DENSITIES IN GRAPHS AND MATROIDS

A Dissertation

by

LAVANYA KANNAN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2007

Major Subject: Mathematics

DENSITIES IN GRAPHS AND MATROIDS

A Dissertation

by

LAVANYA KANNAN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Arthur M. Hobbs
Committee Members,	Itshak Borosh
	Illya Hicks
	Paula Tretkoff
	Catherine Yan
Head of Department,	Al Boggess

December 2007

Major Subject: Mathematics

ABSTRACT

Densities in Graphs and Matroids. (December 2007)

Lavanya Kannan, B.Sc., Madras University, India;

M.Sc., Indian Institute of Technology, Madras, India

Chair of Advisory Committee: Dr. Arthur M. Hobbs

Certain graphs can be described by the distribution of the edges in its subgraphs. For example, a cycle C is a graph that satisfies $\frac{|E(H)|}{|V(H)|} < \frac{|E(C)|}{|V(C)|} = 1$ for all non-trivial subgraphs of C. Similarly, a tree T is a graph that satisfies $\frac{|E(H)|}{|V(H)|-1} \leq \frac{|E(T)|}{|V(T)|-1} = 1$ for all non-trivial subgraphs of T. In general, a balanced graph G is a graph such that $\frac{|E(H)|}{|V(H)|} \leq \frac{|E(G)|}{|V(G)|}$ and a 1-balanced graph is a graph such that $\frac{|E(H)|}{|V(H)|-1} \leq \frac{|E(G)|}{|V(G)|-1}$ for all non-trivial subgraphs of G. Apart from these, for integers k and l, graphs G that satisfy the property $|E(H)| \leq k|V(H)| - l$ for all non-trivial subgraphs H of G play important roles in defining rigid structures.

This dissertation is a formal study of a class of density functions that extends the above mentioned ideas. For a rational number $r \leq 1$, a graph G is said to be *r-balanced* if and only if for each non-trivial subgraph H of G, we have $\frac{|E(H)|}{|V(H)|-r} \leq \frac{|E(G)|}{|V(G)|-r}$. For $r > 1$, similar definitions are given. Weaker forms of r-balanced graphs are defined and the existence of these graphs is discussed. We also define a class of vulnerability measures on graphs similar to the edge-connectivity of graphs and show how it is related to r-balanced graphs. All these definitions are matroidal and the definitions of r-balanced matroids naturally extend the definitions of r-balanced graphs.

The vulnerability measures in graphs that we define are ranked and are lesser than the edge-connectivity. Due to the relationship of the r-balanced graphs with the vulnerability measures defined in the dissertation, identifying r-balanced graphs and calculating the vulnerability measures in graphs prove to be useful in the area of network survivability. Relationships between the various classes of r-balanced matroids and their weak forms are discussed. For $r \in \{0, 1\}$, we give a method to construct big r-balanced graphs from small r-balanced graphs. This construction is a generalization of the construction of Cartesian product of two graphs. We present an algorithmic solution of the problem of transforming any given graph into a 1-balanced graph on the same number of vertices and edges as the given graph. This result is extended to a density function defined on the power set of any set E via a pair of matroid rank functions defined on the power set of E. Many interesting results may be derived in the future by choosing suitable pairs of matroid rank functions and applying the above result.

THE ETERNAL WEALTH

வெள்ளத்தால் அழியாது வெந்தழலால் வேகாது வேந்த ராலுங்
கொள்ளத்தான் முடியாது கொடுத்தாலும் நிறைவன்றிக் குறைவு றாது
கள்ளர்க்கோ பயமில்லை காவலுக்கோ மிக எளிது கல்வி யென்னும்
உள்ளத்தே பொருளிருக்கப் புறம்பாகப் பொருள் தேடி யுழல்கின் றாரே

- Anonymous

Knowledge cannot be erased by floods, cannot be burnt in fire, cannot be stolen. It can only increase with sharing. Guarding and nurturing it is so easy, for it is within you. Why search for any other wealth than knowledge?

To my parents

ACKNOWLEDGMENTS

I express my sincere gratitude to my advisor, Dr. Hobbs, who has spent countless hours of support, advice, encouragement with patience, wisdom and openness. I shall never forget whatever he has taught me.

I thank the members of my committee: Dr. Yan, Dr. Hicks, Dr. Borosh and Dr. Tretkoff for serving my committee. I thank Dr. Rojas who substituted for Dr. Tretkoff during my preliminary examination. Dr. Fleischner provided his enthusiasm and support by attending my preliminary examination.

I would like to thank my professors at Texas A&M University and also the professors outside the university with whom I interacted. In particular, I learned a lot of material in my subject area by the courses that I took from Dr. Hobbs, Dr. McCammond, Dr. Yan and Dr. Hicks. Dr. Hobbs and Dr. Lai provided me with research ideas. The graduate advisors during my studies, Dr. Schlumprecht and Dr. Lima-Filho, gave excellent course suggestions and helped plan my research curriculum. Monique Stewart has been extremely dynamic in getting all the graduate paperwork done on time. I thank the members of the Milner office for their friendly help on my office needs. Ms. Robin Campbell helped me in many of my questions with Latex.

My sincere thanks to Dr. Frank and Dr. Jordán of EGRES, Eötvös Loránd University, Budapest, Hungary. They served as my mentors during my visit to the group in Fall 2004. Under their guidance, I studied constructive characterizations of edge-connectivity and got an exposure to some problems in rigidity theory.

I would like to express my gratitude to Dr. Choudum, a professor in the mathematics department at Indian Institute of Technology, Madras, India who introduced me to graph theory.

I thank my officemates at Milner 012 who made my stay enjoyable. I thank my officemate Ievgen Bondarenko for answering many questions about typesetting this dissertation.

My parents and sisters have been my constant source of support and inspiration all my life. Their emotions, enthusiasm and experiences have gone a long way in shaping me into what I am today. I greatly value their love and support and would like to cherish them for a long time to come. I sincerely dedicate this dissertation to my parents.

I am extremely grateful to my husband Kasthuri Srinivasan for his love, support and encouragement all through my graduate studies. My parents-in-law have played a valuable part in understanding the importance of my studies. I hold my mother-in-law's motherly support, especially during the last few months, very dearly in my heart.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I
INTRODUCTION

In this dissertation, we study a class of measures of edge distributions in graphs which includes several different density functions. The density functions are defined in terms of forests in graphs, with natural generalizations to matroids. We also present some variations of edge-connectivity of graphs and their extensions to matroids. These density functions have proved to be useful in many problems, some of which are discussed here.

In the next section, we give a brief introduction to matroids. For all graph theoretical definitions, we refer the readers to Diestel's book [17].

1.1. Terminology and challenges

1.1.1. A brief introduction to matroid theory

Here we briefly recall the portion of matroid theory used in this dissertation. For a detailed and more thorough introduction, we refer the readers to Oxley's book [64], whose development we follow.

Matroid theory, an abstraction of the theory of graphs, is one of the most beautiful and deepest branches of combinatorics. Whitney (1935) introduced matroids "as a common generalization of graphs and matrices" [89].

A *matroid* is an ordered pair $(E, \mathfrak{I})$ consisting of a finite set E and a collection $\mathfrak{I}$ of subsets of E satisfying the following three conditions:

(I1) $\phi \in \mathfrak{I}$.

The journal model is *Discrete Applied Mathematics*.

(I2) If $I \in \mathfrak{I}$ and $I' \subseteq I$, then $I' \in \mathfrak{I}$.

(I3) If I_1 and I_2 are in $\mathfrak{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathfrak{I}$.

The elements of $\mathfrak{I}$ are called the *independent sets* of M and M is said to be *a matroid on the set* E. The matroid M can also be equivalently described by other means, namely

(i) the set of *bases* of M, which is the collection of all the maximal independent sets of M,

(ii) the *rank function* ρ defined for each $F \subseteq E$ as the size of a maximal independent set present in F, and

(iii) the *closure function* defined for each $F \subseteq E$ as the maximal set containing F with the same rank as that of F.

Each of these concepts has its own list of axioms that describe a matroid completely. We recall the axioms of the closure function since they are used extensively in the last chapter. Let Cl be the function from 2^E into 2^E defined for all $X \subseteq E$ by

$$Cl(X) = \{x \in E : \rho(X \cup \{x\}) = \rho(X)\}.$$

Cl is called the *closure operator* of M, and satisfies the following properties:

(CL1) If $X \subseteq E$, then $X \subseteq Cl(X)$.

(CL2) If $X \subseteq Y \subseteq E$, then $Cl(X) \subseteq Cl(Y)$.

(CL3) If $X \subseteq E$, then $Cl(Cl(X)) = Cl(X)$.

(CL4) If $X \subseteq E$, $x \in E$, and $y \in Cl(X \cup \{x\}) - Cl(X)$, then $x \in Cl(X \cup \{y\})$.

A set $X \subseteq E$ such that $Cl(X) = X$ is called a *flat* of M.

Cycle matroid of a graph: The most well-known example of a matroid is the one that is defined on the edge set of a graph. Let G be a graph and let E be the edge set of G. Let $\mathfrak{I}$ be the collection of edge sets of all the forests in G. Then $\mathfrak{I}$ forms the collection of all independent sets of a matroid, called the *cycle matroid of* G. The set of all flats of the cycle matroid of G is the set of all edge sets of subgraphs of G whose components are induced.

Operations in matroids: The concepts of planar duals, deletion of edges and contraction of edges in a graph have natural extensions in matroid theory, the definitions of which are listed below. Let M be a matroid on a set E.

(i) A matroid M^* on E whose bases are the set of all complements of the bases of M is called the *dual* of M. The concept of the dual of a matroid generalizes the notion of orthogonality in vector spaces and the concept of a planar dual of a plane graph.

(ii) If $X \subseteq E$, the set of independent sets of M that are contained in $E - X$ form the collection of independent sets of a matroid on $E - X$ called the *deletion of* X *from* M. This matroid is denoted as $M - X$ in the dissertation. If X is a singleton set, $\{e\}$, we simply denote $M - X$ as $M - e$.

(iii) If $T \subseteq E$, then the *contraction of* T *from* M is given by $M/T = (M^* - T)^*$.

1.1.2. Survivable networks

In real-world, a network may denote any communication network, a road network *etc.* Graphs represent these networks in a natural way. In practice, these networks are vulnerable to failures, accidents and attacks. The *survivability of a network* is its capability to fulfill its mission in a timely manner, in the presence of attacks, failures, or accidents.

Density functions involving the number of edges and number of vertices can be used to study the edge distribution in graphs. For example, subgraphs H of a graph G with high values of $\frac{|E(H)|}{|V(H)|}$ may denote highly active areas in the graph due to the presence of a large number of edges for the given sizes of the vertex sets. Identifying these active areas and safeguarding them against attacks and failures has been a recurrent challenge due to increase in the size of the networks and the continuous threat against them.

Another well-studied ratio is $\frac{|E(H)|}{|V(H)|-1}$. This quantity is related to the minimum number of edge-disjoint forests in a graph; later in this chapter, we state a result by Nash-Williams [61] which establishes the relation. The ratio $\frac{|E(H)|}{|V(H)|-1}$ has proved to be very useful in network survivability. We refer the reader [37] for a detailed discussion.

An important concept that is relevant to the topic of survivable networks is the notion of edge-connectivity in graphs since it tells us how vulnerable a graph is under deletion of edges. Gusfield [29] and Cunningham [15] introduced a measure that is related to the edge-connectivity of a graph. This measure (which will be discussed later in the dissertation) is related to the density function $\frac{|E(H)|}{|V(H)|-1}$ (See [37]). We discuss these relations in a more generalized setting in the dissertation.

1.1.3. Electrical networks

An electrical network is an interconnection of electrical network elements called *devices* such as resistances, capacitances, inductances, and voltage and current sources. Each device d_j is represented by an edge j and is associated with a voltage $v(j)$ and a current $i(j)$. Since current has a direction, an electrical network is considered as a directed graph.

The relations between $v(j)$ and $i(j)$ of each device d_j is called the *device characteristic*. The device characteristic tells us how a device functions. Apart from this, an electrical network has some topological constraints that are governed by *Kirchoff's laws*, which are linear constraints in terms of $v(.)$ alone or $i(.)$ alone. The problem of network analysis is to solve the network, *i.e.,* to find the set of all ordered pairs $(v(.), i(.))$ which satisfy the above mentioned device characteristics and topological constraints.

The equations which arise from the Kirchoff's laws are algebraic in nature, and they depend only on the way the devices are interconnected and not on the device characteristics. Device characteristics are used to calculate $v(j)$ if $i(j)$ is known, or vice versa.

Let G be the underlying undirected graph of an electrical network. Notice that G is connected. The number of variables in the system of equations can be reduced by considering the linear dependence of $v(.)$ and $i(.)$. Suppose T is a spanning tree of G. Then for each j, using the Kirchoff's law of voltages, the voltage $v(j)$ can be expressed as a linear combination of the voltage associated with the edges of T. Similarly, the current $i(j)$ can be expressed as a linear combination of the currents associated with the edges not in T.

Let us partition the elements of G into two sets E_1 and E_2. Let G_1 be the graph obtained from G by removing E_2 from G and let G_2 be the graph obtained from G by contracting all the elements of E_1. If T_1 is a spanning forest of G_1 with maximum number of edges possible and T_2 is a spanning tree of G_2, then $T_1 \cup T_2$ is a spanning tree of G. Then the system of voltage linear equations of G_1 can be represented in terms of variables corresponding to the edges of T_1 and the system of current linear equations of G_2 can be represented in terms of edges of $G_2 - T_2$ called the *nullity* of G_2. Thus arises the problem of finding the best possible partition (E_1, E_2) of $E(G)$

that minimizes the number of variables in the system of equations required to solve the network.

Kishi and Kajitani [48] and Ohtsuki, Ishizaki and Watanabe [63] solved this problem by considering the density function $\frac{|E(H)|}{|V(H)|-1}$ for each subgraph H of a graph G. They showed that there exists an unique edge-set E_1 of G such that $G[E_1]$ is the unique maximal graph with density greater than 2 and the pair $(E_1, E - E_1)$ is the required partition of the system that minimizes the number of variables needed to solve the network. We refer the readers to [79], [58] and [71] for more thorough discussions of the above topic.

1.1.4. Biological and social systems

Graphs are used in many areas of biology. They are used to represent a metabolic network which is the complete set of metabolic and physical processes that determine the physiological and biochemical properties of a cell. Graphs are also used to model proteins in the study of protein folding. Social networks are networks that represent social systems where the vertices are individuals or organizations and the edges between them represent different types of relations between them.

A common feature of many biological and social networks is called the "community structure", the fact that the vertices divide into groups, with dense connections within groups and only sparser connections between groups. Communities are of interest because they correspond to functional units, including pathways and cycles in metabolic networks and collections of pages that are related to topics in the web. In recent years, many mathematical tools and computer algorithms have been developed to detect and quantify the community structure in networks. We refer the readers to [62] for a survey of some of these methods.

One method to model communities in a network is by defining a density function on the network, for example the ratio between the number of edges and the number of vertices, and using it to identify communities. Subgraphs with high values of the ratio correspond to communities. Depending on the problem at hand, other density functions can be used.

1.1.5. Rigid systems

Engineering problems such as designing of a bridge, a cell phone tower, *etc.*, involve studying the properties of various materials and designing a rigid backbone for these structures. One such backbone is a structure called a *framework* that consists rods and joints such that each end of each rod is attached to a joint. The rods are assumed to be strong and rigid and the joints are allowed to be arbitrarily rotatable. A framework can be represented by a graph whose edges are rods and whose vertices are joints.

When dealing with frameworks, one must specify the dimension of the space in which the joints are embedded and the rods are allowed to move. For example, a framework in a two dimensional setting is one whose joints are embedded on a plane and whose rods are allowed to move only on the plane. The joints of a framework in dimensions m are said to be in *generic* positions if no two joints coincide, no three joints lie on a straight line, no four joints lie on a plane, $\ldots$, and no m joints lie on a $(m-2)$-dimensional subspace.

If an external force acts on a framework, a deformation might arise. The deformation could perhaps be prevented if a large enough number of rods are placed appropriately between the joints. A framework is said to be *rigid* if it admits no deformations, that is, if all its motions are rigid motions. A graph G is said to be *rigid in dimension* m if and only if there is a rigid framework with the underlying graph G such that the joints have a generic embedding in a space of dimension m.

There are combinatorial characterizations of rigid graphs in dimensions 1 and 2.

In the case of one dimension, the underlying graph of a rigid framework is not necessarily a path and, overlaps of edges are permitted. A framework in a one dimensional space is rigid if and only if its underlying graph is connected. In two dimensions, the underlying graph of a rigid framework need not be planar and, crossing edges are allowed. We use the following notation and state a theorem due to Laman [52], which is the first combinatorial characterization of rigid graphs on a plane.

Notation: For a graph G with vertex set V and for $U \subseteq V$, $E(U)$ denotes the set of all edges in G both of whose end-vertices belong to U.

Theorem I.1 (Laman [52]). *A graph is rigid in dimension* 2 *if and only if it has a spanning subgraph* G *that satisfies the following:* $|E(U)| \leq 2|U| - 3$ *for all* $U \subseteq V(G)$ *with* $|U| \geq 2$*, and* $|E(G)| = 2|V(G)| - 3$.

In the case of dimensions greater than two, characterizing a rigid graph via some combinatorial properties has been a long-standing problem. However, the following has been shown:

Theorem I.2 (G. Laman [52]). *A graph that is rigid in dimension* m *has a spanning subgraph* G *that satisfies the following:* $|E(U)| \leq m|U| - \binom{m+1}{2}$ *for all* $U \subseteq V(G)$ *with* $|U| \geq m$*, and* $|E(G)| = m|V(G)| - \binom{m+1}{2}$.

A simple connected graph that satisfies the conditions of the above theorem is called a *Laman graph* of dimension m. For $m \geq 3$, not all Laman graphs of dimension m are rigid. Figure 1 shows a famous example of a graph, referred as "the double banana", which is a Laman graph of dimension 3 that is not rigid in dimension 3. We refer the readers to the study of rigidity theory given in [27] and [28].

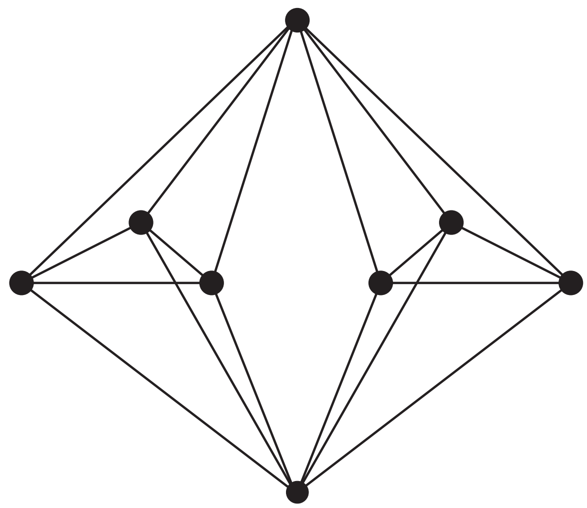

Fig. 1. Double banana

1.1.6. Random graphs

A random graph is obtained by starting with a set of n vertices and adding edges between them at random. Different random graph models produce different probability distributions on graphs. The most commonly studied model is called $G(n, p)$, which forms a graph by including each edge independently with probability p. The theory studies typical properties of random graphs. For example, one might ask, for given values of n and p, what is the probability that $G(n, p)$ is connected? In studying such questions, one often concentrates on the limit behavior of the probabilities as n grows very large.

A graph G is said to be *balanced* if $\frac{|E(H)|}{|V(H)|} \leq \frac{|E(G)|}{|V(G)|}$ for all subgraphs of H of G. Let $b(G) = \frac{|E(G)|}{|V(G)|}$ and $m(G) = \max_{H \subseteq G} b(H)$. The relevance of density functions in the study of random graphs was first identified by Erdős and Rényi [21], where they calculated the probability that a random graph $G(n, p)$ contains a given balanced graph G.

Theorem I.3 (Erdős and Rényi [21]). *If G is a balanced graph, then*

$$\lim_{n\to\infty} Prob(G \subset G(n,p)) = \begin{cases} 0 & if \quad p(n)n^{1/b(G)} \to 0 \quad as \quad n \to \infty, \\ 1 & if \quad p(n)n^{1/b(G)} \to \infty \quad as \quad n \to \infty. \end{cases}$$

Later Bollobás [3] extended this result to any graph.

Theorem I.4 (Bollobás [3]). *If G is a graph, then*

$$\lim_{n\to\infty} Prob(G \subset G(n,p)) = \begin{cases} 0 & if \quad p(n)n^{1/m(G)} \to 0 \quad as \quad n \to \infty, \\ 1 & if \quad p(n)n^{1/m(G)} \to \infty \quad as \quad n \to \infty. \end{cases}$$

Since then, the notion of density appeared in many places in the literature of random graphs in several different contexts. We cite [44], [75] and [54] as examples.

1.2. Some key graph metrics

In this section, we present the two main graph theoretical concepts. This dissertation presents some variations of these concepts and their extensions to matroids.

1.2.1. Average degree

If G is a graph, then the *average degree* of G is defined as the sum of the degrees of the vertices divided by the total number of vertices in G. Since the sum of the degrees of the vertices in G is equal to $2|E(G)|$, the average degree of G is defined as

$$a(G) = \frac{2|E(G)|}{|V(G)|};$$

in this dissertation we use

$$b(G) = \frac{1}{2}a(G)$$

for simplicity.

By calculating $b(H)$ for each subgraph H of G, one can get an idea of which subgraphs of G are densely packed with edges. Thus $b(G)$ is considered as a density measurement in graphs. The collection of values of $b(H)$ of subgraphs H measures how well the edges are distributed inside a graph.

1.2.2. Edge-connectivity

Let k be an integer. A graph G is said to have *edge-connectivity* k if there is a subset F of $E(G)$ of size k such that $G - F$ is disconnected and there is no edge set of size less than k having this property. In other words, G has edge-connectivity k if and only if given any partition P of $V(G)$ of size two, G has at least k *cross-edges*, defined as edges between the partition cells. Edge-connectivity of a graph G is denoted as $\lambda(G)$.

Edge-connectivity is a good measure of vulnerability of a graph since it gives the minimum number of edges to be removed in order to disconnect a graph.

1.3. Balance using graph metrics

In this section, we bring together the measures on graphs that have turned out to be useful in solving the challenges discussed in Section 1.1.

1.3.1. Density functions

An interesting variation of the function $\frac{|E(G)|}{|V(G)|}$ of a graph is the function $\frac{|E(G)|}{|V(G)|-1}$. Since the number $|V(G)|-1$ is the size of a spanning tree in G, the ratio $\frac{|E(G)|}{|V(G)|-1}$ is a bound on the number of edge-disjoint spanning trees in G. This function is well-studied in the literature and has a natural extension to matroids.

A similar function that has a direct connection to the rigidity of bar and joint framework in two dimensions is the function $\frac{|E(G)|}{2|V(G)|-3}$. From Laman's theorem (Theorem I.1), we see that a graph G is rigid in two dimensions if and only if $\frac{|E(H)|}{2|V(H)|-3} \leq 1$

for any non-trivial subgraph H of G and $\frac{|E(G)|}{2|V(G)|-3} = 1$.

Similarly, the function $\frac{|E(G)|}{3|V(G)|-6}$ is related to planar graphs. A simple planar graph G on 3 or more vertices satisfies the property that $\frac{|E(H)|}{3|V(H)|-6} \leq 1$ for any subgraph H of G of order at least 3.

From the above examples we see that some special graphs can be described by ratios involving the number of edges and a linear function on the number of vertices.

1.3.2. An extension of edge-connectivity

The following theorem by Menger is a basic result about edge-connectivity of a graph.

Theorem I.5 (Menger [56]). *For a positive integer k, the edge-connectivity of a graph G is k if and only if every pair of vertices in G is joined by k or more edge-disjoint paths and some pair is joined by exactly k edge-disjoint paths.*

The following theorem extends the idea of edge-connectivity to a characterization of graphs that have k edge-disjoint spanning trees.

Theorem I.6 (Tutte [83], Nash-Williams [60]). *For a positive integer k, a connected graph $G = (V, E)$ contains k edge-disjoint spanning trees if and only if for every partition P of V, the graph G has at least $k(|P| - 1)$ cross-edges.*

1.3.3. (k, l)-sparse graphs

We continue the theme of graph properties that are characterized by bounding functions on the number of edges in each of their subgraphs. For instance, a graph G is a Hamiltonian cycle if and only if $\frac{|E(G)|}{|V(G)|} = 1$ and for each non-trivial subgraph H of G, we have $\frac{|E(H)|}{|V(H)|} < 1$. A non-trivial tree T is characterized by $\frac{|E(T)|}{|V(T)|-1} = 1$ and each induced subgraph F of T with at least two vertices satisfies $\frac{|E(F)|}{|V(F)|-1} \leq 1$. More generally, we have

Theorem I.7 (Nash-Williams [61]). *For a positive integer k, a graph $G = (V, E)$ can be partitioned into k forests if and only if $\frac{|E(U)|}{|U|-1} \leq k$ for every non-empty set $U \subseteq V$.*

Theorem I.7 places a restriction on the number of edges that can be induced by a vertex set of any given size. Note the connection between Theorem I.6 and Theorem I.7: Theorem I.6 requires an lower bound on the number of edges in G, while Theorem I.7 requires a upper bound on the number of edges in G. These theorems are shown by Catlin *et al.* [10] to be related to the density function $\frac{|E(G)|}{|V(G)|-1}$.

To generalize the bound of Theorem I.7, Whiteley in [87] and [88] introduced the following definition: Let k and l be integers. For $l < k$, a graph G which is allowed to have loops is said to be (k, l)-*sparse* if and only if for every non-empty subset $U \subseteq V$, $\frac{|E(U)|}{k|U|-l} \leq 1$. For $k \leq l < 2k$, a loopless graph G is said to be (k, l)-*sparse* if and only if for every subset $U \subseteq V$ with $|U| \geq 2$, we have $\frac{|E(U)|}{k|U|-l} \leq 1$ (or, $\frac{|E(U)|}{|U|-\frac{l}{k}} \leq k$). We call G a *tight* (k, l)-*sparse graph* if G is (k, l)-sparse and $\frac{|E(G)|}{k|V(G)|-l} = 1$ (or, $\frac{|E(U)|}{|U|-\frac{l}{k}} = k$).

Note that if k is an integer, a connected tight (k, k)-sparse graph satisfies the sufficiency conditions of both Theorem I.6 and Theorem I.7. Thus a graph is a tight (k, k)-sparse graph if and only if it is a union of k edge-disjoint spanning trees.

The total number of edges in each subgraph of a tight (k, l)-sparse graph G is bounded while the number of edges in G is maximized. This property is seen in some types of graphs. By the definition of $(2, 3)$-sparse graphs, note that a Laman graph of dimension 2 is a tight $(2, 3)$-sparse graph.

Another place where the above-mentioned property is seen is in the case of planar triangulations. Planar graphs are a class of simple graphs that satisfy $\frac{|E(H)|}{3|V(H)|-6} \leq 1$ (or, $\frac{|E(H)|}{|V(H)|-2} \leq 3$) for all subgraphs H of order at least three. A *plane triangulation* is a plane graph that additionally satisfies $\frac{|E(G)|}{3|V(G)|-6} = 1$ (or, $\frac{|E(G)|}{|V(G)|-2} = 3$). Note that the condition $\frac{|E(H)|}{|V(H)|-2} \leq 3$ can be checked only for subgraphs H with at least three vertices.

1.4. More definitions

Deciding if the edges of a graph are distributed evenly inside the graph may be done by two methods. The first is by defining a density measure applicable to all subgraphs of a graph and checking if the density of each subgraph is not more than that of the whole graph. Given a function f from the subgraphs H of G to the real numbers, we say G is *balanced with respect to* f if and only if $f(H) \leq f(G)$ for all subgraphs H of G. In this section, we define a class of functions f which are related to the average degree of graphs.

The second method is by checking if the graph has certain high edge-connectivity so that the graph does not have unacceptable weakness. We discuss some variations of edge-connectivity.

1.4.1. r-balanced graphs and matroids

Owing to the many kinds of problems that could demand different kinds of density functions, we formally define a broad class of density functions. The reasons for the definition of these density functions are two-fold:

1. Like the study of (k, l)-sparse graphs, we are interested in studying the distribution of edges of a graph in relation to a linear function on the number of vertices in the graph. There is an obvious limitation in the study of (k, l)-sparse graphs. Since an edge with its two end vertices forms a subgraph, any (k, l)-sparse graph satisfies $1 \leq 2k - l$ or $l < 2k$. Thus, there are no (k, l)-sparse graphs for $l \geq 2k$ and so there is a need for an extension of the concept of (k, l)-sparse graph for $l \geq 2k$. One way to address this situation is to avoid the bounding condition for subgraphs of small sizes, as in the case of the bounding condition in simple planar graphs.

2. As we saw in some applications, there are many problems which are expressed in terms of both graphs and matroids. Hence we extend the study of density to matroids. Matroids are an abstraction of graphs that do not have the notion of vertices. However, matroids may be thought of as made up of edges. To address the concept of vertices in matroids, we depend on the rank function of a matroid.

Motivated by the concept of density in graphs and some of their extensions to matroids, we provide a class of density functions in matroids and graphs which include the various classes of (k, l)-sparse graphs.

Let $\omega(G)$ be the number of components of graph G. A *maximal forest* in G is a forest with the maximum number of edges possible. The number $|V(G)| - \omega(G)$ is called the *rank* $\rho(G)$ of G and it is the size of any maximal forest in G. The rank function in matroids is a natural generalization of the rank function in graphs. Suppose a graph G has two components G_1 and G_2. Let G' be a graph obtained by identifying a vertex from G_1 and a vertex from G_2. Thus G' is a connected graph. Also, the number of vertices of G' is one less than that of G and the number of components of G' is one less than that of G. Thus the ranks of G and G' are the same. In fact, the cycle matroids of G and G' are the same. In general, the cycle matroid of any graph G is a cycle matroid of a connected graph formed by identifying one vertex from each component of G as a single vertex. Thus the rank of any graph is the same as the rank of the cycle matroid of the connected graph described above.

For a connected graph G, the rank of the cycle matroid on G is $|V(G)| - 1$. Thus a linear relation in terms of the rank of a matroid may be considered as an extension to matroids of a linear relation in terms of the number of vertices in a graph. Thus, by the definition of the rank of a connected graph, we consider $\rho(M)$ to be equivalent to $|V(G)| - 1$.

Let M be a matroid on a set E with rank function ρ. For a rational number r, we define

$$d_r(F) = \frac{|F|}{\rho(F) - (r-1)}$$

for all subsets F of E such that $\rho(F) > r - 1$. We denote $d_r(E)$ as $d_r(M)$. The matroid M is said to be *r-balanced* if $\rho(M) > r-1$ and $d_r(F) \leq d_r(E)$ for all subsets F of E such that $\rho(F) > r - 1$. A graph G is called *r-balanced* if the cycle matroid of G is r-balanced. For the graph G, we denote $d_r(E(G))$ as $d_r(G)$. Note that for a connected graph G, we have $d_r(G) = \frac{|E(G)|}{|V(G)|-r}$.

1.4.2. (r, s)-balanced matroids

For some values of r, there may not exist an r-balanced graph, especially with small values of $d_r(G)$. For example, there is no 1.75-balanced graph G with $d_{1.75}(G) < 4$. (Suppose G is a graph with $d_{1.75}(G) < 4$. Then for any $e \in E(G)$, we have $d_{1.75}(G[e]) = \frac{1}{2-1.75} = \frac{1}{0.25} = 4 > d_{1.75}(G)$. Thus G is not 1.75-balanced.) Therefore in the notion of r-balanced matroids, we introduce a parameter s and waive the condition $d_r(F) \leq d_r(E)$ for subsets F of rank less than s.

For an integer s such that $s > r - 1$, M is said to be *(r, s)-balanced* if $\rho(M) \geq s$ and $d_r(F) \leq d_r(E)$ for all subsets F of E such that $\rho(F) \geq s$. A graph G is called *(r, s)-balanced* if the cycle matroid of G is (r, s)-balanced. Note that in the above definition, we use the letters "r" and "s" partly to avoid confusing (r, s)-balanced graphs with (k, l)-sparse graphs and partly because r is a rational number and is not necessarily an integer.

1.4.3. Functions extending the idea of edge-connectivity

Here we extend the concept of edge-connectivity. We define the following for a graph G: For each integer $s \geq 0$, let

$$\mu^s(G) := \min_{X \subseteq E(G)} \left\{ \frac{|X|}{\omega(G-X) - \omega(G)} \middle| \omega(G-X) > \omega(G), |V(G)| - \omega(G-X) \geq s \right\}.$$

For a connected graph G, if $s = |V(G)| - 2$ then $\mu^s(G)$ becomes the edge-connectivity of G. The case when $s = 0$, is shown in [10] to be related to Theorem I.6 and $d_1(G)$.

For a matroid M on a non-empty set E, let

$$\mu^s(M) := \min \left\{ \frac{|F|}{\rho(M) - \rho(M-F)} \middle| F \subset E, s \leq \rho(M-F) < \rho(M) \right\}.$$

Suppose M is a cycle matroid of a graph G. Then $\rho(G) = |V(G)| - \omega(G)$ and, for $X \subseteq E(G)$, $\rho(G-X) = |V(G)| - \omega(G-X)$. In the latter formula, we count all the isolated vertices of G that are created by the removal of X from G. Thus $\mu^s(M)$ coincides with the definition of $\mu^s(G)$.

Note that if $F = E$, we have $\frac{|F|}{\rho(E)-\rho(E-F)} = \frac{|E|}{\rho(E)} = d_1(M)$. In Chapter V, we show the connection between $\mu^s(M)$ and $d_r(M)$ for suitable values of r and s.

1.5. Historical background

In 1960, Erdős and Rényi [21] introduced the concept of balanced graphs. In 1961, Tutte [83] and Nash-Williams [60] independently published Theorem I.6. This was followed by Nash-Williams [61](1964) presenting Theorem I.7. Gusfield [29](1983) observed that Theorem I.6 implies that each $2k$-edge-connected graph has k edge-disjoint spanning trees. Theorems I.6 and I.7 are matroidal and their matroidal versions are due to Edmonds [19] and [18](1965). Easier proofs of the above results on matroids were later given by Harary and Welsh [33](1969). This led to the introduction of the quantity $d_1(M)$ for matroids by Kelly and Oxley [46](1982).

The density function $d_1(M)$ was studied in a different setting by a generalized concept called the "principal partitions of a set". Principal partitions of a set are partitions on a set with respect to two submodular functions. The study was initiated by Kishi and Kajitani [48] in 1968, while they were studying the topological degrees of freedom of electrical networks. We give more details in Chapter VI.

In 1970, Laman [52] noticed the first combinatorial characterization of 2-dimensional rigid graphs, given by Theorem I.1.

The number $\mu^1(G)$ is called the *strength* of a matroid and was first introduced by Gusfield [29] in 1983 in reciprocal form. The quantity was generalized to matroids by Cunnigham [15] in 1985 and by Catlin, Grossman, Hobbs and Lai [10] in 1992. It is shown in [10] that the term $\mu^1(M)$ related to $d_1(M)$ as follows: A matroid is 1-balanced if and only if $\mu^1(M) = d_1(M)$.

Whiteley in [87](1988) and [88](1996) defined (k, l)-sparse graphs and showed their relevance in rigidity theory. He also showed that, for each $0 \leq l < 2k$, the collection of all (k, l)-sparse subgraphs of a graph G forms the collection of independent sets of a matroid on $E(G)$.

There has been recent activity on (k, l)-sparse graphs. In 1996, Albertson and Haas [1] showed the existence of (k, l)-sparse graphs for various values of k and l. A method named the "pebble game algorithm" was introduced in [43](1997) to identify $(2, 3)$-sparse graphs. This algorithm was simplified in [2](2003). In [53](2005), a family of pebble game algorithms that identify the (k, l)-sparse graphs for $0 \leq l < 2k$ was given. An application of sparse graphs to protein folding appeared in [16](2001).

A constructive characterization of a graph property is a building procedure consisting of simple operations such that the graphs obtained from some specified initial graph or graphs by these operations are precisely those having the property. This kind of characterization is common in graph theory. For example, a graph is connected if and only if it can be obtained from a vertex by zero or more applications of the operation: add a new edge connecting an existing vertex with either an existing vertex or a new one. Results on ear-decomposition of 2-edge connected graphs [86] and Tutte's characterization of 3-connected graphs [84](1961) are also examples of constructive characterizations.

Several constructive characterizations of (k, l)-sparse graphs have appeared in the literature and constructions are found in the literature. For some examples, we cite the papers by Henneberg [35](1911), Laman [52](1970), Tay [81](1991), Haas [32](2002), Frank and Szegő [23](2003), Szegő [80](2006) and Fekete, Zsolt and Szegő [22](2007).

1.6. Overview

In the next two chapters, we recall some density functions that are already defined in the literature. These functions are the special cases of r-balanced graphs for the values 0 and 1. We provide a generalized Cartesian product of graphs which generates large balanced graphs and 1-balanced graphs from smaller ones.

In Chapter IV, we provide a method of transforming a graph into a 1-balanced graph. This is of high practical importance. This result raises the question of whether the same result is true for other values of r.

In Chapter V, we revisit (r, s)-balanced graphs and address the following types of questions: For what values of r and s do there exist (r, s)-balanced graphs? How are the (r, s)-balanced graphs related to (k, l)-sparse graphs? What are the relationships between the various classes of (r, s)-balanced graphs? Apart from these, we also give some constructions and applications of (r, s)-balanced graphs. The connection between (k, l)-sparse graphs and (r, s)-balanced graph is explored in Chapter V.

In Chapter VI, our result in Chapter IV is extended to principal partitions of a set.

1.7. Preliminaries

Here we provide some basic lemmas which will be used in the rest of the dissertation.

Lemma I.8 (Hardy, Littlewood, Polya [34]). *Let* $p_1/q_1, p_2/q_2, \ldots, p_k/q_k$ *be fractions in which* p_i *is a real number and* q_i *is a positive real number for each* $i \in \{1, 2, \ldots, k\}$. *Then*

$$\min_{1\le i\le k} \frac{p_i}{q_i} \le \frac{p_1 + p_2 + \cdots + p_k}{q_1 + q_2 + \cdots + q_k} \le \max_{1\le i\le k} \frac{p_i}{q_i}$$

with equality on both sides if and only if the fractions p_i/q_i*;* $i \in \{1, 2, \ldots, k\}$ *are all the same.*

The following are some elementary results involving real numbers.

Lemma I.9. *If* a *and* b *are two rational numbers such that* $a < b$*, then any rational number* $r \in [a, b]$ *can be expressed as* $r = \frac{k-l}{k}a + \frac{l}{k}b$ *where* l *and* k *are non-negative integers with* $l \le k$.

Lemma I.10. *If* a*,* b *and* x *are positive real numbers such that* $\frac{a-x}{b-1} \le \frac{a}{b}$*, then* $x \ge \frac{a}{b}$.

We introduce the following notation for convenience.

Notation: If G is a graph and s a positive integer, G^s denotes the graph obtained from G by replacing each edge by s parallel edges.

CHAPTER II

BALANCED GRAPHS

The average degree of the vertices of a graph is perhaps a first natural quantity of measurement to decide if the edges of the graph are distributed nicely in the graph. In this chapter, we discuss a density function that is directly related to average degree.

2.1. Definition and examples

Let G be a graph. The number $\frac{1}{|V(G)|}\sum_{v\in V(G)} d(v) = 2\frac{|E(G)|}{|V(G)|}$ is the *average degree* of G. The average degree is a global quantity measured locally by the vertex degrees. Let

$$b(H) = \frac{|E(H)|}{|V(H)|}$$

for any non-empty subgraph H of graph G. Thus $2b(G)$ is the average degree of G. G is said to be *balanced* if $b(H) \leq b(G)$ for all non-empty subgraphs H and *strictly balanced* if $b(H) < b(G)$ for all non-empty proper subgraphs H. The definition of balanced graphs differs from the definition of 0-balanced graphs, as will be shown in the next section.

Cycles, trees and complete k-partite graphs for any positive integer k are strictly balanced. Regular connected graphs are balanced. Figure 2 shows a graph that is balanced, but not strictly balanced.

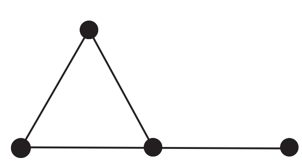

Fig. 2. A balanced graph that is not strictly balanced

2.1.1. Balanced graphs and 0-balanced graphs

Recall that a graph G is said to be 0-balanced if $\frac{|E(H)|}{\rho(H)+1} \leq \frac{|E(G)|}{\rho(G)+1}$ for all subgraphs H of G. Also, $d_0(G) = \frac{|E(G)|}{\rho(G)+1}$ for any graph G. For a connected graph G, we have $\rho(G) + 1 = |V(G)|$ and for any subgraph H of G, we have $\rho(H) + 1 \leq |V(H)|$. We have the following result:

Theorem II.1. *If a connected graph G is 0-balanced, then G is balanced.*

Proof. Since G is connected, we have $\rho(G) + 1 = |V(G)|$. Thus $d_0(G) = \frac{|E(G)|}{|V(G)|}$. Let H be a subgraph of G. Since G is 0-balanced,

$$d_0(H) = \frac{|E(H)|}{\rho(H)+1} \leq d_0(G).$$

But $\rho(H) + 1 \leq |V(H)|$. Thus,

$$\frac{|E(H)|}{|V(H)|} \leq \frac{|E(H)|}{\rho(H)+1} = d_0(H) \leq d_0(G) = \frac{|E(G)|}{|V(G)|}.$$

Therefore, G is balanced. □

However, not all balanced graphs are 0-balanced as Figure 3 shows.

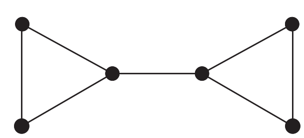

Fig. 3. Example of a balanced graph that is not 0-balanced

2.1.2. (k, l)-sparse graphs and balanced graphs

For integers k and l, recall from Section 1.3.3 that a graph G is (k, l) sparse if and only if every subgraph H of G satisfies $\frac{|E(H)|}{|V(H)|-\frac{l}{k}} \leq k$. A (k, l)-sparse graph is tight if

$\frac{|E(G)|}{|V(G)|-\frac{l}{k}} = k$. Thus a tight $(k, 0)$-sparse graph satisfies $\frac{|E(H)|}{|V(H)|} \leq k = \frac{|E(G)|}{|V(G)|}$. Therefore, a tight $(k, 0)$-sparse graph is balanced.

Suppose G is $(k, 0)$-sparse for some integer k. If l is an integer such that $l \leq 0$, then $|E(H)| \leq k|V(H)| \leq k|V(H)| - l$ and hence G is (k, l)-sparse. But the converse is not true. Figure 4 shows an example of a graph that is $(1, -1)$-sparse but not $(1, 0)$-sparse.

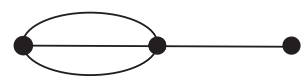

Fig. 4. Example of a $(1, -1)$-sparse graph that is not $(1, 0)$-sparse

2.1.3. Relationships between various notions of balanced graphs

We have the following containment relationship for connected graphs as shown in Figure 5.

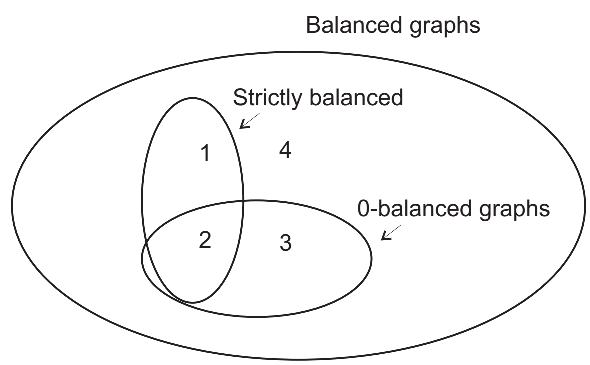

Fig. 5. Relationships between various notions of balanced graphs

The graphs in Figures 3, 6, 7 and 8 are examples of graphs that show that the containment of sets shown in Figure 5 are strict. Table I summarizes the list of examples.

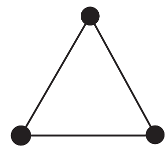

Fig. 6. Example of a strictly balanced graph that is also 0-balanced

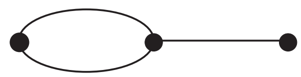

Fig. 7. Example of a 0-balanced graph that is not strictly balanced

2.2. Earlier results

Erdős and Rényi [21] introduced balanced graphs in their work on random graphs. Since then, balanced graphs and strictly balanced graphs have been widely studied in the context of random graphs; for example, see [44], [75], [4], [72], [73], [78], [74], [54], [55] and [3]. Győri, Rothchild and Runcińki [31] proved that every graph G is contained in a balanced graph G' such that $b(G') = \max\{b(H) : H \subseteq G\}$. They also showed that if m and n are positive integers, then there exists a balanced graph on n vertices and m edges. Veerapandiyan and Arumugam [85] proved that for $l \geq 0$,

Table I. Summary of examples of different types of balanced graphs

Set in Figure 5	Example
1	Figure 3
2	Figure 6
3	Figure 7
4	Figure 8

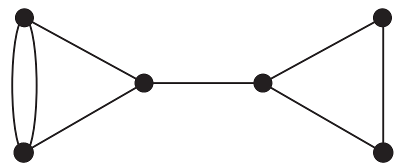

Fig. 8. Example of a balanced graph that is neither 0-balanced nor strictly balanced

every tight (k, l)-sparse graph is balanced[1]. They also showed that all maximal planar graphs and all maximal outer-planar graphs are balanced.

An algorithm using network flows to check if a graph is balanced or not was given by Picard and Queyranne [70]. This was later analyzed and simplified by Penrice [69]. The definition of balanced graphs was generalized by Zhang, Sun and Li [90] to graphs whose edges and vertices have weights. They also extended the algorithm by Penrice to this generalized definition.

A similar density function is related to the edge coloring in graphs. See [77], [25] and [26] for a discussion of the relation between $\chi'(G)$ and the quantity

$$\max_{H \subseteq G} \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil$$

of a graph G.

2.3. Some preliminary results on balanced graphs

The following lemma is useful.

Lemma II.2. *Let G be a non-trivial graph. Then G has a* **connected** *induced subgraph H such that $b(H)$ is the maximum for b over all subgraphs of G.*

Proof. Suppose a subgraph H achieves the maximum value for b over all subgraphs of G. Clearly we may suppose that H is induced and has no isolated vertices. If H is

[1]Their paper claims that the result is true for all values of l, but the result is not true for $l < 0$, as Figure 4 shows.

not connected, let it have components $C_1, \ldots, C_k$ with component C_i having a_i edges and $b_i \geq 2$ vertices for $i = 1, 2, \ldots, k$. Then $b(H) = \frac{\sum a_i}{\sum b_i} \leq \max \frac{a_i}{b_i}$ by Lemma I.8, so there is a connected induced subgraph of G which achieves the maximum value for b. □

2.4. A characterization of balanced graphs

In this section, we see a new characterization of balanced graphs which is used in the next section to construct big balanced graphs. The characterization is also used to show that the Cartesian product of balanced graphs is balanced.

The next theorem is our new characterization of balanced graphs. The characterization involves arbitrary non-negative vertex weights[2]. The result is used in Chapter III.

Theorem II.3 (Kannan in [38]). *Let L be a graph on m vertices $V = \{v_1, \cdots, v_m\}$. Let α be any non-negative integer valued function on the vertex set V. Let*

$$N_\alpha := \sum_{v_i v_j \in E(L)} \Big[\min(\alpha(v_i), \alpha(v_j)) - \frac{1}{m}\sum_{r=1}^{m} \alpha(v_r)\Big].$$

Then L is balanced if and only if $N_\alpha \leq 0$ for all α, and L is strictly balanced if and only if $N_\alpha < 0$ for all non-constant α.

Proof. **(Sufficiency of L balanced)** For a contradiction, suppose L is balanced while there is a non-negative, integer-valued function α on $V(L)$ with $N_\alpha > 0$. Choose α_0 such that $N_{\alpha_0} > 0$ and $s = \max_{1 \leq i \leq m} \alpha_0(v_i)$ is as small as possible. If α_0 were constant on $\{v_1, \ldots, v_m\}$, then $N_{\alpha_0} = 0$. Hence, there is a $j \in \{1, 2, \ldots, m\}$ such that $\alpha_0(v_j) < s$. Then $s \geq 1$ since $\alpha_0(v_j) \geq 0$.

[2]The origin of N_α and the relation of L. Kannan to this origin is described in Chapter III.

Let $S := \{v_i : \alpha_0(v_i) = s\}$. By the definition of s and j, $S \notin \{\emptyset, V\}$. Consider the function α_0' defined by $\alpha_0'(v_i) = \alpha_0(v_i)$ if $v_i \notin S$ and $\alpha_0(v_i) - 1$ if $v_i \in S$. Thus $\max_{1 \le i \le m} \alpha_0(v_i) < s$.

We claim that $N_{\alpha_0'} \ge N_{\alpha_0}$.

Let $L' := L[S]$, and denote $m' := |V(L')| = |S|$ and $\ell' := |E(L')|$. Then

$$\begin{aligned}\frac{1}{m}\sum_{r=1}^{m}\alpha_0'(v_r) &= \frac{1}{m}\Bigg[\sum_{r:v_r\notin S}\alpha_0(v_r) + \sum_{r:v_r\in S}(\alpha_0(v_r)-1)\Bigg]\\ &= \frac{1}{m}\Bigg[\sum_{r:v_r\notin S}\alpha_0(v_r) + \sum_{r:v_r\in S}\alpha_0(v_r) - m'\Bigg]\\ &= \frac{1}{m}\sum_{r=1}^{m}\alpha_0(v_r) - \frac{m'}{m}.\end{aligned}$$

Suppose $v_iv_j \in E(L')$. Then $\min(\alpha_0'(v_i), \alpha_0'(v_j)) = \min(\alpha_0(v_i), \alpha_0(v_j)) - 1$. Therefore, in this case we have

$$\min(\alpha_0'(v_i), \alpha_0'(v_j)) - \frac{1}{m}\sum_{r=1}^{m}\alpha_0'(v_r) = \min(\alpha_0(v_i), \alpha_0(v_j)) - 1 - \frac{1}{m}\sum_{r=1}^{m}\alpha_0(v_r) + \frac{m'}{m}.$$

If $v_iv_j \notin E(L')$, then $\min(\alpha_0'(v_i), \alpha_0'(v_j)) = \min(\alpha_0(v_i), \alpha_0(v_j))$. This is true even if, for example, $v_i \in S$ and $v_j \notin S$, for then $\alpha_0'(v_i) = s-1$ and $\alpha_0'(v_j) = \alpha_0(v_j) \le s-1$, and so $\min(\alpha_0'(v_i), \alpha_0'(v_j)) = \min(s-1, \alpha_0(v_j)) = \alpha_0(v_j) = \min(\alpha_0(v_i), \alpha_0(v_j))$. Thus we have

$$\min(\alpha_0'(v_i), \alpha_0'(v_j)) - \frac{1}{m}\sum_{r=1}^{m}\alpha_0'(v_r) = \min(\alpha_0(v_i), \alpha_0(v_j)) - \frac{1}{m}\sum_{r=1}^{m}\alpha_0(v_r) + \frac{m'}{m}.$$

Therefore,

$$N_{\alpha_0'} - N_{\alpha_0} = \ell'(-1 + \frac{m'}{m}) + (\ell - \ell')\frac{m'}{m} = -\ell' + \frac{\ell m'}{m} = m'(-\frac{\ell'}{m'} + \frac{\ell}{m}) \ge 0$$

since $\frac{\ell'}{m'} = d(L') \le d(L) = \frac{\ell}{m}$ either because L is balanced or because $\ell' = 0$. Hence the claim.

But $N_{\alpha_0'} \geq N_{\alpha_0} > 0$ is a contradiction to the minimality of s by the definition of $N_{\alpha_0'}$. The contradiction proves sufficiency.

(Necessity of L balanced) Suppose $N_\alpha \leq 0$ for all labellings α. Let L' be any non-trivial vertex-induced subgraph of L, and suppose L' has m' vertices and ℓ' edges. Define α on $V(L)$ by letting $\alpha(v) = 1$ if $v \in V(L')$ and 0 if $v \notin V(L')$. Then

$$\frac{1}{m}\sum_{r=1}^{m} \alpha(v_r)) = \frac{m'}{m},$$

and

$$\begin{aligned} 0 &\geq N_\alpha \\ &= \sum_{v_iv_j \in E(L')} (1 - \frac{m'}{m}) + \sum_{v_iv_j \notin E(L')} (-\frac{m'}{m}) \\ &= \ell' - \frac{m'\ell'}{m} - \frac{m'\ell}{m} + \frac{m'\ell'}{m} \\ &= \ell' - \frac{m'\ell}{m} \\ &= m'(\frac{\ell'}{m'} - \frac{\ell}{m}). \end{aligned}$$

Hence we have $\frac{\ell'}{m'} \leq \frac{\ell}{m}$ (*i.e.*, $d(L') \leq d(L)$), so L is balanced.

(Sufficiency of L strictly balanced) We note that $N_\alpha = 0$ for all constant labellings α. For a contradiction, suppose L is strictly balanced while there is a non-constant, non-negative, integer-valued function α on $V(L)$ with $N_\alpha \geq 0$. Choose non-constant α_0 such that $N_{\alpha_0} \geq 0$ and $s = \max_{1 \leq i \leq m} \alpha_0(v_i)$ is as small as possible. Since α_0 is not constant, there is a $j \in \{1, 2, \ldots, m\}$ such that $\alpha_0(v_j) < s$. Then the integer $s \geq 1$ since $\alpha_0(v_j) \geq 0$.

Let $S := \{v_i : \alpha_0(v_i) = s\}$; by definition of s and j, $S \notin \{\emptyset, V\}$. Consider the function α_0' defined by $\alpha_0'(v_i) = \alpha_0(v_i)$ if $v_i \notin S$ and $\alpha_0(v_i) - 1$ if $v_i \in S$. Thus

$\max_{1\le i\le m}\alpha_0(v_i) < s$.

We claim that $N_{\alpha_0'} > N_{\alpha_0}$, and thus α_0' is non-constant.

Let $L' := L[S]$, and denote $m' := |V(L')| = |S|$ and $\ell' := |E(L')|$. Exactly as in the case of non-strictly balanced,

$$N_{\alpha_0'} - N_{\alpha_0} = m'(-\frac{\ell'}{m'} + \frac{\ell}{m}).$$

But this is greater than zero either because L is strictly balanced and so $\frac{\ell'}{m'} = d(L') < d(L) = \frac{\ell}{m}$ or because $\ell' = 0$.

Thus $N_{\alpha_0'} > N_{\alpha_0} \geq 0$, so α_0' is not constant. This is a contradiction to the choice of α_0 and the minimality of s. The contradiction proves sufficiency.

(Necessity of L strictly balanced) Suppose $N_\alpha < 0$ for all non-constant labellings α. Let L' be any non-trivial vertex-induced subgraph of L, $L' \neq L$, and suppose L' has m' vertices and ℓ' edges. Define α on $V(L)$ by letting $\alpha(v) = 1$ if $v \in V(L')$ and 0 if $v \notin V(L')$. Then α is not constant, so

$$\frac{1}{m}\sum_{r=1}^{m}\alpha(v_r)) = \frac{m'}{m},$$

and

$$\begin{aligned}
0 &> N_\alpha \\
&= \sum_{v_iv_j\in E(L')}(1-\frac{m'}{m}) + \sum_{v_iv_j\notin E(L')}(-\frac{m'}{m}) \\
&= \ell' - \frac{m'\ell'}{m} - \frac{m'\ell}{m} + \frac{m'\ell'}{m} \\
&= \ell' - \frac{m'\ell}{m} \\
&= m'(\frac{\ell'}{m'} - \frac{\ell}{m}).
\end{aligned}$$

Since $m' > 0$, we have $\frac{\ell'}{m'} < \frac{\ell}{m}$ (*i.e.*, $d(L') < d(L)$), so L is strictly balanced. □

2.5. Generalized Cartesian products

In this section, we prove that Cartesian products of balanced graphs are balanced. In fact, we will prove an extension of the result. We present a construction of big balanced graphs from small ones by joining some additional edges. The construction resembles that of internet graphs and hence the construction would prove to be useful in practice.

The following definition is given in [38]. Throughout this section, let L be a connected graph with ℓ edges and m vertices, and let the vertices be labeled $v_1, v_2, \ldots, v_m$. Label the edges of L as $e_1, e_2, \ldots, e_\ell$. Let $G_1, G_2, \ldots, G_m$ be vertex-disjoint connected graphs, each having n vertices and e edges. Let k be a positive integer. Let $B_1, B_2, \ldots, B_\ell$ be k-regular bipartite graphs such that, if edge e_i of L joins vertices v_r and v_s, then the two sides of B_i are the vertex sets of G_r and G_s. Let $A_k = A_k(G_1, \ldots, G_m; L) = \left(\bigcup_{i=1}^{m} G_i\right) \cup \left(\bigcup_{i=1}^{\ell} B_i\right)$. When the value of k is already known, we may use $A = A(G_1, \ldots, G_m; L)$ (omitting the subscript). Then A is called a *generalized Cartesian product*. Note that the definition of A is ambiguous, since there are many possible k-regular bipartite graphs B_i. We allow this ambiguity because the choices of the B_i make no difference to our results. Also note that if G and L are graphs, then the Cartesian product $G \times L$ is a generalized Cartesian product with $G_i = G$ for $i = 1, 2, \ldots, m$ and $k = 1$. Figure 9 shows graphs L and G_1, G_2, G_3 and a generalized Cartesian product $A_2(G_1, G_2, G_3; L)$.

Let H be a subgraph of A, and suppose H includes one or more vertices of $G_{i_1}, \ldots, G_{i_{\ell'}}$ and no others of the G_i. Let L' be the subgraph of L generated by the vertices $v_{i_1}, \ldots, v_{i_{\ell'}}$. Then we say that *$L'$ is induced by H*.

The following theorem shows how one may construct big balanced graphs from

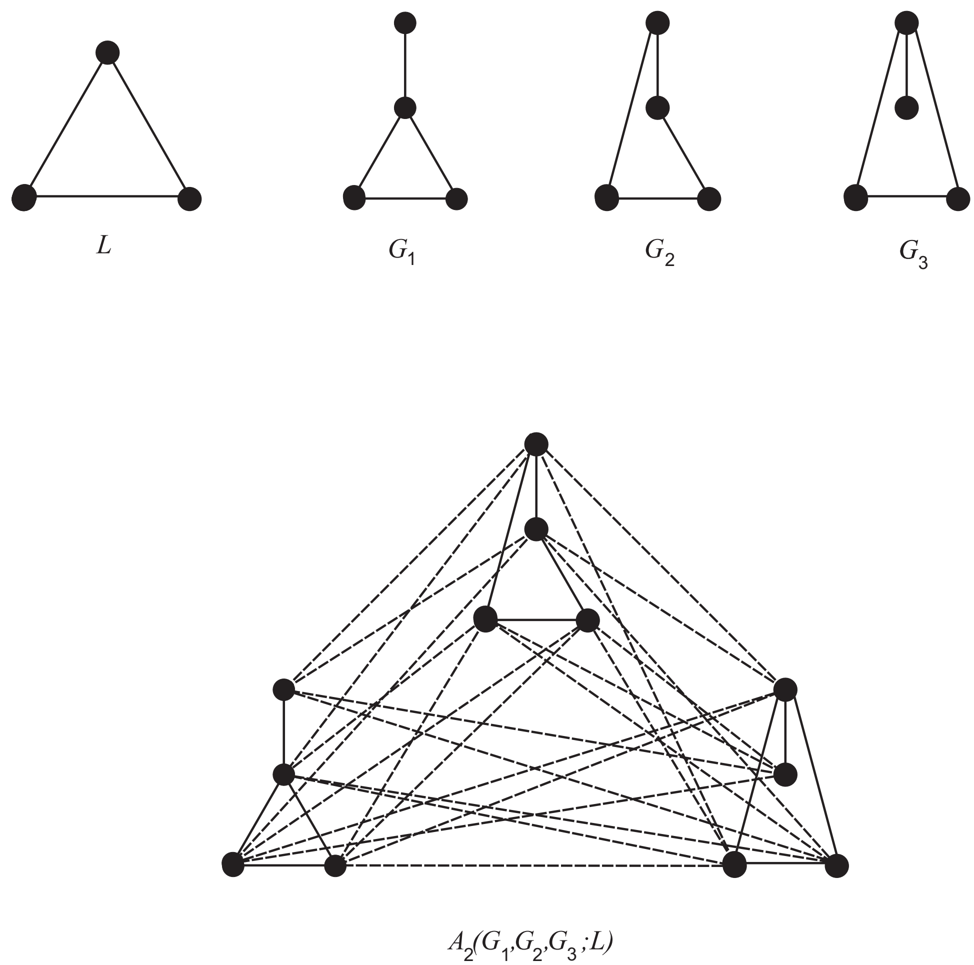

Fig. 9. Example of a generalized Cartesian product

small ones.

Theorem II.4. *Let L be a graph on m vertices. Let k be any positive integer and let $G_1, \ldots, G_m$ be balanced graphs. Then $A = A_k(G_1, \ldots, G_m; L)$ is balanced if and only if L is balanced.*

Proof.

$$b(A) = \frac{nkl + me}{mn} = \frac{kl}{m} + \frac{e}{n}. \tag{2.1}$$

Note that for $i = 1, \cdots, m$,

$$b(G_i) < b(A). \tag{2.2}$$

(Necessity) Suppose A is balanced. Let $V(L) = \{v_1, v_2, \cdots, v_m\}$. Let L' be any subgraph of L, and suppose L' has ℓ' edges and m' vertices. Form A' on L' as A is formed on L. Then,

$$b(A') = \frac{nk\ell' + m'e}{m'n} = \frac{kl'}{m'} + \frac{e}{n}. \tag{2.3}$$

Since A is balanced, we have $b(A') \leq b(A)$, and by (2.1) and (2.3), we have

$$\frac{k\ell'}{m'} \leq \frac{k\ell}{m}. \tag{2.4}$$

Thus

$$\frac{\ell'}{m'} \leq \frac{\ell}{m}. \tag{2.5}$$

Therefore L is balanced.

(Sufficiency) Suppose L is balanced. Let H be a subgraph of A. If H is a subgraph of G_j for some $j = 1, \cdots, m$, then since G_j is balanced, we have $b(H) \leq b(G_j)$ and by (2.2), we have $b(G_j) < b(L)$; thus $b(H) < b(L)$. Otherwise, let $H_i = H \cap G_i$ and $n_i = |V(H_i)|$ for $i = 1, \cdots, m$. Without loss of generality, we may suppose that there is an integer $m' > 0$ such that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_{m'}$ and $n_i = 0$ for $i > m'$. Let L' be the subgraph of L induced by H, and note that $L' = L$ is possible. For each $i \in \{1, 2, \ldots, m'\}$, let $e_i = |E(H_i)|$, and $e' = |E(H) \cap E(\bigcup_{i=1}^{\ell} B_i)|$. Notice that

$$e' \leq k \sum_{v_i v_j \in E(L')} \min(n_i, n_j). \tag{2.6}$$

By Theorem II.3, since L is balanced, we have

$$\sum_{v_i v_j \in E(L')} \min(n_i, n_j) \leq \frac{l}{m} \sum_{i=1}^{m} n_i = \frac{l}{m} \sum_{i=1}^{m'} n_i. \tag{2.7}$$

By (2.6) and (2.7), we get

$$e' \leq \frac{kl}{m} \sum_{i=1}^{m'} n_i. \tag{2.8}$$

Thus,

$$b(H) = \frac{e' + \sum_{i=1}^{m'} e_i}{\sum_{i=1}^{m'} n_i} \tag{2.9}$$

$$\leq \frac{\frac{kl}{m}\sum_{i=1}^{m'} n_i + \sum_{i=1}^{m'} e_i}{\sum_{i=1}^{m'} n_i} \tag{2.10}$$

$$= \frac{kl}{m} + \frac{\sum_{i=1}^{m'} e_i}{\sum_{i=1}^{m'} n_i}. \tag{2.11}$$

By Lemma I.8, we have $\frac{\sum_{i=1}^{m'} e_i}{\sum_{i=1}^{m'} n_i} \leq \max_{1 \leq i \leq m'} \frac{e_i}{n_i} \leq \frac{e}{n}$ since G_i is balanced. Therefore, $b(H) \leq \frac{kl}{m} + \frac{e}{n} = g(A)$ and thus A is balanced. □

The *Cartesian product* $H_1 \times H_2$ of two graphs H_1 and H_2 is the graph on $V(H_1) \times V(H_2)$, each vertex labeled as (v_1, v_2) where $v_i \in H_i$; vertices (u_1, u_2) and (v_1, v_2) are adjacent in $H_1 \times H_2$ if and only if either $u_1 = v_1$ and u_2v_2 is an edge in H_2 or $u_2 = v_2$ and u_1v_1 is an edge in H_1.

Corollary II.5. *The Cartesian product of balanced graphs is balanced.*

Proof. Let G and L be two balanced graphs. Then $G \times L = A_1(G, G, \cdots, G; L)$ with suitable choices of the bipartite graphs B_{ij}. By the above theorem, $G \times L$ is balanced. □

CHAPTER III

1-BALANCED MATROIDS AND 1-BALANCED GRAPHS

A natural extension of balanced graphs is by defining the following density function on a graph with $|V(G)| > 1$, namely $\frac{|E(G)|}{|V(G)|-1}$. If G is connected then the rank of G is $|V(G)| - 1$. Thus the ratio $\frac{|E(G)|}{|V(G)|-1}$ is extendable to matroids. Recall from Chapter I that for a matroid M on a set E with rank function ρ, we have

$$d_1(F) = \frac{|F|}{\rho(F)}$$

for all subsets F of E such that $\rho(F) > 0$. The matroid M is said to be 1-balanced if and only if $d_1(E)$ is the maximum value among $d_1(F)$ where $F \subseteq E$.

1-balanced graphs and matroids are well-known in the literature. In the next section, we survey some earlier results on 1-balanced matroids and graphs. Section 3.1.2 has the lemmas that are used in this chapter and the next. In Section 3.2, we show which regular graphs are 1-balanced. In Section 3.3, we show that with the value of k within two limits, any generalized Cartesian product that is formed by using 1-balanced graphs is 1-balanced.

This chapter originated from an unpublished manuscript by Hobbs, Lai, Lai and Weng [40] in which a version of Theorem III.13 was stated with a faulty proof. The corrected version, including Kannan's proof of Theorem II.3 will appear in [39].

3.1. Earlier results

3.1.1. 1-balanced matroids

As an extension of the concept of balanced graphs, Kelly and Oxley [46] introduced the concept of 1-balanced matroids and called them "balanced matroids". 1-balanced matroids are also referred as "uniformly dense matroids" in [10] and "molecular matroids" in [59], while 1-balanced graphs are referred as "strongly balanced graphs" in [76]. Corp and McNulty [13] used the terminology of "balanced matroids". We introduce the terminology of "1-balanced matroids" for the following reasons: Firstly, in order to reconcile the existing terminology in the literature and secondly, to extend the notions of density functions to what we refer as "r-balanced matroids". By doing so, we are also not mixing the concepts of balanced graphs and 1-balanced graphs.

Another measure related to $\gamma_1(M)$ through the matroid dual is the *strength* of a matroid, defined by Cunningham [15] as

$$\eta_1(M) = \min\left\{\frac{|E|-|F|}{\rho(M)-\rho(F)} : F \subset E \quad \text{such that} \quad \rho(F) < \rho(M)\right\}.$$

For a graph G, the quantity $\eta_1(G)$ was introduced earlier by Gusfield [29] in the reciprocal form.

Suppose that $X \subseteq E(G)$ has the minimum number of edges that disconnect a connected graph G, *i.e.*, $|X| = \lambda(G)$, the edge-connectivity of G. Then, $\omega(G-X) = 2$. Therefore, $\frac{|X|}{\omega(G-X)-\omega(G)} = \frac{|X|}{2-1} = |X|$. By the definition of $\eta_1(G)$, we have $\eta_1(G) \leq |X| = \lambda(G)$. If $U \subseteq V(G)$, then the set of all edges with exactly one end-vertex in U is denoted as $[U, \overline{U}]$. Since $|[U, \overline{U}]| \geq \lambda(G)$, we have $|[U, \overline{U}]| \geq \eta_1(G)$. This fact is used later in this chapter.

In papers [45], [46] and [47], Kelly and Oxley extended the investigation of random graphs to that of random matroids. The theory of random matroids was also studied by many others, including for example, Kordecki and Łuczac [49].

The following relation is immediate:

$$\eta_1(M) \leq d_1(M) \leq \gamma_1(M). \tag{3.1}$$

Theorem III.1 (Catlin, Grossman, Hobbs, Lai [10]). *For a matroid M on a set E,* *the following are equivalent*

1. $\gamma_1(M)\rho(M) = |E|$ *i.e., M is* 1*-balanced;*

2. $\eta_1(M)\rho(M) = |E|$*;*

3. $\eta_1(M) = \gamma_1(M)$*.*

Lai et al. [51] proved that every matroid M is contained in a 1-balanced matroid M' such that $d_1(M') = \max\{d_1(N) : N \subseteq M\}$. Earlier, this result was proved for graphs by Payan [65].

3.1.2. 1-balanced graphs

There are many graphs that are 1-balanced, including trees, cycles, and complete graphs. Ruciński and Vince [76] proved that any 1-balanced graph is balanced and that the converse is not true. In Chapter V, among many other results, we show that any 1-balanced graph is 0-balanced. Figure 10 contains some examples of graphs that are balanced but not 1-balanced. In fact, the first example in Figure 10 is 0-balanced.

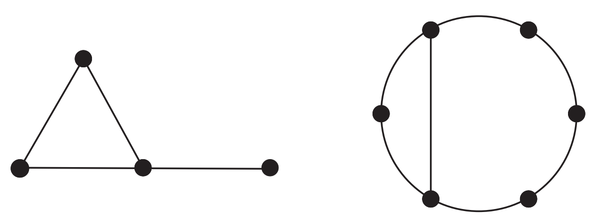

Fig. 10. Examples of balanced graphs that are not 1-balanced

The following lemma is immediate for 1-balanced graphs.

Lemma III.2. *A graph G is* 1*-balanced if and only if for all non-trivial* **connected** *subgraphs H of G, we have $d_1(H) \leq d_1(G)$.*

Proof. The necessity is clear. For sufficiency, suppose for all non-trivial, induced, connected subgraphs H of G, $d_1(H) \leq d_1(G)$. Let H be a disconnected subgraph of G. Let H_i, $1 \leq i \leq \omega(H)$ be the components of G. Clearly, we may assume that H_i for $1 \leq i \leq \omega(H)$ are non-trivial. By hypothesis, $d_1(H_i) \leq d_1(G)$, so $|E(H_i)| \leq d_1(G)(|V(H_i)| - 1)$ for $1 \leq i \leq \omega(H)$. Hence

$$|E(H)| = \sum_{i=1}^{\omega(H)} |E(H_i)| \leq d_1(G) \sum_{i=1}^{\omega(H)} (|V(H_i)| - 1) = d_1(G)(|V(H)| - \omega(H)),$$

and so $d_1(H) \leq d_1(G)$. □

Thus, for a connected graph G, the concept of 1-balanced graphs may also be realized using the fraction $\frac{|E(H)|}{|V(H)|-1}$ instead of $d_1(H)$. For a positive integer k, any non-trivial subgraph H of a (k,k)-sparse graph G satisfies the condition $\frac{|E(H)|}{|V(H)|-1} \leq k$. If G is a tight (k,k)-sparse graph, then $\frac{|E(G)|}{|V(G)|-1} = k$. Therefore, $\frac{|E(H)|}{|V(H)|-1} \leq k = \frac{|E(G)|}{|V(G)|-1}$ and so G is 1-balanced.

The following useful result was proved for matroids by Catlin, Grossman, Hobbs and Lai [10]:

Lemma III.3 (Catlin, Grossman, Hobbs and Lai [10])**.** *Let G be a graph. Suppose $d_1(H_1) = d_1(H_2) = \gamma_1(G)$ for subgraphs H_1, H_2 of G. Then $d_1(H_1 \cup H_2) = \gamma_1(G)$. Furthermore, if $H_1 \cap H_2$ has an edge, then $d_1(H_1 \cap H_2) = \gamma(G)$.*

As an important consequence of Lemma III.3, we note that G has a unique maximal γ-achieving subgraph without isolated vertices and that each component of this subgraph is a maximal, connected γ-achieving subgraph and is vertex induced. This fact is used frequently in this dissertation.

A 1-balanced graph is regarded as describing a minimally vulnerable network since a knowledgeable enemy (ignoring edge-connectivity) would find no edge set attractive to attack; see Cunningham [15] and Hobbs [37]. In fact, they are addressed as *bland graphs* in [37]. Hence constructing and identifying 1-balanced graphs would prove to be useful in many real-world situations.

Ruciński and Vince [76] (and later Catlin *et al.* [9] independently) proved that for any given positive integers m, n with $n-1 \leq m \leq n(n-1)/2$, there is a simple, connected, 1-balanced graph on n vertices and m edges. By gluing two graphs of suitable uniform densities at a vertex, Catlin *et al.* [8] observed that for any rational numbers x and y with $1 \leq x \leq y$, there is a graph G with $\eta_1(G) = x$ and $\gamma_1(G) = y$. Hence there is a large collection of graphs that are not 1-balanced, and the quantity $\gamma(G) - \eta(G)$ can be arbitrarily large. In view of Theorem III.1, $\gamma_1(G) - \eta_1(G) > 0$ if and only if $\gamma_1(G) - d_1(G) > 0$.

In the literature, there are algorithms to check if a given graph G is 1-balanced or not, see for example, Picard and Queyrenne [70], Cunnigham [15], Hobbs [36], Gusfield [30] and Cheng and Cunnigham [11]. Let $|V(G)| = n$ and $|E(G)| = m$. The algorithm of Hobbs [36] finds both $\gamma(G)$ and $\eta(G)$ in $O(m^3n^4)$ computations. In the next chapter, we use the algorithm in [36] to find the maximal γ-achieving subgraph of G.

The following useful result appeared in [10] generalized to matroids:

Theorem III.4 (Catlin, Grossman, Hobbs, Lai [10])**.** *For any connected graph G and any natural numbers s and t,*

1. $\eta_1(G) \geq \frac{s}{t}$ *if and only if there is a family $\mathcal{T}$ of s spanning trees in G such that each edge of G lies in at most t trees of $\mathcal{T}$.*
2. $\gamma_1(G) \leq \frac{s}{t}$ *if and only if there is a family $\mathcal{T}$ of s spanning trees in G such that each edge of G lies in at least t trees of $\mathcal{T}$.*
3. $\eta_1(G) = \frac{s}{t} = \gamma_1(G)$ *if and only if there is a family $\mathcal{T}$ of s spanning trees in G such that each edge of G lies in exactly t trees of $\mathcal{T}$.*

Peng *et al.* [66], [67] and [68], calculated the strength of several graphs. In [68], it is proved that for any graph G with edge-connectivity $\lambda(G)$, the following holds: $d_1(G) = \frac{|V(G)|\lambda(G)}{2(|V(G)|-1)} \leq \eta_1(G) \leq \lambda(G)$.

The *edge arboricity* $a(G)$ of a graph G is the minimum number of acyclic subgraphs of G whose union covers the edges of G. Recall that a random graph $G(n,p)$ on n vertices is formed by choosing an edge between any two pairs of vertices with probability p. Catlin and Chen [6] determined that $a(G) = \lceil \frac{|E(G)|}{|V(G)|-1} \rceil$ for almost all random graphs $G(n,p)$. In the case when p is a function of n, Catlin, Chen and Palmer [7] proved the same result when $p^3 = c \log n$ for a constant $c \geq 28$ and conjectured that their result holds for much lower edge probabilities. Clark [12] verified this conjecture for $432\frac{\log n}{n^{1/2}} < p = p(n) < 1/2$.

The weighted versions of the theories of 1-balanced graphs and matroids are due to Cheng and Cunningham [11] and Hobbs and Petingi [41] respectively.

We now present some preliminary results which are used later in this chapter.

Lemma III.5. *If a graph G is an edge-disjoint union of connected, spanning 1-balanced subgraphs G_1 and G_2, then G is 1-balanced.*

Proof. Since G_1 and G_2 are connected 1-balanced graphs on the same number of vertices, for $s_1 = |E(G_1)|$, $s_2 = |E(G_2)|$ and $t = |V(G)| - 1$, $d_1(G_1) = \frac{s_1}{t}$ and $d_1(G_2) = \frac{s_2}{t}$. Thus $d_1(G) = \frac{s_1+s_2}{t}$. For $i = 1, 2$, since G_i is 1-balanced, G_i has s_i spanning trees such that each edge in G_i is in exactly t of them. Thus G has $s_1 + s_2$ spanning trees such that each edge in G is in exactly t of them. G is 1-balanced by part (iii) of Theorem III.4. □

The following is an easy consequence of Lemma III.5.

Corollary III.6. *If a graph G is an edge-disjoint union of connected, spanning 1-balanced subgraphs $G_1, G_2, \cdots, G_p$ for some integer $p \geq 1$, then G is 1-balanced.*

Lemma III.7. *Let a be a positive integer. A graph G is* 1*-balanced if and only if G^a is* 1*-balanced.*

Proof. **(Necessity)** If G is 1-balanced, then G^a is 1-balanced by Corollary III.6 (by taking $p = a$ and $G_i = G$ for $i = 1, \ldots, p$).

(Sufficiency) Let H be a induced connected subgraph of G. Since G^a is 1-balanced, $d_1(G^a[V(H)]) \leq d_1(G^a)$. But $d_1(G^a[V(H)]) = ad_1(H)$ and $d_1(G^a) = d_1(G)$. Thus, $ad_1(H) \leq ad_1(G)$. Since a is positive, we have $d_1(H) \leq d_1(G)$. By Lemma III.2, G is 1-balanced. $\square$

3.2. Regular 1-balanced graphs

In the previous chapter, we saw that all regular graphs are balanced. But, not all regular graphs are 1-balanced. The graph G in Figure 11 is an example of a 4-regular graph that is not 1-balanced, because $d_1(G) = \frac{20}{9}$ and $d_1(K_5 - e) = \frac{9}{4} > d_1(G)$.

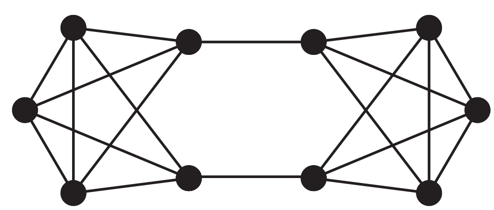

Fig. 11. Example of a regular graph that is not 1-balanced

Next we answer the natural question, "which connected regular graphs are 1-balanced?" For a graph G, the answer for the above question depends on edge-connectivity of G denoted as $\lambda(G)$.

Theorem III.8. *Let G be a connected p-regular graph for an integer $p \geq 2$. G is* 1*-balanced if and only if $\lambda(G) \geq d_1(G)$.*

Proof. **(Necessity)** Since G is p-regular, we have

$$|E(G)| = \frac{p|V(G)|}{2},$$

and so

$$d_1(G) = \frac{|E(G)|}{|V(G)| - 1} = \frac{p|V(G)|}{2(|V(G)| - 1)}.$$

If G is 1-balanced, then

$$\lambda(G) \geq \eta_1(G) = \frac{p|V(G)|}{2(|V(G)| - 1)}.$$

(Sufficiency) Suppose $\lambda(G) \geq d_1(G) = \frac{p|V(G)|}{2(|V(G)|-1)}$ and let U be any proper sub-set of V such that $H = G[U]$ is connected. Then

$$|[U, \overline{U}]| \geq \lambda(G) \geq \frac{p|V(G)|}{2(|V(G)| - 1)}.$$

The number of edges in H is the number of edges with at least one end-vertex in U minus $|[U, \overline{U}]|$. Thus,

$$|E(H)| \leq \frac{p|U|}{2} - \frac{p|V(G)|}{2(|V(G)| - 1)}.$$

Therefore, we have

$$\begin{aligned} d_1(H) &\leq \frac{\frac{p|U|}{2} - \frac{p|V(G)|}{2(|V(G)|-1)}}{|U| - 1} \\ &= \frac{p}{2}\left(\frac{|U|}{|U| - 1} - \frac{|V(G)|}{(|V(G)| - 1)(|U| - 1)}\right) \\ &= \frac{p}{2}\left(\frac{|U||V(G)| - |U| - |V(G)|}{(|U| - 1)(|V(G)| - 1)}\right) \\ &\leq \frac{p}{2}\left(\frac{|V(G)|(|U| - 1)}{(|U| - 1)(|V(G)| - 1)}\right) \\ &= \frac{p}{2}\left(\frac{|V(G)|}{|V(G)| - 1}\right) \\ &= d_1(G) \end{aligned}$$

Thus the theorem follows by Lemma III.2. □

3.3. 1-balanced generalized Cartesian products

The method of generalized Cartesian products defined in Section 2.5 can be used to construct big 1-balanced graphs from small ones. In this section, we prove that 1-balanced generalized Cartesian products can be formed from 1-balanced graphs.

3.3.1. Preliminaries and examples

We first recall the definition of generalized Cartesian products. Let L be a connected graph with ℓ edges and m vertices, and let the vertices be labeled $v_1, v_2, \ldots, v_m$. Label the edges of L as $e_1, e_2, \ldots, e_\ell$. Let $G_1, G_2, \ldots, G_m$ be vertex-disjoint connected graphs, each having n vertices and e edges. Let k be a positive integer. Let $B_1, B_2, \ldots, B_\ell$ be k-regular bipartite graphs such that, if edge e_i of L joins vertices v_r and v_s, then the two sides of B_i are the vertex sets of G_r and G_s. Then, $A_k = A_k(G_1, \ldots, G_m; L) = \left(\bigcup_{i=1}^m G_i\right) \cup \left(\bigcup_{i=1}^\ell B_i\right)$ is called a generalized Cartesian product. When the value of k is already known, we may use $A = A(G_1, \ldots, G_m; L)$ (omitting the subscript). Let $t = d_1(G_i) = \frac{e}{n-1}$ for all $i \in \{1, 2, \ldots, m\}$.

Unlike balanced generalized Cartesian products, the value of k in a 1-balanced generalized Cartesian product has a lower bound as the following Lemma shows.

Lemma III.9. *If A is 1-balanced, then*

$$k \geq \frac{m-1}{\ell}\left(\frac{t}{n}\right) = \frac{d_1(G_i)}{d_1(L)n}.$$

Proof. For each i, we have $|E(B_i)| = 2nk/2 = nk$. Since each G_i is connected and L is connected, A is connected. Hence

$$d_1(A) = \frac{nk\ell + me}{mn-1}.$$

Since A is 1-balanced and G_i is a subgraph of A for each i, we have

$$d_1(A) = \frac{nk\ell + me}{mn-1} \geq \frac{e}{n-1} = d_1(G_i).$$

Solving for k, we get

$$k \geq \frac{m-1}{\ell}\left(\frac{t}{n}\right).$$

□

In this section we prove that A is 1-balanced if $G_1, \ldots, G_m$ and L are 1-balanced and k is a fixed integer such that

$$\frac{m-1}{\ell}\left(\frac{t}{n}\right) \leq k \leq \frac{m-1}{\ell}(mt). \tag{3.2}$$

The above lower bound for k is in view of Lemma III.9. Also, there are examples of Cartesian products A that are not 1-balanced when $k > \frac{m-1}{\ell}(mt)$, even if $G_1, \ldots, G_m$ and L are 1-balanced. Figure 12 shows one. The graph in the figure is $A = A_3(K_2, K_2; K_2)$. Here, $L = K_2$, $t = 1$ and $m = 2$. We have $\frac{m-1}{\ell}(mt) = 2 < 3$. If H denotes the subgraph on 2 vertices and 3 parallel edges, then $d_1(H) = 3$. But, $d_1(A) = \frac{2(3+1)}{3} = \frac{8}{3} < 3 = d_1(H)$. Therefore A is not 1-balanced.

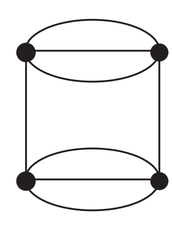

Fig. 12. $A_3(K_2, K_2; K_2)$: Example of a generalized Cartesian product that is not 1-balanced

If a generalized Cartesian product $A = A_k(G_1, \ldots, G_m; L)$ is 1-balanced, it does not imply that any of $G_1, \ldots, G_m$ or L is 1-balanced. The graph in the Figure 13 is an example of a generalized Cartesian product $A_k(G, H; K_2)$ that is 1-balanced, but neither G nor H is 1-balanced.

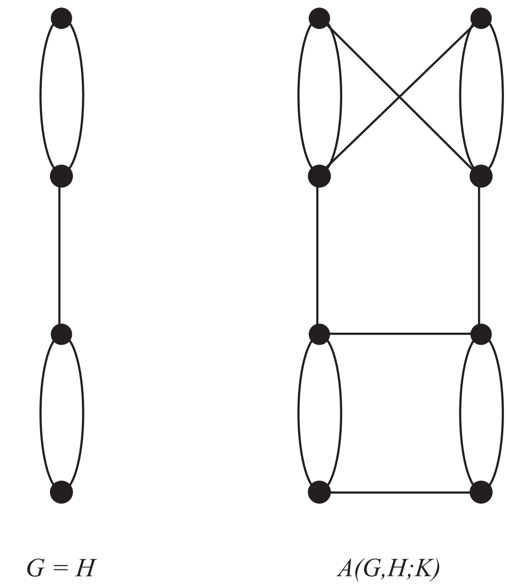

Fig. 13. $A_1(G, H; K_2)$: Example of a generalized Cartesian product that is 1-balanced, but neither G nor H is 1-balanced

It is easy to see that $A_1(G, H; K_2)$ is the union of 2 edge-disjoint spanning trees. Thus A is 1-balanced, by Theorem III.4(iii). But it is easy to see that $\eta_1(G) = 1$ and $\gamma_1(G) = 2$, so G and H are not 1-balanced, by Theorem III.4(iii). Also, note that $\gamma_1(G) = 2$ while $\eta_1(G) = 1$. Thus, G is not 1-balanced. Similarly, H is not 1-balanced.

However, we have this result:

Theorem III.10. *If A is 1-balanced, then L is strictly balanced.*

Proof. Let $V(L) = \{v_1, v_2, \cdots, v_m\}$. Let L' be any proper connected subgraph of L, and suppose L' has ℓ' edges and m' vertices. Form A' on L' as A is formed on L. Then $d_1(A') = \frac{nk\ell'+m'e}{m'n-1}$. Since A is 1-balanced, we have

$$\frac{nk\ell' + m'e}{m'n - 1} = d_1(A') \leq d_1(A) = \frac{nk\ell + me}{mn - 1}.$$

Cross-multiplying and simplifying,

$$mn^2k\ell' + mm'ne - nk\ell' - m'e \leq m'n^2k\ell + mm'ne - nk\ell - me,$$

or

$$\begin{aligned} mn^2k\ell' - nk\ell' - m'e &\leq m'n^2k\ell - nk\ell - me \\ &< m'n^2k\ell - nk\ell' - m'e \end{aligned}$$

since $\ell > \ell'$ and $m > m'$. Hence,

$$mn^2k\ell' - nk\ell' - m'e < m'n^2k\ell - nk\ell' - m'e,$$

which simplifies to $m\ell' < m'\ell$ since $n^2k > 0$. Thus we have $\frac{\ell'}{m'} < \frac{\ell}{m}$ as required. □

The converse of this theorem is false. Every strictly balanced graph which is not 1-balanced serves as a counter-example, since we can let $G_i = K_1$ for every i in constructing A. The graph in Figure 3 is an example of a strictly balanced graph that is not 1-balanced.

3.3.2. Main results

From now on, we assume that $G_1, \ldots, G_m$ are connected 1-balanced graphs. We first show that A is 1-balanced if k is as specified in (3.2) and L is a tree. Our plan of proof is to choose a γ-achieving connected subgraph H of A. We move to the subtree L' of L induced by H and prove $d_1(H) \leq d_1(A')$ in that case. (It is here that we use the new characterization of balanced graphs, namely Theorem II.3.) Using $d_1(A') \leq d_1(A)$, as shown in the next lemma, we conclude that $d_1(H) \leq d_1(A)$. Thus $d_1(A) = \gamma(A)$ and A is 1-balanced.

We start with some lemmas. Let L be any 1-balanced graph, and let L' be a connected induced subgraph of L. Letting A' be constructed from L' as A is constructed from L, we first look at the relationship between $d_1(A)$ and $d_1(A')$ (Lemma III.11) and between $d_1(A)$ and $d_1(G_i)$ (Lemma III.12).

Lemma III.11. *Let $k \geq \frac{t}{d_1(L)n}$, and let L' be a connected induced subgraph of L. Form A' from L' in the same way A is formed from L. If L is 1-balanced, then $d_1(A') \leq d_1(A)$.*

Proof.

$$\begin{aligned} d_1(A) - d_1(A') &= \frac{nk\ell + me}{mn-1} - \frac{nk\ell' + m'e}{m'n-1} \\ &= \frac{nk\ell(m'n-1) + mm'ne - me - nk\ell'(mn-1) - mm'ne + m'e}{(mn-1)(m'n-1)} \\ &= \frac{nk\ell(m'n-1) - me - nk\ell'(mn-1) + m'e}{(mn-1)(m'n-1)}. \end{aligned}$$

Since $\ell = d_1(L)(m-1)$ and $\ell' \leq d_1(L)(m'-1)$, we have

$$\begin{aligned} d_1(A) - d_1(A') &\geq \frac{d_1(L)nk(m-1)(m'n-1) - me - d_1(L)nk(m'-1)(mn-1) + m'e}{(mn-1)(m'n-1)} \\ &= \frac{d_1(L)nk[mm'n - m'n - m + 1 - mm'n + mn + m' - 1] - (m-m')e}{(mn-1)(m'n-1)} \\ &= \frac{d_1(L)nk[-m'n - m + mn + m'] - (m-m')e}{(mn-1)(m'n-1)} \\ &= \frac{d_1(L)nk[(m-m')(n-1)] - (m-m')e}{(mn-1)(m'n-1)} \\ &= (m-m')\frac{d_1(L)nk(n-1) - e}{(mn-1)(m'n-1)} \\ &= (m-m')(n-1)\frac{d_1(L)nk - d_1(G_i)}{(mn-1)(m'n-1)} \\ &\geq 0 \end{aligned}$$

since $k \geq \frac{d_1(G_i)}{d_1(L)n}$. ∎

Lemma III.12. *With* $k \geq \frac{m-1}{\ell}\left(\frac{t}{n}\right)$*, we have* $d_1(G_i) \leq d_1(A)$*.*

Proof. This was noted at the end of the proof of Lemma III.9. ☐

From now on, we assume that k satisfies (3.2).

Theorem III.13. *Let* L *be a tree. Then* A *is* 1*-balanced.*

Proof. If $n = 1$, $A = L$ and since L is 1-balanced, A is 1-balanced. We may assume that $n > 1$.

Suppose, for a contradiction, that A is not 1-balanced. Then by Lemma III.2, there is an induced connected subgraph H of A such that $d_1(H) = \gamma(A) > d_1(A)$. Let $H_i = H \cap G_i$ and $n_i = |V(H_i)|$. Without loss of generality, we may suppose there is an integer $m' > 0$ such that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_{m'}$ and $n_i = 0$ for $i > m'$. Let L' be the subgraph of L induced by H, and note that $L' = L$ is possible. L' is the same subgraph of L which is induced by $\{v_1, \ldots, v_{m'}\}$. For each $i \in \{1, 2, \ldots, m'\}$, let $e_i = |E(H_i)|$, $\omega_i = \omega(H_i)$, and $e' = |E(H) \cap E(\bigcup_{i=1}^{\ell} B_i)|$. Notice that

$$e' \leq k \sum_{v_i v_j \in E(L')} \min(n_i, n_j). \tag{3.3}$$

Since L is a tree and H is connected, L' is a tree. $d_1(L) = 1$. So,

$$\frac{t}{n} \leq k \leq mt. \tag{3.4}$$

Recall that $d_1(A') \leq d_1(A)$ by Lemma III.11. Thus

$$d_1(H) > d_1(A) \geq d_1(A'), \tag{3.5}$$

so A' is also not 1-balanced.

We consider two cases:

Case 1 : $\frac{t}{n} \leq k \leq m't$.

First we show

$$\frac{nk\ell' - (m'-1)t}{m'n-1} < \frac{e' - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1}. \tag{3.6}$$

Since $k \geq \frac{t}{n}$, $d_1(A) \geq d_1(G_i)$ for each i by Lemma III.12, $H \not\subseteq G_i$ for any i, so $m' > 1$.

Recalling that $t = d_1(G_i) = \frac{e}{n-1}$,

$$\left.\begin{aligned} d_1(A') &= \frac{nk\ell' + m'e}{m'n-1} \\ &= \frac{nk\ell' + m'\frac{e}{n-1}(n-1)}{m'n-1} \\ &= \frac{tm'(n-1) + nk\ell'}{m'n-1} \\ &= \frac{m'nt - t + t - m't + nk\ell'}{m'n-1} \\ &= t + \frac{nk\ell' - (m'-1)t}{m'n-1}. \end{aligned}\right\} \tag{3.7}$$

Also,

$$d_1(H) = \frac{\sum_{i=1}^{m'} e_i + e'}{\sum_{i=1}^{m'} n_i - 1}.$$

But, with $i \leq m'$, $n_i \geq 1$. Thus, if $e_i \neq 0$, then

$$e_i = \frac{e_i}{n_i - \omega_i}(n_i - \omega_i) \leq d_1(G_i)(n_i - \omega_i) \leq t(n_i - 1).$$

On the other hand, if $e_i = 0$, then

$$e_i = 0 \leq t(n_i - 1).$$

Thus, from the definitions of the symbols,

$$\left.\begin{aligned} d_1(H) &= \frac{\sum_{i=1}^{m'} e_i + e'}{\sum_{i=1}^{m'} n_i - 1} \\ &\leq \frac{t\sum_{i=1}^{m'}(n_i - 1) + e'}{\sum_{i=1}^{m'} n_i - 1} \\ &= \frac{t\left(\sum_{i=1}^{m'} n_i - 1\right) + t + e' - \sum_{i=1}^{m'} t}{\sum_{i=1}^{m'} n_i - 1} \\ &= t + \frac{e' - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1}. \end{aligned}\right\} \tag{3.8}$$

Since $d_1(A') < d_1(H)$, (3.6) follows from (3.7) and (3.8).

Next we show that

$$\sum_{v_iv_j \in E(L')} \left[\min(n_i, n_j) - \frac{1}{m'}\sum_{r=1}^{m'} n_r\right] > 0 \tag{3.9}$$

follows from (3.6), thus leading to a contradiction. But

$$\frac{nk\ell' - (m'-1)t}{m'n - 1} = \frac{\frac{k\ell'}{m'} - (m'-1)t}{m'n-1} + \frac{k\ell'}{m'}.$$

Replacing the left-hand side of (3.6) with this, moving $\frac{k\ell'}{m'}$ to the other side, and using (3.3),

$$\begin{aligned} & \frac{\frac{k\ell'}{m'} - (m'-1)t}{m'n-1} \\ < \quad & \frac{-\frac{k\ell'}{m'}\sum_{i=1}^{m'} n_i + \frac{k\ell'}{m'} + k\sum_{v_iv_j\in E(L')}\min(n_i,n_j) - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1} \\ = \quad & \frac{\frac{k\ell'}{m'} + k\left(\sum_{v_iv_j\in E(L')}\left[\min(n_i,n_j) - \frac{1}{m'}\sum_{i=1}^{m'} n_i\right]\right) - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1}. \end{aligned}$$

Multiplying through by the denominators and canceling like terms, we get

$$\begin{aligned} &\frac{k\ell'}{m'}\sum_{i=1}^{m'} n_i - (m'-1)t\sum_{i=1}^{m'} n_i \\ < \;&\frac{k\ell'}{m'}(m'n) + k(m'n-1)\Bigg(\sum_{v_iv_j\in E(L')}\Big[\min(n_i,n_j) - \frac{1}{m'}\sum_{i=1}^{m'} n_i\Big]\Bigg) - m'n(m'-1)t. \end{aligned}$$

Thus

$$\begin{aligned} &\frac{k\ell'}{m'}\sum_{i=1}^{m'} n_i - \frac{k\ell'}{m'}(m'n) + m'n(m'-1)t - (m'-1)t\sum_{i=1}^{m'} n_i \\ < \;&k(m'n-1)\Bigg(\sum_{v_iv_j\in E(L')}\Big[\min(n_i,n_j) - \frac{1}{m'}\sum_{i=1}^{m'} n_i\Big]\Bigg). \end{aligned}$$

Gathering together the two terms containing $(m'-1)t$ and then the first two terms of the previous inequality, we get

$$\begin{aligned} &(m'-1)t\sum_{i=1}^{m'}(n-n_i) - \frac{k\ell'}{m'}\Big(m'n - \sum_{i=1}^{m'} n_i\Big) \\ < \;&k(m'n-1)\Bigg(\sum_{v_iv_j\in E(L')}\Big[\min(n_i,n_j) - \frac{1}{m'}\sum_{i=1}^{m'} n_i\Big]\Bigg). \end{aligned}$$

Combining the terms on the left hand side gives us

$$\Big((m'-1)t-\frac{k\ell'}{m'}\Big)\sum_{i=1}^{m'}(n-n_i) < k(m'n-1)\Bigg(\sum_{v_iv_j\in E(L')}\Big[\min(n_i,n_j)-\frac{1}{m'}\sum_{i=1}^{m'} n_i\Big]\Bigg). \tag{3.10}$$

But $\sum_{i=1}^{m'}(n-n_i) \geq 0$ since $n_i \leq n$ for all i. Moreover, since $\ell' = m'-1$, $m' \geq 2$ and $k \leq m't$,

$$(m'-1)t - \frac{k\ell'}{m'} = (m'-1)[t - \frac{k}{m'}] \geq 0$$

Thus the left hand side of (3.10) is non-negative. Since $k(m'n-1)$ is positive, the rest of the right hand side must be positive. Hence the inequality (3.9). But L' is a tree, and so it is 1-balanced and thus balanced. By Theorem II.3, the inequality we have just reached is impossible. Thus A' is 1-balanced, so the proposed subgraph H cannot exist.

Case 2 : $m't \leq k \leq mt$.

For this case, we show that $d_1(A) < d_1(H)$ and $k \geq m't$ together imply that $k > mt$ which is a contradiction.

Using the similar computations we used in (3.7), we obtain

$$\left.\begin{aligned} d_1(A) &= \frac{nk\ell + m'e}{mn-1} \\ &= \frac{nk\ell + m\frac{e}{n-1}(n-1)}{mn-1} \\ &= \frac{tm(n-1)+nk\ell}{mn-1} \\ &= \frac{mnt - t + t - mt + nk\ell}{mn-1} \\ &= t + \frac{nk\ell - (m-1)t}{mn-1}. \end{aligned}\right\} \tag{3.11}$$

From (3.5), we have $d_1(A) < d_1(H)$. Thus by (3.8) and (3.11),

$$\frac{nk\ell - (m-1)t}{mn-1} < \frac{e' - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1}. \tag{3.12}$$

Now, we will get a bound for e'. By (3.3), we have

$$e' \leq k \sum_{v_i v_j \in E(L')} \min(n_i, n_j).$$

By Theorem II.3, since L' is a balanced graph,

$$\sum_{v_i v_j \in E(L')} \min(n_i, n_j) \leq \frac{\ell'}{m'} \sum_{i=1}^{m'} n_i.$$

Since $\ell' = m' - 1$,

$$e' \leq k\left(\sum_{i=1}^{m'} n_i - \frac{1}{m'} \sum_{i=1}^{m'} n_i\right).$$

Substituting this in (3.12) and adding and subtracting k in the numerator of the left hand side, we have

$$\frac{nk\ell - (m-1)t}{mn-1} < \frac{k\left(\sum_{i=1}^{m'} n_i - 1\right) - k\left(\frac{1}{m'}\sum_{i=1}^{m'} n_i - 1\right) - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1}.$$

Using the fact that $k \geq m't$ and simplifying, we have

$$\begin{aligned}
\frac{nk\ell - (m-1)t}{mn-1} &< \frac{k\left(\sum_{i=1}^{m'} n_i - 1\right) - m't\left(\frac{1}{m'}\sum_{i=1}^{m'} n_i - 1\right) - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1} \\
&= \frac{k\left(\sum_{i=1}^{m'} n_i - 1\right) - t\left(\sum_{i=1}^{m'} n_i - m'\right) - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1} \\
&= \frac{k\left(\sum_{i=1}^{m'} n_i - 1\right) - t\left(\sum_{i=1}^{m'} n_i - 1\right)}{\sum_{i=1}^{m'} n_i - 1} \\
&= k - t.
\end{aligned}$$

Substituting $\ell = m - 1$ and cross-multiplying, we have

$$(m-1)(nk - t) < (mn-1)(k-t),$$

which simplifies to

$$-nk - mt < -k - mnt.$$

Thus, $(n-1)(mt) < (n-1)k$. Since $n > 1$, we have $k > mt$ which is a contradiction.

Hence A is 1-balanced. □

Now, we are ready to show that if L is 1-balanced, then A is 1-balanced.

Theorem III.14. *If L is 1-balanced, then A is 1-balanced.*

Proof. Let $d_1(L) = \frac{r}{s}$. Since

$$\frac{t}{d_1(L)n} \leq k \leq \frac{mt}{d_1(L)},$$

substituting $d_1(L) = \frac{r}{s}$, we have

$$\frac{st}{rn} \leq k \leq \frac{mst}{r}, \quad \text{or} \quad \frac{st}{n} \leq kr \leq mst. \tag{3.13}$$

We first prove that A^{rs} is 1-balanced. By Lemma III.7, if A^{rs} is 1-balanced, then A is 1-balanced. To prove A^{rs} is 1-balanced, we will prove that A^{rs} is an edge-disjoint union of r spanning 1-balanced connected subgraphs.

Since L is 1-balanced of density $\frac{r}{s}$, by part (iii) of Theorem III.4, there are r spanning trees $T_1, T_2, \cdots, T_r$ in L such that each edge of L appears in exactly s of the trees. Let us denote by B_e the k-regular bipartite graph that replaces the edge $e \in L$ in A. For $1 \leq j \leq r$, let A_j be the generalized Cartesian product $A_{kr}(G_1^s, \cdots, G_m^s; T_j)$ using the kr-regular graphs B_e^r for each edge e in the tree T_j. Notice that G_i^s is 1-balanced by Lemma III.7 and $d_1(G_i^s) = st$ for $i = 1, \cdots, m$. By (3.13), we have

$$\frac{d_1(G_i^s)}{n} \leq kr \leq md_1(G_i^s).$$

By Theorem III.13, A_j is 1-balanced for $j = 1, 2, \cdots, r$.

Claim: $A^{rs} = \cup_{j=1}^r A_j$.

Proof of claim: Each A_j, $1 \leq j \leq r$ has a copy of G_i^s for each $i \in \{1, 2, \ldots, m\}$. Hence the edges of G_i appear rs times in $\cup_{j=1}^r A_j$.

Now, let $e = (u, v)$ be an edge in L. In A^{rs}, we have B_e^{rs} between G_u and G_v. On the other hand, B_e^r appears in exactly s of $A_1, A_2, \cdots, A_r$ since e appears in exactly s of $T_1, T_2, \cdots, T_r$. Thus we can find B_e^{rs} in $\cup_{j=1}^r A_j$. Hence the claim.

Since A^{rs} is an edge-disjoint union of the connected, spanning, 1-balanced subgraphs $A_j; j = 1, 2, \cdots, r$, by corollary III.6, A^{rs} is 1-balanced. □

Corollary III.15. *If connected graphs G_1 and G_2 are both 1-balanced, then the Cartesian product $G_1 \times G_2$ is 1-balanced.*

Proof. There are two ways to view $G_1 \times G_2$ as a generalized Cartesian product. $G_1 \times G_2 = A_1(G_1, G_1, \cdots, G_1; G_2)$ with suitable choices of the bipartite graphs B_{ij}. Similarly, $G_1 \times G_2 = G_2 \times G_1 = A_1(G_2, G_2, \cdots, G_2; G_1)$ with suitable choices of the bipartite graphs B_{ij}.

We first prove that either

$$\frac{d_1(G_1)}{|V(G_1)|d_1(G_2)} \leq 1, \tag{3.14}$$

or

$$\frac{d_1(G_2)}{|V(G_2)|d_1(G_1)} \leq 1 \tag{3.15}$$

holds. Suppose both (3.14) and (3.15) do not hold. Then we have

$$d_1(G_1) > |V(G_1)|d_1(G_2) > |V(G_1)||V(G_2)|d_1(G_1),$$

a contradiction since $|V(G_1)||V(G_2)| \geq 1$.

Now, if (3.14) holds, then by Theorem III.14, $A_1(G_1, G_1, \cdots, G_1; G_2) = G_1 \times G_2$ is 1-balanced. Similarly, if (3.15) holds, then by Theorem III.14, $A_1(G_2, G_2, \cdots, G_2; G_1) = G_1 \times G_2$ is 1-balanced. □

CHAPTER IV

TRANSFORMING AN ARBITRARY GRAPH INTO A 1-BALANCED GRAPH

In this chapter, we see how we may transform a non 1-balanced graph into a 1-balanced graph.

4.1. Motivation

As we saw in the last chapter, 1-balanced graphs are of practical importance. Hence, constructing and identifying 1-balanced graphs are of interest. For integers n and e with $n-1 \leq e \leq \binom{n}{2}$, there is at least one connected 1-balanced graph on n vertices and e edges. See [76] and [9].

Typically in real-world situations, networks are already in existence and the network owners do not want to dismantle the existing network completely and construct a new network that is 1-balanced. Rather, they are willing to budget modest amounts each year to gradually transform the network into one that is closer in some sense to being 1-balanced. In this chapter, we find a first solution for this problem.

In social networks, it has been shown that the vertices' positions within the communities can affect the role or function they assume. For example, it has long been accepted that individuals who lie on the boundaries of communities, bridging gaps between otherwise unconnected people, enjoy an unusual level of influence as the gatekeepers of information flow between groups. One may notice that increasing the number of gatekeepers would ultimately make the graph more uniformly distributed. We use this intuition to gradually alter the edges of an arbitrary graph and obtain a 1-balanced graph[3].

[3] Earlier, Hong-Jian Lai and Hongyuan Lai [50], in an unpublished manuscript, proved that any graph can be transformed to a graph G with $\gamma_1(G) - \eta_1(G) < 1$.

We measure closeness of a graph G to a 1-balanced graph by the difference $\gamma_1(G) - d_1(G)$ and reduce this difference to 0. The algorithm is carried on by a "think globally, act locally" approach. The main part of the algorithm is in Section 4.4, where we show that if a graph G is not 1-balanced then we can construct a new graph G' by re-defining the adjacency of an edge from G such that either $\gamma_1(G') < \gamma_1(G)$ or $\gamma_1(G') = \gamma_1(G)$ and the maximal γ_1-achieving subgraph of G' is properly contained in the maximal γ_1-achieving subgraph of G. Replacing G by G', we repeat the process at most $O(|E(G)||V(G)|^3)$ times until a 1-balanced graph is obtained. In Section 4.3, we show how the algorithm by Hobbs [36] can be used to find the maximal γ_1-achieving subgraph of a graph.

In Section 4.5, we provide a conjecture whose truth would decrease the number of steps required to transform a graph into a 1-balanced graph. We also present two theorems in support of our conjecture.

4.2. Preliminaries

Lemma IV.1. *If the graph G is not* 1*-balanced, then the rank of the maximal γ_1-achieving subgraph cannot be more than $|V(G)| - 2$.*

Proof. If the rank of the maximal γ_1-achieving subgraph H is $|V(G)| - 1$, then H is spanning and induced. This implies $H = G$, or in other words, G is 1-balanced, which is a contradiction. □

If G is a graph and H is a non-trivial subgraph of G, let G/H be the graph obtained by contracting the edges of H. If the graph H is induced, G/H does not have loops. Moreover, $F \subset E(G)$ such that $F \cap E(H) = \emptyset$ corresponds in a natural way to an edge set in G/H. For convenience, we denote the corresponding edge sets of $G - E(H)$ and G/H as the same, although, if needed in the context, we specify the graph in which the edge set belongs.

The following lemma is proved in [10] in the case that H is a maximal γ_1-achieving subgraph of G. Here, we add the condition "connected" to H.

Lemma IV.2. *Let G be a graph that is not 1-balanced and let H be a maximal connected γ_1-achieving subgraph of G. Let v be the vertex in G/H obtained by contracting the edges of H. If $\widehat{H}$ is a connected subgraph of G/H containing the vertex v, then $d_1(\widehat{H}) < \gamma_1(G)$.*

Proof. Since H is a maximal connected γ_1-achieving subgraph of G and the graph $G[E(H) \cup E(\widehat{H})]$ is a connected subgraph of G strictly containing H, we have

$$d_1(G[E(H) \cup E(\widehat{H})]) < \gamma_1(G). \tag{4.1}$$

On the other hand,

$$|E(H) \cup E(\widehat{H})| = |E(H)| + |E(\widehat{H})|$$

and

$$|V(G[E(H) \cup E(\widehat{H})])| = |V(H)| + |V(\widehat{H})| - 1.$$

Thus

$$\begin{aligned} d_1(G[E(H) \cup E(\widehat{H})]) &= \frac{|E(H)| + |E(\widehat{H})|}{|V(H)| + |V(\widehat{H})| - 2} \\ &\geq \min\left\{\frac{|E(H)|}{|V(H)| - 1}, \frac{|E(\widehat{H})|}{|V(\widehat{H})| - 1}\right\} \end{aligned}$$

by Lemma I.8.

But

$$\frac{|E(H)|}{|V(H)| - 1} = d_1(H) = \gamma_1(G)$$

and

$$\frac{|E(\widehat{H})|}{|V(\widehat{H})| - 1} = d_1(\widehat{H}).$$

Hence

$$d_1(G[E(H) \cup E(\widehat{H})]) \geq \min\{\gamma_1(G), d_1(\widehat{H})\}. \tag{4.2}$$

If $d_1(\widehat{H}) \geq \gamma_1(G)$, then by (6.6) we have $d_1(G[E(H) \cup E(\widehat{H})]) \geq \gamma_1(G)$, a contradiction to (6.5). Hence $d_1(\widehat{H}) < \gamma_1(G)$. □

4.3. Finding the maximal γ_1-achieving subgraph of a graph

At the end of this section, we show a method to find the maximal γ_1-achieving subgraph of a connected graph G. Suppose s, t are integers such that $\gamma_1(G) = \frac{s}{t}$, then in view of part (ii) of Theorem III.4 there is a family $\mathfrak{F}$ of s forests in G such that each edge of G lies in exactly t forests of $\mathfrak{F}$. If H is a subgraph of G, let $\mathfrak{F}_H := \{F \cap H : F \in \mathfrak{F}\}$. A forest F in a graph G is *maximal* if and only if $|V(F)| - \omega(F) = |V(G)| - \omega(G)$.

Theorem IV.3. *Let G be a graph with $\gamma_1(G) = \frac{s}{t}$, where s and t are positive integers. Let $\mathfrak{F}$ be a family of s forests such that each edge of G appears in exactly t forests of $\mathfrak{F}$. Let H be the maximal γ_1-achieving subgraph of G. Then $\mathfrak{F}_H$ is a collection of s maximal forests in H. Moreover, H is the maximal subgraph without isolated vertices satisfying this property.*

Proof. Let $\mathfrak{F} = \{F_1, \ldots, F_s\}$ and let $F_i' := F_i \cap H$ for $i = 1, \ldots, s$. Since $d_1(H) = \gamma_1(G)$, we have

$$d_1(H) = \frac{s}{t}. \tag{4.3}$$

For $i = 1, \ldots, s$, we have $|V(F_i')| - \omega(F_i') \leq |V(H)| - \omega(H)$ since F_i' is a forest in H. Suppose $|V(F_j')| - \omega(F_j') < |V(H)| - \omega(H)$ for some $j \in \{1, \ldots, s\}$, then we have

$$t|E(H)| \leq \sum_{i=1}^{s}(|V(H)| - \omega(F_i')) < s(|V(H)| - \omega(H)).$$

Therefore,

$$d_1(H) = \frac{|E(H)|}{|V(H)| - \omega(H)} < \frac{s}{t},$$

a contradiction to (4.3). Thus, $|V(F_i')| - \omega(F_i') = |V(H)| - \omega(H)$ for $i = 1, \ldots, s$ and so, $F_1', \ldots, F_s'$ are maximal forests in H.

Let H' be a maximal subgraph of G without isolated vertices such that $\mathfrak{F}_{H'}$ is a collection of s maximal forests in H'. Then $H \subseteq H'$ since $\mathfrak{F}_H$ is a collection of maximal forests in H, so

$$t|E(H')| = s(|V(H')| - \omega(H')),$$

implying $d_1(H') = \frac{|E(H')|}{|V(H')| - \omega(H')} = \frac{s}{t} = \gamma_1(G)$. Thus $H' \subseteq H$ since H is the maximal γ_1-achieving subgraph. Therefore, $H = H'$. □

Hobbs' algorithm in [36] finds a family $\mathfrak{F} = \{F_1, \ldots, F_s\}$ as specified in Theorem IV.3. Using this family, the maximal γ_1-achieving subgraph can be found as follows: By Theorem IV.3, the maximal γ_1-achieving subgraph of G is the union of all the non-trivial subgraphs of G that are induced by the vertex sets of the form $\cap_{i=1}^{s} U_i$, where U_i is the vertex set of a component of F_i, for $i = 1, \ldots, s$.

4.4. Transforming a graph into a 1-balanced graph

Theorem IV.4. *If G is a connected graph that is not 1-balanced, then there exists a connected graph G' with the vertex set $V(G)$ such that*

1. *$G - e = G' - e'$ for some $e \in E(G), e' \in E(G')$ such that e and e' have a common end-vertex; and*
2. *$\gamma_1(G') \leq \gamma_1(G)$, and if $\gamma_1(G') = \gamma_1(G)$, then all the γ_1-achieving subgraphs of G' are γ_1-achieving subgraphs of G. Furthermore, the size of the maximal γ_1-achieving subgraph of G' is smaller than that of G.*

Proof. Let H be a maximal connected γ_1-achieving subgraph of G. Then, $H \neq G$ since G is not 1-balanced. Let $f = uv$ be an edge in G with $u \in V(H)$ and $v \notin V(H)$. There is such an edge since $H \neq G$ and G is connected. Let $e = uw$ be an edge in H incident to u. Form a new graph G' from G by removing the edge e and adding a new edge $e' = vw$. Clearly, $V(G) = V(G')$ and (1) is satisfied.

To check (2), in view of Lemma III.3, we show that if H' is a non-trivial, induced, connected subgraph of G', then

$$d_1(H') \leq \gamma_1(G) \quad \text{if } e' \notin E(H') \quad \text{and} \tag{4.4}$$

$$d_1(H') < \gamma_1(G) \quad \text{if } e' \in E(H'). \tag{4.5}$$

Before proving (4.4) and (4.5), we show that if (4.4) and (4.5) are true, then (2) holds: By (4.4), (4.5) and the definition of γ_1, we conclude that $\gamma_1(G') \leq \gamma_1(G)$. Further, if $\gamma_1(G') = \gamma_1(G)$, by (4.5), any connected subgraph of G containing e' cannot be a γ_1-achieving subgraph of G'. Hence, any γ_1-achieving subgraph H' of G' does not contain e' and, being a subgraph of G, H' is a γ_1-achieving subgraph of G. On the other hand, H is a γ_1-achieving subgraph of G; hence the maximal γ_1-achieving subgraph of G contains e. Therefore all γ_1-achieving subgraphs of G' are γ_1-achieving subgraphs in G and thus they do not contain e and e'. We conclude that the maximal γ_1-achieving subgraph of G' is properly contained in the maximal γ_1-achieving subgraph of G.

Proof of (4.4) and (4.5): Let H' be an induced, connected subgraph of G'. If H' does not contain e', then H' is a subgraph of G. Thus $d_1(H') \leq \gamma_1(G)$ and (4.4) is verified.

Let us now suppose that H' contains the edge e'. Then $v, w \in V(H')$.

Case (i): $u \in V(H')$. In this case, $H'' := G[V(H')]$ is connected since any path

P' in H' containing the edge $e' = vw$ can be modified to obtain a walk P'' in H'' by replacing the edge vw by the edges vu, uw in that order. Also, H'' contains the edge e but not e'. Hence, H' and H'' are both connected and have the same number of edges on the same number of vertices, so

$$d_1(H') = d_1(H''). \tag{4.6}$$

But $H'' \neq H$ since H'' contains the edge $f = uv$. H is a maximal connected γ_1-achieving subgraph and H'' is a connected subgraph of G whose vertex set intersects with $V(H)$ and with $V(G - H)$. By Lemma III.3, we have

$$d_1(H'') < \gamma_1(G).$$

Therefore by (4.6),

$$d_1(H') < \gamma_1(G).$$

Case (ii): $u \notin V(H')$. Then $f = uv \notin E(H')$. Let $E_1 = E(H') \cap E(H)$ and $E_2 = E(H') - E_1$. Thus $e' \in E_2$. Note that $f \notin E_2$. Let $H_1 = G[E_1]$.

Let $\widehat{G} = G/H$ and $\widehat{G'} = G'/(H - e)$. Let $H_2 = \widehat{G'}[E_2]$. Then H_2 is connected and isomorphic to $\widehat{H} := \widehat{G}[E_2 - e' + f]$. Thus

$$d_1(H_2) = d_1(\widehat{H}) = \frac{|E_2|}{\rho_{\widehat{G'}}(E_2)}. \tag{4.7}$$

By Lemma IV.2, $d_1(\widehat{H}) < \gamma_1(G)$. Thus

$$d_1(H_2) < \gamma_1(G). \tag{4.8}$$

If H_1 is a graph with no edges, then

$$d_1(H') = \frac{|E_2|}{\rho_{G'}(E_2)}. \tag{4.9}$$

But $\rho_{G'}(E_2) \geq \rho_{\widehat{G'}}(E_2)$. Hence we have

$$d_1(H') \leq \frac{|E_2|}{\rho_{\widehat{G'}}(E_2)} = d_1(H_2) < \gamma_1(G) \tag{4.10}$$

by (4.7) and (4.8). Thus (4.5) holds. Therefore, we assume that $E_1 \neq \emptyset$. We have

$$d_1(H_1) \leq \gamma_1(G) \tag{4.11}$$

since H_1 is a subgraph of G.

Note that $|V(H')| = |V(H_1)| + |V(H_2)| - 1$. By Lemma I.8 we have

$$d_1(H') = \frac{|E_1| + |E_2|}{|V(H_1)| + |V(H_2)| - 2} \leq \max_{i=1,2} \frac{|E_i|}{|V(H_i)| - 1}, \tag{4.12}$$

with equality if and only if $\frac{|E_1|}{|V(H_1)|-1} = \frac{|E_2|}{|V(H_2)|-1}$.

But,

$$\frac{|E_1|}{|V(H_1)| - 1} \leq \frac{|E_1|}{|V(H_1)| - \omega(H_1)} = d_1(H_1) \tag{4.13}$$

and

$$\frac{|E_2|}{|V(H_2)| - 1} = d_1(H_2) \tag{4.14}$$

since H_2 is connected.

Thus by (5.21), (5.22) and (4.14),

$$d_1(H') \leq \max_{i=1,2}\{d_1(H_1), d_1(H_2)\}, \tag{4.15}$$

with equality if and only if $d_1(H_1) = d_1(H_2)$ and $d_1(H_1) = \frac{|E_1|}{|V(H_1)|-1}$. But $d_1(H_2) < \gamma_1(G)$ by (4.8) and $d_1(H_1) \leq \gamma_1(G)$ by (4.11). Thus $d_1(H') < \gamma_1(G)$ by (4.15), and (4.5) holds. □

Now, we describe an algorithm to modify a given graph G that is not uniformly dense into a graph that is uniformly dense. Since G is not uniformly dense, we have $|V(G)| > 2$, for all graphs on 2 vertices are uniformly dense. Let $i = 1$ initially and

$G_1 = G$. The algorithm proceeds as follows: Pick a maximal connected γ_1-achieving subgraph H_i of G_i. Let $e_i = u_i w_i \in E(H_i)$ such that u_i is adjacent to a vertex $v_i \in V(G_i) - V(H_i)$. Let $G_{i+1} = G_i - e_i + v_i w_i$. If G_{i+1} is uniformly dense, we are done. Otherwise, replace i with $i+1$ and repeat the procedure.

The algorithm terminates when a uniformly dense graph is obtained. By Theorem IV.4, for $i \geq 1$, we have

(i) $\gamma_1(G_{i+1}) \leq \gamma_1(G_i)$ and

(ii) if $\gamma_1(G_{i+1}) = \gamma_1(G_i)$ then the size of the maximal γ_1-achieving subgraph of G_{i+1} is less than that of G_i.

Thus, we have an integer k and integers $0 = i_0 < i_1 < \ldots < i_k$ such that

- $\gamma_1(G_{i_j+1}) = \ldots = \gamma_1(G_{i_{(j+1)}})$ for $j = 0, \ldots, k-1$,
- $\gamma_1(G_{i_j}) > \gamma_1(G_{i_j+1})$ for $j = 1, \ldots, k$ and
- $\gamma_1(G_{i_k+1}) = d_1(G)$, and the algorithm terminates.

We now calculate the number of iterations. We find the following numbers:

(1) $l := \max\{i_{(j+1)} - i_j : j = 0, \ldots, k-1\}$ = maximum possible number of consecutive steps i with the same value of $\gamma_1(G_{i+1})$.

(2) k.

The total number of iterations is bounded by $(l+1)k$ since after the last iteration, the value of γ_1 decreases.

(1) For each $j \in \{0, \ldots, k-1\}$, the maximal γ_1-achieving subgraph of $G_{i_{(j+1)}}$ is contained in the maximal γ_1-achieving subgraph of G_{i_j+1}. Thus $i_{(j+1)} - i_j$ is less than the rank of G_{i_j+1}. By Lemma IV.2, the rank of the maximal γ_1-achieving subgraph of

G_{i_j+1} is less than $|V(G)|-2$. Therefore, $i_{(j+1)}-i_j \leq |V(G)|-2$ and by the definition of l, we have $l \leq |V(G)|-2$.

(2) Suppose $\gamma_1(G_{i+1}) < \gamma_1(G_i)$ for some $i \geq 1$. Then

$$\begin{aligned}\gamma_1(G_i)-\gamma_1(G_{i+1}) &= \frac{|E(H_i)|}{|V(H_i)|-1}-\frac{|E(H_{i+1})|}{|V(H_{i+1})|-1} \\ &= \frac{|E(H_i)|(|V(H_{i+1})|-1)-|E(H_{i+1})|(|V(H_i)|-1)}{(|V(H_i)|-1)(|V(H_{i+1})|-1)} \\ &\geq \frac{1}{(|V(G)|-1)(|V(G)|-2)} \qquad (4.16)\end{aligned}$$

since the numerator is greater than 1 and in the denominator, $|V(H_i)| < |V(G)|$ and $|V(H_{i+1})| \leq |V(G)|$. By (4.16) we need at most $(\gamma_1(G)-d_1(G))(|V(G)|-1)(|V(G)|-2)$ such iterations.

If H is a γ_1-achieving subgraph of G, then

$$\gamma_1(G)-d_1(G) = \frac{|E(H)|}{|V(H)|-1}-d_1(G) < |E(G)|. \qquad (4.17)$$

Thus

$$k \leq |E(G)|(|V(G)|-1)(|V(G)|-2).$$

Therefore,

$$(l+1)k \leq |E(G)|(|V(G)|-1)^2(|V(G)|-2) = O(|E(G)||V(G)|^3).$$

The value of γ_1 and the maximal γ_1-achieving subgraphs of the graph obtained after each step is calculated in $O(|E(G)|^3|V(G)|^4)$ time complexity using the algorithm in [36] in view of Section 4.3.

The following corollary proves the existence of 1-balanced graphs on n vertices and m edges for all $m \geq n-1$. This result was obtained by Ruciński and Vince [76] for $n-1 \leq m \leq \frac{n(n-1)}{2}$.

Corollary IV.5. *For integers m, n such that $m \geq n-1$, there is a 1-balanced graph* *on n vertices and m edges.*

Proof. Taking any connected graph on n vertices and m edges and applying IV.4 repetitively, we obtain a 1-balanced graph on n vertices and m vertices. $\square$

4.5. Minimizing the number of steps

In Section 4.3, we saw how one can find the maximal γ_1-achieving subgraph of a graph. A polynomial time algorithm to find all the γ_1-achieving subgraphs of a graph can be found in a more general setting is provided by [58, Pages 412–421]. In this section, we point out how knowing all the γ_1-achieving subgraphs of a graph helps in reducing the number of iterations in achieving a 1-balanced graph.

In the previous section, we showed that at most $|E(G)||V(G)|^3$ iterations are needed to achieve a 1-balanced graph. For most graphs, the number of iterations could be very much less. The estimate is the best we could obtain that could be expressed in terms of known parameters of the graph. The reason for this is that there is no good estimate for k in the discussion. However, in the next paragraph, we discuss how to minimize the number of consecutive iterations with the same γ_1 value. This is of interest for practical purposes.

Suppose a graph G is not 1-balanced and there are two γ_1-achieving subgraphs H_1, H_2 having at least one common edge. Changing one end vertex of an edge from $H_1 \cap H_2$ to a vertex outside $H_1 \cup H_2$ not only decreases the densities of both H_1 and H_2 at the same time, but also decreases the densities of $H_1 \cap H_2$ and $H_1 \cup H_2$, which are also γ_1-achieving. This idea is captured by addressing the case in which there is a nested sequence of γ_1-achieving subgraphs. At any point of the algorithm, the collection of all minimal γ_1-achieving subgraphs is a collection C of pairwise edge-disjoint γ_1-achieving subgraphs. Since each of these subgraphs has to be reduced by an edge move, there have to be at least $|C|$ iterations before the γ_1 value decreases.

Conjecture IV.6. *Exactly $|C|$ iterations are enough to decrease the value of γ_1.*

As a consequence of the following theorem, we show that at most $2|C|$ iterations are enough.

Theorem IV.7. *Let G be a connected non-1-balanced graph and let $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_k$ be a sequence of connected γ_1-achieving subgraphs of G such that H_k is a maximal connected γ_1-achieving subgraph of G. Then after two steps, each consisting of changing one end vertex of one edge, a new graph G' can be obtained with $\gamma_1(G') \leq \gamma_1(G)$, and if $\gamma_1(G') = \gamma_1(G)$, then all γ_1-achieving subgraphs of G' are γ_1-achieving subgraphs in G, and for $i = 1, \ldots, k$, the subgraph $G'[V(H_i)]$ is not γ_1-achieving in G'.*

Proof. Since G is connected, there exists an edge e with end-points $u \in H_k$ and $v \notin H_k$. If $u \in H_1$, the theorem holds by Theorem IV.4. The number of steps is only one. If $u \notin H_1$, let $\overline{u}$ be a vertex in H_1. Let $\overline{G} := G - e + \overline{u}v$. Let $\overline{e}$ be the new edge $\overline{u}v$.

Claim: $\gamma_1(G) = \gamma_1(\overline{G})$ and both G and $\overline{G}$ have the same γ_1-achieving subgraphs.

Proof of claim: Note that H_i with $1 \leq i \leq k$ are subgraphs of $\overline{G}$. Since $d_1(H_i) = \gamma_1(G)$ for $1 \leq i \leq k$, we have

$$\gamma_1(\overline{G}) \geq \gamma_1(G). \tag{4.18}$$

Let $\overline{H}$ be a connected γ_1-achieving subgraph of $\overline{G}$. We show that $\overline{H}$ is a subgraph of G. This proves the claim.

For a contradiction, suppose $\overline{H} \not\subseteq G$. Then $\overline{e} \in E(\overline{H})$. Let $E_1 = E(H_k) \cap E(\overline{H})$

and $E_2 = E(\overline{H}) - E_1$. Then

$$|E(\overline{H})| = |E_1| + |E_2|$$

and

$$|V(\overline{H})| = |V(E_1)| + |V(\overline{H}/E_1)| - \omega(\overline{G}[E_1]).$$

Therefore we have

$$\begin{aligned} d_1(\overline{H}) = \frac{|E(\overline{H})|}{|V(\overline{H})| - 1} &= \frac{|E_1| + |E_2|}{|V(E_1)| + |V(\overline{H}/E_1)| - \omega(\overline{G}[E_1]) - 1} \\ &\leq \max\left\{\frac{|E_1|}{|V(E_1)| - \omega(\overline{G}[E_1])}, \frac{|E_2|}{|V(\overline{H}/E_1)| - 1}\right\}, \end{aligned} \tag{4.19}$$

with equality if and only if $\frac{|E_1|}{|V(E_1)|-\omega(\overline{G}[E_1])} = \frac{|E_2|}{|V(\overline{H}/E_1)|-1}$ by Lemma I.8. But $\omega(\overline{G}[E_1]) = \omega(G[E_1])$, so

$$\frac{|E_1|}{|V(E_1)| - \omega(\overline{G}[E_1])} = \frac{|E_1|}{|V(E_1)| - \omega(G[E_1])} = d_1(G[E_1]). \tag{4.20}$$

But $G[E_1]$ is a subgraph of G, so

$$d_1(G[E_1]) \leq \gamma_1(G). \tag{4.21}$$

By (4.20) and (4.21), we have

$$\frac{|E_1|}{|V(E_1)| - \omega(\overline{G}[E_1])} \leq \gamma_1(G). \tag{4.22}$$

Since $V((\overline{H} \cup H_k)/H_k) \subseteq V(\overline{H}/E_1)$ and $E((\overline{H} \cup H_k)/H_k) = E_2$, we have

$$\frac{|E_2|}{|V(\overline{H}/E_1)| - 1} \leq \frac{|E_2|}{|V((\overline{H} \cup H_k)/H_k)| - 1} = d_1((\overline{H} \cup H_k)/H_k). \tag{4.23}$$

The graph $(\overline{H} \cup H_k)/H_k$ is isomorphic to $G([V(\overline{H} \cup H_k)])/H_k$ which is a connected subgraph of G/H_k containing the vertex obtained from the contracted edges. Thus,

$$d_1((\overline{H} \cup H_k)/H_k) < \gamma_1(G) \tag{4.24}$$

by Lemma IV.2. By (4.23) and (4.24), we have

$$\frac{|E_2|}{|V(\overline{H}/E_1)| - 1} < \gamma_1(G). \tag{4.25}$$

By (4.19), (4.22) and (4.25), $\gamma_1(\overline{G}) = d_1(\overline{H}) < \gamma_1(G)$, a contradiction to (4.18). Thus, H' is a subgraph of G. Thus the claim.

In $\overline{G}$, let $\overline{w}$ be a vertex adjacent to $\overline{u}$. By Theorem IV.4, $G' = \overline{G} - \overline{uw} + \overline{w}v$ is a graph with the vertex set $V(\overline{G})$ such that $\gamma_1(G') \leq \gamma_1(G)$, and if $\gamma_1(G') = \gamma_1(G)$, then all γ_1-achieving subgraphs of G' are γ_1-achieving subgraphs in G, and for $i = 1, \ldots, k$, the subgraph $G'[V(H_i)]$ is not γ_1-achieving in G'. □

Let G be a connected non-1-balanced graph and let $H_1 \subseteq H_2 \subseteq \ldots \subseteq H_k$ be a sequence of connected γ_1-achieving subgraphs of G such that H_k is a maximal connected γ_1-achieving subgraph of G. Since G is connected, there exist vertices $w \in V(H_1)$ and $v \notin V(H_k)$ with a wv path P in G satisfying the following properties:

1. There exists a vertex $u \in V(H_1)$ that is adjacent to w in P, and

2. Vertex v is the only vertex of P outside H_k.

For $1 \leq i \leq k-1$, let $n_i := |V(H_{i+1}) - V(H_i)|$. The following theorem supports Conjecture IV.6.

Theorem IV.8. *Let G be a connected non-1-balanced graph and let $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_k$ be a sequence of connected γ_1-achieving subgraphs of G such that H_k is a maximal connected γ_1-achieving subgraph of G. Let P be a path as described above.*

Also, assume that $n_i \leq 1$ for $1 \leq i \leq k-1$. Then there exists a connected graph G' with the vertex set $V(G)$ such that

1. *$G - e = G' - e'$ for some $e \in E(H_1), e' \in E(G')$ such that e and e' have a common end-vertex; and*

2. *$\gamma_1(G') \leq \gamma_1(G)$ and if $\gamma_1(G') = \gamma_1(G)$, then all γ_1-achieving subgraphs of G' are γ_1-achieving in G and for $1 \leq i \leq k$, the subgraph $G'[V(H_i)]$ is not γ_1-achieving in G'.*

Proof. We use induction on k. Theorem IV.4 proves the case when $k = 1$. Note that the above conditions for P are satisfied in Theorem IV.4 for the proof of the case $k = 1$.

Suppose the theorem is true for all $k \in \{1, 2, \ldots, l-1\}$ for some integer $l \geq 2$. Let $k = l$.

If $n_i = 0$ for some $1 \leq i \leq l-1$, then the collection

$$\mathcal{H} = \{H_i\}_{i \in \{1,2,\cdots,l\} - \{j : n_j = 0\}}$$

forms a smaller nested sequence of γ_1-achieving subgraphs with $H_l \in \mathcal{H}$. Thus, the theorem is true by the induction hypothesis. Hence we may assume that $n_i = 1$ for all $1 \leq i \leq l-1$.

Let the vertices of the path be $P = w, u, u_2, \cdots, u_l, v$. Let the edges in P be labeled as $e, e_1, \cdots e_l$. Using the induction hypothesis with $k = 1$, we see that $\overline{G} := G - e_{l-1} + u_{l-1}v$ is a graph with

3. $\gamma_1(\overline{G}) = \gamma_1(G)$ and

4. all γ_1-achieving subgraphs of $\overline{G}$ are γ_1-achieving subgraphs of G. Moreover, $\overline{G}[V(H_l)]$ is not γ_1-achieving in $\overline{G}$.

For $1 \leq i \leq l-1$, H_i is a γ_1-achieving subgraph of $\overline{G}$. Let $\overline{H_{l-1}}$ be the maximal connected γ_1-achieving subgraph in $\overline{G}$ that contains H_{l-1}. Then $\overline{H_{l-1}}$ does not contain the vertex v, for if $v \in V(\overline{H_{l-1}})$, then by Lemma III.3, we see that $G[V(H_l) \cup \{v\}]$ is a connected γ_1-achieving subgraph in G, a contradiction to the fact that H_l is a maximal connected γ_1-achieving subgraph. Thus $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{l-2} \subseteq \overline{H_{l-1}}$ is a sequence of connected γ_1-achieving subgraphs of $\overline{G}$ such that $\overline{H_{l-1}}$ is a maximal connected γ_1-achieving subgraph of $\overline{G}$. The path $\overline{P} = wPu_{l-1}\overline{e}v$ is a wv path satisfying the conditions of the theorem. By the induction hypothesis, $\widehat{G} := \overline{G} - e + wv$ is such that

6. $\gamma_1(\widehat{G}) \leq \gamma_1(\overline{G}) = \gamma_1(G)$. If $\gamma_1(\widehat{G}) = \gamma_1(G)$, then all γ_1-achieving subgraphs of $\widehat{G}$ are γ_1-achieving subgraphs of $\overline{G}$, and for $1 \leq i \leq l$, the subgraph $\widehat{G}[V(H_i)]$ is not γ_1-achieving in $\widehat{G}$.

For notational simplicity, let e' be the newly added edge wv in $\widehat{G}$ and let $\overline{e}$ be the newly added edge $u_{l-1}v$ in $\overline{G}$. We recall that $\overline{G} = G - e_{l-1} + \overline{e}$ and $\widehat{G} = \overline{G} - e + e'$.

Claim: The graph $G' := \widehat{G} - \overline{e} + e_{l-1} = G - e + e'$ is the required graph.

Proof of claim:

Clearly, $G - e = G' - e'$.

Let H' be a connected subgraph of G' such that $d_1(H') = \gamma_1(G')$. If $e_{l-1} \notin E(H')$, then H' is a subgraph of $\widehat{G}$ and hence we have $\gamma_1(G') \leq \gamma_1(\widehat{G})$. Thus, by (6), $\gamma_1(G') \leq \gamma_1(G)$, with equality only if H' is a γ_1-achieving subgraph of G.

Let us now suppose that $e_{l-1} \in E(H')$.

Case 1: $v \notin V(H')$. In this case, H' is a subgraph of G. Hence $\gamma_1(G') \leq \gamma_1(G)$ with equality only if H' is a γ_1-achieving subgraph of G.

Case 2: $v \in V(H')$. Consider the subgraph $\widehat{H} = \widehat{G}[V(H')] = H' - e_{l-1} + \overline{e}$. Then, $\widehat{H}$ is connected and $|E(\widehat{H})| = |E(H')|$. We have

$$d_1(\widehat{H}) = d_1(H'). \tag{4.26}$$

If $\gamma_1(\widehat{G}) < \gamma_1(G)$, then $d_1(\widehat{H}) < \gamma_1(G)$. Suppose $\gamma_1(\widehat{G}) = \gamma_1(G)$. Thus $\widehat{H}$ is a connected subgraph of $\widehat{G}$ containing the edge e_{l-1} and the vertex v. But all γ_1-achieving subgraphs of $\widehat{G}$ are γ_1-achieving in G by (4) and (6); and the vertex v is not contained in any connected γ_1-achieving subgraph of G that contains e_{l-1}. Therefore, $\widehat{H}$ is not γ_1-achieving in $\widehat{G}$ and so $d_1(\widehat{H}) < \gamma_1(G)$. By (4.26), $d_1(H') < \gamma_1(G)$. Therefore $\gamma_1(G') < \gamma_1(G)$.

Thus, we have $\gamma_1(G') \leq \gamma_1(G)$. For $1 \leq i \leq l$, $G'[V(H_i)]$ is a proper subgraph of H_i since $e \notin E(G')$. Thus $d_1(G'[V(H_i)]) < \gamma_1(G)$. Thus if $\gamma_1(G') = \gamma_1(G)$, then for $i = 1, \ldots, l$, the subgraph $G'[V(H_i)]$ is not γ_1-achieving in G'.

$\square$

CHAPTER V

DENSITIES IN GRAPHS AND MATROIDS

In this chapter, we present our study of (r, s)-balanced matroids for any rational number r and any non-negative integer $s > r - 1$. We provide some examples of (r, s)-balanced graphs. We also give several results concerning (r, s)-balanced graphs and matroids.

5.1. Definition

Throughout this chapter we assume that M is a matroid on a non-empty set E and with rank function ρ. For a rational number r, recall that

$$d_r(F) = \frac{|F|}{\rho(F) - (r-1)}$$

for all subsets F of E such that $\rho(F) > r - 1$. We use the notation $\rho(M)$ in place of $\rho(E)$ and $d_r(M)$ in place of $d_r(E)$. For an integer s such that $s > r - 1$, the matroid M is said to be (r, s)-*balanced* if $\rho(M) \geq s$ and $d_r(F) \leq d_r(M)$ for all subsets F of E such that $\rho(F) \geq s$. If $d_r(F) \leq d_r(M)$ for all subsets F of E such that $\rho(F) > r - 1$, we simply call M as a r-*balanced matroid.* Note that the definition of the r-balanced matroids is the same as that of the (r, s)-balanced matroids if $r - 1 < s \leq r + 1$ or if $s = 0$.

Recall that the *rank* of a graph G, denoted as $\rho(G)$ is the size of the maximal forest present in G and is equal to $|V(G)| - \omega(G)$. The quantity $\rho(G)$ is the rank of the cycle matroid of G. A graph G is (r, s)-*balanced* if and only if its cycle matroid is (r, s)-balanced. A graph is r-*balanced* if its cycle matroid is r-balanced. For a subgraph H of a graph G, we denote $d_r(E(H))$ simply as $d_r(H)$.

Note that a connected graph G on $\lceil r+1 \rceil$ vertices has rank $\lceil r \rceil$ and all its proper induced subgraphs are of rank less than r and the condition $d_r(H) \leq d_r(G)$ is satisfied for all subgraphs of rank at least r and therefore G is r-balanced. Also, a graph G with rank greater than $r-1$ is (r,s)-balanced where s is the rank of G.

5.2. (r,s)-balanced matroids

In this section, we examine some relations between the various classes of (r,s)-balanced matroids.

Let r_1 and r_2 be two rational numbers such that $r_1 < r_2$ and let M be a matroid with $\rho(M) \geq r_2 - 1$. We have

$$\rho(M) - (r_1 - 1) > \rho(M) - (r_2 - 1).$$

Thus,

$$d_{r_1}(M) < d_{r_2}(M). \tag{5.1}$$

Let s be a non-negative integer such that $s > r - 1$. We define

$$\gamma_r^s(M) := \max\{d_r(F) | F \subset E, \rho(F) \geq s\}. \tag{5.2}$$

By the definition of (r,s)-balanced matroids, a matroid M is (r,s)-balanced if and only if $\gamma_r^s(M) = d_r(M)$. Also, let

$$\mu^s(M) := \min\left\{\frac{|E| - |F|}{\rho(M) - \rho(F)} \middle| F \subset E, s \leq \rho(F) < \rho(M)\right\}. \tag{5.3}$$

Note that from the definition of $\eta_1(M)$ in Chapter III, we have

$$\mu^0(M) = \eta_1(M). \tag{5.4}$$

We define

$$\eta_r^s(M) := \min(\mu^s(M), d_r(M)). \tag{5.5}$$

Note that $\eta_1^1(M) = \eta_1(M)$. In general, if $r-1 < s \leq r+1$ or if $s = 0$, we just denote $\gamma_r^s(M)$ as $\gamma_r(M)$ and $\eta_r^s(M)$ as $\eta_r(M)$.

Note that, from (5.2) and (5.5),

$$\eta_r^s(M) \leq d_r(M) \leq \gamma_r^s(M). \tag{5.6}$$

The following result is a generalization of Theorem III.1 which proves the case $r = s = 1$.

Theorem V.1. *Let r be a rational number and s be a non-negative integer such that $s > r-1$. If $\rho(M) \geq s$, the following are equivalent:*

(i) $\gamma_r^s(M) = d_r(M)$ *(i.e., M is (r,s)-balanced),*

(ii) $\eta_r^s(M) = d_r(M)$,

(iii) $\gamma_r^s(M) = \eta_r^s(M)$.

Proof. By the relation (5.6), (iii) implies (i) and (ii).

Let $F \subset E$ such that $\rho(F) < \rho(M)$ and $\rho(F) \geq s$. Then,

$$d_r(M) = \frac{|E|}{\rho(M) - (r-1)} \leq \frac{|E| - |F|}{\rho(M) - \rho(F)}$$

if and only if

$$|E|\big(\rho(M) - \rho(F)\big) \leq |E|\big(\rho(M) - (r-1)\big) - |F|\big(\rho(M) - (r-1)\big).$$

Simplifying the above inequality, we get

$$|F|\big(\rho(M) - (r-1)\big) \leq |E|\big(\rho(F) - (r-1)\big),$$

which is equivalent to $d_r(F) \leq d_r(M)$.

Hence (i) and (ii) are equivalent and together they imply (iii). $\square$

The following is a easy consequence of the above theorem.

Corollary V.2. *Let r be a rational number and s be a non-negative integer such that $s > r - 1$. A matroid M is (r, s)-balanced if and only if $d_r(M) \leq \mu^s(M)$.*

Proof. By Theorem V.1, the matroid M is (r, s)-balanced if and only if $d_r(M) = \eta_r^s(M)$. By the definition of $\eta_r^s(M)$, we have $d_r(M) = \eta_r^s(M)$ if and only if $d_r(M) \leq \mu^s(M)$. □

Lemma V.3. *Let r_1 and r_2 be two rational numbers such that $r_1 < r_2$. Let s be a non-negative integer such that $s > r_2 - 1$. If M is (r_2, s)-balanced, then M is (r_1, s)-balanced.*

Proof. By (5.1), we have $d_{r_1}(M) \leq d_{r_2}(M)$. Suppose M is (r_2, s)-balanced, then by Corollary V.2, we have $d_{r_2}(M) \leq \mu^s(M)$. Thus $d_{r_1}(M) \leq d_{r_2}(M) \leq \mu^s(M)$. Therefore, M is (r_1, s)-balanced. □

Lemma V.4. *Let r be a rational number and let s_1, s_2 be two integers such that $r - 1 \leq s_1 < s_2$. If M is (r, s_1)-balanced, then M is (r, s_2)-balanced.*

Proof. Since M is (r, s_1)-balanced, we have $d_r(F) \leq d_r(M)$ for all $F \subseteq E$ such that $\rho(F) \geq s_1$. In particular, since $s_2 > s_1$, we have $d_r(F) \leq d_r(M)$ for all $F \subseteq E$ such that $\rho(F) \geq s_2$. Therefore, M is (r, s_2)-balanced. □

Corollary V.5. *A loopless* 1*-balanced matroid is r-balanced for any rational number $r < 1$.*

Proof. Let M be a 1-balanced matroid, *i.e.,* M is $(1, 1)$-balanced. By Lemma V.3, M is $(r, 1)$-balanced for any rational number $r < 1$. Hence it is enough to show $d_r(F) \leq d_r(M)$ for $F \subseteq E$ such that $\rho(F) = 0$. Since M is loopless, $\rho(F) = 0$ if and only if $F = \phi$. Therefore, $d_r(F) = 0$ if $\rho(F) = 0$. Thus, $d_r(F) \leq d_r(M)$ for $F \subseteq E$ such that $\rho(F) = 0$. □

Table II. Relationship between the various (r, s)-balanced matroids

$-\frac{1}{2}$-balanced	$\Rightarrow$	$(-\frac{1}{2}, 1)$-balanced	$\Rightarrow$	$(-\frac{1}{2}, 2)$-balanced
$\Uparrow$		$\Uparrow$		$\Uparrow$
0-balanced	$\Rightarrow$	$(0, 1)$-balanced	$\Rightarrow$	$(0, 2)$-balanced
$\Uparrow$		$\Uparrow$		$\Uparrow$
$\frac{1}{2}$-balanced	$\Rightarrow$	$(\frac{1}{2}, 1)$-balanced	$\Rightarrow$	$(\frac{1}{2}, 2)$-balanced
	$\Nwarrow$ *	$\Uparrow$		$\Uparrow$
		1-balanced	$\Rightarrow$	$(1, 2)$-balanced
		$\Uparrow$		$\Uparrow$
		$\frac{3}{2}$-balanced	$\Rightarrow$	$(\frac{3}{2}, 2)$-balanced
				$\Uparrow$
				2-balanced

* if the matroid is loopless

Table II shows the various implications that are proved in Lemma V.3 and Lemma V.4. The number $-\frac{1}{2}$ can be replaced by any number between -1 and 0, the number $\frac{1}{2}$ can be replaced by any number between 0 and 1 and the number $\frac{3}{2}$ can be replaced by any number between 1 and 2.

5.3. Existence of (r,s)-balanced graphs

In this section, we show the existence of (r, s)-balanced graphs for some values of r and s. For this, we prove some key lemmas which are later used to show that (k, l)-sparse graphs and Laman graphs are examples of (r, s)-balanced graphs. The readers will notice that Laman graphs form the basic example of (r, s)-graphs in this dissertation. Apart from the existence proofs, we give some preliminary results concerning (r,s)-balanced graphs.

5.3.1. (k,l)-sparse graphs and (r,s)-balanced graphs

Recall from Section 1.3.3 that for $k \leq l \leq 2k$, a loopless graph G is said to be (k,l)-*sparse* if and only if for every subset $U \subseteq V$ with $|U| \geq 2$, we have $\frac{|E(U)|}{|U|-\frac{l}{k}} \leq k$. We call G a *tight* (k,l)-*sparse graph* if G is (k,l)-sparse and $\frac{|E(G)|}{|V(G)|-\frac{l}{k}} = k$. In this section, we show that a tight (k,l)-sparse graph is $\frac{l}{k}$-balanced.

We first present the following generalization of Lemma III.2.

Lemma V.6. *For $1 \leq r < 2$, a multi-graph G is r-balanced if and only if for all non-trivial* **connected** *subgraphs H of G with $|V(H)| > r$, we have $d_r(H) \leq d_r(G)$.*

Proof. The necessity is trivial.

(Sufficiency) Assume that for all non-trivial connected subgraphs H of G with $|V(H)| > r$, we have $d_r(H) \leq d_r(G)$. Let H' be a non-trivial subgraph of G with $\rho(H') > r-1$. Thus, H' has at least one edge. We also assume that H' has no trivial components. Let $H_1, \ldots, H_t$ be the components of H' with $t \geq 1$. Then $|V(H_i)| \geq 2 > r$. Thus, $|E(H)| = \sum_{i=1}^{t} |E(H_i)|$ and $\rho(H) = \sum_{i=1}^{t} |V(H_i)| - t$.

For $i = 2, \ldots, t$, by the assumption of the theorem, we have

$$d_r(H_i) = \frac{|E(H_i)|}{|V(H_i)| - 1 - (r-1)} \leq d_r(G).$$

Thus,

$$|E(H_i)| \leq d_r(G)\big(|V(H_i)| - 1 - (r-1)\big).$$

Therefore, we have

$$\begin{aligned}
d_r(H) &= \frac{|E(H)|}{\rho(H)-(r-1)}\\
&= \frac{\sum_{i=1}^{t}|E(H_i)|}{\sum_{i=1}^{t}|V(H_i)|-t-(r-1)}\\
&\leq \frac{d_r(G)\sum_{i=1}^{t}\big(|V(H_i)|-1-(r-1)\big)}{\sum_{i=1}^{t}|V(H_i)|-t-(r-1)}\\
&= \frac{d_r(G)\big(\sum_{i=1}^{t}|V(H_i)|-t-t(r-1)\big)}{\sum_{i=1}^{t}|V(H_i)|-t-(r-1)}\\
&\leq \frac{d_r(G)\big(\sum_{i=1}^{t}|V(H_i)|-t-(r-1)\big)}{\sum_{i=1}^{t}|V(H_i)|-t-(r-1)} = d_r(G)
\end{aligned}$$

The last inequality holds since $t \geq 1$ and $r-1 \geq 0$. □

Corollary V.7. *For positive integers k and l such that $k \leq l \leq 2k$, a tight, connected (k,l)-sparse graph is a $\frac{l}{k}$-balanced graph.*

Proof. If G is a tight, connected (k,l)-sparse graph, then $|E(G)| = k|V(G)|-l$. Thus

$$d_{\frac{l}{k}}(G) = \frac{|E(G)|}{|V(G)|-\frac{l}{k}} = \frac{k(|V(G)|-\frac{l}{k})}{|V(G)|-\frac{l}{k}} = k.$$

If H is a connected subgraph of G, then by the definition of (k,l)-sparse graphs, we have $|E(H)| \leq k|V(H)|-l$. Therefore,

$$d_{\frac{l}{k}}(H) = \frac{|E(H)|}{|V(H)|-\frac{l}{k}} \leq \frac{k(|V(H)|-\frac{l}{k})}{|V(H)|-\frac{l}{k}} = k = d_{\frac{l}{k}}(G).$$

By Lemma V.6, G is $\frac{l}{k}$-balanced. □

5.3.2. Laman graphs and (r,s)-balanced graphs

Recall from Section 1.1.5 that a *Laman graph* of dimension m is a simple graph that satisfies the following: $|E(U)| \leq m|U| - \binom{m+1}{2}$ for all $U \subseteq V(G)$ with $|U| \geq m$, and $|E(G)| = m|V(G)| - \binom{m+1}{2}$. In this section, we prove that for any positive integer

m, any Laman graph of dimension m is $(\frac{m+1}{2}, m-1)$-balanced. We first need the following generalization of Lemma V.6.

Lemma V.8. *Let r be a rational number and t be a positive integer. Let G be a graph with $\rho(G) \geq r-1$ and let H be a disconnected subgraph of G with $\rho(H) > r-1$. Let $H_1, \ldots, H_t$ be non-trivial components. If for each $i = 1, \ldots, t$, there exists a rational number r_i such that*

(a) $\rho(H_i) \geq r_i - 1$,

(b) $\sum_{i=1}^{t}(r_i - 1) = r - 1$ *and*

(c) $d_{r_i}(H_i) \leq d_r(G)$,

then $d_r(H) \leq d_r(G)$.

Proof. Suppose there exist rational numbers r_i for $i = 1, \ldots, t$ such that $\sum_{i=1}^{t}(r_i - 1) = r - 1$, then since $\rho(H) = \sum_{i=1}^{t} \rho(H_i)$, we have

$$\rho(H) - (r-1) = \sum_{i=1}^{t} \big(\rho(H_i) - (r_i - 1)\big).$$

Thus

$$\begin{aligned} d_r(H) &= \frac{|E(H)|}{\rho(H) - (r-1)} \\ &= \frac{\sum_{i=1}^{t} |E(H_i)|}{\sum_{i=1}^{t} \big(\rho(H_i) - (r_i - 1)\big)} \\ &\leq \max_{1 \leq i \leq t} \frac{|E(H_i)|}{\rho(H_i) - (r_i - 1)} \end{aligned} \tag{5.7}$$

by Lemma I.8. But for $i = 1, \ldots, t$, we have

$$\frac{|E(H_i)|}{\rho(H_i) - (r_i - 1)} = d_{r_i}(H_i) \leq d_r(G)$$

by the hypothesis (c) of the theorem. Therefore, $d_r(H) \leq d_r(G)$ by (5.7). $\square$

As a corollary to Lemma V.8, we have

Theorem V.9. *A Laman graph of dimension m is an $(\frac{m+1}{2}, m-1)$-balanced graph with density m.*

Proof. Let G be a Laman graph of dimension m. We show that G satisfies the hypothesis of Lemma V.8 with $r = \frac{m+1}{2}$ and $s = m-1$. Since $|E(G)| = m|V(G)| - \binom{m+1}{2}$ and G is connected, we have

$$d_{\frac{m+1}{2}}(G) = \frac{m\big(|V(G)| - \frac{m+1}{2}\big)}{|V(G)| - \frac{m+1}{2}} = m. \tag{5.8}$$

Let H be a subgraph of G with $\rho(H) \geq m-1$ and with no isolated vertices. If H is connected, then since $|V(H)| \geq m$, we have $|E(H)| \leq m\big(|V(H)| - \binom{m+1}{2}\big)$. Therefore,

$$d_{\frac{m+1}{2}}(H) = \frac{|E(H)|}{|V(H)| - \frac{m+1}{2}} \leq m = d_{\frac{m+1}{2}}(G),$$

the result we seek.

Now, we assume that H is disconnected. Let $H_1, \ldots, H_t$ be the components of H with $t > 1$. Let us denote $\rho(H_i)$ by ρ_i for simplicity. Then $\sum_{i=1}^{t} \rho_i = \rho(H)$. Therefore, we have

$$d_{\frac{m+1}{2}}(H) = \frac{|E(H)|}{\rho(H) - (\frac{m+1}{2} - 1)} = \frac{\sum_{i=1}^{t} |E(H_i)|}{\sum_{i=1}^{t} \rho_i - (\frac{m+1}{2} - 1)}. \tag{5.9}$$

Suppose $\rho_i \geq m-1$, for some $i \in \{1, \ldots, m\}$, by the definition of Laman graphs, we have

$$d_{\frac{m+1}{2}}(H_i) \leq m = d_{\frac{m+1}{2}}(G). \tag{5.10}$$

Suppose $\rho_i < m-1$ for some $i \in \{1, \ldots, m\}$, then since G is a simple graph, we have $|E(H_i)| \leq \frac{\rho_i(\rho_i+1)}{2}$. Thus,

$$d_1(H_i) = \frac{|E(H_i)|}{\rho_i} \leq \frac{\rho_i(\rho_i+1)}{2\rho_i} = \frac{\rho_i+1}{2} \leq \frac{m}{2} < m = d_{\frac{m+1}{2}}(G) \tag{5.11}$$

by (5.8). Also, if $\rho_i < m-1$, then

$$d_{\frac{\rho_i}{2}+1}(H_i) = \frac{|E(H_i)|}{\rho_i - \frac{\rho_i}{2}} \leq \frac{\rho_i(\rho_i+1)}{2\left(\rho_i - \frac{\rho_i}{2}\right)} = \frac{\rho_i(\rho_i+1)}{\rho_i} = \rho_i + 1 < m = d_{\frac{m+1}{2}}(G) \tag{5.12}$$

by (5.8). We have two cases to consider.

Case 1: Suppose there exists an integer $j_0 \in \{1, \ldots, m\}$ such that $\rho(H_{j_0}) \geq m-1$. We let $r_{j_0} = \frac{m+1}{2}$ and $r_i = 1$ for $i \neq j_0$. Clearly, for each $i = 1, \ldots, t$, we have $\rho_i \geq r_i - 1$ and $\sum_{i=1}^{t}(r_i - 1) = \frac{m+1}{2} - 1$. Moreover, by (5.10), we have

$$d_{r_{j_0}}(H_{j_0}) \leq d_{\frac{m+1}{2}}(G)$$

and by (5.11), we have

$$d_{r_i}(H_i) = d_1(H_i) \leq d_{\frac{m+1}{2}}(G)$$

for $i \neq j_0$. Thus by Lemma V.8, we get $d_{\frac{m+1}{2}}(H) \leq d_{\frac{m+1}{2}}(G)$.

Case 2: Suppose $\rho(H_i) < m-1$ for all $i \in \{1, \ldots, m\}$. We let $s_i = \frac{\rho_i}{2} + 1$. Then by (5.12), we have

$$d_{s_i}(H_i) \leq d_{\frac{m+1}{2}}(G). \tag{5.13}$$

Also, $\rho_i \geq s_i - 1$ and

$$\sum_{i=1}^{t}(s_i - 1) = \sum_{i=1}^{t}\frac{\rho_i}{2} = \frac{\sum_{i=1}^{t}\rho_i}{2} = \frac{\rho(H)}{2} \geq \frac{m-1}{2} = \frac{m+1}{2} - 1.$$

Therefore, for each $i = 1, \ldots, t$, we can choose $r_i \leq s_i$ such that $\sum_{i=1}^{t}(r_i - 1) = \frac{m+1}{2} - 1$. Since $r_i \leq s_i$, we have $\rho_i \geq r_i - 1$. Also, by (5.1), we have $d_{r_i}(H_i) \leq d_{s_i}(H_i)$. Thus by (5.13), we have $d_{r_i}(H_i) \leq d_{s_i}(H_i) \leq d_{\frac{m+1}{2}}(G)$. By Lemma V.8, we get

$$d_{\frac{m-1}{2}}(H) \leq d_{\frac{m+1}{2}}(G).$$

Thus the theorem follows. □

5.3.3. A degree condition in a (r,s)-balanced graph

The following lemma is useful to check if a graph is (r, s)-balanced. The lemma is used later.

Lemma V.10. *For a rational number r and an integer s with $s > r - 1$, let G be a simple connected (r, s)-balanced graph with $\rho(G) \geq s + 1$. If $v \in V(G)$ with $\rho(G - v) = \rho(G) - 1$, then $deg_G(v) \geq d_r(G)$.*

Proof. Since $\rho(G - v) = \rho(G) - 1$ and $\rho(G) \geq s + 1$, we have $\rho(G - v) \geq s$. Since G is (r, s)-balanced, we have $d_r(G - v) \leq d_r(G)$. Thus

$$\begin{aligned} \frac{|E(G)| - deg_G(v)}{\rho(G - v) - (r - 1)} &\leq \frac{|E(G)|}{\rho(G) - (r - 1)} \\ \frac{|E(G)| - deg_G(v)}{\rho(G) - 1 - (r - 1)} &\leq \frac{|E(G)|}{\rho(G) - (r - 1)} \end{aligned}$$

By Lemma I.10, we have $deg_G(v) \geq \frac{|E(G)|}{\rho(G)-(r-1)} = d_r(G)$. □

5.3.4. Edge-disjoint unions of (r,s)-balanced graphs

In this section, we show that any edge-disjoint union of connected (r, s)-balanced graphs on the same vertex set is also an (r, s)-balanced graph. This is an extension of the Corollary III.6 which proves the below result for $r = 1$ and $s = 1$.

Lemma V.11. *Let r be a rational number and $s > r - 1$ be a non-negative integer. If a connected graph G is an edge-disjoint union of spanning (r, s)-balanced subgraphs G_i, $i = 1, \ldots, t$, then G is (r, s)-balanced.*

Proof. Let $x_i = d_r(G_i)$. Since G is an edge-disjoint union of spanning (r, s)-balanced subgraphs G_i for $i = 1, \ldots, t$, we have

$$d_r(G) = \frac{|E(G)|}{|V(G)| - r} = \frac{\sum_{i=1}^t |E(G_i)|}{|V(G)| - r} = \sum_{i=1}^t x_i. \tag{5.14}$$

Let H be a subgraph of G with $\rho(H) \geq s$. For $i = 1, \ldots, t$, let $H_i = H \cap G_i$. Let H_i' be a subgraph in G_i with $\rho(H_i') = \rho(H)$ such that $H_i \subseteq H_i'$. Such graphs H_i' exist since $\rho(G) \geq s$, some edges can be added to H_i to obtain H_i'.

For each $i = 1, \ldots, t$, since $\rho(H_i) \geq s$ and G_i is (r, s)-balanced, we have

$$d_r(H_i') = \frac{|E(H_i')|}{\rho(H_i') - (r-1)} = \frac{|E(H_i')|}{\rho(H) - (r-1)} \leq d_r(G_i) = x_i. \tag{5.15}$$

Thus,

$$\begin{aligned} d_r(H) &= \frac{|E(H)|}{\rho(H) - (r-1)} \\ &= \sum_{i=1}^t \frac{|E(H_i)|}{\rho(H) - (r-1)} \\ &\leq \sum_{i=1}^t \frac{|E(H_i')|}{\rho(H_i') - (r-1)} \\ &\leq \sum_{i=1}^t x_i = d_r(G) \end{aligned}$$

by (5.15) and (5.14). Hence the lemma follows. □

5.3.5. Existence of Laman graphs with a given degree on a vertex

We saw that Laman graphs of dimension m are $(\frac{m+1}{2}, m-1)$-balanced graphs with density m. Thus Laman graphs are examples of (r, s)-balanced graphs for various values of r and s. In this section, using the results in rigidity theory, we point out that for any given integer $n \geq m$, there exist Laman graphs of dimension m on n vertices.

The following is shown in [27] using a simple graph theoretic argument.

Lemma V.12. *For a positive integer m, let G be a Laman graph of dimension m. Then $\delta(G) \geq m$, where $\delta(G)$ is the minimum degree of G.*

With the help of a special construction of Laman graphs, namely the "Hanneberg's 0-extension", we notice that there exists Laman graphs of dimension m with minimum degree exactly m with any vertex set of size at least m.

The graph K_m is a Laman graph of dimension m. If G is a Laman graph of dimension m, then the *Hanneberg's 0-extension* of G is defined in [27, Page 112] as the graph obtained by adding a vertex v and adding m non-parallel edges incident at v. It can be verified (easily) that the Hanneberg's 0-extension of any Laman graph of dimension m is also a Laman graph of dimension m. Notice that Laman graphs of dimension m that are constructed using only Hanneberg's 0-extensions have a vertex of degree exactly m.

The following result was observed in [27].

Lemma V.13. *For positive integers m and n with $n > m$, there exists Laman graph G of dimension m with $|V(G)| = n$ and a vertex v of degree m. Moreover, the graph $G - v$ is also a Laman graph of dimension m.*

See Figure 14 for examples on Laman graphs of dimension 2 and 3 constructed from K_2 and K_3 respectively via Hanneberg's 0-extension. Note that not all Laman graphs of dimension m can be constructed only by Hanneberg's 0-extensions from K_m. Figure 1 in Chapter I shows an example of a Laman graph of dimension 3 that does not have any vertex of degree 3. Therefore, it cannot be obtained from K_3 via 0-extensions.

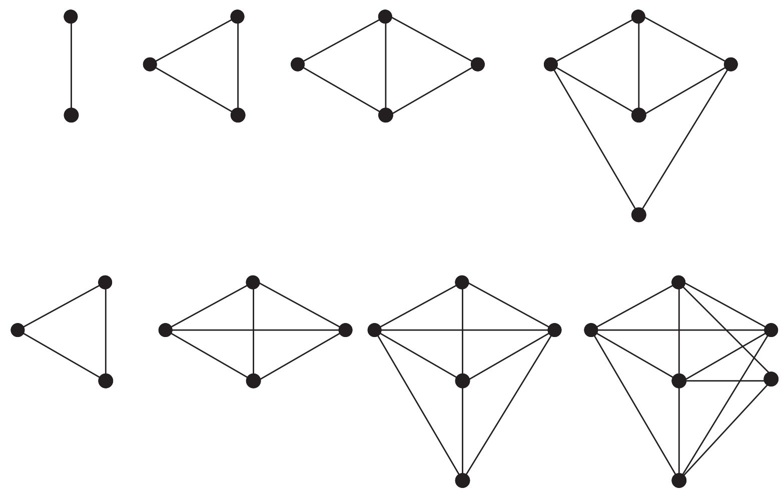

Fig. 14. Laman graphs of dimensions 2 and 3 constructed from K_2 and K_3 respectively via Hanneberg's 0-extensions

5.3.6. Questions on the existence of (r,s)-balanced graphs

As shown by an example in 1.4.2, the existence of a (r, s)-balanced graph G depends on the values of r and s, and also the value of $d_r(G)$. We have the following conjecture:

Conjecture V.14. *For any rational number r and any non-negative integer $s > r-1$ there exists (r, s)-balanced graphs G with arbitrarily large values of $d_r(G)$.*

In view of the above conjecture, we pose the following two questions:

Question 1: For a rational number r, find the smallest possible non-negative integer s such that there exists a (r, s)-balanced graph on a given number of vertices and edges.

For $0 \leq r < 2$, the smallest s is 1 as our examples show. For $r = \frac{m+1}{2}$, a Laman graph of dimension m is an (r, s)-balanced graph when $s = m - 1$.

Question 2: For a rational number r and integers s and n, what is the smallest value $f(r, s, n)$ such that there exists an (r, s)-balanced graph G on n vertices such that $d_r(G) = f(r, s, n)$?

5.4. Examples of ($\mathbf{r}_1$,s)-balanced graphs that are not ($\mathbf{r}_2$,s)-balanced for $\mathbf{r}_1 \leq \mathbf{r}_2$

If r_1 and r_2 are two rational numbers such that $r_1 < r_2$ and s is a non-negative integer such that $s > r_2 - 1$, then in Corollary V.3 we saw that any (r_2, s)-balanced graph is also (r_1, s)-balanced. Here we show that the reverse implication is not true. For any two positive rational numbers r_1, r_2 with $\frac{1}{2} \leq r_1 < r_2$ and for a sufficiently large value of s, we give an example of an (r_1, s)-balanced graph that is not an (r_2, s)-balanced graph. Examples with lesser values of s would be of interest.

Theorem V.15. *Let r_1 and r_2 be rational numbers such that $\frac{1}{2} \leq r_1 < r_2$. Then there exists an integer $s \geq r_2 - 1$ and a (r_1, s)-balanced graph G that is not (r_2, s)-balanced.*

Proof. Let $m \geq 1$ be an integer such that $\frac{m}{2} \leq r_1 < \frac{m+1}{2}$. Then by Lemma I.9, there exist positive integers l and k with $l \leq k$, such that

$$r_1 = \frac{k-l}{k}\binom{m}{2} + \frac{l}{k}\binom{m+1}{2}. \tag{5.16}$$

Let $s = m$. Let G_1 be a Laman graph of dimension $m-1$ and let G_2 be a Laman graph of dimension m with $\rho(G_1) = \rho(G_2) \geq s+1$, on the same vertex set V. Further, let $v \in V$ with $deg_{G_1}v = m-1$ and $deg_{G_2}v = m$ such that $G_1 - v$ and $G_2 - v$ are Laman graphs of dimensions $m-1$ and m respectively. Such graphs, G_1 and G_2 with the presence of the vertex v exist by Lemma V.13. Note that both $G_1 - v$ and $G_2 - v$ are connected because they are Laman graphs.

Let G be the edge-disjoint union of $G_1^{m(k-l)}$ and $G_2^{l(m-1)}$. The graph $G - v$ is connected since both $G_1 - v$ and $G_2 - v$ are connected. Thus $\rho(G - v) = \rho(G) - 1$. This fact will be used later.

Let us calculate the degree of the vertex v in G.

$$deg_G(v) = (k-l)m(m-1) + lm(m-1) = km(m-1). \tag{5.17}$$

Claim: G is (r_1, s)-balanced but not (r_2, s)-balanced. Since G_1 is a $(\frac{m}{2}, m-2)$-balanced graph and G_2 is a $(\frac{m+1}{2}, m-1)$-balanced graph, we have

$$\begin{aligned}
|E(G)| &\leq (k-l)m(m-1)\big(|V(G)| - \frac{m}{2}\big) + lm(m-1)\big(|V(G)| - \frac{m+1}{2}\big) \\
&= m(m-1)\Big(k|V(G)| - (k-l)\frac{m}{2} - l\frac{m+1}{2}\Big) \\
&= km(m-1)\Big(|V(G)| - \Big(\frac{k-l}{k}(\frac{m}{2}) + \frac{l}{k}(\frac{m+1}{2})\Big)\Big) \\
&= km(m-1)(|V(G)| - r_1)
\end{aligned}$$

by (5.16). Thus,

$$d_{r_1}(G) = m(m-1)k. \tag{5.18}$$

For $H \subseteq G$ with $\rho(H) \geq s$, we have

$$\begin{aligned}
|E(H)| &\leq (k-l)m(m-1)\Big(\rho(H) - (\frac{m}{2} - 1)\Big) + lm(m-1)\Big(\rho(H) - (\frac{m+1}{2} - 1)\Big) \\
&= m(m-1)\Big(k\rho(H) - (k-l)(\frac{m}{2} - 1) - l(\frac{m+1}{2} - 1)\Big) \\
&= km(m-1)\Big(\rho(H) - \Big(\frac{k-l}{k}(\frac{m}{2}) + \frac{l}{k}(\frac{m+1}{2}) - 1\Big)\Big) \\
&= km(m-1)\big(\rho(H) - (r_1 - 1)\big).
\end{aligned}$$

This implies that $d_{r_1}(H) \leq km(m-1) = d_{r_1}(G)$. Thus G is (r_1, s)-balanced.

By Lemma V.10, since $\rho(G) \geq s+1$ and $\rho(G-v) = \rho(G) - 1$, we notice that if G is (r_2, s)-balanced, then we must have $deg_G(v) \geq d_{r_2}(G)$. But from (5.17) and (5.18), we have $deg_G(v) = m(m-1)k = d_{r_1}(G)$. Also, since $r_1 < r_2$, we have $d_{r_1}(G) < d_{r_2}(G)$ by (5.1). Thus, $deg_G(v) = d_{r_1}(G) < d_{r_2}(G)$ and therefore G is not (r_2, s)-balanced.

Hence the theorem follows. □

5.5. An application

In this section, we provide an application of (r, s)-balanced graphs. This is an extension of the idea presented in [37] in the context network survivability.

We first recall the definitions in terms of graphs. Let r be a rational number and s a non-negative integer such that $s > r - 1$. Also, let G be a graph with $\rho(G) \geq s$. For each subgraph H of G with $\rho(H) \geq s$, we have

$$d_r(H) = \frac{|E(H)|}{\rho(H) - (r-1)}.$$

Also,

$$\gamma_r^s(G) = \max\{d_r(H) : H \subseteq G, \rho(H) \geq s\}$$

and

$$\mu^s(G) := \min_{X \subseteq E(G)} \left\{ \frac{|X|}{\omega(G-X) - \omega(G)}; \omega(G-X) > \omega(G), |V(G)| - \omega(G-X) \geq s \right\}.$$

By Theorem V.1, G is (r, s)-balanced if $\gamma_r^s(G) = d_r(G)$ and/or $d_r(G) \leq \mu^s(G)$.

Recall that the quantity $\eta_1(G)$ is the minimum among the ratios $\frac{|F|}{\omega(G-F)-\omega(G)}$ for all $F \subseteq E(G)$ with $\omega(G-F) > \omega(G)$. Thus $\eta_1(G)$ gives the minimum among the number of edges that can be removed from G per number of additional components that is formed by the removal. Because of the relation between $\eta_1(G)$ and $\gamma_1(G)$, the concept of 1-balanced graphs is related with the strength of graphs. We present an extension of this idea to (r, s)-balanced graphs.

As we mentioned before, a network may represent a communication network or a road network between cities. These networks are highly susceptible to attacks and problems such as clustering in the case of communication network or traffic failures in the case of road networks. As noticed in [37], the most vulnerable parts of these networks are represented by the edges of the graph since vertices may represent the command centers of the network and may have high security.

A common motive of an enemy trying to attack a network is not to destroy the whole network, but rather to disrupt some areas so that the normal functioning of the system is affected for a certain amount of time. This is a typical consequence of any traffic blockage in a road network.

The following are some key features that network planners may consider while constructing their networks.

1. The edges of the network must be uniformly distributed so that there is no special subgraph that looks "busy", or in other words, has a lot of edges. If there is such a special subgraph, it may be easily possible to isolate this subgraph when some edges fail to function. While designing a network, the network owners already take some security measures in safeguarding the command units. One form of security in command modules may be achieved by grouping a certain number of vertices and distributing the workload to all the vertices. The group of vertices selected for a command unit must be highly reliable. Therefore, the number of vertices that form a command unit is not more than a certain integer chosen by the network designers. On the other hand, communications between various vertices of a command unit is high and thus the number of edges between them is high. Requiring that the network be a balanced graph is impractical because the command units may have higher average degree compared to the whole graph. However, constructing a network that is r-balanced for a sufficiently large rational number r, may be practical and may prove to be useful in the context of network survivability.

2. Let us represent the network by a graph G. We may measure the effort involved in deleting some edges in a network as the number of edges removed divided by the number of additional components produced by the erasure of the edges in the network. Suppose an enemy tries to erase some edges in the network only to leave

behind a subnetwork of at least a reasonable size. Such a knowledgeable enemy is aware of the cost involved in their mission and wants to reduce the effort involved in attacking. Thus, the minimal effort that is required is $\mu^s(G)$ for some positive integer s, rather than just $\mu^0(G)$ (or, $\eta_1(G)$).

Now, the effort involved in attacking all the edges of the graph G may be measured as $\frac{|E(G)|}{|V(G)|-1}$. But, due to several reasons, this effort may be taken as $d_r(G) = \frac{|E(G)|}{|V(G)|-r} \geq \frac{|E(G)|}{|V(G)|-1}$, where r is some positive rational number suitably chosen by the network designers. The network owners may try to distract the enemies by setting $d_r(G)$ smaller than $\mu^s(G)$, so that the effort involved in destroying all the edges of the graph is lesser than destroying part of the edges of the graph.

Thus, constructing an (r, s)-balanced graph G answers the above concerns of the network owner. On one hand, since $d_r(H) \leq d_r(G)$ for all subgraphs of rank at least s, we see that the edges of the graph are spread out evenly, while the number of vertices in any command module is not more than s . On the other hand, we have $d_r(G) \leq \mu^s(G)$ and thus the network is not vulnerable to limited attacks.

5.6. A characterization

In this section, we present a characterization of r-balanced matroids when $0 \leq r \leq 1$. This characterization involves matroid duals and the quantity $\gamma_1(M)$.

Since $d_1(F)$ is defined only for $F \subseteq E$ with $\rho(F) > 0$, to calculate $\gamma_1(M)$ of a matroid M, it does not matter if M is loopless or not. We let $\eta_1(M) = \gamma_1(M) = \infty$ if $\rho(M) = 0$.

If M^* denotes the dual of M, then by the definition of M^*, we have

$$\rho(M^*) = |E| - \rho(M). \tag{5.19}$$

Therefore we have

Lemma V.16. *For a matroid M, $\gamma_1(M) = 1$ if and only if $\eta_1(M^*) = \infty$ and $\gamma_1(M^*) = \infty$ if and only if $\eta_1(M) = 1$.*

Proof. $\gamma_1(M) = 1$ if and only if $\rho(M) = |E|$, or in other words, by (5.19), we have $\rho(M^*) = 0$ which is equivalent to $\eta_1(M^*) = \infty$. Similarly, $\gamma_1(M^*) = \infty$ if and only if $\rho(M^*) = 0$, so that, by (5.19), we have $\rho(M) = |E|$. Therefore, $\rho(F) = |F|$ for all $F \subseteq E$. Thus $\frac{|E|-|F|}{\rho(M)-\rho(F)} = 1$ and so $\eta_1(M) = 1$. □

For a loopless matroid M having a loopless dual M^*, the quantities $\eta_1(M^*)$ and $\gamma_1(M^*)$ were calculated in [10] as follows:

Theorem V.17 (Catlin, Grossman, Hobbs and Lai)**.** *For any loopless matroid M on the set E, having a loopless dual M^*,*

$$\eta_1(M^*) = \frac{\gamma_1(M)}{\gamma_1(M) - 1},$$

and equivalently,

$$\gamma_1(M^*) = \frac{\eta_1(M)}{\eta_1(M) - 1}.$$

Combining Lemma V.16 and Theorem V.17, we have

Theorem V.18. *For any matroid M on the set E and with dual M^*,*

$$\eta_1(M^*) = \frac{\gamma_1(M)}{\gamma_1(M) - 1}, \quad \text{if} \quad \gamma_1(M) \neq 1, \infty;$$

$\eta_1(M^*) = \infty$ if $\gamma_1(M) = 1$ and $\eta_1(M^*) = 1$ if $\gamma_1(M) = \infty$. Also,

$$\gamma_1(M^*) = \frac{\eta_1(M)}{\eta_1(M) - 1} \quad \text{if} \quad \eta_1(M) \neq 1, \infty;$$

$\gamma_1(M^*) = \infty$ if $\eta_1(M) = 1$ and $\gamma_1(M^*) = 1$ if $\eta_1(M) = \infty$.

Corollary V.19. *A matroid M is 1-balanced if and only if M^* is 1-balanced.*

Proof. Using the computation of $\eta_1(M^*)$ and $\gamma_1(M^*)$ in Theorem V.18, we have $\gamma_1(M) = \eta_1(M)$ if and only if $\gamma_1(M^*) = \eta_1(M^*)$ *i.e.,* M^* is 1-balanced. □

It is apparent that the dual of a 1-balanced matroid is balanced since the dual is 1-balanced. However, for example, it is not true that duals of 0-balanced matroids are 0-balanced. The graphs in Figure 15 are both 0-balanced and the duals of both the graphs have density $d_0 = \frac{9}{3} = 3$. It can be checked that the matroid dual of the first graph is balanced. But the matroid dual of the second graph has 7 parallel edges whose density d_0 is $\frac{7}{2} > 3$. Hence the matroid dual of the second matroid is not balanced.

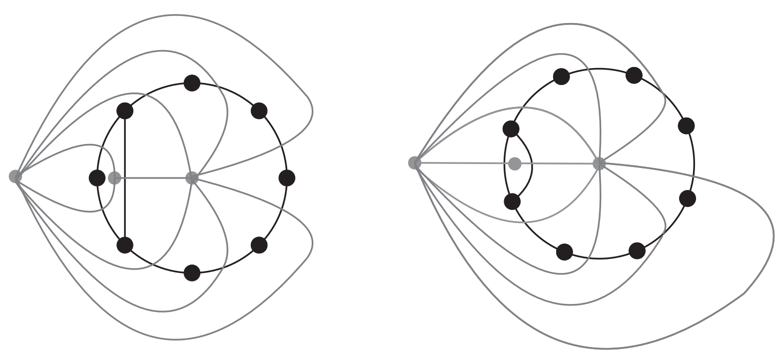

Fig. 15. Duals of balanced graphs

The following theorem classifies all matroids whose dual matroids are r-balanced when $0 \leq r < 1$.

Theorem V.20. *Let M be a matroid on a set E. For $0 \leq r < 1$, M^* is r-balanced if and only if either $\rho(M) \leq 1 - r$ or $\gamma_1(M)(\rho(M) + r - 1) \leq |E|$.*

Proof. For $0 \leq r < 1$, we have $\eta_r(M) = \min\{\mu^0(M), d_r(M)\}$ by the definition of $\eta_r(M)$. But $\mu^0(M) = \eta_1(M)$.

Thus the matroid M^* is r-balanced if and only if

$$\eta_1(M^*) \geq d_r(M^*) = \frac{|E|}{\rho(M^*) - (r-1)}. \tag{5.20}$$

From (5.20) and (5.19), we have

$$\eta_1(M^*) \geq \frac{|E|}{|E| - \rho(M) - (r-1)}. \tag{5.21}$$

Case (i) If $\eta_1(M^*) = \infty$, then by the definition of η_1, we have $\rho(M^*) = 0$, and hence M^* is r-balanced. In this case, we have $\gamma_1(M) = 1$ by Lemma V.16. Since $\rho(M^*) = 0$, by (5.19) we have $\rho(M) = |E|$. Hence $\gamma_1(M)(\rho(M) + r - 1) = 1(|E| + r - 1) < |E|$ since $r < 1$.

Case (ii) If $\eta_1(M^*) = 1$, then by (5.21), M^* is r-balanced if and only if

$$\frac{|E|}{|E| - \rho(M) - (r-1)} \leq 1$$

i.e., $|E| \leq |E| - \rho(M) - (r-1)$ or $\rho(M) \leq 1 - r$.

Case (iii) We may now assume that $\eta_1(M^*) \neq \infty$ and $\eta_1(M^*) \neq 1$. Hence by Lemma V.16, $\gamma_1(M) \neq 1$ and $\gamma_1(M) \neq \infty$. By Theorem V.18,

$$\eta_1(M^*) = \frac{\gamma_1(M)}{\gamma_1(M) - 1}. \tag{5.22}$$

Noting $|E| - \rho(M) + 1 > 0$, by (5.21) and (5.22), we have

$$\frac{\gamma_1(M)}{\gamma_1(M) - 1} \geq \frac{|E|}{|E| - \rho(M) - (r-1)}$$

which is equivalent to

$$\gamma_1(M)|E| - \gamma_1(M)\rho(M) - \gamma_1(M)(r-1) \geq \gamma_1(M)|E| - |E|,$$

i. e., $\gamma_1(M)(\rho(M) + r - 1) \leq |E|$. □

Since $(M^*)^* = M$, Theorem V.20 may be considered as a characterization of r-balanced matroids for $0 \leq r < 1$.

The following are algorithms to check if a matroid M is r-balanced for $0 \leq r \leq 1$.

The first algorithm works by using (5.4) and Theorem V.1.

Algorithm 1:

Step 1 : Find $\eta_1(M)$.

Step 2 : If $d_r(M) \leq \eta_1(M)$, then M is r-balanced. Else, M is not r-balanced.

The next algorithm follows from Theorem V.20.

Algorithm 2:

Step 1 : If $\rho(M^*) \leq 1 - r$, then M is r-balanced. Else,

Step 2 : Find $\gamma_1(M^*)$.

Step 3 : If $\gamma_1(M^*) \leq \frac{|E|}{\rho(M^*)-1}$, then M is r-balanced. Else, M is not r-balanced.

Since the time it takes to find $\eta_1(M)$ and $\gamma_1(M)$ is polynomial in the input size, the time to find if M is r-balanced or not is polynomial in the input size.

5.7. Further questions

Question 3: If $r > 1$ and s is an integer greater than $r - 1$, is there an efficient algorithm to check if a given graph is (r, s)-balanced?

Question 4: Given a graph G, find the minimum values of r and s such that G is (r, s)-balanced.

CHAPTER VI

PAIRS OF SUBMODULAR FUNCTIONS, BALANCED SETS AND DENSITY

The density functions on a graph give information about which subgraphs are densely packed. For instance, the sets with high values of $d_1(G)$ correspond to subgraphs where we can pack the largest (fractional) number of edge-disjoint forests. In the literature, the density function $d_1(G)$ is generalized and defined in terms of a pair of "submodular functions". In this chapter, we show how our results presented in Chapter IV extend to this generalized setting.

In Section 6.1 we recall some definitions and provide a brief survey of the generalized density function. In Section 6.2 some useful results are derived. In Section 6.3, we recall the definitions of "matroid extensions" from matroid theory. This will be used in Section 6.4 to show our main result.

6.1. Background

A real-valued function f on the power set of a set E is said to be *submodular* if and only if for all $X, Y \subseteq E$,

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y).$$

If we have

$$f(X \cup Y) + f(X \cap Y) \geq f(X) + f(Y),$$

then f is said to be *supermodular*. For the purpose of this chapter, we assume that all submodular functions used in this chapter take non-zero values on all non-empty sets. The rank function of a loopless matroid is an example of such a submodular function.

Throughout this chapter, we assume that the set E is non-empty. A submodular function f on the power set of a set E is a *polymatroid function* if and only if it takes the value zero on ϕ and is non-negative and increasing, *i.e.,* $f(A) \leq f(B)$ if $A \subseteq B$. We call an integer-valued polymatroid function an *integer polymatroid function.* Edmonds and Rota [20] showed that if f is an integer polymatroid function, then f induces a matroid on E whose rank function is $r_f(X) := \min\{f(Y)+|X-Y| : Y \subseteq X\}$ for $X \subseteq E$. See [64, Chapter 12] for a proof of this result. On the other hand, the rank function of a matroid is an integer polymatroid function whose values do not exceed the value one for any singleton set. If f is a rank function of a matroid, then $r_f = f$. Thus, an integer polymatroid function can be regarded as a generalization of a matroid rank function. There are a number of instances where integer polymatroid functions are found in the field of graph theory. We refer the readers to [64, Chapter 12] and [58, Chapter 9] for some examples and also for the study of submodular functions in this context.

The following definition of "density" with respect to two submodular functions is given by Narayanan [57], [58]. Let f_1 and f_2 be two positive-valued submodular functions defined on the subsets of a set E. Let $\mathcal{F} = \{f_1, f_2\}$. The *density* of $X \subseteq E(X \neq \phi)$ with respect to $\mathcal{F}$ is the ratio

$$d_{\mathcal{F}}(X) = \frac{f_2(E) - f_2(E - X)}{f_1(X) - f_1(\phi)}.$$

The set E is said to be *balanced with respect to* $\mathcal{F}$ if and only if E has the highest density among all its subsets. If f_1 is the rank function of a matroid on E and if f_2 is the cardinality function defined for all subsets of E, then the definition of $d_{\mathcal{F}}$ coincides with that of the density function d_1.

Since f_2 is submodular, the numerator of $d_{\mathcal{F}}(X)$, namely $f_2(E)-f_2(E-X)$, is less than or equal to $f_2(X)$. Thus, if the value of $d_{\mathcal{F}}(X)$ is high, then $f_2(X)/f_1(X)$ is also high. We conclude that the density function $d_{\mathcal{F}}$ gives useful information about the relative value of $f_2(X)$ with respect to $f_1(X)$. In some cases, these relative values have special meanings. For instance, if f_1 is the rank function of a graph G and f_2 is the cardinality function defined on the edge sets of G, then the sets of the highest density $d_{\mathcal{F}}$ (or, d_1) correspond to the subgraphs where we can pack the largest (fractional) number of edge-disjoint forests.

Density for a pair of submodular functions has been studied in a different setting through a concept called "principal partitions". The subject of principal partitions began with the study of graphs by Kishi and Kajitani [48], continued with matroids by Bruno and Weinberg [5], Tomizawa [82] and Narayanan [57], and was subsequently generalized to a pair of polymatriod functions by Iri [42]. The literature of the study of principal partitions is large. We refer the readers to [58, Chapter 10] and [24, Chapter IV, Section 7] for detailed treatments of the subject.

Let f_1 and f_2 be submodular functions on the subsets of a set E. The collection of all sets which minimize $\lambda f_1(X)+f_2(E-X)$ over subsets of E, for any possible value of $\lambda \geq 0$, is called the *principal partition of* $\{f_1, f_2\}$. In [58], the function f_2 is chosen to be strictly increasing in order to achieve a nice containment relation between the subsets that form the principal partition. The number λ_0 is said to be a *critical value of* $\{f_1, f_2\}$ if there is more than one subset that minimizes $\lambda_0 f_2(X) + f_1(E - X)$. It is shown in [58] that E is balanced with respect to $\{f_1, f_2\}$ if and only if the number of critical values of $\{f_1, f_2\}$ is one and is equal to $f_2(E)/f_1(E)$.

For integers m, n with $m \geq n$, the question of whether there is a 1-balanced matroid on a set of m elements such that the rank of the whole matroid is n, is a trivial one. The matroid $U_{n,m}$, the uniform matroid of rank n on a m-element set, is an example of a 1-balanced matroid with density $\frac{m}{n}$.

In this chapter, we show that for any two positive numbers m, n, and a finite set E such that $m \leq |E|$ and $n \leq |E|$, there exist two matroids on E with rank function

ρ_1 and ρ_2 such that $\rho_1(E) = n$ and $\rho_2(E) = m$ and if $\mathcal{F} = \{\rho_1, \rho_2\}$, then E is balanced with respect to $\mathcal{F}$. This result is similar to Corollary IV.5 and the connection between these two results is discussed in the last section of this chapter. Since we deal with rank functions of matroids, some matroidal notations and concepts are recalled in Section 6.3.

6.2. Lemmas

Let f be a function on the set E and let the function t be defined as $t(X) := f(E-X)$ for all $X \in E$. Then, we have

Lemma VI.1. *The function f is submodular if and only if t is submodular.*

Proof. Suppose f is submodular. Let $X_1, X_2 \subseteq E$. Then

$$E - (X_1 \cup X_2) = (E - X_1) \cap (E - X_2)$$

and

$$E - (X_1 \cap X_2) = (E - X_1) \cup (E - X_2).$$

By the submodularity of f, we have

$$f\big((E - X_1) \cap (E - X_2)\big) + f\big((E - X_1) \cup (E - X_2)\big) \leq f(E - X_1) + f(E - X_2).$$

Therefore,

$$\begin{aligned} t(X_1 \cup X_2) + t(X_1 \cup X_2) &= f\big(E - (X_1 \cup X_2)\big) + f\big(E - (X_1 \cap X_2)\big) \\ &= f\big((E - X_1) \cap (E - X_2)\big) + f\big((E - X_1) \cup \big(E - X_2)\big) \\ &\leq f(E - X_1) + f(E - X_2) \\ &= t(X_1) + t(X_2). \end{aligned}$$

Thus t is submodular. The sufficiency of the theorem is proved by interchanging the roles of f and t in the proof for necessity. □

Let f_1, f_2 be two polymatroid functions defined on a set E. Let $\mathcal{F} = \{f_1, f_2\}$. The denominator of $d_{\mathcal{F}}(X)$ is $f_1(X)$ since $f_1(\phi) = 0$. For convenience, we denote the numerator of $d_{\mathcal{F}}(X)$ as $h_2(X)$ throughout the chapter, *i.e.*, $h_2(X) = f_2(E) - f_2(E - X)$. Thus $d_{\mathcal{F}}(X) = h_2(X)/f_1(X)$. Note that $h_2(X)$ is a supermodular function since by Lemma VI.1, it is clear that $f_2(E - X)$ is a submodular function.

Let

$$\gamma_{\mathcal{F}}(E) = \max_{X \subseteq E} d_{\mathcal{F}}(X).$$

We call a set $X \subseteq E$ a $\gamma_{\mathcal{F}}(E)$-*achieving set* if $d_{\mathcal{F}}(X) = \gamma_{\mathcal{F}}(E)$.

Lemma VI.2. *Let $\mathcal{F} = \{f_1, f_2\}$ be a set of two polymatroid functions on a set E. Let X_1, X_2 be two non-empty $\gamma_{\mathcal{F}}(E)$-achieving sets. Then $X_1 \cup X_2$ is a $\gamma_{\mathcal{F}}(E)$-achieving set and if $X_1 \cap X_2 \neq \phi$, then $X_1 \cap X_2$ is also a $\gamma_{\mathcal{F}}(E)$-achieving set.*

Proof. Since X_1 and X_2 are $\gamma_{\mathcal{F}}(E)$-achieving, we have

$$\frac{h_2(X_1)}{f_1(X_1)} = \gamma_{\mathcal{F}}(E) = \frac{h_2(X_2)}{f_1(X_2)}. \tag{6.1}$$

Since h_2 is supermodular and f_1 is submodular, we have

$$h_2(X_1 \cup X_2) + h_2(X_1 \cap X_2) \geq h_2(X_1) + h_2(X_2)$$

and

$$f_1(X_1 \cup X_2) + f_1(X_1 \cap X_2) \leq f_1(X_1) + f_1(X_2).$$

Thus,

$$\frac{h_2(X_1 \cup X_2) + h_2(X_1 \cap X_2)}{f_1(X_1 \cup X_2) + f_1(X_1 \cap X_2)} \geq \frac{h_2(X_1) + h_2(X_2)}{f_1(X_1) + f_1(X_2)}. \tag{6.2}$$

Using Lemma I.8 on the right-hand side of (6.2) and then using (6.1), we have

$$\begin{aligned} \frac{h_2(X_1 \cup X_2) + h_2(X_1 \cap X_2)}{f_1(X_1 \cup X_2) + f_1(X_1 \cap X_2)} &\geq \frac{h_2(X_1) + h_2(X_2)}{f_1(X_1) + f_1(X_2)} \\ &\geq \min\left\{\frac{h_2(X_1)}{f_1(X_1)}, \frac{h_2(X_2)}{f_1(X_2)}\right\} \\ &= \gamma_{\mathcal{F}}(E). \end{aligned} \tag{6.3}$$

Now, if $X_1 \cap X_2 = \phi$, we have $h_2(X_1 \cap X_2) = 0 = f_1(X_1 \cap X_2)$ and so by (6.3) we have

$$d_{\mathcal{F}}(X_1 \cup X_2) = \frac{h_2(X_1 \cup X_2)}{f_1(X_1 \cup X_2)} \geq \gamma_{\mathcal{F}}(E).$$

But $d_{\mathcal{F}}(X_1 \cup X_2) \leq \gamma_{\mathcal{F}}(E)$ and therefore $d_{\mathcal{F}}(X_1 \cup X_2) = \gamma_{\mathcal{F}}(E)$, *i.e.*, $X_1 \cup X_2$ is $\gamma_{\mathcal{F}}(E)$-achieving.

Suppose $X_1 \cap X_2 \neq \phi$. Applying Lemma I.8 on the left-hand side of (6.3), we have

$$\begin{aligned} \max\left\{\frac{h_2(X_1 \cup X_2)}{f_1(X_1 \cup X_2)}, \frac{h_2(X_1 \cap X_2)}{f_1(X_1 \cap X_2)}\right\} &\geq \frac{h_2(X_1 \cup X_2) + h_2(X_1 \cap X_2)}{f_1(X_1 \cup X_2) + f_1(X_1 \cap X_2)} \\ &\geq \gamma_{\mathcal{F}}(E). \end{aligned} \tag{6.4}$$

But

$$\frac{h_2(X_1 \cup X_2)}{f_1(X_1 \cup X_2)} \leq \gamma_{\mathcal{F}}(E)$$

and

$$\frac{h_2(X_1 \cap X_2)}{f_1(X_1 \cap X_2)} \leq \gamma_{\mathcal{F}}(E).$$

Thus, we have

$$\gamma_{\mathcal{F}}(E) \geq \max\left\{\frac{h_2(X_1 \cup X_2)}{f_1(X_1 \cup X_2)}, \frac{h_2(X_1 \cap X_2)}{f_1(X_1 \cap X_2)}\right\} \geq \gamma_{\mathcal{F}}(E)$$

by (6.4). But the above condition holds if only if

$$\max\left\{\frac{h_2(X_1 \cup X_2)}{f_1(X_1 \cup X_2)}, \frac{h_2(X_1 \cap X_2)}{f_1(X_1 \cap X_2)}\right\} = \gamma_{\mathcal{F}}(E)$$

and

$$\frac{h_2(X_1 \cup X_2)}{f_1(X_1 \cup X_2)} = \gamma_{\mathcal{F}}(E) = \frac{h_2(X_1 \cap X_2)}{f_1(X_1 \cap X_2)}$$

by Lemma I.8. Thus $X_1 \cup X_2$ and $X_1 \cap X_2$ are $\gamma_{\mathcal{F}}(E)$-achieving. □

As an important consequence of Lemma VI.2, we note that there is a unique maximal $\gamma_{\mathcal{F}}(E)$-achieving set. This fact will be used frequently in the chapter.

Lemma VI.3. *Let $\mathcal{F} = \{\rho_1, \rho_2\}$ be a set of two rank functions on a set E. If F_0 is the maximal $\gamma_{\mathcal{F}}(E)$-achieving subset of E, then F_0 is a flat in the matroid induced by ρ_1 on E.*

Proof. Let Cl_1 denote the closure function of the matroid induced by ρ_1 on E. By (CL1), $F_0 \subseteq Cl_1(F_0)$. Since ρ_2 is increasing and $E - Cl_1(F_0) \subseteq E - F_0$, we have

$$\rho_2(E - F_0) \geq \rho_2(E - Cl_1(F_0)).$$

Also, $\rho_1(F_0) = \rho_1(Cl_1(F_0))$. Therefore, we have

$$\begin{aligned} \gamma_{\mathcal{F}}(E) &= d_{\mathcal{F}}(F_0) \\ &= \frac{\rho_2(E) - \rho_2(E - F_0)}{\rho_1(F_0)} \\ &\leq \frac{\rho_2(E) - \rho_2(E - Cl_1(F_0))}{\rho_1(Cl_1(F_0))} \\ &= d_{\mathcal{F}}(Cl_1(F_0)) \\ &\leq \gamma_{\mathcal{F}}(E). \end{aligned}$$

Thus the above inequality is an equality and since there is only one maximal $\gamma_{\mathcal{F}}(E)$-achieving subset of E, we have $F_0 = Cl_1(F_0)$, *i.e.*, F_0 is closed with respect to ρ_1. □

Let f be a real-valued function defined on the subsets of a non-empty set E. Given $X \subseteq E$, we denote by $f^{E/X}$ the function $f(X \cup Y) - f(X)$ for all $Y \subseteq E - X$, and call $f^{E/X}$ the *contraction* of f to $E - X$.

Lemma VI.4. *Let $\mathcal{F} = \{f_1, f_2\}$ be a set of two polymatroid functions on a set E, such that E is not balanced with respect to $\mathcal{F}$ and let F_0 be the maximal $\gamma_{\mathcal{F}}(E)$-achieving subset of E. Let $\mathcal{F}^{E/F_0} := \{f_1^{E/F_0}, f_2^{E/F_0}\}$. If $F \subseteq E - F_0$ is a non-empty set, then $d_{\mathcal{F}^{E/F_0}}(F) < \gamma_{\mathcal{F}}(E)$.*

Proof. Since F_0 is the maximal $\gamma_{\mathcal{F}}(E)$-achieving set and since $F_0 \cup F$ is a subset of E strictly containing F_0, we have

$$d_{\mathcal{F}}(F_0 \cup F) < \gamma_{\mathcal{F}}(E). \tag{6.5}$$

On the other hand, by the definition of contraction, we have

$$h_2(F_0 \cup F) = h_2(F_0) + h_2^{E/F_0}(F)$$

and

$$f_1(F_0 \cup F) = f_1(F_0) + f_1^{E/F_0}(F).$$

Thus,

$$\begin{aligned} d_{\mathcal{F}}(F_0 \cup F) &= \frac{h_2(F_0 \cup F)}{f_1(F_0 \cup F)} \\ &= \frac{h_2(F_0) + h_2^{E/F_0}(F)}{f_1(F_0) + f_1^{E/F_0}(F)} \\ &\geq \min\left\{\frac{h_2(F_0)}{f_1(F_0)}, \frac{h_2^{E/F_0}(F)}{f_1^{E/F_0}(F)}\right\} \end{aligned}$$

by Lemma I.8.

But

$$\frac{h_2(F_0)}{f_1(F_0)} = d_{\mathcal{F}}(F_0) = \gamma_{\mathcal{F}}(E)$$

and

$$\frac{h_2^{E/F_0}(F)}{f_1^{E/F_0}(F)} = d_{\mathcal{F}^{E/F_0}}(F).$$

Hence,

$$d_{\mathcal{F}}(F_0 \cup F) \geq \min\{\gamma_{\mathcal{F}}(E), d_{\mathcal{F}^{E/F_0}}(F)\}. \tag{6.6}$$

Now, if $d_{\mathcal{F}^{E/F_0}}(F) \geq \gamma_{\mathcal{F}}(E)$, then by (6.6), we have $d_{\mathcal{F}}(F_0 \cup F) \geq \gamma_{\mathcal{F}}(E)$, a contradiction to (6.5). Hence $d_{\mathcal{F}^{E/F_0}}(F) < \gamma_{\mathcal{F}}(E)$. □

6.3. Matroid extensions

We refer the readers to Section 1.1.1 for the matroidal terms that appear in this section.

Let M be a matroid on the set E with rank function ρ. If M is obtained from a matroid N by deleting a non-empty subset T of $E(N)$, then N is called an *extension* of M. In particular, if $|T| = 1$, then N is a *single-element extension* of M. Crapo [14] characterized all single-element extensions of a matroid.

The following Lemma is proved in the literature; see [64, Page 35] for example.

Lemma VI.5. *If X and Y are two flats of a matroid M, then $X \cap Y$ is a flat.*

A pair of flats (X, Y) *is a modular pair of flats* if

$$\rho(X) + \rho(Y) = \rho(X \cup Y) + \rho(X \cap Y).$$

An arbitrary set $\mathcal{C}$ of flats of a matroid M is called a *modular cut* if it satisfies the following:

(1) If $F \in \mathcal{C}$ and F' is a flat of M containing F, then $F' \in \mathcal{C}$.

(2) If F_1, $F_2 \in \mathcal{C}$ and (F_1, F_2) is a modular pair, then $F_1 \cap F_2 \in \mathcal{C}$.

A trivial example of a modular cut in a matroid M is the set of all flats that contain all the elements of F, for a fixed $F \subseteq E$. (This type of collection satisfies (1) clearly and (2) follows by Lemma VI.5.)

We now provide a non-trivial example of a modular cut that is used in the next section.

Lemma VI.6. *Let M be a matroid on a set E and let $e, f \in E$. Let $\mathcal{C}$ be the set of all flats F of M such that either $\{e,f\} \subseteq F$ or $\{e,f\} \subseteq E - F$ but $f \in Cl(F \cup \{e\})$ (equivalently, by (CL4), $e \in Cl(F \cup \{f\})$). Then, $\mathcal{C}$ is a modular cut of M.*

Proof. (1) Let $F \in \mathcal{C}$ and let F' be a flat in M containing F. If $\{e,f\} \subseteq F$, then since $F \subseteq F'$, we have $\{e,f\} \subseteq F'$. Therefore $F' \in \mathcal{C}$. Suppose $\{e,f\} \subseteq E - F$ but $f \in Cl(F \cup \{e\})$. If $\{e,f\} \subseteq F'$, then $F \in \mathcal{C}$. Hence, we may assume that $\{e,f\} \not\subseteq F'$. If $e \in F'$, then by (CL2) and (CL3), since $F \cup \{e\} \subseteq F'$, we have $Cl(F \cup \{e\}) \subseteq Cl(F') = F'$. But $f \in Cl(F \cup \{e\})$ and thus $f \in F'$, a contradiction since $\{e,f\} \not\subseteq F'$. Thus $e \notin F'$. By a similar (symmetric) argument, we conclude that $f \notin F'$. Thus $\{e,f\} \subseteq E - F'$. By (CL2), we have $Cl(F \cup \{e\}) \subseteq Cl(F' \cup \{e\})$. Since $f \in Cl(F \cup \{e\})$, we see that $f \in Cl(F' \cup \{e\})$ and therefore $F' \in \mathcal{C}$.

(2) Let $F_1, F_2 \in \mathcal{C}$ such that (F_1, F_2) is a modular pair. Then by Theorem VI.5, $F_1 \cap F_2$ is a flat. If for each $i = 1, 2$, we have $\{e,f\} \subseteq F_i$, then $\{e,f\} \subseteq F_1 \cap F_2$ and therefore $F_1 \cap F_2 \in \mathcal{C}$. Thus, we assume without loss of generality that $\{e,f\} \subseteq E - F_1$. Then, $\{e,f\} \subseteq E - (F_1 \cap F_2)$. Since $F_1 \in \mathcal{C}$, we have $f \in Cl(F_1 \cup \{e\})$. To show $f \in Cl\big((F_1 \cap F_2) \cup \{e\}\big)$, we show that $\rho\big((F_1 \cap F_2) \cup \{e,f\}\big) = \rho(F_1 \cap F_2) + 1$. We have two cases:

(2.1) If $\{e, f\} \subseteq F_2$, then $(F_1 \cap F_2) \cup \{e, f\} = (F_1 \cup \{e, f\}) \cap F_2$. But,

$$\begin{aligned}
\rho\big((F_1 \cup \{e, f\}) \cap F_2\big) \quad + \quad & \rho\big((F_1 \cup \{e, f\}) \cup F_2\big) \\
\leq \quad & \rho\big(F_1 \cup \{e, f\}\big) + \rho(F_2), \\
& \quad \text{since } \rho \text{ is submodular,} \\
= \quad & \rho(F_1) + 1 + \rho(F_2), \\
& \quad \text{since } \rho(F_1 \cup \{e, f\}) = \rho(F_1) + 1, \\
= \quad & \rho(F_1 \cap F_2) + 1 + \rho(F_1 \cup F_2), \\
& \quad \text{since } (F_1, F_2) \text{ is a modular pair.}
\end{aligned}$$

Also, $(F_1 \cup \{e, f\}) \cup F_2 = F_1 \cup F_2$. Thus the above inequality is in fact an equality. Therefore, $\rho\big((F_1 \cap F_2) \cup \{e, f\}\big) = \rho(F_1 \cap F_2) + 1$.

(2.2) If $\{e, f\} \subseteq E - F_2$, then $(F_1 \cap F_2) \cup \{e, f\} = (F_1 \cup \{e, f\}) \cap (F_2 \cup \{e, f\})$. By a similar reasoning as the above case, we have

$$\begin{aligned}
\rho\big((F_1 \cup \{e, f\}) \cap (F_2 \cup \{e, f\})\big) \quad + \quad & \rho\big((F_1 \cup \{e, f\}) \cup (F_2 \cup \{e, f\})\big) \\
\leq \quad & \rho\big(F_1 \cup \{e, f\}\big) + \rho\big(F_2 \cup \{e, f\}\big) \\
= \quad & \rho(F_1) + 1 + \rho(F_2) + 1 \\
= \quad & \rho(F_1 \cap F_2) + 1 + \rho(F_1 \cup F_2) + 1 \\
= \quad & \rho(F_1 \cap F_2) + 1 + \rho\big((F_1 \cup F_2) \cup \{e, f\}\big),
\end{aligned}$$

since $F_1 \cup F_2 \in \mathcal{C}$ and $\{e, f\} \subseteq E - (F_1 \cup F_2)$. Also, $(F_1 \cup \{e, f\}) \cup (F_2 \cup \{e, f\}) = (F_1 \cup F_2) \cup \{e, f\})$. Thus the above inequality is in fact an equality. Thus $\rho\big((F_1 \cap F_2) \cup \{e, f\}\big) = \rho(F_1 \cap F_2) + 1$.

The lemma follows from (1) and (2). $\square$

Notation: For a matroid M, we denote the rank function of a matroid as ρ_M

and the closure function as Cl_M.

If M is a matroid on a set E, all the possible single-element extensions of M via addition of a new element e' to E is described by the following result by Crapo [14].

Theorem VI.7 (Crapo [14]). *Let $\mathcal{C}$ be a modular cut of a matroid M on a set E. Then there is a unique single-element extension N of M on $E \cup \{e'\}$ such that $\mathcal{C}$ consists of those flats F of M for which $F \cup \{e'\}$ is a flat of N having the same rank as F. Moreover, for all subsets X of E,*

$$\rho_N(X) = \rho_M(X) \quad \text{and}$$

$$\rho_N(X \cup \{e'\}) = \begin{cases} \rho_M(X), & \text{if} \quad Cl_M(X) \in \mathcal{C}, \\ \rho_M(X) + 1, & \text{if} \quad Cl_M(X) \notin \mathcal{C}. \end{cases} \tag{6.7}$$

Notation: The matroid N of the above theorem is denoted as $M +_{\mathcal{C}} e'$ and if $\mathcal{C}$ is understood from context, we denote N simply by $M + e'$.

The following was also observed by Crapo [14]. (See [64, Page 255] for a proof.)

Corollary VI.8. *If M is a matroid and $\mathcal{C}$ is a modular cut in M, then $\rho(M +_{\mathcal{C}} e') = \rho(M)$ if and only if $\mathcal{C} \neq \phi$.*

6.4. Transforming a set into a balanced set with respect to two matroid rank functions

This section has our main result, namely, for any given positive integers, m and n, we show the existence of two rank functions on a non-empty set E such that $m \leq |E|$, $n \leq |E|$ and E is balanced with respect to the functions with density m/n. The method we use to prove this result is to take any two rank functions ρ_1, ρ_2 on the set E and transform them into a new pair of rank functions so that E is balanced with respect to the new pair if E is not balanced with respect to ρ_1, ρ_2. Thus, this is a generalization of our main result in Chapter IV. The following theorem is a generalization of Theorem IV.4. Let e' be a new element not in E.

Theorem VI.9. *Let $\mathcal{F} = \{\rho_1, \rho_2\}$ be a set of two rank functions of two matroids on a set E such that E is not balanced with respect to $\mathcal{F}$. Then, there exists $\mathcal{F}' = \{\rho_1', \rho_2'\}$ whose elements are rank functions of matroids on the set $E' = (E \cup \{e'\}) - \{e\}$, for some $e \in E$ and $e' \notin E$ such that*

1. *$\rho_i'(E') = \rho_i(E)$ for $i = 1, 2$, and*
2. *$\gamma_{\mathcal{F}'}(E') \leq \gamma_{\mathcal{F}}(E)$. Further, if $\gamma_{\mathcal{F}'}(E') = \gamma_{\mathcal{F}}(E)$, then the maximal $\gamma_{\mathcal{F}'}$-achieving subset of E' is properly contained in the maximal $\gamma_{\mathcal{F}}$-achieving subset of E.*

Proof. Let M_1 and M_2 be the matroids induced by ρ_1 and ρ_2 respectively. We denote Cl_{M_i} simply as Cl_i. Let F_0 be the maximal $\gamma_{\mathcal{F}}$-achieving set of E. By Lemma VI.3, F_0 is a flat in M_1. Since E is not balanced with respect to $\mathcal{F}$, we have $F_0 \neq E$. Let $e \in F_0$ and $f \notin F_0$.

Let $\mathcal{C}_1$ be the set of all flats F in M_1 such that either $\{e, f\} \subseteq F$ or $\{e, f\} \subseteq E - F$ but $\rho_1(F \cup \{e, f\}) = \rho_1(F) + 1$ and let $\mathcal{C}_2$ be the set of all flats in M_2 that contain e. By Lemma VI.6 and the discussion preceding it, we see that $\mathcal{C}_1$ is a modular cut in M_1 and $\mathcal{C}_2$ is a modular cut in M_2. Note that if $F \subseteq F_0$, then $\rho_1(F \cup \{e, f\}) > \rho_1(F) + 1$ since $f \notin F_0$ and F_0 is a flat in M_1. Thus $Cl_1(F) \notin \mathcal{C}_1$. This fact is used later in the proof.

Let $i \in \{1, 2\}$. By Theorem VI.7, $M_i + e'$ is a matroid on $E + e'$. Let ρ_i' be the rank function of $M_i + e'$. Then, for $X \subseteq E$, we have

$$\rho_i'(X) = \rho_i(X), \tag{6.8}$$

and

$$\rho_i'(X \cup \{e'\}) = \begin{cases} \rho_i(X), & \text{if} \quad Cl_i(X) \in \mathcal{C}_i, \\ \rho_i(X) + 1, & \text{if} \quad Cl_i(X) \notin \mathcal{C}_i. \end{cases} \tag{6.9}$$

Consider the matroid $M_i + e' - e$, which is obtained from $M_i + e'$ by deleting the element e. The rank function of $M_i + e' - e$ is also ρ_i'. Notice that $M_1 + e' - e$ is isomorphic to M_1 with e' corresponding to e.

Let $i \in \{1, 2\}$. By Corollary VI.8,

$$\rho_i'(E \cup \{e'\}) = \rho_i'(E). \tag{6.10}$$

Thus $e \in Cl_{M_i+e'-e}(E \cup \{e'\})$ and therefore

$$\rho_i'(E \cup \{e'\} - \{e\}) = \rho_i'(E \cup \{e'\}). \tag{6.11}$$

By (6.10) and (6.11), we have

$$\rho_i'(E \cup \{e'\} - \{e\}) = \rho_i'(E). \tag{6.12}$$

To prove the theorem, we prove:

(I) If $F \subseteq E'$, then $d_{\mathcal{F}'}(F) \leq \gamma_{\mathcal{F}}(E)$ with equality only if $F \subseteq E$ and F is a $\gamma_{\mathcal{F}}(E)$-achieving subset of E.

If we show (I), then either $\gamma_{\mathcal{F}'}(E') < \gamma_{\mathcal{F}}(E)$ or $\gamma_{\mathcal{F}'}(E') = \gamma_{\mathcal{F}}(E)$, and if the latter holds then the maximal $\gamma_{\mathcal{F}'}(E')$-achieving set is strictly contained in the maximal $\gamma_{\mathcal{F}}(E)$-achieving set since e' is not contained in the maximal $\gamma_{\mathcal{F}'}(E')$-achieving set. Hence the theorem follows.

Now, we prove (I). Let $F \subseteq E'$. By (6.12), we have $\rho_2'(E') = \rho_2(E)$, so

$$d_{\mathcal{F}'}(F) = \frac{\rho_2'(E') - \rho_2'(E' - F)}{\rho_1'(F)} = \frac{\rho_2(E) - \rho_2'(E' - F)}{\rho_1'(F)}. \tag{6.13}$$

We have two cases to consider:

Case (1): Suppose $e' \notin F$. Then by the definition of ρ_1', we have

$$\rho_1'(F) = \rho_1(F). \tag{6.14}$$

Also, since $e' \in E' - F$ and $E' - F = \big(E - (F \cup \{e\})\big) \cup \{e'\}$, by the definition of ρ_2' we have

$$\rho_2'(E' - F) = \begin{cases} \rho_2\big(E - (F \cup \{e\})\big), & \text{if} \quad e \in Cl_2\big(E - (F \cup \{e\})\big), \\ \rho_2\big(E - (F \cup \{e\})\big) + 1, & \text{if} \quad e \notin Cl_2\big(E - (F \cup \{e\})\big). \end{cases}$$

If $e \in Cl_2\big(E - (F \cup \{e\})\big)$, then $Cl_2\big(E - (F \cup \{e\})\big) = Cl_2(E - F)$ and thus $\rho_2\big(E - (F \cup \{e\})\big) = \rho_2(E - F)$. If $e \notin Cl_2\big(E - (F \cup \{e\})\big)$, then $\rho_2(E - F) = \rho_2\big(E - (F \cup \{e\})\big) + 1$ and therefore, $\rho_2'(E' - F) = \rho_2(E - F)$. Thus,

$$\rho_2'(E' - F) = \rho_2(E - F). \tag{6.15}$$

Substituting (6.14) and (6.15) in (6.13), we get

$$d_{\mathcal{F}'}(F) = \frac{\rho_2(E) - \rho_2'(E' - F)}{\rho_1'(F)} = \frac{\rho_2(E) - \rho_2(E - F)}{\rho_1(F)} = d_{\mathcal{F}}(F).$$

Therefore,

$$d_{\mathcal{F}'}(F) \leq \gamma_{\mathcal{F}}(F),$$

with equality only if F is a γ-achieving subset in E.

Case (2): Suppose $e' \in F$, then since $E' - F = E - (F \cup \{e\} - \{e'\})$ and $e' \notin E' - F$, we have

$$\rho_2'(E' - F) = \rho_2\big(E - (F \cup \{e\} - \{e'\})\big). \tag{6.16}$$

Subcase (2.1): Suppose $Cl_1(F) \notin \mathcal{C}_1$. Using the definition of ρ_1' and then using the submodularity of ρ_1, we have

$$\rho'_1(F) = \rho_1(F - \{e'\}) + 1 \geq \rho_1(F \cup \{f\} - \{e'\}). \tag{6.17}$$

Also, since $E - (F \cup \{e, f\} - \{e'\}) \subset E - (F \cup \{e\} - \{e'\})$, we have

$$\rho_2\big(E - (F \cup \{e\} - \{e'\})\big) \geq \rho_2\big(E - (F \cup \{e, f\} - \{e'\})\big). \tag{6.18}$$

By (6.16) and (6.18), we get

$$\rho'_2(E' - F) \geq \rho_2\big(E - (F \cup \{e, f\} - \{e'\})\big). \tag{6.19}$$

Using (6.19) and (6.17) in (6.13), we have

$$\begin{aligned} d_{\mathcal{F}'}(F) = \frac{\rho_2(E) - \rho'_2(E' - F)}{\rho'_1(F)} &\leq \frac{\rho_2(E) - \rho_2\big(E - (F \cup \{e, f\} - \{e'\})\big)}{\rho_1(F \cup \{f\} - \{e'\})} \\ &= \frac{h_2(F \cup \{e, f\} - \{e'\})}{\rho_1(F \cup \{f\} - \{e'\})}, \end{aligned} \tag{6.20}$$

where $h_2(X)$ denotes $\rho_2(E) - \rho_2(E - X)$. Let $F_1 := (F \cup \{f\} - \{e'\}) - F_0$. Since $f \notin F_0$, we have $F_1 \neq \phi$. Since h_2 is supermodular and ρ_1 is submodular, we have

$$\begin{aligned} h_2(F \cup \{e, f\} - \{e'\}) &\leq h_2\big(F_0 \cup (F \cup \{e, f\} - \{e'\})\big) - h_2(F_0) \\ &= h_2(F_0 \cup F_1) - h_2(F_0) \\ &= h_2^{E/F_0}(F_1), \end{aligned} \tag{6.21}$$

and

$$\begin{aligned} \rho_1(F \cup \{f\} - \{e'\}) &\geq \rho_1\big(F_0 \cup (F \cup \{f\} - \{e'\})\big) - \rho_1(F_0) \\ &= \rho_1(F_0 \cup F_1) - \rho_1(F_0) \\ &= \rho_1^{E/F_0}(F_1). \end{aligned} \tag{6.22}$$

Substituting (6.21) and (6.22) in (6.20) and then using Lemma VI.4, we have

$$d_{\mathcal{F}'}(F) \leq \frac{h_2^{E/F_0}(F_1)}{\rho_1^{E/F_0}(F_1)} = d_{\mathcal{F}^{E/F_0}}(F_1) < \gamma_{\mathcal{F}}(E). \tag{6.23}$$

Subcase (2.2): Suppose $Cl_1(F) \in \mathcal{C}_1$. As noted before, $F \not\subseteq F_0$. Thus, $F_2 := (F - \{e'\}) - F_0 \neq \phi$. By the definition of ρ_1', we have

$$\rho_1'(F) = \rho_1(F - \{e'\}). \tag{6.24}$$

Substituting (6.16) and (6.24) in (6.13),

$$\begin{aligned} d_{\mathcal{F}'}(F) = \frac{\rho_2(E) - \rho_2'(E' - F)}{\rho_1'(F)} \quad &\leq \quad \frac{\rho_2(E) - \rho_2\big(E - (F \cup \{e\} - \{e'\})\big)}{\rho_1(F - \{e'\})} \\ &= \quad \frac{h_2(F \cup \{e\} - \{e'\})}{\rho_1(F - \{e'\})}. \end{aligned} \tag{6.25}$$

Since h_2 is supermodular and f_1 is submodular, we have

$$\begin{aligned} h_2(F \cup \{e\} - \{e'\}) \quad &\leq \quad h_2\big(F_0 \cup (F \cup \{e\} - \{e'\})\big) - h_2(F_0) \\ &= \quad h_2(F_0 \cup F_2) - h_2(F_0) \\ &= \quad h_2^{E/F_0}(F_2), \end{aligned} \tag{6.26}$$

and

$$\begin{aligned} \rho_1(F \cup \{e\} - \{e'\}) \quad &\geq \quad \rho_1\big(F_0 \cup (F \cup \{e\} - \{e'\})\big) - \rho_1(F_0) \\ &= \quad \rho_1(F_0 \cup F_2) - \rho_1(F_0) \\ &= \quad \rho_1^{E/F_0}(F_2). \end{aligned} \tag{6.27}$$

Using (6.26) and (6.27) in (6.25), and then using Lemma VI.4, we get

$$d_{\mathcal{F}'}(F) \leq \frac{h_2^{E/F_0}(F_2)}{\rho_1^{E/F_0}(F_2)} = d_{\mathcal{F}^{E/F_0}}(F_2) < \gamma_{\mathcal{F}}(E). \tag{6.28}$$

Thus, (I) is true and the theorem follows. □

Corollary VI.10. *Let m, n be two positive integers and let E be a non-empty set such that $|E| \geq m$ and $|E| \geq n$. Then, there exists two matroid rank functions ρ_1 and ρ_2, such that $\rho_1(E) = n$ and $\rho_2(E) = m$ and E is balanced with respect to $\{\rho_1, \rho_2\}$.*

Proof. Let ρ_1 and ρ_2 be the rank functions of the matroids, $U_{n,|E|}$ and $U_{m,|E|}$ respectively, defined on the set E. If E is balanced with respect to $\{\rho_1, \rho_2\}$, we are done. Otherwise, we apply Theorem VI.9 repeatedly, replacing ρ_i by ρ'_i after each application, until E is balanced. □

6.5. Application to graphic matroids

We saw earlier (when we defined $d_{\mathcal{F}}$) that $d_{\mathcal{F}}$ is a direct extension of the function $d_1(M)$, where M is a matroid. Theorem VI.9 can easily be adapted to the case when ρ_2 is the cardinality function defined on the power set of E. In this case, the function ρ'_2 turns out to be the cardinality function defined on the power set of E'. Thus, repeated application of the theorem proves the existence of 1-balanced matroid on a given number of elements with any given rank. If G is a graph, one may notice that the Theorem VI.9 is a natural extension of Theorem IV.4.

If a matroid M is a cycle matroid of a graph, then M is called a *graphic matroid.* Restating Theorem IV.4 in terms of rank functions, Theorem IV.4 shows that if ρ_2 is the cardinality function and ρ_1 is a rank function of a cycle matroid on a non-empty set E, then ρ'_2 and ρ'_1 can be chosen as a cardinality function and a rank function of a cycle matroid, respectively, of another set E' with $E = E'$. On close examination, we can notice that Theorem VI.9 is an extension of Theorem IV.4. However, Theorem IV.4 cannot be derived from Theorem VI.9 since a matroid has to satisfy some conditions in order to be graphic.

CHAPTER VII
SUMMARY AND FUTURE WORK

(r, s)-balanced graphs for various different values of r and s have been found in many places in the literature; balanced graphs, Laman graphs, (k, l)-sparse graphs, 1-balanced graphs are some them. The dissertation is a collective study of these graphs with natural extensions to matroids.

In Chapters II and III, we provided constructions of large balanced and 1-balanced graphs. These graph constructions are generalizations of the Cartesian product of two graphs. An algorithmic method of transforming any given graph to a 1-balanced graph is presented in Chapter IV. In Chapter VI, this result is extended to a density defined on a set by a pair of rank functions.

Our study of (r, s)-balanced graphs and matroids appears in Chapter V. The study consisted of proving the existence of (r, s)-balanced graphs for various values of r and s. The examples are constructed from Laman graphs of different dimensions. The study of (r, s)-balanced graphs is extended naturally to matroids and some relations between different classes of (r, s)-balanced matroids are shown. A nice connection between (r, s)-balanced graphs and some vulnerability measures similar to edge-connectivity is established. For $0 \leq r < 1$, we found a nice characterization of r-balanced matroids using matroid duals, which also gave some useful algorithms to decide if a given matroid is r-balanced. These algorithms are presented at the end of Chapter V.

There are a number of directions to continue our study of (r, s)-balanced graphs. These directions include proving the existence of (r, s)-balanced graphs (some questions are mentioned in Chapter V), finding algorithms to identify (r, s)-balanced graphs.

Another interesting set of questions originate from Theorem VI.9. Suppose in Theorem VI.9, we restrict each of ρ_1 and ρ_2 to be a rank function of certain particular type of matroid on E, say, for example a graphic matroid, a uniform matroid, a transversal matroid, *etc.* It would be worthwhile to derive results as similar to Theorem VI.9, but with the restriction that ρ'_i is also of the same type as that of ρ_i for $i = 1, 2$. Theorem IV.4 is one such result where ρ_1 is the rank function of a graphic matroid and ρ_2 is the rank function of a uniform matroid.

REFERENCES

[1] M. O. Albertson, R. Haas, Bounding functions and rigid graphs, SIAM J. Discrete Math. 9 (2) (1996) 269–273.

[2] A. R. Berg, T. Jordán, Algorithms for graph rigidity and scene analysis, in: Algorithms—ESA 2003, vol. 2832 of Lecture Notes in Comput. Sci., Springer, Berlin, 2003, pp. 78–89.

[3] B. Bollobás, Random Graphs, vol. 73 of Cambridge Studies in Advanced Mathematics, 2nd ed., Cambridge University Press, Cambridge, 2001.

[4] B. Bollobás, J. C. Wierman, Subgraph counts and containment probabilities of balanced and unbalanced subgraphs in a large random graph, in: M. F. Capobianco, M. Guan, F. D. Hsu, F. Tian (eds.), Graph Theory and Its Applications: East and West (Jinan, 1986), vol. 576 of Ann. New York Acad. Sci., New York Acad. Sci., New York, 1989, pp. 63–70.

[5] J. Bruno, L. Weinberg, The principal minors of a matroid, Linear Algebra and Appl. 4 (1971) 17–54.

[6] P. A. Catlin, Z.-H. Chen, The arboricity of the random graph, in: Y. Alavi, et al. (eds.), Graph Theory, Combinatorics, Algorithms, and Applications (San Francisco, CA, 1989), SIAM, Philadelphia, PA, 1991, pp. 119–124.

[7] P. A. Catlin, Z.-H. Chen, E. M. Palmer, On the edge arboricity of a random graph, Ars Combin. 35 (A) (1993) 129–134.

[8] P. A. Catlin, K. C. Foster, J. W. Grossman, A. M. Hobbs, Graphs with specified edge-toughness and fractional arboricity, unpublished manuscript, Department of Mathematics, Texas A&M University (1989).

[9] P. A. Catlin, J. W. Grossman, A. M. Hobbs, Graphs with uniform density, Congr. Numer. 65 (1988) 281–285.

[10] P. A. Catlin, J. W. Grossman, A. M. Hobbs, H.-J. Lai, Fractional arboricity, strength, and principal partitions in graphs and matroids, Discrete Appl. Math. 40 (3) (1992) 285–302.

[11] E. Cheng, W. H. Cunningham, A faster algorithm for computing the strength of a network, Inf. Process. Lett. 49 (4) (1994) 209–212.

[12] L. Clark, The edge arboricity of a random graph, Congr. Numer. 103 (1994) 123–128.

[13] J. Corp, J. McNulty, On a characterization of balanced matroids, Ars Combin. 58 (2001) 111–112.

[14] H. H. Crapo, Single-element extensions of matroids, J. Res. Nat. Bur. Standards Sect. B 69B (1965) 55–65.

[15] W. H. Cunningham, Optimal attack and reinforcement of a network, J. Assoc. Comput. Mach. 32 (3) (1985) 549–561.

[16] M. T. D. J. Jacobs, A. Rader, L. A. Kuhn, Protein flexibility predictions using graph theory, Proteins: Structure, Function, and Genetics 44 (2001) 150–165.

[17] R. Diestel, Graph Theory, vol. 173 of Graduate Texts in Mathematics, 3rd ed., Springer-Verlag, Berlin, 2005.

[18] J. Edmonds, Lehman's switching game and a theorem of Tutte and Nash-Williams, J. Res. Nat. Bur. Standards Sect. B 69B (1965) 73–77.

[19] J. Edmonds, Minimum partition of a matroid into independent subsets, J. Res. Nat. Bur. Standards Sect. B 69B (1965) 67–72.

[20] J. Edmonds, G.-C. Rota, Submodular set functions (abstract), Waterloo Conference on Combinatorics, Waterloo, Canada, 1968.

[21] P. Erdős, A. Rényi, On the evolution of random graphs, Bull. Inst. Internat. Statist. 38 (1961) 343–347.

[22] Z. Fekete, L. Szegő, A note on $[k,l]$-sparse graphs, in: A. Bondy, J. Fonlupt, J.-L. Fouquet, J.-C. Fournier, J. L. R. Alfonsín (eds.), Graph Theory in Paris, Trends Math., Birkhäuser, Basel, 2007, pp. 169–177.

[23] A. Frank, L. Szegő, Constructive characterizations for packing and covering with trees, Discrete Appl. Math. 131 (2) (2003) 347–371.

[24] S. Fujishige, Submodular Functions and Optimization, vol. 58 of Annals of Discrete Mathematics, 2nd ed., Elsevier B. V., Amsterdam, 2005.

[25] M. K. Goldberg, On multigraphs of almost maximal chromatic class (in Russian), Discret. Analiz 23 (1973) 3–7.

[26] M. K. Goldberg, Clusters in a multigraph with elevated density, Electron. J. Combin. 14 (1) (2007) 1–9.
URL `http://www.combinatorics.org/Volume_14/PDF/v14i1r10.pdf`

[27] J. Graver, B. Servatius, H. Servatius, Combinatorial Rigidity, vol. 2 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1993.

[28] J. E. Graver, Counting on Frameworks, vol. 25 of The Dolciani Mathematical Expositions, Mathematical Association of America, Washington, DC, 2001.

[29] D. Gusfield, Connectivity and edge-disjoint spanning trees, Inform. Process. Lett. 16 (2) (1983) 87–89.

[30] D. Gusfield, Computing the strength of a graph, SIAM J. Comput. 20 (4) (1991) 639–654.

[31] E. Győri, B. Rothschild, A. Ruciński, Every graph is contained in a sparsest possible balanced graph, Math. Proc. Cambridge Philos. Soc. 98 (3) (1985) 397–401.

[32] R. Haas, Characterizations of arboricity of graphs, Ars Combin. 63 (2002) 129–137.

[33] F. Harary, D. Welsh, Matroids versus graphs, in: G. Chartrand, S. F. Kapoor (eds.), The Many Facets of Graph Theory (Proc. Conf., Western Mich. Univ., Kalamazoo, Mich., 1968), Springer, Berlin, 1969, pp. 155–170.

[34] G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988, reprint of the 1952 edition.

[35] L. Henneberg, Die graphische Statik der starren Systeme, Johnson Reprint 1968, Leipzig, Germany, 1911.

[36] A. M. Hobbs, Computing edge-toughness and fractional arboricity, in: B. Richter (ed.), Graphs and Algorithms (Boulder, CO, 1987), vol. 89 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1989, pp. 89–106.

[37] A. M. Hobbs, Network survivability, in: J. G. Michaels, K. H. Rosen (eds.), Applications of Discrete Mathematics, McGraw-Hill, New York, 1991, pp. 332–353.

[38] A. M. Hobbs, L. Kannan, H.-J. Lai, H. Lai, Stepwise transformations to uniformly dense graphs, submitted for publication (2007).

[39] A. M. Hobbs, L. Kannan, H.-J. Lai, H. Lai, G. Weng, Balanced graphs and uniformly dense constructions, in preparation for publication (2007).

[40] A. M. Hobbs, H.-J. Lai, H. Lai, G. Weng, Constructing uniformly dense graphs, unpublished manuscript, Department of Mathematics, Texas A&M University (1994).

[41] A. M. Hobbs, L. Petingi, The weighted-element case of strength and fractional arboricity in matroids, Congr. Numer. 149 (2001) 211–222.

[42] M. Iri, A review of recent work in Japan on principal partitions of matroids and their applications, in: A. Gewirtz, L. V. Quintas (eds.), Second International Conference on Combinatorial Mathematics (New York, 1978), vol. 319 of Ann. New York Acad. Sci., New York Acad. Sci., New York, 1979, pp. 306–319.

[43] D. J. Jacobs, B. Hendrickson, An algorithm for two-dimensional rigidity percolation: the pebble game, J. Comput. Phys. 137 (2) (1997) 346–365.

[44] M. Karoński, A. Ruciński, On the number of strictly balanced subgraphs of a random graph, in: M. Borowiecki, J. W. Kennedy, M. M. Syslo (eds.), Graph Theory (Łagów, 1981), vol. 1018 of Lecture Notes in Math., Springer, Berlin, 1983, pp. 79–83.

[45] D. G. Kelly, J. G. Oxley, Asymptotic properties of random subsets of projective spaces, Math. Proc. Cambridge Philos. Soc. 91 (1) (1982) 119–130.

[46] D. G. Kelly, J. G. Oxley, Threshold functions for some properties of random subsets of projective spaces, Quart. J. Math. Oxford Ser. Second Series 33 (132)

(1982) 463–469.

[47] D. G. Kelly, J. G. Oxley, On random representable matroids, Stud. Appl. Math. 71 (3) (1984) 181–205.

[48] G. Kishi, Y. Kajitani, Maximally distant trees and principal partition of a linear graph, IEEE Trans. Circuit Theory CT-16 (1969) 323–330.

[49] W. Kordecki, T. Łuczak, On random subsets of projective spaces, Colloq. Math. 62 (2) (1991) 353–356.

[50] H.-J. Lai, H. Lai, On strongly reduced graphs, unpublished manuscript, Department of Mathematics, West Virginia University (June 1992).

[51] H.-J. Lai, H. Lai, Every matroid is a submatroid of a uniformly dense matroid, Discrete Appl. Math. 63 (2) (1995) 151–160.

[52] G. Laman, On graphs and rigidity of plane skeletal structures, J. Engrg. Math. 4 (1970) 331–340.

[53] A. Lee, I. Streinu, Pebble game algorithms and (k, l)-sparse graphs, in: S. Felsner (ed.), 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), vol. AE of DMTCS Proceedings, Discrete Mathematics and Theoretical Computer Science, 2005.
URL `http://www.dmtcs.org/proceedings/html/dmAE0136.abs.html`

[54] T. Łuczak, A. Ruciński, Balanced extensions of spare graphs, in: J. Nešetřil, M. Fiedler (eds.), Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity (Prachatice, 1990), vol. 51 of Ann. Discrete Math., North-Holland, Amsterdam, 1992, pp. 191–203.

[55] T. Łuczak, A. Ruciński, Convex hulls of dense balanced graphs, J. Comput. Appl. Math. 41 (1-2) (1992) 205–213.

[56] K. Menger, Über reguläre Baumkurven, Math. Ann. 96 (1) (1927) 572–582.

[57] H. Narayanan, Theory of matroids and network analysis, Ph.D. thesis, Department of Electical Engineering, Indian Institute of Technology, Bombay, Mumbai, India, 1974.

[58] H. Narayanan, Submodular Functions and Electrical Networks, vol. 54 of Annals of Discrete Mathematics, North-Holland Publishing Co., Amsterdam, 1997.

[59] H. Narayanan, M. N. Vartak, On molecular and atomic matroids, in: S. B. Rao (ed.), Combinatorics and Graph Theory (Calcutta, 1980), vol. 885 of Lecture Notes in Math., Springer, Berlin, 1981, pp. 358–364.

[60] C. S. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc. 36 (1961) 445–450.

[61] C. S. J. A. Nash-Williams, Decomposition of finite graphs into forests, J. London Math. Soc. 39 (1964) 12.

[62] M. E. J. Newman, Finding community structure in networks using the eigenvectors of matrices, Physical Review E 74 (Sept.) (2006) 1–22.

[63] T. Ohtsuki, Y. Ishizaki, H. Watanabe, Topological degrees of freedom and mixed analysis of electrical networks, IEEE Trans. Circuit Theory CT-17 (1970) 491–499.

[64] J. G. Oxley, Matroid Theory, The Clarendon Press, Oxford University Press, New York, 1992.

[65] C. Payan, Graphes équilibrés et arboricité rationnelle, European J. Combin. 7 (3) (1986) 263–270.

[66] Y.-H. Peng, C. C. Chen, K. M. Koh, On the edge-toughness of a graph. I, Southeast Asian Bull. Math. 12 (2) (1988) 109–122.

[67] Y.-H. Peng, C. C. Chen, K. M. Koh, On the existence of balanced graphs with given edge-toughness and edge-connectivity, Ars Combin. 33 (1992) 129–143.

[68] Y.-H. Peng, T.-S. Tay, On the edge-toughness of a graph. II, J. Graph Theory 17 (2) (1993) 233–246.

[69] S. G. Penrice, Balanced graphs and network flows, Networks 29 (2) (1997) 77–80.

[70] J.-C. Picard, M. Queyranne, A network flow solution to some nonlinear $0-1$ programming problems, with applications to graph theory, Networks 12 (2) (1982) 141–159.

[71] A. Recski, Matroid Theory and Its Applications in Electric Network Theory and in Statics, vol. 6 of Algorithms and Combinatorics, Springer-Verlag, Berlin, 1989.

[72] A. Ruciński, When are small subgraphs of a random graph normally distributed?, Probab. Theory Related Fields 78 (1) (1988) 1–10.

[73] A. Ruciński, Small subgraphs of random graphs—a survey, in: M. Karoński, J. Jaworski, A. Ruciński (eds.), Random Graphs '87 (Poznań, 1987), Wiley, Chichester, 1990, pp. 283–303.

[74] A. Ruciński, On convex hulls of graphs, Ars Combin. 32 (1991) 293–300.

[75] A. Ruciński, A. Vince, Balanced graphs and the problem of subgraphs of random graphs, in: Proceedings of the Sixteenth Southeastern International Conference

on Combinatorics, Graph theory and Computing (Boca Raton, FL, 1985), vol. 49, 1985.

[76] A. Ruciński, A. Vince, Strongly balanced graphs and random graphs, J. Graph Theory 10 (2) (1986) 251–264.

[77] P. Seymour, Some unsolved problems on one-factorizations of graphs, in: J. A. Bondy, U. S. R. Murty (eds.), Graph Theory and Related Topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977), Academic Press, New York, 1979, pp. 367–368.

[78] W.-C. S. Suen, A correlation inequality and a Poisson limit theorem for nonoverlapping balanced subgraphs of a random graph, Random Structures Algorithms 1 (2) (1990) 231–242.

[79] M. N. S. Swamy, K. Thulasiraman, Graphs, Networks, and Algorithms, John Wiley & Sons Inc., New York, 1981.

[80] L. Szegő, On constructive characterizations of (k, l)-sparse graphs, European J. Combin. 27 (7) (2006) 1211–1223.

[81] T.-S. Tay, La méthode de Henneberg appliquée aux charpentes de barres et de corps rigides, Structural Topology (17) (1991) 53–58, dual French-English text.

[82] N. Tomizawa, Strongly irreducible matroids and principal partition of a matroid into strongly irreducible minors, Electron. Commun. Japan 59 (2) (1976) 1–10.

[83] W. T. Tutte, On the problem of decomposing a graph into n connected factors, J. London Math. Soc. 36 (1961) 221–230.

[84] W. T. Tutte, A theory of 3-connected graphs, Nederl. Akad. Wetensch. Proc. Ser. A 64 = Indag. Math. 23 (1961) 441–455.

[85] N. Veerapandiyan, S. Arumugam, On balanced graphs, Ars Combin. 32 (1991) 221–223.

[86] D. B. West, Introduction to Graph Theory, Prentice Hall Inc., Upper Saddle River, NJ, 1996.

[87] W. Whiteley, The union of matroids and the rigidity of frameworks, SIAM J. Discrete Math. 1 (2) (1988) 237–255.

[88] W. Whiteley, Some matroids from discrete applied geometry, in: J. Bonin, J. Oxley, B. Servatius (eds.), Matroid Theory (Seattle, WA, 1995), vol. 197 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1996, pp. 171–311.

[89] H. Whitney, On the abstract properties of linear dependence, Amer. J. Math. 57 (3) (1935) 509–533.

[90] S. Zhang, H. Sun, X. Li, w-density and w-balanced property of weighted graphs, Appl. Math. J. Chinese Univ. Ser. B 17 (3) (2002) 355–364.

VITA

Lavanya Kannan attended her undergraduate college at the Ethiraj College for Women, affiliated to Madras University, Chennai, India during 1995 - 98 and received her Bachelor of Science (B.Sc.) in Mathematics. She got her Master of Science (M.Sc.) from Indian Institute of Technology (I.I.T.) Madras, India in 2000. During 2000 - 01, she attended the mathematics department at Ohio-State University, Ohio, USA as a graduate student. She then joined the mathematics department at Texas A&M University for her doctoral work.

After the completion of her Ph.D., she will join the Stowers Institute of Medical Research located at Kansas City, Missouri to do her postdoctoral work in bioinformatics. She can be reached at lavanya.kannan@gmail.com. Her permanent address at her home country (India) is

Plot number 5, Second Street,
Ramagiri Nagar, Velacheri, Chennai - 600 042
Tamil Nadu, India.