compute 74 1-step ahead forecasts, down to 67 8-step ahead forecast.

Estimation of models

Based on the estimated models in Appendix E, Table 4.22 shows that, in general, the TAR, SETAR and AR models show a reasonable in-sample fit.

[27] Cecchetti et al. (2000), Stock and Watson (2003) and Banerjee and Marcellino (2006) use quarterly data for analyzing the inflation.

Model adequacy	AR	TAR	SETAR	STAR	MSAR
White noise	1	1	1	1	1
Model specification	1	1	1	1	3
Homoscedasticity	1	1	1	2	3

Cells with number 1 and highlighted green represent good adequacy, 2 and highlighted orange represent regular adequacy, and 3 and highlighted red represent bad adequacy.

Table 4.22.: Model adequacy. U.S. CPI case.

Forecasting evaluation

Table 4.23 shows the HP unbiased test and Ljung-Box's Q correlation test. At the 10% significance level, only forecast errors of the TAR model are unbiased throughout all the forecast horizon, followed by the MSAR, AR, SETAR, and STAR models that exhibit unbiased forecast errors at the beginning of the forecast horizon. Regarding the Ljung Box test, the SETAR and STAR are the only models whose forecast errors do not have serial correlation at horizons greater than 2 and 1 periods, respectively.

(a)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.482	0.926	0.586	0.132	0.550
2	0.295	0.783	0.415	0.148	0.739
3	0.178	0.692	0.126	0.034	0.932
4	0.058	0.609	0.036	0.012	0.570
5	0.013	0.633	0.006	0.000	0.277
6	0.005	0.423	0.015	0.000	0.129
7	0.001	0.360	0.005	0.000	0.051
8	0.000	0.309	0.001	0.000	0.020

(b)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.038	0.001	0.015	0.098	0.000
2	0.000	0.000	0.001	0.786	0.001
3	0.015	0.004	0.244	0.691	0.002
4	0.018	0.000	0.473	0.171	0.003
5	0.000	0.000	0.067	0.291	0.003
6	0.002	0.000	0.400	0.235	0.004
7	0.006	0.001	0.276	0.826	0.004
8	0.000	0.001	0.640	0.287	0.005

Cells highlighted green have a p–value less than 0.1.

Table 4.23.: p–values of the (a) unbiased test and (b) correlation test for the first 4 lags. U.S. CPI case.

Now, we present the forecast comparison, also at the 10% significance level. Table 4.24 shows that the TAR model has smaller MSE than the benchmark linear model at horizons up until 4 periods, but the MSAR model has the smallest MSE among all the estimated models.

Horizon	MSE	AR	TAR	SETAR	STAR	MSAR
1	0.402	1.000	0.944	0.962	1.117	0.884
2	0.506	1.000	0.895	1.089	0.921	0.700
3	0.520	1.000	0.978	1.081	0.968	0.675
4	0.471	1.000	0.991	1.007	1.167	0.740
5	0.409	1.000	1.184	1.149	1.321	0.853
6	0.421	1.000	1.170	1.281	1.371	0.833
7	0.461	1.000	1.215	1.379	1.275	0.773
8	0.469	1.000	1.282	1.299	1.668	0.777

(1) The column marked by MSE shows the MSE of forecasts from the AR model. (2) Cells highlighted green are the smallest values for each forecasts horizon.

Table 4.24.: Relative MSE of forecasts. U.S. CPI case.

Henceforth, we define the TAR model as the benchmark model. Table 4.25, that shows the results of the DM and MDM tests, indicates in general that forecasts from the TAR model are more accurate than those from the linear model at horizons up until 2 periods, and there is no significant difference with forecasts from the SETAR and STAR models. However, forecast from the MSAR model are more accurate than those from the TAR model.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.942	0.641	0.820	0.046
2	0.969	0.960	0.652	0.059
3	0.635	0.827	0.473	0.007
4	0.547	0.545	0.947	0.020
5	0.002	0.413	0.725	0.015
6	0.014	0.774	0.808	0.007
7	0.013	0.812	0.614	0.003
8	0.025	0.527	0.874	0.008

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.058	0.359	0.180	0.954
2	0.031	0.040	0.348	0.941
3	0.365	0.173	0.527	0.993
4	0.453	0.455	0.053	0.980
5	0.998	0.587	0.275	0.985
6	0.986	0.226	0.192	0.993
7	0.987	0.188	0.386	0.997
8	0.975	0.473	0.126	0.992

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.943	0.642	0.822	0.045
2	0.970	0.961	0.653	0.057
3	0.643	0.983	0.472	0.006
4	0.547	0.546	0.949	0.020
5	0.002	0.413	0.727	0.014
6	0.014	0.775	0.810	0.007
7	0.013	0.814	0.615	0.003
8	0.024	0.528	0.875	0.008

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.057	0.358	0.178	0.955
2	0.030	0.039	0.347	0.943
3	0.357	0.017	0.528	0.994
4	0.453	0.454	0.051	0.980
5	0.998	0.587	0.273	0.986
6	0.986	0.225	0.190	0.993
7	0.987	0.186	0.385	0.997
8	0.976	0.472	0.125	0.992

Cells highlighted green have a p–value less than 0.1.

Table 4.25.: DM test when (a) H_1 : Forecasts from competing model (F2) are better than forecasts from TAR model (F1) and (b) H_1 : F1 are better than F2; MDM test when (c) H_1 : F2 are better than F1 and (d) H_1 : F1 are better than F2. U.S. CPI case.

Table 4.26 shows, under the CH test, that the TAR model could encompass the STAR model. But under the ER and HLN tests and for the overall comparison, the TAR could encompass the linear model at horizons up until 3 periods, while the TAR, SETAR and STAR do not encompass each other, and the MSAR model could encompass the TAR model at short horizons.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.134	0.206	0.166	0.187
2	0.021	0.021	0.142	0.454
3	0.057	0.044	0.133	0.344
4	0.033	0.084	0.315	0.934
5	0.085	0.207	0.572	0.897
6	0.024	0.035	0.448	0.932
7	0.014	0.064	0.359	0.756
8	0.006	0.037	0.113	0.673

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.027	0.059	0.079	0.115
2	0.007	0.033	0.021	0.174
3	0.009	0.004	0.008	0.222
4	0.001	0.001	0.027	0.808
5	0.000	0.000	0.002	0.435
6	0.000	0.001	0.000	0.177
7	0.000	0.002	0.000	0.272
8	0.000	0.000	0.000	0.052

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.365	0.008	0.525	0.003
2	0.610	0.569	0.000	0.000
3	0.255	0.410	0.027	0.000
4	0.072	0.019	0.000	0.000
5	0.000	0.002	0.000	0.000
6	0.000	0.016	0.000	0.000
7	0.000	0.013	0.000	0.000
8	0.000	0.005	0.031	0.000

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.000	0.001	0.000	0.818
2	0.003	0.000	0.000	0.569
3	0.064	0.009	0.038	0.559
4	0.042	0.010	0.000	0.069
5	0.791	0.006	0.000	0.032
6	0.865	0.000	0.000	0.009
7	0.899	0.000	0.000	0.151
8	0.892	0.003	0.000	0.171

(e)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.213	0.009	0.207	0.000
2	0.701	0.708	0.001	0.023
3	0.112	0.221	0.053	0.002
4	0.054	0.019	0.011	0.000
5	0.000	0.001	0.000	0.000
6	0.000	0.011	0.000	0.000
7	0.000	0.011	0.000	0.000
8	0.000	0.002	0.026	0.000

(f)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.003	0.055	0.099	0.413
2	0.001	0.008	0.033	0.278
3	0.068	0.003	0.006	0.721
4	0.024	0.006	0.000	0.079
5	0.605	0.022	0.002	0.033
6	0.433	0.000	0.000	0.021
7	0.450	0.002	0.001	0.090
8	0.446	0.015	0.000	0.087

Cells highlighted green have a p–value less than 0.1.

Table 4.26.: CH test when (a) H_0 : F1 encompasses F2 and (b) H_0: F2 encompasses F1; ER test when (c) H_0 : F1 encompasses F2 and (d) H_0: F2 encompasses F1; HLN test when (e) H_0 : F1 encompasses F2 and (f) H_0: F2 encompasses F1. U.S. CPI case.

Table 4.27 shows that forecasts from the MSAR model are more accurate than those from the naïve model, for all the forecast horizon.

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.926	0.900	0.909	0.979	0.871
2	1.032	0.976	1.077	0.990	0.863
3	1.039	1.028	1.080	1.022	0.854
4	0.982	0.978	0.986	1.061	0.845
5	0.909	0.990	0.974	1.045	0.840
6	0.918	0.993	1.039	1.075	0.838
7	0.954	1.051	1.120	1.077	0.839
8	0.954	1.080	1.087	1.232	0.841

Cells highlighted green represent the lowest value for each forecast horizon.

Table 4.27.: Theil's U statistic. U.S. CPI case.

As a summary, it is observed that only the TAR model has unbiased forecasts, but the SETAR and STAR models have uncorrelated forecasts errors for some forecast periods. Additionally, Table 4.28 indicates a satisfactory performance of the TAR model, in general. According to the Relative MSE of forecast, the TAR model has better MSE of forecasts than the AR models at short horizons, and the SETAR and STAR models in general. The

DM and MDM tests suggest that forecasts from the TAR model are more accurate than those form the lineal model at short horizons. Regarding the encompassing tests, the TAR model could encompass the linear model at horizons up until 3 periods, and is encompassed by the MSAR at long horizons. Additionally, forecasts from the MSAR model are more accurate than those from the naïve model. However, the TAR model presents a Theil's U statistic smaller than the SETAR, STAR and AR models.

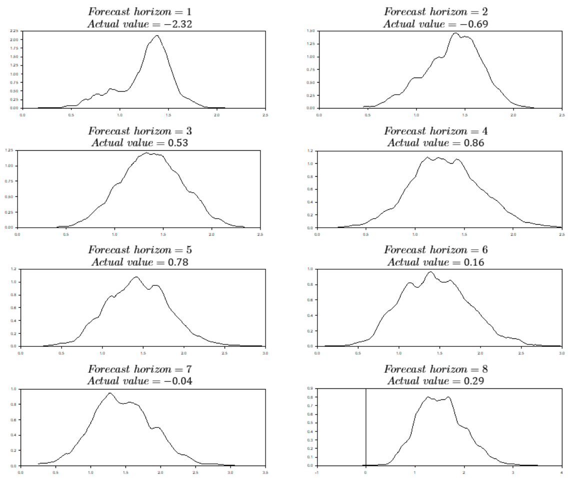
Therefore, the results of this out-of-sample forecasts comparison weakly favor the TAR model for forecasting the growth rate of the CPI. Besides, the MSAR model seems to be more competitive, according with these tests. Nevertheless, the TAR model appears to be marginally preferred to the other competing modes at short horizons (no more that 3-step ahead forecasts).

Horizon	Relative MSE	DM - MDM	Encompassing tests	U-Theil
1	3 (AR, SETAR, STAR)	1 (AR)	1 (AR)	3 (AR, SETAR, STAR)
2	3 (AR, SETAR, STAR)	2 (AR, SETAR)	1 (AR)	2 (AR, SETAR)
3	2 (AR, SETAR)	0	1 (AR)	2 (AR, SETAR)
4	3 (AR, SETAR, STAR)	1 (STAR)	0	3 (AR, SETAR, STAR)
5	1 (STAR)	0	0	1 (STAR)
6	2 (SETAR, STAR)	0	0	2 (SETAR, STAR)
7	2 (SETAR, STAR)	0	0	2 (SETAR, STAR)
8	2 (SETAR, STAR)	0	0	2 (SETAR, STAR)

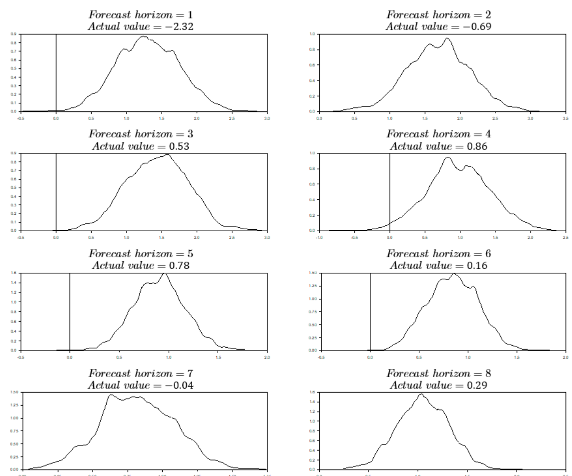
(1) Cells highlighted green report that the TAR model outperforms 3 or 4 models, highlighted orange report that the TAR model outperforms 1 or 2 models, and highlighted red report that the TAR model does not outperform any model. (2) Models in parenthesis are outperformed by the TAR model.

Table 4.28.: Summary of the forecasting performance of the TAR model. U.S. CPI case

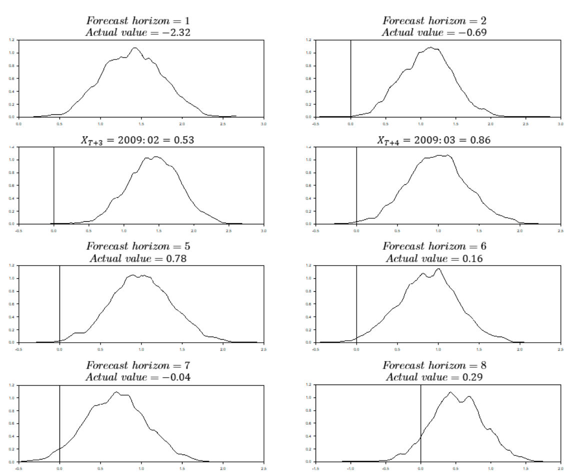
Finally, we compare the TAR model with the SETAR and STAR models based on their ability to describe the predictive distribution of the growth rate of the CPI, in terms of the capacity to handle cycles. Figure 4.18 shows the predictive distributions for horizons 1 to 8, when the forecast origin is 2008:03. Based on the business contraction from 2007:04 to 2009:02, the predictive distributions of the TAR model tend to be bimodal when the time series has negative values and become more unimodal when the time series is increasing. The predictive distributions of the SETAR and STAR models slightly capture this behavior. This pattern of the predictive distributions is also observed in other parts of the time series. This suggests that the predictive distributions of the TAR model seem to handle reasonably well cycles of the times series in a better way than those of the other models.



(a)



(b)



(c)

Figure 4.18.: 1 to 8-step predictive distributions of the change in the unemployment rate series, for the (a) TAR, (b) SETAR and (c) STAR models. U.S. CPI case.

As a global summary for the U.S. case, this empirical evaluation shows that forecasts from the TAR model outperform forecasts from the competing models at different h-step ahead horizons, for all the considered economic times series, although in general, the forecasting performance of this model is not better than that of the MSAR model. According to all the used evaluation criteria, the TAR model forecasts the unemployment rate better than the other models, at forecasts horizons greater than 2 periods ahead, and also, it's the model with the best in-sample properties. Regarding the GDP, industrial production index and the CPI time series, forecasts from the TAR model outperform forecasts from the other models, only according the relative MSE and Theil's U statistic and for horizons between 1 and 5 periods ahead. However, it prevails as the model with great in-sample properties. It is also relevant to remark that the TAR model shows a shape changing characteristic in the Bayesian predictive distributions of this model that may capture the cycles in the economic time series.

4.2. Empirical results for the Colombian economic time series

4.2.1. Unemployment rate

Description of the data

The data consists on the change in the seasonally adjusted Colombian monthly unemployment rate[28] (the first difference of the time series) and, from 2002:02 to 2016:09 (189 observations), and the growth rate of the seasonally adjusted Colombian monthly ISE index data (the first difference of the logarithm of the time series) over the same period. This period is chosen due to institutional constraints and the data is respectively obtained from both the Central Bank of Colombia databases and the DANE. Nonetheless, this is an interesting period to analyze because the Colombian economy began a recovery period, with a modified macroeconomic policy, after the financial crisis of the late 90s.

In Figure 4.19 we observe that changes in the unemployment rate are less volatile since 2008 than at the beginning of the analyzed period, while the ISE index allows us to identify the recovery of the Colombian economy in 2002 after the crisis of the late 90s and the posterior contraction in 2009 because of the U.S. financial crisis of 2007. For comparison, we use the business cycle determined by Alfonso et al. (2012), which are the grid areas that denote the contractions from peak to trough.

[28] The Colombian monthly unemployment rate, retrieved from the Central Bank of Colombia, has seasonal patterns, thus this time series was seasonally adjusted using the X13-ARIMA program of the U.S. Census Bureau of the U.S. Department of commerce.

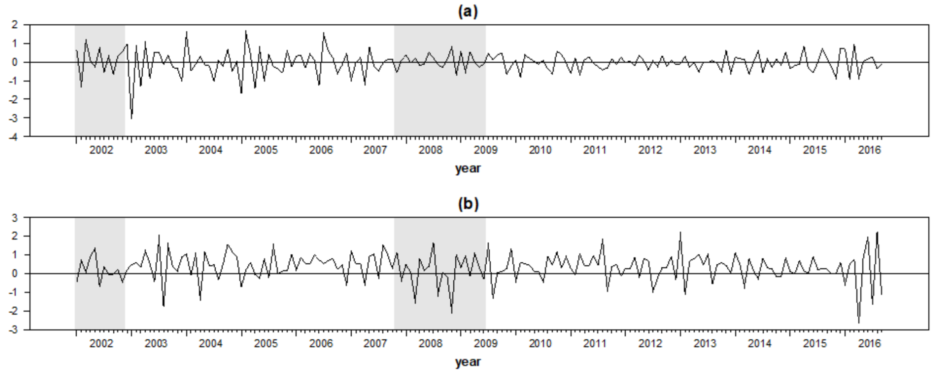


Figure 4.19.: (a) Time plot of the change in the the Colombian monthly unemployment rate and (b) time plot of growth rate of Colombian ISE monthly index.

Then, we use as the training subsample all the observations from 2001:02 to 2011:12 (131 observations). The remaining observations are reserved for the out-of-sample forecasting comparison. The selection of the forecast horizon for monthly time series is based on the literature review. Then, a sequence of 1 to 12-step ahead forecasts are generated from the forecast origin 2011:12. After that, the forecast origin is moved one period ahead and forecasts are calculated again. This procedure is repeated until we compute 53 1-step ahead forecasts, 57 2-step ahead forecasts, and so on until 46 12-step ahead forecasts.

Estimation of models

Based on the estimation procedure in Appendix F, Table 4.29 shows a summary of the model adequacy of the analyzed models. Globally, it is observed that the TAR and SETAR models exhibit a reasonable in-sample fit, compared with the other models.

Model adequacy	AR	TAR	SETAR	STAR	MSAR
White noise	1	1	1	1	2
Model specification	2	1	1	2	3
Homoscedasticity	3	3	3	3	3

Cells with number 1 and highlighted green represent good adequacy, 2 and highlighted orange represent regular adequacy, and 3 and highlighted red represent bad adequacy.

Table 4.29.: Model adequacy. Colombian unemployment rate case.

Forecasting evaluation

Table 4.30 shows the HP unbiased test and Ljung-Box's Q correlation test. In general, at the 10% significance level, forecasts errors for all the models are unbiased, and only the TAR, SETAR and STAR shows uncorrelated forecasts errors at horizons greater than 7, 4 and 1 periods, respectively.

(a)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.408	0.390	0.547	0.447	0.308
2	0.704	0.791	0.807	0.788	0.188
3	0.625	0.836	0.234	0.965	0.142
4	0.662	0.810	0.740	0.913	0.106
5	0.591	0.834	0.990	0.473	0.114
6	0.573	0.849	0.679	0.720	0.132
7	0.671	0.788	0.227	0.493	0.107
8	0.657	0.813	0.668	0.158	0.094
9	0.713	0.726	0.573	0.628	0.079
10	0.633	0.856	0.335	0.692	0.098
11	0.688	0.815	0.970	0.805	0.103
12	0.664	0.809	0.590	0.647	0.115

(b)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.011	0.000	0.035	0.000	0.010
2	0.000	0.118	0.042	0.460	0.011
3	0.002	0.003	0.108	0.898	0.011
4	0.023	0.018	0.082	0.808	0.009
5	0.004	0.001	0.578	0.190	0.010
6	0.000	0.003	0.674	0.461	0.014
7	0.000	0.023	0.134	0.341	0.014
8	0.000	0.100	0.116	0.329	0.014
9	0.000	0.115	0.260	0.769	0.016
10	0.000	0.122	0.531	0.093	0.018
11	0.000	0.652	0.374	0.775	0.020
12	0.004	0.114	0.069	0.278	0.020

Cells highlighted green have a p–value less than 0.1.

Table 4.30.: p–values of the (a) unbiased test and (b) correlation test for the first 4 lags. Colombian unemployment rate case.

Next, we present the forecast comparison, at the 10% significance level. Table 4.31 shows that in general, the TAR model and the linear model are very close in MSE. We observe that the linear model has the smallest MSE among all the estimated models, except for the horizons 6, 7, 8 and 10, where the MSE of the TAR model is much better.

Horizon	MSE	AR	TAR	SETAR	STAR	MSAR
1	0.195	1.000	1.264	1.123	1.160	1.026
2	0.141	1.000	1.658	1.364	2.093	1.519
3	0.180	1.000	1.185	1.354	1.644	1.247
4	0.182	1.000	1.028	1.172	1.402	1.287
5	0.186	1.000	1.010	1.100	1.179	1.299
6	0.189	1.000	0.996	1.163	1.578	1.314
7	0.190	1.000	0.998	1.271	1.688	1.303
8	0.194	1.000	0.998	1.302	1.467	1.308
9	0.194	1.000	1.017	1.208	1.188	1.326
10	0.198	1.000	0.997	1.441	1.749	1.328
11	0.201	1.000	1.000	1.596	1.008	1.338
12	0.203	1.000	1.005	1.139	1.729	1.339

(1) The column marked by MSE shows the MSE of forecasts from the AR model. (2) Cells highlighted green are the smallest values for each forecasts horizon.

Table 4.31.: Relative MSE of forecasts. Colombian unemployment rate case.

Henceforth, we define the TAR model as the benchmark model. Table 4.32 shows, for the overall results, that forecasts from the TAR model are more accurate than those from the SETAR at horizon greater than 5 periods and STAR models at horizon greater than 1 period.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.008	0.065	0.100	0.004
2	0.003	0.066	0.990	0.292
3	0.007	0.905	0.960	0.626
4	0.206	0.883	0.933	0.903
5	0.323	0.795	0.795	0.911
6	0.570	0.883	1.000	0.931
7	0.533	0.982	1.000	0.918
8	0.556	0.981	0.979	0.911
9	0.183	0.994	0.941	0.901
10	0.535	0.994	0.990	0.917
11	0.497	1.000	0.518	0.927
12	0.436	0.809	0.998	0.914

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.992	0.935	0.900	0.996
2	0.997	0.934	0.010	0.708
3	0.993	0.095	0.040	0.374
4	0.795	0.117	0.067	0.097
5	0.677	0.205	0.205	0.089
6	0.430	0.117	0.000	0.069
7	0.467	0.018	0.000	0.082
8	0.444	0.019	0.021	0.089
9	0.817	0.006	0.059	0.099
10	0.465	0.006	0.010	0.083
11	0.503	0.000	0.482	0.073
12	0.564	0.191	0.002	0.086

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.020	0.052	0.029	0.009
2	0.015	0.122	1.000	0.263
3	0.029	0.861	0.948	0.637
4	0.269	0.885	0.968	0.888
5	0.271	0.797	0.774	0.886
6	0.559	0.970	0.999	0.909
7	0.530	0.969	0.999	0.892
8	0.546	0.950	0.950	0.879
9	0.032	1.000	0.943	0.865
10	0.535	0.995	0.991	0.886
11	0.497	1.000	0.519	0.897
12	0.435	0.811	0.998	0.877

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.980	0.948	0.971	0.991
2	0.985	0.878	0.000	0.737
3	0.971	0.139	0.052	0.363
4	0.731	0.115	0.032	0.112
5	0.729	0.203	0.226	0.114
6	0.441	0.030	0.001	0.091
7	0.470	0.031	0.001	0.108
8	0.454	0.050	0.050	0.121
9	0.968	0.000	0.057	0.135
10	0.465	0.005	0.009	0.114
11	0.503	0.000	0.481	0.103
12	0.565	0.189	0.002	0.123

Cells highlighted green have a p–value less than 0.1.

Table 4.32.: DM test when (a) H_1 : Forecasts from competing model (F2) are better than forecasts from TAR model (F1) and (b) H_1 : F1 are better than F2; MDM test when (c) H_1 : F2 are better than F1 and (d) H_1 : F1 are better than F2. Colombian unemployment rate case.

Table 4.33 shows the forecast encompassing tests of Chong and Hendry (1986) (CH), Ericsson (1992) (ER) and Harvey et al. (1998) (HLN) that evaluate the null hypothesis of forecast encompassing. At the 10% significance level, we find that the TAR model encompasses the SETAR, STAR and MSAR models for some forecast periods.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.136	0.099	0.017	0.060
2	0.000	0.573	0.233	0.615
3	0.922	0.199	0.426	0.629
4	0.319	0.511	0.705	0.578
5	0.959	0.563	0.045	0.630
6	0.140	0.364	0.045	0.640
7	0.272	0.817	0.020	0.540
8	0.355	0.610	0.711	0.557
9	0.707	0.510	0.119	0.538
10	0.737	0.292	0.269	0.534
11	0.256	0.016	0.010	0.624
12	0.930	0.814	0.032	0.593

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000
4	0.880	0.319	0.300	0.069
5	0.765	0.742	0.544	0.076
6	0.484	0.143	0.737	0.281
7	0.524	0.330	0.453	0.193
8	0.431	0.715	0.042	0.075
9	0.876	0.139	0.211	0.020
10	0.665	0.565	0.516	0.230
11	0.695	0.812	0.804	0.085
12	0.794	0.222	0.634	0.085

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.000	0.000	0.348	0.000
2	0.000	0.002	0.745	0.018
3	0.000	0.592	0.572	0.098
4	0.164	0.304	0.033	0.413
5	0.214	0.399	0.020	0.516
6	0.660	0.349	0.571	0.576
7	0.630	0.736	0.688	0.479
8	0.607	0.671	0.046	0.525
9	0.092	0.755	0.336	0.454
10	0.798	0.284	0.010	0.506
11	0.683	0.027	0.050	0.569
12	0.557	0.718	0.014	0.541

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.004	0.921	0.000	0.000
2	0.000	0.804	0.000	0.226
3	0.001	0.003	0.000	0.021
4	0.790	0.000	0.000	0.000
5	0.721	0.016	0.000	0.000
6	0.429	0.000	0.000	0.000
7	0.516	0.000	0.000	0.000
8	0.426	0.000	0.000	0.000
9	0.875	0.000	0.000	0.000
10	0.665	0.000	0.008	0.000
11	0.695	0.000	0.000	0.000
12	0.794	0.016	0.595	0.000

(e)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.003	0.008	0.029	0.003
2	0.003	0.007	0.201	0.037
3	0.003	0.302	0.371	0.059
4	0.086	0.164	0.280	0.200
5	0.090	0.218	0.020	0.250
6	0.329	0.161	0.975	0.282
7	0.317	0.371	0.717	0.229
8	0.306	0.664	0.344	0.250
9	0.052	0.625	0.043	0.213
10	0.399	0.862	0.817	0.243
11	0.342	0.979	0.026	0.275
12	0.283	0.352	0.954	0.260

(f)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.973	0.460	0.693	0.994
2	0.998	0.597	0.000	0.110
3	0.984	0.001	0.001	0.027
4	0.605	0.000	0.003	0.011
5	0.365	0.004	0.001	0.010
6	0.217	0.009	0.000	0.007
7	0.255	0.001	0.000	0.010
8	0.208	0.002	0.000	0.013
9	0.563	0.000	0.000	0.013
10	0.334	0.000	0.001	0.010
11	0.347	0.000	0.006	0.010
12	0.396	0.050	0.000	0.012

Cells highlighted green have a p–value less than 0.1.

Table 4.33.: CH test when (a) H_0 : F1 encompasses F2 and (b) H_0: F2 encompasses F1; ER test when (c) H_0 : F1 encompasses F2 and (d) H_0: F2 encompasses F1; HLN test when (e) H_0 : F1 encompasses F2 and (f) H_0: F2 encompasses F1. Colombian unemployment rate case.

Table 4.34 shows that forecasts from all models are more accurate than those from the naïve model, where the AR and TAR model have the smallest value of the test.

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.629	0.707	0.667	0.678	0.637
2	0.532	0.685	0.621	0.770	0.656
3	0.596	0.649	0.693	0.764	0.665
4	0.594	0.602	0.643	0.704	0.674
5	0.596	0.599	0.625	0.647	0.679
6	0.596	0.595	0.643	0.749	0.683
7	0.595	0.594	0.671	0.773	0.679
8	0.596	0.595	0.680	0.722	0.681
9	0.592	0.597	0.651	0.646	0.682
10	0.595	0.594	0.714	0.787	0.686
11	0.594	0.594	0.751	0.597	0.687
12	0.594	0.596	0.634	0.782	0.688

Cells highlighted green represent the lowest value for each forecast horizon.

Table 4.34.: Theil's U statistic. Colombian unemployment rate case.

As a summary, it is observed that forecasts errors for all the models are unbiased, and only the TAR, SETAR and STAR shows uncorrelated forecasts errors at different horizons. Additionally, Table 4.35 indicates a satisfactory performance of the TAR model. According to the relative MSE of forecasts, we observe that the linear model has the smallest MSE at short horizons, and the TAR model has better MSE at middle horizons. The DM and MDM test suggest that forecasts from the TAR model are more accurate than those from the SETAR and STAR models. Besides, the TAR model encompasses the SETAR, STAR and MSAR models for some horizons. Following the Theil's U statistic, forecasts from AR and TAR models are more accurate from those from the naïve model.

Therefore, from this out-of-sample forecasts comparison, we can conclude that the TAR model could forecasts the change in the unemployment rate with a satisfactory performance, given that of the alternatives, the TAR model appears to be marginally preferred to the competing modes.

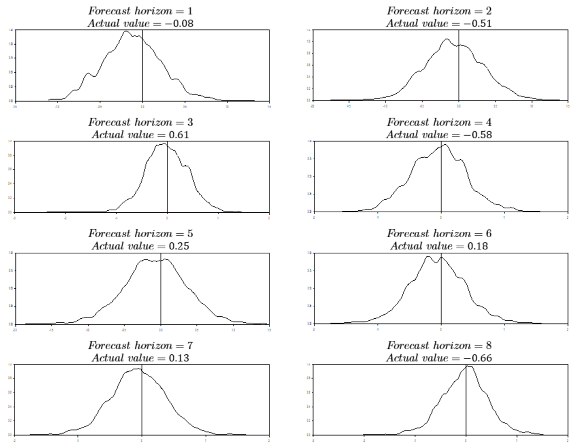
Horizon	Relative MSE	DM - MDM	Encompassing tests	U-Theil
1	0	0	0	0
2	1 (STAR)	1 (STAR)	1 (STAR)	1 (STAR)
3	3 (SETAR, STAR, MSAR)	2 (SETAR, STAR)	3 (SETAR, STAR, MSAR)	3 (SETAR, STAR, MSAR)
4	3 (SETAR, STAR, MSAR)	2 (SETAR, STAR)	3 (SETAR, STAR, MSAR)	3 (SETAR, STAR, MSAR)
5	3 (SETAR, STAR, MSAR)	2 (SETAR, STAR)	3 (SETAR, STAR, MSAR)	3 (SETAR, STAR, MSAR)
6	4 (AR SETAR, STAR, MSAR)	2 (SETAR, STAR)	3 (SETAR, STAR, MSAR)	4 (AR SETAR, STAR, MSAR)
7	4 (AR SETAR, STAR, MSAR)	2 (SETAR, STAR)	3 (SETAR, STAR, MSAR)	4 (AR SETAR, STAR, MSAR)
8	4 (AR SETAR, STAR, MSAR)	2 (SETAR, STAR)	3 (SETAR, STAR, MSAR)	4 (AR SETAR, STAR, MSAR)
9	3 (SETAR, STAR, MSAR)	2 (SETAR, STAR)	2 (SETAR, MSAR)	3 (SETAR, STAR, MSAR)
10	4 (AR SETAR, STAR, MSAR)	2 (SETAR, STAR)	2 (SETAR, MSAR)	4 (AR SETAR, STAR, MSAR)
11	3 (SETAR, STAR, MSAR)	2 (SETAR, STAR)	1 (MSAR)	3 (SETAR, STAR, MSAR)
12	3 (SETAR, STAR, MSAR)	2 (SETAR, STAR)	2 (SETAR, MSAR)	3 (SETAR, STAR, MSAR)

(1) Cells highlighted green report that the TAR model outperforms 3 or 4 models, highlighted orange report that the TAR model outperforms 1 or 2 models, and highlighted red report that the TAR model does not outperform any model. (2) Models in parenthesis are outperformed by the TAR model.

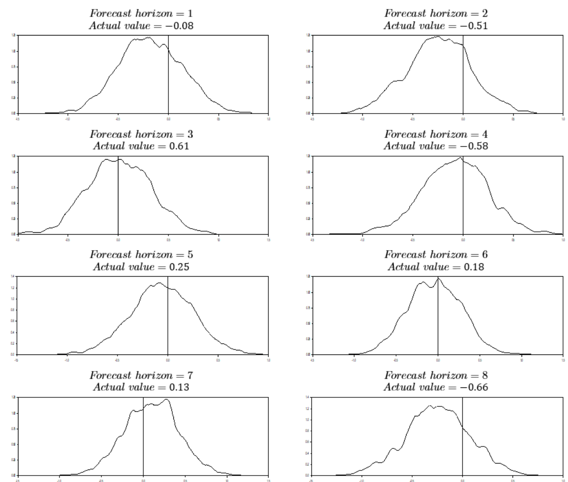
Table 4.35.: Summary of the forecasting performance of the TAR model. Colombian unemployment rate case.

Finally, we compare the TAR model with the SETAR and STAR models based on their ability to describe the predictive distribution of the economic time series. Given that, according to the literature review, in Colombia there has not been determined yet a business cycle during the analyzed forecasting period, we use the Bry and Boschan (1971) business cycle dating algorithm[29], using the annual growth rate of the Colombian ISE index as a measure of economic activity. This algorithm determines a contraction phase from 2013:10 to 2015:02. However, during this period, it is observed a real activity slow down but not a economic hardship with negative values, in fact, all values of the growth rate of the Colombian GDP during the forecasting subsample, are positive.

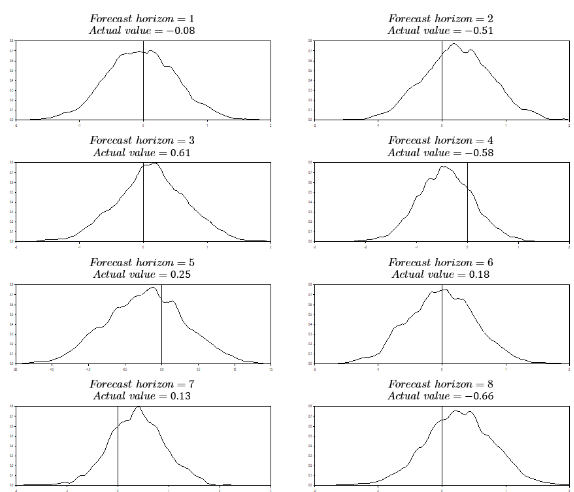
[29] This algorithm is the best known algorithm to detect turning points in the monthly time series, which are used to determine periods of expansions and contractions (Harding and Pagan, 2002). In general, the Bry-Boschan Algorithm: i) Replaces outliers in a preliminary trend-cycle; ii) Selects preliminary turning points by finding local maxima and minima of the adjusted trend-cycle; iii) Eliminates consecutive "peaks" and consecutive "troughs", by keeping the most extreme in a sequence.



(a)



(b)



(c)

Figure 4.20.: 1 to 8-step predictive distributions of the change in the unemployment rate series, for the (a) TAR, (b) SETAR and (c) STAR models. Colombian unemployment rate case.

Figure 4.20 shows the predictive distributions for horizons 1 to 8, when the forecast origin is 2013:08. The predictive distributions of the TAR model tend to be unimodal when the time series is ascending to a peak (negative increments of the unemployment rate), although it does not capture the bimodal shape in unstable periods. That behavior could be explained in the sense that, there is no a strong contraction phase in the Colombian real activity during the forecasting period, as it was mentioned above. The predictive distributions of the SETAR and STAR models have the same performance. This pattern of the predictive distributions is also observed in other parts of the time series.

4.2.2. Gross domestic product

Description of the data

We analyze, as a proxy of the GDP, the annual growth rate of the seasonally adjusted Colombian monthly indicator of the real economic activity ISE, from 2003:01 to 2016:09 (165 observations), which was retrieved from the DANE. As the threshold value, we use the term spread defined in Section 3.2 over the same period, that is, the difference between the ten-year Treasury bonds (TES) as the long-term government bond rate, and the inter-bank interest rate as the overnight rate. Both series were retrieved from the Central Bank of Colombia.

For the forecast comparison, we denote $X_t = [\log(ISE_t) - \log(ISE_{t-12})] * 100$ as the annual growth rate of the ISE, and Z_t, the spread term defined above. Both series are plotted in Figure 4.21 that shows a similar behavior during contractions periods of these series. The shading areas denote the business cycle contractions from peak to trough based on Alfonso et al. (2012).

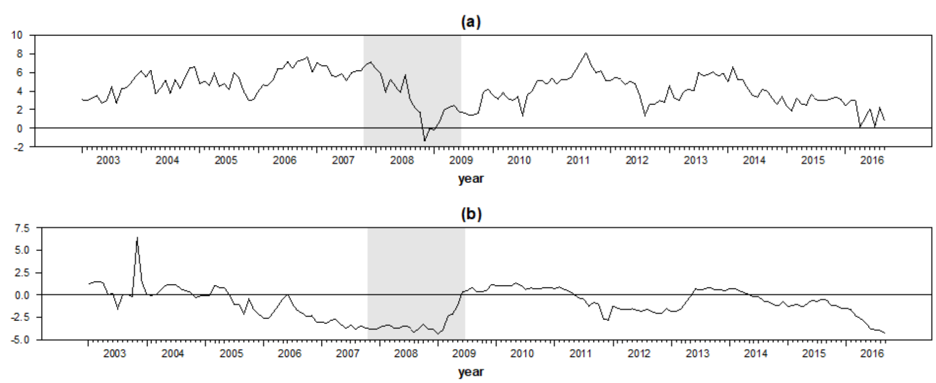


Figure 4.21.: (a) Time plot of the annual growth rate of Colombian GDP and (b) time plot of Colombian term spread.

We use as the training subsample, the data until 2012:06 (114 observations). The remaining observations are reserved for the out-of-sample forecasting evaluation. By using the procedure mentioned at the begining of this Chapter, a sequence of 1 to 12-step ahead forecasts are generated until we compute 51 1-step ahead forecasts, down to 40 12-step ahead forecast.

Estimation of models

Based on results in Appendix G, Table 4.36 shows that globally, the STAR model presents the best reasonable in-sample fit, followed by the TAR, SETAR and AR models.

Model adequacy	AR	TAR	SETAR	STAR	MSAR
White noise	2	2	2	1	1
Model specification	1	1	1	1	3
Homoscedasticity	1	1	1	1	2

Cells with number 1 and highlighted green represent good adequacy, 2 and highlighted orange represent regular adequacy, and 3 and highlighted red represent bad adequacy.

Table 4.36.: Model adequacy. Colombian GDP case.

Forecasting evaluation

Table 4.37 shows the HP unbiased test and Ljung-Box's Q correlation test. Globally, at the 10% significance level, only forecasts errors of the SETAR model are unbiased, and only the SETAR and STAR exhibit uncorrelated forecasts errors at some forescasts peroiods.

(a)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.076	0.007	0.611	0.048	0.000
2	0.017	0.001	0.456	0.041	0.000
3	0.004	0.000	0.105	0.028	0.000
4	0.004	0.000	0.306	0.009	0.000
5	0.002	0.000	0.686	0.016	0.000
6	0.001	0.000	0.848	0.007	0.000
7	0.001	0.000	0.909	0.001	0.000
8	0.001	0.000	0.955	0.001	0.000
9	0.001	0.000	0.997	0.001	0.000
10	0.001	0.000	0.900	0.001	0.000
11	0.001	0.000	0.730	0.000	0.000
12	0.001	0.000	0.862	0.000	0.000

(b)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.009	0.000
4	0.000	0.000	0.000	0.205	0.000
5	0.000	0.000	0.000	0.796	0.000
6	0.000	0.000	0.001	0.951	0.000
7	0.000	0.000	0.021	0.529	0.000
8	0.000	0.002	0.128	0.629	0.000
9	0.000	0.010	0.123	0.257	0.000
10	0.000	0.039	0.469	0.056	0.000
11	0.000	0.068	0.579	0.000	0.000
12	0.000	0.076	0.701	0.002	0.000

Cells highlighted green have a p–value less than 0.1.

Table 4.37.: p–values of the (a) unbiased test and (b) correlation test for the first 4 lags. Colombian GDP case.

Now, we present the forecast comparison, also at the 10% significance level. Table 4.38 shows in general that the AR model has the smallest MSE among all the estimated models, follow by the MSAR and TAR models.

Horizon	MSE	AR	TAR	SETAR	STAR	MSAR
1	0.893	1.000	1.061	1.118	0.984	2.850
2	1.133	1.000	1.257	1.528	1.430	2.699
3	1.242	1.000	1.311	2.082	2.142	2.418
4	1.679	1.000	1.256	2.549	2.148	1.850
5	1.993	1.000	1.229	2.289	2.157	1.592
6	2.239	1.000	1.270	1.767	1.383	1.427
7	2.465	1.000	1.250	2.442	1.367	1.308
8	2.805	1.000	1.237	2.638	1.574	1.172
9	3.047	1.000	1.213	2.912	1.517	1.066
10	3.201	1.000	1.169	2.861	1.658	1.004
11	3.371	1.000	1.138	3.011	1.923	0.976
12	3.490	1.000	1.141	3.374	1.795	0.962

(1) The column marked by MSE shows the MSE of forecasts from the AR model. (2) Cells highlighted green are the smallest values for each forecasts horizon.

Table 4.38.: Relative MSE of forecasts. Colombian GDP case.

Hereafter, we define the TAR model as the benchmark model. Table 4.39 shows that for the overall comparison, forecasts from the TAR model are more accurate than those from the SETAR and STAR models at long horizons, and more accurate than those from the MSAR model at short horizons, although forecasts from the linear model are more accurate than those from the TAR model, in general.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.168	0.684	0.283	1.000
2	0.003	0.878	0.804	1.000
3	0.002	0.930	0.990	1.000
4	0.020	0.995	0.998	0.967
5	0.007	0.996	0.994	0.943
6	0.015	0.770	0.637	0.678
7	0.045	0.967	0.629	0.561
8	0.042	0.977	0.883	0.428
9	0.064	0.985	0.809	0.342
10	0.131	0.983	0.946	0.327
11	0.157	0.993	1.000	0.328
12	0.137	0.994	0.999	0.315

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.832	0.316	0.717	0.000
2	0.997	0.122	0.196	0.000
3	0.998	0.070	0.010	0.000
4	0.980	0.005	0.002	0.033
5	0.993	0.004	0.006	0.057
6	0.985	0.230	0.363	0.322
7	0.955	0.033	0.371	0.439
8	0.958	0.023	0.117	0.572
9	0.936	0.015	0.191	0.658
10	0.869	0.017	0.054	0.673
11	0.843	0.007	0.000	0.672
12	0.863	0.006	0.001	0.685

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.166	0.686	0.281	1.000
2	0.003	0.880	0.806	1.000
3	0.001	0.932	0.990	0.999
4	0.019	0.995	0.998	0.929
5	0.006	0.996	0.995	0.944
6	0.030	0.715	0.631	0.641
7	0.077	0.923	0.608	0.548
8	0.086	0.952	0.846	0.446
9	0.132	0.961	0.753	0.384
10	0.186	0.960	0.872	0.370
11	0.221	0.975	0.999	0.369
12	0.198	0.982	1.000	0.363

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.834	0.314	0.719	0.000
2	0.997	0.120	0.194	0.000
3	0.999	0.068	0.010	0.001
4	0.981	0.005	0.002	0.071
5	0.994	0.004	0.005	0.056
6	0.970	0.286	0.369	0.359
7	0.923	0.077	0.392	0.452
8	0.914	0.048	0.154	0.554
9	0.868	0.039	0.247	0.616
10	0.814	0.040	0.128	0.630
11	0.779	0.025	0.001	0.631
12	0.802	0.018	0.000	0.637

Cells highlighted green have a p–value less than 0.1.

Table 4.39.: DM test when (a) H_1 : Forecasts from competing model (F2) are better than forecasts from TAR model (F1) and (b) H_1 : F1 are better than F2; MDM test when (c) H_1 : F2 are better than F1 and (d) H_1 : F1 are better than F2. Colombian GDP case.

Table 4.40 shows that the linear model encompasses the TAR model at long horizons, but under the ER and HLN tests, the TAR model encompasses the SETAR and STAR models at horizons greater than 5 and 8 periods respectively, while the TAR and MSAR model do not encompass each other.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.011	0.011	0.020	0.361
2	0.007	0.015	0.011	0.009
3	0.001	0.006	0.001	0.021
4	0.000	0.000	0.000	0.274
5	0.000	0.000	0.000	0.287
6	0.070	0.059	0.106	0.028
7	0.063	0.037	0.136	0.332
8	0.048	0.024	0.097	0.339
9	0.047	0.009	0.080	0.325
10	0.049	0.013	0.066	0.354
11	0.045	0.005	0.054	0.383
12	0.041	0.002	0.058	0.371

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.058	0.345	0.047	0.000
2	0.018	0.186	0.024	0.000
3	0.007	0.052	0.029	0.000
4	0.004	0.218	0.008	0.000
5	0.002	0.530	0.015	0.000
6	0.118	0.804	0.085	0.000
7	0.127	0.855	0.016	0.000
8	0.105	0.887	0.013	0.000
9	0.107	0.928	0.028	0.000
10	0.117	0.995	0.039	0.000
11	0.111	0.800	0.015	0.000
12	0.101	0.906	0.017	0.000

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.070	0.140	0.033	0.016
2	0.000	0.004	0.084	0.896
3	0.000	0.115	0.681	0.014
4	0.000	0.288	0.532	0.006
5	0.000	0.095	0.505	0.000
6	0.000	0.128	0.010	0.000
7	0.007	0.259	0.017	0.001
8	0.008	0.340	0.044	0.000
9	0.014	0.541	0.132	0.001
10	0.058	0.477	0.358	0.000
11	0.068	0.876	0.934	0.001
12	0.042	0.993	0.672	0.001

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.951	0.013	0.333	0.000
2	0.001	0.000	0.000	0.000
3	0.058	0.000	0.000	0.000
4	0.136	0.000	0.000	0.000
5	0.079	0.000	0.000	0.000
6	0.114	0.003	0.001	0.000
7	0.222	0.000	0.002	0.000
8	0.233	0.000	0.000	0.002
9	0.298	0.000	0.001	0.008
10	0.508	0.000	0.000	0.007
11	0.625	0.000	0.000	0.009
12	0.594	0.000	0.000	0.016

(e)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.033	0.048	0.027	0.994
2	0.001	0.003	0.046	0.448
3	0.000	0.056	0.339	0.057
4	0.006	0.149	0.268	0.044
5	0.001	0.071	0.247	0.004
6	0.003	0.089	0.004	0.013
7	0.014	0.134	0.000	0.006
8	0.012	0.180	0.003	0.004
9	0.023	0.261	0.038	0.001
10	0.059	0.225	0.162	0.000
11	0.071	0.437	0.533	0.001
12	0.056	0.504	0.335	0.001

(f)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.525	0.036	0.166	0.000
2	0.986	0.002	0.003	0.000
3	0.974	0.001	0.000	0.000
4	0.907	0.000	0.000	0.002
5	0.946	0.000	0.000	0.000
6	0.931	0.002	0.028	0.017
7	0.870	0.000	0.025	0.019
8	0.868	0.000	0.004	0.031
9	0.832	0.001	0.008	0.056
10	0.732	0.001	0.001	0.045
11	0.678	0.001	0.000	0.042
12	0.691	0.002	0.000	0.051

Cells highlighted green have a p–value less than 0.1.

Table 4.40.: CH test when (a) H_0 : F1 encompasses F2 and (b) H_0: F2 encompasses F1; ER test when (c) H_0 : F1 encompasses F2 and (d) H_0: F2 encompasses F1; HLN test when (e) H_0 : F1 encompasses F2 and (f) H_0: F2 encompasses F1. Colombian GDP case.

Table 4.41 shows that only at horizons of 1 period, forecasts from all models, except the MSAR model, are more accurate than those from the naïve model. Globally, the AR model has the smallest value of the test, and the TAR model has the second-smallest value of the test at horizons up until 7 periods.

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.935	0.963	0.989	0.928	1.579
2	1.063	1.192	1.314	1.271	1.747
3	1.149	1.316	1.658	1.681	1.786
4	1.344	1.506	2.145	1.970	1.828
5	1.449	1.606	2.192	2.128	1.828
6	1.522	1.715	2.023	1.789	1.818
7	1.580	1.767	2.469	1.847	1.807
8	1.727	1.920	2.804	2.166	1.869
9	1.818	2.002	3.103	2.239	1.877
10	1.843	1.992	3.117	2.373	1.847
11	1.890	2.016	3.280	2.622	1.868
12	1.901	2.030	3.492	2.547	1.865

Cells highlighted green represent the lowest value for each forecast horizon.

Table 4.41.: Theil's U statistic. Colombian GDP case.

As a summary, it is observed that only the SETAR model has unbiased forecasts, and with the STAR model, are the only models that present uncorrelated forecasts errors for some forecast periods. Additionally, Table 4.42 let us observe a satisfactory performance of the TAR model. According to the Relative MSE of forecast and the DM and MDM tests, the TAR model has better MSE of forecasts than the SETAR, STAR and MSAR models, for some horizons. Regarding the encompassing tests, only the TAR model encompasses the SETAR model at all horizons, encompasses the STAR at long horizons and the MSAR at short horizons. Additionally, forecasts from all models are not more accurate than those from the naïve model.

Therefore, from this out-of-sample forecasts comparison, we can conclude that the TAR model could forecasts the growth rate of the real GDP with a good performance, given that of the alternatives, the TAR model appears to be marginally preferred to the competing modes.

Horizon	Relative MSE	DM - MDM	Encompassing tests	U-Theil
1	2 (SETAR, MSAR)	1 (MSAR)	1 (MSAR)	2 (SETAR, MSAR)
2	3 (SETAR, STAR, MSAR)	1 (MSAR)	1 (MSAR)	3 (SETAR, STAR, MSAR)
3	3 (SETAR, STAR, MSAR)	3 (SETAR, STAR, MSAR)	1 (STAR)	3 (SETAR, STAR, MSAR)
4	3 (SETAR, STAR, MSAR)	3 (SETAR, STAR, MSAR)	2 (SETAR, STAR)	3 (SETAR, STAR, MSAR)
5	3 (SETAR, STAR, MSAR)	3 (SETAR, STAR, MSAR)	1 (STAR)	3 (SETAR, STAR, MSAR)
6	3 (SETAR, STAR, MSAR)	0	1 (SETAR)	3 (SETAR, STAR, MSAR)
7	3 (SETAR, STAR, MSAR)	1 (SETAR)	1 (SETAR)	3 (SETAR, STAR, MSAR)
8	2 (SETAR, STAR)	1 (SETAR)	1 (SETAR)	2 (SETAR, STAR)
9	2 (SETAR, STAR)	1 (SETAR)	1 (SETAR)	2 (SETAR, STAR)
10	2 (SETAR, STAR)	1 (SETAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
11	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	1 (SETAR)
12	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	1 (SETAR)

(1) Cells highlighted green report that the TAR model outperforms 3 or 4 models, highlighted orange report that the TAR model outperforms 1 or 2 models, and highlighted red report that the TAR model does not outperform any model. (2) Models in parenthesis are outperformed by the TAR model.

Table 4.42.: Summary of the forecasting performance of the TAR model. Colombian GDP case.

Finally, we compare the TAR model with the SETAR and STAR models based on their ability to describe the predictive distribution of the economic time series. As it was observed in the Colombian unemployment rate case, the forecasting period of the time series do not show a strong contraction phase (there are only positive values, that is, positive growth rates of the GDP). Thus, the 1 to 8-step predictive distributions of the TAR, SETAR and STAR models tend to be unimodal, capturing the stability of the time series.

4.2.3. Industrial production index

Description of the data

We analyze the seasonally adjusted Colombian biannual growth rate of Colombian industrial production index, from 2003:01 to 2016:11 (167 observations), which was retrieved from the DANE. As the threshold value, we use the term spread defined in Section 3.2 over the same period and mentioned above.

For the forecast comparison, we denote $X_t = \left[\log(Indpro_t) - \log(Indpro_{t-6})\right] * 100$ as the annual growth rate of the Colombian industrial production index (indpro), and Z_t, the spread term. Both series are plotted in Figure 4.22 that shows a similar behavior during contractions periods of these series. The shading areas denote the business cycle contractions from peak to trough.

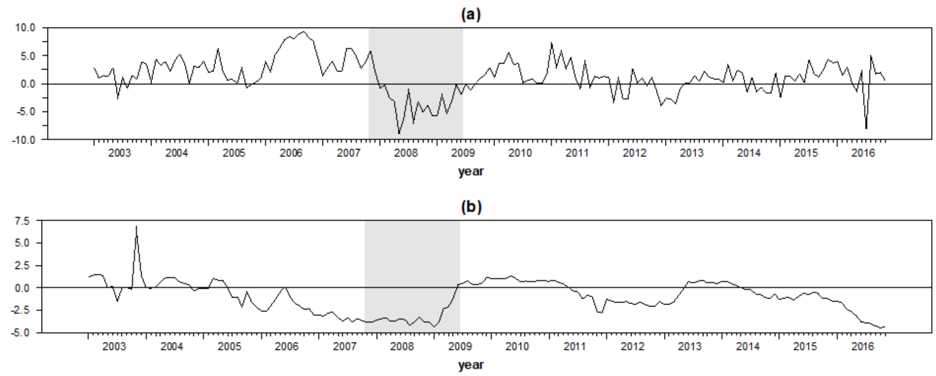


Figure 4.22.: (a) Time plot of the biannual growth rate of Colombian industrial production index and (b) time plot of Colombian term spread.

We use as the training subsample, the data until 2012:06 (114 observations). The remaining observations are reserved for the out-of-sample forecasting evaluation. By using the procedure mentioned in Section 4.1.1, a sequence of 1 to 12-step ahead forecasts are generated until we compute 53 1-step ahead forecasts, down to 42 12-step ahead forecast.

Estimation of models

Based on results in Appendix H, Table 4.43 shows that, in general, the TAR and STAR models present the best reasonable in-sample fit.

Model adequacy	AR	TAR	SETAR	STAR	MSAR
White noise	2	1	2	1	1
Model specification	1	1	1	1	3
Homoscedasticity	1	1	1	1	1

Cells with number 1 and highlighted green represent good adequacy, 2 and highlighted orange represent regular adequacy, and 3 and highlighted red represent bad adequacy.

Table 4.43.: Model adequacy. Colombian indpro case.

Forecasting evaluation

Table 4.44 shows the HP unbiased test and Ljung-Box's Q correlation test. At the 10% significance level, forecasts errors of the SETAR model are unbiased, followed by the MSAR, STAR and AR models for horizons up to 9. Regarding the Ljung Box test, in general, the TAR, SETAR and STAR are the only models whose forecast errors do not have serial correlation at horizons greater than 5 periods.

(a)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.814	0.045	0.706	0.865	0.613
2	0.657	0.047	0.898	0.673	0.776
3	0.546	0.014	0.946	0.752	0.940
4	0.465	0.002	0.882	0.257	0.872
5	0.537	0.000	0.916	0.549	0.783
6	0.444	0.000	0.423	0.178	0.598
7	0.259	0.000	0.240	0.153	0.347
8	0.165	0.000	0.159	0.019	0.238
9	0.083	0.002	0.207	0.036	0.152
10	0.028	0.003	0.117	0.024	0.073
11	0.020	0.002	0.354	0.042	0.056
12	0.013	0.010	0.466	0.169	0.047

(b)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.000	0.000	0.000	0.010	0.000
2	0.000	0.000	0.297	0.000	0.000
3	0.000	0.000	0.400	0.002	0.000
4	0.000	0.001	0.250	0.002	0.000
5	0.000	0.026	0.098	0.010	0.000
6	0.000	0.105	0.739	0.185	0.000
7	0.000	0.282	0.833	0.019	0.000
8	0.000	0.263	0.162	0.279	0.000
9	0.000	0.332	0.004	0.273	0.000
10	0.000	0.361	0.266	0.430	0.000
11	0.000	0.655	0.188	0.603	0.000
12	0.000	0.280	0.482	0.766	0.001

Cells highlighted green have a p–value less than 0.1.

Table 4.44.: p–values of the (a) unbiased test and (b) correlation test for the first 4 lags. Colombian indpro case.

Now, we present the forecast comparison, also at the 10% significance level. Table 4.45 shows, for the overall comparison, the TAR model and the linear model are very close in MSE, but the MSAR model is the one that has the smallest MSE. The TAR model has the third-smallest MSE.

Horizon	MSE	AR	TAR	SETAR	STAR	MSAR
1	5.284	1.000	1.017	1.137	1.034	0.991
2	4.649	1.000	1.093	1.369	1.588	1.115
3	5.920	1.000	1.100	1.705	2.182	0.868
4	6.706	1.000	1.150	1.454	2.237	0.767
5	6.353	1.000	1.219	1.874	2.574	0.817
6	7.341	1.000	1.155	1.852	2.643	0.711
7	6.387	1.000	1.244	1.956	2.719	0.784
8	6.574	1.000	1.140	1.755	3.114	0.772
9	6.389	1.000	1.356	1.658	2.507	0.804
10	5.928	1.000	1.032	1.666	2.933	0.858
11	6.378	1.000	1.070	1.617	2.586	0.820
12	6.423	1.000	1.003	1.764	2.747	0.837

(1) The column marked by MSE shows the MSE of forecasts from the AR model. (2) Cells highlighted green are the smallest values for each forecasts horizon.

Table 4.45.: Relative MSE of forecasts. Colombian indpro case.

Hereafter, we define the TAR model as the benchmark model. Table 4.46 shows in general, that forecasts from the TAR model are more accurate than those from the SETAR and STAR at horizon greater than 2 and 1 periods respectively, but forecasts from the MSAR models are more accurate than those from the TAR model at horizon over 3 and 9 periods. We also find no significant difference between the TAR and AR MSE of the forecasts.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.422	0.861	0.543	0.370
2	0.213	0.885	1.000	0.539
3	0.189	0.988	0.996	0.087
4	0.120	0.869	0.991	0.031
5	0.067	0.939	1.000	0.034
6	0.153	0.991	1.000	0.011
7	0.071	0.961	0.993	0.020
8	0.195	0.977	1.000	0.038
9	0.137	0.911	0.975	0.057
10	0.448	0.959	1.000	0.251
11	0.364	0.981	1.000	0.124
12	0.493	0.993	0.998	0.185

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.578	0.139	0.457	0.630
2	0.787	0.115	0.000	0.462
3	0.811	0.012	0.004	0.913
4	0.880	0.131	0.009	0.969
5	0.933	0.061	0.000	0.966
6	0.847	0.009	0.000	0.989
7	0.929	0.039	0.007	0.980
8	0.805	0.023	0.000	0.962
9	0.863	0.089	0.025	0.943
10	0.552	0.041	0.000	0.749
11	0.636	0.019	0.000	0.876
12	0.507	0.007	0.002	0.815

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.422	0.861	0.543	0.369
2	0.218	0.827	1.000	0.548
3	0.186	0.989	0.995	0.085
4	0.118	0.872	0.992	0.030
5	0.065	0.941	1.000	0.032
6	0.150	0.992	1.000	0.010
7	0.068	0.963	0.993	0.019
8	0.192	0.978	1.000	0.036
9	0.134	0.913	0.976	0.055
10	0.448	0.961	1.000	0.248
11	0.363	0.982	1.000	0.121
12	0.493	0.993	0.998	0.182

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.578	0.139	0.457	0.631
2	0.782	0.173	0.000	0.452
3	0.814	0.011	0.005	0.915
4	0.882	0.128	0.008	0.970
5	0.935	0.059	0.000	0.968
6	0.850	0.008	0.000	0.990
7	0.932	0.037	0.007	0.981
8	0.808	0.022	0.000	0.964
9	0.866	0.087	0.024	0.945
10	0.552	0.039	0.000	0.752
11	0.637	0.018	0.000	0.879
12	0.507	0.007	0.002	0.818

Cells highlighted green have a p–value less than 0.1.

Table 4.46.: DM test when (a) H_1 : Forecasts from competing model (F2) are better than forecasts from TAR model (F1) and (b) H_1 : F1 are better than F2; MDM test when (c) H_1 : F2 are better than F1 and (d) H_1 : F1 are better than F2. Colombian indpro case.

Table 4.47 shows, under the CH test, that the TAR model is encompassed by all the competing models, although the TAR model also encompasses the STAR and MSAR models at some horizons. However, under the ER and HLN tests and for the overall comparison, none of the models encompasses each other, except at some horizons where the TAR could encompass the SETAR and STAR models, and could be encompassed at middle horizons by the MSAR model.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.034	0.024	0.186	0.015
2	0.037	0.212	0.231	0.187
3	0.011	0.064	0.003	0.676
4	0.004	0.042	0.075	0.864
5	0.013	0.880	0.022	0.741
6	0.016	0.159	0.504	0.730
7	0.058	0.053	0.611	0.914
8	0.087	0.075	0.670	0.804
9	0.036	0.148	0.493	0.465
10	0.009	0.026	0.970	0.479
11	0.010	0.097	0.789	0.062
12	0.009	0.059	0.929	0.061

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.355	0.276	0.052	0.276
2	0.672	0.946	0.132	0.127
3	0.532	0.696	0.063	0.670
4	0.700	0.450	0.296	0.812
5	0.734	0.621	0.825	0.347
6	0.488	0.358	0.297	0.358
7	0.497	0.053	0.294	0.382
8	0.167	0.190	0.018	0.153
9	0.597	0.155	0.299	0.680
10	0.044	0.321	0.072	0.065
11	0.026	0.454	0.105	0.052
12	0.007	0.689	0.197	0.022

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.007	0.327	0.081	0.013
2	0.000	0.034	0.032	0.024
3	0.000	0.273	0.699	0.000
4	0.000	0.035	0.369	0.000
5	0.000	0.007	0.575	0.000
6	0.000	0.072	0.006	0.000
7	0.000	0.201	0.031	0.000
8	0.000	0.070	0.005	0.000
9	0.001	0.012	0.003	0.000
10	0.003	0.410	0.022	0.002
11	0.000	0.036	0.002	0.000
12	0.000	0.330	0.048	0.000

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.020	0.003	0.050	0.071
2	0.064	0.000	0.000	0.014
3	0.071	0.000	0.000	0.149
4	0.037	0.000	0.000	0.548
5	0.046	0.000	0.000	0.321
6	0.005	0.000	0.000	0.397
7	0.039	0.000	0.000	0.468
8	0.001	0.000	0.000	0.187
9	0.267	0.000	0.000	0.739
10	0.006	0.000	0.000	0.063
11	0.001	0.000	0.000	0.063
12	0.000	0.000	0.000	0.033

(e)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.014	0.185	0.061	0.027
2	0.005	0.014	0.069	0.024
3	0.001	0.138	0.647	0.001
4	0.000	0.019	0.186	0.000
5	0.000	0.006	0.292	0.000
6	0.000	0.068	0.009	0.000
7	0.000	0.098	0.016	0.000
8	0.000	0.050	0.007	0.000
9	0.002	0.018	0.003	0.001
10	0.001	0.194	0.027	0.000
11	0.000	0.038	0.005	0.000
12	0.001	0.174	0.024	0.000

(f)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.010	0.001	0.028	0.026
2	0.031	0.001	0.000	0.044
3	0.043	0.000	0.000	0.080
4	0.026	0.001	0.000	0.283
5	0.033	0.001	0.000	0.176
6	0.007	0.000	0.000	0.205
7	0.026	0.000	0.000	0.235
8	0.003	0.000	0.000	0.097
9	0.133	0.000	0.000	0.368
10	0.008	0.003	0.000	0.062
11	0.001	0.000	0.000	0.040
12	0.000	0.000	0.000	0.021

Cells highlighted green have a p–value less than 0.1.

Table 4.47.: CH test when (a) H_0 : F1 encompasses F2 and (b) H_0: F2 encompasses F1; ER test when (c) H_0 : F1 encompasses F2 and (d) H_0: F2 encompasses F1; HLN test when (e) H_0 : F1 encompasses F2 and (f) H_0: F2 encompasses F1. Colombian indpro case.

Table 4.48 shows that the MSAR model has the smallest value of the Theil's U statistic, followed by the AR and TAR model, and only forecasts from these three models are more accurate than those from the naïve model at all horizons.

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.765	0.771	0.815	0.778	0.761
2	0.715	0.748	0.837	0.901	0.755
3	0.800	0.839	1.045	1.182	0.746
4	0.845	0.906	1.019	1.264	0.740
5	0.816	0.901	1.117	1.309	0.737
6	0.873	0.938	1.188	1.419	0.736
7	0.812	0.906	1.136	1.339	0.719
8	0.817	0.872	1.082	1.441	0.718
9	0.796	0.927	1.026	1.261	0.714
10	0.759	0.771	0.980	1.300	0.703
11	0.786	0.813	0.999	1.263	0.711
12	0.780	0.781	1.035	1.292	0.713

Cells highlighted green represent the lowest value for each forecast horizon.

Table 4.48.: Theil's U statistic. Colombian indpro case.

As a summary, it is observed that only the TAR model does not have unbiased forecasts, and only the TAR, SETAR and STAR models have uncorrelated forecasts errors at long horizons. Additionally, Table 4.49 indicates a satisfactory performance of the TAR model, in general. According to the Relative MSE of forecast, the TAR model has better MSE of forecasts than the SETAR and STAR models, and the MSAR has the smallest MSE. The DM and MDM tests suggest that forecasts from the TAR model are more accurate than those form the SETAR and STAR models, and forecasts from the MSAR are more accurate than those from the TAR model at middle horizon. Regarding the encompassing tests, at some horizons the TAR could encompass the SETAR and STAR models, and could be encompassed at middle horizons by the MSAR model. Additionally, forecasts from the MSAR model are more accurate than those from the naïve model. However, the TAR model presents a Theil's U statistic smaller than the SETAR and STAR models.

Therefore, from this out-of-sample forecasts comparison, we can conclude that the TAR model could forecasts the growth rate of the industrial production index well, given that of the alternatives, the TAR model appears to be marginally preferred to the competing modes, except the MSAR model that seems to be more competitive according with these tests.

Horizon	Relative MSE	DM - MDM	Encompassing tests	U-Theil
1	2 (SETAR, STAR)	0	1 (SETAR)	2 (SETAR, STAR)
2	2 (SETAR, STAR)	1 (STAR)	0	2 (SETAR, STAR)
3	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
4	2 (SETAR, STAR)	1 (STAR)	1 (STAR)	2 (SETAR, STAR)
5	2 (SETAR, STAR)	2 (SETAR, STAR)	1 (STAR)	2 (SETAR, STAR)
6	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)
7	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)
8	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)
9	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)
10	2 (SETAR, STAR)	2 (SETAR, STAR)	1 (SETAR)	2 (SETAR, STAR)
11	2 (SETAR, STAR)	2 (SETAR, STAR)	0	2 (SETAR, STAR)
12	2 (SETAR, STAR)	2 (SETAR, STAR)	1 (SETAR)	2 (SETAR, STAR)

(1) Cells highlighted green report that the TAR model outperforms 3 or 4 models, highlighted orange report that the TAR model outperforms 1 or 2 models, and highlighted red report that the TAR model does not outperform any model. (2) Models in parenthesis are outperformed by the TAR model.

Table 4.49.: Summary of the forecasting performance of the TAR model. Colombian indpro case.

Finally, we compare the TAR model with the SETAR and STAR models based on their ability to describe the predictive distribution of the economic time series, in terms of the capacity to handle cycles. As it was observed in the Colombian unemployment rate case, the forecasting period of the time series do not show strong contraction business cycle phase. Thus, the 1 to 8-step predictive distributions of the TAR, SETAR and STAR models are unimodal, capturing the stability of the time series. However, the TAR model slightly shows bimodal shapes for the lowest values of the growth rate of the industrial production index, traying to capture descending periods better than the other models.

4.2.4. Inflation

Description of the data

We analyze, as a proxy of the inflation, the seasonally adjusted Colombian monthly Consumer Price Index (CPI), from 2003:01 to 2016:09 (165 observations), which was retrieved from the DANE. As the threshold value, we use the term spread defined in Section 3.2 over the same period.

For the forecast comparison, we denote $X_t = [\log(CPI_t) - \log(CPI_{t-1})]*100$ as the growth rate of the CPI, and Z_t, the spread term defined above. Both series are plotted in Figure 4.23 that shows a similar behavior during contractions periods of these series. The shading areas denote the business cycle contractions from peak to trough.

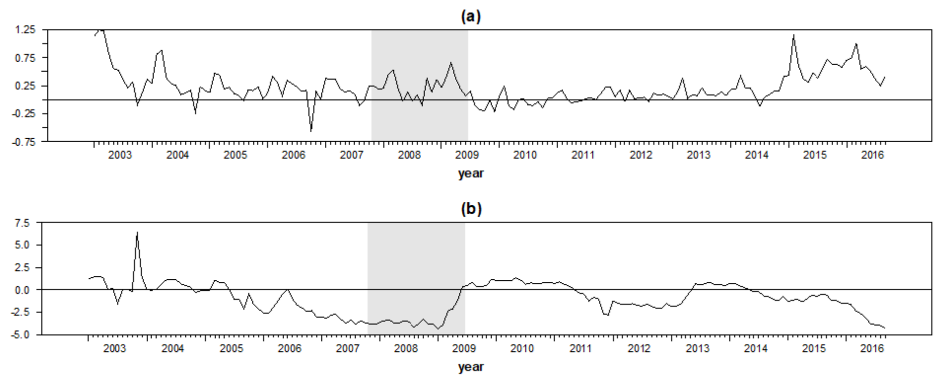


Figure 4.23.: (a) Time plot of the monthly growth rate of Colombian CPI and (b) time plot of Colombian term spread.

We use as the training subsample, the data from 2003:01 to 2012:06 (114 observations). The remaining observations are reserved for the out-of-sample forecasting evaluation. By using the procedure mentioned at the begining of this Chapter, a sequence of 1 to 12-step ahead forecasts are generated until we compute 53 1-step ahead forecasts until 42 12-step ahead forecast.

Estimation of models

Based on the estimation procedure in Appendix I, Table 4.50 shows that globally, the SETAR model presents the best reasonable in-sample fit.

Model adequacy	AR	TAR	SETAR	STAR	MSAR
White noise	1	2	1	1	1
Model specification	1	1	1	1	2
Homoscedasticity	3	3	2	3	3

Cells with number 1 and highlighted green represent good adequacy, 2 and highlighted orange represent regular adequacy, and 3 and highlighted red represent bad adequacy.

Table 4.50.: Model adequacy. Colombian CPI case.

Forecasting evaluation

Table 4.51 shows the HP unbiased test and Ljung-Box's Q correlation test. Globally at the 10% significance level, forecasts errors for all models are unbiased, and only the TAR, SETAR and STAR exhibit uncorrelated forecasts errors at horizons greater than 3 periods, in general.

(a)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.009	0.005	0.000	0.001	0.003
2	0.001	0.001	0.000	0.000	0.003
3	0.000	0.000	0.000	0.000	0.002
4	0.000	0.000	0.000	0.000	0.003
5	0.000	0.000	0.000	0.000	0.003
6	0.000	0.000	0.000	0.000	0.003
7	0.000	0.000	0.000	0.000	0.002
8	0.000	0.000	0.000	0.000	0.002
9	0.000	0.000	0.000	0.000	0.002
10	0.000	0.000	0.000	0.000	0.003
11	0.000	0.000	0.000	0.000	0.001
12	0.000	0.000	0.000	0.000	0.001

(b)

Horizon	AR	TAR	SETAR	STAR	MSAR
1	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.769	0.000	0.000
3	0.000	0.001	0.305	0.646	0.000
4	0.000	0.126	0.265	0.908	0.000
5	0.000	0.442	0.173	0.906	0.000
6	0.000	0.575	0.866	0.832	0.000
7	0.000	0.705	0.087	0.243	0.000
8	0.000	0.563	0.087	0.393	0.000
9	0.000	0.826	0.021	0.699	0.000
10	0.000	0.811	0.935	0.549	0.000
11	0.000	0.631	0.861	0.359	0.000
12	0.000	0.695	0.668	0.733	0.000

Cells highlighted green have a p–value less than 0.1.

Table 4.51.: p–values of the (a) unbiased test and (b) correlation test for the first 4 lags. Colombian CPI case.

Now, we present the forecast comparison, also at the 10% significance level. Table 4.52 shows that the MSAR model has smaller MSE than the benchmark linear model, and the TAR model has the third-smallest MSE.

Horizon	MSE	AR	TAR	SETAR	STAR	MSAR
1	0.039	1.000	1.172	1.209	0.950	1.032
2	0.062	1.000	1.119	1.603	1.209	0.653
3	0.080	1.000	1.025	1.321	1.280	0.515
4	0.089	1.000	0.993	1.283	1.378	0.467
5	0.097	1.000	1.022	1.372	1.355	0.440
6	0.101	1.000	1.041	1.422	1.448	0.429
7	0.105	1.000	1.032	1.378	1.578	0.422
8	0.107	1.000	1.019	1.427	1.374	0.419
9	0.110	1.000	1.048	1.535	1.400	0.416
10	0.112	1.000	1.020	1.570	1.295	0.406
11	0.114	1.000	1.030	1.869	1.242	0.397
12	0.117	1.000	1.007	2.092	1.414	0.396

(1) The column marked by MSE shows the MSE of forecasts from the AR model. (2) Cells highlighted green are the smallest values for each forecasts horizon.

Table 4.52.: Relative MSE of forecasts. Colombian CPI case.

Hereafter, we define the TAR model as the benchmark model. Table 4.53 shows for the overall comparison, that forecasts from the TAR model are more accurate than those from the SETAR and STAR models at horizons greater than 2 periods, and there is no significant difference with the linear model. However, forecast from the MSAR model are more accurate than those from the TAR model.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.016	0.759	0.031	0.016
2	0.057	1.000	0.721	0.001
3	0.324	0.999	0.898	0.000
4	0.560	0.958	0.993	0.000
5	0.185	0.975	0.989	0.000
6	0.121	0.999	0.998	0.000
7	0.111	0.986	1.000	0.000
8	0.234	0.997	0.996	0.000
9	0.069	0.993	1.000	0.000
10	0.229	0.992	0.997	0.000
11	0.109	0.995	0.849	0.000
12	0.382	1.000	0.988	0.000

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.984	0.241	0.969	0.985
2	0.943	0.000	0.279	0.999
3	0.676	0.001	0.102	1.000
4	0.440	0.042	0.007	1.000
5	0.815	0.025	0.011	1.000
6	0.879	0.001	0.002	1.000
7	0.889	0.014	0.000	1.000
8	0.766	0.003	0.004	1.000
9	0.931	0.007	0.000	1.000
10	0.771	0.008	0.003	1.000
11	0.891	0.005	0.151	1.000
12	0.618	0.000	0.012	1.000

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.015	0.761	0.030	0.015
2	0.055	1.000	0.723	0.001
3	0.323	0.999	0.900	0.000
4	0.561	0.960	0.993	0.000
5	0.182	0.976	0.980	0.000
6	0.119	0.999	0.998	0.000
7	0.109	0.987	1.000	0.000
8	0.232	0.967	0.992	0.000
9	0.066	0.994	1.000	0.000
10	0.226	0.975	0.992	0.000
11	0.106	0.995	0.780	0.000
12	0.380	1.000	0.948	0.000

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.985	0.239	0.970	0.985
2	0.945	0.000	0.277	0.999
3	0.677	0.001	0.100	1.000
4	0.439	0.040	0.007	1.000
5	0.818	0.024	0.020	1.000
6	0.881	0.001	0.002	1.000
7	0.891	0.013	0.000	1.000
8	0.768	0.033	0.008	1.000
9	0.934	0.006	0.000	1.000
10	0.774	0.025	0.008	1.000
11	0.894	0.005	0.220	1.000
12	0.620	0.000	0.052	1.000

Cells highlighted green have a p–value less than 0.1.

Table 4.53.: DM test when (a) H_1 : Forecasts from competing model (F2) are better than forecasts from TAR model (F1) and (b) H_1 : F1 are better than F2; MDM test when (c) H_1 : F2 are better than F1 and (d) H_1 : F1 are better than F2. Colombian CPI case.

Table 4.54 shows that under the CH test, all the models do not encompass each other. However, under the ER and HLN tests, the TAR model encompasses the SETAR and STAR models, but it is encompassed by the linear model at some horizons, in general. Under the HLN test, the MSAR model could encompass the TAR model.

(a)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.001	0.001	0.001	0.000
2	0.000	0.013	0.000	0.000
3	0.000	0.000	0.001	0.000
4	0.000	0.000	0.000	0.000
5	0.000	0.004	0.031	0.000
6	0.000	0.023	0.000	0.000
7	0.000	0.001	0.059	0.000
8	0.000	0.009	0.032	0.000
9	0.000	0.076	0.036	0.000
10	0.000	0.080	0.079	0.000
11	0.000	0.961	0.020	0.000
12	0.000	0.418	0.123	0.000

(b)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.006	0.000	0.024	0.001
2	0.000	0.000	0.001	0.001
3	0.000	0.000	0.000	0.001
4	0.000	0.000	0.000	0.001
5	0.000	0.000	0.007	0.007
6	0.000	0.000	0.000	0.003
7	0.000	0.000	0.000	0.003
8	0.000	0.002	0.002	0.002
9	0.000	0.000	0.000	0.004
10	0.000	0.001	0.006	0.004
11	0.000	0.000	0.008	0.001
12	0.000	0.000	0.008	0.000

(c)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.001	0.768	0.000	0.000
2	0.000	0.026	0.337	0.000
3	0.419	0.138	0.922	0.000
4	0.868	0.661	0.384	0.000
5	0.170	0.769	0.437	0.000
6	0.128	0.064	0.174	0.000
7	0.088	0.550	0.010	0.000
8	0.245	0.268	0.225	0.000
9	0.070	0.098	0.078	0.000
10	0.269	0.177	0.307	0.000
11	0.120	0.002	0.821	0.000
12	0.483	0.000	0.159	0.000

(d)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.076	0.090	0.543	0.029
2	0.020	0.000	0.034	0.021
3	0.915	0.000	0.015	0.029
4	0.637	0.007	0.000	0.012
5	0.684	0.000	0.000	0.001
6	0.405	0.000	0.000	0.006
7	0.433	0.000	0.000	0.007
8	0.782	0.000	0.000	0.008
9	0.245	0.000	0.000	0.005
10	0.736	0.000	0.000	0.007
11	0.431	0.000	0.065	0.005
12	0.906	0.000	0.001	0.007

(e)

Horizon	TAR - AR	TAR - SETAR	TAR - STAR	TAR - MSAR
1	0.005	0.385	0.002	0.007
2	0.033	0.991	0.178	0.001
3	0.213	0.932	0.539	0.000
4	0.434	0.676	0.810	0.000
5	0.087	0.615	0.784	0.000
6	0.069	0.956	0.921	0.000
7	0.052	0.727	0.995	0.000
8	0.124	0.871	0.903	0.000
9	0.038	0.929	0.970	0.000
10	0.128	0.899	0.858	0.000
11	0.053	0.979	0.590	0.000
12	0.227	0.999	0.919	0.000

(f)

Horizon	AR - TAR	SETAR - TAR	STAR - TAR	MSAR - TAR
1	0.946	0.047	0.725	0.961
2	0.903	0.000	0.018	0.988
3	0.543	0.000	0.009	0.991
4	0.324	0.002	0.000	0.998
5	0.660	0.001	0.000	0.999
6	0.799	0.000	0.000	0.998
7	0.783	0.001	0.000	0.997
8	0.609	0.000	0.000	0.997
9	0.880	0.001	0.000	0.998
10	0.634	0.001	0.000	0.998
11	0.795	0.002	0.036	0.999
12	0.454	0.000	0.001	0.998

Cells highlighted green have a p–value less than 0.1.

Table 4.54.: CH test when (a) H_0 : F1 encompasses F2 and (b) H_0: F2 encompasses F1; ER test when (c) H_0 : F1 encompasses F2 and (d) H_0: F2 encompasses F1; HLN test when (e) H_0 : F1 encompasses F2 and (f) H_0: F2 encompasses F1. Colombian CPI case.

Table 4.55 shows that the MSAR model has the smallest value of the Theil's U statistic, followed by the AR and TAR models, and forecasts from the TAR models are more accurate than those from the SETAR and STAR models.

Horizon	AR	TAR	SETAR	STAR	MSAR
1	1.036	1.122	1.140	1.010	1.053
2	1.297	1.373	1.643	1.427	1.048
3	1.454	1.472	1.671	1.645	1.044
4	1.534	1.529	1.738	1.801	1.048
5	1.579	1.597	1.850	1.839	1.047
6	1.598	1.630	1.906	1.923	1.047
7	1.610	1.636	1.890	2.022	1.046
8	1.613	1.628	1.927	1.890	1.044
9	1.624	1.663	2.013	1.922	1.048
10	1.643	1.660	2.059	1.870	1.047
11	1.706	1.731	2.332	1.901	1.074
12	1.708	1.714	2.469	2.030	1.075

Cells highlighted green represent the lowest value for each forecast horizon.

Table 4.55.: Theil's U statistic. Colombian CPI case.

As a summary, it is observed that forecasts errors from all models are not unbiased, and only the TAR, SETAR and STAR exhibit uncorrelated forecasts errors at horizons greater than 3 periods. Additionally, Table 4.56 let us observe a satisfactory performance of the TAR model. According to the Relative MSE of forecasts and the DM and MDM tests, the TAR model has better MSE of forecasts than the SETAR and STAR models. Regarding the encompassing tests, the TAR model encompasses the SETAR and STAR models at all horizons, but it could be compassed by the MSAR model. Additionally, forecasts from all models are not more accurate than those from the naïve model.

Therefore, from this out-of-sample forecasts comparison, we can conclude that the TAR model could forecasts the growth rate of the CPI with a reasonable performance, given that of the alternatives, the TAR model appears to be marginally preferred to the competing modes.

Horizon	Relative MSE	DM - MDM	Encompassing tests	U-Theil
1	1 (SETAR)	0	1 (SETAR)	1 (SETAR)
2	2 (SETAR, STAR)	1 (SETAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
3	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
4	3 (AR, SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
5	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
6	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
7	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
8	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
9	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
10	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
11	2 (SETAR, STAR)	1 (SETAR)	2 (SETAR, STAR)	2 (SETAR, STAR)
12	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)	2 (SETAR, STAR)

(1) Cells highlighted green report that the TAR model outperforms 3 or 4 models, highlighted orange report that the TAR model outperforms 1 or 2 models, and highlighted red report that the TAR model does not outperform any model. (2) Models in parenthesis are outperformed by the TAR model.

Table 4.56.: Summary of the forecasting performance of the TAR model. Colombian CPI case.

Finally, we compare the TAR model with the SETAR and STAR models based on their ability to describe the predictive distribution of the economic time series. As it was mentioned before, given that there is not a strong contraction phase during the forecasting subsample, the 1 to 8-step predictive distributions of the TAR, SETAR and STAR models tend to be unimodal, capturing the stability of the time series. However, the TAR model slightly shows bimodal shapes for the lowest values of the CPI, traying to capture descending periods better than the other models.

As a global summary for the Colombian case, this empirical evaluation shows that forecasts from the TAR model outperform forecasts from some of the competing models at different step ahead horizons, for all considered economic times series, although in general, the forecasting performance of this model is not better than that of the linear model and the MSAR model. The evaluation criteria let us observe that forecasts from the TAR model, when using the unemployment rate, outperform those from the competing models at forecasts horizons greater than 3 periods ahead, and also, it is the model with the best in-sample properties along with the SETAR model. Regarding the GDP, forecasts from the TAR model outperform those from the competing models at forecasts horizons between 2 and 7 periods ahead. When using the industrial production index and the CPI time series, forecasts from the TAR model only show a better performance than that from the SETAR and STAR models at all the analyzed forecast horizon, in general. Additionally, given that the Colombian economic time series do not show a strong contraction phase during the forecasting period, it was not possible to evaluate the shape changing characteristic in the Bayesian predictive distributions of the TAR model.

Conclusions

The purpose of this Thesis has been to assess the forecasting performance of the TAR model. In that sense, we contributed to the literature firstly, by finding the Bayesian predictive distribution of the TAR model, using the joint predictive distribution that makes easier the computation of this distributions and reduces the complexity of the definitions of other approaches; and secondly, by finding important signals about the forecasting ability of a TAR model in the economic field.

To get those results, we compared the forecasting performance of the TAR model with that of a linear autoregressive model and nonlinear SETAR, STAR and MSAR models, using different economic time series of the United States and Colombian economies: the unemployment rate, the GDP, the industrial production index and the inflation. Therefore, we estimated 40 models and computed forecasts from all these models.

The results show, globally, good in-sample properties of the TAR model, which means that the TAR model describes well the characteristics of the economic time series. Additionally, regarding the forecast evaluation, we found a satisfactory performance of the TAR model in forecasting all the economic time series, since it outperformed the SETAR and STAR nonlinear models and the linear model, according to the used evaluation criteria. Regarding the MSAR model, although it has the worst in-sample properties, forecasts from the MSAR model shows better properties than forecasts from the TAR model. In general, the TAR model seems marginally preferred to the SETAR, STAR models and AR models at different forecast horizons, and in some particular cases, to the MSAR model, for forecasting economic time series, especially, the unemployment rate.

Finally, we found that the Bayesian predictive distributions of the TAR model shows a shape changing characteristic, with which the TAR model appear to capture business cycles features of the considered time series better than the other competing models do. This shape changing quality may suggest that the TAR model can manage cycles in the economic field and can forecast much better economic time series in contraction periods, which is important, because it is during this periods that decision and policy makers are more aware of economic forecasts.

This findings are the base to new fields for research. Some of them are the use of forecasting combination methods in order to improve forecasts from the TAR model, forecast evaluation during contraction and expansionary periods and reestimating the models throughout the rolling forecasting procedure, among others.

Appendix A

Theoretical Background

A.1. Gibbs Sampler

Let (X, Y) be a pair of random variables. We assume that the conditional distributions $f\,(x|y)$ and $f\,(y|x)$ are known, so we can generate a sample from $f(x)$ by sampling these conditional distributions. Then, we select an arbitrary starting value of $Y_0 = y_0$ from the Gibbs sequence $Y_0, X_0, Y_1, X_1, \ldots, Y_m, X_m$. The other values of the sequence are obtained iteratively from $X_j \sim (x|Y_j = y_j)$, where the new value $X_j = x_j$ is then used to obtain $Y_{j+1} \sim (y|X_j = x_j)$, for $j = 0, 1, .., m$. This iterative process is called the Gibbs sampling (Casella and George, 1992). Under some regularity conditions, if m is large enough, the distribution of X_m converges to $f(x)$ and the final observation of the Gibbs sequence, $X_m = x_m$, is approximately a random draw from $f(x)$.

The Gibbs sampler proceeds as follows:

I) Take $X^{(t)} = (x_1^{(t)}, x_2^{(t)}, \ldots, x_p^{(t)})$. We define $X^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_p^{(0)})$ as the arbitrary starting values.

II) Draw a random sample $x_1^{(t+1)}$ from $f_1\left(x_1 \middle| x_2^{(t)}, \ldots, x_p^{(t)}\right)$.

III) Draw a random sample $x_2^{(t+1)}$ from $f_2\left(x_2 \middle| x_1^{(t+1)}, x_3^{(t)}, \ldots, x_p^{(t)}\right)$. Note that we use the value $x_1^{(t+1)}$.

IV) Draw a random sample $x_3^{(t+1)}$ from $f_3\left(x_3 \middle| x_1^{(t+1)}, x_2^{(t+1)}, x_4^{(t)}, \ldots, x_p^{(t)}\right)$. Note that we use the values $x_1^{(t+1)}$ and $x_2^{(t+1)}$. Repeat until we draw a random sample $x_p^{(t+1)}$ from $f_p\left(x_p \middle| x_1^{(t+1)}, x_2^{(t+1)}, \ldots, x_{p-1}^{(t+1)}\right)$.

V) Generate the burn-in sample (optional).

By repeating that iteration m times, we obtain the Gibbs sequence of random variables

$$\left(x_1^{(1)}, \ldots, x_p^{(1)}\right), \ldots, \left(x_1^{(m)}, \ldots, x_p^{(m)}\right).$$

For m large enough, $\left(x_1^{(m)}, \ldots, x_p^{(m)}\right)$ is approximately a random draw from the desired distribution. By using Monte Carlo Integration on those draws, we can obtain the quantities of interest.

A.2. Bayesian Predictive Distributions

A feature of Bayesian inference are the predictive distributions which allow us to make inferences about new observations. Let $\boldsymbol{x}$ and $\boldsymbol{\theta}$ be the vector of observed data and parameters respectively, where $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. For a given model and before the data $\boldsymbol{x}$ is seen, the distribution of the unknown observable $\boldsymbol{x}$ is defined as (Gelman et al., 2009):

$$p(\boldsymbol{x}) = \int_{\boldsymbol{\Theta}} f(\boldsymbol{x}, \boldsymbol{\theta})\, d\boldsymbol{\theta} = \int_{\boldsymbol{\Theta}} f(\boldsymbol{x}|\boldsymbol{\theta})\, p(\boldsymbol{\theta})\, d\boldsymbol{\theta}. \tag{A.1}$$

This distribution is called the prior predictive distribution. Once the data $\boldsymbol{x}$ has been observed, we can predict a new observation $\widetilde{x}$ from the model given $\boldsymbol{x}$:

$$\begin{aligned} p(\widetilde{x}|\boldsymbol{x}) &= \int_{\boldsymbol{\Theta}} f(\widetilde{x}, \boldsymbol{\theta}|\boldsymbol{x})\, d\boldsymbol{\theta} \\ &= \int_{\boldsymbol{\Theta}} f(\widetilde{x}|\boldsymbol{x}, \boldsymbol{\theta})\, f(\boldsymbol{\theta}|\boldsymbol{x})\, d\boldsymbol{\theta}, \end{aligned} \tag{A.2}$$

which is defined as the posterior predictive distribution (Gelman et al., 2009). Therefore, that predictive distribution integrates uncertainty about $\boldsymbol{\theta}$ and the future value $\widetilde{x}$, both conditional on $\boldsymbol{x}$ and the assumptions of the proposed model (Geweke and Amisano, 2010).

We note that the predictive distributions do not depend on any unknown quantities but on the observed data, thus $\boldsymbol{x}$ gives information about $\boldsymbol{\theta}$ which gives information about $\widetilde{x}$ (Hoff, 2009).

Appendix B

General review of the models estimation for the change in the U.S. unemployment rate

B.1. Estimation of the TAR model

We use as the variable of interest the change in the U.S. unemployment rate, that is $X_t = u_t - u_{t-1}$, where u_t is the unemployment rate. As the threshold variable, we use the growth rate of the U.S. real GDP, that is $Z_t = [\log(GDP_t) - \log(GDP_{t-1})] * 100$. Figure B.1 and B.2 show that both series have significant autocorrelations for the first number of lags.

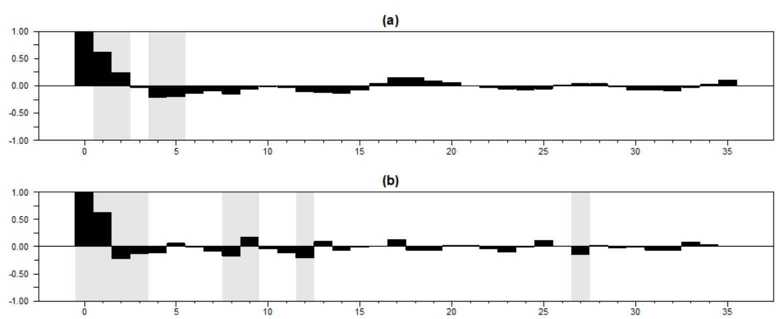


Figure B.1.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the change in the U.S. unemployment rate.

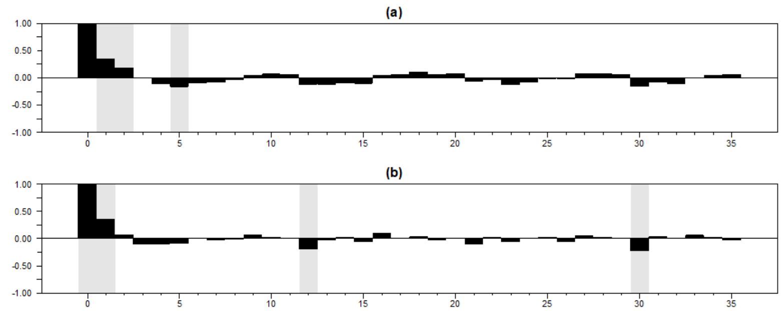


Figure B.2.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the growth rate of U.S. real GDP.

Next, we check the nonlinearity of the series based on Nieto and Hoyos's (2011) test which is an extension of Tsay's (1998) statistic. We test the null hypothesis of AR linearity against the alternative of bivariate TAR nonlinearity. Under the AIC criterion, we found that $\bar{k} = 12$ is a reasonable autoregressive order for X_t. Under the null hypothesis, the results of the test let us define the number 0 as the delay parameter of the threshold variable, thus the input variable for the dynamic system is Z_t.

Figure B.3 shows the change in the U.S. quarterly unemployment rate and the growth rate of the U.S. quarterly real GDP from 1948:02 to 1995:04. The shading areas denote the business cycle contractions from peak to trough based on NBER.

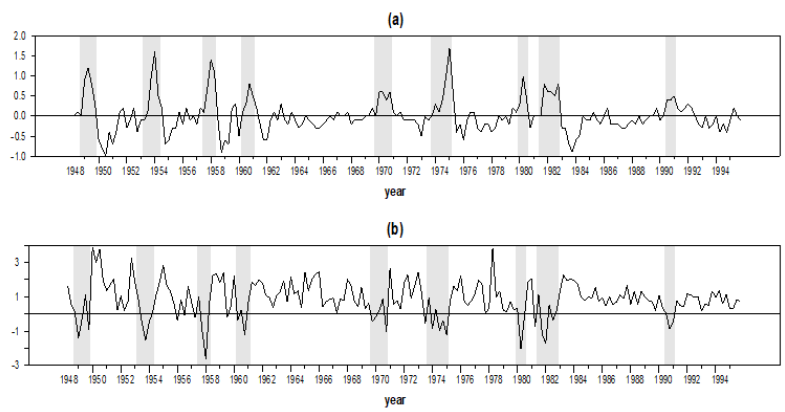


Figure B.3.: (a) Time plot of the change in the U.S. quarterly unemployment rate and (b) Time plot of the growth rate of U.S. quarterly real GDP.

We proceed to identify the number of thresholds for the TAR model as indicated in Section 1.1. Thus, we specify the maximum number of regimes l_0 by means of a regression function between X_t and Z_t that is estimated using a nonparametric kernel approach, which is presented in Figure B.4. We observe that 3 could be postulated as the possible maximum regimes for the TAR model, with possible threshold values -1.5 and 1.0 which could represent periods in the economy of i) contraction, that generates increases in the unemployment, ii) stabilization, where there is no destruction or creation of employment, and iii) expansion, where there is a decrease in unemployment.

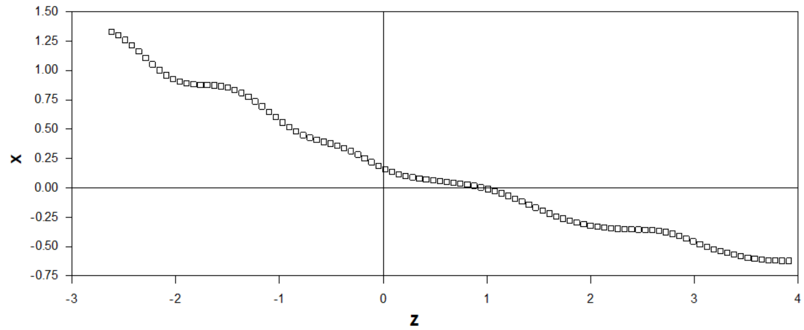


Figure B.4.: Nonparametric regression between the change in the U.S. unemployment rate (X) and the growth rate of U.S. real GDP (Z).

Once we have defined l_0, we select the appropriate thresholds for each possible regimen $l = 2, \ldots, l_0$. That requires to generate intermediate draws of the nonstructural parameters. Thus, we specify the prior densities for the nonstructural parameters according to Section 1.1. In that sense, we define the prior densities for $\boldsymbol{\theta}_x$ where $\boldsymbol{\theta}_{0,j} = \bar{\mathbf{0}}, V_{0,j}^{-1} = 0.01\mathbf{I}$ with $\mathbf{I}$ the identity matrix, $\gamma_{0,j} = 1.5$ and $\beta_{0,j} = \frac{\widetilde{\sigma}^2}{2}$ with $j = 1, 2, 3$ and $\widetilde{\sigma}^2 = 0.084$ that is the residual variance of the AR(12) that was fitted to the change in the unemployment rate. The maximum autoregressive order for all regimes is $\bar{k} = 12$, the same value fitted to the change in the unemployment rate. The prior distributions for both the number of regimes and the autoregressive orders that we use in the identification of l are $\pi_2 = \pi_3 = 0.5$ and $p(k_{il}|l) = 0.076$ for $k_{il} = 0, 1, \ldots, 12; i = 1, .., l$, respectively.

With that information, we identify the thresholds $r_i; i = 1, \ldots, l-1$ for the $l_0 - 1$ possible models that we denote from now on $M_j; j = 2, \ldots, l_0$ (model M with j regimes). Thus, to look for the location of thresholds in each possible regime, we choose the percentiles $5k$ where $k = 1, 2, \ldots, 19$, with respective values -1.00, -0.44, -0.12, 0.19, 0.32, 0.43, 0.56, 0.70, 0.75, 0.85, 0.94, 1.00, 1.13, 1.33, 1.58, 1.74, 1.93, 2.05, 2.42. Then, we choose the thresholds of the model M_2 and M_3 by searching among the set of all possible combinations of autoregressive orders. The possible thresholds and autoregressive orders for each possible regime are presented in Table B.1.

ℓ	**Thresholds**	**Autoregressive orders**	**Minimum NAIC**
2	0.75	1, 4	1.16734
3	0.19 1.00	11, 1, 4	0.43985

Table B.1.: Set of possible number of regimes for the real data. U.S. unemployment rate case.

With those possible thresholds and the information mentioned above, we compute the posterior probability distribution for the number of regimes. 3000 iterates were generated with a burn-in point of 10% of the draws[30]. The results showed in Table B.2 allow us to set $\hat{l} = 3$ as the appropriate number of regimes.

ℓ	$\hat{p}_\ell$
2	0.0026
3	0.9974

Table B.2.: Posterior probability function for the number of regimes for the real data. U.S. unemployment rate case.

Thus, conditional on $\hat{l} = 3$, we estimate the autoregressive orders for k_1, k_2 and k_3. Based on Table B.3, the identified autoregressive orders are $\hat{k}_1 = 11$, $\hat{k}_2 = 3$ and $\hat{k}_3 = 0$. As before, convergence of the Gibbs sampler was checked via stationarity, and for 5000 iterates with a burn-in point of 10% of the draws, it is observed that the sample autocorrelations functions decay quickly[31].

Autoregressive Order	**Regime**		
	1	**2**	**3**
0	8.13×10^{-5}	2.55×10^{-11}	**0.4194**
1	0.0317	0.1585	0.1923
2	0.0213	0.1604	0.1522
3	0.0134	**0.1765**	0.1311
4	0.0121	0.1150	0.0159
5	0.0098	0.0536	0.0127
6	0.0065	0.0734	0.0189
7	0.0077	0.0746	0.0164
8	0.0239	0.0604	0.0122
9	0.0997	0.0417	0.0049
10	0.2378	0.0439	0.0080
11	**0.2912**	0.0164	0.0086
12	0.2451	0.0257	0.0075

Table B.3.: Posterior probabilities for the autoregressive orders in the real data. U.S. unemployment rate case.

This concludes the identification stage of the TAR model. Consequently, we fit a TAR(3;11, 3,0) with thresholds values $r_1 = 0.19$ and $r_2 = 1.00$, which respectively are the 20th and 60th percentiles of the growth rate of the real GDP. Table B.4 shows the estimates for

[30] The convergence of the Gibbs sampler was checked via the stationarity approach, and it was observed that the sample autocorrelation functions of the sequences $\left\{\hat{p}_l^{(i)}\right\}$ for $l = 2, 3$ decay quickly.

[31] We also performed a sensibility analysis where we changed the prior densities of the autoregressive orders and the nonstructural parameters, and we found that the estimated autoregressive orders are the same for different priors of the nonstructural parameters, although some of them changed for different priors of the autoregressive orders.

the nonstructural parameters, with their respective posterior standard error in parenthesis and 90% credible interval in brackets[32]. These results show that not all the coefficients are significant at the 5% significance level. However, we decided to estimate the model with all the coefficients, given that that improves the final estimation.

	Regime		
Parameter	**1**	**2**	**3**
$a_0^{(j)}$	0.32 (0.08)	0.03 (0.02)	-0.25 (0.03)
	$[0.20, 0.45]$	$[0.00, 0.07]$	$[-0.30, -0.20]$
$a_1^{(j)}$	0.51 (0.21)	0.51 (0.07)	
	$[0.17, 0.86]$	$[0.40, 0.62]$	
$a_2^{(j)}$	0.10 (0.31)	-0.04 (0.08)	
	$[-0.42, 0.61]$	$[-0.18, 0.09]$	
$a_3^{(j)}$	-0.21 (0.27)	-0.04 (0.07)	
	$[-0.65, 0.24]$	$[-0.15, 0.08]$	
$a_4^{(j)}$	-0.24 (0.25)		
	$[-0.64, 0.17]$		
$a_5^{(j)}$	0.10 (0.31)		
	$[-0.40, 0.61]$		
$a_6^{(j)}$	0.18 (0.25)		
	$[-0.24, 0.60]$		
$a_7^{(j)}$	-0.17 (0.28)		
	$[-0.62, 0.30]$		
$a_8^{(j)}$	-0.95 (0.30)		
	$[-1.46, -0.46]$		
$a_9^{(j)}$	0.76 (0.31)		
	$[0.26, 1.25]$		
$a_{10}^{(j)}$	-0.16 (0.28)		
	$[-0.61, 0.30]$		
$a_{11}^{(j)}$	-0.36 (0.24)		
	$[-0.74, 0.02]$		
$h^{(j)}$	0.12 (0.04)	0.03 (0.004)	0.07 (0.01)
	$[0.07, 0.19]$	$[0.02, 0.03]$	$[0.05, 0.09]$

Table B.4.: Parameter estimates for the TAR model. U.S. unemployment rate case.

B.2. Estimation of the SETAR model

To fit a SETAR model for the change in the unemployment rate, as it is mentioned in Section 1.2, and based on the autocorrelation functions in Figure B.1 and the AIC and BIC criterion, we determine, in the identification stage, $p = 3$ as the autoregressive order

[32] 5000 iterates were generated with a burn-in point of 10% of the draws and it was found that the autocorrelation functions for all the parameters decay quickly, indicating the convergence of the Gibbs sampler.

but with the second lag restricted to zero.

Once we have selected the autoregressive model, we check the nonlinearity of the series based on Tsay's (1989) test and find that with $d = 6$, where d is the delay parameter, it is obtained the minimum p-value = 0.029 of the F statistic $F = 3.603$, rejecting the linearity of the series at the 5% significance level.

Then, to determine the number of regimes and the threshold values, we use Figure B.5 that shows the sequence of the t ratios of a lag-3 AR coefficient versus the threshold variable X_{t-6} in an arranged autoregression of order 3, and we identify that the data can be divided into two regimes with a possible threshold at $X_{t-d} = -0.2$, because of the change on the slope at approximately this point.

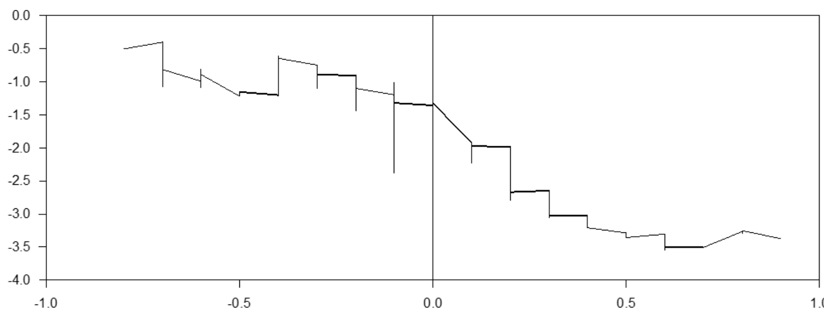


Figure B.5.: Time plot of t-ratio of recursive estimates of the AR-3 coefficient in an arranged autoregression of order 3 and delay parameter 6. U.S. unemployment rate case.

We also use the NAIC criterion of Tong (1990), which suggests a threshold value of $r_j = -0.2$ and autoregressive orders AR(1) and AR(3) for each regime, with a NAIC=-439.858. Therefore, we fit a SETAR(2;1,3) for the change in the US unemployment rate from 1948:2 to 1995:4, with threshold value X_{t-6}. The estimated parameters and their standard errors in parenthesis for each regime are shown in Table B.5, where all estimates are significant at the 5% level.

	Regime	
Parameter	**1**	**2**
$\Phi_1^{(j)}$	0.21 (0.09)	0.67 (0.06)
$\Phi_3^{(j)}$		-0.21 (0.06)

Table B.5.: Parameter estimates for the SETAR model. U.S. unemployment rate case.

B.3. Estimation of the STAR model

We estimate the STAR model based on Section 1.3. With the identified $p = 12$ autoregressive order, we evaluate the nonlinearity of the series based on Teräsvirta's (1994) test and find that with the delay parameter $d = 1$, we obtain the minimum p-value = 0.000 of the F statistic $F = 2.6148$, which rejects the linearity of the series.

Then, as we mentioned in Section 1.3, we choose between the LSTAR and the ESTAR models through a sequence of tests of nested hypothesis (Teräsvirta, 1994), with which we find that H_{01} and H_{03} are rejected with $F - stat = 2.0518(0.0246)$ and $F - stat = 3.4756(0.0001)$ respectively, while H_{02} with F-stat=31.6644(0.0808) is not rejected at 5% significance level (the p-values are in parenthesis). According to Teräsvirta (1994), these results suggest that we must estimate a LSTAR model.

In Table B.6, we show the estimates for the parameters and their respective standard error in parenthesis, where there are some coefficients that are not significant at the 5% level.

	Regime	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	-0.09 (0.10)	4.24 (5.67)
$\Phi_1^{(j)}$	0.43 (0.16)	-0.56 (1.41)
$\Phi_2^{(j)}$		-3.37 (3.90)
$\Phi_4^{(j)}$	-0.20 (0.05)	
γ		1.39 (0.61)
c		1.26 (0.61)

Table B.6.: Parameter estimates for the STAR model. U.S. unemployment rate case.

B.4. Estimation of the MSAR model

Based on Section 1.4, we estimate the MSAR model. Table B.7 shows the estimates of the parameters of the model and their respective standard errors in parenthesis.

	state	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	-0.07 (0.04)	0.09 (0.06)
$\Phi_1^{(j)}$	0.28 (0.12)	0.79 (0.11)
$\Phi_2^{(j)}$	0.09 (0.11)	-0.28 (0.12)
$\Phi_4^{(j)}$	-0.12 (0.09)	-0.24 (0.11)

Table B.7.: Parameter estimates for the MSAR model. U.S. unemployment rate case.

Appendix C

General review of the models estimation for the annual growth rate of the U.S. real GDP

C.1. Estimation of the TAR model

We use as the variable of interest the annual growth rate of the U.S. real GDP, and as the threshold variable, we use the spread term defined in Section 3.2. Figure C.1 and Figure C.2 show that the autocorrelations are significant for a large number of lags.

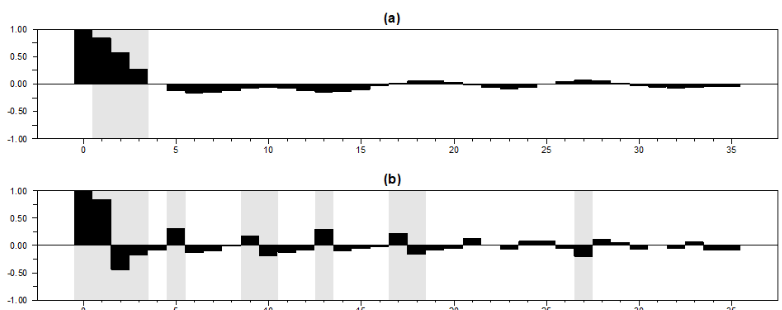


Figure C.1.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the annual growth rate of U.S. real GDP.

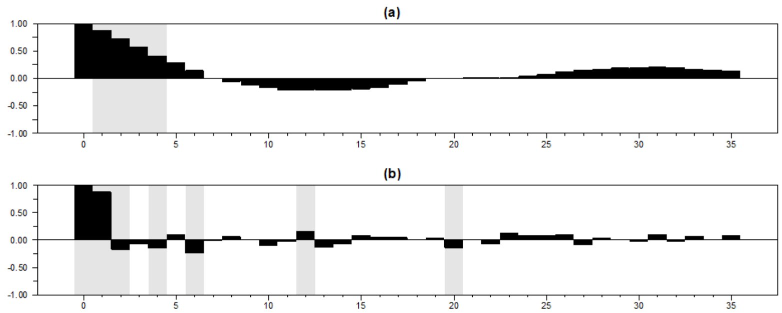


Figure C.2.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the U.S. spread term.

Then, we test the null hypothesis of AR linearity against the alternative of bivariate TAR nonlinearity. Under the AIC criterion, we found that $\bar{k} = 14$ is a reasonable autoregressive order for X_t. Under the null hypothesis, the results of the test let us define a delay parameter of 2 for the threshold variable, thus, the input variable for the dynamic system is Z_{t-2}.

Figure C.3 shows the annual growth rate of the U.S. real GDP and spread term from 1956:01 to 1998:02. The shading areas denote the business cycle contractions from peak to trough based on NBER.

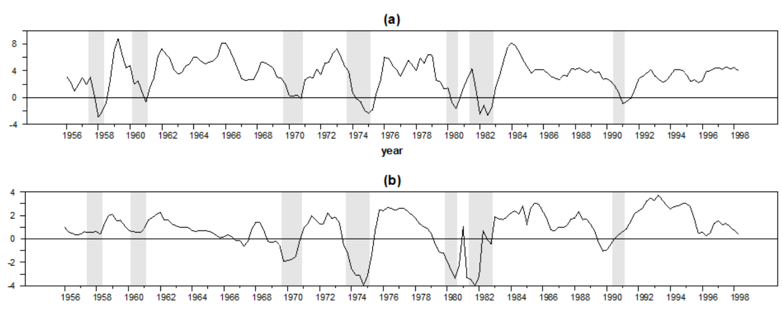


Figure C.3.: (a) Time plot of the annual growth rate of U.S. real GDP and (b) Time plot of U.S. spread term.

We specify the maximum number of regimes l_0 by means of a regression function between X_t and Z_t, that is estimated using a nonparametric kernel approach and that is presented in Figure C.4. We observe that 3 could be postulated as the possible maximum regimes for the TAR model.

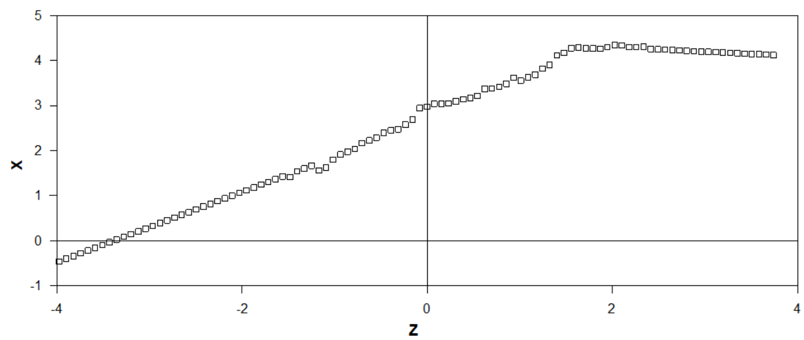


Figure C.4.: Nonparametric regression between the growth rate of U.S. real GDP (X) and U.S. spread term (Z).

Next, to specify the prior densities for the nonstructural parameters, we define the prior densities for $\boldsymbol{\theta}_x$ where $\boldsymbol{\theta}_{0,j} = \bar{\mathbf{0}}$, $V_{0,j}^{-1} = 0.01\mathbf{I}$ with $\mathbf{I}$ the identity matrix, $\gamma_{0,j} = 1.5$ and $\beta_{0,j} = \frac{\widetilde{\sigma}^2}{2}$ with $j = 1, 2, 3$ and $\widetilde{\sigma}^2 = 0.819$ that is the residual variance of the AR(14) that was fitted to the annual growth of the real GDP. The maximum autoregressive order for all regimes is $\bar{k} = 14$, the same value fitted to the variable of interest. The prior distributions for both the number of regimes and the autoregressive orders that we use in the identification of l are $\pi_2 = \pi_3 = 0.5$ and $p\left(k_{il}|l\right) = 0.066 for k_{il} = 0, 1, \ldots, 14; i = 1, .., l$, respectively.

Then, to look for the location of thresholds in each possible regime, we choose the percentiles $5k$ where $k = 1, 2, \ldots, 19$, with respective values -2.90, -1.51, -0.49, -0.16, 0.28, 0.46, 0.59, 0.68, 0.78, 0.96, 1.12, 1.31, 1.52, 1.66, 1.81, 2.09, 2.39, 2.60, 2.91. The possible thresholds and autoregressive orders for each possible regime are presented in Table C.1.

ℓ	**Thresholds**	**Autoregressive orders**	**Minimum NAIC**
2	0.68	5, 5	3.70792
3	0.59 0.78	11, 13, 12	3.08316

Table C.1.: Set of possible number of regimes for the real data. U.S. GDP case.

With those possible thresholds and the information mentioned above, we compute the posterior probability distribution for the number of regimes. 3000 iterates were generated with a burn-in point of 10% of the draws[33]. The results showed in Table C.2 allow us to set $\hat{l} = 3$ as the appropriate number of regimes.

[33] The convergence of the Gibbs sampler was checked via the stationarity approach, where it was observed that the sample autocorrelation functions of the sequences $\left\{\hat{p}_l^{(i)}\right\}$ for $l = 2, 3$ decay quickly.

ℓ	$\hat{p}_\ell$
2	0.41520
3	0.58480

Table C.2.: Posterior probability function for the number of regimes for the real data. U.S. GDP case.

Thus, conditional on $\hat{l} = 3$, we estimate the autoregressive orders for k_1, k_2 and k_3. Based on Table C.3, the identified autoregressive orders are $\hat{k}_1 = 6, \hat{k}_2 = 2$ and $\hat{k}_3 = 1$. As before, convergence of the Gibbs sampler was checked via stationarity, and for 5000 iterates with a burn-in point of 10% of the draws, it is observed that the sample autocorrelations functions decay quickly[34].

Autoregressive Order	**Regime** 1	2	3
0	6.10×10^{-48}	0.0024	1.23×10^{-49}
1	0.0652	0.0468	**0.7534**
2	0.0243	**0.2936**	0.2222
3	0.0110	0.1595	0.0195
4	0.0001	0.0520	2.80×10^{-25}
5	0.1145	0.0458	0.0001
6	**0.2416**	0.0237	0.0007
7	0.0979	0.0214	0.0003
8	0.0053	0.0133	9.42×10^{-18}
9	0.0671	0.0928	0.0018
10	0.0874	0.0622	0.0005
11	0.0984	0.0362	0.0009
12	0.0003	0.0300	0.0001
13	0.0945	0.0559	0.0005

Table C.3.: Posterior probabilities for the autoregressive orders in the real data. U.S. GDP case.

Consequently, we fit a TAR(3;6,2,1) with thresholds values $r_1 = 0.59$ and $r_2 = 0.78$, which respectively are the 35th and 45th percentiles of the spread term. However, the fitted model was not appropriate to explain the marginal heteroscedasticity of the data[35]. Therefore,

[34] We also performed a sensibility analysis where we changed the prior densities of the autoregressive orders and the nonstructural parameters, and we found that the estimated autoregressive orders are the same for different priors of the nonstructural parameters, although some of them changed for different priors of the autoregressive orders.

[35] When we checked the residuals of the estimated TAR(3;6,2,1), we observed that the standardized and squared standardized residuals of the model slightly signal that some nonlinear structure in the data is not explained by the model. The Ljung-Box statistics for checking "whiteness" are, respectively, $Q(8) = 19.796(0.011)$ and $Q(8) = 16.365(0.037)$ with the number in parenthesis denoting the p-value.

based on the NAIC criterion[36], the estimated model for the change in the annual growth rate of the U.S. real GDP is a TAR(3;11,13,12) with thresholds values $r_1 = 0.59$ and $r_2 = 0.78$.

	Regime		
Parameter	**1**	**2**	**3**
$a_0^{(j)}$	-0.66 (0.60)	-2.95 (3.21)	1.64 (0.26)
	$[-1.63, 0.35]$	$[-8.14, 2.25]$	$[1.22, 2.06]$
$a_1^{(j)}$	1.17 (0.16)	0.84 (0.54)	0.98 (0.10)
	$[0.90, 1.43]$	$[-0.02, 1.73]$	$[0.82, 1.14]$
$a_2^{(j)}$	-0.08 (0.22)	0.86 (1.87)	-0.05 (0.15)
	$[-0.43, 0.28]$	$[-2.25, 3.93]$	$[-0.29, 0.19]$
$a_3^{(j)}$	-0.14 (0.22)	-0.97 (3.94)	-0.18 (0.15)
	$[-0.50, 0.22]$	$[-7.38, 5.50]$	$[-0.44, 0.07]$
$a_4^{(j)}$	-0.44 (0.23)	0.71 (2.67)	-0.55 (0.12)
	$[-0.82, -0.06]$	$[-3.67, 5.06]$	$[-0.75, -0.35]$
$a_5^{(j)}$	0.47 (0.23)	0.04 (0.74)	0.62 (0.14)
	$[0.10, 0.84]$	$[-1.12, 1.21]$	$[0.40, 0.85]$
$a_6^{(j)}$	0.09 (0.24	0.13 (1.48)	-0.01 (0.14)
	$[-0.29, 0.48]$	$[-2.27, 2.49]$	$[-0.25, 0.23]$
$a_7^{(j)}$	-0.16 (0.27)	-0.25 (1.86)	-0.03 (0.14)
	$[-0.60, 0.28]$	$[-3.24, 2.85]$	$[-0.27, 0.20]$
$a_8^{(j)}$	-0.07 (0.27)	-0.87 (1.83)	-0.42 (0.12)
	$[-0.50, 0.37]$	$[-3.76, 2.08]$	$[-0.62, -0.22]$
$a_9^{(j)}$	0.21 (0.24)	1.77 (2.74)	0.36 (0.12)
	$[-0.19, 0.61]$	$[-2.71, 6.23]$	$[0.16, 0.55]$
$a_{10}^{(j)}$	0.05 (0.25)	-0.49 (1.17)	-0.04 (0.13)
	$[-0.36, 0.45]$	$[-2.42, 1.41]$	$[-0.24, 0.17]$
$a_{11}^{(j)}$	-0.09 (0.16)	-0.61 (1.26)	0.07 (0.13)
	$[-0.36, 0.17]$	$[-2.60, 1.46]$	$[-0.14, 0.29]$
$a_{12}^{(j)}$		0.07 (1.23)	-0.14 (0.08)
		$[-1.94, 2.09]$	$[-0.27, -0.01]$
$a_{13}^{(j)}$		0.48 (0.57)	
		$[-0.47, 1.43]$	
$h^{(j)}$	1.32 (0.29)	0.40 (0.47)	0.53 (0.09)
	$[0.92, 1.86]$	$[0.09, 1.18]$	$[0.41, 0.69]$

Table C.4.: Parameter estimates for the TAR model. U.S. GDP case.

Table C.4 shows the estimates for the nonstructural parameters, with their respective

The CUSUM and CUSUMSQ indicated that there was no statistical evidence for model misspecification but there was some heteroscedasticity in $\{\varepsilon_t\}$.

[36] The NAIC criterion for the estimated TAR(3;6,2,1) is 3.718, while for the estimated TAR(3;11,13,12) is 3.022.

posterior standard error in parenthesis and 90% credible interval in brackets[37]. These results show that not all the coefficients are significant at the 5% level. However, we decided to estimate the model with all the coefficients, given that that improves the final estimation.

The fitted TAR model for the change in the annual growth rate of the U.S. real GDP is given by:

$$X_t = \begin{cases} -0.66 + 1.17X_{t-1} - 0.08X_{t-2} - 0.14X_{t-3} - 0.44X_{t-4} & \\ +0.47X_{t-5} - 0.09X_{t-6} - 0.16X_{t-7} - 0.07X_{t-8} + 0.21X_{t-9} & \\ +0.05X_{t-10} - 0.09X_{t-11} + 1.32\varepsilon_t, & \text{if } Z_{t-2} \leq 0.59 \\ -2.95 + 0.84X_{t-1} + 0.86X_{t-2} - 0.97X_{t-3} + 0.71X_{t-4} & \\ +0.04X_{t-5} + 0.13X_{t-6} - 0.25X_{t-7} - 0.87X_{t-8} + 1.77X_{t-9} & \\ -0.49X_{t-10} - 0.61X_{t-11} + 0.07X_{t-12} + 0.48X_{t-13} + 0.40\varepsilon_t, & \text{if } 0.59 < Z_{t-2} \leq 0.78 \\ 1.64 + 0.98X_{t-1} - 0.05X_{t-2} - 0.18X_{t-3} - 0.55X_{t-4} & \\ +0.62X_{t-5} - 0.01X_{t-6} - 0.03X_{t-7} - 0.42X_{t-8} + 0.36X_{t-9} & \\ -0.04X_{t-10} + 0.07X_{t-11} - 0.14X_{t-12} + 0.53\varepsilon_t, & \text{if } Z_{t-2} > 0.78 \end{cases}$$

This model could represent periods in the economy of i) contraction, given that this regime presents the greatest decreases in the growth rate of the real GDP, when spreads are low due to contractionary monetary policies; ii) stabilization, where stable spread values are related to minor variations in the growth rate of the real GDP; and iii) expansion, where this last regime is associated with the greatest increases in the growth rate of real GDP, when the monetary policy is expansioning.

When we check the residuals, in Figure C.5 we observe that the standardized and squared standardized residuals signal that the noise process is white, and the Ljung-Box statistics for checking "whiteness" are, respectively, $Q(8) = 3.805(0.874)$ and $Q(8) = 10.351(0.241)$. Figure C.6 reports that the CUSUM and CUSUMSQ behave well, which indicates that there is no statistical evidence for model misspecification but slightly, some heteroscedasticity in $\{\varepsilon_t\}$.

[37] 5000 iterates were generated with a burn-in point of 10% of the draws, and it was found that the autocorrelation functions for all the parameters decay quickly, indicating the convergence of the Gibbs sampler.

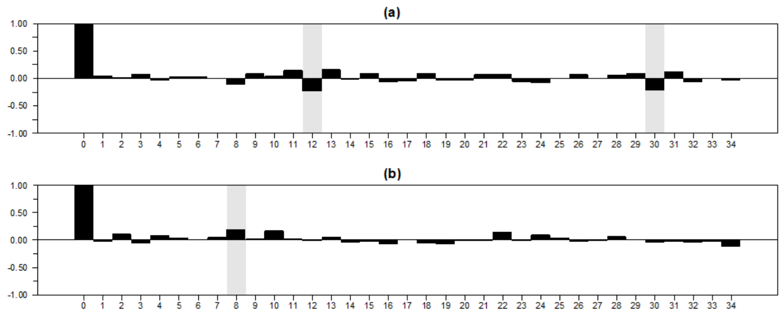


Figure C.5.: Partial autocorrelation function of the fitted TAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. GDP case.

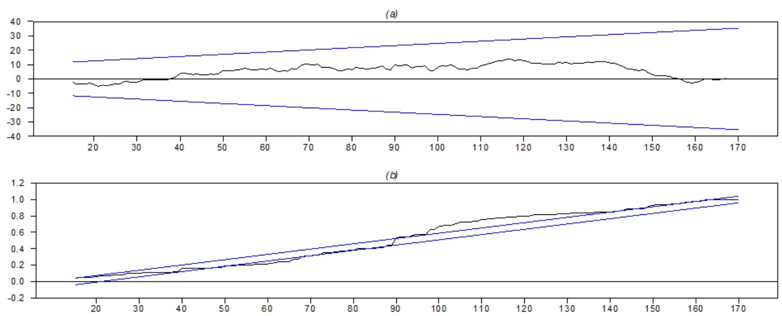


Figure C.6.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted TAR model. U.S. GDP case.

C.2. Estimation of the SETAR model

Based on the autocorrelation functions in Figure C.1 and the AIC and BIC criterion, we identify $p = 5$ as the final autoregressive order to fit a SETAR model. Under the Tsay's (1989) test we find that with $d = 2$, it is obtained the minimum p-value $= 0.001$ of the F statistic $F = 3.834$, rejecting the linearity of the series at the 5% significance level.

Figure C.7 shows the sequence of the t ratios of a lag-2 AR coefficient versus the threshold variable X_{t-d} in an arranged autoregression of order 5. We identify that data can be divided into two regimes with a possible threshold at $X_{t-d} = 2.5$, because of the change on the slope at approximately this point.

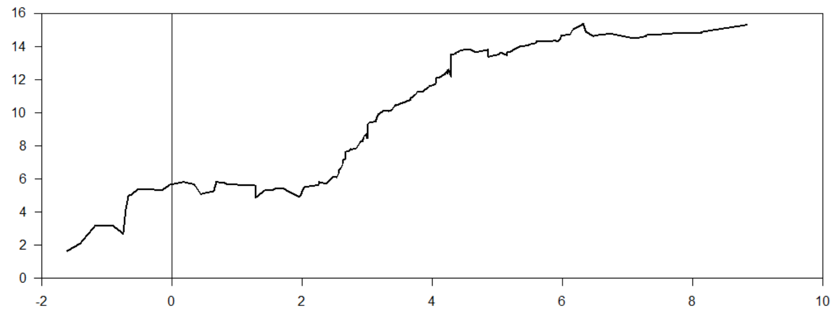


Figure C.7.: Time plot of t-ratio of recursive estimates of the AR-2 coefficient in an arranged autoregression of order 5 and delay parameter 2. U.S. GDP case.

We also use the NAIC criterion of Tong (1990), which suggests a threshold value of $r_j = 2.29$ and autoregressive orders AR(4) and AR(5) for each regime, with a NAIC = 20.769. Therefore, we fit a SETAR(2;4,5) for the annual growth of the real GDP, with threshold value X_{t-2}. The estimated parameters and their standard errors in parenthesis for each regime are shown in Table C.5 where all estimates are significant at the 5% level, except the lag-3 AR coefficient of the second regime.

	Regime	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	1.60 (0.23)	
$\Phi_1^{(j)}$	0.86 (0.10)	1.23 (0.08)
$\Phi_2^{(j)}$		-0.31 (0.13)
$\Phi_3^{(j)}$		0.16 (0.13)
$\Phi_4^{(j)}$	-0.55 (-0.55)	-0.50 (0.13)
$\Phi_5^{(j)}$		0.38 (0.08)

Table C.5.: Parameter estimates for the SETAR model. U.S. GDP case.

In that sense, the estimated SETAR model for the annual growth rate of the U.S. real GDP, is given by:

$$X_t = \begin{cases} 1.60 + 0.86X_{t-1} - 0.55X_{t-4} + \varepsilon_t, & \text{if } X_{t-2} \leq 2.29 \\ 1.23X_{t-1} - 0.31X_{t-2} + 0.16X_{t-3} - 0.50X_{t-4} + 0.38X_{t-5} + \varepsilon_t, & \text{if } X_{t-2} > 2.29 \end{cases}$$

This model could represent periods in the economy of i) contraction, where the first regime contains the decreases in the growth rate of the real GDP, that are signaled in the business cycle contractions of the NBER, and ii) expansion, where the second regime shows the greatest increases in the growth rate of the real GDP.

Finally, we evaluate the adequacy of the model. Figure C.8 let us observe that the standardized and squared standardized residuals of the model slightly signal that some nonlinear structure in the data is not explained by the model, and the Ljung-Box statistics are, respectively, $Q(8) = 23.636(0.003)$ and $Q(8) = 19.355(0.013)$. Figure C.9 presents the CUSUM and CUSUMSQ, which indicate that there is no statistical evidence for model misspecification but heteroscedasticity in $\{\varepsilon_t\}$.

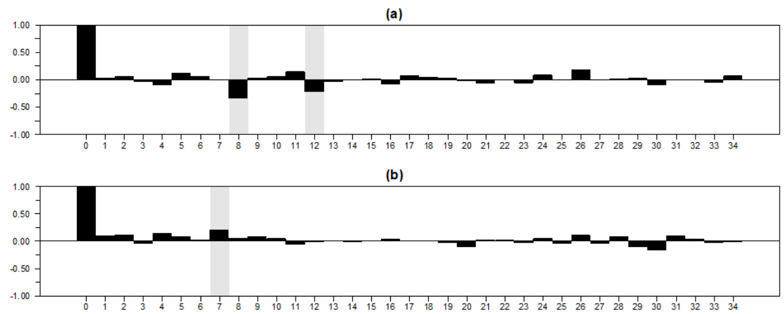


Figure C.8.: Partial autocorrelation function of the fitted SETAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. GDP case.

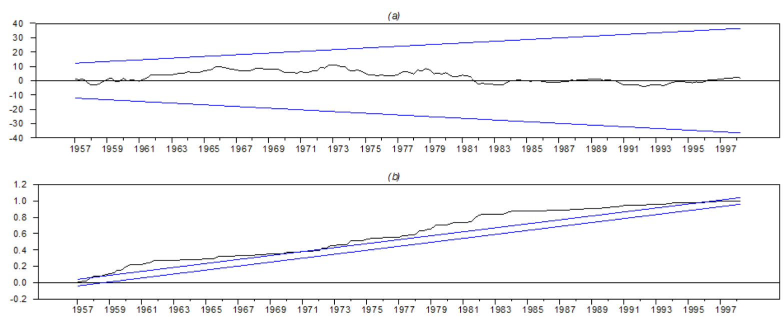


Figure C.9.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted SETAR model. U.S. GDP case.

C.3. Estimation of the STAR model

With the identified p=12 autoregressive order, we evaluate the nonlinearity of the series based on Teräsvirta's (1994) test and find that with the delay parameter $d = 4$, we obtain the p-value $= 0.0074$ for the F statistic $F = 1.8646$, which rejects the linearity of the series.

When we compute the tests of nested hypothesis of Teräsvirta (1994), we find that at the 5% significance level, H_{01} is rejected, while H_{02} is not, with a respective p - value of 0.0199 and 0.0714. H_{03} is also not rejected with a p-value $= 0.2223$. These results suggest that we must estimate a LSTAR model.

Table C.6 shows the estimates for the parameters and their respective standard error in parenthesis, which are significant at the 5% level, except for the lag-1 AR coefficient of the second regime and the parameter γ.

Parameter	Regime 1	Regime 2
$\Phi_0^{(j)}$	0.91 (0.23)	
$\Phi_1^{(j)}$	1.21 (0.13)	-0.18 (0.18)
$\Phi_2^{(j)}$	-0.50 (0.18)	0.56 (0.22)
$\Phi_4^{(j)}$	-0.54 (0.09)	
$\Phi_5^{(j)}$	0.25 (0.07)	
γ		3.29 (2.27)
c		2.57 (0.61)

Table C.6.: Parameter estimates for the STAR model. U.S. GDP case.

Consequently, the estimated STAR model for the annual growth rate of the U.S. real GDP is given by:

$$\begin{aligned} X_t = &\ 0.91 + 1.21X_{t-1} - 0.50X_{t-2} - 0.54X_{t-4} + 0.25X_{t-5} \\ &+ F\left(X_{t-4}\right)\left(-0.18X_{t-1} + 0.56X_{t-2}\right) + \varepsilon_t, \end{aligned}$$

where

$$F\left(X_{t-4}\right) = \left(1 + \exp\left\{-3.29\left(X_{t-4} - 2.57\right)\right\}\right)^{-1}.$$

When we check the residuals, Figure C.10 shows that the standardized and squared standardized residuals of the estimated model signal that some nonlinear structure in the data is not explained by the model. Furthermore, the Ljung-Box statistics are, respectively, $Q\left(8\right) = 26.906(0.000)$ and $Q\left(8\right) = 35.247(0.000)$. Figure C.11 reports the CUSUM and CUSUMSQ, which indicate that there is no statistical evidence for model misspecification but heteroscedasticity in $\{\varepsilon_t\}$.

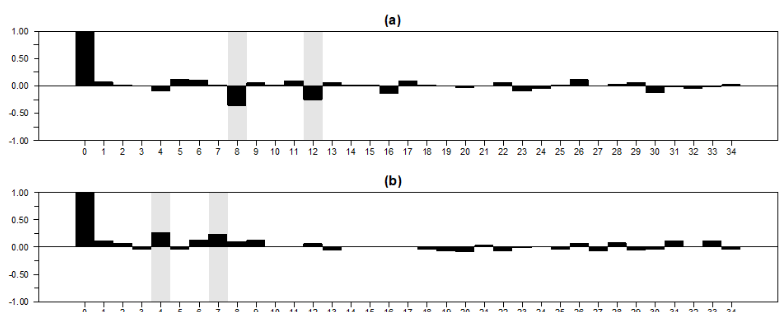


Figure C.10.: Partial autocorrelation function of the fitted STAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. GDP case.

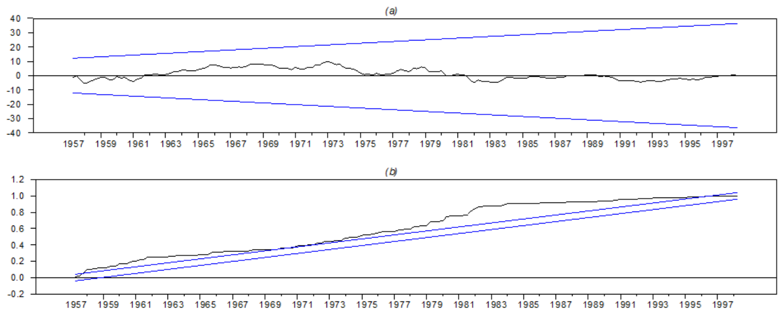


Figure C.11.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted STAR model. U.S. GDP case.

C.4. Estimation of the MSAR model

Table C.7 shows the estimates of the parameters of the model and their respective standard errors in parenthesis.

	state	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	0.42 (0.20)	0.89 (0.29)
$\Phi_1^{(j)}$	1.26 (0.10)	1.09 (0.11)
$\Phi_2^{(j)}$	-0.14 (0.15)	-0.19 (0.16)
$\Phi_3^{(j)}$	-0.42 (0.12)	-0.24 (0.13)
$\Phi_5^{(j)}$	0.18 (0.06)	0.01 (0.08)

Table C.7.: Parameter estimates for the MSAR model. U.S. GDP case.

Therefore, the estimated MSAR model for the annual growth rate of the U.S. real GDP is given by:

$$X_t = \begin{cases} 0.42 + 1.26X_{t-1} - 0.14X_{t-2} - 0.42X_{t-4} + 0.18X_{t-5} + \varepsilon_{1t}, & \text{if } s_t = 1 \\ 0.89 + 1.09X_{t-1} - 0.19X_{t-2} - 0.24X_{t-4} + 0.01X_{t-5} + \varepsilon_{2t}, & \text{if } s_t = 2 \end{cases}$$

The conditional mean of X_t for regime 1 is -1.59 and for regime 2 is -9.59. The sample variances of ε_{1t} and ε_{2t} are 0.33 and 2.25, respectively[38]. Hence, the first state could represent the stable periods with minor fluctuations in the U.S. economy, and the second state represents an unstable economy with sharp fluctuations.

Figure C.12 shows that the standardized residuals of the model of the model slightly signal that some nonlinear structure in the data is not explained by the model. The Ljung-Box

[38] The probabilities of moving from one state to the other are $p\left(s_t = 2|s_{t-1} = 1\right) = 0.02(0.02)$ and $p\left(s_t = 1|s_{t-1} = 2\right) = 0.02(0.02)$, where the number in parenthesis is the standard error. Additionally, the unconditional probability that the process is in regime 1 is 0.52 and that it is in regime 2 is 0.48.

statistics are, respectively, $Q(8) = 92.505(0.000)$ and $Q(8) = 11.006(0.201)$. Figure C.13 reports the CUSUM and CUSUMSQ, indicating that there is statistical evidence for model misspecification and heteroscedasticity in $\{\varepsilon_t\}$.

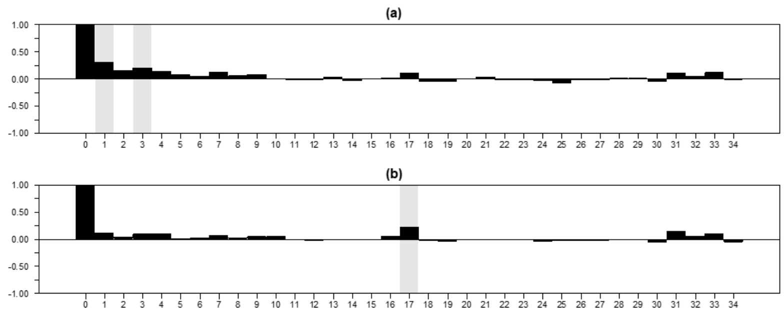


Figure C.12.: Partial autocorrelation function of the fitted MSAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. GDP case.

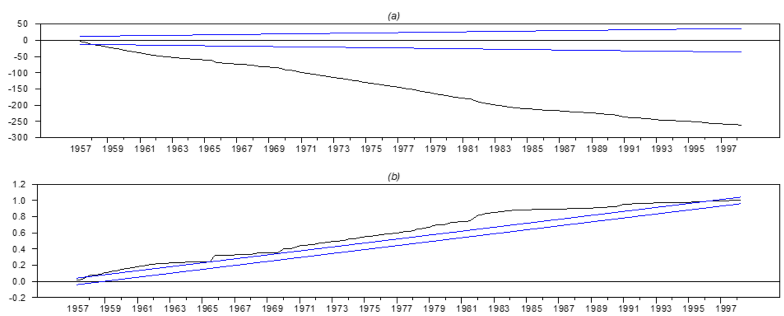


Figure C.13.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted MSAR model. U.S. GDP case.

C.5. Estimation of the AR model

We estimate an AR(5) model for the annual growth rate of the U.S. real GDP[39], which is given by:

$$\left(1 - 1.15B + 0.23B^2 - 0.02B^3 + 0.45B^4 - 0.30B^5\right) X_t = 3.35 + a_t, \quad \hat{\sigma}_a^2 = 0.10.$$

The standard errors of the coefficients are 0.07, 0.11, 0.11, 0.11, 0.07 and 0.39, respectively. When we check the residuals, Figure C.14 shows that the standardized residuals and the squared standardized residuals signal that some linear structure in the data is not explained by the model, and the Ljung-Box statistics are, respectively, $Q(8) = 27.477(0.001)$ and $Q(8) = 37.612(0.000)$. Figure C.15 shows the CUSUM indicating that there is no statistical

[39] As reported by the Augmented Dickey-Fuller (ADF) ($p-$value $= 0.028$), Phillips Perron (PP) ($p-$value $= 0.001$) and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) (Test statistic $= 0.091$ and Critical value $= 0.463$, the growth rate of the U.S. real GDP is stationary at the 5% significance level.

evidence for model misspecification and CUSUMSQ indicating some statistical evidence for heteroscedasticity in $\{\varepsilon_t\}$.

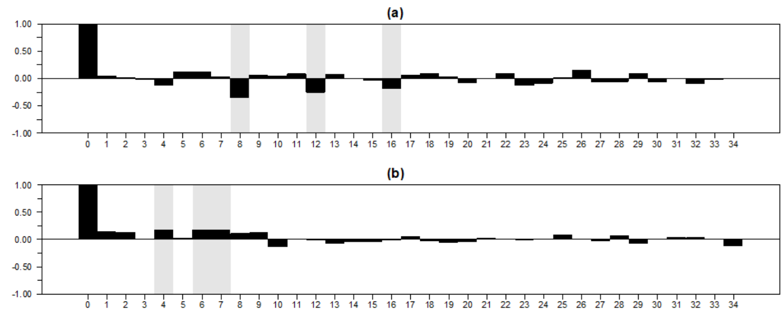


Figure C.14.: Partial autocorrelation function of the fitted AR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. GDP case.

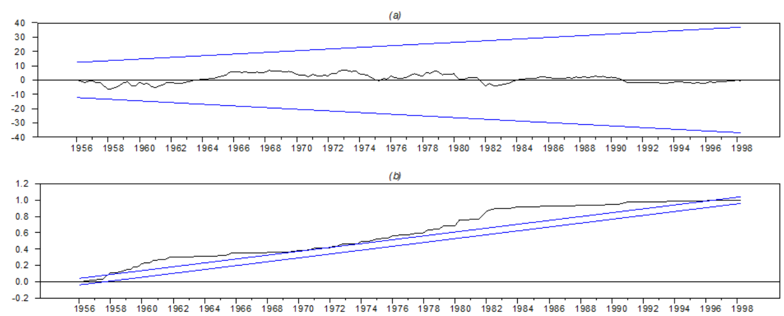


Figure C.15.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted AR model. U.S. GDP case.

Appendix D

General review of the models estimation for the annual growth rate of the U.S. Industrial Production Index

D.1. Estimation of the TAR model

We use the annual growth rate of the U.S. industrial production index as the variable of interest and the U.S. spread term defined in Section 3.2 as the threshold variable. Figure D.1 and Figure D.2 show that the autocorrelations are significant for a large number of lags.

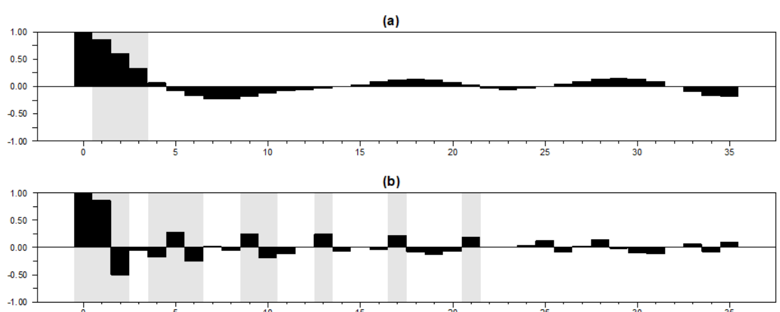


Figure D.1.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the annual growth rate of U.S. industrial production index.

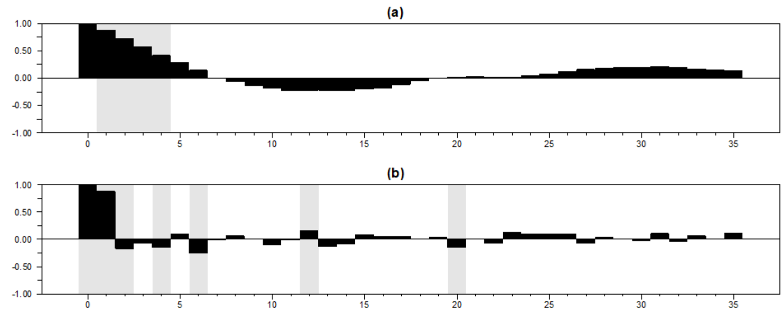


Figure D.2.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the U.S. spread term.

Then, we test the null hypothesis of AR linearity against the alternative of bivariate TAR nonlinearity. Under the AIC criterion, we found that $\bar{k} = 13$ is a reasonable autoregressive order for X_t. Under the null hypothesis, the results of the test let us define as the delay parameter of 2 for the threshold variable, thus, the input variable for the dynamic system is Z_{t-2}.

Figure D.3 shows the annual growth rate of the U.S. industrial production index and the spread term from 1960:03 to 1999:04. The shading areas denote the business cycle contractions from peak to trough based on NBER.

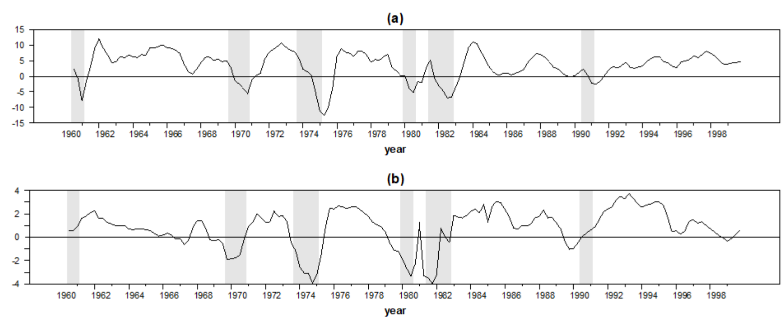


Figure D.3.: (a) Time plot of the annual growth rate of U.S. industrial production index and (b) Time plot of U.S. spread term.

We specify the maximum number of regimes l_0 by means of a regression function between X_t and Z_t, that is estimated using a nonparametric kernel approach that is presented in Figure D.4. We observe that 3 could be postulated as the possible maximum regimes for the TAR model.

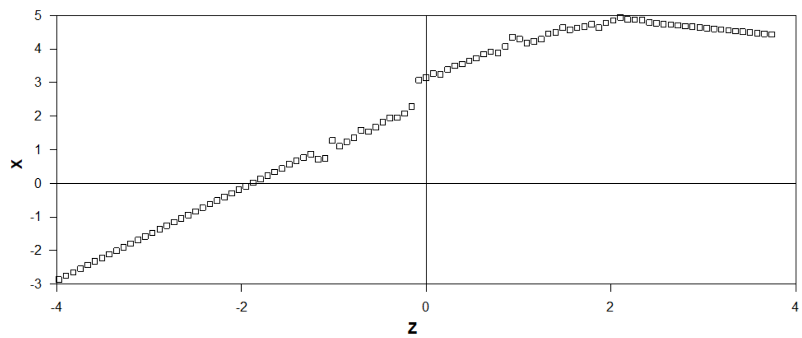


Figure D.4.: Nonparametric regression between the annual growth rate of the U.S. industrial production index (X) and U.S. spread term (Z).

To specify the prior densities for the nonstructural parameters, we define the prior densities for $\boldsymbol{\theta}_x$ where $\boldsymbol{\theta}_{0,j} = \bar{\mathbf{0}}$, $V_{0,j}^{-1} = 0.01\mathbf{I}$ with $\mathbf{I}$ the identity matrix, $\gamma_{0,j} = 1.5$ and $\beta_{0,j} = \frac{\widetilde{\sigma}^2}{2}$ with $j = 1, 2, 3$ and $\widetilde{\sigma}^2 = 1.981$ that is the residual variance of the AR(13) that was fitted to the annual growth of the industrial production index. The maximum autoregressive order for all regimes is $\bar{k} = 13$, the same value fitted to the variable of interest. The prior distributions for both the number of regimes and the autoregressive orders that we use in the identification of l are $\pi_2 = \pi_3 = 0.5$ and $p\,(k_{il}|l) = 0.0714$ for $k_{il} = 0, 1, \ldots, 13; i = 1, .., l$, respectively.

Then, to look for the location of thresholds in each possible regime, we choose the percentiles $5k$ where $k = 1, 2, \ldots, 19$, with respective values -3.07, -1.60, -0.59, -0.28, -0.05, 0.28, 0.54, 0.67, 0.76, 0.95, 1.12, 1.32, 1.53, 1.68, 1.82, 2.19, 2.40, 2.64, 2.95. The possible thresholds and autoregressive orders for each possible regime are presented in Table D.1.

ℓ	**Thresholds**	**Autoregressive orders**	**Minimum NAIC**
2	0.76	2, 2	4.57512
3	0.76 1.12	6, 12, 13	3.95755

Table D.1.: Set of possible number of regimes for the real data. U.S. indpro case.

With those possible thresholds and the information mentioned above, we compute the posterior probability distribution for the number of regimes. 3000 iterates were generated with a burn-in point of 10% of the draws[40]. The results showed in Table D.2 allow us to set $\hat{l} = 3$ as the appropriate number of regimes.

[40] The convergence of the Gibbs sampler was checked via the stationarity approach, where it was observed that the sample autocorrelation functions of the sequences $\left\{\hat{p}_l^{(i)}\right\}$ for l=2,3 decay quickly.

ℓ	$\hat{p}_\ell$
2	0.02460
3	0.97540

Table D.2.: Posterior probability function for the number of regimes for the real data. U.S. indpro case.

Thus, conditional on $\hat{l} = 3$, we estimate the autoregressive orders for k_1, k_2 and k_3. Based on Table D.3, the identified autoregressive orders are $\hat{k}_1 = 5$, $\hat{k}_2 = 5$ and $\hat{k}_3 = 13$. As before, convergence of the Gibbs sampler was checked via stationarity, and for 5000 iterates with a burn-in point of 10% of the draws, it is observed that the sample autocorrelations functions decay quickly[41].

Autoregressive Order	**Regime**		
	1	**2**	**3**
0	$1.39{\times}10^{-50}$	$2.67{\times}10^{-04}$	$1.41{\times}10^{-42}$
1	0.0855	0.0199	$2.61{\times}10^{-11}$
2	0.0225	0.0155	0.0022
3	0.1106	0.0155	0.0026
4	$6.75{\times}10^{-04}$	0.0033	$3.15{\times}10^{-06}$
5	**0.3064**	**0.2116**	$2.11{\times}10^{-04}$
6	0.0103	0.1590	0.0019
7	0.2304	0.1266	$9.55{\times}10^{-04}$
8	$4.62{\times}10^{-06}$	0.1473	$2.40{\times}10^{-04}$
9	0.07764	0.0457	0.0135
10	0.1115	0.0642	0.0058
11	0.0356	0.0214	0.0065
12	$5.00{\times}10^{-04}$	0.0764	$5.68{\times}10^{-04}$
13	0.0084	0.0936	**0.9654**

Table D.3.: Posterior probabilities for the autoregressive orders in the real data. U.S. indpro case.

Consequently, we fit a TAR(3;5,5,13) with thresholds values $r_1 = 0.76$ and $r_2 = 1.12$, which respectively are the 45th and 55th percentiles of the spread term. Table D.4 shows the estimates for the nonstructural parameters, with their respective posterior standard error in parenthesis and 90% credible interval in brackets[42]. These results show that not all the coefficients are significant at the 5% level. However, we decided to estimate the model with

[41] We also perform a sensibility analysis where we changed the prior densities of the autoregressive orders and the nonstructural parameters, and we found that the estimated autoregressive orders are the same for different priors of the nonstructural parameters and for different priors of the autoregressive orders.

[42] 5000 iterates were generated with a burn-in point of 10% of the draws, and we find that the autocorrelation functions for all the parameters decay quickly, indicating the convergence of the Gibbs sampler.

all the coefficients, given that that improves the final estimation.

	Regime		
Parameter	**1**	**2**	**3**
$a_0^{(j)}$	-0.45 (0.40)	0.97 (0.52)	1.04 (0.27)
	$[-1.10, 0.21]$	$[0.14, 1.80]$	$[0.60, 1.48]$
$a_1^{(j)}$	1.38 (0.12)	1.05 (0.26)	1.33 (0.13)
	$[1.18, 1.59]$	$[0.63, 1.47]$	$[1.12, 1.53]$
$a_2^{(j)}$	-0.57 (0.21)	-0.25 (0.47)	-0.47 (0.20)
	$[-0.92 - 0.23]$	$[-1.00, 0.49]$	$[-0.80, -0.15]$
$a_3^{(j)}$	0.41 (0.21)	0.30 (0.48)	0.09 (0.20)
	$[0.07, 0.75]$	$[-0.49, 1.07]$	$[-0.24, 0.42]$
$a_4^{(j)}$	-0.53 (0.21)	-0.83 (0.43)	-0.70 (0.21)
	$[-0.86, -0.18]$	$[-1.53, -0.13]$	$[-1.05, -0.35]$
$a_5^{(j)}$	0.27 (0.14)	0.64 (0.27)	0.85 (0.23)
	$[0.03, 0.51]$	$[0.20, 1.08]$	$[0.46, 1.23]$
$a_6^{(j)}$			-0.34 (0.25)
			$[-0.75, 0.08]$
$a_7^{(j)}$			0.19 (0.26)
			$[-0.23, 0.60]$
$a_8^{(j)}$			-0.78 (0.25)
			$[-1.20, -0.37]$
$a_9^{(j)}$			0.88 (0.24)
			$[0.48, 1.27]$
$a_{10}^{(j)}$			-0.29 (0.22)
			$[-0.65, 0.06]$
$a_{11}^{(j)}$			0.17 (0.20)
			$[-0.17, 0.50]$
$a_{12}^{(j)}$			-0.52 (0.17)
			$[-0.80, -0.24]$
$a_{13}^{(j)}$			0.34 (0.10)
			$[0.19, 0.50]$
$h^{(j)}$	3.21 (0.58)	1.27 (0.75)	1.02 (0.21)
	$[2.40, 4.25]$	$[0.55, 2.66]$	$[0.73, 1.39]$

Table D.4.: Parameter estimates for the TAR model. U.S. indpro case.

Therefore, the estimated TAR model for the annual growth rate of the U.S. industrial production index is given by:

$$X_t = \begin{cases} -0.45 + 1.38X_{t-1} - 0.57X_{t-2} + 0.41X_{t-3} - 0.53X_{t-4} & \\ +0.27X_{t-5} \ + 3.21\varepsilon_t, & \text{if } Z_{t-2} \leq 0.76 \\ 0.97 + 1.05X_{t-1} - 0.25X_{t-2} + 0.30X_{t-3} - 0.83X_{t-4} & \\ +0.64X_{t-5} \ + 1.27\varepsilon_t, & \text{if } 0.76 < Z_{t-2} \leq 1.12 \\ +0.85X_{t-5} - 0.34X_{t-6} + 0.19X_{t-7} - 0.78X_{t-8} + 0.889X_{t-9} & \\ -0.29X_{t-10} + 0.17X_{t-11} - 0.52X_{t-12} + 0.34X_{t-13} + 1.02\varepsilon_t, & \text{if } Z_{t-2} > 1.12 \end{cases}$$

This model could represent periods in the economy of i) contraction, given that this regime contains the greatest decreases in the growth rate of the industrial production index, when spreads are low due to contractionary monetary policies; ii) stabilization, where stable spread values are related to minor fluctuations in the growth rate of the industrial production index; and iii) expansion, where this last regime exhibits the greatest increases in the growth rate of the industrial production index, when the monetary policy is expansioning.

When we check the residuals, in Figure D.5 we observe that the standardized residuals and the squared standardized residuals of the model slightly signal that some nonlinear structure in the data is not explained by the model. Moreover, the Ljung-Box statistics are, respectively, $Q\,(8) = 11.554(0.172)$ and $Q\,(8) = 23.988(0.002)$. Figure D.6 reports that the CUSUM and CUSUMSQ behave well, indicating that there is no statistical evidence for model misspecification but there is some heteroscedasticity in $\{\varepsilon_t\}$.

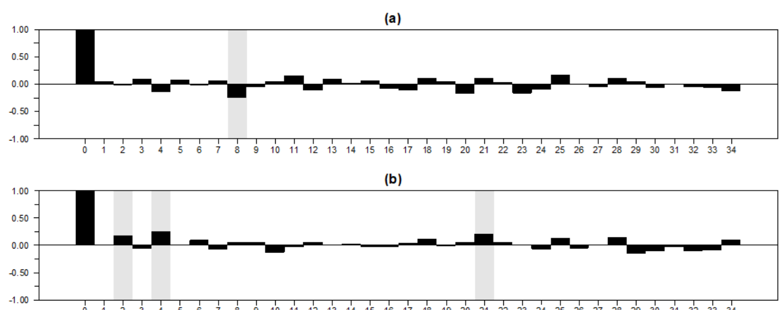


Figure D.5.: Partial autocorrelation function of the fitted TAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. indpro case.

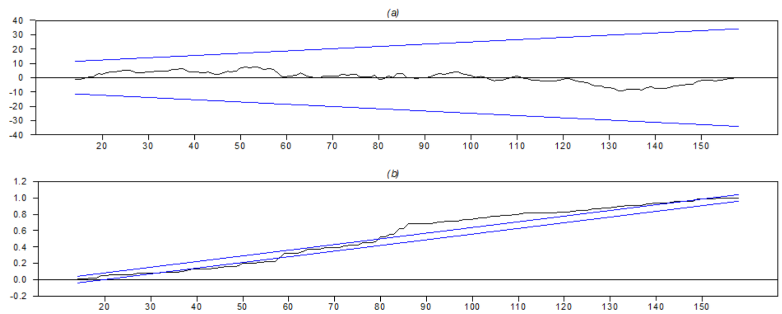


Figure D.6.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted TAR model. U.S. indpro case.

D.2. Estimation of the SETAR model

Based on the autocorrelation functions in Figure D.1 and the AIC and BIC criterion, we identify $p = 9$ as the final autoregressive order to fit a SETAR model. Under the Tsay's (1989) test we find that with $d = 5$, it is obtained the minimum p-value $= 0.0004$ of the F statistic $F = 3.534$, rejecting the linearity of the series at the 5% significance level.

Figure D.7 shows the sequence of the t ratios of a lag-5 AR coefficient versus the threshold variable X_{t-d} in an arranged autoregression of order 9, and we identify that the data can be divided into two regimes with a possible threshold at $X_{t-d} = 5.0$, because of the change on the slope at approximately this point.

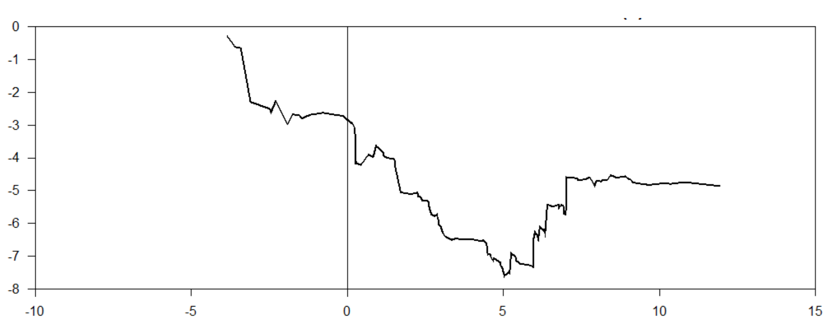


Figure D.7.: Time plot of t-ratio of recursive estimates of the AR-5 coefficient in an arranged autoregression of order 9 and delay parameter 5. U.S. indpro case.

We also use the NAIC criterion of Tong (1990), which suggests a threshold value of $r_j = 5.08$ and autoregressive orders AR(9) and AR(2) for each regime, with a $NAIC = 108.918$. Based on that, we fit a SETAR(2;5,2) for the annual growth rate of the industrial production index, with threshold value X_{t-5}. The estimated parameters and their standard errors in parenthesis for each regime are shown in Table D.5, where all estimates are significant at the 5% level, although the lag-3 AR coefficient of the first regime is significant at the 10% level.

	Regime	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	0.72 (0.20)	
$\Phi_1^{(j)}$	1.35 (0.10)	1.44 (0.12)
$\Phi_2^{(j)}$	-0.59 (0.16)	-0.49 (0.11)
$\Phi_3^{(j)}$	0.26 (0.15)	
$\Phi_4^{(j)}$	-0.55 (0.13)	
$\Phi_5^{(j)}$	0.35 (0.09)	

Table D.5.: Parameter estimates for the SETAR model. U.S. indpro case.

Thus, the estimated SETAR model for the annual growth rate of the U.S. industrial production index, is given by:

$$X_t = \begin{cases} 0.72 + 1.35X_{t-1} - 0.59X_{t-2} + 0.26X_{t-3} & \\ -0.55X_{t-4} + 0.35X_{t-5} + \varepsilon_t, & \text{if } X_{t-5} \leq 5.08 \\ 1.44X_{t-1} - 0.49X_{t-2} + \varepsilon_t, & \text{if } X_{t-5} > 5.08 \end{cases}$$

This model could represent periods in the economy of i) contraction, where the first regime contains mostly the decreases in the growth rate of the industrial production index, and ii) expansion, where the second regime shows mostly the increases in the growth rate of the industrial production index.

Figure D.8 let us observe that the standardized and squared standardized residuals of the model signal that some nonlinear structure in the data is not explained by the model. Furthermore, the Ljung-Box statistics are, respectively, $Q(8) = 28.391(0.000)$ and $Q(8) = 23.980(0.002)$. Figure D.9 presents the CUSUM and CUSUMSQ, indicating that there is no statistical evidence for model misspecification but some heteroscedasticity in $\{\varepsilon_t\}$.

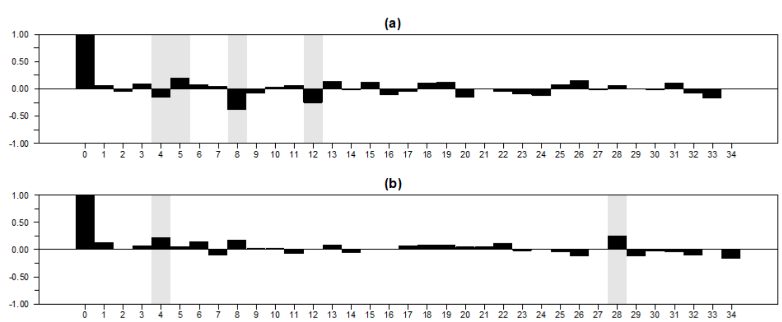


Figure D.8.: Partial autocorrelation function of the fitted SETAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. indpro case.

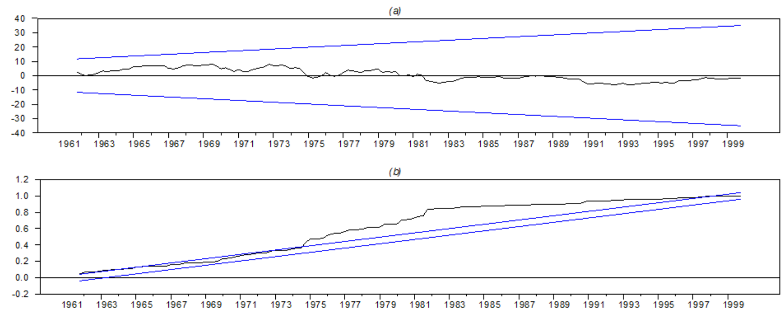


Figure D.9.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted SETAR model. U.S. indpro case.

D.3. Estimation of the STAR model

With the identified $p = 11$ autoregressive order, we evaluate the nonlinearity of the series based on Teräsvirta's (1994) test and find that, with a delay parameter of $d = 1$, we obtain the p-value $= 0.0067$ for the F statistic $F = 1.9261$, which rejects the linearity of the series.

When we compute the tests of nested hypothesis of Teräsvirta (1994), we find that at the 10% significance level, H_{01} and H_{03} are rejected, while H_{02} is not, where the respective p-value is 0.0605, 0.0116 and 0.2517. These results suggest that we must estimate a LSTAR model.

In Table D.6, we show the estimates for the parameters and their respective standard error in parenthesis, which are significant at the 5% significance level, except the parameter γ.

	Regime	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	0.46 (0.24)	
$\Phi_1^{(j)}$	1.79 (0.16)	-0.45 (0.18)
$\Phi_2^{(j)}$	-1.26 (0.21)	0.67 (0.20)
$\Phi_3^{(j)}$	0.33 (0.13)	
$\Phi_4^{(j)}$	-0.52 (0.12)	
$\Phi_5^{(j)}$	0.44 (0.11)	
$\Phi_6^{(j)}$	-0.17 (0.07)	
γ		16.55 (44.73)
c		-2.93 (1.45)

Table D.6.: Parameter estimates for the STAR model. U.S. indpro case.

Consequently, the estimated STAR model for the annual growth rate of the U.S. industrial production index is given by:

$$
\begin{aligned}
X_t = \; & 0.46 + 1.79X_{t-1} - 1.26X_{t-2} + 0.33X_{t-3} - 0.52X_{t-4} + 0.44X_{t-5} - 0.17X_{t-6} \\
& + F\left(X_{t-5}\right)\left(-0.45X_{t-1} + 0.67X_{t-2}\right) + \varepsilon_t,
\end{aligned}
$$

where

$$
F\left(X_{t-5}\right) = \left(1 + \exp\left\{-16.55\left(X_{t-5} + 2.93\right)\right\}\right)^{-1}.
$$

When checking the residuals, Figure D.10 shows that the standardized and squared standardized residuals of the model signal that some nonlinear structure in the data is not explained by the model. Moreover, the Ljung-Box statistics are, respectively, $Q(8) = 21.376(0.006)$ and $Q(8) = 28.559(0.000)$. Figure D.11 presents the CUSUM and CUSUMSQ, which indicate that there is no statistical evidence for model misspecification but some heteroscedasticity in $\{\varepsilon_t\}$.

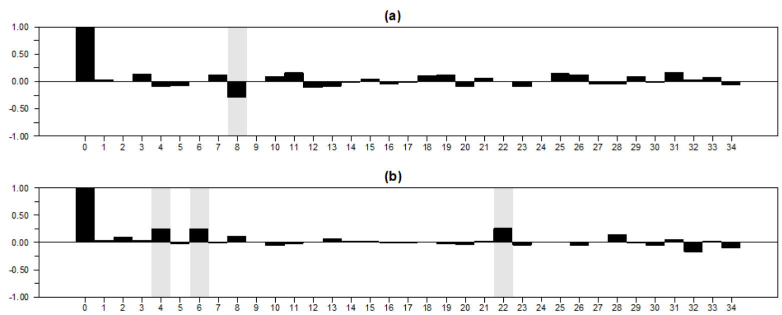


Figure D.10.: Partial autocorrelation function of the fitted STAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. indpro case.

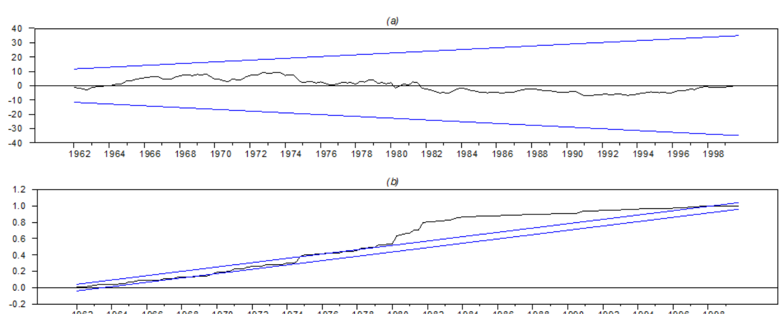


Figure D.11.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted STAR model. U.S. indpro case.

D.4. Estimation of the MSAR model

Table D.7 shows the estimates of the parameters of the model and their respective standard errors in parenthesis.

	state	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	0.71 (0.18)	0.36 (0.37)
$\Phi_1^{(j)}$	1.42 (0.08)	1.32 (0.12)
$\Phi_2^{(j)}$	-0.47 (0.10)	-0.49 (0.14)
$\Phi_4^{(j)}$	-0.33 (0.09)	-0.18 (0.15)
$\Phi_5^{(j)}$	0.23 (0.07)	0.15 (0.12)

Table D.7.: Parameter estimates for the MSAR model. U.S. indpro case.

Therefore, the estimated MSAR model for the annual growth rate of the U.S. industrial production index is given by:

$$X_t = \begin{cases} 0.71 + 1.42X_{t-1} - 0.47X_{t-2} - 0.33X_{t-4} + 0.23X_{t-5} + \varepsilon_{1t}, & \text{if } s_t = 1 \\ 0.36 + 1.32X_{t-1} - 0.49X_{t-2} - 0.18X_{t-4} + 0.15X_{t-5} + \varepsilon_{2t}, & \text{if } s_t = 2 \end{cases}$$

The conditional mean of X_t for regime 1 is -1.66 and for regime 2 is -1.11. The sample variances of ε_{1t} and ε_{2t} are 0.76 and 5.31, respectively[43]. Hence, the first state could represent a contractionary economy with reductions in the growth rate of the industrial production index, and the second state could represent a more stable economy with major increases in the growth rate of this macroeconomic indicator.

Figure D.12 shows that the standardized and squared standardized residuals of the model signal that some nonlinear structure in the data is not explained by the model, and the Ljung-Box statistics are, respectively, $Q(8) = 57.790(0.000)$ and $Q(8) = 20.506(0.009)$. Figure D.13 presents the CUSUM and CUSUMSQ, which indicate that there is statistical evidence for model misspecification and some heteroscedasticity in $\{\varepsilon_t\}$.

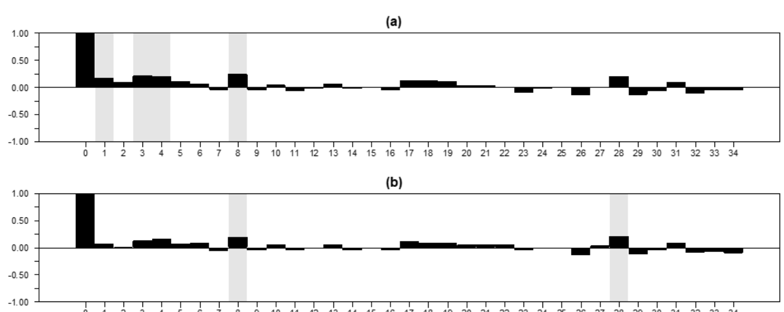


Figure D.12.: Partial autocorrelation function of the fitted MSAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. indpro case.

[43] The probabilities of moving from one state to the other are $p(s_t = 2|s_{t-1} = 1) = 0.05(0.03)$ and $p(s_t = 1|s_{t-1} = 2) = 0.06(0.05)$, where the number in parenthesis is the standard error. Additionally, the unconditional probability that the process is in regime 1 is 0.59 and that it is in regime 2 is 0.41.

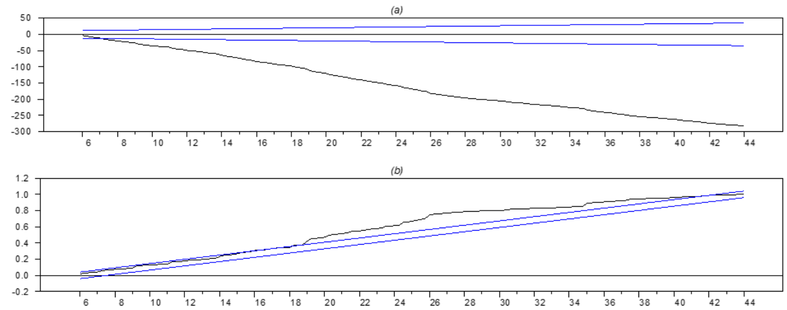


Figure D.13.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted MSAR model. U.S. indpro case.

D.5. Estimation of the AR model

We estimate a SARIMA(2,0,0)×(1,0,0) model for the annual growth rate of the U.S. industrial production index[44], which is given by:

$$\left(1 - 1.42B + 0.50B^2\right)\left(1 + \ 0.39B^4\right)X_t = a_t, \quad \hat{\sigma}_a^2 = 0.10,$$

The standard errors of the coefficients are 0.07, 0.07 and 0.08, respectively. When we check the residuals, Figure D.14 shows that the standardized and the squared standardized residuals slightly signal that some linear structure in the data is not explained by the model, and the Ljung-Box statistics are, respectively, $Q\,(8) = 33.708(0.000)$ and $Q\,(8) = 17.445(0.026)$. Figure D.15 shows the CUSUM and CUSUMSQ, which respectively indicate that there is no statistical evidence for model misspecification but there is statistical evidence for some heteroscedasticity in $\{\varepsilon_t\}$.

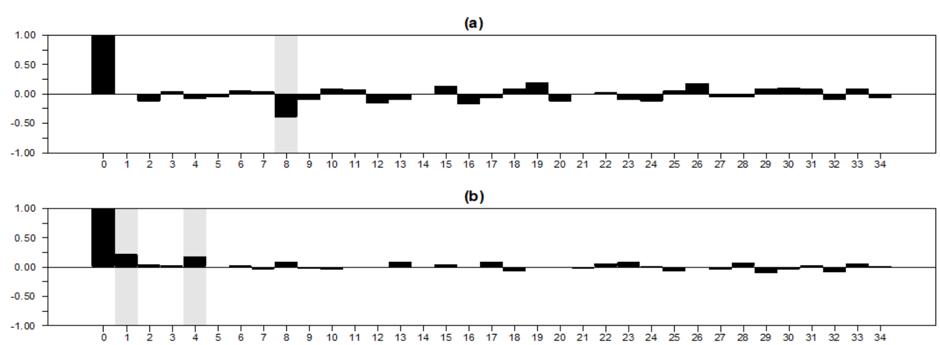


Figure D.14.: Partial autocorrelation function of the fitted AR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. indpro case.

[44] As reported by the Augmented Dickey-Fuller (ADF) ($p-$value $= 0.000$), Phillips Perron (PP) ($p-$value $= 0.001$) and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) (Test statistic $= 0.132$ and Critical value $= 0.463$, the growth rate of the U.S. industrial production index is stationary at the 5% significance level.

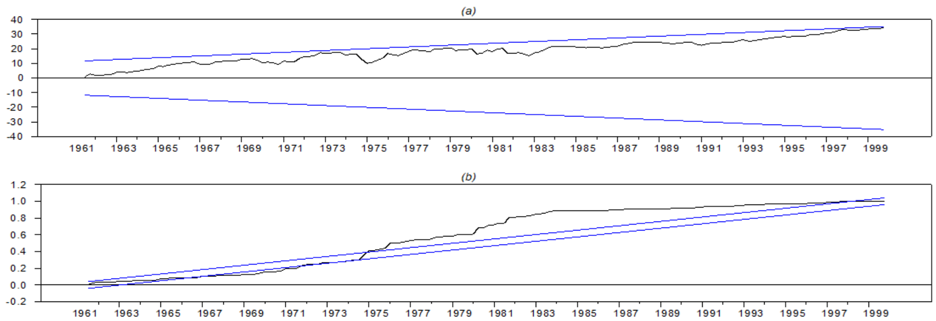


Figure D.15.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted AR model. U.S. indpro case.

Appendix E

General review of the models estimation for the growth rate of the U.S. quarterly CPI

E.1. Estimation of the TAR model

We use as the variable of interest the growth rate of the U.S. CPI, and as the threshold variable, we use the U.S. spread term mentioned above. Figure E.1 and Figure E.2 show that the series have significant autocorrelations for a large number of lags.

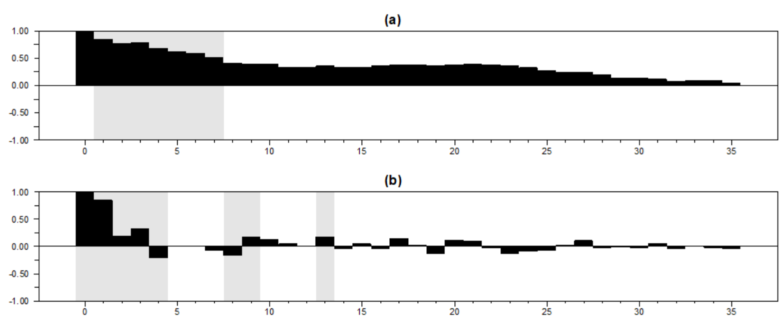


Figure E.1.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the growth rate of U.S. CPI.

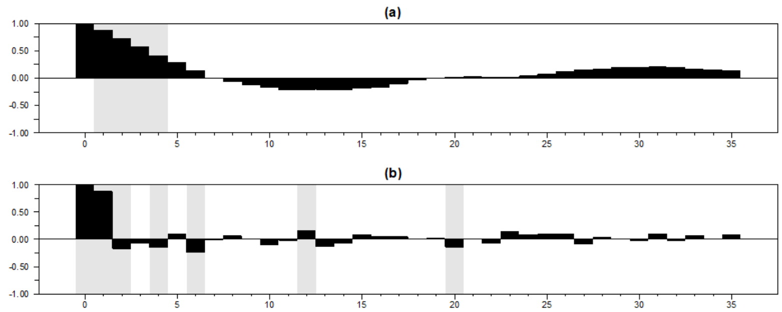


Figure E.2.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the U.S. spread term.

Then, we test the null hypothesis of AR linearity against the alternative of bivariate TAR nonlinearity. Under the AIC criterion, we found that $\bar{k} = 9$ is a reasonable autoregressive order for X_t. Under the null hypothesis, the results of the test let us define as the delay parameter of 7 for the threshold variable, thus, the input variable for the dynamic system is Z_{t-7}. This is in accordance of Schuh (2001, p. 40), who says that *"(...) changes in monetary policy typically affect the economy with a lag of six to 18 months, policymakers must evaluate economic activity in the future to determine the appropriate monetary conditions today"*.

Figure E.3 shows the growth rate of the U.S. CPI and spread term from 1956:03 to 1998:01. The shading areas denote the business cycle contractions from peak to trough based on NBER.

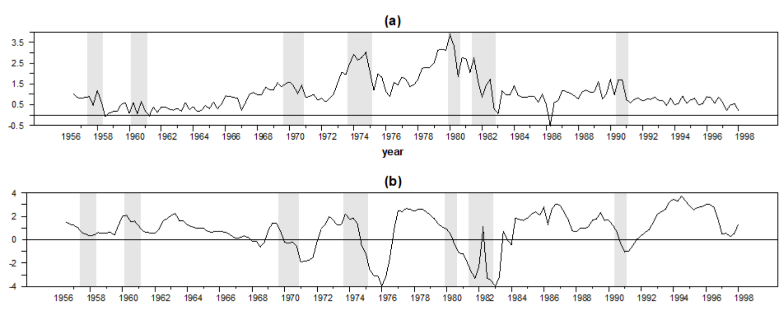


Figure E.3.: (a) Time plot of the annual growth rate of U.S. CPI and (b) Time plot of U.S. spread term.

We specify the maximum number of regimes l_0 by means of a regression function between X_t and Z_t, that is estimated using a nonparametric kernel approach and that is presented in Figure E.4. We observe that 3 could be postulated as the possible maximum regimes for the TAR model.

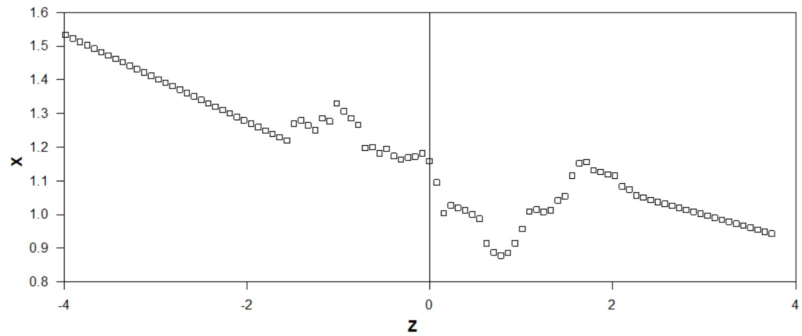


Figure E.4.: Nonparametric regression between the growth rate of U.S. CPI (X) and the U.S. spread term (Z).

Next, to specify the prior densities for the nonstructural parameters, we define the prior densities for $\boldsymbol{\theta}_x$ where $\boldsymbol{\theta}_{0,j} = \bar{\mathbf{0}}, V_{0,j}^{-1} = 0.01\mathbf{I}$ with $\mathbf{I}$ the identity matrix, $\gamma_{0,j} = 1.5$ and $\beta_{0,j} = \frac{\widetilde{\sigma}^2}{2}$ with $j = 1, 2, 3$ and $\widetilde{\sigma}^2 = 0.133$ that is the residual variance of the AR(9) that was fitted to the annual growth of the CPI. The maximum autoregressive order for all regimes is $\bar{k} = 9$, the same value fitted to the variable of interest. The prior distributions for both the number of regimes and the autoregressive orders that we use in the identification of l are $\pi_2 = \pi_3 = 0.5$ and $p\,(k_{il}|l) = 0.1$ for $k_{il} = 0, 1, \ldots 9; i = 1, .., l$, respectively.

Then, to look for the location of thresholds in each possible regime, we choose the percentiles $5k$ where $k = 1, 2, \ldots, 19$, with respective values -2.95, -1.51, -0.50, -0.18, 0.23, 0.46, 0.59, 0.68, 0.81, 0.96, 1.18, 1.32, 1.57, 1.67, 1.82, 2.10, 2.39, 2.61, 2.92. The possible thresholds and autoregressive orders for each possible regime are presented in Table E.1.

ℓ	**Thresholds**	**Autoregressive orders**	**Minimum NAIC**
2	0.68	3, 3	1.62567
3	-0.18 1.18	3, 2, 3	1.16958

Table E.1.: Set of possible number of regimes for the real data. U.S. CPI case.

With those possible thresholds and the information mentioned above, we compute the posterior probability distribution for the number of regimes. 3000 iterates were generated with a burn-in point of 10% of the draws[45]. The results showed in Table E.2 allow us to set $\hat{l} = 3$ as the appropriate number of regimes.

ℓ	$\hat{p}_\ell$
2	0.3068
3	0.6932

Table E.2.: Posterior probability function for the number of regimes for the real data. U.S. CPI case.

[45] The convergence of the Gibbs sampler was checked via the stationarity approach, where it was observed that the sample autocorrelation functions of the sequences $\left\{\hat{p}_l^{(i)}\right\}$ for $l = 2, 3$ decay quickly.

Thus, conditional on $\hat{l} = 3$, we estimate the autoregressive orders for k_1, k_2 and k_3. Based on Table E.3, the identified autoregressive orders are $\hat{k}_1 = 7, \hat{k}_2 = 7$ and $\hat{k}_3 = 6$. As before, convergence of the Gibbs sampler was checked via stationarity, and for 5000 iterates with a burn-in point of 10% of the draws, it is observed that the sample autocorrelations functions decay quickly[46].

Autoregressive Order	**Regime**		
	1	**2**	**3**
0	5.73×10^{-22}	3.33×10^{-47}	4.30×10^{-48}
1	7.55×10^{-04}	4.28×10^{-08}	1.19×10^{-05}
2	1.40×10^{-04}	0.0041	0.0037
3	0.1908	0.0481	0.1805
4	0.0955	0.0016	0.0109
5	0.1247	0.1994	0.0590
6	0.2506	0.1275	**0.3037**
7	**0.2797**	**0.5044**	0.3012
8	0.0336	0.0878	0.0729
9	0.0242	0.0270	0.0681

Table E.3.: Posterior probabilities for the autoregressive orders in the real data. U.S. CPI case.

Consequently, we fit a TAR(3;7,7,6) with thresholds values $r_1 = -0.18$ and $r_2 = 1.18$, which respectively are the 20th and 55th percentiles of the spread term. Table E.4 shows the estimates for the nonstructural parameters, with their respective posterior standard error in parenthesis and 90% credible interval in brackets[47]. These results show that not all the coefficients are significant at the 5% level. However, we decided to estimate the model with all the coefficients, given that that improves the final estimation.

[46] We also performed a sensibility analysis where we changed the prior densities of the autoregressive orders and the nonstructural parameters, and it was found that the estimated autoregressive orders are the same for different priors of the nonstructural parameters and for different priors of the autoregressive orders.

[47] 5000 iterates were generated with a burn-in point of 10% of the draws, and it was found that the autocorrelation functions for all the parameters decay quickly, indicating the convergence of the Gibbs sampler.

Parameter	**Regime** 1	2	3
$a_0^{(j)}$	0.13 (0.22)	-0.04 (0.06)	-0.06 (0.09)
	$[-0.22, 0.49]$	$[-0.14, 0.07]$	$[-0.22, 0.10]$
$a_1^{(j)}$	0.48 (0.17)	0.61 (0.12)	0.60 (0.13)
	$[0.21, 0.75]$	$[0.42, 0.80]$	$[0.39, 0.81]$
$a_2^{(j)}$	-0.37 (0.20)	0.24 (0.13)	0.21 (0.16)
	$[-0.69, -0.04]$	$[0.02, 0.45]$	$[-0.05, 0.46]$
$a_3^{(j)}$	0.67 (0.21)	0.13 (0.11)	0.29 (0.16)
	$[0.31, 1.01]$	$[-0.06, 0.32]$	$[0.03, 0.56]$
$a_4^{(j)}$	0.004 (0.22)	-0.13 (0.12)	-0.17 (0.17)
	$[-0.3, 0.36]$	$[-0.33, 0.08]$	$[-0.43, 0.11]$
$a_5^{(j)}$	-0.03 (0.22)	0.19 (0.11)	0.10 (0.16)
	$[-0.39, 0.33]$	$[0.01, 0.38]$	$[-0.17, 0.37]$
$a_6^{(j)}$	0.16 (0.22)	-0.01 (0.10)	0.15 (0.15)
	$[-0.21, 0.53]$	$[-0.18, 0.15]$	$[-0.11, 0.40]$
$a_7^{(j)}$	-0.18 (0.17)	0.05 (0.10)	
	$[-0.46, 0.10]$	$[-0.12, 0.21]$	
$h^{(j)}$	0.19 (0.05)	0.06 (0.01)	0.12 (0.02)
	$[0.12, 0.29]$	$[0.04, 0.09]$	$[0.09, 0.16]$

Table E.4.: Parameter estimates for the TAR model. U.S. CPI case.

The estimated TAR model for the growth rate of the U.S. CPI is given by:

$$X_t = \begin{cases} 0.13 + 0.48X_{t-1} - 0.37X_{t-2} + 0.67X_{t-3} + 0.004X_{t-4} & \\ -0.03X_{t-5} + 0.16X_{t-6} - 0.18X_{t-7} + 0.19\varepsilon_t, & \text{if } Z_{t-7} \leq -0.08 \\ -0.04 + 0.61X_{t-1} + 0.24X_{t-2} + 0.13X_{t-3} - 0.13X_{t-4} & \\ +0.19X_{t-5} - 0.01X_{t-6} + 0.05X_{t-7} + 0.06\varepsilon_t, & \text{if } -0.08 < Z_{t-7} \leq 1.18 \\ -0.06 + 0.60X_{t-1} + 0.21X_{t-2} + 0.29X_{t-3} - 0.17X_{t-4} & \\ +0.10X_{t-5} + 0.15X_{t-6} + 0.12\varepsilon_t, & \text{if } Z_{t-7} > 1.18 \end{cases}$$

This model could represent i) a first regime characterized by a real activity tightening associated with minor fluctuations in the inflation; ii) a second regime with an economic transition with a stable behavior of the inflation; and iii) a third regime with a real activity increasing associated with major fluctuations in the inflation.

When we check the residuals, in Figure E.5 we observe that the standardized residuals and squared standardized residuals could signal that the noise process is white, and the Ljung-Box statistics are, respectively, $Q(8) = 6.407(0.601)$ and $Q(8) = 12.677(0.123)$. Figure E.6 reports that the CUSUM and CUSUMSQ behave well, which indicates that there is no statistical evidence for model misspecification or strong heteroscedasticity in $\{\varepsilon_t\}$.

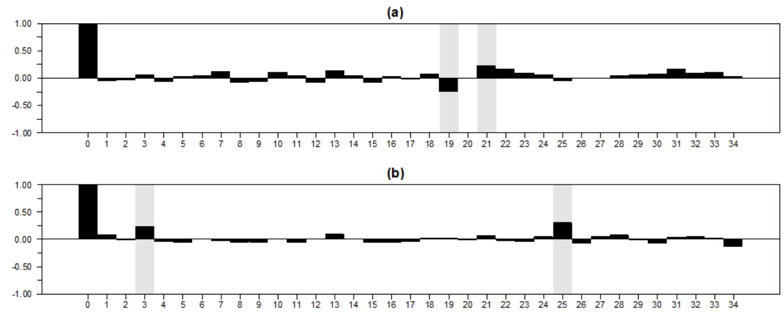


Figure E.5.: Partial autocorrelation function of the fitted TAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. CPI case.

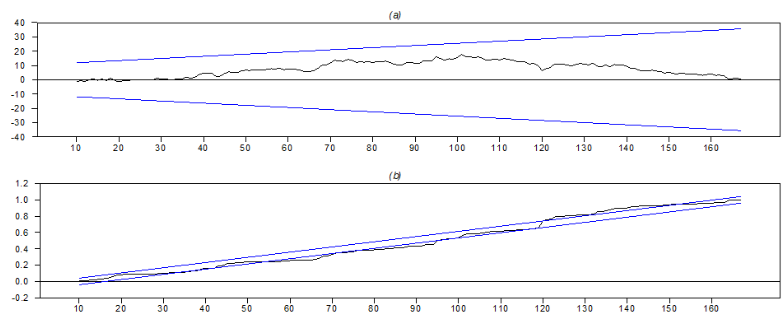


Figure E.6.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted TAR model. U.S. CPI case.

E.2. Estimation of the SETAR model

Based on the autocorrelation functions in Figure E.1 and the AIC and BIC criterion, we identify $p = 9$ as the final autoregressive order to fit a SETAR model. Under the Tsay's (1989) test and we find that with $d = 2$, it is obtained the minimum p-value $= 0.0035$ of the F statistic $F = 2.795$, rejecting the linearity of the series at the 5% significance level.

Figure E.7 shows the sequence of the t ratios of a lag-9 AR coefficient versus the threshold variable X_{t-d} in an arranged autoregression of order 9. We identify that data can be divided into two regimes with a possible threshold at $X_{t-d} = 1.0$, because of the change on the slope at approximately this point.

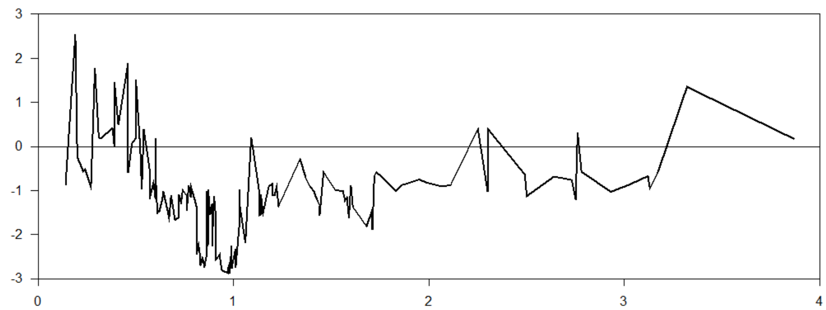


Figure E.7.: Time plot of t-ratio of recursive estimates of the AR-9 coefficient in an arranged autoregression of order 9 and delay parameter 2. U.S. CPI case.

We also use the NAIC criterion of Tong (1990), which suggests a threshold value of $r_j = 0.98$ and autoregressive orders AR(3) and AR(4) for each regime, with a NAIC = -345.436. Based on that, we fit a SETAR(2;3,4) for the growth rate of the CPI, with threshold value X_{t-2}. The estimated parameters and their standard errors in parenthesis for each regime are shown in Table E.5 where all estimates are significant at the 5% level, except the constant coefficient of both regimes.

	Regime	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	0.10 (0.07)	-0.02 (0.14)
$\Phi_1^{(j)}$	0.45 (0.09)	0.83 (0.10)
$\Phi_2^{(j)}$	0.20 (0.12)	
$\Phi_3^{(j)}$	0.28 (0.08)	0.65 (0.13)
$\Phi_4^{(j)}$		-0.53 (0.14)

Table E.5.: Parameter estimates for the SETAR model. U.S. CPI case.

Thus, the estimated SETAR model for the growth rate of the U.S. CPI, is given by:

$$X_t = \begin{cases} 0.10 + 0.45X_{t-1} + 0.20X_{t-2} + 0.28X_{t-3} + \varepsilon_t, & \text{if } X_{t-2} \leq 0.98 \\ -0.02 + 0.83X_{t-1} + 0.65X_{t-3} - 0.53X_{t-4} + \varepsilon_t, & \text{if } X_{t-2} > 0.98 \end{cases}$$

This model could represent i) a first regime characterized by a real activity tightening associated with minor fluctuations in the inflation; and ii) a second regime with a real activity increasing associated with major fluctuations in the inflation.

Figure E.8 shows that the standardized and squared standardized residuals of the model could signal that the noise process is white. Furthermore, the Ljung-Box statistics are, respectively, $Q(8) = 11.517(0.174)$ and $Q(8) = 7.190(0.516)$. Figure E.9 presents the CUSUM and CUSUMSQ, which indicate that there is no statistical evidence for model misspecification or strong heteroscedasticity in $\{\varepsilon_t\}$.

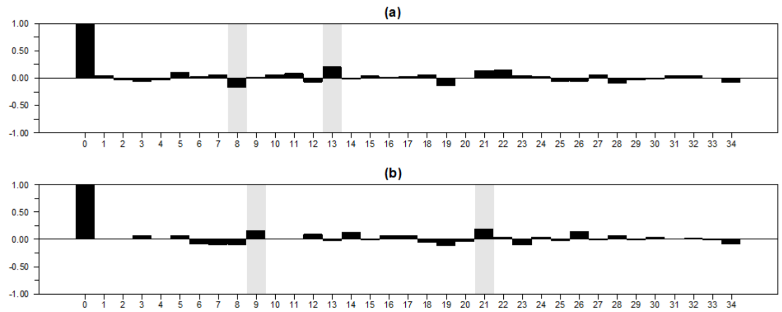


Figure E.8.: Partial autocorrelation function of the fitted SETAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. CPI case.

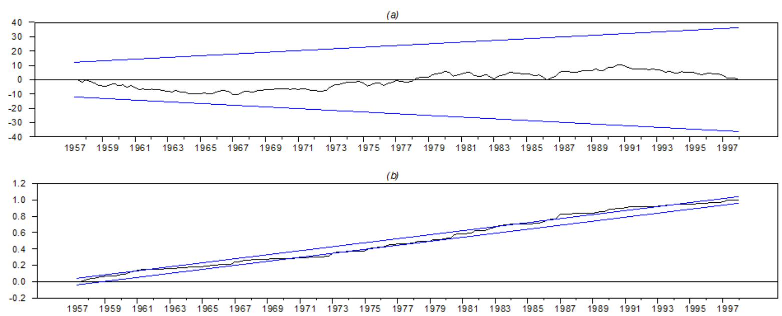


Figure E.9.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted SETAR model. U.S. CPI case.

E.3. Estimation of the STAR model

With the identified $p = 11$ autoregressive order, we evaluate the nonlinearity of the series based on Teräsvirta's (1994) test and find that, with a delay parameter of $d = 1$, we obtain the p-value $= 0.0011$ for the F statistic $F = 2.2235$, which rejects the linearity of the series.

When we compute the tests of nested hypothesis of Teräsvirta (1994), we find that at the 10% significance level, H_{01} and H_{03} are rejected, while H_{02} is not, where the respective p-value are $0.0870, 0.0016$ and 0.1056. These results suggest that we must estimate a LSTAR model.

In Table E.6 we show the estimates for the parameters and their respective standard error in parenthesis, where only the lag-3 and lag-4 AR coefficient of the first regime and the parameter γ are significant at the 5% level.

Parameter	Regime 1	Regime 2
$\Phi_0^{(j)}$		0.70 (0.80)
$\Phi_1^{(j)}$	-0.60 (1.09)	1.13 (1.27)
$\Phi_3^{(j)}$	0.41 (0.08)	
$\Phi_4^{(j)}$	-0.22 (0.08)	
γ		1.29 (0.35)
c		0.32 (1.07)

Table E.6.: Parameter estimates for the STAR model. U.S. CPI case.

Consequently, the estimated STAR model for the growth rate of the U.S. CPI is given by:

$$X_t = -0.60X_{t-1} + 0.41X_{t-3} - 0.22X_{t-4} + F\left(X_{t-1}\right)\left(070 + 1.13X_{t-1}\right) + \varepsilon_t,$$

where

$$F\left(X_{t-1}\right) = \left(1 + \exp\left\{-1.29\left(X_{t-1} - 0.32\right)\right\}\right)^{-1}.$$

When we check the residuals, Figure E.10 shows that the standardized and squared standardized residuals signal that some nonlinear structure in the data is not explained by the model. Additionally, the Ljung-Box statistics are, respectively, $Q\left(8\right) = 13.957(0.083)$ and $Q\left(8\right) = 17.068(0.029)$. Figure E.11 presents the CUSUM and CUSUMSQ, which indicate that there is no statistical evidence for model misspecification but some heteroscedasticity in $\{\varepsilon_t\}$.

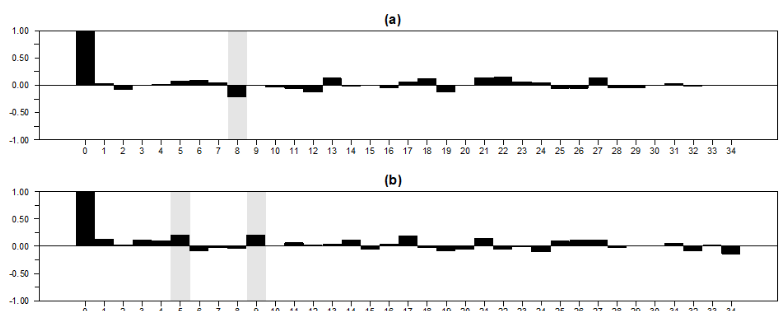


Figure E.10.: Partial autocorrelation function of the fitted STAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. CPI case.

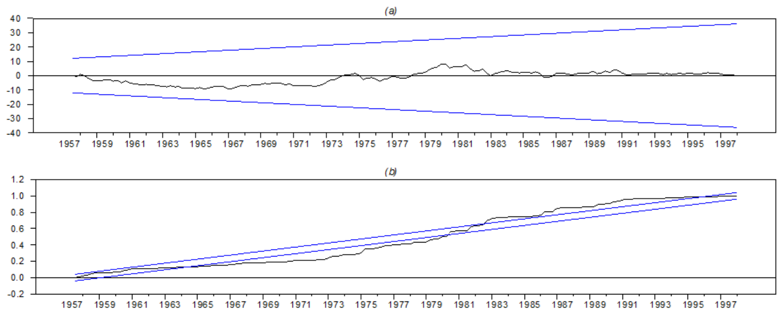


Figure E.11.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted STAR model. U.S. CPI case.

E.4. Estimation of the MSAR model

Based on Section 1.4, we estimate the MSAR model. Table E.7 shows the estimates of the parameters of the model and their respective standard errors in parenthesis.

	state	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	0.04 (0.05)	0.17 (0.11)
$\Phi_1^{(j)}$	0.57 (0.08)	0.43 (0.12)
$\Phi_2^{(j)}$	0.33 (0.11)	-0.43 (0.12)
$\Phi_3^{(j)}$	0.16 (0.09)	0.61 (0.12)

Table E.7.: Parameter estimates for the MSAR model. U.S. CPI case.

Therefore, the estimated MSAR model for the growth rate of the U.S. CPI is given by:

$$X_t = \begin{cases} 0.04 + 0.57X_{t-1} + 0.33X_{t-2} + 0.16X_{t-3} + \varepsilon_{1t}, & \text{if } s_t = 1 \\ 0.17 + 0.43X_{t-1} - 0.43X_{t-2} + 0.61X_{t-3} + \varepsilon_{2t}, & \text{if } s_t = 2 \end{cases}$$

The conditional mean of X_t for regime 1 is 0.09 and for regime 2 is 0.29. The sample variances of ε_{1t} and ε_{2t} are 0.08 and 0.09, respectively[48]. Hence, the first state could represent a minor stable economy with minor fluctuations in the CPI, and the second state could represent a stable economy with increases in the CPI.

When we check this model, Figure E.12 shows that the standardized and squared standardized residuals of the model signal that the noise process is white, and the Ljung-Box statistics are, respectively, $Q(8) = 12.027(0.150)$ and $Q(8) = 7.980(0.435)$. Figure E.13 presents the CUSUM and CUSUMSQ, which indicate that there is statistical evidence for model misspecification and heteroscedasticity in $\{\varepsilon_t\}$.

[48] The probabilities of moving from one state to the other are $p(s_t = 2|s_{t-1} = 1) = 0.18(0.09)$ and $p(s_t = 1|s_{t-1} = 2) = 0.38(0.15)$, where the number in parenthesis is the standard error. Additionally, the unconditional probability that the process is in regime 1 is 0.68 and that it is in regime 2 is 0.32.

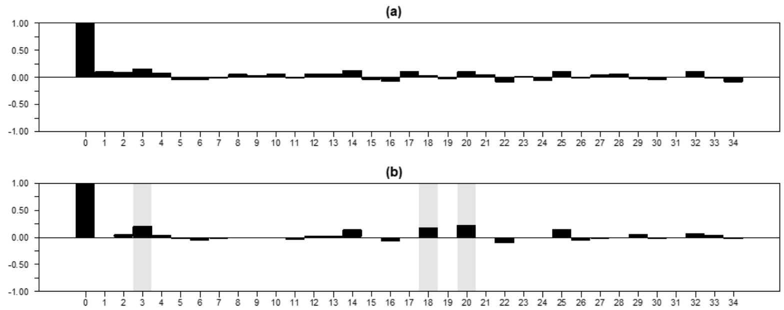


Figure E.12.: Partial autocorrelation function of the fitted MSAR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. CPI case.

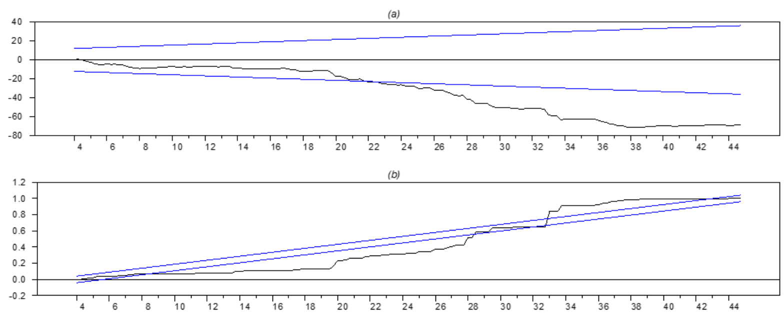


Figure E.13.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted MSAR model. U.S. CPI case.

E.5. Estimation of the AR model

We estimate an AR(4) model for the growth rate of the U.S. CPI[49], which is given by:

$$\left(1 - 0.70B + 0.06B^2 - 0.46B^3 + 0.21B^4\right) X_t = 1.03 + a_t, \quad \hat{\sigma}_a^2 = 0.14,$$

The standard errors of the coefficients are 0.08, 0.09, 0.09, 0.09 and 0.26, respectively. When we check the residuals, Figure E.14 shows that the standardized and the squared standardized residuals slightly signal that some linear structure in the data is not explained by the model, and the Ljung-Box statistics are, respectively, $Q\,(8) = 10.004(0.265)$ and $Q\,(8) = 17.396(0.026)$. Figure E.15 shows the CUSUM and CUSUMSQ, which respectively indicate that there is no statistical evidence for model misspecification, but there is statistical evidence for some heteroscedasticity in ε_t.

[49] As reported by the Augmented Dickey-Fuller (ADF) ($p-$value $= 0.089$), Phillips Perron (PP) ($p-$value $= 0.005$) and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) (Test statistic $= 0.374$ and Critical value $= 0.463$, the growth rate of the growth rate of the U.S. CPI is stationary at the 5% significance level.

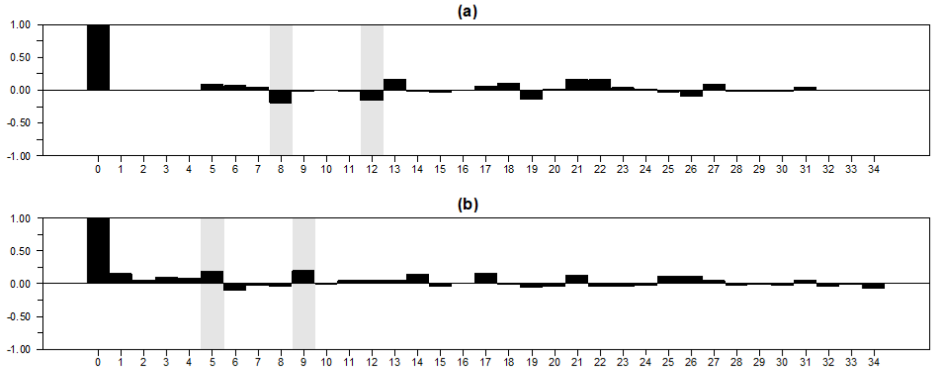


Figure E.14.: Partial autocorrelation function of the fitted AR model. (a) Standardized residuals and (b) squared standardized residuals. U.S. CPI case.

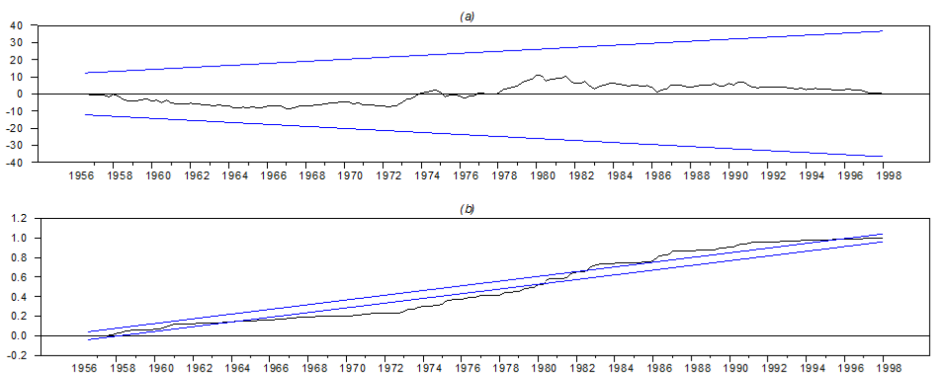


Figure E.15.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted AR model. U.S. CPI case.

Appendix F

General review of the model estimation for the change in the Colombian unemployment rate

F.1. Estimation of the TAR model

We use as the variable of interest the change in the Colombian unemployment rate, that is $X_t = u_t - u_{t-1}$, where u_t is the unemployment rate. As the threshold variable, we use the growth rate of the ISE index, that is $Z_t = [\log(ISE_t) - \log(ISE_{t-1})] * 100$.

Figure F.1 shows that the series have significant autocorrelations at the first two lags. Figure F.2 shows that series have significant autocorrelations at lags 1, 4, 9 and 10.

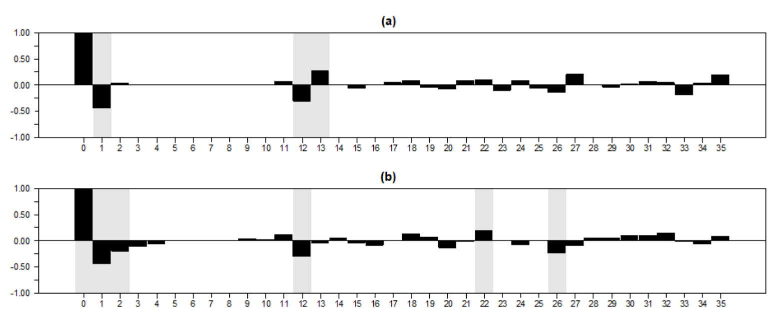


Figure F.1.: (a) Autocorrelation function and (b) partial autocorrelation function for change in the Colombian unemployment rate.

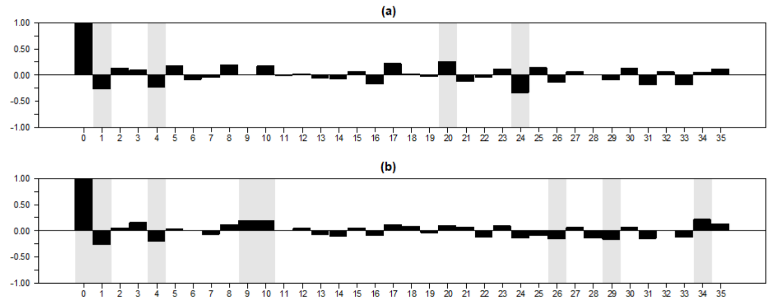


Figure F.2.: (a) Autocorrelation function and (b) partial autocorrelation function for the growth rate of Colombian ISE index.

After that, we test the null hypothesis of AR process linearity against the alternative of bivariate TAR nonlinearity. We apply the statistic to different lags of the threshold variable, that is, $Z_{t-d}, d = 1, \ldots, 10$. Under the AIC criterion, we found that $\bar{k} = 3$ is a reasonable autoregressive order for X_t. The results of the test let us define $d = 2$ as the delay parameter of the threshold variable, consequently the input variable for the dynamic system is Z_{t-2}.

Figure F.3 shows the change in the Colombian unemployment rate and in the growth rate of the ISE index, where the latter is lagged 2 months. We can observe in some periods the countercyclical behavior between both series, that is, fallings in the ISE index with raisings in the unemployment rate.

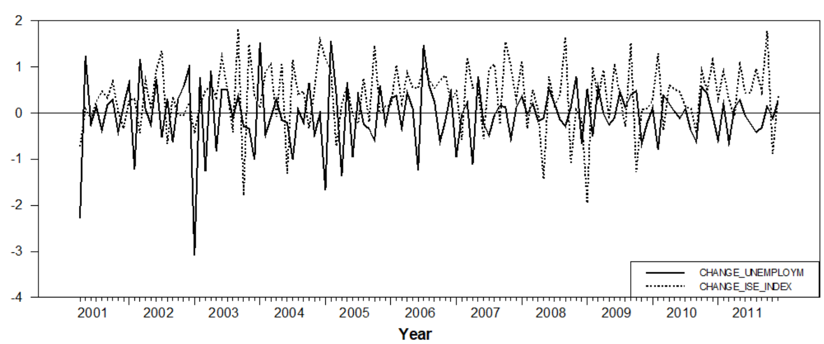


Figure F.3.: Growth rate of Colombian ISE index lagged 2 months and change in the Colombian unemployment rate.

The next step is identifying the number of thresholds for the TAR model as indicated in Section 1.1. Thus, we specify the maximum number of regimes l_0 by means of a regression function between X_t and Z_t, that is estimated using a nonparametric kernel approach and that is presented in Figure F.4. We observe that 3 could be postulated as the possible maximum regimes for the TAR model, with possible threshold values -0.90 and 0.10. Those possible regimes could represent periods in the economy of i) contraction that generates increases of unemployment, ii) stabilization where there is no destruction or creation of employment, and iii) expansion where there is a decrease of unemployment.

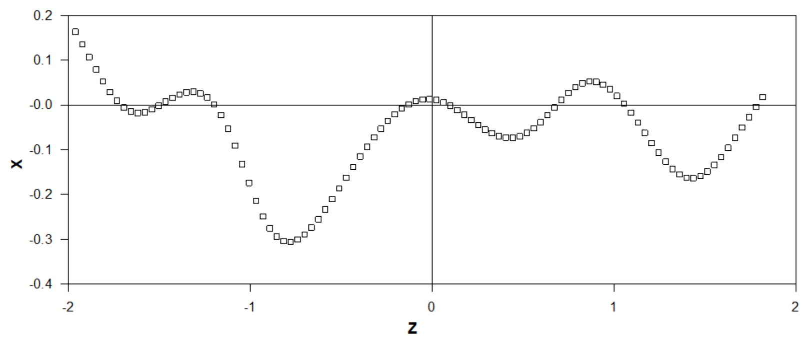


Figure F.4.: Nonparametric regression between the change in the Colombian unemployment rate (X) and the growth rate of Colombian ISE index (Z).

Once we have defined l_0, we select the appropriate thresholds for each possible regimen $l = 2, \ldots, l_0$, which requires to generate intermediate draws of the nonstructural parameters. Thus, we specify the prior densities for the nonstructural parameters as stated in Section 1.1. In that sense, we define the prior densities for $\boldsymbol{\theta}_x$, with $\boldsymbol{\theta}_{0,j} = \bar{\mathbf{0}}, V_{0,j}^{-1} = 0.01\mathbf{I}$ where $\mathbf{I}$ is the identity matrix, $\gamma_{0,j} = 1.5$ and $\beta_{0,j} = \frac{\widetilde{\sigma}^2}{2}$ with $j = 1, 2, 3$ and $\widetilde{\sigma}^2 = 0.31$ is the residual variance of the AR(3) fitted to the change in the Colombian unemployment rate. The maximum autoregressive order for all regimes is $\bar{k} = 3$, the same order fitted to the change in the Colombian unemployment rate. The prior distributions for both the number of regimes and the autoregressive orders that we use in the identification of l are $\pi_2 = \pi_3 = 0.5$ and $p\left(k_{il}|l\right) = 0.25$ for $k_{il} = 0, 1, 2, 3$, and $i = 1, \ldots, l$.

With that information, we identify the thresholds $r_i; i = 1, \ldots, l-1$ for the $l_0 - 1$ possible M models with j regimes, denoted $M_j; j = 2, \ldots, l_0$. Then, to look for the location of thresholds in each possible regime, we choose the percentiles $5k$ where $k = 1, 2, \ldots, 19$, with respective values -0.83, -0.44, -0.33, -0.13, -0.02, 0.10, 0.15, 0.24, 0.32, 0.36, 0.45, 0.50, 0.54, 0.70, 0.89, 0.94, 1.05, 1.15, 1.47. Then, we choose the threshold of model M_2 and M_3 by searching among the set of all possible combinations of autoregressive orders. The possible thresholds and autoregressive orders for each possible regime are presented in Table F.1.

ℓ	**Thresholds**	**Autoregressive orders**	**Minimum NAIC**
2	0.32	2, 1	2.37152
3	0.15 0.54	2, 1, 2	1.95958

Table F.1.: Set of possible number of regimes for the real data. Colombian unemployment rate case.

With those possible thresholds and the information mentioned above, we compute the posterior probability distribution for the number of regimes. 3000 iterates were generated with a burn-in point of 10% of the draws[50]. The results showed in Table F.2 allow us to

[50] The convergence of the Gibbs sampler was checked via stationarity approach, where it was observed that sample autocorrelation functions of the sequences $\left\{\hat{p}_l^{(i)}\right\}$ for $l = 2, 3$, decay quickly.

set $\hat{l} = 3$ as the appropriate number of regimes.

ℓ	$\hat{p}_\ell$
2	0.4298
3	0.5702

Table F.2.: Posterior probability function for the number of regimes for the real data. Colombian unemployment rate case.

Thus, conditional on $\hat{l} = 3$, we estimate the autoregressive orders for k_1, k_2 and k_3. Based on Table F.3, the identified autoregressive orders are $\hat{k}_1 = 3, \hat{k}_2 = 0$ and $\hat{k}_3 = 2$. Convergence of the Gibbs sampler was checked via stationarity, and for 5000 iterates with a burn-in point of 10% of the draws, it is observed that the sample autocorrelations functions decay quickly[51].

Autoregressive Order	**Regime**		
	1	**2**	**3**
0	0.0134	**0.3067**	0.0021
1	0.0220	0.2931	0.1550
2	0.2854	0.2147	**0.5280**
3	**0.6792**	0.1856	0.3149

Table F.3.: Posterior probabilities for the autoregressive orders in the real data. Colombian unemployment rate case.

This concludes the identification stage of the TAR model. Consequently, we fit a TAR(3;3,0,), with thresholds values $r_1 = 0.15$ and $r_2 = 0.54$, which respectively are the 35th and 65th percentiles of the growth rate of the ISE index. The estimated nonstructural parameters are showed in Table F.4, with their respective posterior standard error in parenthesis and 90% credible interval in brackets[52].

[51] We also performed a sensibility analysis where we changed the prior densities for autoregressive orders and the nonstructural parameters, and we found that autoregressive orders estimated are the same for different priors of the nonstructural parameters and for different priors of the autoregressive orders.

[52] 5000 iterates were generated with a burn-in point of 10% of the draws and it was found that the autocorrelation functions for all the parameters decay quickly, indicating the convergence of the Gibbs sampler.

	Regime		
Parameter	**1**	**2**	**3**
$a_0^{(j)}$	-0.04 (0.08) $[-0.17, 0.11]$	-0.19 (0.10) $[-0.35, -0.02]$	-0.02 (0.08) $[-0.16, 0.11]$
$a_1^{(j)}$	-0.84 (0.16) $[-1.10, -0.58]$		-0.54 (0.15) $[-0.78, -0.30]$
$a_2^{(j)}$	-0.63 (0.18) $[-0.93, -0.34]$		-0.25 (0.17) $[-0.53, 0.02]$
$a_3^{(j)}$	-0.24 (0.17) $[-0.52, 0.03]$		
$h^{(j)}$	0.28 (0.07) $[0.20, 0.40]$	0.39 (0.09) $[0.27, 0.56]$	0.29 (0.06) $[0.20, 0.41]$

Table F.4.: Parameter estimates for the TAR model. Colombian unemployment rate case.

The fitted TAR model for the change in the Colombian unemployment rate, is given by:

$$X_t = \begin{cases} -0.04 - 0.84X_{t-1} - 0.63X_{t-2} - 0.24X_{t-3} + 0.28\varepsilon_t, & \text{if } Z_{t-2} \leq 0.15 \\ -0.19 + 0.39, & \text{if } 0.15 < Z_{t-2} \leq 0.54 \\ -0.02 - 0.54X_{t-1} - 0.25X_{t-2} + 0.29\varepsilon_t, & \text{if } Z_{t-2} > 0.54 \end{cases}$$

Each regime of this model could represent periods in the economy of i) contraction, where the regime shows sharp fluctuations in the unemployment rate when the GDP presents low growth rates; ii) stabilization, with minor fluctuations in the unemployment rate and GDP; and iii) expansion, where the regime exhibits several decreases in the unemployment rate, when the growth rate of the GDP is increasing.

When we check the residuals, in Figure F.5 we observe that the standardized and squared standardized residuals signal that the noise process is white, and the Ljung-Box statistics are, respectively, $Q\,(12) = 13.323(0.346)$ and $Q\,(12) = 7.892(0.794)$. Figure F.6 reports that the CUSUM and CUSUMSQ behave well, which indicates that there is no statistical evidence for model misspecification but some heteroscedasticity in $\{\varepsilon_t\}$.

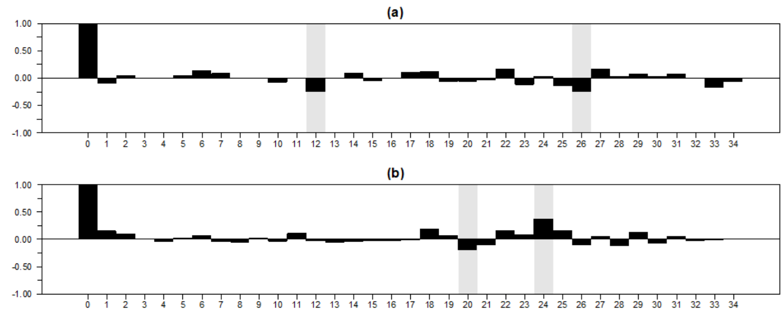


Figure F.5.: Partial autocorrelation function of the fitted TAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian unemployment rate case.

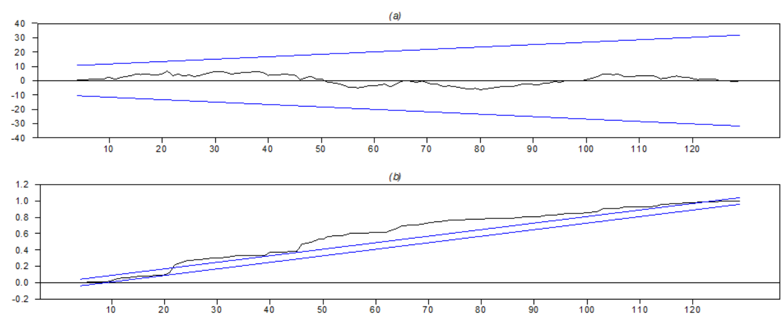


Figure F.6.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted TAR model. Colombian unemployment rate case.

F.2. Estimation of the SETAR model

Based on the autocorrelation functions in Figure F.1 and the AIC and BIC criterion, we identify $p = 2$ as the final autoregressive order to fit a SETAR model. Once we have selected the autoregressive model, we check the nonlinearity of the series based on Tsay's (1989) test and find that with $d = 2$, where d is the delay parameter, it is obtained the minimum p-value $= 0.012$ of the F statistic $F = 4.594$, rejecting the linearity of the series at the 5% significance level.

Then, to determine the number of regimes and the threshold values, we use Figure F.7 that shows the sequence of the t ratios of a lag-2 AR coefficient versus the threshold variable X_{t-2} in an arranged autoregression of order 2, and we identify that the data can be divided into two regimes with a possible threshold at $X_{t-d} = -0.4$, because of the change on the slope at approximately this point.

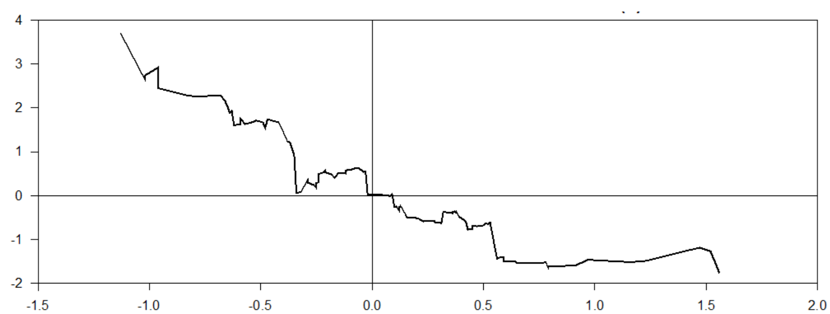


Figure F.7.: Time plot of t-ratio of recursive estimates of the AR-2 coefficient in an arranged autoregression of order 2 and delay parameter 2. Colombian unemployment rate case.

We also use the NAIC criterion of Tong (1990), which suggests a threshold value of $r_j = -0.48$ and autoregressive orders AR(2) and AR(1) for each regime, with a NAIC = -148.819. Therefore, we fit a SETAR(2;2,1) for the change in the Colombian unemployment rate, with threshold value X_{t-2}. The estimated parameters and their standard errors in parenthesis for each regime are shown in Table F.5, where only the lag-1 AR coefficient of the second regime is not significant at the 10% level.

	Regime	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	0.30 (0.18)	-0.14 (0.06)
$\Phi_1^{(j)}$	0.11 (0.21)	-0.60 (0.10)
$\Phi_2^{(j)}$	0.44 (0.19)	

Table F.5.: Parameter estimates for the SETAR model. Colombian unemployment rate case.

Thus, the estimated SETAR model for the change in the Colombian unemployment rate, is given by:

$$X_t = \begin{cases} 0.30 + 0.11X_{t-1} + 0.44X_{t-2} + \varepsilon_t, & \text{if } X_{t-2} \leq -0.48 \\ -0.14 - 0.60X_{t-1} + \varepsilon_t, & \text{if } X_{t-2} > -0.48 \end{cases}$$

This model could represent periods in the economy of i) stability, where the first regime contains minor variations in the unemployment rate, and ii) instability, where the second regime shows several increases in the unemployment rate.

Figure F.8 shows that the standardized and squared standardized residuals of the model signal that the noise process is white. Moreover, the Ljung-Box statistics are, respectively, $Q\,(12) = 15.583(0.211)$ and $Q\,(12) = 21.397(0.045)$. Figure F.9 presents the CUSUM and CUSUMSQ, indicating that there is no statistical evidence for model misspecification but some heteroscedasticity in $\{\varepsilon_t\}$.

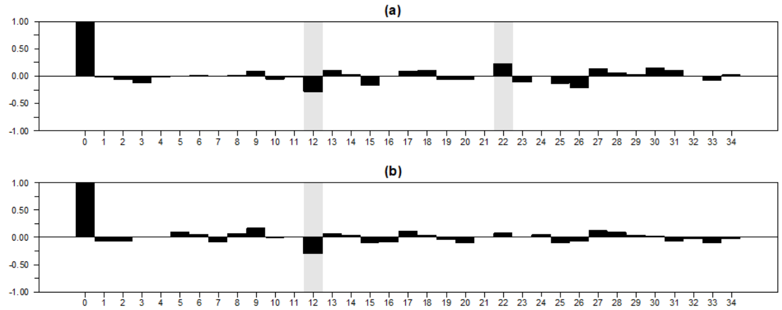


Figure F.8.: Partial autocorrelation function of the fitted SETAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian unemployment rate case.

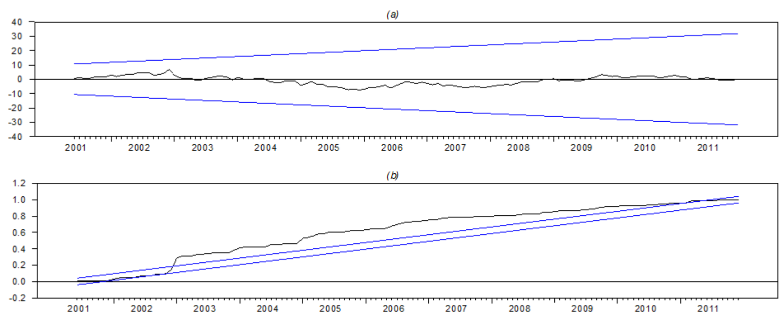


Figure F.9.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted SETAR model. Colombian unemployment rate case.

F.3. Estimation of the STAR model

We estimate the STAR model based on Section 1.3. With the identified $p = 3$ autoregressive order, we evaluate the nonlinearity of the series based on Teräsvirta's (1994) test and find that with a delay parameter of 1, $d = 1$, we obtain a p-value $= 0.072$ of the F statistic $F = 1.820$, which rejects the linearity of the series.

Then, we choose between the LSTAR and the ESTAR models through a sequence of tests of nested hypothesis (Teräsvirta, 1994). We find that H_{01}, H_{02} and H_{03} with $F - stat = 0.648(0.586), F - stat = 2.407(0.071)$ and $F - stat = 2.365(0.074)$ respectively, are not rejected at the 5% significance level (the p-values are in parenthesis). Thus, the model we choose is the LSTAR.

In Table F.6, we show the estimates for the parameters and their respective standard error in parenthesis, where all estimates, except the parameter γ and c, are significant at the 5% level.

	Regime	
Parameter	**1**	**2**
$\Phi_1^{(j)}$		-0.65 (0.13)
$\Phi_2^{(j)}$		-0.41 (0.15)
$\Phi_3^{(j)}$	-0.17 (0.08)	
γ		2.79 (4.09)
c		-0.60 (0.43)

Table F.6.: Parameter estimates for the STAR model. Colombian unemployment rate case.

Consequently, the estimated STAR model for the change in the Colombian unemployment rate is given by:

$$X_t = -0.17X_{t-3} + F\left(X_{t-2}\right)\left(-0.65X_{t-1} - 0.41X_{t-2}\right) + \varepsilon_t,$$

where

$$F\left(X_{t-2}\right) = \left(1 + \exp\left\{-2.79 \times 1.48\left(X_{t-2} + 0.60\right)\right\}\right)^{-1}.$$

When we check the residuals, Figure F.10 shows that the standardized and squared standardized residuals of the model that the noise process could be white, and the Ljung-Box statistics are, respectively, $Q\left(12\right) = 17.611(0.128)$ and $Q\left(12\right) = 13.889(0.308)$. Figure F.11 presents the CUSUM and CUSUMSQ, indicating that there is no statistical evidence for model misspecification but some heteroscedasticity in $\{\varepsilon_t\}$.

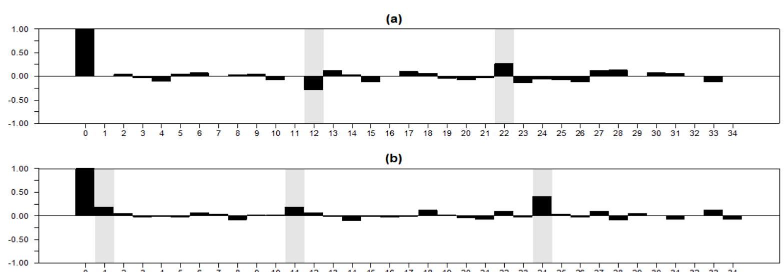


Figure F.10.: Partial autocorrelation function of the fitted STAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian unemployment rate case.

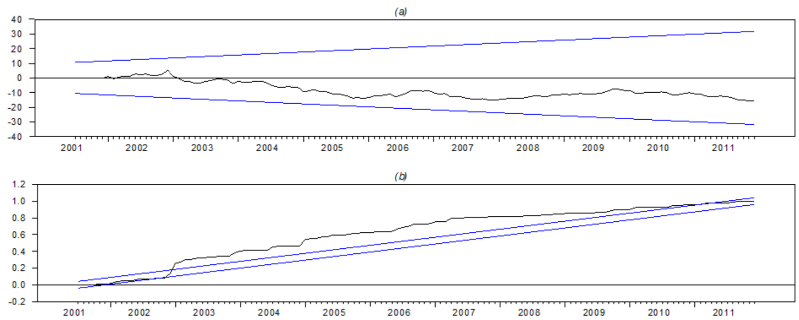


Figure F.11.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted STAR model. Colombian unemployment rate case.

F.4. Estimation of the MSAR model

Based on Section 1.4, we estimate the MSAR model. Table F.7 shows the estimates of the parameters of the model and their respective standard errors in parenthesis.

	state	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	-0.03 (0.05)	-0.19 (0.22)
$\Phi_1^{(j)}$	-0.44 (0.14)	-0.61 (0.24)
$\Phi_2^{(j)}$	-0.11 (0.15)	-0.24 (0.28)

Table F.7.: Parameter estimates for the MSAR model. Colombian unemployment rate case.

Therefore, the estimated MSAR model for the change in the Colombian unemployment rate is given by:

$$X_t = \begin{cases} -0.03 - 0.44X_{t-1} - 0.11X_{t-2} + \varepsilon_{1t}, & \text{if } s_t = 1 \\ -0.19 - 0.61X_{t-1} - 0.24X_{t-2} + \varepsilon_{2t}, & \text{if } s_t = 2 \end{cases}$$

The conditional mean of X_t for regime 1 is -0.02 and for regime 2 is -0.12. Hence, the first state represents the stable periods in the Colombian economy, with no considerable movements in the unemployment rate, and the second state represents the expansionary periods with more decreases in this economic time series[53].

When we evaluate this model, Figure F.12 shows that the standardized and squared standardized residuals signal that the noise process could be white, and the Ljung-Box statistics

[53] The probabilities of moving from one state to the other are $p(s_t = 2|s_t = 1) = 0.93(0.05)$ and $p(s_t = 1|s_t = 2) = 0.28(0.15)$, where the number in parenthesis is the standard error. Additionally, the unconditional probability that the process is in regime 1 is 0.79 and that it is in regime 2 is 0.21. The coefficients allow us to know that the probability that expansion is followed by another expansion period is $p(1,1) = 0.93$, and that the probability that contraction is followed by another contraction period is $p(2,2) = 0.92$.

are, respectively, $Q(12) = 18.921(0.090)$ and $Q(12) = 15.483(0.216)$. Figure F.13 presents the CUSUM and CUSUMSQ, indicating that there is statistical evidence for model misspecification and heteroscedasticity in $\{\varepsilon_t\}$.

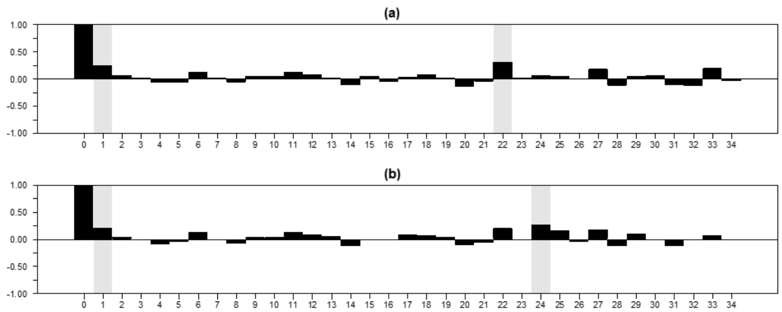


Figure F.12.: Partial autocorrelation function of the fitted MSAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian unemployment rate case.

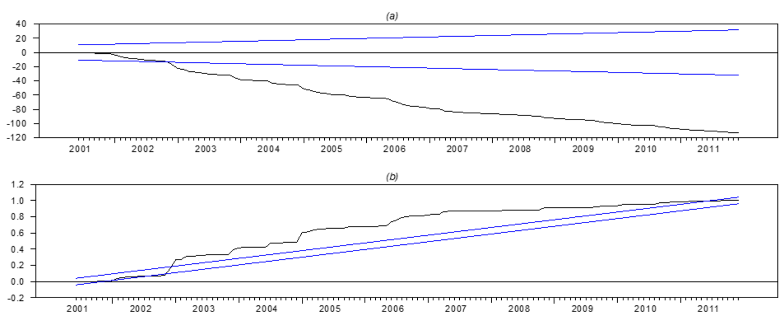


Figure F.13.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted MSAR model. Colombian unemployment rate case.

F.5. Estimation of the AR model

We estimate an AR(2) model for the change in the Colombian unemployment rate[54]:

$$\left(1 + +0.56B + 0.16B^2\right) X_t = a_t, \quad \hat{\sigma}_a^2 = 0.3,$$

The standard error of both coefficients is 0.09. Figure F.14 shows that the standardized residuals and the squared standardized residuals signal that the noise process could be white. Fourthermore, the Ljung-Box statistics are, respectively, $Q(12) = 18.098(0.113)$ and $Q(12) = 13.589(0.328)$. Figure F.15 shows the CUSUM and CUSUMSQ, indicating

[54] As reported by the Augmented Dickey-Fuller (ADF) ($p-$value $= 0.000$), Phillips Perron (PP) ($p-$value $= 0.000$) and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) (Test statistic $= 0.085$ and Critical value $= 0.463$, the growth rate of the Colombian unemployment rate is stationary at the 5% significance level.

that there is no statistical evidence for model misspecification but some heteroscedasticity in $\{\varepsilon_t\}$.

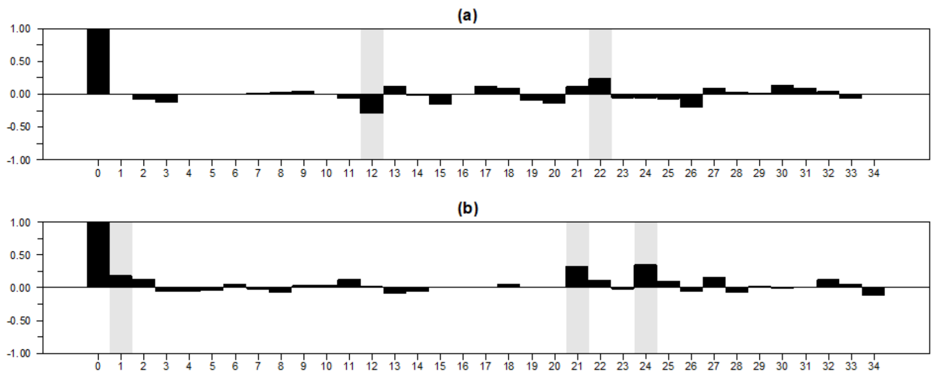


Figure F.14.: Partial autocorrelation function of the fitted AR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian unemployment rate case.

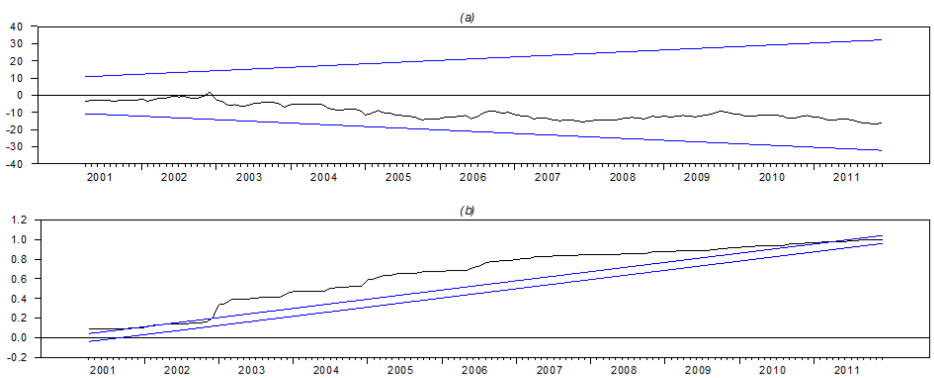


Figure F.15.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted AR model. Colombian unemployment rate case.

Appendix G

General review of the model estimation for the annual growth rate of the Colombian GDP

G.1. Estimation of the TAR model

We use as the variable of interest the annual growth rate of the ISE, as a proxi of the GDP, and as the threshold variable we use the Colombian spread term defined in Section 3.2. Figure G.1 and Figure G.2 show that both series have significant autocorrelations for a large number of lags.

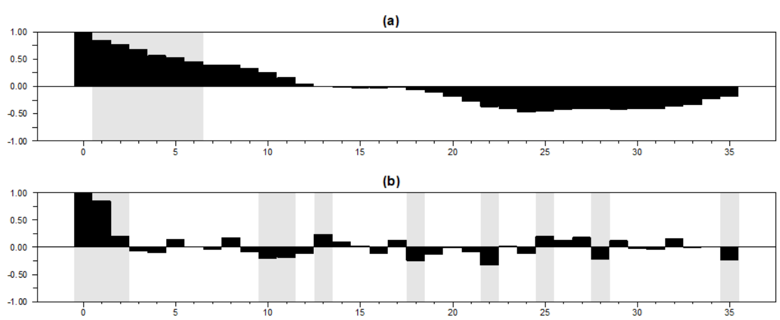


Figure G.1.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the annual growth rate of Colombian GDP.

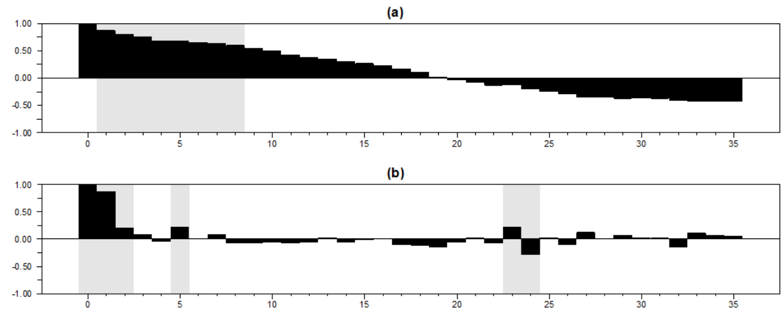


Figure G.2.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the Colombian spread term.

Then, we test the null hypothesis of AR linearity against the alternative of bivariate TAR nonlinearity. Under the AIC criterion, we found that $\bar{k} = 2$ is a reasonable autoregressive order for X_t. Under the null hypothesis, the results of the test let us define as the delay parameter of 5 for the threshold variable, thus, the input variable for the dynamic system is Z_{t-5}.

Figure G.3 shows the annual growth rate of the Colombian GDP and the spread term from 2003:06 to 2012:06. The shading areas denote the business cycle contractions from peak to trough based on Alfonso et al. (2012).

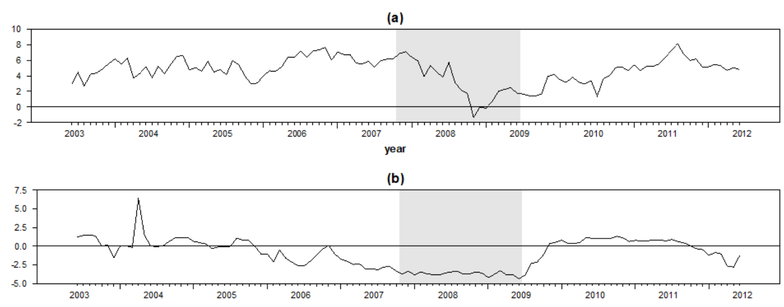


Figure G.3.: (a) Time plot of the annual growth rate of Colombian GDP and (b) Time plot of the Colombian spread term.

We specify the maximum number of regimes l_0 by means of a regression function between X_t and Z_t, that is estimated using a nonparametric kernel approach and that is presented in Figure G.4. We observe that 3 could be postulated as the possible maximum regimes for the TAR model.

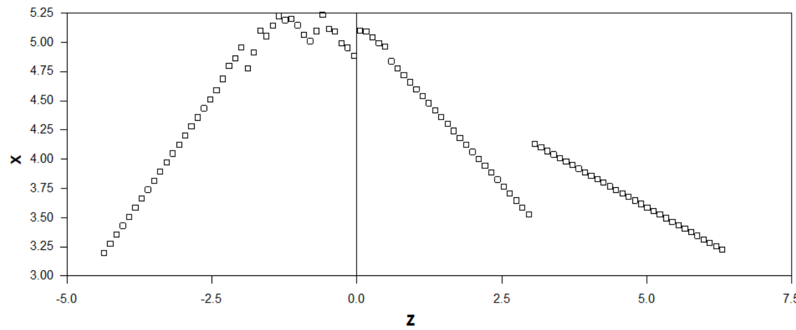


Figure G.4.: Nonparametric regression between the annual growth rate of the Colombian GDP (X) and Colombian spread term (Z).

Next, to specify the prior densities for the nonstructural parameters, we define the prior densities for $\boldsymbol{\theta}_x$ where $\boldsymbol{\theta}_{0,j} = \bar{\mathbf{0}}, V_{0,j}^{-1} = 0.01\mathbf{I}$ with $\mathbf{I}$ the identity matrix, $\gamma_{0,j} = 1.5$ and $\beta_{0,j} = \frac{\widetilde{\sigma}^2}{2}$ with $j = 1, 2, 3$ and $\widetilde{\sigma}^2 = 0.871$ that is the residual variance of the AR(2) that was fitted to the annual growth rate of the ISE. The maximum autoregressive order for all regimes is $\bar{k} = 2$, the same value fitted to the variable of interest. The prior distributions for both the number of regimes and the autoregressive orders that we use in the identification of l are $\pi_2 = \pi_3 = 0.5$ and $p\,(k_{il}|l) = 0.333$ for $k_{il} = 0, 1, 2; i = 1, .., l$, respectively.

Then, to look for the location of thresholds in each possible regime, we choose the percentiles $5k$ where $k = 1, 2, \ldots, 19$, with respective values -3.82, -3.70, -3.44, -3.20, -2.75, -2.33, -2.01, -1.24, -1.02, -0.34, -0.03, 0.04, 0.30, 0.49, 0.66, 0.79, 0.96, 1.08, 1.28. The possible thresholds and autoregressive orders for each possible regime are presented in Table G.1.

ℓ	**Thresholds**	**Autoregressive orders**	**Minimum NAIC**
2	-0.34	1, 1	3.25830
3	-3.20 -0.34	1, 1, 1	2.80465

Table G.1.: Set of possible number of regimes for the real data. Colombian GDP case.

With those possible thresholds and the information mentioned above, we compute the posterior probability distribution for the number of regimes. 3000 iterates were generate, with a burn-in point of 10% of the draws[55]. The results showed in Table G.2 allow us to set $\hat{l} = 3$ as the appropriate number of regimes.

[55] The convergence of the Gibbs sampler was checked via the stationarity approach, where it was observed that the sample autocorrelation functions of the sequences $\left\{\hat{p}_l^{(i)}\right\}$ for $l = 2, 3$ decay quickly.

ℓ	$\hat{p}_\ell$
2	0.3062
3	0.6938

Table G.2.: Posterior probability function for the number of regimes for the real data. Colombian GDP case.

Thus, conditional on $\hat{l} = 3$, we estimate the autoregressive orders for k_1, k_2 and k_3. Based on Table G.3, the identified autoregressive orders are $\hat{k}_1 = 2,\ \hat{k}_2 = 2$ and $\hat{k}_3 = 2$. As before, convergence of the Gibbs sampler was checked via stationarity, and for 5000 iterates with a burn-in point of 10% of the draws, it is observed that the sample autocorrelations functions decay quickly[56].

Autoregressive Order	**Regime 1**	**Regime 2**	**Regime 3**
0	$1.54{\times}10^{-11}$	$1.01{\times}10^{-50}$	$1.87{\times}10^{-35}$
1	0.2140	0.1537	0.0597
2	**0.7860**	**0.8463**	**0.9403**

Table G.3.: Posterior probabilities for the autoregressive orders in the real data. Colombian GDP case.

Consequently, we fit a TAR(3;2,2,2) with thresholds values $r_1 = -3.20$ and $r_2 = -0.34$, which respectively are the 20th and 50th percentiles of the spread term. Table G.4 shows the estimates for the nonstructural parameters, with their respective posterior standard error in parenthesis and 90% credible interval in brackets[57]. These results show that not all the coefficients are significant at the 5% level. However, we decide to estimate the model with all the coefficients, given that that improves the final estimation.

[56] We performed a sensibility analysis where we changed the prior densities of the autoregressive orders and the nonstructural parameters, and we found that the estimated autoregressive orders are the same for different priors of the nonstructural parameters and for different priors of the autoregressive orders.

[57] 5000 iterates were generated, with a burn-in point of 10% of the draws, and it was found that the autocorrelation functions for all the parameters decay quickly, indicating the convergence of the Gibbs sampler.

	Regime		
Parameter	**1**	**2**	**3**
$a_0^{(j)}$	0.20 (0.47)	0.64 (0.40)	1.43 (0.53)
	$[-0.57, 0.98]$	$[-0.04, 1.28]$	$[0.56, 2.29]$
$a_1^{(j)}$	0.70 (0.22)	0.73 (0.17)	0.51 (0.14)
	$[0.34, 1.05]$	$[0.44, 1.01]$	$[0.28, 0.74]$
$a_2^{(j)}$	0.18 (0.22)	0.17 (0.17)	0.20 (0.13)
	$[-0.18, 0.53]$	$[-0.12, 0.45]$	$[-0.02, 0.42]$
$h^{(j)}$	1.51 (0.51)	0.41 (0.11)	0.94 (0.19)
	$[0.89, 2.44]$	$[0.27, 0.61]$	$[0.67, 1.28]$

Table G.4.: Parameter estimates for the TAR model. Colombian GDP case.

The fitted TAR model for the annual growth rate of the Colombian GDP is given by:

$$X_t = \begin{cases} 0.20 + 0.70X_{t-1} + 0.18X_{t-2} + 1.51\varepsilon_t, & \text{if } Z_{t-5} \leq -3.20 \\ 0.64 + 0.73X_{t-1} + 0.17X_{t-2} + 0.41\varepsilon_t, & \text{if } -3.20 < Z_{t-5} \leq -0.34 \\ 1.43 + 0.51X_{t-1} + 0.20X_{t-2} + 0.94\varepsilon_t, & \text{if } Z_{t-5} > -0.34 \end{cases}$$

This model could represent periods in the economy of i) contraction, given that this regime presents the lowest values of the growth rate of the real GDP, when spreads are low due to contractionary monetary policies; ii) transition, where the regime has minor but positive fluctuations in the growth rate of the real GDP; during a stable behavior of the spread term and iii) expansion, where this last regime is associated with the important increases in the growth rate of real GDP, when the monetary policy is expansioning.

When we check the residuals, in Figure G.5 we observe that the standardized and squared standardized residuals slightly signal that some nonlinear structure in the data is not explained by the model, and the Ljung-Box statistics are, respectively, $Q\,(12) = 27.613(0.006)$ and $Q\,(12) = 18.715(0.095)$. Figure G.6 shows that the CUSUM and CUSUMSQ behave well, which indicates that there is neither statistical evidence for model misspecification nor heteroscedasticity in $\{\varepsilon_t\}$.

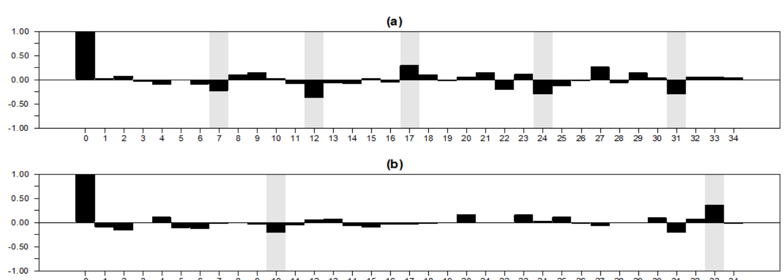


Figure G.5.: Partial autocorrelation function of the fitted TAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian GDP case.

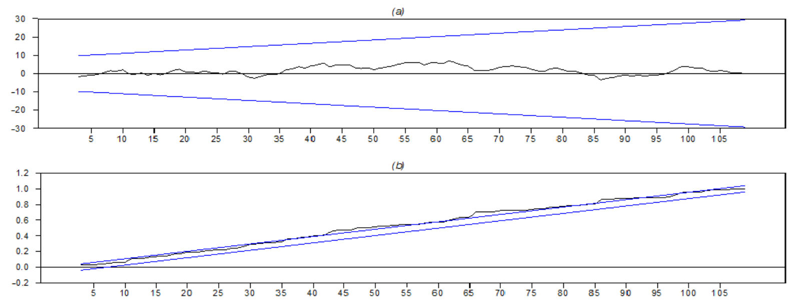


Figure G.6.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted TAR model. Colombian GDP case.

G.2. Estimation of the SETAR model

Based on the autocorrelation functions in Figure G.1 and the AIC and BIC criterion, we identify $p = 2$ as the final autoregressive order to fit a SETAR model. Under the Tsay's (1989) test and find that with $d = 4$, it is obtained the minimum p-value $= 0.0279$ of the F statistic $F = 3.711$, rejecting the linearity of the series at the 5% significance level.

Figure G.7 shows the sequence of the t ratios of a lag-2 AR coefficient versus the threshold variable X_{t-d} in an arranged autoregression of order 2, and we identify that the data can be divided into two regimes with a possible threshold at $X_{t-d} = 3.0$, because of the change on the slope at approximately this point.

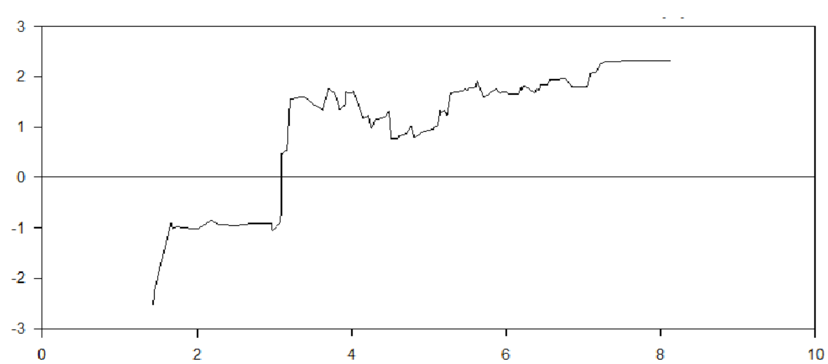


Figure G.7.: Time plot of t-ratio of recursive estimates of the AR-2 coefficient in an arranged autoregression of order 2 and delay parameter 4. Colombian GDP case.

We also use the NAIC criterion of Tong (1990), which suggests a threshold value of $r_j = 3.19$ and autoregressive orders AR(1) and AR(2) for each regime, with a NAIC = -12.958. Based on that, we fit a SETAR(2;1,2) for the annual growth rate of the GDP, with threshold value X_{t-4}. The estimated parameters and their standard errors in parenthesis for each regime are shown in Table G.5, where all estimates are significant at the 5% level.

	Regime	
Parameter	**1**	**2**
$\Phi_1^{(j)}$	1.04 (0.05)	0.67 (0.12)
$\Phi_2^{(j)}$	0.67 (0.12)	0.31 (0.12)

Table G.5.: Parameter estimates for the SETAR model. Colombian GDP case.

Thus, the estimated SETAR model for the annual growth rate of the Colombian GDP is given by:

$$X_t = \begin{cases} 1.04X_{t-1} + \varepsilon_t, & \text{if } X_{t-4} \leq 3.19 \\ 0.67X_{t-1} + 0.31X_{t-2} + \varepsilon_t, & \text{if } X_{t-4} > 3.19 \end{cases}$$

This model could represent periods in the economy of i) stability, where the first regime contains the minor increases in the growth rate of the real GDP, and ii) expansion, where the second regime shows the greatest increases in the growth rate of the real GDP.

Figure G.8 shows that the standardized and squared standardized residuals of the model slightly signal that some nonlinear structure in the data is not explained by the model. The Ljung-Box statistics are, respectively, $Q\,(12) = 29.561(0.003)$ and $Q\,(12) = 22.395(0.033)$. Figure G.9 presents the CUSUM and CUSUMSQ, which indicate that there is no statistical evidence for model misspecification but some heteroscedasticity in $\{\varepsilon_t\}$.

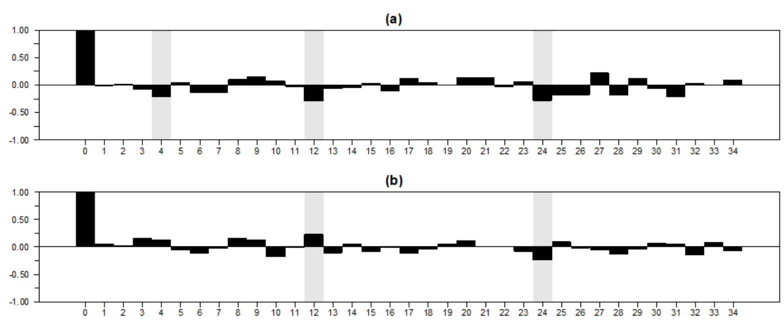


Figure G.8.: Partial autocorrelation function of the fitted SETAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian GDP case.

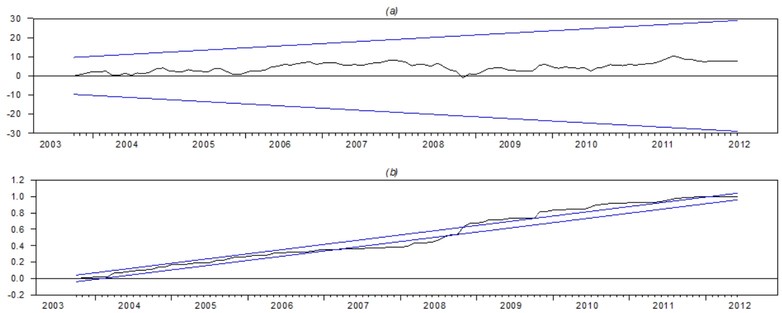


Figure G.9.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted SETAR model. Colombian GDP case.

G.3. Estimation of the STAR model

With an identified $p = 13$ autoregressive order, we evaluate the nonlinearity of the series based on Teräsvirta's (1994) test and find that, with a delay parameter of $d = 13$, we obtain the p-value $= 0.0064$ for the F statistic $F = 2.1967$, which rejects the linearity of the series.

When we compute the tests of nested hypothesis of Teräsvirta (1994), we find that at the 10% significance level, H_{01} is rejected, while H_{02} and H_{03} are not, where the respective p-value is 0.0018, 0.2014 and 0.6787. These results suggest that we must estimate a LSTAR model.

Table G.6 shows the estimates for the parameters and their respective standard error in parenthesis, where only the lag-4 AR coefficient of the second regime and the parameter γ are not significant at the 5% level.

	Regime	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	0.90 (0.38)	
$\Phi_1^{(j)}$	0.40 (0.16)	0.35 (0.16)
$\Phi_2^{(j)}$	0.31 (0.11)	
$\Phi_4^{(j)}$		-0.15 (0.10)
$\Phi_9^{(j)}$	0.46 (0.17)	-0.42 (0.18)
$\Phi_{12}^{(j)}$	-0.36 (0.09)	
$\Phi_{13}^{(j)}$		0.23 (0.10)
γ		11.49 (17.76)
c		3.98 (0.30)

Table G.6.: Parameter estimates for the STAR model. Colombian GDP case.

Consequently, the estimated STAR model for the annual growth rate of the Colombian GDP is given by:

$$X_t = 0.90 + 0.40X_{t-1} + 0.43X_{t-2} + 0.46X_{t-9} - 0.36X_{t-12} \\ + F(X_{t-13})(0.35X_{t-1} - 0.15X_{t-4} - 0.42X_{t-9} + 0.23X_{t-13}) + \varepsilon_t,$$

where

$$F(X_{t-13}) = (1 + \exp\{-11.49(X_{t-13} - 3.98)\})^{-1}.$$

When we check the residuals, Figure G.10 shows that the standardized and squared standardized residuals of the model signal that the noise process is white, and the Ljung-Box statistics are, respectively, $Q(12) = 13.437(0.338)$ and $Q(12) = 5.006(0.958)$. Figure G.11 presents the CUSUM and CUSUMSQ, which indicate that there is neither statistical evidence for model misspecification nor heteroscedasticity in $\{\varepsilon_t\}$.

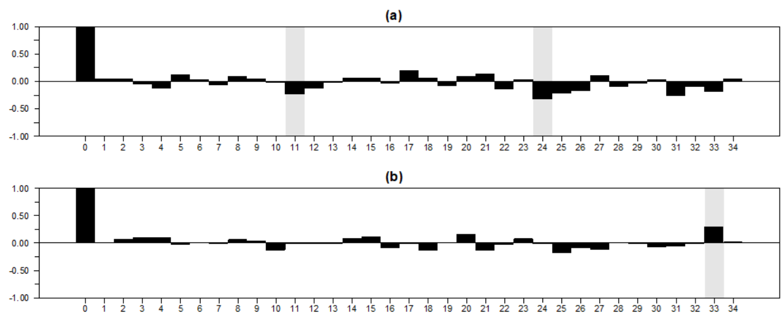


Figure G.10.: Partial autocorrelation function of the fitted STAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian GDP case.

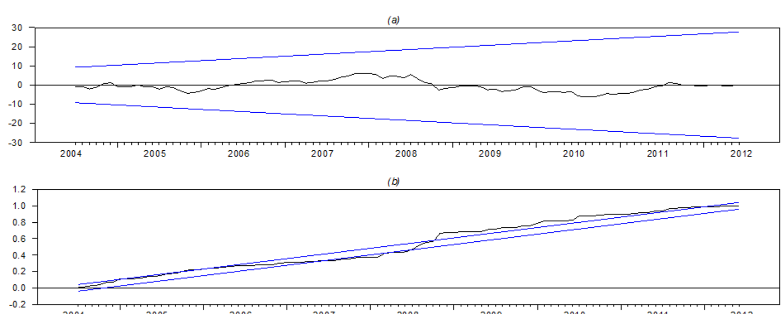


Figure G.11.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted STAR model. Colombian GDP case.

G.4. Estimation of the MSAR model

Table G.7 shows the estimates of the parameters of the model and their respective standard errors in parenthesis.

	state	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	-0.32 (0.10)	0.83 (0.34)
$\Phi_1^{(j)}$	0.71 (0.04)	0.71 (0.11)
$\Phi_2^{(j)}$	0.64 (0.05)	0.16 (0.13)
$\Phi_3^{(j)}$	-0.51 (0.04)	-0.04 (0.14)
$\Phi_5^{(j)}$	0.31 (0.04)	-0.02 (0.10)

Table G.7.: Parameter estimates for the MSAR model. Colombian GDP case.

Therefore, the estimated MSAR model for the annual growth rate of the Colombian GDP is given by:

$$X_t = \begin{cases} -0.32 + 0.71X_{t-1} + 0.64X_{t-2} - 0.51X_{t-3} + 0.31X_{t-5} + \varepsilon_{1t}, & \text{if } s_t = 1 \\ 0.83 + 0.71X_{t-1} + 0.16X_{t-2} - 0.04X_{t-3} - 0.02X_{t-5} + \varepsilon_{2t}, & \text{if } s_t = 2 \end{cases}$$

The conditional mean of X_t for regime 1 is -1.09 and for regime 2 is 2.89. The sample variances of ε_{1t} and ε_{2t} are 0.01 and 0.97, respectively[58]. Hence, the first state could represent unstable periods with minor increases in the real output, and the second state represents a stable economy with strong increases I the real output.

When we check this model, Figure G.12 shows that the standardized and squared standardized residuals of the model signal that the noise process is white. Moreover, the Ljung-Box statistics are, respectively, $Q\,(12) = 12.068(0.440)$ and $Q\,(12) = 4.225(0.979)$. Figure G.13 presents the CUSUM and CUSUMSQ, which indicate that there is statistical evidence for model misspecification and some heteroscedasticity in $\{\varepsilon_t\}$.

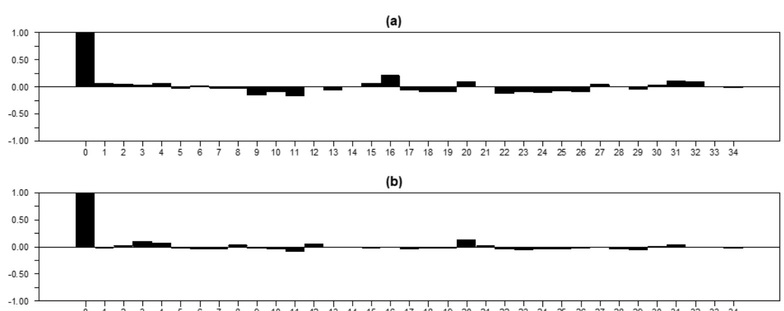


Figure G.12.: Partial autocorrelation function of the fitted MSAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian GDP case.

[58] The probabilities of moving from one state to the other are $p\,(s_t = 2|s_{t-1} = 1) = 0.59(0.15)$ and $p\,(s_t = 1|s_{t-1} = 2) = 0.17(0.07)$, where the number in parenthesis is the standard error. Additionally, the unconditional probability that the process is in regime 1 is 0.22 and that it is in regime 2 is 0.78.

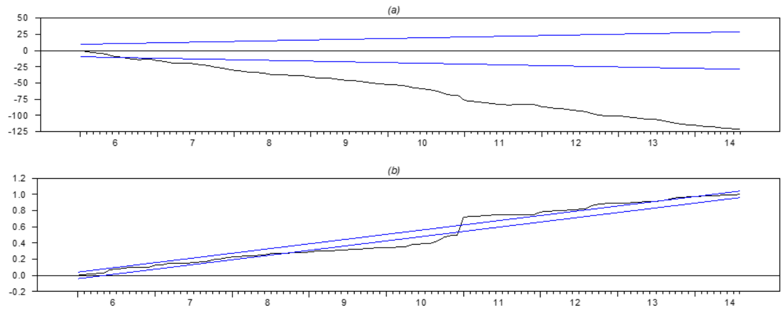


Figure G.13.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted MSAR model. Colombian GDP case.

G.5. Estimation of the AR model

We estimate an AR(2) model for the annual growth rate of the Colombian GDP[59], which is given by:

$$\left(1 - 0.66B - 0.21B^2\right) X_t = 4.56 + a_t, \quad \hat{\sigma}_a^2 = 0.89,$$

The standard errors of the coefficients are 0.09, 0.09 and 0.66, respectively. When we check the residuals, Figure G.14 shows that the standardized residuals and the squared standardized residuals slightly signal that some linear structure in the data is not explained by the model, and the Ljung-Box statistics are, respectively, $Q\,(12) = 31.400(0.002)$ and $Q\,(12) = 13.494(0.334)$. Figure G.15 shows the CUSUM that indicates there is no statistical evidence for model misspecification, and the CUSUMSQ that indicates some heteroscedasticity in $\{\varepsilon_t\}$.

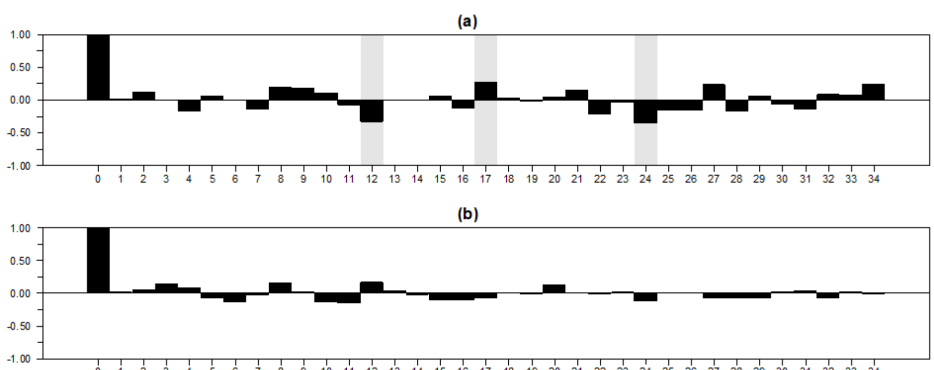


Figure G.14.: Partial autocorrelation function of the fitted AR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian GDP case.

[59] As reported by the Augmented Dickey-Fuller (ADF) ($p-$value = 0.032), Phillips Perron (PP) ($p-$value = 0.040) and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) (Test statistic = 0.131 and Critical value = 0.463, the growth rate of the Colombian GDP is stationary at the 5% significance level.

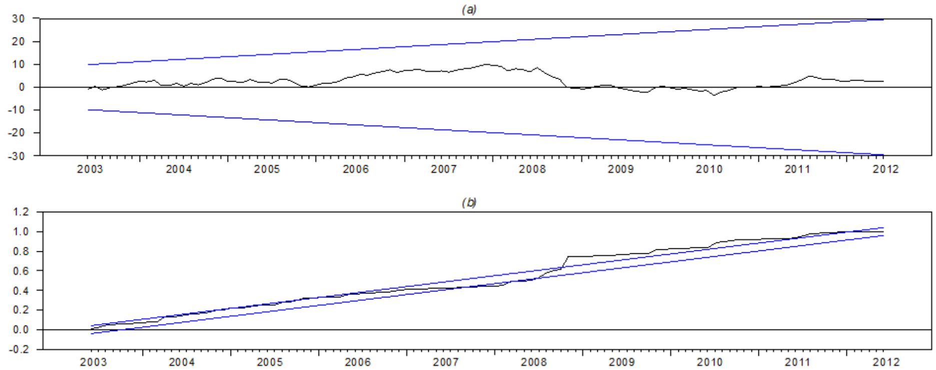


Figure G.15.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted AR model. Colombian GDP case.

Appendix H

General review of the models estimation for the biannual growth rate of the Colombian Industrial Production Index

H.1. Estimation of the TAR model

We use as the variable of interest the biannual growth rate of the Colombian industrial production index, and as the threshold variable, we use the spread term mentioned above. Figure H.1 and Figure H.2 show that both series have significant autocorrelations for a large number of lags.

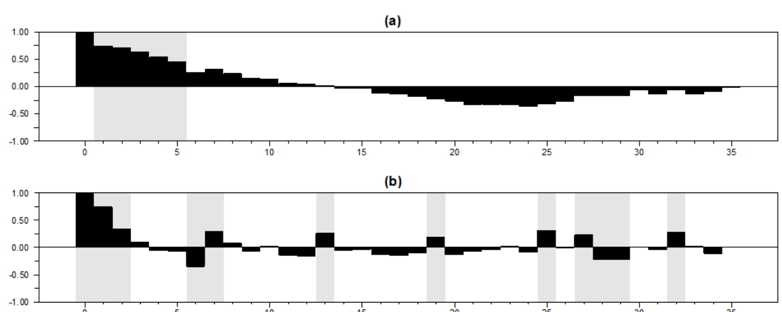


Figure H.1.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the biannual growth rate of Colombian industrial production index.

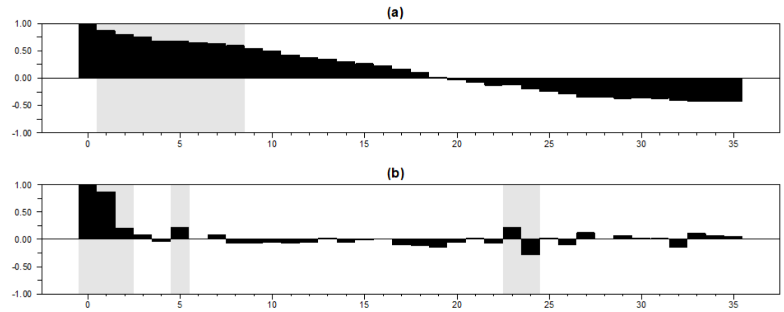


Figure H.2.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the Colombian spread term.

We test the null hypothesis of AR linearity against the alternative of bivariate TAR non-linearity. Under the AIC criterion, we found that $\bar{k} = 7$ is a reasonable autoregressive order for X_t. Under the null hypothesis, the results of the test let us define as the delay parameter of 5 for the threshold variable, thus, the input variable for the dynamic system is Z_{t-5}.

Figure H.3 shows the biannual growth rate of the industrial production index and the spread term from 2003:06 to 2012:06. The shading areas denote the business cycle contractions from peak to trough based on Alfonso et al. (2012).

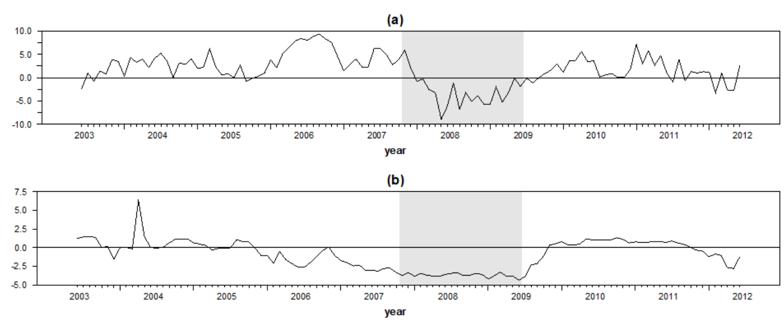


Figure H.3.: (a) Time plot of the biannual growth rate of Colombian industrial production index and (b) Time plot of Colombian spread term.

Then, we specify the maximum number of regimes l_0 by means of a regression function between X_t and Z_t, that is estimated using a nonparametric kernel approach and that is presented in Figure H.4. We observe that 3 could be postulated as the possible maximum regimes for the TAR model.

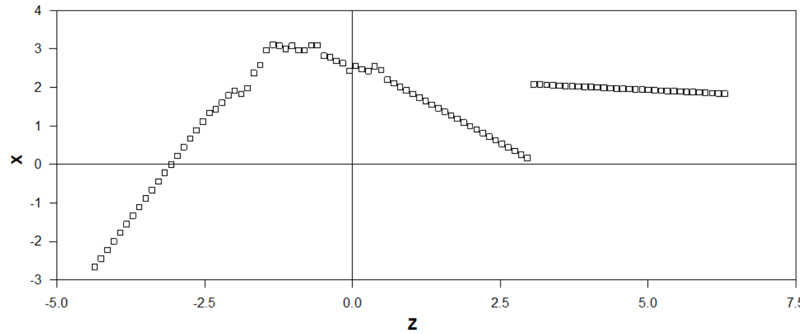


Figure H.4.: Nonparametric regression between the biannual growth rate of Colombian industrial production (X) and Colombian spread term (Z).

Then, to specify the prior densities for the nonstructural parameters, we define the prior densities for $\boldsymbol{\theta}_x$ where $\boldsymbol{\theta}_{0,j} = \bar{\mathbf{0}}, V_{0,j}^{-1} = 0.01\mathbf{I}$ with $\mathbf{I}$ the identity matrix, $\gamma_{0,j} = 1.5$ and $\beta_{0,j} = \frac{\widetilde{\sigma}^2}{2}$ with $j = 1, 2, 3$ and $\widetilde{\sigma}^2 = 3.809$ that is the residual variance of the AR(7) that was fitted to the biannual growth rate of the industrial production index. The maximum autoregressive order for all regimes is $\bar{k} = 7$, the same value fitted to the variable of interest. The prior distributions for both the number of regimes and the autoregressive orders that we use in the identification of l are $\pi_2 = \pi_3 = 0.5$ and $p\,(k_{il}|l) = 0.125$ for $k_{il} = 0, 1, \ldots, 7; i = 1, .., l$, respectively.

Then, to look for the location of thresholds in each possible regime, we choose the percentiles $5k$ where $k = 1, 2, \ldots, 19$, with respective values -3.82, -3.70, -3.44, -3.20, -2.75, -2.33, -2.01, -1.24, -1.02, -0.34, -0.03, 0.04, 0.30, 0.49, 0.66, 0.79, 0.96, 1.08, 1.28. The possible thresholds and autoregressive orders for each possible regime are presented in Table H.1.

ℓ	**Thresholds**	**Autoregressive orders**	**Minimum NAIC**
2	-0.34	1, 1	4.96476
3	-1.02 -0.03	7, 7, 7	4.38697

Table H.1.: Set of possible number of regimes for the real data. Colombian indpro case.

With those possible thresholds and the information mentioned above, we compute the posterior probability distribution for the number of regimes. 3000 iterates were generated with a burn-in point of 10% of the draws[60]. The results showed in Table H.2 allow us to set =3 as the appropriate number of regimes.

[60] The convergence of the Gibbs sampler was checked via the stationarity approach, where it was observed that the sample autocorrelation functions of the sequences $\left\{\hat{p}_l^{(i)}\right\}$ for $l = 2, 3$ decay quickly.

ℓ	$\hat{p}_\ell$
2	0.1095
3	0.8905

Table H.2.: Posterior probability function for the number of regimes for the real data. Colombian indpro case.

Thus, conditional on $\hat{l} = 3$, we estimate the autoregressive orders for k_1, k_2 and k_3. Based on Table H.3, the identified autoregressive orders are $\hat{k}_1 = 7$, $\hat{k}_2 = 3$ and $\hat{k}_3 = 5$. As before, convergence of the Gibbs sampler was checked via stationarity, and for 5000 iterates with a burn-in point of 10% of the draws, it is observed that the sample autocorrelations functions decay quickly[61].

Autoregressive Order	**Regime** 1	2	3
0	2.80×10^{-12}	0.0018	1.51×10^{-04}
1	0.0347	0.0987	0.0052
2	0.0499	0.1445	0.2346
3	0.1246	**0.3307**	0.1804
4	0.1323	0.1223	0.1870
5	0.1241	0.0530	**0.2692**
6	0.0089	0.0095	0.0040
7	**0.5256**	0.2395	0.1195

Table H.3.: Posterior probabilities for the autoregressive orders in the real data. Colombian indpro case.

Consequently, we fit a TAR(3;7,3,5) with thresholds values $r_1 = -1.02$ and $r_2 = -0.03$, which respectively are the 45th and 55th percentiles of the spread term. Table H.4 shows the estimates for the nonstructural parameters, with their respective posterior standard error in parenthesis and 90% credible interval in brackets[62]. These results show that not all the coefficients are significant at the 5% level. However, we decide to estimate the model with all the coefficients, given that that improves the final estimation.

[61] We performed a sensibility analysis where we changed the prior densities of the autoregressive orders and the nonstructural parameters, and we found that the estimated autoregressive orders are the same for different priors of the nonstructural parameters and for different priors of the autoregressive orders.

[62] 5000 iterates were generated with a burn-in point of 10% of the draws, and it was found that the autocorrelation functions for all the parameters decay quickly, indicating the convergence of the Gibbs sampler.

Parameter	Regime 1	Regime 2	Regime 3
$a_0^{(j)}$	0.14 (0.32)	-0.67 (1.34)	1.26 (0.63)
	$[-0.39, 0.66]$	$[-2.89, 1.47]$	$[0.22, 2.29]$
$a_1^{(j)}$	0.73 (0.15)	0.67 (0.39)	0.17 (0.17)
	$[0.48, 0.97]$	$[0.03, 1.30]$	$[-0.11, 0.45]$
$a_2^{(j)}$	0.09 (0.16)	0.16 (0.36)	0.34 (0.18)
	$[-0.18, 0.35]$	$[-0.42, 0.72]$	$[0.04, 0.63]$
$a_3^{(j)}$	0.40 (0.16)	0.18 (0.38)	0.04 (0.19)
	$[0.14, 0.67]$	$[-0.42, 0.80]$	$[-0.26, 0.34]$
$a_4^{(j)}$	-0.21 (0.16)		-0.04 (0.18)
	$[-0.47, 0.06]$		$[-0.33, 0.25]$
$a_5^{(j)}$	0.12 (0.16)		-0.04 (0.16)
	$[-0.14, 0.37]$		$[-0.31, 0.22]$
$a_6^{(j)}$	-0.62 (0.16)		
	$[-0.89, -0.35]$		
$a_7^{(j)}$	0.38 (0.16)		
	$[0.12, 0.64]$		
$h^{(j)}$	4.61 (1.05)	6.00 (3.71)	4.26 (0.99)
	$[3.20, 6.57]$	$[2.53, 12.70]$	$[2.92, 6.09]$

Table H.4.: Parameter estimates for the TAR model. Colombian indpro case.

Thus, the fitted TAR model for the biannual growth rate of the Colombian industrial production index is given by:

$$X_t = \begin{cases} 0.14 + 0.73X_{t-1} + 0.09X_{t-2} + 0.40X_{t-3} - 0.21X_{t-4} & \\ +0.12X_{t-5} - 0.62X_{t-6} + 0.38X_{t-7} + 4.61\varepsilon_t, & \text{if } Z_{t-5} \leq -1.02 \\ -0.67 + 0.67X_{t-1} + 0.16X_{t-2} + 0.18X_{t-3} + 6.00\varepsilon_t, & \text{if } -1.02 < Z_{t-5} \leq -0.03 \\ 1.26 + 0.17X_{t-1} + 0.34X_{t-2} + 0.04X_{t-3} - 0.04X_{t-4} & \\ -0.04X_{t-5} + 4.26\varepsilon_t, & \text{if } Z_{t-5} > -0.03 \end{cases}$$

This model could represent periods in the economy of i) contraction, given that this regime contains the greatest decreases in the growth rate of the industrial production index, when there is a low spread due to contractionary monetary policies; ii) stabilization, where the regime shows minor increases in the growth rate of the industrial production index; and iii) expansion, where the regime exhibits the greatest increases in the growth rate of the industrial production index, when there is high spread because of expansionary monetary policies.

When we check the residuals, in Figure H.5 it is observed that the standardized residuals and squared standardized residuals of the model signal that the noise process could be white, and the Ljung-Box statistics are, respectively, $Q\,(12) = 11.647(0.474)$ and $Q\,(12) = 16.512(0.169)$. Figure H.6 reports that the CUSUM and CUSUMSQ behave well, which indicates that there is neither statistical evidence for model misspecification nor heteroscedasticity in $\{\varepsilon_t\}$.

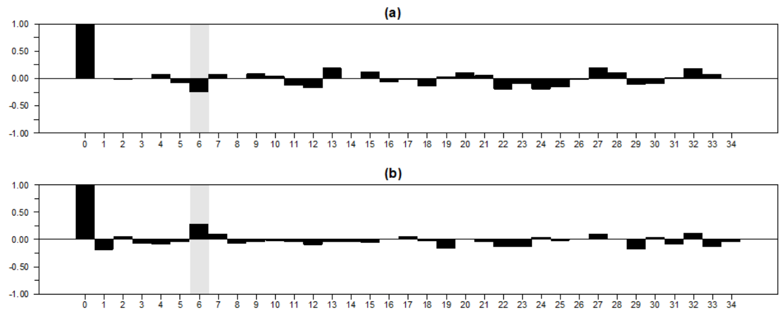


Figure H.5.: Partial autocorrelation function of the fitted TAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian indpro case.

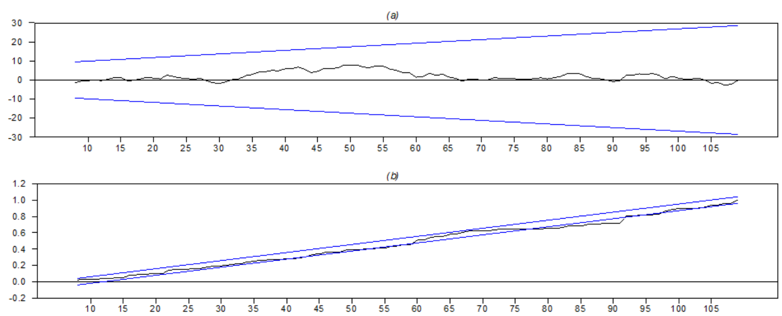


Figure H.6.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted TAR model. Colombian indpro case.

H.2. Estimation of the SETAR model

Based on the autocorrelation functions in Figure H.1 and the AIC and BIC criterion, we identify $p = 3$ as the final autoregressive order to fit a SETAR model. Once we select the autoregressive model, we check the nonlinearity of the series based on Tsay's (1989) test and find that with $d = 2$, it is obtained the minimum p-value $= 0.0352$ of the F statistic $F = 2.696$, rejecting the linearity of the series at the 5% significance level.

Figure H.7 shows the sequence of the t ratios of a lag-2 AR coefficient versus the threshold variable X_{t-d} in an arranged autoregression of order 3, and we identify that the data can be divided into two regimes with a possible threshold at $X_{t-d} = -0.5$, because of the change on the slope at approximately this point.

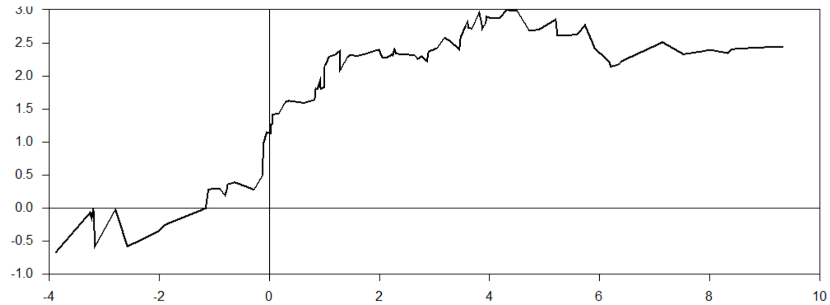


Figure H.7.: Time plot of t-ratio of recursive estimates of the AR-2 coefficient in an arranged autoregression of order 3 and delay parameter 2. Colombian indpro case.

We also use the NAIC criterion of Tong (1990), which suggests a threshold value of $r_j = -0.79$ and autoregressive orders AR(1) and AR(2) for each regime, with a NAIC = 168.239. Based on that, we fit a SETAR(2;1,2) for the biannual growth rate of the industrial production index, with threshold value X_{t-2}. The estimated parameters and their standard errors in parenthesis for each regime are shown in Table H.5, where all estimates are significant at the 5% level.

	Regime	
Parameter	**1**	**2**
$\Phi_1^{(j)}$	0.66 (0.16)	0.62 (0.10)
$\Phi_2^{(j)}$		0.27 (0.11)

Table H.5.: Parameter estimates for the SETAR model. Colombian indpro case.

Therefore, the estimated SETAR model for the biannual growth rate of the Colombian industrial production index is given by:

$$X_t = \begin{cases} 0.66X_{t-1} + \varepsilon_t, & \text{if } X_{t-2} \leq -0.79 \\ 0.62X_{t-1} + 0.27X_{t-2} + \varepsilon_t, & \text{if } \ X_{t-2} > -0.79 \end{cases}$$

This model could represent periods in the economy of i) contraction, where the first regime contains mostly the decreases in the growth rate of the industrial production index, and ii) expansion, where the second regime shows mostly the increases in the growth rate of the industrial production index.

Figure H.8 shows that the standardized and squared standardized residuals of the model signal that some nonlinear structure in the data is not explained by the model. Furthermore, the Ljung-Box statistics are, respectively, $Q\,(12) = 40.662(0.000)$ and $Q\,(12) = 13.061(0.365)$. Figure H.9 presents the CUSUM and CUSUMSQ, which indicate that there is neither statistical evidence for model misspecification nor heteroscedasticity in $\{\varepsilon_t\}$.

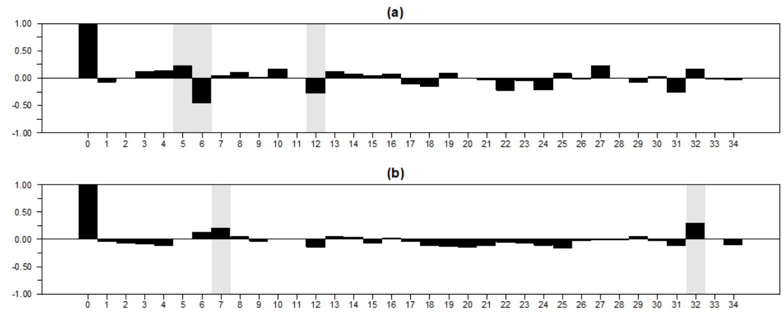


Figure H.8.: Partial autocorrelation function of the fitted SETAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian indpro case.

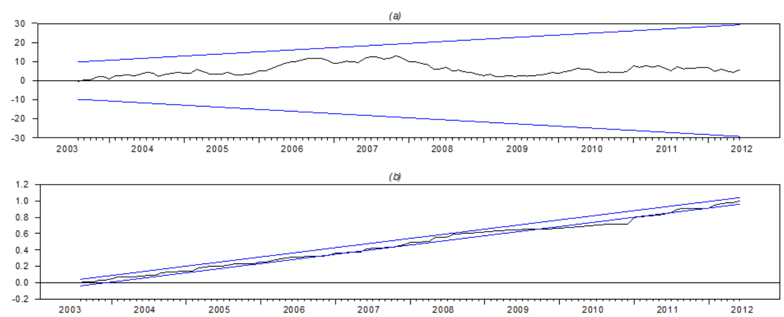


Figure H.9.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted SETAR model. Colombian indpro case.

H.3. Estimation of the STAR model

With an identified $p = 7$ autoregressive order, we evaluate the nonlinearity of the series based on Teräsvirta's test and find that, with a delay parameter of $d = 2$, we obtain the p-value $= 0.0041$ for the F statistic $F = 2.3340$, which rejects the linearity of the series.

When we compute the tests of nested hypothesis of Teräsvirta (1994), we find that at the 5% significance level, H_{01} and H_{03} are rejected, while H_{02} is not, where the respective p-value is 0.0013, 0.0431 and 0.8840. These results suggest that we must estimate a LSTAR model.

In Table H.6, we show the estimates for the parameters and their respective standard error in parenthesis, where only the lag-4 AR coefficient of the first regime and the parameter γ and c are not significant at the 5% level.

	Regime	
Parameter	**1**	**2**
$\Phi_1^{(j)}$	0.75 (0.09)	
$\Phi_3^{(j)}$	0.56 (0.16)	-0.59 (0.26)
$\Phi_4^{(j)}$	-0.28 (0.18)	0.66 (0.28)
$\Phi_6^{(j)}$	-0.62 (0.10)	
$\Phi_7^{(j)}$	0.45 (0.10)	
γ		5.16 (7.89)
c		1.10 (1.08)

Table H.6.: Parameter estimates for the STAR model. Colombian indpro case.

Consequently, the estimated STAR model for the biannual growth rate of the Colombian industrial production index is given by:

$$\begin{aligned}X_t &= 0.75X_{t-1} + 0.56X_{t-3} - 0.28X_{t-4} - 0.62X_{t-6} + 0.45X_{t-7}\\ &\quad + F\left(X_{t-2}\right)\left(-0.59X_{t-3} + 0.66X_{t-4}\right) + \varepsilon_t,\end{aligned}$$

where

$$F\left(X_{t-2}\right) = \left(1 + \exp\left\{-5.16\left(X_{t-2} - 1.10\right)\right\}\right)^{-1}.$$

When we check the adequacy of the estimated model, Figure H.10 shows that the standardized and squared standardized residuals of the estimated model signal that the noise process is white. Moreover, the Ljung-Box statistics are, respectively, $Q\left(12\right) = 13.315(0.347)$ and $Q\left(12\right) = 16.618(0.165)$. Figure H.11 presents the CUSUM and CUSUMSQ, which indicate that there is neither statistical evidence for model misspecification nor heteroscedasticity in $\{\varepsilon_t\}$.

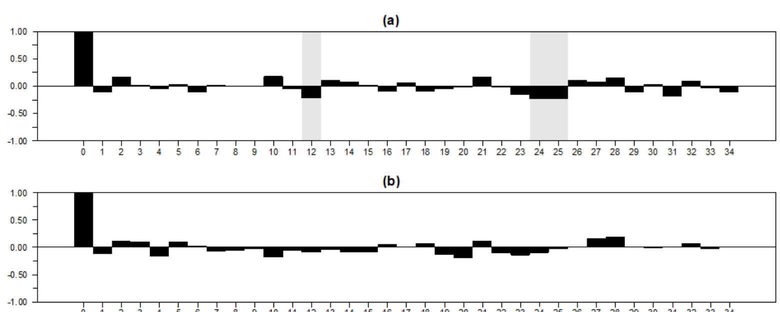


Figure H.10.: Partial autocorrelation function of the fitted STAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian indpro case.

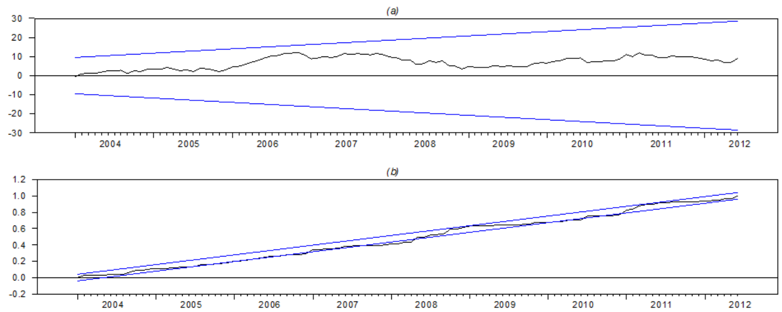


Figure H.11.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted STAR model. Colombian indpro case.

H.4. Estimation of the MSAR model

Table H.7 shows the estimates of the parameters of the model and their respective standard errors in parenthesis.

	state	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	-0.24 (0.28)	0.89 (0.42)
$\Phi_1^{(j)}$	0.31 (0.10)	0.80 (0.19)
$\Phi_2^{(j)}$	0.70 (0.10)	-0.12 (0.22)

Table H.7.: Parameter estimates for the MSAR model. Colombian indpro case.

Therefore, the estimated MSAR model for the biannual growth rate of the Colombian industrial production index is given by:

$$X_t = \begin{cases} -0.24 + 0.31X_{t-1} + 0.70X_{t-2} + \varepsilon_{1t}, & \text{if } s_t = 1 \\ 0.89 + 0.80X_{t-1} - 0.12X_{t-2} + \varepsilon_{2t}, & \text{if } s_t = 2 \end{cases}$$

The conditional mean of X_t for regime 1 is -0.35 and for regime 2 is 4.52. The sample variances of ε_{1t} and ε_{2t} are 1.62 and 5.75, respectively[63]. Hence, the first state could represent an unstable economy with decreases in the growth rate of the industrial production index in average, and the second state could represent a stable economy with increases in the growth rate of this macroeconomic indicator in average.

When we evaluate this model, Figure H.12 shows that the standardized and squared standardized residuals of the model signal that the noise process is white. Moreover, the Ljung-Box statistics are, respectively, $Q(12) = 4.767(0.965)$ and $Q(12) = 2.065(0.999)$.

[63] The probabilities of moving from one state to the other are $p(s_t = 2|s_{t-1} = 1) = 0.70(0.24)$ and $p(s_t = 1|s_{t-1} = 2) = 0.60(0.25)$, where the number in parenthesis is the standard error. Additionally, the unconditional probability that the process is in regime 1 is 0.46 and that it is in regime 2 is 0.54.

Figure H.13 presents the CUSUM and CUSUMSQ, which indicate that there is statistical evidence for model misspecification and homoscedasticity in $\{\varepsilon_t\}$.

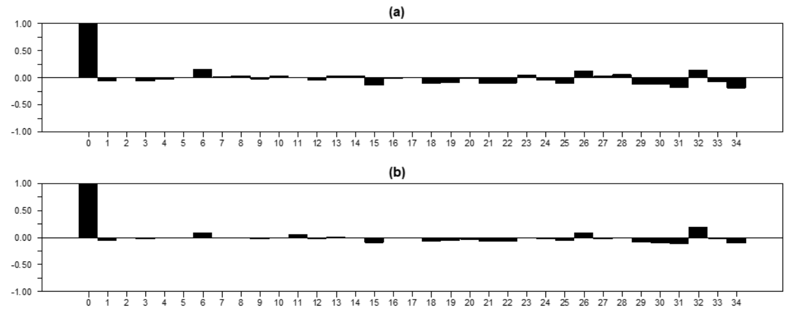


Figure H.12.: Partial autocorrelation function of the fitted MSAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian indpro case.

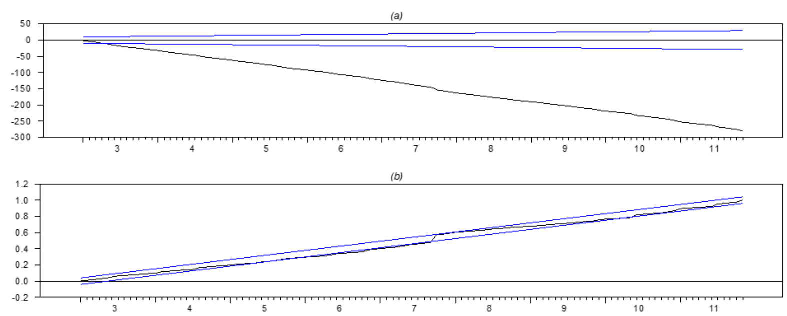


Figure H.13.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted MSAR model. Colombian indpro case.

H.5. Estimation of the AR model

We estimate an AR(2) model for the biannual growth rate of the Colombian industrial production index[64], which is given by:

$$\left(1 - 0.52B - 0.34B^2\right) X_t = a_t, \quad \hat{\sigma}_a^2 = 4.91.$$

The standard error of both coefficients is 0.09. When we check the residuals, Figure H.14 shows that the standardized residuals of the model slightly signal that some linear structure in the data is not explained by the model, and the Ljung-Box statistics are, respectively, $Q\,(12) = 31.400(0.002)$ and $Q\,(12) = 13.494(0.334)$. Figure H.15 shows the CUSUM and

[64] As reported by the Phillips Perron (PP) ($p-$ value $= 0.002$) and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) (Test statistic $= 0.205$ and Critical value $= 0.463$, the growth rate of the Colombian industrial production index is stationary at the 5% significance level. The Augmented Dickey-Fuller (ADF) $p-$ value is 0.143.

CUSUMSQ, indicating that there is no statistical evidence for model misspecification or heteroscedasticity in $\{\varepsilon_t\}$.

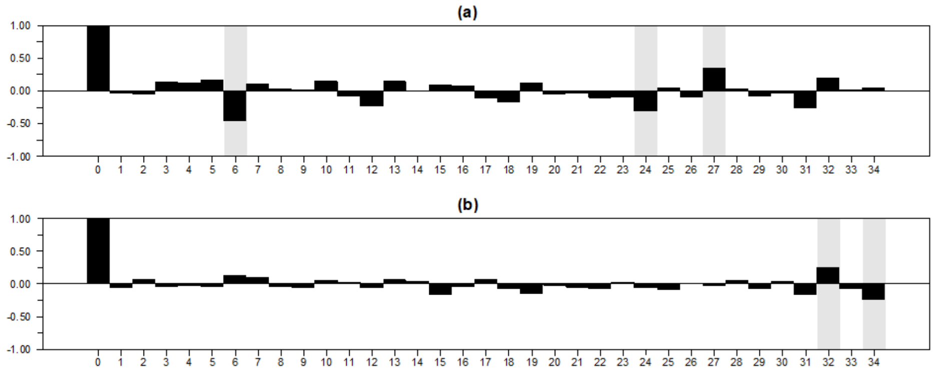


Figure H.14.: Partial autocorrelation function of the fitted AR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian indpro case.

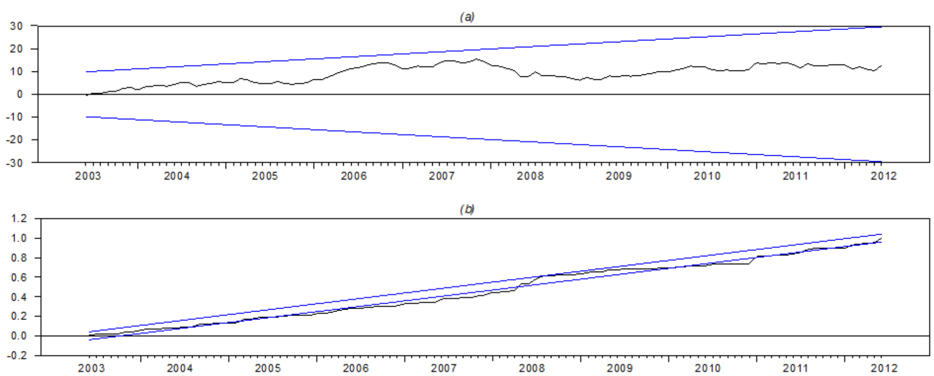


Figure H.15.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted AR model. Colombian indpro case.

Appendix I

General review of the models estimation for the growth rate of the Colombian CPI

I.1. Estimation of the TAR model

We use as the variable of interest, the growth rate of the Colombian CPI, and as the threshold variable, we use the spread term mentioned above. Figure I.1 and Figure I.2 show that both series have significant autocorrelations for a large number of lags.

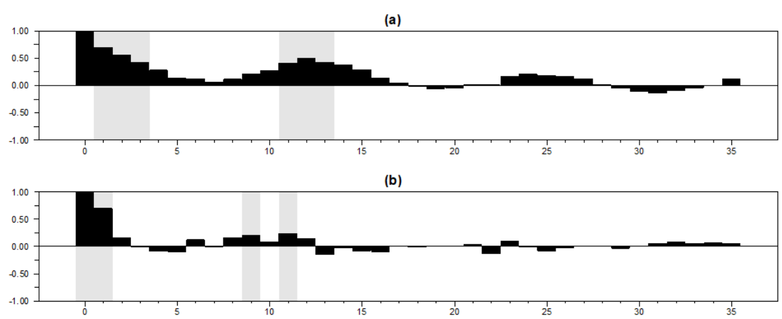


Figure I.1.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the growth rate of Colombian CPI.

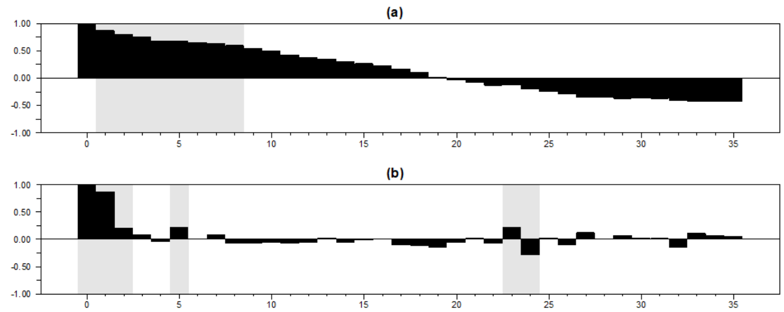


Figure I.2.: (a) Sample autocorrelation function and (b) sample partial autocorrelation function for the Colombian spread term.

Then, we test the null hypothesis of AR linearity against the alternative of bivariate TAR nonlinearity. Under the AIC criterion, we found that $\bar{k} = 3$ is a reasonable autoregressive order for X_t. Under the null hypothesis, the results of the test let us define as the delay parameter of 3 for the threshold variable, thus, the input variable for the dynamic system is Z_{t-3}.

Figure I.3 shows the growth rate of the Colombian CPI and spread term from 2003:04 to 2012:06. The shading areas denote the business cycle contractions from peak to trough based on Alfonso et al. (2012).

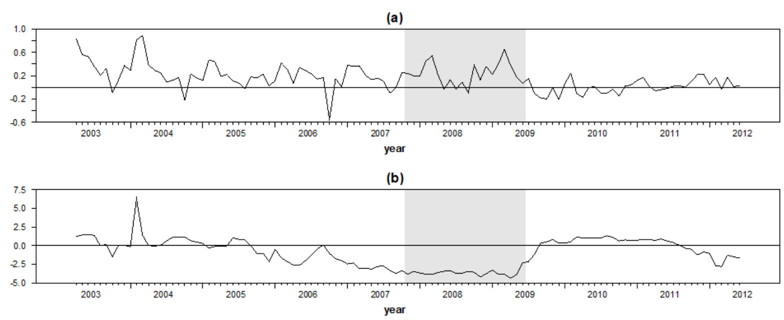


Figure I.3.: (a) Time plot of the growth rate of Colombian CPI and (b) Time plot of Colombian spread term.

We specify the maximum number of regimes l_0 by means of a regression function between X_t and Z_t, that is estimated using a nonparametric kernel approach and that is presented in Figure I.4. We observe that 3 could be postulated as the possible maximum regimes for the TAR model.

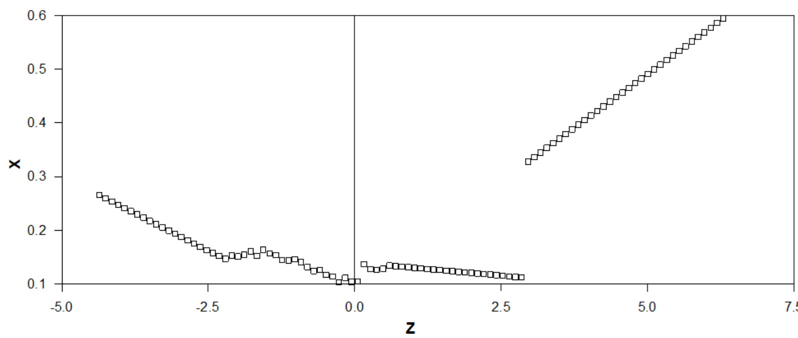


Figure I.4.: Nonparametric regression between the monthly growth rate of the CPI (X) and the spread term (Z).

Next, to specify the prior densities for the nonstructural parameters, we define the prior densities for $\boldsymbol{\theta}_x$ where $\boldsymbol{\theta}_{0,j} = \bar{\mathbf{0}}, V_{0,j}^{-1} = 0.01\mathbf{I}$ with $\mathbf{I}$ the identity matrix, $\gamma_{0,j} = 1.5$ and $\beta_{0,j} = \frac{\widetilde{\sigma}^2}{2}$ with $j = 1, 2, 3$ and $\widetilde{\sigma}^2 = 0.032$ that is the residual variance of the AR(3) that was fitted to the growth rate of the CPI. The maximum autoregressive order for all regimes is $\bar{k} = 3$, the same value fitted to the variable of interest. The prior distributions for both the number of regimes and the autoregressive orders that we use in the identification of l are $\pi_2 = \pi_3 = 0.5$ and $p\left(k_{il}|l\right) = 0.25$ for $k_{il} = 0, 1, 2, 3; i = 1, .., l$, respectively.

Then, to look for the location of thresholds in each possible regime, we choose the percentiles $5k$ where $k = 1, 2, \ldots, 19$, with respective values -3.82, -3.69, -3.44, -3.16, -2.71, -2.32, -1.87, -1.26, -1.03, -0.40, -0.07, 0.02, 0.22, 0.47, 0.66, 0.78, 0.93, 1.08, 1.27. The possible thresholds and autoregressive orders for each possible regime are presented in Table I.1.

ℓ	**Thresholds**	**Autoregressive orders**	**Minimum NAIC**
2	-1.03	1, 1	-0.12288
3	-1.03 0.22	1, 3, 1	-0.58614

Table I.1.: Set of possible number of regimes for the real data. Colombian CPI case.

With those possible thresholds and the information mentioned above, we compute the posterior probability distribution for the number of regimes. 3000 iterates were estimated with a burn-in point of 10% of the draws[65]. The results showed in Table I.2 allow us to set $\hat{l} = 3$ as the appropriate number of regimes.

[65] The convergence of the Gibbs sampler was checked via the stationarity approach, where it was observed that the sample autocorrelation functions of the sequences $\left\{\hat{p}_l^{(i)}\right\}$ for $l = 2, 3$ decay quickly.

ℓ	$\hat{p}_\ell$
2	0.2173
3	0.7827

Table I.2.: Posterior probability function for the number of regimes for the real data. Colombian CPI case.

Thus, conditional on $\hat{l} = 3$, we estimate the autoregressive orders for k_1, k_2 and k_3. Based on Table I.3, the identified autoregressive orders are $\hat{k}_1 = 2$, $\hat{k}_2 = 1$ and $\hat{k}_3 = 2$. As before, convergence of the Gibbs sampler was checked via stationarity, and for 5000 iterates with a burn-in point of 10% of the draws, it is observed that the sample autocorrelations functions decay quickly[66].

Autoregressive Order	**Regime** 1	**2**	**3**
0	0.0252	0.0047	7.54×10^{-06}
1	0.2118	**0.5358**	0.2129
2	**0.6848**	0.0313	**0.3954**
3	0.0782	0.4282	0.3917

Table I.3.: Posterior probabilities for the autoregressive orders in the real data. Colombian CPI case.

Consequently, we fit a TAR(3;2,1,2) with thresholds values $r_1 = -1.03$ and $r_2 = 0.22$, which respectively are the 45th and 65th percentiles of the spread term. Table I.4 shows the estimates for the nonstructural parameters, with their respective posterior standard error in parenthesis and 90% credible interval in brackets[67]. These results show that not all the coefficients are significant at the 5% level. However, we decide to estimate the model with all the coefficients, given that that improves the final estimation.

[66] We performed a sensibility analysis where we changed the prior densities of the autoregressive orders and the nonstructural parameters, and it was found that the estimated autoregressive orders are the same for different priors of the nonstructural parameters and for different priors of the autoregressive orders.

[67] 5000 iterates were generated with a burn-in point of 10% of the draws, and it was found that the autocorrelation functions for all the parameters decay quickly, indicating the convergence of the Gibbs sampler.

Parameter	Regime 1	Regime 2	Regime 3
$a_0^{(j)}$	0.11 (0.04)	0.16 (0.03)	0.01 (0.03)
	$[0.04, 0.18]$	$[0.10, 0.21]$	$[-0.04, 0.07]$
$a_1^{(j)}$	0.22 (0.15)	0.32 (0.11)	0.69 (0.17)
	$[-0.03, 0.47]$	$[0.14, 0.50]$	$[0.40, 0.97]$
$a_2^{(j)}$	0.12 (0.15)		0.15 (0.20)
	$[-0.13, 0.37]$		$[-0.18, 0.48]$
$h^{(j)}$	0.04 (0.01)	0.01 (0.00)	0.03 (0.01)
	$[0.03, 0.05]$	$[0.01, 0.02]$	$[0.02, 0.05]$

Table I.4.: Parameter estimates for the TAR model. Colombian CPI case.

The fitted TAR model for the growth rate of the Colombian CPI is given by:

$$X_t = \begin{cases} 0.11 + 0.22X_{t-1} + 0.12X_{t-2} + 0.04\varepsilon_t, & \text{if } Z_{t-3} \leq -1.03 \\ 0.16 + 0.32X_{t-1} + 0.01\varepsilon_t, & \text{if } -1.03 < Z_{t-3} \leq 0.22 \\ 0.01 + 0.69X_{t-1} + 0.15X_{t-2} + 0.03\varepsilon_t, & \text{if } Z_{t-3} > 0.22 \end{cases}$$

This model could represent i) a first regime characterized by negative spreads, that respond to contractionary monetary policies, and generate a slowdown real activity with minor fluctuations in the inflation; ii) a second regime with an economic transition with a stable behavior of the inflation; and iii) a third regime with positive spreads, that respond to expansionary monetary policies, and produce real activity increases associated with major fluctuations in the inflation.

When we check the residuals, in Figure I.5 we observe that the standardized and squared standardized residuals signal that some nonlinear structure in the data is not explained by the model. The Ljung-Box statistics are, respectively, $Q\,(12) = 34.736(0.001)$ and $Q\,(12) = 8.216(0.768)$. Figure I.6 shows the CUSUM and CUSUMSQ, that indicate that there is no statistical evidence for model misspecification but there is heteroscedasticity in $\{\varepsilon_t\}$.

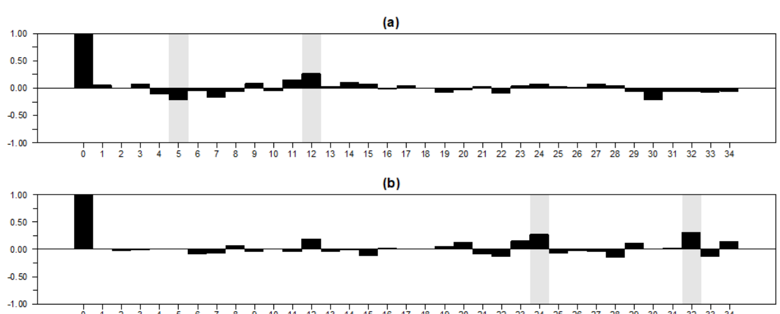


Figure I.5.: Partial autocorrelation function of the fitted TAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian CPI case.

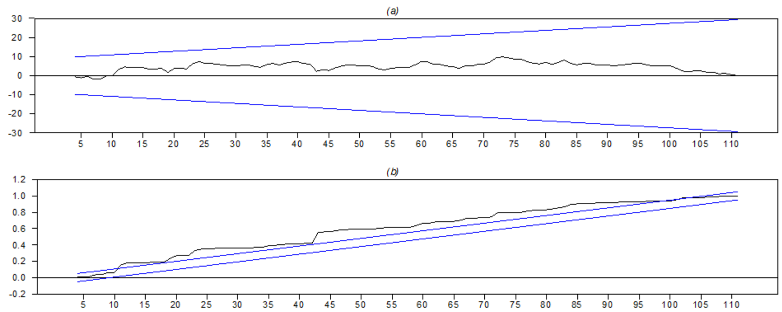


Figure I.6.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted TAR model. Colombian CPI case.

I.2. Estimation of the SETAR model

Based on the autocorrelation functions in Figure I.1 and the AIC and BIC criterion, we identify p=3 as the final autoregressive order to fit a SETAR model. Once we select the autoregressive model, we check the nonlinearity of the series based on Tsay's (1989) test and find that with $d = 3$, it is obtained the p-value $= 0.0578$ of the F statistic $F = 2.577$, rejecting the linearity of the series at the 10% significance level.

Figure I.7 shows the sequence of the t ratios of a lag-3 AR coefficient versus the threshold variable X_{t-d} in an arranged autoregression of order 3, and we identify that the data can be divided into two regimes with a possible threshold at $X_{t-d} = 0.09$, because of the change on the slope at approximately this point.

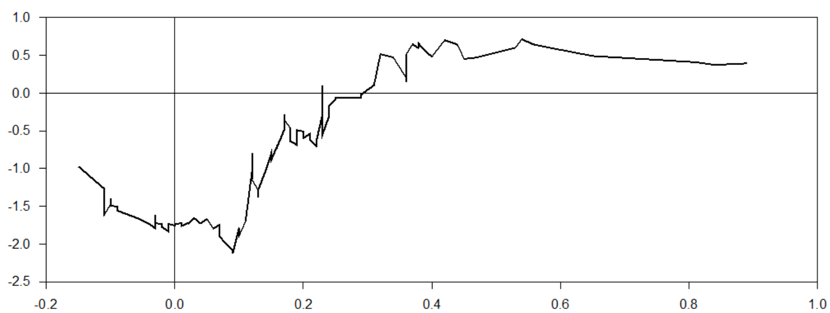


Figure I.7.: Time plot of t-ratio of recursive estimates of the AR-3 coefficient in an arranged autoregression of order 3 and delay parameter 3. Colombian CPI case.

We also use the NAIC criterion of Tong (1990), which suggests a threshold value of $r_j = 0.07$ and autoregressive orders AR(2) and AR(1) for each regime, with a NAIC = -370.198. Based on that, we fit a SETAR(2;2,1) for the monthly growth rate of the CPI, with threshold value X_{t-3}. The estimated parameters and their standard errors in parenthesis for each regime are shown in Table I.5, where all estimates are significant at the 5% level.

	Regime	
Parameter	**1**	**2**
$\Phi_0^{(j)}$		0.08 (0.03)
$\Phi_1^{(j)}$	0.54 (0.16)	0.46 (0.11)
$\Phi_2^{(j)}$	0.34 (0.17)	

Table I.5.: Parameter estimates for the SETAR model. Colombian CPI case.

In that sense, the fitted SETAR model for the growth rate of the Colombian CPI is given by:

$$X_t = \begin{cases} 0.54X_{t-1} + 0.34X_{t-2} + \varepsilon_t, & \text{if } X_{t-3} \leq 0.07 \\ 0.08 + 0.46X_{t-1} + \varepsilon_t, & \text{if } X_{t-3} > 0.07 \end{cases}$$

This model could represent i) a first regime characterized by a real activity tightening associated with minor fluctuations in the inflation; and ii) a second regime with a real activity increasing associated with major fluctuations in the inflation.

Figure I.8 shows that the correlations of the standardized and squared standardized residuals of the model could signal that the noise process is white, and the Ljung-Box statistics are, respectively, $Q(12) = 17.405(0.135)$ and $Q(12) = 7.777(0.802)$. Figure I.9 presents the CUSUM and CUSUMSQ, indicating that there is no statistical evidence for model misspecification and there is some heteroscedasticity in $\{\varepsilon_t\}$.

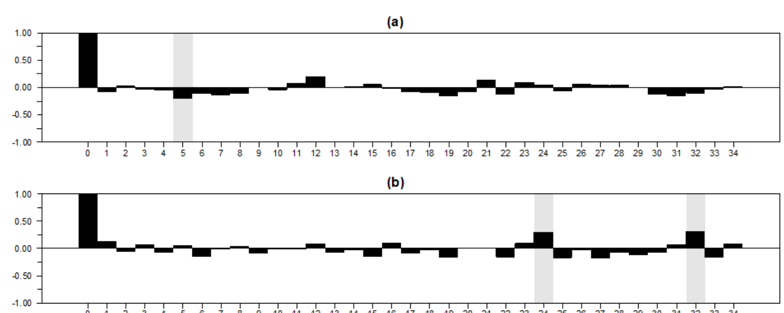


Figure I.8.: Partial autocorrelation function of the fitted SETAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian CPI case.

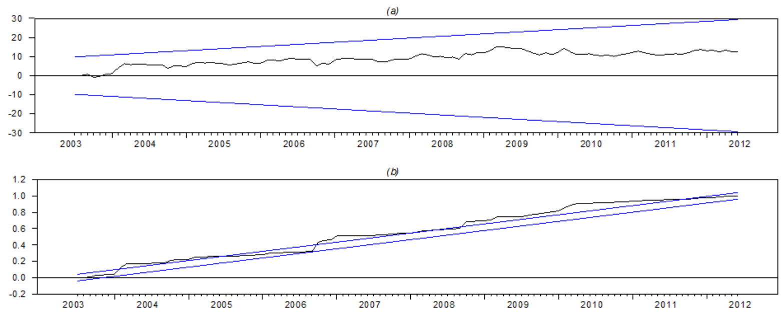


Figure I.9.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted SETAR model. Colombian CPI case.

I.3. Estimation of the STAR model

With an identified $p = 2$ autoregressive order, we evaluate the nonlinearity of the series based on Teräsvirta's test and find that, with a delay parameter of $d = 1$, we obtain the p-value $= 0.0648$ for the F statistic $F = 2.0589$, which rejects the linearity of the time series.

When we compute the tests of nested hypothesis of Teräsvirta (1994), we find that at the 5% significance level, H_{01} and H_{02} are not rejected, while H_{03} is not, where the respective p-value is 0.7155, 0.4696 and 0.0072. These results suggest that we must estimate a LSTAR model.

In Table I.6, we show the estimates for the parameters and their respective standard error in parenthesis, where only the lag-4 AR coefficient of the first regime and the parameter γ and c are not significant at the 5% level.

	Regime	
Parameter	**1**	**2**
$\Phi_1^{(j)}$	-0.29 (0.33)	0.93 (0.35)
$\Phi_2^{(j)}$	0.11 (0.10)	
γ		5.08 (10.84)
c		-0.14 (0.10)

Table I.6.: Parameter estimates for the STAR model. Colombian CPI case.

Consequently, the fitted STAR model for the growth rate of the Colombian CPI is given by:

$$X_t = -0.29X_{t-1} + 0.11X_{t-2} + F\left(X_{t-1}\right)\left(0.93X_{t-1}\right) + \varepsilon_t,$$

where

$$F\left(X_{t-1}\right) = \left(1 + \exp\left\{-5.08\left(X_{t-1} + 0.14\right)\right\}\right)^{-1}.$$

When we check the residuals, Figure I.10 shows that the standardized and squared standardized residuals of the model could signal that the noise process is white. Furthermore, the Ljung-Box statistics are, respectively, $Q(12) = 17.640(0.127)$ and $Q(12) = 11.140(0.517)$. Figure I.11 presents the CUSUM and CUSUMSQ, which indicate that there is no statistical evidence for model misspecification but there is heteroscedasticity in $\{\varepsilon_t\}$.

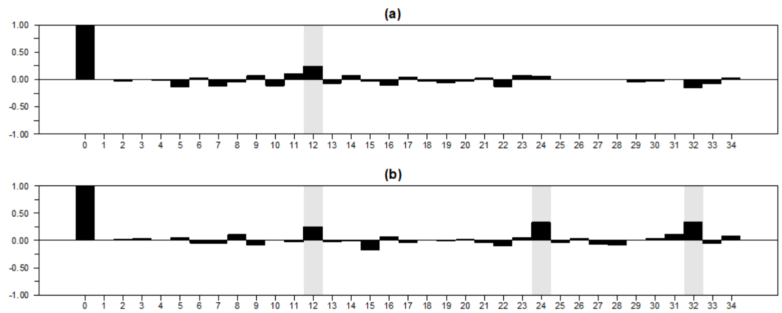


Figure I.10.: Partial autocorrelation function of the fitted STAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian CPI case.

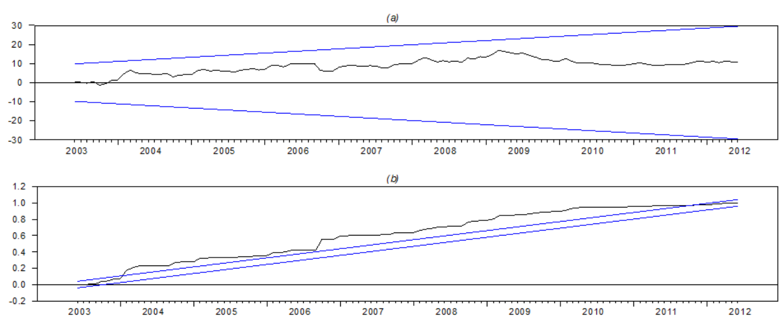


Figure I.11.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted STAR model. Colombian CPI case.

I.4. Estimation of the MSAR model

Table I.7 shows the estimates of the parameters of the model and their respective standard errors in parenthesis.

	state	
Parameter	**1**	**2**
$\Phi_0^{(j)}$	0.02 (0.02)	0.11 (0.04)
$\Phi_1^{(j)}$	0.43 (0.15)	0.39 (0.15)
$\Phi_2^{(j)}$	0.10 (0.13)	0.20 (0.17)

Table I.7.: Parameter estimates for the MSAR model. Colombian CPI case.

Therefore, the fitted MSAR model for the monthly growth rate of the Colombian CPI is given by:

$$X_t = \begin{cases} 0.02 + 0.43X_{t-1} + 0.10X_{t-2} + \varepsilon_{1t}, & \text{if } s_t = 1 \\ 0.11 + 0.39X_{t-1} + 0.20X_{t-2} + \varepsilon_{2t}, & \text{if } s_t = 2 \end{cases}$$

The conditional mean of X_t for regime 1 is 0.04 and for regime 2 is 0.18 The sample variances of ε_{1t} and ε_{2t} are 0.01 and 0.05, respectively[68]. Hence, the first state could represent a minor stable economy with minor fluctuations in the CPI, and the second state could represent a stable economy with increases in the CPI.

When we check the residuals, Figure I.12 shows that the standardized and squared standardized residuals of the model signal that the noise process is white, and the Ljung-Box statistics are, respectively, $Q(12) = 11.220(0.510)$ and $Q(12) = 0.865(0.999)$. Figure I.13 presents the CUSUM and CUSUMSQ, which indicates that there is statistical evidence for some model misspecification and heteroscedasticity in $\{\varepsilon_t\}$.

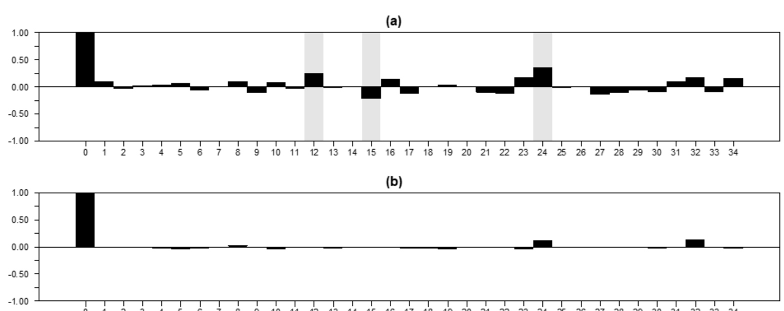


Figure I.12.: Partial autocorrelation function of the fitted MSAR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian CPI case.

68 The probabilities of moving from one state to the other are $p(s_t = 2|s_{t-1} = 1) = 0.18(0.10)$ and $p(s_t = 1|s_{t-1} = 2) = 0.78(0.14)$, where the number in parenthesis is the standard error. Additionally, the unconditional probability that the process is in regime 1 is 0.55 and that it is in regime 2 is 0.45.

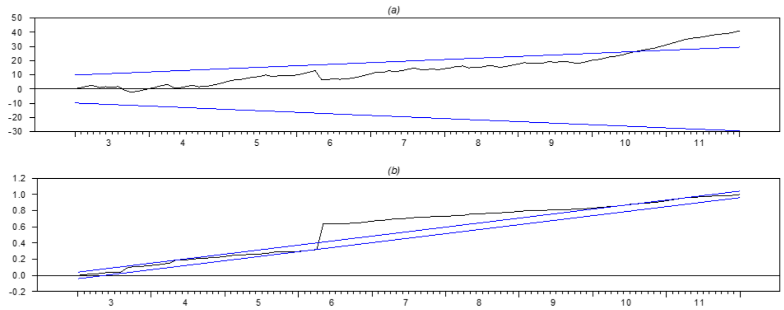


Figure I.13.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted MSAR model. Colombian CPI case.

I.5. Estimation of the AR model

We estimate an AR(1) for the growth rate of the Colombian CPI[69], which is given by:

$$(1 - 0.56B)\, X_t = 0.16 + a_t, \quad \hat{\sigma}_a^2 = 0.04.$$

The standard errors of the coefficients are 0.08 and 0.09, respectively. When we check the residuals, Figure I.14 shows that the standardized and the squared standardized residuals of the model signal that the noise process is white. Moreover, the Ljung-Box statistics are, respectively, $Q\,(12) = 20.517(0.058)$ and $Q\,(12) = 8.388(0.754)$. Figure I.15 shows the CUSUM, which indicates that there is no statistical evidence for model misspecification, and the CUSUMSQ that indicates some statistical evidence for heteroscedasticity in $\{\varepsilon_t\}$.

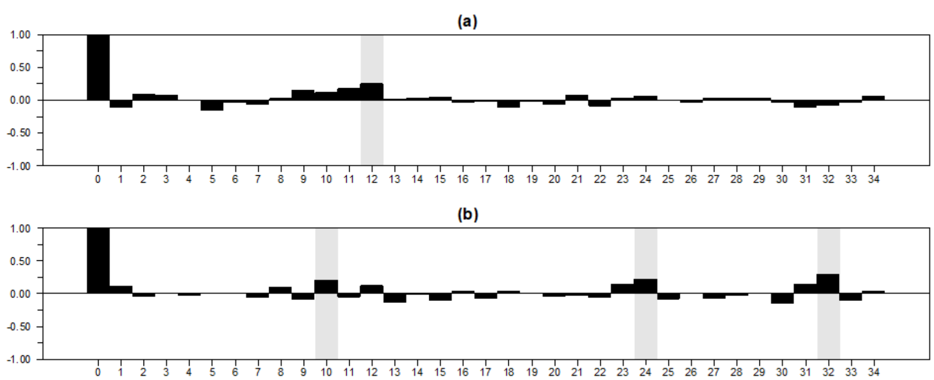


Figure I.14.: Partial autocorrelation function of the fitted AR model. (a) Standardized residuals and (b) squared standardized residuals. Colombian CPI case.

[69] As reported by the Augmented Dickey-Fuller (ADF) ($p-$value $= 0.000$), Phillips Perron (PP) ($p-$value $= 0.000$) and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) (Test statistic $= 0.119$ and Critical value $= 0.463$, the Colombian CPI is stationary at the 5% significance level.

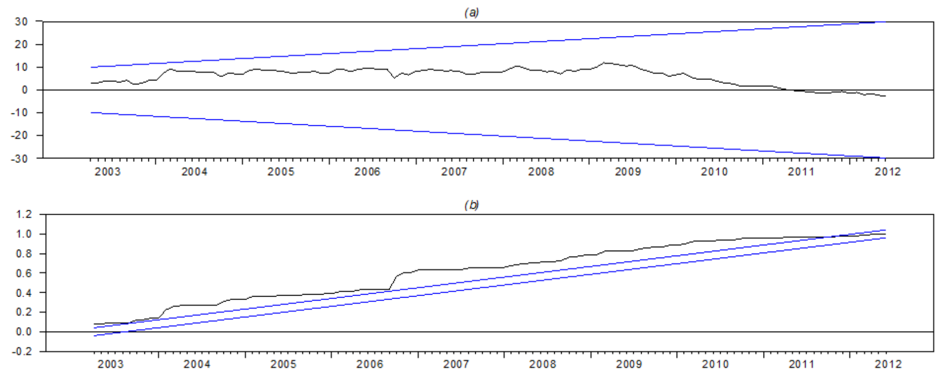


Figure I.15.: (a) CUSUM and (b) CUSUMSQ charts for the residuals of the fitted AR model. Colombian CPI case.

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