

JOINT OPTIMIZATION OF LOCATION AND INVENTORY DECISIONS  
FOR IMPROVING SUPPLY CHAIN COST PERFORMANCE

A Dissertation

by

BURCU BARIŞ KESKİN

Submitted to the Office of Graduate Studies of  
Texas A&M University  
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2007

Major Subject: Industrial Engineering

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Approved by:

Co-Chairs of Committee,	Halit Üster Sıla Çetinkaya
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## ABSTRACT

Joint Optimization of Location and Inventory Decisions  
for Improving Supply Chain Cost Performance. (December 2007)

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This dissertation is focused on investigating the integration of inventory and facility location decisions in different supply chain settings. Facility location and inventory decisions are interdependent due to the economies of scale that are inherent in transportation and replenishment costs. The facility location decisions have an impact on the transportation and replenishment costs which, in turn, affect the optimal inventory policy. On the other hand, the inventory policy dictates the frequency of shipments to replenish inventory which, in turn, affects the number of deliveries, and, hence, the transportation costs, between the facilities. Therefore, our main research objectives are to:

- compare the optimal facility location, determined by minimizing total transportation costs, to the one determined by the models that also consider the timing and quantity of inventory replenishments and corresponding costs,
- investigate the effect of facility location decisions on optimal inventory decisions, and
- measure the impact of integrated decision-making on overall supply chain cost

performance.

Placing a special emphasis on the explicit modeling of transportation costs, we develop several novel models in mixed integer linear and nonlinear optimization programming. Based on how the underlying facility location problem is modeled, these models fall into two main groups: 1) continuous facility location problems, and 2) discrete facility location problems. For the stylistic models, the focus is on the development of analytical solutions. For the more general models, the focus is on the development of efficient algorithms. Our results demonstrate

- the impact of *explicit* transportation costs on integrated decisions,
- the impact of *different* transportation cost functions on integrated decisions in the context of continuous facility location problems of interest,
- the value of integrated decision-making in different supply chain settings, and
- the performance of solution methods that jointly optimize facility location and inventory decisions.

To My Family

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## CHAPTER I

### INTRODUCTION

Interest in supply chain management (SCM) has grown in both industry and academia over the past two decades. The main factors that have contributed to this trend are i) an aspiration for cost reduction, ii) the rise of the systems concept, and iii) the impact of the core-competency strategy.

**Aspiration for Cost Reduction** Since the 1980s, several important SCM initiatives, such as just-in-time manufacturing, lean production, and total quality management, have improved production processes and reduced the associated production costs. However, the constantly increasing level of competition among companies has created an extensive need for improvements in logistics in terms of cost reductions and/or service level increases. To further reduce costs, companies have begun to investigate supply chain initiatives focusing on inbound and outbound logistics.

In 2004, logistics costs were equivalent to approximately 8.6% of the U.S. gross domestic product (GDP), which corresponds to \$1,015 billion. This is an increase from \$528 billion in 1984 (Wilson, 2005) and a jump of \$71 billion from 2003 (an increase of 7.5%). The largest part of the increase can be attributed to rising transportation costs (particularly in the trucking industry), which represents over 50% of the total cost of logistics. In Table 1, we present the rise in total logistics cost and its components (inventory, transportation, and administrative costs) in relation to GDP (Wilson, 2005). This trend of increasing total logistics planning costs, including transportation, inventory holding, and

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This dissertation follows the style and format of *Operations Research*.

administrative costs, has triggered more industry interest in supply chain practices due to the potential of substantial savings that can be achieved through better planning and management of complex logistics systems.

**The Rise of the Systems Concept in SCM** Since the industrial revolution, achieving best practice has focused managerial attention on functional specialization. The prevailing belief was that better performance of a specific function led to greater efficiency of the overall process. Over the past few decades, it has become increasingly apparent that functions, although individually performed best in their class, do not necessarily combine or aggregate to achieve the lowest total cost or most effective processes. Hence, many companies have started breaking traditional organizational barriers to cooperation and coordination among different functional departments. However, it has been challenging for companies to redirect their traditional emphasis on functionality in an effort to focus on process improvement.

This challenge can be overcome by adopting a *systems concept*, which is an analytical framework that seeks total integration of the components essential to achieving the stated objectives. The goal of the systems analysis methodology is to create a whole or integrated effort, which is greater than the sum of the individual parts or components. Such integration creates a synergistic interrelationship between components in pursuit of higher overall achievement.

The components of a logistical system are typically called functions. The logistical functions are order processing, transportation and distribution, inventory control, warehousing, material handling, and packaging. The systems analysis perspective in logistics transforms traditional supply chain arrangements from loosely linked groups of independent businesses that buy and sell inventory to



TABLE 1. The Cost of the U.S. Business Logistics System in Relation to the Gross Domestic Product (GDP)

Year	GDP	Inventory	Transportation	Administrative	Total	Cost
	\$ Tr.	Costs	Costs	Costs	Cost	% of GDP
1984	3.93	240	268	20	528	13.4%
1985	4.22	227	274	20	521	12.3%
1986	4.46	217	281	20	518	11.6%
1987	4.74	225	294	21	540	11.4%
1988	5.10	251	313	23	587	11.5%
1989	5.48	282	329	24	635	11.6%
1990	5.80	283	351	25	659	11.4%
1991	6.00	256	355	24	635	10.6%
1992	6.34	237	375	24	636	10.0%
1993	6.66	239	396	25	660	9.9%
1994	7.07	265	420	27	712	10.1%
1995	7.40	302	441	30	773	10.4%
1996	7.82	303	467	31	801	10.2%
1997	8.30	314	503	33	850	10.2%
1998	8.75	321	529	34	884	10.1%
1999	9.27	333	554	35	922	9.9%
2000	9.82	374	594	38	1006	10.2%
2001	10.13	320	609	37	966	9.5%
2002	10.49	301	582	35	918	8.8%
2003	11.00	301	607	36	944	8.6%
2004	11.74	332	644	39	1015	8.6%

each other into a managerially coordinated initiative to emphasize *market impact, overall efficiency, continuous improvement, and overall competitiveness*. Hence, in recent years, the objective of SCM has shifted to increase the efficiency and effectiveness across the entire supply chain network by taking a *systems approach* (Simchi-Levi et al., 2007). However, there is still a widely recognized lack of tools for achieving the *systems approach* in logistics.

**The Impact of Core-Competency Strategy** Prahalad and Hamel (1990) argue that a number of companies have achieved significantly better results than their competitors by focusing on only a few competencies, or so-called core-competencies, and by outsourcing other non-core activities to companies that have a core competence in those activities. This strategy has gained a lot of attention from large, highly vertically-integrated companies, such as Philips Electronics, Unilever, Procter & Gamble, General Motors, etc. and has been adopted at a fast pace (de Kok and Graves, 2003). In the early nineties, a number of companies recognized SCM as one of their core competencies. This led to the creation of Vendor Managed Inventory (VMI) concepts. On the other hand, outsourcing of the physical distribution and its increased impact on customer service have stimulated the emergence of third party logistics (3PL) service providers. Furthermore, deregulation of the transportation industry has led to the development of a variety of transportation modes and reduced transportation costs, helping to improve the performance of VMI and 3PL applications (Muriel and Simchi-Levi, 2003).

In light of all these trends, both the theory and practice of SCM concentrate heavily on three important logistical components- facility location, inventory, and transportation decisions (Ballou, 1998). Focusing on these components, we integrate

the supply chain to achieve a better overall cost performance and take advantage of new supply chain initiatives such as VMI and 3PL.

Traditionally, facility location, inventory, and transportation decisions are treated separately or pairwise. For instance, given inventory decisions, the traditional facility location problem focuses on the trade-off between transportation and fixed facility costs (Daskin, 1995). On the other hand, inventory policy decisions are optimized to balance the trade-off between inventory holding and fixed replenishment costs, assuming that the underlying supply chain network structure and the transportation costs in the system are known. In general, these replenishment costs represent the ordering costs as well as the transportation costs inherent in the system.

In the last decade, the integration of inventory and transportation decisions has received increasing attention. Several researchers including Federgruen and Zipkin (1984); Yano and Gerschak (1989); Gallego and Simchi-Levi (1990); Anily and Federgruen (1993); Federgruen and Simchi-Levi (1995); Viswanathan and Mathur (1997); Chan et al. (2002); Çetinkaya and Lee (2002); Toptal et al. (2003); Çetinkaya et al. (2006) analyze integrated inventory and transportation models and provide managerial insights. Bertazzi and Speranza (2000) and Çetinkaya (2004) provide detailed reviews on integrated inventory and transportation models.

The interaction between facility location and inventory decisions dates back to the 1960s (Baumol and Wolfe, 1958; Heskett, 1966). However, until recently, existing logistics models ignored this interaction. One stream of research addresses this interaction by placing a particular emphasis on the inclusion of inventory costs, without explicitly optimizing and coordinating inventory decisions, in network design problems (Croxtton and Zinn, 2005; Daskin et al., 2002; Erlebacher and Meller, 2000; Jayaraman, 1998; Nozick and Turnquist, 1998, 2001; Shen et al., 2003; Shen and Daskin, 2005; Shu et al., 2005). Another stream of research proposes integration,

i.e., simultaneous optimization, of location and inventory decisions (Drezner et al., 2003; McCann, 1993; Romeijn et al., 2007; Teo and Shu, 2004). However, research in integrated location-inventory models is still new, and the existing literature does not explicitly consider the impact of *realistic transportation costs* in multi-stage distribution systems.

Facility location and inventory decisions are connected through the transportation costs in the system. Facility location decisions have an impact on transportation, and, hence, on replenishment costs which, in turn, affect the optimal inventory policy. On the other hand, the inventory policy dictates the frequency of shipments to replenish inventory, which, in turn, affects the number of deliveries, and, hence, the transportation costs, between facilities. As a result, facility location and inventory decisions are interdependent due to the economies of scale inherent in transportation and replenishment costs. Therefore, for improved supply chain cost performance, we need to optimize these problems simultaneously in an integrated manner. The absence of optimization models that can do so motivated this dissertation research.

In response to the lack of integrated facility location and inventory models in the current literature, in this dissertation, we investigate the following research questions:

- How does an optimal facility location that is determined by minimizing total transportation and fixed facility costs differ from one that is determined by models that also take into account the timing and quantity of inventory replenishments and the corresponding costs?
- How do facility location decisions affect the optimal inventory policy parameters?
- What is the benefit of integrating location and inventory decisions in terms of overall supply chain cost performance?

Facility location decisions are strategic level decisions, whereas inventory decisions are tactical level decisions. Therefore, addressing these research questions contributes to the literature by investigating opportunities for the coordination of strategic and tactical decisions.

### **I.1. Scope of the Dissertation**

In order to address these research questions, in this dissertation, we analyze the integration of facility location and inventory in both continuous and discrete location models with the following primary objectives:

- Analyze the interaction between facility location and inventory policy decisions under explicit transportation costs in different supply chain settings,
- Develop novel models and solution methods that rely on mixed integer programming, nonlinear optimization, heuristics, and meta-heuristics for the joint optimization of location and inventory decisions while placing an emphasis on the explicit consideration of realistic transportation cost functions in different supply chain settings,
- Evaluate the impact of joint optimization on the supply chain cost performance in different supply chain settings, and
- Provide managerial insights to improve system-wide efficiency.

More specifically, the models proposed fall into two main groups based on the underlying facility location problem modeled, that is i) *continuous facility location problems with inventory considerations* and ii) *discrete facility location problems with inventory considerations*.

## Continuous Facility Location Problems with Inventory Considerations

In continuous facility location problems, the objective is to find the coordinates of the new facility while minimizing the weighted distances between the new facility and the existing facility locations. In a typical continuous facility location problem, the weight of an existing location is a coefficient that converts the distance (between that facility and the new facility) into cost by considering the annual demand of that facility. In this research, however, we utilize inventory-policy-parameter-dependent weights by incorporating inventory decisions into the continuous facility location problem.

We consider two-stage and three-stage continuous location problems with inventory considerations. In the former, the distribution network consists of an uncapacitated central DC (the location of which is to be determined) and a set of geographically dispersed retailers whose locations are known. Each retailer faces a constant (i.e., deterministic and stationary) retailer-specific demand for a single product that must be met without shortage or backlogging. To satisfy the demand in a timely manner, the retailers hold inventory, and hence, incur inventory holding costs. Furthermore, retailers incur fixed replenishment costs as well as transportation costs each time they replenish their stock from the central DC whose location is to be determined. The central DC does not hold inventory and satisfies the retailers' order quantity via *direct shipments* at every replenishment instant. The problem is to determine the inventory policy parameters of the retailers and the location of the central DC simultaneously while minimizing transportation, inventory holding, and fixed replenishment costs.

The three-stage distribution system of interest is a generalization of the two-stage distribution system. The three-stage distribution system consists of a

supplier, a central DC to be located, and a set of retailers that face deterministic and stationary demand. In this system, the central DC is now also an inventory holding location that incurs fixed replenishment and inventory holding costs. Hence, we explicitly model the outside supplier to account for the transportation costs between the supplier and the central DC. The problem is to determine the optimum location of the central DC, as well as the *coordinated* parameters of the inventory policies of the retailers *and the central DC* simultaneously, so that the system-wide transportation, holding, and replenishment costs are minimized.

**Discrete Facility Location Problems with Inventory Considerations** In discrete facility location models, instead of calculating the coordinates of the new facility location, we evaluate a list of candidate facilities to determine which, and how many, facilities to select. In the discrete facility location setting, we consider two-stage and three-stage distribution systems with inventory considerations.

In a two-stage distribution system, at the first stage, we consider a set of geographically dispersed and established retailers with deterministic stationary demand. They incur inventory holding and inventory replenishment costs and replenish their inventory from a specific DC periodically. At the second stage, we have a set of candidate facilities for central DCs. Each DC serves a pool of retailers; however, the number and the locations of the DCs are not given *a priori*.

The three-stage distribution system generalizes the two-stage distribution system. In this system, similarly as in the continuous setting, we take inventory considerations at the DCs into account. Hence, we explicitly model the suppliers at the third stage, and we explicitly consider the assignment decisions

among the suppliers and the DCs as well as the transportation costs among these facilities.

The models in the discrete facility location setting are generalizations of the models considered in the continuous facility location setting for two reasons. First, we consider selecting *multiple* DCs from a list of candidate facilities. Second, along with specifying the number of DCs and their locations, we make *DC-retailer assignment decisions*, i.e., which retailers are served by which DC, in both two-stage and three-stage models and *supplier-DC assignment decisions*, i.e., which DCs are served by which supplier, in three-stage models. These two generalizations bring additional *combinatorial* complexity into the models.

Next, we present some brief information about the specific problems we investigate under these two general headings.

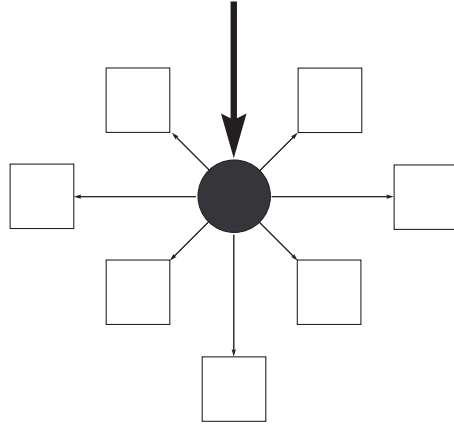
### **I.1.1. Continuous Facility Location Problems with Inventory Considerations in Two-Stage Supply Chains**

We consider a two-stage distribution system consisting of a central DC, the location of which is to be determined, and a set of geographically dispersed retailers as shown in Figure 1. Each retailer faces a constant (i.e., deterministic and stationary) retailer-specific demand for a single product that must be met without shortages or backlogging. To satisfy the demand in a timely manner, the retailers hold inventory, and hence, incur inventory holding costs. Furthermore, the retailers incur fixed replenishment costs as well as transportation costs each time they replenish their stock from the central DC whose location is to be determined.

In this problem, the central DC does not carry inventory. This situation commonly occurs when the central DC does not incur any fixed or transportation costs



FIGURE 1. Two-Stage Continuous Facility Location Problems with Inventory Considerations



when receiving replenishments from an outside supplier but rather incurs a per unit cost for each item purchased. For this system, we do not need to explicitly model the link between the outside supplier and the DC since the costs associated with the inclusion of this link are sunk costs. This problem setting is clearly applicable when the “DC” is a manufacturer that performs production on a lot-for-lot basis, and, hence, does not carry any finished goods inventory. Consequently, the distribution system associated with this case is a *two-stage* distribution system where the main concern is to minimize the inventory replenishment and holding costs at the retailers and the transportation costs from the central DC to the retailers. In order to balance the trade-off between transportation costs and inventory replenishment and holding costs, we determine the central DC location and the retailers’ inventory policy parameters simultaneously.

An important novelty of problems in the continuous facility location setting is the explicit consideration of realistic transportation costs. In this problem, we consider quantity-based, distance-based, and quantity-and-distance-based transportation

costs. Depending on how the transportation costs are modeled, the integrated facility location-inventory problem in two-stage distribution systems leads to different solutions as we discuss in Chapter III.

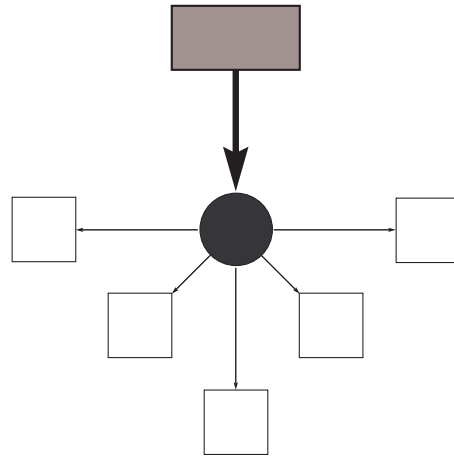
We also investigate the impact of integrated decision-making by comparing the cost of joint optimization with the cost of a benchmark solution, which is called the sequential solution, where the central DC location and inventory decisions are specified in a sequential order. That is, we first determine the location of the central DC by considering the annual demands of the retailers as the weights in the location problem. Next, we calculate the value of the inventory policy parameters by minimizing the inventory replenishment and holding costs and the transportation costs between the retailers and the central DC. Note that the sequential solution approach does not consider the impact of inventory policy decisions on the location of the central DC. This is the main distinction between the joint optimization and the sequential solution approaches. We report the results based on this comparison in Chapter III.

### **I.1.2. Continuous Facility Location Problems with Inventory Considerations in Three-Stage Supply Chains**

The three-stage distribution system of interest, given in Figure 2, is a generalization of the two-stage distribution system discussed in Section I.1.1 since the central DC, whose location is to be determined, is now an inventory keeping location and incurs fixed replenishment and inventory holding costs. Hence, the inventory policy of the central DC must be coordinated with the inventory policies of the retailers. This characteristic brings additional complexity into the models and solution techniques. Under this setting, our goal is to find the optimum location of the central DC as well as the parameters of the inventory policies of the retailers *and the central DC* simultaneously so that the system-wide transportation, holding, and replenishment

costs are minimized.

FIGURE 2. Three-Stage Continuous Facility Location Problems with Inventory Considerations



In Chapter IV, considering quantity-based, distance-based, and quantity-and-distance-based transportation cost functions, we analyze the properties of the integrated facility location-inventory problem in three-stage systems and develop efficient solution procedures. It is important to note that, given the central DC location, the proposed problem reduces to the single warehouse multi-retailer (SWMR) lot-sizing problem with deterministic stationary demand. As we will discuss in Chapter II, Roundy (1985) shows that the power-of-two policies are 94% and 98% effective in solving the single warehouse multi-retailer (SWMR) lot-sizing problem with deterministic and stationary demand when the base period is fixed and variable, respectively. Hence, while studying the integrated facility location and inventory problem in the three-stage distribution system, we restrict ourselves to the power-of-two policies for coordinating the inventory decisions of the central DC and the retailers.

As a special case, we also consider a distribution system with a single supplier, a

central DC, and a single retailer. For this case, given the central DC location, we have the single warehouse single retailer (SWSR) lot-sizing problem with deterministic stationary demand as a subproblem. Integer ratio policies are effective in solving the SWSR lot-sizing problem with deterministic stationary demand (Goyal, 1976; Roundy, 1984). Hence, for problems with a single supplier, a central DC, and a single retailer, we consider only integer-ratio policies. We also discuss the solution of this special case in Chapter IV.

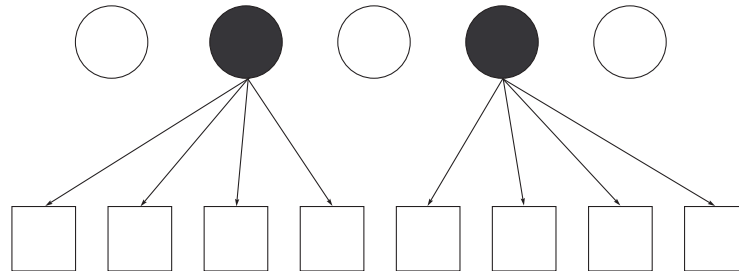
In order to quantify the value of integrated decision-making, we develop several benchmark models that reflect certain real-life practices, including VMI applications. We compare each benchmark model with the integrated facility location-inventory model and report the results, i.e. savings or losses, of this comparative analysis in Chapter IV.

### **I.1.3. Fixed Charge Facility Location Problems with Inventory Considerations**

We consider fixed charge facility location problems (FCFLP) with inventory decisions as given in Figure 3. From a candidate set, we consider establishing a number of DCs to serve geographically dispersed retailers with deterministic stationary demand. In this setting, retailers hold inventory, but DCs do not. Hence, we explicitly account for the inventory holding and replenishment costs at the retailers and the transportation costs between the retailers and their respective DCs. For transportation costs, we focus on distance-based transportation costs. This type of transportation cost function is a generalization of a per mile per unit transportation cost and represents the interaction between the facility location and the inventory problems clearly.

Furthermore, each DC has a facility-specific fixed operational cost. Due to the existence of this fixed cost, there is a trade-off between the tactical and strategic costs.

FIGURE 3. The Problem Setting for Fixed Charge Facility Location Problems with Inventory Considerations



Under this setting, the goal is to minimize the total costs in the system, including the total inventory replenishment and holding costs, the total transportation costs between the DCs and the retailers, and the fixed operational costs of the selected DCs by determining

- the number and locations of DCs,
- the assignment of each selected DC to a retailer, and
- the inventory decisions of each retailer.

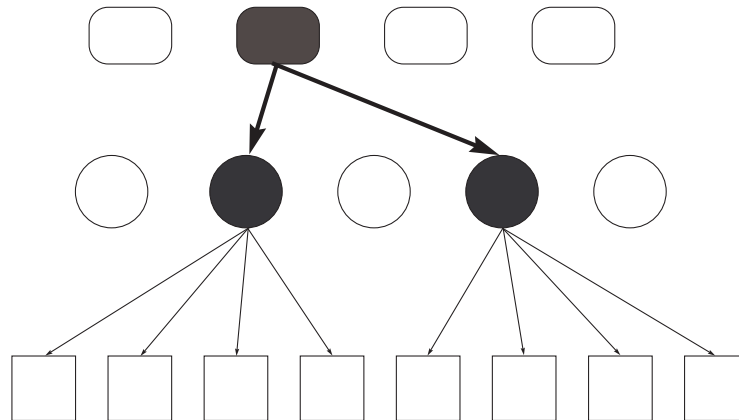
With this objective in mind, in Chapter V, we model the FCFLP problem with inventory considerations. We first consider the case where the DCs do not have any capacity restrictions. For this problem, we develop a Lagrangian-relaxation based heuristic. We provide computational results that depict the performance of these algorithms as well as the value of integrated decision-making in the discrete setting. Next, we consider generalizations of the problem for different real-life capacity restrictions. We then examine the influence of each capacity restriction on the structural properties of the problem formulation and its solution approaches. Finally, via computational tests, we measure the value of integrated decision-making under capacity

restrictions.

#### I.1.4. Production Distribution System Design Problems with Inventory Considerations

Production-distribution networks provide an effective tool to model the manufacturing and logistics activities of a firm. In general, a production distribution system design (PDS) involves the determination of the best configuration relating to the locations and sizes of the suppliers (plants) and DCs (warehouses), their technology content, their product offerings, and the transportation decisions required to achieve a firm's long term goals. Generalizing the problem settings in the continuous facility location problem and FCFLP, we consider a three-stage PDS problem with inventory considerations as given in Figure 4. In the first stage, there are retailers

FIGURE 4. The Problem Setting for Production Distribution System Design Problems with Inventory Considerations



(customers) with stationary and deterministic demand at established locations. The second stage consists of candidate locations for the DCs. Each retailer replenishes its inventory from a particular established DC at the second stage via *direct shipments*.

DCs are also inventory holding points, and they replenish their inventory from the capacitated suppliers located at the third stage via *direct shipments*. Hence, the coordination of inventory issues raised in the three-stage continuous facility location problem are also relevant for this problem setting. At the third stage, there are a set of potential capacitated suppliers. We assume that capacities at the supplier locations are known. Under this setting, the goal is to minimize the total costs in the system, including the total inventory replenishment and holding costs of the retailers and the DCs, the total transportation costs between the DCs and the retailers and between the suppliers and DCs, and the fixed operational costs of selected DCs and suppliers by determining

- the number and locations of DCs,
- the number and locations of suppliers,
- the assignment of each selected DC to a retailer,
- the assignment of each selected supplier to each selected DC, and
- the inventory decisions of each retailer and each selected DC.

While it is a general design concern, a PDSD problem with the above described characteristics is also related to the redesign of the existing system of a firm where suppliers with certain capacity limitations are mainly set-up with their corresponding technologies; however, the outbound supply chain needs to be evaluated for increased service efficiency and cost effectiveness.

In Chapter VI, we explicitly model the problem formulation and develop efficient solution procedures to

- evaluate the impact of inventory decisions on the number and selection of the

potential DCs as well as the retailer-to-DC and DC-to-supplier assignment decisions,

- analyze the impact of DC locations on inventory decisions, and
- quantify the benefits of joint optimization.

## **I.2. Organization of the Dissertation**

The dissertation is organized as follows. In Chapter II, we present a brief overview of the literature in location and inventory theory in relation to the models discussed in this dissertation. In this chapter, we also review the existing work on joint location-inventory models. Chapters III and IV concentrate on continuous location models in two- and three-stage distribution systems. In Chapter III, we consider a generalized Weber problem with inventory considerations. On the other hand, in Chapter IV, we address the issues regarding inventory policy coordination as well as facility location. In Chapters V and VI, we focus on discrete facility location problems in two- and three-stage distribution systems, respectively. In Chapter V, we discuss FCFLP problems with inventory considerations. This is followed by a discussion of production distribution system design problems with inventory considerations in VI. Finally, in Chapter VII, we conclude with a brief summary of the research results and a discussion of the potential impact of this dissertation on future research and practice.



## CHAPTER II

### LITERATURE REVIEW

Most logistics research has treated location theory and inventory theory separately. The traditional literature in location theory focuses on modeling the trade-off between an annual fixed facility location and transportation costs, and location models typically do not include inventory related costs and/or decisions. For a summary of location models, the reader is referred to texts by Daskin (1995); Drezner (1995); Drezner and Hamacher (2004); Hurter and Martinich (1989); and Love et al. (1988). On the other hand, the existing literature in inventory theory, which assumes that strategic location decisions are made and the corresponding supply chain network structure is given, offers a variety of models for computing the optimal inventory policy parameters based on the trade-off between inventory replenishment and holding costs. For a detailed review of inventory models, the reader is referred to the texts by Axsäter (2006); Graves and Kan (1993); Nahmias (2004); and Zipkin (2000).

After the advent of the *systems concept*, increased attention has been paid to optimizing the supply chain as a whole. The systems concept is an analytical framework that seeks the total integration of those supply chain components essential for achieving stated, but often conflicting, objectives. Texts by Chopra and Meindl (2003); Shapiro (2006); and Simchi-Levi et al. (2007) provide reviews of the systems concept and its uses in supply chain management.

In this chapter, we review the logistics literature relevant to the models introduced in this dissertation. In Sections II.1 and II.2, we present overviews of location and inventory theory, respectively. In Section II.3, we provide a critical review of integrated location-inventory models.

## II.1. Overview of Location Theory

Location theory, as applied to SCM, focuses on mathematical models for determining the number of facilities (e.g., DCs) and their locations as well as the facility (e.g., DC-retailer) assignments. There are four components that characterize location problems (ReVelle and Eiselt, 2005): (1) the customers (e.g., retailers) who are already presumed to be located at certain points, (2) the facilities that will be located, (3) a space in which customers and facilities are/will be located, and (4) a metric that indicates the distances between the facilities and the customers. Location problems are also classified as continuous and discrete facility location problems. In this section, we provide a brief background of the models in continuous and discrete location theory that will be introduced in later chapters.

In continuous location problems, the facilities can generally be located anywhere on the plane. The single facility continuous location problem was first introduced by Fermat in the 17<sup>th</sup> century, and it arises frequently in many real-life situations where a central facility is to be located so as to minimize the travel time or costs of serving a geographically dispersed set of existing locations. The work by Weber (1929) is the first known research in this category and is considered the formal origin of location theory. The single facility continuous location problem appears in the literature under different names including the Weber problem, the general Fermat's problem (Kuhn, 1973), the generalized Weber problem (Morris, 1981), the Fermat-Weber Problem (Brimberg and Love, 1993), and the single facility location problem (Rosen and Xue, 1991). In the remainder of the dissertation, we refer to this problem as the Weber problem.

### II.1.1. The Weber Problem

The objective of the Weber problem is to determine the coordinates of a single facility on a plane such that the weighted sum of the distances to given demand points on the plane are minimized. Finding the optimal location of this new facility is equivalent to solving the following optimization problem (Love et al., 1988):

$$\min_{\mathbf{X}} W(\mathbf{X}) = \sum_{j=1}^n w_j d(\mathbf{X}, \mathbf{A}_j),$$

where

- $n$  is the number of existing facilities (or, “demand points”),
- $w_j$  converts the distance between the new facility and existing facility  $j$  into cost, and  $w_j > 0$ ,
- $\mathbf{X} = (x_1, x_2)$  is the location of the new facility,
- $\mathbf{A}_j = (a_{j1}, a_{j2})$  is the location of the existing facility,
- $d(\mathbf{X}, \mathbf{A}_j)$  is the distance between the new facility and existing facility  $j$ .

The solution of the continuous location problem depends on the distance norm utilized in the model. In this dissertation, we consider two commonly employed norms in continuous facility location studies: namely, the squared Euclidean distance norm given by

$$d(\mathbf{X}, \mathbf{A}_j) = (x_1 - a_{j1})^2 + (x_2 - a_{j2})^2, \quad \forall j = 1, \dots, n,$$

and the more general  $\ell_p$  distance norm given by

$$d(\mathbf{X}, \mathbf{A}_j) = (|x_1 - a_{j1}|^p + |x_2 - a_{j2}|^p)^{1/p}, \quad \forall j = 1, \dots, n, \quad p \geq 1,$$

where  $p = 1$  and  $p = 2$  represent the well-known rectangular and Euclidean distances, respectively. The single facility continuous location problem with Squared Euclidean

and Euclidean distance appear as subproblems in our joint optimization models.

The solution of the Weber problem with the squared Euclidean distance leads to a simple, closed-form solution known as the center-of-gravity formula (Love et al., 1988):

$$x_1 = \frac{\sum_{j=1}^n w_j a_{j1}}{\sum_{j=1}^n w_j} \quad \text{and} \quad x_2 = \frac{\sum_{j=1}^n w_j a_{j2}}{\sum_{j=1}^n w_j}. \quad (2.1)$$

The Euclidean distance,  $\ell_2$ , refers to the straight line distance and is commonly used in continuous facility location problems. The single facility location problem with Euclidean distance is solved by a steepest-descent algorithm known as the Weiszfeld Algorithm (Weiszfeld, 1937). Many researchers have investigated the convergence properties of the Weiszfeld Algorithm including Brimberg and Love (1993); Kuhn (1973); Morris (1981); Ostresh (1978); Üster and Love (2000).

On the other hand, if more general  $\ell_p$  distances are employed,  $\mathbf{X} = (x_1, x_2)$  is computed using the well-known generalization of the iterative Weiszfeld procedure developed by Morris and Verdini (1979). That is, due to discontinuities in the derivatives at the existing demand locations under  $\ell_p$  distances, the following hyperbolic approximation, denoted by  $\tilde{\ell}_p(\mathbf{u}, \mathbf{v})$ , is utilized to approximate the  $\ell_p$  distance between two locations  $\mathbf{X} = (x_1, x_2)$  and  $\mathbf{A}_j = (a_{j1}, a_{j2}), j = 1, \dots, n$ :

$$\tilde{\ell}_p(\mathbf{X}, \mathbf{A}_j) = [((x_1 - a_{j1})^2 + \epsilon)^{p/2} + ((x_2 - a_{j2})^2 + \epsilon)^{p/2}]^{1/p}, \quad p \geq 1, \epsilon > 0. \quad (2.2)$$

Although this approximation is not a norm, since it lacks the stationarity property, i.e.,  $\tilde{\ell}_p(\mathbf{0}) \neq 0$ , it is still a convex function of  $\mathbf{X}$  as shown by Morris and Verdini (1979). Hence, the iterative procedure for computing  $\mathbf{X}$  is derived using the first

order conditions which lead to

$$x_1^{k+1} = \frac{\sum_{j=1}^n w_j ((x_1^k - a_{j1})^2 + \epsilon)^{\frac{p}{2}-1} \left( \tilde{\ell}_p(\mathbf{X}^k, \mathbf{A}_j) \right)^{1-p} a_{j1}}{\sum_{j=1}^n w_j ((x_1^k - a_{j1})^2 + \epsilon)^{\frac{p}{2}-1} \left( \tilde{\ell}_p(\mathbf{X}^k, \mathbf{A}_j) \right)^{1-p}} \quad \text{and} \quad (2.3)$$

$$x_2^{k+1} = \frac{\sum_{\mathcal{I} \cup \{0\}} w_j ((x_2^k - a_{j2})^2 + \epsilon)^{\frac{p}{2}-1} \left( \tilde{\ell}_p(\mathbf{X}^k, \mathbf{A}_j) \right)^{1-p} a_{j2}}{\sum_{\mathcal{I} \cup \{0\}} w_j ((x_2^k - a_{j2})^2 + \epsilon)^{\frac{p}{2}-1} \left( \tilde{\ell}_p(\mathbf{X}^k, \mathbf{A}_j) \right)^{1-p}}. \quad (2.4)$$

Here,  $\mathbf{X}^k = (x_1^k, x_2^k)$  denotes the location at iteration  $k = 1, 2, \dots$ . Note that this procedure is convergent for  $p \geq 1$ , (Üster and Love, 2000), and there exists another very efficient algorithm for the case of rectangular distances, i.e., if  $p = 1$ , (Love et al., 1988).

In discrete facility location problems, there are a set of demand points (retailers) and a set of potential (candidate) facility sites whose locations are known. Typically, the objective is to select a number of facilities from the candidate set so that the sum of fixed facility costs and transportation costs are minimized. These problems are generally represented with mixed integer programming formulations. Considerable attention has been devoted to discrete models for the location of plants and warehouses in different supply chain settings. In this review, we pay particular attention to the FCFLP problem with, and without, capacities and the production distribution system design problem.

### II.1.2. The Fixed Charge Facility Location Problem

The FCFLP problem determines the number and location of facilities, to be located among a set of potential facility sites, to serve a set of known demand locations so that the total cost, including the fixed charge of locating the facilities and the transportation costs is minimized. The mathematical models for these problems involve two sets of decision variables. The first set includes the *location variables* that

determine whether a facility should be located at a candidate facility site. The second set contains the *assignment (allocation) variables* that determine the assignment or allocation of customers to the open facilities.

The FCFLP problem is classified as the uncapacitated facility location problem or the capacitated facility location problem based on the capacity restrictions of the potential facility sites. The classical uncapacitated FCFLP represents the foundation on which all other facility location problems are based. It was first formulated by Balinski (1964). Since then, a great deal of research has been carried out to develop models and algorithms in the area of facility location including the works by Aikens (1985); Brimberg and Love (1994); Davis and Ray (1969); Efraymson and Ray (1966); Ellwein (1970); Ghosh (2003); Hajiaghayi et al. (2003); Harkness and ReVelle (2003); Hoeyer (2003); Jaramillo et al. (2002); Jones et al. (1994); Krarup and Pruzan (1983); Kuehn and Hamburger (1963); Manne (1964); McGinnis and White (1983); and Soland (1974).

Before providing the classical formulation of the FCFLP, we introduce the following notation for the problem parameters:

- $\mathcal{I}$  set of demand points (retailers), indexed by  $i$ .
- $\mathcal{J}$  set of potential facility locations, indexed by  $j$ .
- $f_j$  fixed cost of locating a facility at site  $j \in \mathcal{J}$ .
- $D_i$  annual demand at demand point  $i \in \mathcal{I}$ .
- $c_{ij}$  cost per unit to ship from facility site  $j \in \mathcal{J}$  to demand point  $i \in \mathcal{I}$ .

We introduce the following decision variables:

$$x_j = \begin{cases} 1, & \text{if we locate a facility at site } j \in \mathcal{J}, \\ 0, & \text{otherwise.} \end{cases}$$

$$y_{ij} = \begin{cases} 1, & \text{if demand point } i \in \mathcal{I} \text{ is assigned to a facility at candidate site } j \in \mathcal{J}, \\ 0, & \text{otherwise.} \end{cases}$$

We now formulate the FCFLP as follows:

$$\text{Minimize} \quad \sum_{j \in \mathcal{J}} f_j x_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} D_i c_{ij} y_{ij} \quad (\text{FCFLP})$$

subject to

$$\sum_{j \in \mathcal{J}} y_{ij} = 1, \quad \forall i \in \mathcal{I}. \quad (2.5)$$

$$y_{ij} \leq x_j, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (2.6)$$

$$x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (2.7)$$

$$y_{ij} \in \{0, 1\}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (2.8)$$

The objective function minimizes the sum of the fixed cost of locating facilities and the total transportation cost from the facilities to demand points. Constraints (2.5) ensure that each demand point is assigned to exactly one facility. Constraints (2.6) state that demand points can only be assigned to the facilities that are located or opened. Constraints (2.7) and (2.8) are the integrality constraints. Constraints (2.8) are referred to as *single-sourcing* constraints since they assure that each demand point is served by only one facility. Note that, even if the assignment variables were defined as continuous, due to the structure of the FCFLP model, these variables would be binary. Another important property of the FCFLP is that since the facilities are uncapacitated, each demand point will be assigned to the nearest open facility.

It should be noted that the FCFLP is NP-hard (Krarup and Pruzan, 1983). To solve the FCFLP problem efficiently, different solution methods, both exact and

heuristic, have been suggested since the 1960s. Kuehn and Hamburger (1963), in their heuristic, use only an *add routine*. However, Feldman et al. (1966) use both *add and drop routines* to improve the computation time. The first algorithm to guarantee an optimal solution for the uncapacitated case is presented by Efroymsen and Ray (1966). Later, a branch and bound algorithm is provided by Soland (1974) for identifying an optimal solution. Erlenkotter (1978) develops a dual-based algorithm for the FCFLP that is very effective. Some approximation algorithms also exist for the FCFLP (see Youssef and Mahmoud (1994), Shmoys et al. (1997)).

The capacitated facility location problem is one of the main extensions of the classical FCFLP. We define the following parameters in addition to the ones defined for the FCFLP:

$W_j$  maximum annual capacity of a candidate facility  $j$ ,  $\forall j \in \mathcal{J}$ .

Then, the formulation of the capacitated FCFLP differs from the uncapacitated FCFLP by the addition of the following constraint:

$$\sum_{j \in \mathcal{J}} D_i y_{ij} \leq W_j x_j, \quad \forall j \in \mathcal{J}. \quad (2.9)$$

Note that with the inclusion of the capacity constraints (2.9), the constraints (2.6) become logically redundant. However, keeping these constraints strengthens the linear relaxation of the problem. The Lagrangian relaxation method has been applied to solve the capacitated FCFLP by relaxing the assignment constraints or the capacity constraints (Daskin (1995)). Van-Roy (1986) proposes a Cross Decomposition algorithm that combines the dual-based method of Erlenkotter (1978) and the Lagrangian relaxation methods.

For extended reviews on the FCFLP problem, one can refer to papers by Branda and Chiu (1989), Kilkenny and Thisse (1999), Krarup and Pruzan (1983), Aikens



(1985), and McGinnis and White (1983).

### **II.1.3. The Production Distribution System Design Problem**

Geoffrion and Graves (1974) were the first researchers to formally define and discuss a production distribution system design problem. They point out that a commonly occurring problem in distribution system design is the optimal location of intermediate distribution facilities between plants (supply points) and customers (demand points). In their paper, a multi-commodity capacitated single period version of this problem is formulated as a mixed integer linear program. A solution technique based on Benders Decomposition is developed, implemented, and successfully applied to a real problem.

Several characteristics of the problem, such as locating DCs and/or plants, and capacity considerations at the plants and/or warehouses help us classify different production distribution system design problems. Some PDSD problems consider three-stage networks including plants, DCs, and customers, whereas some others consider four-stage networks including suppliers or raw material providers to the three-stage network. Another important classification in PDSD research is the stage(s) at which the facilities are located. There are studies that consider locating only DCs, only plants, or both. Most researchers (Kuehn and Hamburger, 1963; Geoffrion and Graves, 1974; Kaufman et al., 1977; Ro and Tcha, 1984; Lee, 1991; Cohen and Moon, 1991; Hindi and Basta, 1994) do not limit the number of facilities to be located. On the other hand, in some studies (Pirkul and Jayaraman, 1996, 1998; Keskin and Üster, 2007a,b) the number of facilities is limited, similar to the  $p$ -median problem. Considering capacity at different levels is another classification criterion. Capacity limitations can be only at the DCs, only at the plants, or at both. The capacity limitations at the plants are handled in two ways. Some researchers (Geoffrion and Graves, 1974; Cohen and Moon, 1991; Hindi and Basta, 1994; Jayaraman and Pirkul,

2001) consider product specific capacities at the plants, i.e., each plant has a particular capacity for each product, and several others consider an overall production capacity (Pirkul and Jayaraman, 1996, 1998; Keskin and Üster, 2007a,b) for all products. Another important distinction in PDSO studies is the distribution sourcing, which can be either single sourcing or multi-sourcing. In single sourcing, each customer is assigned to only one warehouse at the upper level, but in multi-sourcing, there is no such restriction. Single-sourcing and multi-sourcing assumptions are more important for capacitated problems since, in the absence of the capacity constraints, multi-sourcing is effectively the same as single-sourcing. Finally, another significant difference in PDSO models is the fixed cost considerations at the facilities.

In Table 2, we provide a summary of the research on PDSO studies related to our problem. Solution approaches for these problems are optimization algorithms within the framework of Benders' Decomposition (Geoffrion and Graves, 1974; Lee, 1991; Cohen and Moon, 1991); heuristics based on branch-and-bound (Kaufman et al., 1977; Ro and Tcha, 1984; Hindi and Basta, 1994); and Lagrangian relaxation (Pirkul and Jayaraman, 1996, 1998; Jayaraman and Pirkul, 2001). However, these techniques consume extensive amounts of time and effort to find an optimal solution for realistic sized problems. On the other hand, several metaheuristic approaches, including scatter search, tabu search, and genetic algorithms, have been shown to be effective in solving relatively large size problems (Keskin and Üster, 2007a,b; Syarif et al., 2002). These results suggest that it is worthwhile to utilize metaheuristic approaches for PDSO problems.

There exists a large body of research on the modeling and design of various components of integrated production distribution systems. For comprehensive review papers in this area, see Erengüç et al. (1999), Geoffrion and Powers (1995), Goetschalckx et al. (2002), Sarmiento and Nagi (1999), and Thomas and Griffin (1996).

## II.2. Overview of the Inventory Theory

The control and maintenance of inventories of physical goods is a problem common to all enterprises in all sectors of a given economy. According to Hadley and Whitin (1963), an *inventory policy* is defined as any solution of the inventory problem that specifies when to replenish the inventory and how much to order for replenishment. These are the two fundamental issues that must be addressed in managing inventories. Hence, the inventory theory literature focuses on developing and evaluating policies for effective management of inventories in the supply chain.

There are tremendous differences between existing inventory systems due to the nature of demand and number of inventory keeping facilities, and, hence, complexity and size. For example, demand can be deterministic or stochastic depending on the uncertainty inherent in the system. Also, demand can be treated as stationary or time-varying (dynamic). Our focus, in this dissertation, is on deterministic stationary demand, and, hence, the following review concentrates on this particular case. In particular, we review the Economic Order Quantity (EOQ) model and the multi-stage inventory models. As a special case of the latter, we introduce the Single Warehouse Single-Retailer (SWSR) lot-sizing problem.

### II.2.1. The Economic Order Quantity (EOQ) Model

The EOQ model is the simplest and the most fundamental of all inventory models (Harris, 1915; Nahmias, 2004). It deals with the ordering and storage of a single product by a single facility. The goal is to decide how much to order per shipment such that the fixed ordering (setup) and holding costs are minimized. Hence, the EOQ model describes an important trade-off between fixed ordering and holding costs, and it is the basis for the analysis of many complex systems.

In the EOQ model, the demand rate at the facility is deterministic and stationary with  $D$  units per unit time. The demand must be met without shortages or backlogs. Furthermore, there is no lead time. The costs include

- $K$  fixed ordering (setup) cost,
- $c$  direct unit ordering cost,
- $h$  inventory holding cost per unit of product per year.

Then, we formulate the total average annual cost  $C(Q)$  as a function of the order quantity  $Q$ :

$$C(Q) = K\frac{D}{Q} + cD + \frac{hQ}{2}.$$

The three terms composing  $C(Q)$  are annual ordering cost, annual purchase cost, and annual inventory holding cost, respectively. Since  $C(Q)$  is a convex function of order quantity  $Q$ , the optimal order quantity  $Q^*$  is found by taking the first derivative of  $C(Q)$  with respect to  $Q$  and setting it equal to zero. Doing so we obtain:

$$Q^* = \sqrt{\frac{2KD}{h}}.$$

Then, the optimal cost per year  $C^*$  is:

$$C^* = \sqrt{2KDh} + cD.$$

### II.2.2. Multi-stage Inventory Models

If there is more than a single inventory keeping facility, the interaction between these facilities may require the coordination of replenishments. The integrated models that optimize inventory decisions of several vertically connected facilities in a supply chain are known as multi-stage inventory models. Interest in multi-stage models with deterministic stationary demand was initiated by Goyal (1976) and Schwarz (1973). Papers in this area focus on the interaction between one inventory keeping

facility (namely, a supplier, a distribution center, or a warehouse) and one or more stocking facilities (namely, retailers) under deterministic stationary demand. These papers establish the foundations for the single warehouse single retailer (SWSR) and the single warehouse multi-retailer (SWMR) deterministic lot-sizing literature. Both SWSR and SWMR lot-sizing models, under deterministic stationary demand with infinite planning horizon and instantaneous replenishments, appear as subproblems in this dissertation. Hence, we refer in detail to these models and their solution approaches in our analysis.

The SWSR lot-sizing problem under deterministic stationary demand is the basis for the multi-stage inventory models since it considers two stock-keeping (inventory-keeping) locations, i.e., a single warehouse and a single retailer. The problem is to find the order quantities of the warehouse and the retailer,  $Q_w$  and  $Q_r$ , respectively, so that the total cost at the warehouse and the retailer, including the inventory ordering and inventory holding costs, is minimized. The traditional SWSR lot-sizing problem assumes that the warehouse and the retailer cooperate and determine their inventory replenishment strategies using a centralized approach. Using this centralized approach, it is desirable to coordinate the inventory decisions of the warehouse and the retailer. For this purpose, many researchers consider lot-for-lot and integer-ratio policies. The lot-for-lot policy refers to an integrated replenishment strategy for the warehouse and the retailer where the order quantity  $Q_w$  of the warehouse is equivalent to the order quantity  $Q_r$  of the retailer, i.e.,  $Q_w = Q_r$ . This policy definitely coordinates the inventory replenishment decisions in the supply chain.

On the other hand, the integer ratio policy states that the warehouse's order quantity is an integer-multiple  $n$  of the retailer's order quantity, that is  $Q_w = nQ_r$ . Since its origination (Goyal, 1976), this model has been referred to as the basic deterministic model in multi-stage inventory systems in many of the inventory books

(e.g. Silver et al., 1998, p.477). We introduce the following notation for this problem:

- $D$  deterministic, constant demand rate at the retailer, in units/unit time.
- $K_w$  fixed ordering (setup) cost associated with a replenishment at the warehouse.
- $K_r$  fixed ordering cost associated with a replenishment at the retailer.
- $h'_w$  the inventory holding cost per unit per unit time at the warehouse.
- $h'_r$  the inventory holding cost per unit per unit time at the retailer,  $h'_r > h'_w$ .
- $T_w$  the reorder interval of the warehouse,  $T_w = Q_w/D$ .
- $T_r$  the reorder interval of the retailer,  $T_r = Q_r/D$ .

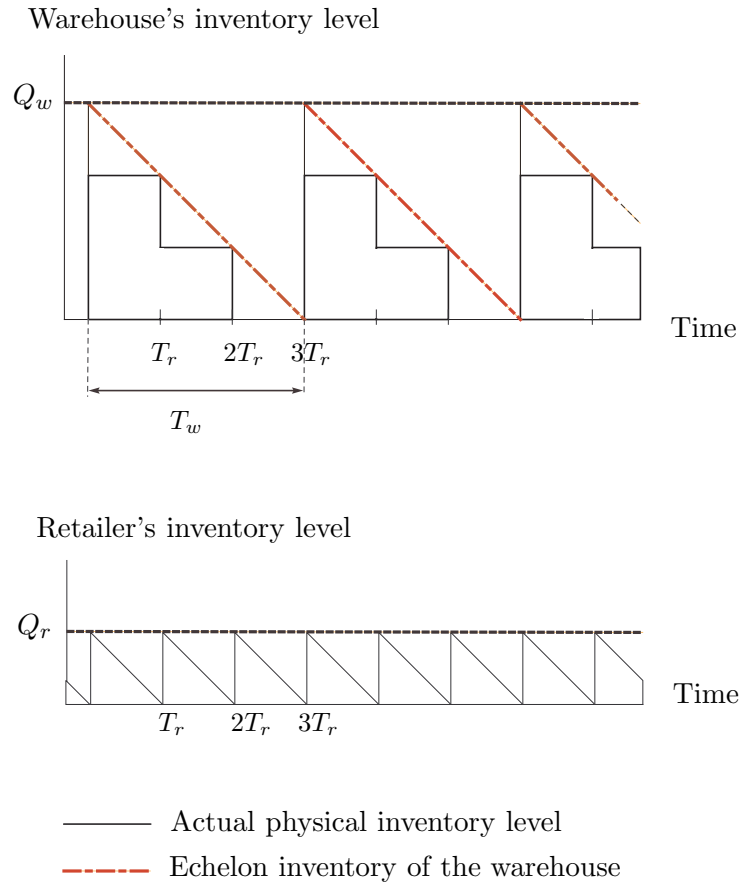
Under integer ratio policies, the main objective of the SWSR lot-sizing problem is to determine the integer ratio  $n$  and the retailer's order quantity  $Q_r$  while minimizing the total average annual cost in the system. The total average annual cost for the system under integer ratio policies is given as

$$\frac{K_r D}{Q_r} + \frac{h'_r Q_r}{2} + \frac{K_w D}{n Q_r} + \frac{h'_w (n-1) Q_r}{2},$$

where the first two terms are average ordering and inventory holding costs at the retailer and the last two terms are the similar costs incurred by the warehouse.

Note from Figure 5 that the inventory profile at the warehouse does not follow the usual sawtooth pattern, even though the demand at the retailer is deterministic and constant. This is due to withdrawals of size  $Q_r$  every  $T_r$  time units from the warehouse's inventory. With conventional definitions of inventories, the determination of average inventory levels become more complicated than the sawtooth pattern as shown with the above formulation. Therefore, many researchers prefer to use a concept known as *echelon inventory*, introduced by Clark and Scarf (1960). The echelon inventory of stage  $j$  (in a general multi-stage setting) is defined as the number of units in the system that are at, or have passed through, stage  $j$  but have as yet

FIGURE 5. Inventory Levels for the SWMR Lot-sizing Problem.



not been specifically committed to the external demand. In Figure 5, we provide the inventory profiles of a warehouse and the retailer when  $n = 3$ . This figure also illustrates the echelon inventory at the warehouse. As can be seen from the figure, it is simple to compute the average echelon inventory since the sawtooth pattern at the warehouse re-emerges. However, in order to estimate inventory holding costs at the warehouse and the retailer, we should define the echelon inventory holding costs as the incremental cost of moving the product from the warehouse to the retailer so that the inventory costs are not double-counted. For instance, for the SWSR problem, the

echelon holding cost for the warehouse is  $h_s = h'_s$ , and the echelon holding cost for the retailer is  $h_r = h'_r - h_s$ . With the echelon inventory concept, the total average annual cost for the system under integer ratio policies is given as

$$\frac{K_r D}{Q_r} + \frac{h_r Q_r}{2} + \frac{K_s D}{n Q_r} + \frac{h_s n Q_r}{2}.$$

Since the SWSR lot-sizing problem appears as a subproblem in several integrated location-inventory models in Chapter IV, the solution methods for this problem are further discussed in that chapter.

The SWMR lot-sizing problem under deterministic stationary demand is a generalization of several classical inventory models including the SWSR lot-sizing problem. Arkin, Joneja, and Roundy (1989) show that the SWMR lot-sizing problem is NP-hard. An optimal inventory policy for the SWMR lot-sizing problem has not been found. However, considering an infinite horizon and instantaneous deliveries, Schwarz (1973) proves that if an optimal policy for the deterministic stationary demand SWMR lot-sizing problem exists, it has the following properties:

- *Zero Inventory Ordering*: Each facility orders when its inventory is zero.
- *Last Minute Ordering*: The warehouse orders only when at least one retailer orders.
- *Stationarity Between Orders*: At each retailer, all orders placed between two successive orders at the warehouse are of equal size.

Still, the structure of the optimal inventory policy for the deterministic SWMR problem may be exceedingly complex. Even if it could be computed efficiently, its complexity would make it unattractive to implement in practice (Graves and Schwarz, 1977). However, in a seminal paper, Roundy (1985) shows that the best power-of-two



policy, where the replenishment intervals are chosen as power-of-two multiples of a base period, has an average cost that is within either 2% (when the base period is variable) or 6% (when the base period is fixed) of a lower bound of the minimum cost value. The SWMR inventory models have been studied extensively since this seminal work; for details, see the review paper by Muckstadt and Roundy (1993). Simchi-Levi et al. (2004) summarizes the classical SWMR model and its solution under power-of-two policies. The notation for the SWMR problem is as follows:

- $\mathcal{I}$  set of retailers, indexed by  $i \in \mathcal{I}$ .
- $D_i$  deterministic and stationary demand rate at retailer  $i \in \mathcal{I}$ .
- $K_0$  fixed ordering (setup) cost associated with a replenishment at the warehouse.
- $K_i$  fixed ordering cost associated with a replenishment at retailer  $i \in \mathcal{I}$ .
- $h'_0$  the inventory holding cost per unit per unit time at the warehouse.
- $h'_i$  the inventory holding cost per unit per unit time at retailer  $i \in \mathcal{I}$ ,  $h'_i \geq h'_0$ .
- $h_0$  echelon holding cost rate at the warehouse,  $h_0 = h'_0$ .
- $h_i$  echelon holding cost rate at retailer  $i \in \mathcal{I}$ ,  $h_i = h'_i - h_0$ .
- $T_b$  base planning period.
- $T_0$  the reorder interval of the warehouse.
- $T_i$  the reorder interval of the retailer  $i \in \mathcal{I}$ .
- $\mathbf{T}$  the reorder interval's vector,  $\mathbf{T} = (T_0, T_1, \dots, T_n)$ .

Using echelon inventory, the SWMR problem is formulated as follows:

$$\text{Min } Z(\mathbf{T}) = \frac{K_0}{T_0} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} + \sum_{i \in \mathcal{I}} \frac{K_i}{T_i} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i \quad (\text{SWMR})$$

subject to

$$T_i = 2^{v_i} T_b \quad \text{and} \quad v_i \in \mathbb{Z}, \text{ for } i = 0, \dots, n. \quad (2.10)$$

$$\mathbf{T} \in \mathbb{R}_+^{n+1}. \quad (2.11)$$

In the objective function of the SWMR problem, the first two terms represent the average annual ordering and holding costs at the warehouse, respectively, and the last two terms represent the total average annual ordering and holding costs at the retailers, respectively. Note that, with constraint (2.10), the reorder intervals of the retailers and the warehouse are restricted to a value that is a power-of-two multiple of a base period,  $T_b$ .

In this dissertation, the SWMR problem appears as a subproblem in several integrated location-inventory models, especially in the models discussed in Chapters IV and VI.

### **II.3. Overview of the Joint Location-Inventory Models**

As early as the 1960s (Heskett, 1966; Ballou, 1998), the lack of research investigating the interaction between facility location and inventory decisions was recognized as one of the deficiencies of existing logistics models. Heskett (1966) was one of the first to point out the interaction between the inventory and location models. However, he suggests that the dimensional inconsistency of location and inventory problems prevents integrated location-inventory research since the location problem is a strategic decision and the inventory problem is an operational decision. Ballou (1998, p.39) states that logistics planning deals with four major problems regarding customer service levels, facility location, inventory, and transportation decisions. These problem areas are interrelated and should be planned as a whole, although the common approach is to plan them separately. Each one of these problems has an important impact on supply chain system design.

A recent line of work, including the research by Barahona and Jensen (1998); Croxton and Zinn (2005); Daskin et al. (2002); Erlebacher and Meller (2000); Jayara-

man (1998); Miranda and Garrido (2004); Nozick and Turnquist (1998, 2001); Shen et al. (2003); Shen and Daskin (2005); Shu et al. (2005); Snyder et al. (2003); Teo et al. (2001), presents a remedy to this deficiency in integrated location-inventory theory by placing particular emphasis on the inclusion of inventory costs in network design problems. An even better approach for eliminating this deficiency is to consider joint optimization of facility location and inventory decisions. However, research in this area is still limited (Drezner et al., 2003; McCann, 1993; Romeijn et al., 2007; Teo and Shu, 2004). This dissertation contributes to this line of work in the integrated location-inventory theory literature. In the next two sections, we discuss these two streams of research in detail.

### **II.3.1. Inclusion of Inventory Costs in Facility Location Models**

The first stream of research considers inventory costs (e.g., order, holding, backlog and shortage costs) in the context of distribution system design while ignoring the effects of the inventory ordering policies on the optimal network design.

In one of the early papers in this context, Barahona and Jensen (1998) consider a two-stage location model with fixed inventory costs and develop a solution method based on Dantzig-Wolfe Decomposition. For a three-stage network design model, Jayaraman (1998) models in-transit inventory and linear cycle stock costs together with fixed facility location costs and unit-based transportation costs under deterministic demand to determine the number and location of plants and distribution centers (DCs).

Under stochastic demand, Teo et al. (2001) consider a two-stage location model with fixed inventory costs; however they ignore transportation costs. They develop a  $\sqrt{2}$ -approximation algorithm. Considering stochastic demand, Nozick and Turnquist (1998) analyze the impact of integrating inventory costs into a two-stage fixed-charge

facility location model. A basic premise of the work by Nozick and Turnquist (1998) is the consideration of safety stock costs to provide a desired level of service, together with other fixed location and transportation costs, in determining the optimal number of DCs and their locations. This paper provides a linear approximation for safety stock costs as a function of the number of DCs. In a more recent paper, Nozick and Turnquist (2001) extend the analysis done by Nozick and Turnquist (1998) to consider a two-stage multi-product system under stochastic demand where safety stocks are considered at both the DC and plant level. Under this setting, they determine the number of DCs located and what products to stock at each level. They also show that the demand for individual products at the DCs affect which products each DC should stock.

Erlebacher and Meller (2000) revisit the idea of including inventory costs in a two-stage distribution system with stochastic demand. The differences between the models by Nozick and Turnquist (1998) and Erlebacher and Meller (2000) are the consideration of continuously represented customer locations and the rectilinear distances between the DCs and the customer locations in the latter. For this setting, Erlebacher and Meller develop a nonlinear mixed integer programming formulation to determine the number and location of the DCs that serve a number of customers (e.g., retailers) and the allocations between the DCs and the customers. However, their model is NP-hard and cannot be solved using exact methods. Hence, Erlebacher and Meller develop an enumeration procedure to determine the number of DCs and a heuristic for allocating the DCs to the customers.

Croxton and Zinn (2005) extend the analysis by Nozick and Turnquist (1998) to consider a three-stage multi-product network design problem where the number and location of DCs are determined while minimizing total transportation, fixed location, and safety stock costs. This model is tested based on data from a national retailer,

and due to explicit consideration of inventory costs, an immediate result is a reduced number of DCs.

Daskin et al. (2002), Shen et al. (2003), and Shu et al. (2005) also study a two-stage network design problem under stochastic demand. However, these papers are motivated by the distribution of perishable and expensive blood products to local hospitals, and their goal is to locate regional centers for blood platelets in the first stage and assign these centers to local hospitals by considering the fixed locations and transportation costs as well as safety stock costs. For this specific application, different formulations and solution approaches are presented including a Lagrangian relaxation algorithm (Daskin et al., 2002) and a set-covering problem with branch-and-price approach (Shen et al., 2003; Shu et al., 2005). The main result, other than the theoretical and algorithmic contributions in these papers, is that integrating facility location decisions with the cost of inventory risk-pooling explicitly has an impact on the number of regional centers located.

Several researchers extend the research by Daskin et al. (2002); Shen et al. (2003); Shu et al. (2005) in different directions considering similar problem settings. Snyder et al. (2003) describe the random parameters of the model by using discrete scenarios. Each scenario dictates the demand and cost information that drives the supply chain model. They minimize the expected cost of the system across all the scenarios. Ozsen (2004) considers capacity restrictions at the first stage. The capacity constraints are defined based on how the inventory is managed. Hence, her model evaluates the trade-off between having more DCs in order to have sufficient system capacity versus ordering more frequently through the definition of capacity. Shen and Daskin (2005) consider a customer service element and develop practical methods for the evaluation of cost/service trade-offs.

### II.3.2. Joint Optimization of Facility Location and Inventory Decisions

In this proposed dissertation, our focus is not only on incorporating inventory costs in the facility location problem but also on determining inventory policy parameters together with facility location decisions. Hence, the limited existing research by Drezner et al. (2003); McCann (1993); Romeijn et al. (2007); Teo and Shu (2004) on joint optimization of facility location and inventory policy parameters is closely related to our research problems.

McCann (1993) considers a two-stage supply chain that consists of a warehouse and *two* markets (e.g., retailers) where the only inventory keeping point is the warehouse and the location of the warehouse is unknown. Hence, the problem is to find the optimum location and the optimum order quantity of the warehouse while minimizing total inventory and transportation costs in the system. The location problem studied by McCann (1993) is a continuous Weber problem. Moreover, the inventory problem is a single-facility lot-sizing problem that can be solved using the EOQ formula. McCann shows that the location of the warehouse, obtained using constant transportation costs, does not coincide with the location obtained using total logistics costs. As an extension of McCann's work, Drezner et al. (2003) consider the problem of locating a central warehouse given the locations of a fixed number ( $\geq 2$ ) of multiple local warehouses where the central warehouse does not keep inventory, but the local warehouses do. They show that the solution determined by the traditional approach, that minimizes the total transportation costs only, differs from the one determined by the approach that also takes into account the inventory and service costs.

Finally, considering the discrete facility location problem setting, Teo and Shu (2004) study a warehouse-retailer network design problem that incorporates transportation and inventory cost functions under deterministic stationary demand over

an infinite planning horizon in a two-stage distribution system. Their goal is to determine how many warehouses to setup, where to locate them, how to serve the retailers using these warehouses, and the optimal inventory policies for the warehouses and the retailers so that the total of transportation, fixed facility, inventory replenishment, and holding costs is minimized. In their model, the transportation costs are per unit per mile costs. They do not explicitly consider the impact of trip distances and frequencies on the transportation costs, and hence, they do not account for the interdependency of the inventory and the facility location problem explicitly. They show that the network design problem can be modeled approximately (to within 98% accuracy) as a set-partitioning problem that can be efficiently solved using the column generation method. Romeijn et al. (2007) extend the work by Teo and Shu (2004) to consider demand variability and capacity congestion by including safety stock and congestion costs in the model. They also formulate the problem as a set-partitioning problem and solve it using a column generation approach.

TABLE 2. Summary of Related PDSO Studies

Legend: Y: Yes, N: No, U: Unlimited, L: Limited, I: Individual, C: Overall, M: Multi, S: Single

	Kuehn and Hamburger (1963)	Geoffrion and Graves (1974)	Kaufman et al. (1977)	Ro and Tcha (1984)
Stages	3	3	3	3
DC Location	Y	Y	Y	Y
Plant Location	N	N	Y	Y
Number of Locations	U	U	U	U
DC Capacity	N	Y	N	N
Plant Capacity	N	I	N	N
Sourcing (S/M)	M	S	M	M
DC Fixed Cost	Y	Y	Y	Y
Plant Fixed Cost	N	N	Y	Y
	Lee (1991)	Cohen and Moon (1991)	Hindi and Basta (1994)	Pirkul and Jayaraman (1996)
Stages	3	3	3	3
DC Location	Y	N	Y	Y
Plant Location	N	Y	N	Y
Number of Locations	U	U	U	L
DC Capacity	Y	N	Y	Y
Plant Capacity	N	I	I	C
Sourcing (S/M)	M	M	M	S
DC Fixed Cost	Y	N	Y	Y
Plant Fixed Cost	N	Y	N	Y
	Pirkul and Jayaraman (1998)	Jayaraman and Pirkul (2001)	Syarif et al. (2002)	Keskin and Üster (2007a); (2007b)
Stages	3	4	4	3
DC Location	Y	Y	Y	Y
Plant Location	Y	Y	Y	N
Number of Locations	L	L	L	L
DC Capacity	Y	Y	Y	Y
Plant Capacity	C	I	C	C
Sourcing (S/M)	M	S	M	M
DC Fixed Cost	Y	Y	Y	Y
Plant Fixed Cost	Y	Y	Y	N



## CHAPTER III

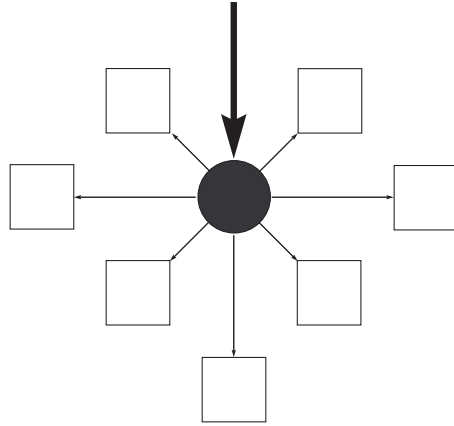
CONTINUOUS FACILITY LOCATION PROBLEMS  
IN TWO-STAGE SUPPLY CHAINS

In this chapter, we develop and analyze an integrated location and inventory model for a two-stage supply chain in a continuous facility location problem setting. The underlying location problem in this model is similar to the Weber problem discussed in Section II.1.1. We consider a set of geographically dispersed retailers whose locations are known, and we locate a central DC to serve these retailers. Each retailer operates under the assumptions of the EOQ model discussed in Section II.2.1. That is, each retailer faces a constant (i.e., deterministic and stationary) retailer-specific demand for a single product that must be met without shortage or backlogging. To satisfy the demand in a timely manner, the retailers hold inventory, and, hence, incur inventory holding costs. Furthermore, the retailers incur fixed replenishment costs as well as transportation costs each time they replenish their stock from the central DC whose location is to be determined. The central DC does not have any capacity restrictions, and it satisfies the retailers' order quantity via *direct shipments* at every replenishment instant.

The main assumption regarding this problem is that the central DC does not carry inventory. Typically, the central DC does not carry inventory when there is a per unit cost for each item purchased and no cost to receive replenishment from the outside supplier. For this system, we do not need to explicitly model the link between the outside supplier and the DC since the costs associated with the inclusion of this link are sunk costs. This problem setting is clearly applicable if the "DC" is a manufacturer that performs production on a lot-for-lot basis and, hence, does not carry any finished goods inventory. Consequently, the distribution system associated

with this problem is a *two-stage* distribution system as given in Figure 6.

FIGURE 6. Two-Stage Continuous Facility Location Problems with Inventory Considerations



In this problem, the main concern is to minimize the inventory replenishment and holding costs at the retailers as well as the transportation cost from the central DC to the retailers. In order to balance the trade-off between transportation costs and inventory replenishment and holding costs, we determine the central DC location and the retailers' inventory policy parameters simultaneously. Both the modeling and the solution of this problem depend heavily on the estimation of the distance between facilities and the transportation cost structures inherent in the system. In order to estimate the distance, we use two distance norms, squared Euclidean distance and Euclidean distance.

This chapter is organized as follows. The next section introduces the notation in this chapter as well as the general model with generic transportation costs and distance estimates. In Section III.2, we discuss the transportation cost structures considered in this integrated location-inventory problem. We examine the specific models resulting from different transportation costs in Sections III.3, III.4, and III.5.

Problems where transportation costs are modeled as a function of distance pose the most challenge in this chapter. We present numerical results regarding the solution of these models as well as the value of integrated decision-making in two-stage distribution systems in Section III.6. We conclude by summarizing the key results in Section III.7.

### III.1. General Model and Notation

In this section, we define the notation to facilitate the technical discussion. First, we define the following parameters:

$\mathcal{I}$	set of retailers, $i = 1, \dots, n$ .
$P_i$	location of retailer $i$ , $P_i = (a_i, b_i)$ , $\forall i \in \mathcal{I}$ .
$D_i$	constant demand rate at retailer $i$ , $\forall i \in \mathcal{I}$ .
$D$	total demand at the central DC, $D = \sum_i D_i$ .
$h'_i$	inventory holding cost rate for each unit of inventory at each retailer $i$ , $\forall i \in \mathcal{I}$ .
$K_i$	fixed ordering cost of each retailer $i$ , $\forall i \in \mathcal{I}$ .
$d(A, B)$	the distance between points A and B.

The decision variables of the problem are:

$Q_i$	order quantity of retailer $i$ from the central DC, $\forall i \in \mathcal{I}$ .
$\mathbf{Q}$	vector of order quantities, i.e., $\mathbf{Q} = \{Q_1, \dots, Q_n\}$ .
$T_i$	reorder interval of retailer $i$ , $T_i = Q_i/D_i$ , $\forall i \in \mathcal{I}$ .
$\mathbf{T}$	vector of reorder intervals, i.e., $\mathbf{T} = \{T_1, \dots, T_n\}$ .
$X$	location of the central DC, $X = (x, y)$ .
$d_i$	the distance between the central DC and retailer $i$ , i.e., $d(X, P_i)$ .

The order quantity and the reorder interval of retailer  $i$ , i.e.,  $\{Q_i, T_i\}$  for  $i \in \mathcal{I}$ , dictates the inventory policy of that retailer. We denote the transportation cost per

replenishment from the central DC to retailer  $i$ ,  $\forall i \in \mathcal{I}$ , with  $\alpha_i$ , where  $\alpha_i$  can be a function of order quantity  $Q_i$ , or distance  $d_i$  between that retailer and the DC, or both  $Q_i$  and  $d_i$ ,  $\forall i \in \mathcal{I}$ , as we discuss in Section III.2.

We observe that the transportation cost for each order impacts the inventory policy as an additional order setup cost. Furthermore, the reorder interval implies an order frequency that determines the number of trips between the retailer and the central DC, which in turn, affects the transportation cost. Hence, there is a strong interrelation between the inventory policy of the retailers and the location of the central DC. Solving for these decisions separately, or in a sequential manner, is suboptimal. The best approach is to optimize these decisions simultaneously. For this reason, we minimize the total average annual cost for the integrated location-inventory problem:

$$\min_{\mathbf{X}, \mathbf{T}} Z(\mathbf{X}, \mathbf{T}) = \sum_{i=1}^n \frac{\alpha_i}{T_i} + \sum_{i=1}^n \left\{ \frac{K_i}{T_i} + \frac{1}{2} h'_i T_i D_i \right\}. \quad (3.1)$$

In this formulation, the first term is the transportation cost for each order, and the second term is the inventory policy related costs, including the fixed ordering cost and the inventory holding cost at each retailer location. Note that in Equation 3.1, the total average annual cost for the integrated location-inventory problem is modeled in terms of the reorder intervals of the retailers. However, since the demand is constant and stationary, the reorder quantities can be calculated using the relation

$$Q_i = T_i D_i, \quad \forall i \in \mathcal{I}.$$

Using the relation between the reorder quantity and the reorder intervals, we can rewrite the total average annual cost for the integrated location-inventory problem in

terms of reorder quantities as follows:

$$\min_{\mathbf{X}, \mathbf{Q}} Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n \frac{\alpha_i D_i}{Q_i} + \sum_{i=1}^n \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}. \quad (3.2)$$

The solution of this problem depends heavily on the structure of the transportation costs. When the transportation cost structures include distance related terms, the solution of the problem is influenced by how the distance between two points is modeled. For this problem, we use the distance norms that are most common in continuous facility location problems: the squared Euclidean distance and the Euclidean distance (see, Section II.1.1). In the next section, we discuss the form of the transportation costs considered in this research.

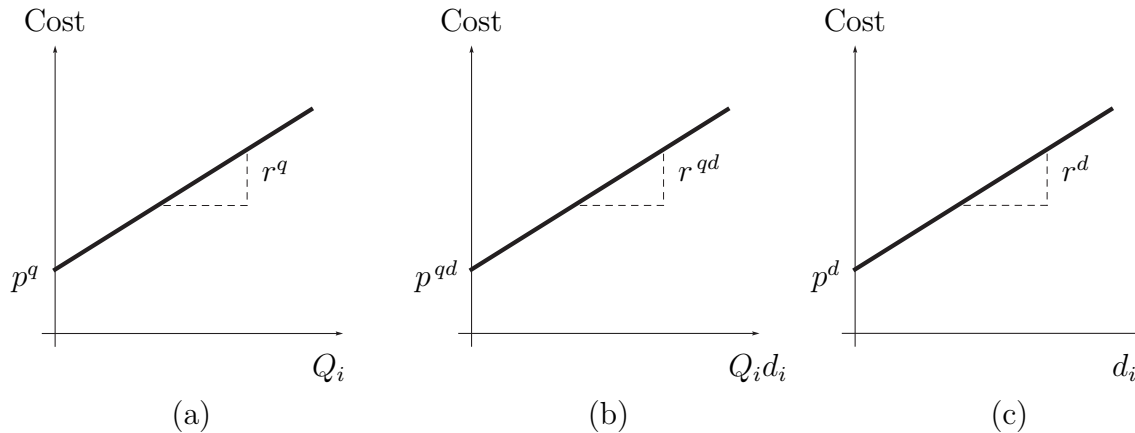
### III.2. Transportation Costs

An important novelty of the problems in this chapter and in Chapter IV is the explicit consideration of realistic transportation costs. A transportation service incurs a number of costs such as labor, fuel, maintenance, terminal, roadway, administrative, and others. In general, transportation costs can be represented by two components. The first component comprises fixed costs, including terminal charges, roadway acquisition, transport equipment, and carrier administration. The second component depends on volume, distance, and services provided, and it includes line-haul costs, such as fuel, labor, handling, pickup, and delivery. Line-haul transport rates are generally based on two important dimensions: distance and shipping quantity (volume). More specifically, we concentrate on the following three transportation cost functions (see Figure 7):

- As a function of quantity shipped, Figure 7a:

$$\alpha_i(Q_i) = p^q + r^q Q_i, \quad i \in \mathcal{I}. \quad (3.3)$$

FIGURE 7. Transportation Cost Functions



Here,  $r^q > 0$  represents the delivery cost per unit item weight, and  $p^q \geq 0$  represents the bundling cost. If  $p^q > 0$ , there are both economies of scale and economies of distance associated with (3.3). Otherwise, there are no economies of scale, but only economies of distance. In this context, economies of scale refers to a diminishing unit transportation cost with increased quantity, and economies of distance refers to a diminishing unit transportation cost with increased distance. An example of (3.3) arises in the context of the regular postal service (USPS) where the rate of the transportation cost depends on the weight (or size) of the package but not the distance in domestic delivery.

- As a function of quantity and distance, Figure 7b:

$$\alpha_i(Q_i, d_i) = p^{qd} + r^{qd}Q_i d_i, \quad i \in \mathcal{I}. \quad (3.4)$$

In (3.4),  $r^{qd} > 0$  represents the delivery cost per unit per mile, and  $p^{qd} \geq 0$  represents the carrier administration costs. If  $p^{qd} > 0$ , there are both economies

of scale and economies of distance associated with (3.4). Otherwise, there are neither economies of scale nor economies of distance. This class of transportation cost functions arises in common carrier delivery services such as FedEx and DHL. The cost of delivery via FedEx not only depends on the size (dimensions and weight) of the package but also depends on the distance  $d_i$ .

- As a function of distance traveled, Figure 7c:

$$\alpha_i(d_i) = p^d + r^d d_i, \quad i \in \mathcal{I}. \quad (3.5)$$

In (3.5),  $r^d > 0$  denotes the delivery cost per mile, and  $p^d \geq 0$  represents the fixed cost of loading/unloading the truck destined to retailer  $i$ . If  $p^d > 0$ , there are both economies of scale and economies of distance associated with (3.5). Otherwise, there are only economies of scale. This class of transportation cost functions arises in railroad delivery using a very spacious cargo container, i.e., an uncapacitated truck. The rate of the transportation cost depends on the distance traveled (mileage cost) but not the quantity shipped.

### III.3. Models with Quantity-based Transportation Costs (*PI-Q*)

In this section, we discuss the analysis of the two-stage integrated location inventory model with quantity-based transportation costs. In other words, the transportation cost per replenishment  $\alpha_i$ , for  $i \in \mathcal{I}$ , given in the general formulation (3.2), is a function of the order quantity and follows the structure of Equation 3.3. Then, the model is re-written as

$$\min_{\mathbf{X}, \mathbf{Q}} Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n \frac{(p_i^q + r_i^q Q_i) D_i}{Q_i} + \sum_{i=1}^n \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}.$$

**LEMMA 1** *For the two-stage integrated location-inventory model with quantity-based transportation costs as in Equation 3.3, the central DC location and the reorder quantities of the retailers are independent of each other. The central DC location can be anywhere on the plane. The reorder quantity of each retailer is given by*

$$Q_i = \sqrt{\frac{2(K_i + p_i^q)D_i}{h_i'}}, \quad \forall i \in \mathcal{I}.$$

Regarding the analysis of this problem, we have the following remarks:

- Since the transportation costs do not depend on the distance, the central DC can be located anywhere on the plane regardless of how the distance is modeled.
- The variable portion of the transportation cost per replenishment  $r_i^q$ ,  $i \in \mathcal{I}$ , has no impact on either the location of the central DC or on the reorder quantities of the retailers.
- The fixed portion of the transportation cost per replenishment  $p_i^q$ ,  $i \in \mathcal{I}$ , acts as an additional setup (ordering) cost for each retailer and has an impact on the reorder quantity.

#### III.4. Models with Quantity- and Distance-based Transportation Costs (PI-Qd)

In this section, we discuss the analysis of the two-stage integrated location inventory model with quantity-based and distance-based transportation costs. Specifically,  $\alpha_i$ , for  $i \in \mathcal{I}$ , given in the general formulation (3.2), is a function of order quantity and distance and follows the structure of 3.4. Then, the model is re-written as

$$\min_{\mathbf{X}, \mathbf{Q}} Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n \frac{(p_i^{qd} + r_i^{qd} Q_i d_i) D_i}{Q_i} + \sum_{i=1}^n \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h_i' Q_i \right\}.$$



We reorganize this cost function:

$$\min_{\mathbf{X}, \mathbf{Q}} Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n w_i d_i + \sum_{i=1}^n \left\{ \frac{(K_i + p_i^{qd}) D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\},$$

where  $w_i = r_i^{qd} D_i$ , the weight of each facility in the location problem.

Note that based on this formulation, the first term in this formulation denotes the Weber problem where we determine the location of the central DC by minimizing weighted distances to the existing retailer locations. As discussed in Section II.1.1, the solution of the Weber problem depends on the distance norm assumptions. For squared Euclidean distance, we have a closed form solution for the location of the central DC given by 2.1. For Euclidean distance,  $\ell_2$ , we employ the Weiszfeld algorithm to locate the central DC. On the other hand, the second and third terms under the summation represent a modified EOQ formulation for each retailer. Hence, the order quantity of each retailer can be calculated by a modified EOQ formula.

The following lemma summarizes the previous technical discussion.

**LEMMA 2** *For the two-stage integrated location-inventory model with transportation costs structures (3.4), the facility location and the inventory problems are separable:*

- *The location of the central DC depends on the solution of the following Weber problem:*

$$\min_{\mathbf{X}} \sum_{i=1}^n w_i d_i,$$

where  $w_i = r_i^{qd} D_i$  and  $d_i = d(\mathbf{P}_i, \mathbf{X})$  for  $i \in \mathcal{I}$ .

- *The reorder quantity of each retailer is given by*

$$Q_i = \sqrt{\frac{2(K_i + p_i^{qd}) D_i}{h'_i}}, \quad \forall i \in \mathcal{I}.$$

### III.5. Models with Distance-based Transportation Costs

In this section, we discuss the analysis of the two-stage integrated location inventory model with distance-based transportation costs. In particular,  $\alpha_i$  for  $i \in \mathcal{I}$ , given in the general formulation (3.2), is a function of distance and follows the structure of 3.5. Then, the total cost function is expressed as

$$\min_{\mathbf{X}, \mathbf{Q}} Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n \frac{(p_i^d + r_i^d d_i) D_i}{Q_i} + \sum_{i=1}^n \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}. \quad (3.6)$$

Note that even after we reorganize this cost function, the facility location and the inventory problems are not separable due to the first term:

$$\min_{\mathbf{X}, \mathbf{Q}} Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n \frac{r_i^d D_i d_i}{Q_i} + \sum_{i=1}^n \left\{ \frac{(K_i + p_i^d) D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}.$$

The first term contains location and order quantity related decision variables. Hence, the solution of the two-stage integrated facility location problem with distance-based transportation costs is more complicated. Furthermore, it depends on how the distance is modeled. In the rest of this section, we analyze this problem with respect to the squared Euclidean and Euclidean distances which we call Model 1 and Model 2, respectively.

#### III.5.1. Model 1: PI-d-SE

In this section, we analyze problem (3.6) under squared Euclidean distances:

$$\min_{\mathbf{X}, \mathbf{Q}} Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n \frac{r_i^d D_i [(x - a_i)^2 + (y - b_i)^2]}{Q_i} + \sum_{i=1}^n \left\{ \frac{(K_i + p_i) D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}.$$

Before discussing a solution technique, we analyze the properties of the cost function.

**PROPERTY 1** *The cost function,  $Z(\mathbf{X}, \mathbf{Q})$ , is jointly convex in  $\mathbf{X}$  and  $\mathbf{Q}$ .*

**Proof:** For each retailer  $i \in \mathcal{I}$ , consider the function

$$Z_i(\mathbf{X}, Q_i) = \frac{r_i^d D_i [(x - a_i)^2 + (y - b_i)^2]}{Q_i} + \frac{(K_i + p_i^d) D_i}{Q_i} + \frac{1}{2} h'_i Q_i.$$

Then, the overall cost function can be rewritten as  $Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n Z_i(\mathbf{X}, Q_i)$ . We first show that  $Z_i(\mathbf{X}, Q_i)$  is jointly convex in  $\mathbf{X}$  and  $Q_i$ ,  $\forall i \in \mathcal{I}$ . Then, since the summation of convex functions is convex, the result follows.

Consider the Hessian of  $Z_i(\mathbf{X}, Q_i)$ ,  $H_i$ , for any retailer  $i \in \mathcal{I}$ .

$$\begin{aligned} H_i &= \begin{bmatrix} \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial Q_i^2} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial Q_i \partial x} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial Q_i \partial y} \\ \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial x \partial Q_i} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial x^2} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial x \partial y} \\ \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial y \partial Q_i} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial y \partial x} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial y^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2D_i((K_i + p_i^d) + r_i^d[(x - a_i)^2 + (y - b_i)^2])}{Q_i^3} & \frac{-2r_i^d D_i(x - a_i)}{Q_i^2} & \frac{-2r_i^d D_i(y - b_i)}{Q_i^2} \\ \frac{-2r_i^d D_i(x - a_i)}{Q_i^2} & \frac{2r_i^d D_i}{Q_i} & 0 \\ \frac{-2r_i^d D_i(y - b_i)}{Q_i^2} & 0 & \frac{2r_i^d D_i}{Q_i} \end{bmatrix}. \end{aligned} \quad (3.7)$$

We check the principal minors of  $H_i$ :

- $\Delta_1 = \frac{2D_i((K_i + p_i^d) + r_i^d[(x - a_i)^2 + (y - b_i)^2])}{Q_i^3} > 0.$
- $\Delta_2 = \frac{4r_i^d D_i}{Q_i^4} \{(K_i + p_i^d) + r_i^d(y - b_i)^2\} > 0.$
- $\Delta_3 = \frac{8D_i^3 (r_i^d)^2 [K_i + p_i^d]}{Q_i^5} > 0.$

Based on these results,  $H_i$  is positive definite. Even if the fixed portion of the transportation cost  $p_i^d$  and the fixed replenishment cost  $K_i$  were to be zero,  $H_i$  would be positive semi-definite. Hence,  $Z_i(\mathbf{X}, Q_i)$  is a convex function  $\forall i \in \mathcal{I}$ . Therefore,  $Z(\mathbf{X}, \mathbf{Q})$  is jointly convex in  $\mathbf{X}$  and  $\mathbf{Q}$ . ■

**PROPERTY 2** *Given the central DC location  $\mathbf{X}$ , the optimal order quantity for*

retailer  $i$ ,  $i \in \mathcal{I}$ , is given as

$$Q_i^*(\mathbf{X}) = \sqrt{\frac{2D_i((K_i + p_i^d) + r_i^d[(x - a_i)^2 + (y - b_i)^2])}{h'_i}}. \quad (3.8)$$

**Proof:** Given the central DC location, the average annual cost function becomes only a function of the retailers' order quantity  $\mathbf{Q}$ . By updating the setup cost using the transportation cost, we obtain

$$Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n \left\{ \frac{((K_i + p_i^d) + r_i^d[(x - a_i)^2 + (y - b_i)^2])D_i}{Q_i} + \frac{1}{2}h'_i Q_i \right\}.$$

This formulation is the summation of  $n$ -EOQ formulations with the updated setup cost  $(K_i + p_i^d) + r_i^d[(x - a_i)^2 + (y - b_i)^2]$ . Therefore, each retailer's order quantity is given with the modified EOQ-formula:

$$Q_i^*(\mathbf{X}) = \sqrt{\frac{2D_i((K_i + p_i^d) + r_i^d[(x - a_i)^2 + (y - b_i)^2])}{h'_i}}, \quad \forall i \in \mathcal{I}. \quad \blacksquare$$

**PROPERTY 3** Given the order quantity for retailers  $\tilde{\mathbf{Q}}$ , the location of the central DC is calculated using the closed form formula:

$$x^*(\tilde{\mathbf{Q}}) = \frac{\sum_{i=1}^n \frac{D_i a_i r_i^d}{\tilde{Q}_i}}{\sum_{i=1}^n \frac{D_i r_i^d}{\tilde{Q}_i}} \quad \text{and} \quad y^*(\tilde{\mathbf{Q}}) = \frac{\sum_{i=1}^n \frac{D_i b_i r_i^d}{\tilde{Q}_i}}{\sum_{i=1}^n \frac{D_i r_i^d}{\tilde{Q}_i}}. \quad (3.9)$$

**Proof:** When the order quantity for retailers  $\tilde{\mathbf{Q}}$  is given, the second and third terms in the average annual cost function become constant parameters. Furthermore, let  $w_i = \frac{r_i^d D_i}{\tilde{Q}_i}$  be the weight for each retailer  $i \in \mathcal{I}$ . Then, the average annual cost function is expressed as

$$C(\mathbf{X}) = Z(\mathbf{X}, \tilde{\mathbf{Q}}) = \sum_{i=1}^n \{w_i[(x - a_i)^2 + (y - b_i)^2] + A_i\},$$

where  $A_i = \frac{(K_i + p_i^d)D_i}{\tilde{Q}_i} + \frac{1}{2}h'_i \tilde{Q}_i$ . In this formulation, the first term is the single facility

location problem with squared Euclidean distance using the updated weights  $w_i$ . Therefore, the solution of this problem is given with the closed-form formula of the center-of-gravity of the retailer locations, i.e.,

$$x^*(\tilde{\mathbf{Q}}) = \frac{\sum_{i=1}^n w_i a_i}{\sum_{i=1}^n w_i} = \frac{\sum_{i=1}^n \frac{D_i a_i r_i^d}{\tilde{Q}_i}}{\sum_{i=1}^n \frac{D_i r_i^d}{\tilde{Q}_i}} \quad \text{and} \quad y^*(\tilde{\mathbf{Q}}) = \frac{\sum_{i=1}^n w_i b_i}{\sum_{i=1}^n w_i} = \frac{\sum_{i=1}^n \frac{D_i b_i r_i^d}{\tilde{Q}_i}}{\sum_{i=1}^n \frac{D_i r_i^d}{\tilde{Q}_i}}$$

■

Using the relations given in property 2 and property 3, we observe that

- the distance between the central DC and the retailers, hence the central DC location, impacts the setup cost of the inventory problem, which in turn affects the order quantity of the retailers,
- the economic order quantity of each retailer impacts the number of trips between the retailer and the central DC. The trip number is given as  $\frac{D_i}{Q_i}$ . When  $Q_i$  is lower for some retailer  $i$ , the number of trips between the central DC and that retailer are higher. This results in a higher weight for that retailer in the location problem.

Using these properties, we reorganize the cost function  $Z(\mathbf{X}, \mathbf{Q})$ , just in terms of the central DC location:

$$C(\mathbf{X}) = Z(\mathbf{X}, \mathbf{Q}^*(\mathbf{X})) = \sum_{i=1}^n \sqrt{2D_i h'_i ((K_i + p_i^d) + r_i^d [(x - a_i)^2 + (y - b_i)^2])}.$$

Note that  $C(\mathbf{X})$  is convex in  $\mathbf{X}$ , and there is a unique minimizer of this function,  $\mathbf{X}^* = (x^*, y^*)$ . Whenever  $\mathbf{X}^*$  is calculated, the optimal order quantity for each retailer is given by Property 2.

Next, we determine the unique minimizer  $\mathbf{X}^*$ . For this purpose, we define the

first order conditions for  $C(\mathbf{X})$ . The gradient of  $C(\mathbf{X})$  is given as

$$\nabla C(\mathbf{X}) = \begin{pmatrix} \frac{\partial C(\mathbf{X})}{\partial x} \\ \frac{\partial C(\mathbf{X})}{\partial y} \end{pmatrix} \quad (3.10)$$

$$= \begin{pmatrix} \sum_{i=1}^n \frac{\sqrt{2D_i h'_i r_i^d (x-a_i)}}{\sqrt{(K_i+p_i^d)+r_i^d[(x-a_i)^2+(y-b_i)^2]}} \\ \sum_{i=1}^n \frac{\sqrt{2D_i h'_i r_i^d (y-b_i)}}{\sqrt{(K_i+p_i^d)+r_i^d[(x-a_i)^2+(y-b_i)^2]}} \end{pmatrix}. \quad (3.11)$$

Although equating the first order conditions to zero to solve for the minimizer of  $C(\mathbf{X})$  is necessary and sufficient, it is not possible to solve  $\nabla C(\mathbf{X}) = \mathbf{0}$  in a closed form formula. Instead, we develop the following iterative algorithm given in Display III.5.1.

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**DISPLAY 1** The iterative algorithm for squared Euclidean Distance

---

STEP 0: Set *iterationNo*  $\leftarrow 0$ . Set  $\epsilon$  to a predetermined small number.

Initialize  $(x^k, y^k)$  by setting it to any point on the plane  $\mathbb{R}^2$ .

STEP  $k+1$ :

$$(x^{k+1}, y^{k+1}) = \left( \frac{\sum_{i=1}^n \frac{\sqrt{D_i h'_i a_i}}{\sqrt{(K_i+p_i^d)+r_i^d[(x^k-a_i)^2+(y^k-b_i)^2]}}}{\sum_{i=1}^n \frac{\sqrt{D_i h'_i}}{\sqrt{(K_i+p_i^d)+r_i^d[(x^k-a_i)^2+(y^k-b_i)^2]}}}, \frac{\sum_{i=1}^n \frac{\sqrt{D_i h'_i b_i}}{\sqrt{(K_i+p_i^d)+r_i^d[(x^k-a_i)^2+(y^k-b_i)^2]}}}{\sum_{i=1}^n \frac{\sqrt{D_i h'_i}}{\sqrt{(K_i+p_i^d)+r_i^d[(x^k-a_i)^2+(y^k-b_i)^2]}}} \right)$$

STOPPING CONDITION: The procedure stops when  $\|\mathbf{X}^{k+1} - \mathbf{X}^k\| < \epsilon$ .

---

In the iterative algorithm, starting with a random location, we update the location of the central DC using the formula given in Step  $(k + 1)$ . This algorithm attempts to solve the first order conditions,  $\nabla C(\mathbf{X}) = \mathbf{0}$ , iteratively. Our aim is to show that this is a steepest descent algorithm that converges to the optimal solution for the two-stage integrated facility location-inventory problem with squared Euclidean distance-based transportation costs.

Note that the Weiszfeld Algorithm for the Weber problem with Euclidean dis-

tance norms is very similar to the iterative algorithm given in Display III.5.1. Many researchers have investigated the convergence properties of the Weiszfeld Algorithm, including Brimberg and Love (1993); Kuhn (1973); Morris (1981); Ostresh (1978); Üster and Love (2000). Ostresh (1978) proved that for any starting point  $\mathbf{X}^0$ , the iterative algorithm converges to  $\mathbf{X}^*$ . We generalize these results to our problem to prove the convergence. As a remark, when  $(K_i + p_i^d) = 0$  and the weight  $w_i$  is defined as  $\sqrt{2D_i h_i' r_i^d}$  for all  $i \in \mathcal{I}$ , the problem reduces to the Weber problem as follows:

$$C(\mathbf{X}) = \sum_{i=1}^n \sqrt{2D_i h_i' r_i^d} \sqrt{(x - a_i)^2 + (y - b_i)^2} = \sum_{i=1}^n w_i \sqrt{(x - a_i)^2 + (y - b_i)^2}.$$

Hence, the Weiszfeld algorithm is valid for this special case.

For  $(K_i + p_i^d) > 0$  for all  $i \in \mathcal{I}$ , to prove the convergence, we define the following mapping:

$$T : \mathbf{X} \rightarrow T(\mathbf{X}) : \left( \frac{\sum_{i=1}^n \frac{\sqrt{D_i h_i' a_i}}{\sqrt{(K_i + p_i^d) + r_i^d [(x - a_i)^2 + (y - b_i)^2]}}}{\sum_{i=1}^n \frac{\sqrt{D_i h_i'}}{\sqrt{(K_i + p_i^d) + r_i^d [(x - a_i)^2 + (y - b_i)^2]}}}, \frac{\sum_{i=1}^n \frac{\sqrt{D_i h_i' b_i}}{\sqrt{(K_i + p_i^d) + r_i^d [(x - a_i)^2 + (y - b_i)^2]}}}{\sum_{i=1}^n \frac{\sqrt{D_i h_i'}}{\sqrt{(K_i + p_i^d) + r_i^d [(x - a_i)^2 + (y - b_i)^2]}}} \right).$$

Let  $\mathbf{X}^*$  be the unique minimum for  $C(\mathbf{X})$ . By convexity and the differentiability of  $C(\mathbf{X})$ , the first order conditions are necessary and sufficient for the minimizer. Therefore,  $\nabla C(\mathbf{X}^*) = \mathbf{0}$ . In other words, the point  $\mathbf{X} = \mathbf{X}^*$  if and only if  $\nabla C(\mathbf{X}) = \mathbf{0}$ .

An immediate result of the definition of mapping  $T(\mathbf{X})$  and  $\nabla C(\mathbf{X}^*) = \mathbf{0}$  is the following corollary.

**COROLLARY 1** *If  $\mathbf{X} = \mathbf{X}^*$ , then  $T(\mathbf{X}) = \mathbf{X}$ .*

**LEMMA 3** *The minimizer  $\mathbf{X}^*$  is in the convex hull of the existing retailer locations,  $\Omega$ .*

**Proof:** If  $\mathbf{X}^*$  is an existing retailer location, by definition, it is in the convex hull  $\Omega$ .

Otherwise, using  $\nabla C(\mathbf{X}) = 0$ , we have the following equations:

$$\begin{aligned} x^* &= \frac{\sum_{i=1}^n \frac{\sqrt{D_i h'_i} a_i}{\sqrt{(K_i + p_i^d) + r_i^d [(x^* - a_i)^2 + (y^* - b_i)^2]}}{\sum_{i=1}^n \frac{\sqrt{D_i h'_i}}{\sqrt{(K_i + p_i^d) + r_i^d [(x^* - a_i)^2 + (y^* - b_i)^2]}}}, \\ y^* &= \frac{\sum_{i=1}^n \frac{\sqrt{D_i h'_i} b_i}{\sqrt{(K_i + p_i^d) + r_i^d [(x^* - a_i)^2 + (y^* - b_i)^2]}}{\sum_{i=1}^n \frac{\sqrt{D_i h'_i}}{\sqrt{(K_i + p_i^d) + r_i^d [(x^* - a_i)^2 + (y^* - b_i)^2]}}}. \end{aligned}$$

For each  $i \in \mathcal{I}$ , define

$$\lambda_i = \frac{\frac{\sqrt{D_i h'_i}}{\sqrt{(K_i + p_i^d) + r_i^d [(x^* - a_i)^2 + (y^* - b_i)^2]}}{\sum_{i=1}^n \frac{\sqrt{D_i h'_i}}{\sqrt{(K_i + p_i^d) + r_i^d [(x^* - a_i)^2 + (y^* - b_i)^2]}}}$$

By definition,  $\lambda_i > 0, \forall i \in \mathcal{I}$ , and  $\sum_{i=1}^n \lambda_i = 1$ . Then,  $x^* = \sum_{i=1}^n \lambda_i a_i$  and  $y^* = \sum_{i=1}^n \lambda_i b_i$ . Hence,  $\mathbf{X}^* = (x^*, y^*)$  is given by the weighted sum of the existing retailer locations with positive weights that sum to 1. Hence,  $\mathbf{X}^*$  is in the convex hull  $\Omega$ . ■

The next lemma proves that the iterative algorithm does not overshoot the optimal DC location.

**LEMMA 4** *If  $T(\mathbf{X}) \neq \mathbf{X}$ , then  $C(T(\mathbf{X})) < C(\mathbf{X})$ .*

**Proof:** Considering the definition of  $T(\mathbf{X})$ , it can be shown that  $T(\mathbf{X})$  is the unique minimum of strictly convex function

$$f(\bar{\mathbf{X}}) = \sum_{i=1}^n \frac{\sqrt{D_i h'_i}}{\sqrt{(K_i + p_i^d) + r_i^d [(x - a_i)^2 + (y - b_i)^2]}} ((K_i + p_i^d) + r_i^d [(\bar{x} - a_i)^2 + (\bar{y} - b_i)^2]).$$

Since  $\mathbf{X} \neq T(\mathbf{X})$ ,

$$\begin{aligned} f(T(\mathbf{X})) &< f(\mathbf{X}) \\ &= \sum_{i=1}^n \frac{\sqrt{D_i h'_i} ((K_i + p_i^d) + r_i^d [(x - a_i)^2 + (y - b_i)^2])}{\sqrt{(K_i + p_i^d) + r_i^d [(x - a_i)^2 + (y - b_i)^2]}} \\ &= C(\mathbf{X}). \end{aligned} \tag{3.12}$$



On the other hand,  $f(\bar{\mathbf{X}})$  can be rewritten as

$$f(\bar{\mathbf{X}}) = \sum_{i=1}^n \frac{\sqrt{2D_i h'_i}}{g_i(\bar{\mathbf{X}})} (g_i(\bar{\mathbf{X}}))^2,$$

where  $g_i(\mathbf{X}) = \sqrt{(K_i + p_i^d) + r_i^d[(x - a_i)^2 + (y - b_i)^2]}$ , for all  $i \in \mathcal{I}$ . Using this revised form, consider  $f(T(\mathbf{X}))$ :

$$\begin{aligned} f(T(\mathbf{X})) &= \sum_{i=1}^n \frac{\sqrt{2D_i h'_i}}{g_i(\mathbf{X})} (g_i(T(\mathbf{X})) - g_i(\mathbf{X}) + g_i(\mathbf{X}))^2 \\ &= \sum_{i=1}^n \frac{\sqrt{2D_i h'_i}}{g_i(\mathbf{X})} \{ (g_i(\mathbf{X}))^2 + 2g_i(\mathbf{X})[g_i(T(\mathbf{X})) - g_i(\mathbf{X})] + [g_i(T(\mathbf{X})) - g_i(\mathbf{X})]^2 \} \\ &= \sum_{i=1}^n \sqrt{2D_i h'_i} g_i(\mathbf{X}) + \sum_{i=1}^n 2\sqrt{2D_i h'_i} g_i(T(\mathbf{X})) - \sum_{i=1}^n 2\sqrt{2D_i h'_i} g_i(\mathbf{X}) \\ &\quad + \sum_{i=1}^n \frac{\sqrt{2D_i h'_i}}{g_i(\mathbf{X})} [g_i(T(\mathbf{X})) - g_i(\mathbf{X})]^2 \\ &= C(\mathbf{X}) + 2C(T(\mathbf{X})) - 2C(\mathbf{X}) + \sum_{i=1}^n \frac{\sqrt{2D_i h'_i}}{g_i(\mathbf{X})} [g_i(T(\mathbf{X})) - g_i(\mathbf{X})]^2. \end{aligned} \quad (3.13)$$

By using Equations 3.12 and 3.13, we obtain

$$f(T(\mathbf{X})) = 2C(T(\mathbf{X})) - C(\mathbf{X}) + \sum_{i=1}^n \frac{\sqrt{2D_i h'_i}}{g_i(\mathbf{X})} [g_i(T(\mathbf{X})) - g_i(\mathbf{X})]^2 < C(\mathbf{X}).$$

Organizing the second and third parts of the above inequality,  $2C(T(\mathbf{X})) < 2C(\mathbf{X})$  is acquired, proving the required result. ■

### **THEOREM 1 (Convergence Theorem)**

Given any  $\mathbf{X}^0$ , define  $\mathbf{X}^k = T^k(\mathbf{X}^0)$  for  $k = 1, \dots, n$ . Then,  $\lim_{k \rightarrow \infty} \mathbf{X}^k = \mathbf{X}^*$ .

**Proof:** The properties of  $C(\mathbf{X})$  are proven to be aligned with the properties of the Weber problem with one difference. In  $C(X)$ , there are no discontinuities at the existing retailer locations. Hence, the convergence theorem for Weber problem by Kuhn (1973) applies to  $C(X)$ . We outline the proof here for completeness.

The sequence  $\mathbf{X}^k$  lies in the convex hull  $\Omega$ , which is a compact set. By the

Bolzano-Weierstrass theorem, every bounded infinite set has an accumulation point. Hence, the bounded sequence  $\{\mathbf{X}^k\}$  must have a monotonic subsequence  $\{\mathbf{X}^{k_l}\}$  which must converge because it is monotonic and bounded. Furthermore, since the convex hull  $\Omega$  is a closed set, it contains the limit of  $\{\mathbf{X}^{k_l}\}$  such that

$$\lim_{l \rightarrow \infty} \mathbf{X}^{k_l} = \bar{\mathbf{X}}.$$

We need to show that  $\bar{\mathbf{X}} = \mathbf{X}^*$ . If  $\mathbf{X}^{k+1} = T^{k+1}(\mathbf{X}^0) = \mathbf{X}^k$ , then the sequence will repeat itself, and  $\mathbf{X}^* = \mathbf{X}^k$  by Corollary 1. Otherwise, by Lemma 4, we have

$$C(\mathbf{X}^0) > C(\mathbf{X}^1) > \dots > C(\mathbf{X}^k) > \dots > C(\mathbf{X}^*).$$

Hence,  $\lim_{k \rightarrow \infty} C(\mathbf{X}^{k_l}) - C(T(\mathbf{X}^{k_l})) = 0$ . Then, due to the continuity of  $T$ , we have  $C(\bar{\mathbf{X}}) - C(T(\bar{\mathbf{X}})) = 0$ . Therefore,  $\bar{\mathbf{X}} = T(\bar{\mathbf{X}})$ , and by Corollary 1,  $\bar{\mathbf{X}} = \mathbf{X}^*$ . ■

### III.5.2. Model 2: PI-d-E

Using the Euclidean distance norm, problem (3.6) is expressed as

$$Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n \frac{D_i r_i^d \sqrt{(x - a_i)^2 + (y - b_i)^2}}{Q_i} + \sum_{i=1}^n \left\{ \frac{(K_i + p_i^d) D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}.$$

In order to solve for the central DC location and the retailers' inventory policies, we investigate the properties of this cost function.

**PROPERTY 4** *Given  $\mathbf{X} = \tilde{\mathbf{X}}$ ,  $Z(\tilde{\mathbf{X}}, \mathbf{Q})$  is a convex function of  $\mathbf{Q}$ , and each order quantity is given as*

$$Q_i^*(\tilde{\mathbf{X}}) = \sqrt{\frac{2D_i[(K_i + p_i^d) + r_i^d \sqrt{(\tilde{x} - a_i)^2 + (\tilde{y} - b_i)^2}]}{h'_i}}, \quad \forall i \in \mathcal{I}.$$

**Proof:** Given the DC location  $\mathbf{X}$ , the first term of the total average annual cost function regarding the transportation costs between the retailers and the DC

acts as an additional replenishment cost on each order. Hence, the average annual cost function is the summation of  $n$ -EOQ cost formulations with the modified setup cost  $(K_i + p_i^d) + r_i^d \sqrt{(\tilde{x} - a_i)^2 + (\tilde{y} - b_i)^2}$  for each retailer  $i \in \mathcal{I}$ . Therefore, the result follows. ■

**PROPERTY 5** *Given  $\mathbf{Q} = \tilde{\mathbf{Q}}$ ,  $Z(\mathbf{X}, \tilde{\mathbf{Q}})$  is a convex function of  $\mathbf{X}$ , and  $\mathbf{X}$  is obtained using the Weiszfeld algorithm.*

**Proof:** When the inventory policy parameters of the retailers are known, the total average annual cost function is given as

$$Z(\mathbf{X}, \tilde{\mathbf{Q}}) = \sum_{i=1}^n \frac{D_i r_i^d \sqrt{(x - a_i)^2 + (y - b_i)^2}}{\tilde{Q}_i} + A,$$

where  $A = \sum_{i=1}^n \left\{ \frac{(K_i + p_i^d) D_i}{\tilde{Q}_i} + \frac{1}{2} h'_i \tilde{Q}_i \right\}$  is a constant. Furthermore, let  $w_i = \frac{r_i^d D_i}{\tilde{Q}_i}$ .

Then, the average annual cost function can be represented as

$$Z(\mathbf{X}, \tilde{\mathbf{Q}}) = \sum_{i=1}^n w_i \sqrt{(x - a_i)^2 + (y - b_i)^2} + A,$$

that is the shifted Weber problem with Euclidean distances. Hence,  $\mathbf{X}$  is solved by the Weiszfeld Algorithm. ■

**PROPERTY 6**  *$Z(\mathbf{X}, \mathbf{Q})$  is neither convex nor concave jointly in  $\mathbf{X}$  and  $\mathbf{Q}$ .*

**Proof:** Similar to the proof of Property 1, we consider the convexity of the function

$$Z_i(\mathbf{X}, Q_i) = \frac{r_i^d D_i \sqrt{(x - a_i)^2 + (y - b_i)^2}}{Q_i} + \frac{(K_i + p_i^d) D_i}{Q_i} + \frac{1}{2} h'_i Q_i,$$

for each retailer  $i \in \mathcal{I}$ , since the overall cost function can be rewritten as  $Z(\mathbf{X}, \mathbf{Q}) = \sum_{i=1}^n Z_i(\mathbf{X}, Q_i)$ .

Consider the Hessian of  $Z_i(\mathbf{X}, Q_i)$ ,  $H_i$ , for any retailer  $i \in \mathcal{I}$ .

$$\begin{aligned}
H_i &= \begin{bmatrix} \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial Q_i^2} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial Q_i \partial x} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial Q_i \partial y} \\ \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial x \partial Q_i} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial x^2} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial x \partial y} \\ \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial y \partial Q_i} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial y \partial x} & \frac{\partial^2 Z_i(\mathbf{X}, Q_i)}{\partial y^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2D_i((K_i + p_i^d) + r_i^d \sqrt{(x-a_i)^2 + (y-b_i)^2})}{Q_i^3} & \frac{-r_i^d D_i(x-a_i)}{Q_i^2 \sqrt{(x-a_i)^2 + (y-b_i)^2}} & \frac{-r_i^d D_i(y-b_i)}{Q_i^2 \sqrt{(x-a_i)^2 + (y-b_i)^2}} \\ \frac{-r_i^d D_i(x-a_i)}{Q_i^2 \sqrt{(x-a_i)^2 + (y-b_i)^2}} & \frac{r_i^d D_i(y-b_i)^2}{Q_i [(x-a_i)^2 + (y-b_i)^2]^{3/2}} & \frac{-r_i^d D_i(x-a_i)(y-b_i)}{Q_i [(x-a_i)^2 + (y-b_i)^2]^{3/2}} \\ \frac{-r_i^d D_i(y-b_i)}{Q_i^2 \sqrt{(x-a_i)^2 + (y-b_i)^2}} & \frac{-r_i^d D_i(x-a_i)(y-b_i)}{Q_i [(x-a_i)^2 + (y-b_i)^2]^{3/2}} & \frac{r_i^d D_i(x-a_i)^2}{Q_i [(x-a_i)^2 + (y-b_i)^2]^{3/2}} \end{bmatrix}.
\end{aligned}$$

We check the principal minors of  $H_i$  using Maple's<sup>1</sup> built-in function 'positive-semidef.' We observe that the first principal minor of the Hessian matrix is positive; however, the other principal minors are negative. Hence,  $Z_i(\mathbf{X}, Q_i)$  is neither convex nor concave for  $i \in \mathcal{I}$ . Therefore, the result follows.  $\blacksquare$

Figure 8 illustrates the average annual cost function for a particular problem instance. Although the overall shape of the function looks like a bowl, there are definite cusps that disturb the convexity.

Using Property 5, we organize the cost function as

$$C(\mathbf{X}) = Z(\mathbf{X}, \mathbf{Q}^*(\mathbf{X})) = \sum_{i=1}^n \sqrt{2D_i h_i' [(K_i + p_i^d) + r_i^d \sqrt{(x-a_i)^2 + (y-b_i)^2}]}$$

Unfortunately, due to Property 6,  $C(\mathbf{X})$  is not convex in  $\mathbf{X}$ . In order to solve for the optimal DC location and the corresponding inventory policy parameters, we further investigate the structural properties of  $C(\mathbf{X})$ .

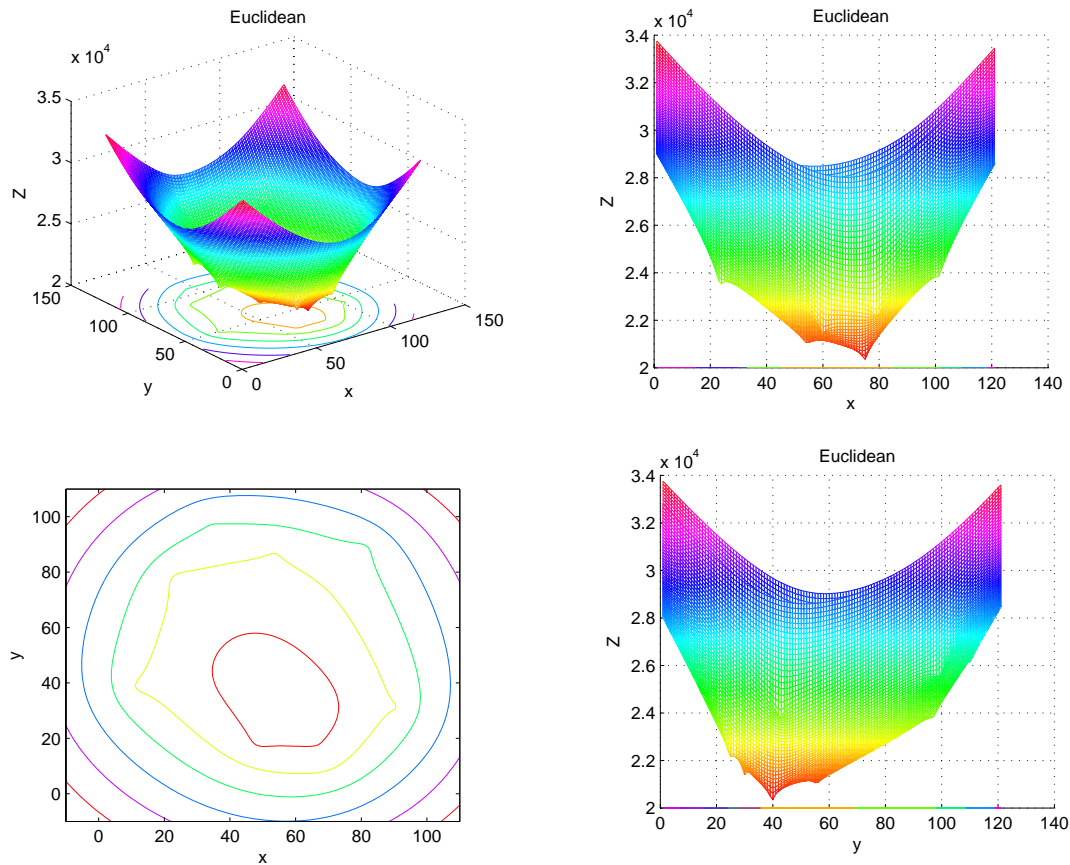
If  $K_i + p_i^d = 0$ , for all  $i \in \mathcal{I}$ , then

$$C(\mathbf{X}) = \sum_{i=1}^n w_i \sqrt{\ell_2(\mathbf{X}, \mathbf{P}_i)}, \quad (3.14)$$

---

<sup>1</sup>Maple is a mathematics software by Maplesoft, Inc.

FIGURE 8. The Average Annual Cost Function with Euclidean Distances



where  $w_i = \sqrt{2D_i h_i^d r_i^d}$  and  $\ell_2(\mathbf{X}, \mathbf{P}_i)$  is the Euclidean distance between  $\mathbf{X}$  and  $\mathbf{P}_i$ . Hence, this is a generalized Weber problem. For such a generalized Weber problem,  $\sum_{i=1}^n w_i [\ell_2(\mathbf{X}, \mathbf{P}_i)]^k$ , with  $0 < k < 1$ , Cooper (1968) showed that the existing facility locations  $\mathbf{P}_i$ ,  $i \in \mathcal{I}$ , are the local minima. In Equation 3.14, since  $k = 1/2$ , the result applies for this special case.

The following theorem extends these results to the case where the summation of fixed costs  $(K_i + p_i^d)$  are positive for some retailer  $i$ .

**THEOREM 2** *If  $K_i + p_i^d > 0$ ,  $\forall i$ , the existing retail locations,  $\mathbf{P}_i$ , are the local minima of  $C(\mathbf{X})$ .*

**Proof:** To simplify the equations, for all  $i \in \mathcal{I}$ , define  $A_i = 2(K_i + p_i^d)h'_i D_i$ ,  $B_i = 2D_i h'_i r_i^d$ , and  $C_i(\mathbf{X}) = \sqrt{A_i + B_i \sqrt{(x - a_i)^2 + (y - b_i)^2}}$ . Then,  $C(\mathbf{X}) = \sum_{i=1}^n C_i(\mathbf{X})$ . The gradient of  $C(\mathbf{X})$  is

$$\begin{aligned} \nabla C(\mathbf{X}) &= \begin{pmatrix} \frac{\partial C(\mathbf{X})}{\partial x} \\ \frac{\partial C(\mathbf{X})}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^n \frac{B_i(x-a_i)}{2C_i(\mathbf{X})\sqrt{(x-a_i)^2+(y-b_i)^2}} \\ \sum_{i=1}^n \frac{B_i(y-b_i)}{2C_i(\mathbf{X})\sqrt{(x-a_i)^2+(y-b_i)^2}} \end{pmatrix}. \end{aligned} \quad (3.15)$$

Since  $\nabla C(\mathbf{X})$  is not defined for existing retailer locations, we cannot use the first and second order conditions to prove the local minima. Instead, we use a directional derivative. For a point  $(x, y)$  to be a local minimum for every unit vector  $\mathbf{U} = (u_1, u_2)$ , the directional derivative of  $C(\mathbf{X})$  in this direction must be positive.

For retailer  $j \in \mathcal{I}$ , consider a neighborhood around  $\mathbf{P}_j$  such that for all  $0 < t < \epsilon_j$ ,

$$\max_{i \neq j} \frac{d}{dt} C_i(\mathbf{P}_j + t\mathbf{U}) > -M_j,$$

where  $M_j$  is a positive real number. Consider

$$\begin{aligned} \left(\frac{d}{dt}\right)C(\mathbf{P}_j + t\mathbf{U}) &= \left(\frac{d}{dt}\right) \left(\sum_{i=1}^n C_i(\mathbf{P}_j + t\mathbf{U})\right) \\ &= \sum_{\substack{i=1 \\ i \neq j}}^n \left(\frac{d}{dt}\right)C_i(\mathbf{P}_j + t\mathbf{U}) + \frac{d}{dt}C_j(\mathbf{P}_j + t\mathbf{U}) \\ &> -\sum_{\substack{i=1 \\ i \neq j}}^n M_j + \frac{d}{dt}\sqrt{A_j + B_j t}, \quad \text{since } \|\mathbf{U}\| = 1, \\ &= -M_j(n-1) + \frac{B_j}{2\sqrt{A_j + B_j t}}. \end{aligned}$$

Now, choose  $\bar{t} < \epsilon_j$  such that

$$\frac{B_j}{2\sqrt{A_j + B_j\bar{t}}} > M_j(n-1).$$

Then, for  $0 < t < \bar{t}$ ,  $\frac{d}{dt}C(\mathbf{P}_j + t\mathbf{U}) > 0$ , which establishes the result. ■

After proving Theorem 2, we revisit Figure 8. The well defined-dips in the total cost function with Euclidean distance are due to the existing retailer locations. As the facility location  $\mathbf{X} = (x, y)$  approaches each retailer location  $\mathbf{P}_i = (a_i, b_i)$ , the Euclidean distance portion of the total cost function will approach zero, producing a dip in the total cost function. In our graphical analysis, we observed that the dips are well-defined and sharp when the fixed costs ( $K_i + p_i^d$ , for  $i \in \mathcal{I}$ ) are close to zero. On the other hand, when the fixed costs are substantial, the dips are smoother and the overall shape of the cost function resembles a bowl shape.

We modified the iterative algorithm developed for the squared Euclidean distance to take advantage of the structural properties of the cost function with the Euclidean distances. The algorithm given in Display 2 has two main steps. In the first step, we check the existing retailer locations as to their potential for being the DC location and calculate the corresponding total cost. The retailer location with the lowest total cost is recorded as the best DC location. Then, in the second step, for a fixed number of iterations, we loop through the iterative algorithm. At each iteration, starting with a random DC location, we update the DC location using the formula given in Step (2.b) until we cannot improve it any further. At this point, we calculate the cost associated with this location. If this cost is better than the best cost ( $C^*$ ) we have obtained thus far, we update the best cost and best DC location. Otherwise, we repeat the iterative algorithm until a fixed number of iterations is satisfied.

### III.6. Numerical Results

In this section, our aim is to demonstrate the impact of joint decision-making regarding DC location and inventory policy parameters. For this purpose, we develop a benchmark model to solve for the DC location and the inventory policy parameters sequentially. We compare the cost of the benchmark model with joint facility location-inventory models using two different experiments. In the first experiment, we test the impact of the randomness of the problem parameters among different retailers on the joint decision-making. In the second experiment, we test the impact of different problem parameters via a factorial design.

In the remainder of this section, we first describe the benchmark model in detail. In Section III.6.2, we present the configuration of the two numerical experiments. In Sections III.6.3 and III.6.4, we discuss the results of the numerical test with respect to the squared Euclidean and Euclidean distances. Finally, we conclude with some key points from our analysis in Section III.6.5.

#### III.6.1. A Benchmark Model: Sequential Approach (BM-SA)

In real world applications, it is quite typical to decide on the strategic problem variables first. Afterwards, based on the strategic decisions, tactical and operational decisions are determined. In developing our sequential approach BM-SA, we follow this simple logic. We first locate the central DC by solving the following Weber problem:

$$\min_{\mathbf{X}} \sum_{i \in \mathcal{I}} w_i d(\mathbf{X}, \mathbf{P}_i),$$

where  $w_i = cD_i$ , and  $c$  is the per unit per mile transportation cost. As we discussed earlier in Chapter II, the solution of this problem depends on the modeling of the distance. For squared Euclidean distance, the location of the DC  $X^*$  is given with a



closed formula 2.1. On the other hand, for the Euclidean distance,  $X^*$  is found by implementing the Weiszfeld Algorithm (Weiszfeld, 1937).

Next, inventory decisions are addressed by solving the corresponding multi-retailer EOQ problem with the updated setup costs:

$$\sum_{i=1}^n \left\{ \frac{(K_i + p_i^d + r_i^d d(\mathbf{X}^*, \mathbf{P}_i)) D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}.$$

The reorder quantity  $Q_i$  of each retailer  $i$  is given by the closed form formulas in Properties 2 and 4 for squared Euclidean and Euclidean distances, respectively.

Then, the cost of the benchmark model (BM-SA) is given as

$$Z_{BM-SA} = \sqrt{2(K_i + p_i^d + r_i^d d(\mathbf{X}^*, \mathbf{P}_i)) D_i h'_i}.$$

Recall that for the models where the transportation cost is modeled as a function of quantity ( $PI-Q$ ) and as a function of both quantity and distance ( $PI-Qd$ ), the facility location and the inventory problems are separable. Therefore, for those models, the joint model and the sequential benchmark model would return the same answer. Hence, we compare only the sequential benchmark model with models ( $PI-d-SE$ ) and ( $PI-d-E$ ).

### III.6.2. Experiments

We measure the value of the integrated framework by comparing the cost of the  $BM-SA$  with the costs of the  $PI-d-SE$  for the squared Euclidean distance and the  $PI-d-E$  for the Euclidean distance. For this purpose, we develop two experimental settings.

In the first experiment, we analyze detailed results based on four data groups that consist of 5, 20, 35, and 50 retailers. In each group, we have 500 problem instances, generated randomly using the uniform distributions in Table 3, resulting in a total of 2,000 problem instances.

TABLE 3. Experiment 1: Parameter Values,  $i \in \mathcal{I}$ .

Parameters	Range
$D_i$	U[350,1400]
$K_i$	U[75, 300]
$h'_i$	U[5,10]
$p_i$	U[425,1700]
$r_i$	U[1.35,1.65]
$\mathbf{P}_i$	U[0,100] $\times$ U[0,100]
$ \mathcal{I} $	5, 20, 35, 50

The value of integrated decision-making is given by the percentage gain due to using  $PI-d$  instead of  $BM-SA$ :

$$\text{Percentage gain from integrated decision-making (\%)} = \frac{Z_{BM-SA} - Z_{PI-d}}{Z_{BM-SA}} \times 100,$$

where  $Z_{PI-d}$  is the cost of the solution suggested by the relevant distance and  $Z_{BM-SA}$  is the cost of the benchmark model. While reporting the results, we present the minimum, average, and maximum percentage gains in the objective function value.

In the second experiment, our goal is to infer the impact of problem parameters on the value of the integrated decision-making. For this purpose, we analyze a total of 32,000 problem instances obtained via a factorial design of the demand and cost parameters in Table 4. We considered 4 factors: demand, fixed replenishment and

TABLE 4. Experiment 2: Bounds on the Parameter Values,  $i \in \mathcal{I}$ 

Parameters	$D_i$	$h'_i$	$K_i + p_i^d$	$r_i^d$
Low (L)	350	5	500	0.75
High (H)	1400	10	2000	3

transportation costs, holding cost, and mileage cost. Hence, the factorial design consists of 16 combinations as shown in Table 5. For each combination, we generate 500 instances by changing the retailer locations randomly using the uniform distribution

TABLE 5. Factorial Design

	Parameters			
	$D_i$	$h_i$	$K_i + p_i^d$	$r_i^d$
1	L	L	L	L
2	L	L	L	H
3	L	L	H	L
4	L	L	H	H
5	L	H	L	L
6	L	H	L	H
7	L	H	H	L
8	L	H	H	H
9	H	L	L	L
10	H	L	L	H
11	H	L	H	L
12	H	L	H	H
13	H	H	L	L
14	H	H	L	H
15	H	H	H	L
16	H	H	H	H

$U[0,100] \times U[0,100]$ .

We test the results of the factorial design four data groups that consist of 5, 20, 35, and 50 retailers. In each data group, there are 8,000 problem instances. We report the average percentage gain for each data group and each factorial combination.

### III.6.3. Results: *PI-d-SE*

In this section, we report the results for the *PI-d-SE* regarding the value of integrated decision-making based on the two numerical experiments explained in the previous section.

The results of the first experiment are given in Table 6. This table summarizes the minimum, average, and maximum percentage gains for 500 problem instances for each data group. In all of the data groups, there are several instances where the solution of the benchmark model returns the same solution as the *PI-d-SE*, and hence,

TABLE 6. Results of Experiment 1 for *PI-d-SE*

$ \mathcal{I} $	Minimum gain (%)	Average gain (%)	Maximum gain (%)
5	0.000	0.416	5.236
20	0.000	0.126	1.242
35	0.000	0.076	1.032
50	0.000	0.046	0.565

the minimum gain between these two solutions is zero. The average percentage gain is less than 0.5%, however it can be as high as 5.2%. When we consider the dollar amount associated with these percentage gains, these differences are significant.

An important observation from the results presented in Table 6 is that the average and maximum percentage gains decrease as the number of retailers in the system increase. In other words, the benefits of integrated decision-making is off-set by the introduction of additional retailers in the system.

TABLE 7. Results of Experiment 2 for *PI-d-SE*

	Parameters				Average Gains for Data groups			
	$D_i$	$h'_i$	$K_i + p_i^d$	$r_i^d$	5	20	35	50
1	L	L	L	L	0.56	0.13	0.07	0.05
2	L	L	L	H	1.35	0.35	0.19	0.13
3	L	L	H	L	0.06	0.02	0.01	0.01
4	L	L	H	H	0.47	0.12	0.07	0.05
5	L	H	L	L	0.45	0.13	0.08	0.05
6	L	H	L	H	1.35	0.33	0.20	0.14
7	L	H	H	L	0.06	0.02	0.01	0.01
8	L	H	H	H	0.45	0.13	0.07	0.05
9	H	L	L	L	0.47	0.14	0.08	0.05
10	H	L	L	H	1.31	0.33	0.20	0.14
11	H	L	H	L	0.06	0.02	0.01	0.01
12	H	L	H	H	0.43	0.13	0.07	0.05
13	H	H	L	L	0.47	0.13	0.07	0.06
14	H	H	L	H	1.38	0.33	0.19	0.14
15	H	H	H	L	0.06	0.02	0.01	0.01
16	H	H	H	H	0.45	0.13	0.07	0.05

In the second experiment, we test the impact of problem parameters on the value of integrated decision-making. In Table 7, we report the average percentage gains for each factorial combination. The average gains are the highest when the mileage cost ( $r_i^d, \forall i \in \mathcal{I}$ ) is set at a low value and the sum of fixed replenishment and transportation costs ( $K_i + p_i^d, \forall i \in \mathcal{I}$ ) is set at a high value. Furthermore, the average gains are the lowest for high values of  $r_i^d$  and low values of  $K_i + p_i^d, \forall i \in \mathcal{I}$ . On the other hand, changes in demand ( $D_i, \forall i \in \mathcal{I}$ ) and holding costs ( $h'_i, \forall i \in \mathcal{I}$ ) do not influence the value of integrated decision-making.

#### III.6.4. Results: *PI-d-E*

In this section, we report the results for the *PI-d-E* regarding the value of integrated decision-making based on the two numerical experiments.

In Table 8, we present the results for the first experiment: the minimum, average, and maximum gains between the *PI-d-E* and *BM-SA* for 500 problem instances. For all data groups, the average gain is less than 0.03%. Even the maximum gain is less than 0.6%. Hence, the main finding is that the value of integrated decision-making with Euclidean distances is not as significant as the value of integrated decision-making with squared Euclidean distances. Furthermore, the pattern of percentage gains decreasing as the number of retailers increases is also observed in the comparative results of the *PI-d-E* and *BM-SA*.

TABLE 8. Results of Experiment 1 for *PI-d-E*

$ \mathcal{I} $	Minimum gain (%)	Average gain (%)	Maximum gain (%)
5	0	0.022	0.596
20	0	0.007	0.081
35	0	0.004	0.071
50	0	0.002	0.023

Although the percentage gains between the *PI-d-E* and *BM-SA* are low, we are still interested in how the problem parameters contribute to the percentage gains. In Table 9, we present the average percentage gains for each factorial combination. The findings are similar to the findings from the comparison of the *PI-d-SE* and the *BM-SA*. The average gains are the highest when  $r_i^d$ ,  $\forall i \in \mathcal{I}$  is set at a low value and  $K_i + p_i^d$ ,  $\forall i \in \mathcal{I}$  is set at a high value. Furthermore, the difference between the benchmark model and the *PI-d-E* is insignificant for high values of  $r_i^d$  and low values of  $K_i + p_i^d$ ,  $\forall i \in \mathcal{I}$ . On the other hand, changes in  $D_i$ ,  $\forall i \in \mathcal{I}$ , and  $h'_i$ ,  $\forall i \in \mathcal{I}$ , do not influence the value of integrated decision-making.

TABLE 9. Results of Experiment 2 for *PI-d-E*

	Parameters				Average Gains for Data sets			
	$D_i$	$h'_i$	$K_i + p_i^d$	$r_i^d$	5	20	35	50
1	L	L	L	L	0.0001	0.0000	0.0000	0.0000
2	L	L	L	H	0.0048	0.0012	0.0007	0.0004
3	L	L	H	L	0.0000	0.0000	0.0000	0.0000
4	L	L	H	H	0.0001	0.0000	0.0000	0.0000
5	L	H	L	L	0.0001	0.0000	0.0000	0.0000
6	L	H	L	H	0.0046	0.0012	0.0007	0.0005
7	L	H	H	L	0.0000	0.0000	0.0000	0.0000
8	L	H	H	H	0.0001	0.0000	0.0000	0.0000
9	H	L	L	L	0.0001	0.0000	0.0000	0.0000
10	H	L	L	H	0.0049	0.0012	0.0007	0.0005
11	H	L	H	L	0.0000	0.0000	0.0000	0.0000
12	H	L	H	H	0.0001	0.0000	0.0000	0.0000
13	H	H	L	L	0.0001	0.0000	0.0000	0.0000
14	H	H	L	H	0.0043	0.0012	0.0007	0.0005
15	H	H	H	L	0.0000	0.0000	0.0000	0.0000
16	H	H	H	H	0.0001	0.0000	0.0000	0.0000

### III.6.5. Concluding Remarks

In this section, our main goal is to shed some light on the value of integrated decision-making for problems *PI-d-SE* and *PI-d-E*. We compared the cost of the *PI-d-SE* and

the *PI-d-E* with the cost of a benchmark model (*BM-SA*) in two different numerical experimental settings. The main findings are as follows:

- Integrated decision-making is more influential when squared Euclidean distances are used rather than Euclidean distances.
- The savings due to integrated decision-making are more pronounced when there is a lesser number of retailers.
- The most influential problem parameters are fixed replenishment and transportation costs as well as mileage cost.

These findings provide insights and outline the conditions for when it is beneficial to consider integrated location-inventory models instead of the traditional models.

### III.7. Summary and Conclusions

In this chapter, we analyzed the integration of facility location and inventory decisions in a continuous facility location problem setting for a two-stage supply chain. We showed that the underlying location problem in this model is similar to the classical Weber problem and the underlying inventory problem is a multi-retailer EOQ problem. We also argued that the link between these classical location and inventory problems is the transportation costs.

We discussed three main transportation cost structures that arise in the context of real-life applications. Considering these different transportation cost structures, we developed and analyzed three different integrated location and inventory models, namely, *PI-Q*, *PI-Qd*, and *PI-d*. The analysis of *PI-Q* and *PI-Qd* are trivial mainly due to the simplifications in the average annual cost function for these models. In *PI-Q*, the central DC can be located anywhere on  $\mathbb{R}^2$ , and the order quantity of

each retailer is determined using a modified EOQ formula. Furthermore, *PI-Qd* is separable in terms of location and inventory decisions, i.e., the location of the central DC has no impact on the inventory policy parameters of each retailer, and vice versa.

For *PI-d*, we investigated the problem using squared Euclidean and Euclidean distance norms, i.e. *PI-d-SE* and *PI-d-E*. For the *PI-d-SE* and *PI-d-E*, the transportation cost associated with each retailer's replenishment has an impact on the inventory policy parameters of the retailer as an additional replenishment setup cost. On the other hand, the corresponding replenishment interval affects the number of trips between the retailer and the central DC, which in turn impacts the annual transportation cost. Hence, there is a strong relation between the optimal inventory policy parameters and the location of the central DC. Hence, the sequential solution is suboptimal. For the *PI-d-SE* and *PI-d-E*, the technical solution contributions and the practical insights for the value of integrated decision-making are explained below.

The *PI-d-SE* is a nonlinear optimization problem with an objective function that is jointly convex in location and inventory decisions. Using this characteristic of the objective function, we devise an iterative algorithm that converges to the optimal central DC location and parameters of the inventory policies of the retailers. The *PI-d-E* is also a nonlinear optimization problem. However, the objective function of the *PI-d-E* problem is neither convex nor concave in location and inventory decisions. On the other hand, we prove that the existing retail locations,  $\mathbf{P}_i$ , are local minima for the central DC location, and we propose an effective search algorithm.

In order to measure the value of integrated decision-making, we developed a benchmark model that we refer to as the *BM-SA*. We reported the average percentage gain due to using the *PI-d-SE* and *PI-d-E* for 34,000 problem instances under two different experiment settings. Our findings show that integrated decision-making is more influential when squared Euclidean distances are used rather than Euclidean



distances. Furthermore, the percentage gain is influenced the most due to the trade-off between the fixed replenishment plus the fixed transportation cost and the variable transportation cost.

The results of this chapter can be extended in many ways. One significant generalization is to consider inventory decisions and the coordination of inventory policies of the central DC and the retailers. We will analyze this generalization in Chapter IV. Another significant generalization is to consider multiple potential DC sites in a discrete facility location problem setting, which we analyze in Chapter V. Another noteworthy extension, that is beyond the scope of this dissertation, is to consider more generalized transportation costs, including quantity- and distance-based discounted freight rates, as well as capacitated truck-load and less-than-truck-load rates.

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**DISPLAY 2** The algorithm for *PI-d-E*


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STEP 1: $C^* \leftarrow \infty.$ **for**  $i \in \mathcal{I}$  **do** $\mathbf{X} \leftarrow \mathbf{P}_i.$  $C = C(\mathbf{X}).$ **if**  $C < C^*$  **then** $C^* \leftarrow C.$  $\mathbf{X}^* \leftarrow \mathbf{X}.$ **end if****end for**STEP 2: $iterNo \leftarrow 1.$ Set  $maxIterNo$ .**for**  $iterNo < maxIterNo$  **do** $k \leftarrow 0.$ Set  $\epsilon$  to a predetermined small number.STEP 2.a: Initialize  $\mathbf{X}^k = (x^k, y^k)$  by setting it to any point on the plane  $\mathbb{R}^2$ .STEP 2.b: Calculate

$$\mathbf{X}^{k+1} = \left( \frac{\sum_{i=1}^n \frac{\sqrt{D_i h'_i a_i}}{\sqrt{(K_i + p_i^d) + r_i^d \sqrt{(x^k - a_i)^2 + (y^k - b_i)^2}}}}{\sum_{i=1}^n \frac{\sqrt{D_i h'_i}}{\sqrt{(K_i + p_i^d) + r_i^d \sqrt{(x^k - a_i)^2 + (y^k - b_i)^2}}}}, \frac{\sum_{i=1}^n \frac{\sqrt{D_i h'_i b_i}}{\sqrt{(K_i + p_i^d) + r_i^d \sqrt{(x^k - a_i)^2 + (y^k - b_i)^2}}}}{\sum_{i=1}^n \frac{\sqrt{D_i h'_i}}{\sqrt{(K_i + p_i^d) + r_i^d \sqrt{(x^k - a_i)^2 + (y^k - b_i)^2}}} \right)$$

**while**  $\|\mathbf{X}^{k+1} - \mathbf{X}^k\| > \epsilon$  **do** $k \leftarrow k + 1.$ Repeat *STEP 2.b*.**end while** $C^{iterNo} = C(\mathbf{X}^k).$ **if**  $C^{iterNo} < C^*$  **then** $C^* \leftarrow C^{iterNo}.$  $\mathbf{X}^* \leftarrow \mathbf{X}^k.$ **end if****end for**STEP 3: Return  $\mathbf{X}^*$  as the DC location.

## CHAPTER IV

CONTINUOUS FACILITY LOCATION PROBLEMS IN THREE-STAGE  
SUPPLY CHAINS

In this chapter, we investigate the integration of facility location and inventory decisions in a continuous facility location problem setting in three-stage supply chains. In particular, we consider a three-stage distribution system consisting of a single supplier at a known location, a central DC whose location is to be determined, and multiple geographically dispersed retailers at known locations.

We examine the case where the DC's (once established) and the retailers' inventory systems are operated under the assumptions of the classical single-warehouse multi-retailer (SWMR) problem studied by Roundy (1985). That is, the retailers face deterministic constant demand rates for a single product that must be met without shortage or backlogging, and they receive supply from the central DC (once established) via direct shipments at the expense of replenishment costs, which include retailer-specific fixed ordering (setup) costs as in Roundy (1985) as well as *additional* transportation costs. Inventory held at each retailer accrue holding costs.

The central DC is also a potential inventory keeping location subject to holding costs, and it replenishes from the supplier by incurring replenishment costs, which also consist of a fixed ordering cost as in Roundy (1985) as well as an *additional* transportation cost. The supplier does not carry any inventory and replenishes on a lot-for-lot basis to fulfill orders from the DC, and, hence, inventory related costs at the supplier are sunk costs and immaterial for our purposes, as in Roundy (1985). Clearly, this setting is applicable when the supplier is a manufacturer who incurs linear production costs only or inventory holding is prohibited at the supplier. Also, as in Roundy (1985), replenishment lead times are assumed negligible and there are

no capacity restrictions. Although per-unit costs (of production/replenishment and transportation) and a fixed cost of establishing the warehouse can be incorporated in the proposed model in a trivial fashion, they are omitted without loss of generality since they are, in fact, immaterial for our purposes.

This problem of interest is a generalization of the two-stage distribution system discussed in Chapter III since the central DC, whose location is to be determined, is now an inventory keeping location and incurs a fixed replenishment and inventory holding cost. Hence, the inventory policy of the central DC must be coordinated with the inventory policies of the retailers. This characteristic brings additional complexity to the models and solution techniques. Considering the setting described above, the problem is the simultaneous optimization of central DC location and inventory replenishment decisions with the objective of minimizing the *system-wide* transportation, fixed ordering, and inventory holding costs. Since we consider the case where the DC location is represented by two continuous decision variables corresponding to the unknown coordinates of the DC as in Chapter III, the underlying problem is a generalization of *both*

- the Weber problem (Love et al., 1988), to explicitly consider the inventory decisions along with transportation, fixed ordering and inventory holding costs, *and*
- the SWMR problem (Roundy, 1985), to explicitly consider the DC location decision along with transportation costs.

The resulting model is applicable in **traditional distribution**, **crossdocking**, and **vendor-managed inventory (VMI)** applications as discussed next.

In the context of a **traditional distribution** application, suppose that the supplier is a manufacturer who does not have an on-site inventory keeping facility, and,

hence, is interested in establishing an intermediate distribution center (DC) for serving multiple smaller warehouses called bins, i.e., “retailers,” at existing geographically dispersed locations, say, throughout a state, via direct shipments from the DC. If the distribution operations, i.e., warehousing and transportation, are managed centrally by the supplier, it may make economical sense to establish a central DC for coordinating outbound shipments in order to realize economies of scale inherent in bulk transportation from the manufacturer to the DC while at the same time minimizing the system-wide distribution costs including transportation, fixed ordering, and inventory holding costs.

If economical, the proposed model leads to a solution where the DC does not carry inventory, and, hence, is operated as a **cross-docking** facility that receives consolidated loads of shipments, from the manufacturer, where each individual shipment is eventually destined to a specific bin via direct delivery from the cross-docking facility.

Now, suppose that the supplier is a distributor who is interested in implementing a **VMI** program in collaboration with its retailers for replenishing its inventory at retail locations. In a typical VMI application, the inventory in the entire system, including inventory at the retailers, is owned by the supplier who is the sole decision maker designing the replenishment plan and overseeing the transportation operations (Bernstein et al., 2007; Çetinkaya and Lee, 2000). Hence, the distributor is, in fact, interested in minimizing the system-wide distribution costs which, in this case, include inventory related costs. If each retailer requests a direct delivery for prompt and hassle-free replenishments, then the supplier should evaluate two options:

- O1. Developing a replenishment plan with retailer-specific deliveries directly from the supplier location.

- O2. Developing an integrated location-inventory strategy by establishing a central DC to *coordinate* (i) consolidated loads of shipments to the DC from the supplier location and (ii) direct shipments from the central DC to the retailers.

The first option can be evaluated using the existing literature whereas in order to evaluate the latter option the distributor is faced with the problem of interest in this chapter.

We emphasize that our focus here is on the impact of coordinated replenishments considering the case of *direct deliveries* between the successive tiers of the underlying distribution system. Incorporation of cost savings associated with vehicle routing for simultaneous optimization of location and inventory decisions in the setting we consider remains an important and challenging practical problem for future investigation. Vehicle routing related generalizations of joint location inventory model are aimed at extending Herer and Roundy's (Herer and Roundy, 2000) results developed considering the case where the central DC location is known, and our analysis provides a foundation for these generalizations as well.

In summary, considering a basic setting with practical motivations, we investigate the impact of location decisions on inventory decisions and vice versa. A careful investigation of these impacts requires integrated decision-making, a task which leads to an interesting optimization problem generalizing the basic models of location theory and inventory theory. As we demonstrate later, the solution of the problem obtained using integrated decision-making differs greatly from the one obtained using the traditional sequential decision-making—where the location decision precedes the inventory decisions—as far as the system-wide costs are concerned.

The organization of this chapter is as follows. In the next section, we present the problem notation as well as the general problem formulation with generic trans-

portation costs. In Sections IV.2 and IV.3, we investigate the general problem with quantity-based  $PII-Q$  and quantity- and distance-based  $PII-Qd$  transportation costs in three-stage supply chains, respectively. In Section IV.4, we analyze the problem with distance-based transportation costs,  $PII-d$ . This section presents the most difficult cases of the problem. In the following sections IV.5 and IV.6, we consider analytical and structural properties of the problem  $PII-d$  that leads to efficient algorithms. In Section IV.7, we present numerical results regarding the effectiveness of the algorithmic approaches as well as the value of integrated decision-making. Finally, Section IV.8 summarizes our key findings from the analysis of continuous facility location problems in three-stage supply chains and concludes the chapter.

#### IV.1. General Model and Notation

As mentioned in the introduction, our modeling assumptions are similar to those in the SWMR problem setting except for the explicit consideration of (i) the central DC location decision and (ii) transportation costs associated with deliveries from the supplier to the central DC and from the central DC to the retailers. In particular, we consider quantity-based (3.3), quantity- and distance-based (3.4), and distance-based (3.5) transportation costs as in Chapter III. For this reason, unlike the SWMR problem, we take into account the facility locations—representing the supplier, DC, and retailers—explicitly.

More specifically, we consider a total of  $n \geq 2$  retailers. The known locations of the supplier and the retailers are denoted by  $\mathbf{P}_0 = (a_0, b_0)$  and  $\mathbf{P}_i = (a_i, b_i), i \in \mathcal{I} = \{1, \dots, n\}$ , respectively. The unknown DC location is denoted by  $\mathbf{X} = (x, y)$ . We treat  $x$  and  $y$  as decision variables.

To proceed with the mathematical model, we summarize the notation introduced

so far and define some additional parameters:

- $\mathcal{I}$  set of retailers,  $\mathcal{I} = \{1, \dots, n\}$ .
- $\mathbf{P}_0$  location of the supplier,  $\mathbf{P}_0 = (a_0, b_0)$ .
- $\mathbf{P}_i$  location of retailer  $i$ ,  $\mathbf{P}_i = (a_i, b_i)$ ,  $\forall i \in \mathcal{I}$ .
- $D_i$  demand rate faced by retailer  $i$ ,  $\forall i \in \mathcal{I}$ .
- $D$  total demand rate,  $D = \sum_i D_i$ .
- $h'_0$  inventory holding cost rate for each unit of inventory at the DC.
- $h'_i$  inventory holding cost rate for each unit of inventory at retailer  $i$ ,  
 $h'_i \geq h'_0$ ,  $\forall i \in \mathcal{I}$ .
- $h_0$  echelon holding cost rate at the DC,  $h_0 = h'_0$ .
- $h_i$  echelon holding cost rate at retailer  $i$ ,  $h_i = h'_i - h'_0$ ,  $\forall i \in \mathcal{I}$ .
- $K_0$  fixed ordering cost of the DC.
- $K_i$  fixed ordering cost of retailer  $i$ ,  $\forall i \in \mathcal{I}$ .
- $T_b$  fixed base period (set a priori).
- $\alpha_0$  transportation cost per replenishment to the DC from the supplier.
- $\alpha_i$  transportation cost per replenishment to retailer  $i$ ,  $\forall i \in \mathcal{I}$ , from the DC.

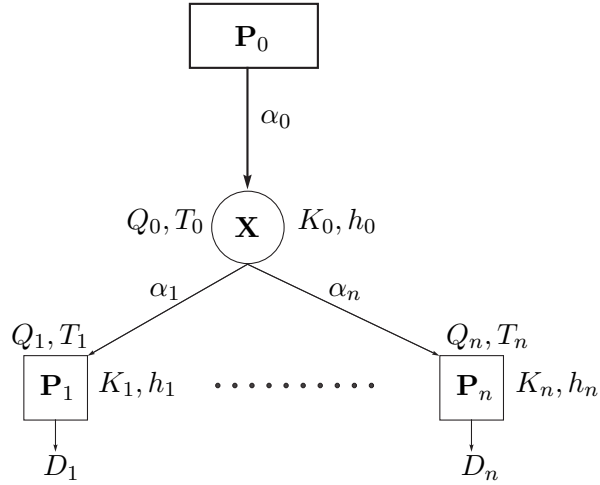
The decision variables of the problem are:

- $T_0$  reorder interval of the DC.
- $T_i$  reorder interval of retailer  $i$ ,  $\forall i \in \mathcal{I}$ .
- $\mathbf{T}$  vector of reorder intervals,  $\mathbf{T} = (T_0, \dots, T_n)$ .
- $Q_0$  order quantity of the DC from the supplier.
- $Q_i$  order quantity of retailer  $i$  from the DC,  $\forall i \in \mathcal{I}$ .
- $\mathbf{Q}$  vector of order quantities,  $\mathbf{Q}$ .
- $\mathbf{X}$  vector of unknown DC location,  $\mathbf{X} = (x, y)$ .

Before constructing the mathematical model, let us recall the problem setting



FIGURE 9. Problem Setting.



described in detail in previous section and illustrated by Figure 9. The main assumptions of this setting that relate to the decision variables and cost parameters can be summarized as follows:

- The DC is replenished at successive reorder intervals of  $T_0$  incurring  $\alpha_0 + K_0$ , which represents the total costs of transportation and ordering per DC replenishment.
- Retailer  $i \in \mathcal{I}$  is replenished at successive reorder intervals of  $T_i$  incurring  $\alpha_i + K_i$ ,  $i \in \mathcal{I}$ , which represents the total costs of transportation and ordering per retailer replenishment.
- Echelon holding costs accumulate at rate  $h_0$  at the DC over  $T_0$  and at rate  $h_i$ , at retailer  $i$  over  $T_i$ .
- The reorder intervals,  $T_0, \dots, T_n$ , are chosen as power-of-two multiples of a fixed base period,  $T_b$ .

Under these assumptions, the system-wide average *annual* total costs of transportation, fixed ordering, and inventory holding, denoted by  $Z(\mathbf{X}, \mathbf{T})$ , can be expressed as

$$\begin{aligned} Z(\mathbf{X}, \mathbf{T}) &= \frac{\alpha_0}{T_0} + \frac{K_0}{T_0} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} \\ &+ \sum_{i \in \mathcal{I}} \frac{\alpha_i}{T_i} + \sum_{i \in \mathcal{I}} \frac{K_i}{T_i} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i. \end{aligned} \quad (4.1)$$

In (4.1), the first three terms represent the average annual costs at the DC, and the last three terms represent the average annual costs at the retailers. In particular, the first term is the average annual transportation cost from the supplier to the DC, the second and third term are the average annual ordering and holding costs at the DC, respectively. The fourth term is the total average annual transportation cost from the DC to retailers, and, finally, the fifth and sixth terms represent the total average annual ordering and holding costs at the retailers, respectively. We note that (4.1) does not include an annual fixed cost for establishing the DC as we assume that this cost is independent of the DC location, and therefore, a sunk cost that is immaterial for our purposes. Likewise, the per-unit costs (of production/replenishment and transportation) are omitted without loss of generality since it is trivial to show that they lead to constant annual cost terms in (4.1), and, hence, they are also immaterial for our purposes. Treating the DC location,  $\mathbf{X} = (x, y)$ , and reorder intervals of the warehouse and retailers,  $\mathbf{T} = (T_0, \dots, T_n)$ , as our decision variables, the integrated location-inventory problem of interest, denoted by  $(PII)$ , can be formulated as the following mixed integer nonlinear program:

$$\text{Min } Z(\mathbf{X}, \mathbf{T}) \tag{PII}$$

subject to

$$T_i = 2^{v_i} T_b \quad \text{and} \quad v_i \in \mathbb{Z}, \quad \text{for } i = 0, \dots, n. \tag{4.2}$$

$$\mathbf{T} \in \mathbb{R}_+^{n+1}, \quad \mathbf{X} \in \mathbb{R}^2. \tag{4.3}$$

The solution of *PII* is greatly influenced by the structure of the transportation costs  $(\alpha_i, i = 0, \dots, n)$  in the problem. Hence, we analyze this integrated location-inventory problem of interest under three different transportation cost structures introduced in Chapter III. In particular, we will consider quantity-based (3.3), quantity- and distance-based (3.4), and distance-based (3.5) transportation costs in (4.1).

#### IV.2. Models with Quantity-based Transportation Costs (*PII-Q*)

This section focuses on the analysis of the integrated location-inventory problem in continuous facility location problem setting in three-stage distribution systems where the transportation costs from the supplier to the DC  $(\alpha_0)$  and from the DC to the retailers  $(\alpha_i, \forall i \in \mathcal{I})$  are modeled as a function of quantity as in (3.3). With this transportation cost structure, the general model (4.1) is specialized as

$$\begin{aligned} Z(\mathbf{X}, \mathbf{T}) &= \frac{p_0^q + r_0^q Q_0}{T_0} + \frac{K_0}{T_0} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} \\ &+ \sum_{i \in \mathcal{I}} \frac{p_i^q + r_i^q Q_0}{T_i} + \sum_{i \in \mathcal{I}} \frac{K_i}{T_i} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i. \end{aligned} \tag{4.4}$$

It is important to note that the model does not contain any cost term related to the location of the central DC. In other words, there are no terms that restrict the location of the DC, or penalize the distance between the DC and the other facilities. The location of the DC can be anywhere on the plane. Therefore, the location problem

and the inventory policy decisions are independent of each other. Moreover, after re-organizing the terms of the cost function, we obtain a shifted SWMR lot sizing problem:

$$Z(\mathbf{X}, \mathbf{T}) = \sum_{i \in \mathcal{I} \cup \{0\}} r_i^q D_i + \frac{K'_0}{T_0} + \sum_{i \in \mathcal{I}} \frac{K'_i}{T_i} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i,$$

where  $w_0 = D$ ,  $K'_0 = K_0 + p_0^q$ , and  $K'_i = K_i + p_i^q$ , for all  $i \in \mathcal{I}$ . In this formulation, the first term is a constant, and the remaining cost terms are equivalent to the SWMR lot sizing problem. Hence, the inventory policy parameters are determined by solving this corresponding SWMR lot sizing problem as explained in Section II.2.2 using the algorithm developed by Roundy (1985).

The following lemma summarizes this important result.

- LEMMA 5**    1. *When the transportation costs in (4.1) are modeled as in (3.3), the inventory policy problem and the location problem are independent.*
2. *The DC can be located anywhere on the plane.*
3. *The inventory policy decisions can be found by solving the modified SWMR model with new setup costs  $K'_0 = K_0 + p_0^q$  and  $K'_i = K_i + p_i^q$  at the warehouse and at retailer  $i \in \mathcal{I}$ , respectively. The overall cost of the SWMR is augmented by total variable transportation cost, i.e.,  $\sum_{i=0}^n r_i^q w_i$ .*

As a special case, we investigate this problem for a single supplier, central DC, and a single retailer distribution system. Then, the underlying inventory problem share the similar characteristics of the (SWSR) lot sizing problem discussed in Section II.2.2. For this special case, we consider the following objective function:

$$\begin{aligned} Z(\mathbf{X}, \mathbf{Q}) &= D(r_0^q + r_1^q) + (p_0^q + K_0) \frac{D}{Q_0} + \frac{1}{2} h_1 Q_1 + (p_1^q + K_1) \frac{D}{Q_1} + \frac{1}{2} h_1 Q_1 \\ &= \min_{\mathbf{Q}} c + k'_0 \frac{D}{Q_0} + \frac{1}{2} h_0 Q_0 + k'_1 \frac{D}{Q_1} + \frac{1}{2} h_1 Q_1, \end{aligned}$$

where  $D = D_1$ ,  $c = D(r_0^q + r_1^q)$  and  $K'_i = p_i^q + K_i$  for  $i = 0, 1$ . Note that, again in this special case, there are no restrictions regarding the location of the central DC. Hence, the problem is reduced to determining inventory policy parameters, which, in this case, are  $Q_0$  and  $Q_1$ . Recalling that the underlying inventory problem is equivalent to the SWSR lot sizing problem, this problem can be solved effectively by restricting ourselves to integer ratio policies. Let  $Q_1 = Q$  and  $Q_0 = m * Q_1 := m * Q$ , where  $m$  is an integer multiple. With this change in the objective function, we obtain

$$Z = c + K'_0 \frac{D}{mQ} + K'_1 \frac{D}{Q} + \frac{1}{2} h_0 m Q + \frac{1}{2} h_1 Q.$$

In this formulation, the decision variables are  $Q$  and  $m$ . The following lemma outlines how to compute these variables:

**LEMMA 6** 1. *Optimal  $m^*$  value is given with the equation*

$$m^* = \arg \min\{Z(\lfloor m_0 \rfloor), Z(\lceil m_0 \rceil)\}$$

where

$$m_0 = \sqrt{\frac{K'_0 h_1}{K'_1 h_0}}.$$

2. *The optimal value of the cost  $Z$  and order quantity  $Q$  as a function of  $m^*$  is given as*

$$Z^*(m^*) = D(r_0^q + r_1^q) + \sqrt{2D(K'_0 h_0 + K'_1 h_1) + 2DK'_1 h_0 m^* + 2D \frac{K'_0 h_1}{m^*}}$$

and

$$Q^*(m^*) = \sqrt{\frac{2(K'_0 + m^* K'_1)D}{m^*(m^* h_0 + h_1)}}.$$

3. *The inventory policy parameters  $(Q^*, m^*)$  do not have an effect on location decision.*

**Proof:** The proof of the first two parts directly follows from Çetinkaya and Lee

(2002), p. 551.

For the last part, since there is no restriction on the DC location regarding the costs or constraints, the DC can be located anywhere on the line from the supplier to the retailer, independent of the value of  $m^*$ . ■

### IV.3. Models with Quantity- and Distance-based Transportation Costs (*PII-Qd*)

In this section, we analyze the case where the transportation costs in the three-stage distribution system are a function of quantity and distance as in (3.4). With this type of transportation cost structures, the general cost function (4.1) is re-written as

$$\begin{aligned} Z(\mathbf{X}, \mathbf{T}) &= \frac{p_0^{qd} + r_0^{qd} Q_0 d(\mathbf{X}, \mathbf{P}_0)}{T_0} + \frac{K_0}{T_0} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} \\ &+ \sum_{i \in \mathcal{I}} \frac{p_i^{qd} + r_i^{qd} Q_i d(\mathbf{X}, \mathbf{P}_i)}{T_i} + \sum_{i \in \mathcal{I}} \frac{K_i}{T_i} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i. \end{aligned} \quad (4.5)$$

Using the properties of the SWMR lot sizing problem including zero inventory ordering property and the relation  $Q_i = T_i * D_i$ <sup>1</sup>, we re-organize (4.5):

$$\begin{aligned} Z(\mathbf{X}, \mathbf{T}) &= \sum_{i \in \mathcal{I} \cup \{0\}} r_i^{qd} D_i d_i + \frac{K_0 + p_0^{qd}}{T_0} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} \\ &+ \sum_{i \in \mathcal{I}} \frac{K_i + p_i^{qd}}{T_i} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i, \end{aligned} \quad (4.6)$$

where  $d_i = d(\mathbf{X}, \mathbf{P}_i)$  for  $i \in \mathcal{I} \cup \{0\}$ .

In this formulation, the first and fourth cost terms contain *only* DC-location related decision variables, and the rest of the cost terms contain *only* inventory-related decision variables. Hence, the location or distance related decision variables do not interact with the decision variables regarding the inventory policy parameters.

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<sup>1</sup> $D_0 = D$

With this characteristic property, the problem reduces to finding a solution to the location problem (defined by the first cost term) and a solution to the inventory problem (defined by the rest of the cost terms) separately. In fact, letting the weight of each facility  $w_i = r_i^{qd} D_i$ , for  $i \in \mathcal{I} \cup \{0\}$ , the location problem is a Weber problem. The following lemma summarizes this structural result.

**LEMMA 7** *For PII-Qd, the facility location and the inventory problems are separable, i.e.,*

$$Z(\mathbf{X}, \mathbf{T}) = Z_L(\mathbf{X}) + Z_I(\mathbf{T}),$$

where

$$\begin{aligned} Z_L(\mathbf{X}) &= \sum_{i \in \mathcal{I} \cup \{0\}} r_i^{qd} D_i d(\mathbf{X}, \mathbf{P}_i) \text{ and} \\ Z_I(\mathbf{T}) &= \sum_{i \in \mathcal{I} \cup \{0\}} \frac{K'_i}{T_0} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i, \end{aligned}$$

where  $K'_i = K_i + p_i^{qd}$  for  $i \in \mathcal{I} \cup \{0\}$ .

In Lemma 7,  $Z_L(\mathbf{X})$  is the Weber problem, and its solution depends on the distance norm as explained in Section II.1.1. On the other hand,  $Z_I(\mathbf{T})$  is the SWMR lot-sizing problem and is solved by Roundy's algorithm (see Section II.2.2).

#### IV.3.1. Single-Retailer Case (*PII-Qd-SR*)

In this subsection, we consider a special case of *PII-Qd* with only one retailer. Hence, Lemma 7 holds, and the average annual cost for *PII-Qd-SR* is given as

$$Z(\mathbf{X}, \mathbf{Q}) = Z_L(\mathbf{X}) + Z_I(\mathbf{Q}),$$

where

$$\begin{aligned} Z_L(\mathbf{X}) &= Dr_0^{qd}d_0 + Dr_1^{qd}d_1, \\ Z_I(\mathbf{Q}) &= K'_0\frac{D}{Q_0} + \frac{1}{2}h_0Q_0 + K'_1\frac{D}{Q_1} + \frac{1}{2}h_1Q_1, \end{aligned}$$

where  $K'_i = p_i^{qd} + K_i$ , for  $i = 0, 1$ .

In this formulation,  $Z_L(\mathbf{X})$  is a special Weber problem with only two existing facilities, and  $Z_I(\mathbf{Q})$  is the SWSR lot sizing model. To solve  $Z_I(\mathbf{Q})$ , we follow an analysis similar to the one in Section IV.2. The optimal order quantity of the retailer  $Q$  and integer multiple  $m^*$  is estimated with the closed-form formula in Lemma 6.

The following lemma gives us the optimal location of the central DC  $\mathbf{X}^* = (x^*, y^*)$ :

**LEMMA 8** *Optimal location  $\mathbf{X}^* = (x^*, y^*)$  of the central DC satisfies one of the three conditions depending on the relation between the unit transportation costs from the supplier to the DC ( $r_0^{qd}$ ) and from the DC to the retailer ( $r_1^{qd}$ ):*

1. If  $r_0^{qd} > r_1^{qd}$ ,  $X^* = (x^*, y^*) = P_0 = (a_0, b_0)$ .
2. If  $r_1^{qd} > r_0^{qd}$ ,  $X^* = (x^*, y^*) = P_1 = (a_1, b_1)$
3. Otherwise,  $r_1^{qd} = r_0^{qd}$ , the warehouse can be anywhere on the line connecting the retailer and the supplier.

**Proof:** Since  $Z_L(\mathbf{X})$  is a special Weber problem with only two existing facilities, the feasible region for  $Z_L(\mathbf{X})$  has only two extreme points. These are the location of the supplier,  $\mathbf{P}_0 = (a_0, b_0)$ , and the location of the retailer,  $\mathbf{P}_1 = (a_1, b_1)$ . Furthermore, the convex hull of the two extreme points is a line. We assume that a norm is used to represent the distances so that the triangular inequality holds. Thus, we only consider the points on the line (or the roadway) that connects the supplier and the retailer as



potential DC locations. We let  $d$  represent the total distance between the supplier and the retailer and  $d_0$  represent the distance between the DC and the supplier. Then, the distance between the DC and the retailer is given by  $d - d_0$ .

Assume that  $r_0^{qd} > r_1^{qd}$ . Let the optimum location of the facility be  $\mathbf{X}^* = (x^*, y^*)$ , some point on the line  $|\mathbf{P}_0\mathbf{P}_1|$  other than  $\mathbf{P}_0 = (a_0, b_0)$ .

Let  $Z_I^*$  and  $Z_L^*$  be the optimum cost of  $Z_L(\mathbf{X})$  and  $Z_I(\mathbf{Q})$ , respectively. Since  $Z_L(\mathbf{X})$  and  $Z_I(\mathbf{Q})$  are separable, the optimal cost of *PII-Qd-SR*,  $Z^*$ , is equal to the optimal cost of the inventory problem  $Z_I^*$  and the cost of the location problem:

$$\begin{aligned} Z^* &= Z_I^* + D \left\{ r_0^{qd} d_0 + r_1^{qd} d_1 \right\} \\ &= Z_I^* + D \left\{ r_0^{qd} d_0 + r_1^{qd} [d - d_0] \right\} \\ &> Z_I^* + D r_1^{qd} d \end{aligned}$$

where the last inequality follows from the assumption  $r_0^{qd} > r_1^{qd}$  and locating the DC at  $\mathbf{P}_0 = (a_0, b_0)$ . However, since  $Z^*$  is the optimal solution, this is a contradiction. Hence, the DC is optimally located at the supplier,  $x^* = a_0$  and  $y^* = b_0$ .

Similarly, for the second case, when  $r_0^{qd} < r_1^{qd}$ , the minimum cost facility location occurs when  $d_0 = d$  and  $d_1 = 0$ . Therefore, the optimal DC location is at  $\mathbf{P}_1$ . In the third case, the general cost function does not change by changing the location of the DC. Therefore, both of the extreme points,  $\mathbf{P}_0$  and  $\mathbf{P}_1$ , and, in fact, any convex combination of these extreme points which is actually the whole convex hull for this case, are optimal. ■

#### IV.4. Models with Distance-based Transportation Costs (*PII-d*)

In this section, we analyze the integrated location and inventory problem in three stage distribution systems where the transportation costs are a function of distance

as in (3.5). With this type of transportation cost structure, the general cost function (4.1) is re-written as

$$\begin{aligned}
Z(\mathbf{X}, \mathbf{T}) = & \frac{p_0^d + r_0^d d(\mathbf{X}, \mathbf{P}_0)}{T_0} + \frac{K_0}{T_0} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} \\
& + \sum_{i \in \mathcal{I}} \frac{p_i^d + r_i^d d(\mathbf{X}, \mathbf{P}_i)}{T_i} + \sum_{i \in \mathcal{I}} \frac{K_i}{T_i} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i. \quad (4.7)
\end{aligned}$$

In this formulation, second, third, fifth, and sixth cost terms contain *only* inventory-related decision variables. However, the first and fourth terms not only contain inventory-related decision variables *but also* DC-location related decision variables. Hence, the location or distance related decision variables heavily interact with the decision variables regarding the inventory policy parameters. This interaction demonstrates the need for simultaneous optimization of location and inventory decisions. However, due to this strong interaction, the solution *PII-d-MR* is quite complicated. In order to be able to understand this problem and provide a solution framework, in the next two sections, Sections IV.5 and IV.6, we investigate analytical and algorithmic properties of the problem.

Before concluding this section, we present an analytical solution for a special case of the problem with distance-based transportation cost where there is only one retailer in the three-stage distribution system.

#### IV.4.1. Single Retailer Case (*PII-d-SR*)

In this section, we consider a special case of *PII-d* where there is a single retailer whose demand rate is denoted by  $D$ . Since the integer ratio policies are effective in solving the single retailer lot-sizing problem (Goyal, 1976; Roundy, 1985), we restrict the inventory policy to this class of policies, i.e., we assume that the warehouse's order quantity,  $Q_0 = T_0 D$ , is an integer multiple  $m$  of the retailer's order quantity

$Q_1 = T_1 D$  so that  $Q_0 = mQ_1$ . As in Section IV.3.1, we assume that a norm is used to represent the distances so that the triangular inequality holds and we let  $d$  represent the total distance between the supplier and the retailer and  $d_0$  represent the distance between the warehouse and the supplier. Then, the distance between the warehouse and the retailer is given by  $d - d_0$ . Also, letting  $K'_i = p_i + K_i$ , for  $i = 0, 1$  and rewriting (4.7), the single-retailer problem is given by

$$\begin{aligned} \text{Min } Z(d_0, Q_1, m) &= r_0^d D \frac{d_0}{mQ_1} - r_1^d D \frac{d_0}{Q_1} + r_1^d D \frac{d}{Q_1} + K'_0 \frac{D}{mQ_1} + K'_1 \frac{D}{Q_1} \\ &\quad + \frac{1}{2} h_0 m Q_1 + \frac{1}{2} h_1 Q_1 \end{aligned} \quad (4.8)$$

subject to

$$0 \leq d_0 \leq d, \quad Q_1 \in \mathbb{R}_+, \quad m \in \mathbb{Z}_+.$$

Here,  $Z(d_0, Q_1, m)$  represents the average *annual* total costs of transportation, fixed ordering, and inventory holding as a function of the decision variables  $d_0$ , the distance between the warehouse and the supplier;  $Q_1$ , the order quantity of the retailer; and  $m$ , the number of retailer reorder intervals within the warehouse reorder interval. Although the first term of (4.8) contains all three decision variables in a nonlinear fashion, the optimal solution, denoted by  $(d_0^*, Q_1^*, m^*)$ , can be computed easily using Theorem 3. Before we proceed with this result, let us define

$$Z_1(m) = \sqrt{2D \left( r_1^d d h_1 + K'_0 h_0 + K'_1 h_1 + (r_1^d d h_0 + K'_1 h_0) m + \frac{K'_0 h_1}{m} \right)}, \quad (4.9)$$

$$Z_2(m) = \sqrt{2D \left( r_0^d d h_0 + K'_0 h_0 + K'_1 h_1 + K'_1 h_0 m + \frac{(r_0^d d h_1 + K'_0 h_1)}{m} \right)}, \quad (4.10)$$

$$m_1 = \sqrt{\frac{K'_0 h_1}{K'_1 h_0 + r_1^d d h_0}}, \quad \text{and} \quad m_2 = \sqrt{\frac{r_0^d d h_1 + K'_0 h_1}{K'_1 h_0}}. \quad (4.11)$$

**THEOREM 3** *Based on the ratio  $r_0^d/r_1^d$ ,  $m^*$  is given by one of the following three cases:*

- If  $r_0^d/r_1^d < m_1$ , then  $m^* = \arg \min\{Z_2(\lfloor m_2 \rfloor), Z_2(\lceil m_2 \rceil)\}$ .
- If  $m_1 \leq r_0^d/r_1^d \leq m_2$ , then  $m^* = \arg \min\{Z_1(\lfloor m_1 \rfloor), Z_1(\lceil m_1 \rceil), Z_2(\lfloor m_2 \rfloor), Z_2(\lceil m_2 \rceil)\}$ .
- If  $r_0^d/r_1^d > m_2$ , then  $m^* = \arg \min\{Z_1(\lfloor m_1 \rfloor), Z_1(\lceil m_1 \rceil)\}$ .

In all cases,

$$d_0^* = \begin{cases} 0, & \text{if } m^* \leq r_0^d/r_1^d \\ d, & \text{otherwise} \end{cases} \quad \text{and} \quad (4.12)$$

$$Q_1^* = \sqrt{\frac{2D(d_0^*(r_0^d - m^*r_1^d) + m^*r_1^d d + K'_0 + K'_1)}{m^*(h_0 m^* + h_1)}},$$

i.e., in the case  $m^* \leq r_0^d/r_1^d$ , the optimal warehouse location coincides with the supplier, and in the case  $m^* > r_0^d/r_1^d$ , the optimal warehouse location coincides with the retailer location. Furthermore, when  $r_0^d/r_1^d$  is an integer and  $m^* = r_0^d/r_1^d$ , then any point on the line between the supplier and the retailer, including the end points, is optimal.

**Proof:** For fixed values of  $m$  and  $d_0$ , we consider the problem  $\min_{Q_1 > 0} \{Z(d_0, Q_1, m)\}$  for which it is easy to show that the unique optimal solution, denoted by  $Q_1^*(d_0, m)$ , is given by

$$Q_1^*(d_0, m) = \sqrt{\frac{2D(d_0[r_0^d - mr_1^d] + mr_1^d d + K'_0 + K'_1)}{m(h_0 m + h_1)}}. \quad (4.13)$$

Substituting (4.13) in (4.8), we have

$$Z(d_0, Q_1^*(d_0, m), m) = \sqrt{\frac{2D(d_0[r_0^d - mr_1^d] + mr_1^d d + K'_0 + K'_1 m)(h_0 m + h_1)}{m}}. \quad (4.14)$$

Next, for a fixed value of  $m$ , we consider the problem  $\min_{0 \leq d_0 \leq d} \{Z(d_0, Q_1^*(d_0, m), m)\}$ . We observe that  $Z(d_0, Q_1^*(d_0, m), m)$  is an increasing function of  $d_0$  if  $m < r_0^d/r_1^d$  whereas it is a decreasing function of  $d_0$  if  $m > r_0^d/r_1^d$ . Furthermore, it is independent of  $d_0$  when  $m = r_0^d/r_1^d$ . It follows that the optimal solution of this problem,

denoted by  $d_0^*(m)$ , is

$$d_0^*(m) = \begin{cases} 0, & \text{if } m < r_0^d/r_1^d, \\ [0, d], & \text{if } m = r_0^d/r_1^d, \\ d, & \text{if } m > r_0^d/r_1^d. \end{cases} \quad (4.15)$$

Then using (4.14), it is easy to show that

$$Z(d_0^*(m), Q_1^*(d_0, m), m) = \begin{cases} Z_1(m), & m \leq r_0^d/r_1^d, \\ Z_2(m), & m > r_0^d/r_1^d, \end{cases}$$

where  $Z_1(m)$  and  $Z_2(m)$  are given by (4.9) and (4.10), respectively.

In order to compute  $m^*$ , we now consider the problem  $\min_{m \in \mathbb{Z}_+} \{Z(d_0^*(m), Q_1^*(d_0, m), m)\}$ . Observe that  $Z(d_0^*(m), Q_1^*(d_0, m), m)$  is continuous, and both functions  $Z_1(m)$  and  $Z_2(m)$  are of the form  $\sqrt{\alpha + \beta m + \gamma/m}$  where  $\alpha$ ,  $\beta$ , and  $\gamma$  are positive constants. Although  $\sqrt{\alpha + \beta m + \gamma/m}$  is not convex, it is straightforward to show that (see Çetinkaya and Lee (2002), p. 551 for a proof), the unique relaxed (non-integer) minimizer  $\sqrt{\gamma/\beta}$  of  $\alpha + \beta m + \gamma/m$  also minimizes  $\sqrt{\alpha + \beta m + \gamma/m}$ . Further,  $\sqrt{\alpha + \beta m + \gamma/m}$  is strictly decreasing over  $m < \sqrt{\gamma/\beta}$  and strictly increasing over  $m > \sqrt{\gamma/\beta}$ . Using these results, it is now easy to verify that  $Z_1(m)$  is strictly decreasing over  $m < m_1$  and strictly increasing over  $m > m_1$ . Likewise,  $Z_2(m)$  is strictly decreasing over  $m < m_2$  and strictly increasing over  $m > m_2$ . Noting that  $m_1 < m_2$ , either one of the following three cases holds:

- If  $r_0^d/r_1^d < m_1$  then  $Z(d_0^*(m), Q_1^*(d_0, m), m)$  is a continuous, unimodal function with a relaxed minimum at  $m = m_2 > m_1$  so that

$$m^* = \arg \min \{Z_2(\lfloor m_2 \rfloor), Z_2(\lceil m_2 \rceil)\}.$$

- If  $m_1 \leq r_0^d/r_1^d \leq m_2$ ,  $Z(d_0^*(m), Q_1^*(d_0, m), m)$  is a continuous bimodal function

with only two local relaxed minimums at  $m_1$  and  $m_2$  so that

$$m^* = \arg \min\{Z_1(\lfloor m_1 \rfloor), Z_1(\lceil m_1 \rceil), Z_2(\lfloor m_2 \rfloor), Z_2(\lceil m_2 \rceil)\}.$$

- If  $m_2 < r_0^d/r_1^d$  then  $Z(d_0^*(m), Q_1^*(d_0, m), m)$  is a continuous, unimodal function with a relaxed minimum at  $m = m_1 < m_2$  so that

$$m^* = \arg \min\{Z_1(\lfloor m_1 \rfloor), Z_1(\lceil m_1 \rceil)\}.$$

Once  $m^*$  is computed  $d^*$  and  $Q^*$  are computed by substituting  $m^*$  in (4.13) and (4.15). ■

#### IV.5. Analytical Analysis for *PII-d-MR*

In this section, we analyze the important characteristics of *PII-d-MR* that are exploited for developing algorithmic approaches in Section IV.6. These characteristics relate to (i) the relationship between our problem and the SWMR and Weber problems, (ii) the impact of distance modeling assumptions, and (iii) the implications of the use of power-of-two policies for coordinating replenishments.

##### IV.5.1. Properties of *PII-d-MR* that Relate to the SWMR and Weber Problems

Upon a closer examination of  $Z(\mathbf{X}, \mathbf{T})$  in (4.7), we observe that the first and fourth terms contain both location and inventory decision variables demonstrating the interrelationship between, and, hence, the need for simultaneous optimization of, these two sets of decision variables. Also, these two terms demonstrate that location and inventory decisions are interrelated through transportation costs. Letting  $S_i(\mathbf{X}) =$

$K_i + p_i + r_i^d d(\mathbf{X}, \mathbf{P}_i)$ ,  $i \in \mathcal{I} \cup \{0\}$ , in (4.7) leads to

$$Z(\mathbf{X}, \mathbf{T}) = \sum_{i \in \mathcal{I} \cup \{0\}} \frac{S_i(\mathbf{X})}{T_i} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i. \quad (4.16)$$

Examining the terms of  $Z(\mathbf{X}, \mathbf{T})$  in (4.16), it is easy to observe that *PII-d-MR* resembles the SWMR problem where the “inventory replenishment” costs  $S_i(\mathbf{X})$ ,  $i \in \mathcal{I} \cup \{0\}$ , change with respect to the unknown DC location. This is because consideration of the per-mile transportation cost,  $r_i^d$ ,  $i \in \mathcal{I} \cup \{0\}$ , leads to a *DC-location-dependent replenishment cost* term,  $r_i^d d(\mathbf{X}, \mathbf{P}_i)$ , in  $S_i(\mathbf{X})$ . This observation, in turn, clearly demonstrates the unique feature of the integrated location-inventory problem at hand and the complication associated with the underlying coordinated replenishment problem.

Likewise, letting  $w_i(T_i) = r_i^d / T_i$ ,  $i \in \mathcal{I} \cup \{0\}$ , in (4.7) leads to

$$\begin{aligned} Z(\mathbf{X}, \mathbf{T}) = & \sum_{i \in \mathcal{I} \cup \{0\}} w_i(T_i) d(\mathbf{X}, \mathbf{P}_i) + \sum_{i \in \mathcal{I} \cup \{0\}} \frac{p_i + K_i}{T_i} \\ & + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i. \end{aligned} \quad (4.17)$$

Observe that the first term of  $Z(\mathbf{X}, \mathbf{T})$  in (4.17) resembles the objective function of the Weber problem where the traditional weights associated with the existing locations are now given by the “implied weights”  $w_i(T_i)$ ,  $i \in \mathcal{I} \cup \{0\}$ , that change with respect to the to-be-computed inventory policy parameters. Noting that,  $T_i = Q_i / D_i$ ,  $i \in \mathcal{I} \cup \{0\}$ , where  $Q_i$  is the replenishment quantity corresponding to the reorder interval  $T_i$ , we have  $w_i(T_i) = T_i / r_i^d = (r_i^d / Q_i) D_i$ . That is, from the facility location perspective, explicit consideration of reorder interval  $T_i$ ,  $i \in \mathcal{I} \cup \{0\}$ , associated with location  $\mathbf{P}_i$ , results in an *inventory-policy-parameter-dependent weight* given by  $r_i^d / T_i$ , or, an *implied per-unit-per-mile cost* given by  $r_i^d / Q_i$ . This observation, also, clearly demonstrates the unique feature of the integrated location-inventory at hand and the complication associated with the underlying facility location problem.

Finally, using (4.16) and (4.17), it is easy to verify the following remark.

**REMARK 1** *Given the DC location  $\mathbf{X}$ , PII-d-MR reduces to the SWMR problem, and, given the inventory policy  $\mathbf{T}$ , PII-d-MR reduces to the Weber problem.*

#### IV.5.2. Properties of *PII-d-MR* Based on Distance Functions

In order to represent the distances as a function of the unknown DC location, we consider two commonly employed norms in continuous facility location studies; namely, the squared Euclidean distance norm and the more general  $\ell_p$  distance norm where  $p = 1$  and  $p = 2$  represent the well-known rectangular and Euclidean distances, respectively. It is clear that independent of the distance function, each of the first and fourth terms of  $Z(\mathbf{X}, \mathbf{T})$  in (4.7) is a function of both the location of the DC,  $\mathbf{X} = (x, y)$ , and the inventory policy parameters,  $\mathbf{T}$ .

The following remark is based on the well-known results, in inventory theory (Roundy, 1985) and continuous facility location theory (Love et al., 1988), that relate to the observation in Remark 1.

**REMARK 2** *Considering the squared Euclidean or  $\ell_p$  distances, for a fixed  $\mathbf{X}$ ,  $Z(\mathbf{X}, \mathbf{T})$  in (4.7) is convex in  $\mathbf{T}$ , and for a fixed  $\mathbf{T}$ ,  $Z(\mathbf{X}, \mathbf{T})$  in (4.7) is convex in  $\mathbf{X}$ .*

Let  $\mathbf{X}^o(\mathbf{T}) = (x^o(\mathbf{T}), y^o(\mathbf{T}))$  denote the solution of *PII-d-MR* for a fixed  $\mathbf{T}$ . Remark 1 implies that if squared Euclidean distances are employed then

$$x^o(\mathbf{T}) = \frac{\sum_{i \in \mathcal{I} \cup \{0\}} w_i(T_i) a_i}{\sum_{i \in \mathcal{I} \cup \{0\}} w_i(T_i)} \quad \text{and} \quad y^o(\mathbf{T}) = \frac{\sum_{i \in \mathcal{I} \cup \{0\}} w_i(T_i) b_i}{\sum_{i \in \mathcal{I} \cup \{0\}} w_i(T_i)}, \quad (4.18)$$

whereas if  $\ell_p$  distances are employed then  $\mathbf{X}^o(\mathbf{T})$  is computed using the well-known generalization of the iterative Weiszfeld procedure developed by Morris and Verdini (1979) as explained in Section II.1.1. Next, we summarize the properties of *PII-d-MR*



regarding the distance functions by presenting formal results without explicit proofs.

We note that

- Lemma 9 is based on well-known results in location theory (Brimberg and Love, 1995),
- Lemma 10 can be proved in a straightforward fashion using the definiteness characteristics of the associated Hessians, and
- the convexity properties of  $Z(\mathbf{X}, \mathbf{T})$  in Theorem 4 follows from Lemma 10.

**LEMMA 9** *Letting  $\Omega$  denote the convex hull of the existing locations  $i \in \mathcal{I} \cup \{0\}$ , we have  $\mathbf{X}^o(\mathbf{T}) \in \Omega$  under both the squared Euclidean and  $\ell_p$  distances.*

**LEMMA 10** *Let  $f_{se}, f_p : \mathbb{R} * \mathbb{R} * \mathbb{R}^+ \rightarrow \mathbb{R}$  where*

$$f_{se}(x, y, T) = \frac{(x - a)^2 + (y - b)^2}{T} \quad \text{and} \quad f_p(x, y, T) = \frac{(|x - a|^p + |y - b|^p)^{1/p}}{T}.$$

*The function  $f_{se}$  is jointly convex and  $f_p$  is neither convex nor concave in  $(x, y, T)$ .*

**THEOREM 4** *Under the squared Euclidean distances, the cost function  $Z(\mathbf{X}, \mathbf{T})$  is jointly convex in  $\mathbf{X}$  and  $\mathbf{T}$ . Under the  $\ell_p$  distances, however,  $Z(\mathbf{X}, \mathbf{T})$  is neither convex nor concave in  $\mathbf{X}$  and  $\mathbf{T}$ .*

### IV.5.3. Properties of *PII-d-MR* Based on Power-of-Two Policies

We denote the optimum solution of *PII-d-MR* by  $(\mathbf{X}^P, \mathbf{T}^P)$  where  $\mathbf{X}^P = (x^P, y^P)$  and  $\mathbf{T}^P = (T_0^P, \dots, T_n^P)$ . If the first constraint (4.2) of *PII-d-MR* is ignored, we obtain a relaxed problem which we denote by *R-d-MR*. We represent the optimum solution of *R-d-MR* by  $(\mathbf{X}^*, \mathbf{T}^*)$  where  $\mathbf{X}^* = (x^*, y^*)$  and  $\mathbf{T}^* = (T_0^*, \dots, T_n^*)$ . Clearly,  $Z(\mathbf{X}^*, \mathbf{T}^*)$  provides a lower bound on  $Z(\mathbf{X}^P, \mathbf{T}^P)$ , and, hence, a careful analysis of *R-d-MR* is

useful for obtaining a good quality solution for *PII-d-MR*. Next, we present some properties regarding the relationship between these two problems and their solutions (in Section IV.5.3.1) followed by an observation (in Section IV.5.3.2) and a formal result (in Section IV.5.3.3).

#### IV.5.3.1. *R-d-MR* as a Lower Bound

Following an approach similar to the one in Roundy (1985) (also summarized in (Simchi-Levi et al., 2004, Chapter 6)) while at the same time incorporating the DC location  $\mathbf{X}$  and introducing some necessary notation, we first represent the *R-d-MR* in a compact form with the eventual goal of illustrating some characteristics of its solution. To this end, we define  $Z_i(\mathbf{X}, T_0, T_i) = S_i(\mathbf{X})/T_i + (1/2)h_0D_i \max\{T_0, T_i\} + (1/2)h_iD_iT_i$ ,  $i \in I$ , and rewrite  $Z(\mathbf{X}, \mathbf{T})$  given by (4.16) as

$$Z(\mathbf{X}, \mathbf{T}) = \frac{S_0(\mathbf{X})}{T_0} + \sum_{i \in \mathcal{I}} Z_i(\mathbf{X}, T_0, T_i).$$

For a fixed DC location,  $\mathbf{X} \in \Omega$ , and a fixed DC reorder interval,  $T_0$ , let us first consider the problem  $\min_{T_i > 0} \{Z_i(\mathbf{X}, T_0, T_i)\}$ ,  $i \in I$ . Defining  $g_i = (1/2)h_iD_i$  and  $g^i = (1/2)h_0D_i$ ,  $i \in \mathcal{I}$ , this problem can be stated as

$$Z_i^*(\mathbf{X}, T_0) \triangleq Z_i(\mathbf{X}, T_0, T_i^*(\mathbf{X}, T_0)) = \inf_{T_i > 0} \left\{ \frac{S_i(\mathbf{X})}{T_i} + g_iT_i + \sum_{i \in \mathcal{I}} g^i \max\{T_0, T_i\} \right\}.$$

where  $T_i^*(\mathbf{X}, T_0)$  is the corresponding solution. In order to solve this particular problem, we define  $\tau_i'(\mathbf{X})$  and  $\tau_i(\mathbf{X})$ —called the breakpoints for reasons that will become obvious shortly—as follows:

$$\tau_i'(\mathbf{X}) = \sqrt{\frac{S_i(\mathbf{X})}{g_i + g^i}}, \quad \forall \mathbf{X} \in \Omega, \quad i \in \mathcal{I}, \quad \text{and} \quad (4.19)$$

$$\tau_i(\mathbf{X}) = \sqrt{\frac{S_i(\mathbf{X})}{g_i}}, \quad \forall \mathbf{X} \in \Omega, \quad i \in \mathcal{I}. \quad (4.20)$$

For any fixed DC location  $\mathbf{X} \in \Omega$  and for all  $i \in \mathcal{I}$ , it is easy to observe that  $\tau'_i(\mathbf{X}) \leq \tau_i(\mathbf{X})$ . Then, for  $i \in \mathcal{I}$ , one can show that (Roundy, 1985)

$$Z_i^*(\mathbf{X}, T_0) = \begin{cases} \sqrt{2[S_i(\mathbf{X})(g_i + g^i)]}, & \text{if } T_0 < \tau'_i(\mathbf{X}), \\ S_i(\mathbf{X})/T_0 + (g_i + g^i)T_0, & \text{if } \tau'_i(\mathbf{X}) \leq T_0 \leq \tau_i(\mathbf{X}), \\ \sqrt{2[S_i(\mathbf{X})g_i]} + g^i T_0, & \text{if } \tau_i(\mathbf{X}) < T_0, \end{cases} \quad (4.21)$$

and, the value of  $T_i$  that minimizes  $Z_i(\mathbf{X}, T_0, T_i)$ , for  $i \in \mathcal{I}$ , can be expressed as a function  $\mathbf{X}$  and  $T_0$  as follows:

$$T_i^*(\mathbf{X}, T_0) = \begin{cases} \tau'_i(\mathbf{X}), & \text{if } T_0 < \tau'_i(\mathbf{X}), \\ T_0, & \text{if } \tau'_i(\mathbf{X}) \leq T_0 \leq \tau_i(\mathbf{X}), \\ \tau_i(\mathbf{X}), & \text{if } \tau_i(\mathbf{X}) < T_0. \end{cases}$$

Based on this result, for any given  $T_0$  and  $\mathbf{X}$ , the retailers can be classified into the following three sets:

- $G(\mathbf{X}, T_0) \triangleq \{i \in \mathcal{I} : T_0 < \tau'_i(\mathbf{X})\}$ ,
- $E(\mathbf{X}, T_0) \triangleq \{i \in \mathcal{I} : \tau'_i(\mathbf{X}) \leq T_0 \leq \tau_i(\mathbf{X})\}$ , and
- $L(\mathbf{X}, T_0) \triangleq \{i \in \mathcal{I} : \tau_i(\mathbf{X}) < T_0\}$ .

Note that, unlike the three sets proposed by Roundy (1985), these sets do not only depend on  $T_0$  but they also depend on  $\mathbf{X}$ .

Defining

$$z(\mathbf{X}, T_0) = \frac{S_0(\mathbf{X})}{T_0} + \sum_{i \in \mathcal{I}} Z_i^*(\mathbf{X}, T_0),$$

we now consider the problem  $\min_{T_0 > 0} \{z(\mathbf{X}, T_0)\}$  for the fixed DC location  $\mathbf{X}$ . To this

end, we also define

$$\begin{aligned}
S(\mathbf{X}, T_0) &= S_0(\mathbf{X}) + \sum_{i \in E(\mathbf{X}, T_0)} S_i(\mathbf{X}), \\
H(\mathbf{X}, T_0) &= \sum_{i \in E(\mathbf{X}, T_0)} (g_i + g^i) + \sum_{i \in L(\mathbf{X}, T_0)} g_i, \quad \text{and} \\
M(\mathbf{X}, T_0) &= \sum_{i \in G(\mathbf{X}, T_0)} \sqrt{2S_i(\mathbf{X})(g_i + g^i)} + \sum_{i \in L(\mathbf{X}, T_0)} \sqrt{2S_i(\mathbf{X})g_i}.
\end{aligned}$$

Then, using (4.21), we have

$$z(\mathbf{X}, T_0) = \frac{S(\mathbf{X}, T_0)}{T_0} + H(\mathbf{X}, T_0)T_0 + M(\mathbf{X}, T_0).$$

When  $\mathbf{X}$  is fixed, it follows from the analysis by Roundy (1985) that  $S(\cdot, \cdot)$ ,  $H(\cdot, \cdot)$ , and  $M(\cdot, \cdot)$  are constant on those intervals where  $G(\mathbf{X}, T_0)$ ,  $E(\mathbf{X}, T_0)$  and  $L(\mathbf{X}, T_0)$  do not change, i.e., the values of  $S(\cdot, \cdot)$ ,  $H(\cdot, \cdot)$ , and  $M(\cdot, \cdot)$  change only when  $T_0$  crosses a breakpoint,  $\tau_i(\mathbf{X})$  or  $\tau'_i(\mathbf{X})$  for some  $i \in I$ , and  $z(\mathbf{X}, T_0)$  attains its minimum at the unique positive number, denoted by  $T_0^*(\mathbf{X})$ , satisfying  $\partial z(\mathbf{X}, T_0)/\partial T_0 = 0$ . Specifically, when  $\mathbf{X}$  is fixed,  $T_0 = T_0^*(\mathbf{X})$  if

$$T_0 = \sqrt{\frac{S(\mathbf{X}, T_0)}{H(\mathbf{X}, T_0)}}. \quad (4.22)$$

The above expression illustrates the relationship between the relaxed DC reorder interval and  $\mathbf{X}$ . Then, when  $\mathbf{X}$  is fixed, we can utilize the approach derived by Roundy (1985) to minimize  $z(\mathbf{X}, T_0)$  by beginning at the rightmost piece of the  $2n+1$  pieces resulting from the breakpoints, and decrease  $T_0$  from piece to piece to find the one containing  $T_0^*(\mathbf{X})$  which, in turn, gives the corresponding retailers sets and order intervals.

The analysis we have presented above adopts the approach presented by Roundy (1985) for computing the solution of *R-d-M* **assuming that  $\mathbf{X}$  is given**. It is

important to note that although the above discussion is useful from an expository perspective for demonstrating the impact of  $\mathbf{X}$  on the analysis by Roundy (1985), it does not immediately lead to an effective way of computing  $(\mathbf{X}^*, \mathbf{T}^*)$  because treating  $\mathbf{X}$  as a decision variable ruins the order-preserving property without which it is not straightforward to construct an efficient search procedure for computing  $(\mathbf{X}^*, \mathbf{T}^*)$  due to the explicit consideration of distance functions discussed in Section IV.5.2. However, if  $(\mathbf{X}^*, \mathbf{T}^*)$  could have been computed effectively then this would have provided a moderate lower bound on the value of  $Z(\mathbf{X}^P, \mathbf{T}^P)$  as we prove in Theorem 5.

**THEOREM 5**  $Z(\mathbf{X}^*, \mathbf{T}^*) \leq Z(\mathbf{X}^P, \mathbf{T}^P) \leq 1.06 Z(\mathbf{X}^*, \mathbf{T}^*)$ .

**Proof:** Suppose  $(\mathbf{X}^*, \mathbf{T}^*)$  is known. Fixing the DC location at  $\mathbf{X}^*$ , we can obtain a feasible solution for *PII-d-MR* by using the approach in Roundy (1985) to transform  $\mathbf{T}^*$  to its corresponding vector of power-of-two intervals. Letting  $\mathbf{T}^{\bar{P}}$  denote this vector, we then have  $Z(\mathbf{X}^*, \mathbf{T}^{\bar{P}})/Z(\mathbf{X}^*, \mathbf{T}^*) \leq 1.06$  due to Remark 1 and the main result in Roundy (1985). The feasible solution  $(\mathbf{X}^*, \mathbf{T}^{\bar{P}})$  may be improved further by finding the best location corresponding to  $\mathbf{T}^{\bar{P}}$  using the appropriate technique that depends on the distance function employed, as described in Section IV.5.2. Letting  $\mathbf{X}^{\bar{P}}$  denote this new location,  $Z(\mathbf{X}^{\bar{P}}, \mathbf{T}^{\bar{P}})$  is clearly an upper bound for  $Z(\mathbf{X}^P, \mathbf{T}^P)$ . Thus,  $Z(\mathbf{X}^{\bar{P}}, \mathbf{T}^{\bar{P}})/Z(\mathbf{X}^*, \mathbf{T}^*) \leq 1.06$ , and the result follows. ■

Unfortunately,  $(\mathbf{X}^*, \mathbf{T}^*)$  is difficult to compute for realistic problems with large number of retailers. Hence, our main goal is to develop algorithmic approaches that do not necessarily rely on  $(\mathbf{X}^*, \mathbf{T}^*)$ , but, yet, are capable of producing solutions with similar quality.

### IV.5.3.2. On Transforming $(\mathbf{X}^*, \mathbf{T}^*)$ to $(\mathbf{X}^P, \mathbf{T}^P)$

It is easy to observe that the proof of Theorem 5 suggests a way to generate *feasible* power-of-two reorder intervals for our problem under which the error bound of 6% holds *provided that* we start with  $(\mathbf{X}^*, \mathbf{T}^*)$ . The approach relies on the use of  $(\mathbf{X}^*, \mathbf{T}^*)$  along with the solution approaches for the Weber and SWMR problems. For the sake of discussion, let us momentarily assume that  $(\mathbf{X}^*, \mathbf{T}^*)$  is known and consider the following generic steps of this approach where we assume assume a base period of 1 without loss of generality:

- (1) Start with  $(\mathbf{X}^*, \mathbf{T}^*)$  with  $Z(\mathbf{X}^*, \mathbf{T}^*)$ .
- (2) Transform  $\mathbf{T}^*$  to  $\mathbf{T}^{\bar{P}}$  to obtain the corresponding power-of-two reorder intervals using  $T_i^{\bar{P}} = 2^{\lfloor \log_2(T_i^*) + 0.5 \rfloor}$ .  
(We have  $Z(\mathbf{X}^*, \mathbf{T}^*) \leq Z(\mathbf{X}^*, \mathbf{T}^{\bar{P}})$ .)
- (3) Find  $\mathbf{X}^{\bar{P}} = \arg \min_{\mathbf{X}} Z(\mathbf{X}, \mathbf{T}^{\bar{P}})$  using the appropriate technique that depends on the distance function employed, as described in Section IV.5.2.  
(We have  $Z(\mathbf{X}^*, \mathbf{T}^*) \leq Z(\mathbf{X}^{\bar{P}}, \mathbf{T}^{\bar{P}}) \leq Z(\mathbf{X}^*, \mathbf{T}^{\bar{P}})$ .)
- (4) Find  $\mathbf{T}^{\bar{P}*} = \arg \min_{\mathbf{T}} Z(\mathbf{X}^{\bar{P}}, \mathbf{T})$ .  
Transform  $\mathbf{T}^{\bar{P}*}$  to  $\mathbf{T}^{\bar{\bar{P}}}$  using  $T_i^{\bar{\bar{P}}} = 2^{\lfloor \log_2(T_i^{\bar{P}*}) + 0.5 \rfloor}$ .  
(We have  $Z(\mathbf{X}^*, \mathbf{T}^*) \leq Z(\mathbf{X}^{\bar{P}}, \mathbf{T}^{\bar{\bar{P}}}) \leq Z(\mathbf{X}^{\bar{P}}, \mathbf{T}^{\bar{P}*}) \leq Z(\mathbf{X}^*, \mathbf{T}^{\bar{P}})$ .)

At this point, we may observe that  $\mathbf{T}^{\bar{\bar{P}}} = \mathbf{T}^{\bar{P}}$ . Hence, a fifth step, that generates the optimum DC location for fixed  $\mathbf{T}^{\bar{P}}$  does not result in a new DC location as it leads to  $\mathbf{X}^{\bar{P}}$  which has already been obtained in Step 3. Thus, the transformation process may terminate with no guarantee on the optimality of the power-of-two reorder intervals implied by  $(\mathbf{X}^{\bar{P}}, \mathbf{T}^{\bar{P}})$  for *PII-d-MR* as illustrated below.

**EXAMPLE 1** Consider a two-retailer system where  $\mathbf{P}_0 = (0, 0)$ ;  $\mathbf{P}_1 = (0, 10)$ ;  $\mathbf{P}_2 = (10, 0)$ ;  $D_1 = D_2 = 100$ ;  $K_0 = 50$ ;  $K_1 = K_2 = 25$ ;  $h_0 = 1$ ;  $h_1 = h_2 = 3$ ;  $p_0^d = 1$ ;  $p_1^d = p_2 = 5$ ;  $r_0^d = 3$ ; and  $r_1^d = r_2^d = 5$ . Also, for illustrative purposes, suppose that the squared Euclidean distances are employed.

- (1) The optimal solution for R-d-MR is  $\mathbf{X}^* = (3.944, 3.944)$  and  $\mathbf{T}^* = (1.393, 1.393, 1.393)$  with  $Z(\mathbf{X}^*, \mathbf{T}^*) = 1078.84$ .
- (2) Transforming  $\mathbf{T}^*$  to  $\mathbf{T}^{\bar{P}}$ , we obtain  $\mathbf{T}^{\bar{P}} = (1, 1, 1)$  and  $Z(\mathbf{X}^*, \mathbf{T}^{\bar{P}}) = 1126.63 < 1.06 Z(\mathbf{X}^*, \mathbf{T}^*) = 1143.57$ .
- (3) Then  $\mathbf{X}^{\bar{P}} = \arg \min Z(\mathbf{X}, \mathbf{T}^{\bar{P}}) = (3.846, 3.846)$ , and  $Z(\mathbf{X}^{\bar{P}}, \mathbf{T}^{\bar{P}}) = 1126.38 < Z(\mathbf{X}^*, \mathbf{T}^{\bar{P}}) = 1126.63$ .
- (4) For  $\mathbf{X}^{\bar{P}}$  fixed, the solutions for R-d-MR and PII-d-MR are  $\mathbf{T}^{\bar{P}*} = (1.347, 1.347, 1.347)$  and  $\mathbf{T}^{\bar{P}} = (1, 1, 1)$ , respectively. Furthermore,  $\mathbf{T}^{\bar{P}} = \mathbf{T}^P$ , and, consequently,  $Z(\mathbf{X}^{\bar{P}}, \mathbf{T}^{\bar{P}}) = 1126.38 = Z(\mathbf{X}^{\bar{P}}, \mathbf{T}^P)$ .

However, the optimal solution is given by  $\mathbf{T}^P = (2, 1, 2)$  and  $\mathbf{X}^P = (2.778, 5.556)$  with a cost of  $Z(\mathbf{T}^P, \mathbf{X}^P) = 1123.28 < 1126.38 = Z(\mathbf{X}^{\bar{P}}, \mathbf{T}^{\bar{P}})$ .

As Example 1 shows, although the rounding scheme is still effective with a 6% error bound provided that we start with  $(\mathbf{X}^*, \mathbf{T}^*)$ , it does not guarantee the optimum power-of-two reorder intervals for PII-d-MR. Since our main goal is to develop algorithmic approaches that do not rely on  $(\mathbf{X}^*, \mathbf{T}^*)$ , this observation must be considered carefully for our purposes. Upon a closer look at the case illustrated above, we conclude that the premature termination is due to the fact that the implied best power-of-two reorder intervals for fixed DC locations  $\mathbf{X}^*$  and  $\mathbf{X}^{\bar{P}}$  are the same. It is this particular conclusion that we exploit later in Section IV.6.2.1 where we develop an alternative iterative approach, called the *Perturb* algorithm, for computing a solution of PII-d-MR that does not necessarily rely on  $(\mathbf{X}^*, \mathbf{T}^*)$ .

### IV.5.3.3. Bounding the Power-of-Two Reorder Intervals

Now, we prove a formal result that we utilize for bounding the power-of-two reorder intervals in solving *PII-d-MR*, and we note that we exploit this result later in Section IV.6.2.2 where we develop an alternative approach, called the *Circle* algorithm, for computing a solution of *PII-d-MR* that does not rely on  $(\mathbf{X}^*, \mathbf{T}^*)$ .

Using Lemma 9, it is possible to bound  $S_i(\mathbf{X})$ ,  $i \in \mathcal{I} \cup \{0\}$ , such that  $S_i^{\min} \leq S_i(\mathbf{X}) \leq S_i^{\max}$  where

$$S_i^{\min} = K_i + p_i \quad \text{and} \quad S_i^{\max} = K_i + p_i + r_i^d \max_{j \in \mathcal{I} \cup \{0\}} \{d(\mathbf{P}_i, \mathbf{F}_j)\}.$$

Let  $\mathbf{T}^{P \min} = (T_0^{P \min}, \dots, T_n^{P \min})$  denote the power-of-two reorder intervals representing the solution of the SWMR problem (obtained using the approach in Roundy (1985)) with setup costs  $\mathbf{S}^{\min} = (S_0^{\min}, \dots, S_n^{\min})$ . Likewise, let  $\mathbf{T}^{P \max} = (T_0^{P \max}, \dots, T_n^{P \max})$  denote the power-of-two reorder intervals representing the solution of the SWMR problem with setup costs  $\mathbf{S}^{\max} = (S_0^{\max}, \dots, S_n^{\max})$ . Also let  $\mathbf{T}^{R \min} = (T_0^{R \min}, \dots, T_n^{R \min})$  and  $\mathbf{T}^{R \max} = (T_0^{R \max}, \dots, T_n^{R \max})$  denote the relaxed reorder intervals leading to  $\mathbf{T}^{P \min} = (T_0^{P \min}, \dots, T_n^{P \min})$  and  $\mathbf{T}^{P \max} = (T_0^{P \max}, \dots, T_n^{P \max})$ , respectively. Recalling that we denote the optimal solution of *PII-d-MR* by  $(\mathbf{X}^P, \mathbf{T}^P)$  where  $\mathbf{T}^P = (T_0^P, \dots, T_n^P)$ , we have the following theorem.

**THEOREM 6**  $T_i^P \in [T_i^{P \min}, T_i^{P \max}]$ ,  $\forall i \in \mathcal{I} \cup \{0\}$ .

**Proof:** Let  $\mathbf{S}^P = (S_0^P, \dots, S_n^P)$  where  $S_i^P = K_i + p_i + r_i d(\mathbf{X}^P, \mathbf{P}_i)$  for  $i \in \mathcal{I} \cup \{0\}$ . Hence,  $\mathbf{S}^P$  represents the setup cost vector corresponding to  $\mathbf{X}^P$ . Also, let  $\mathbf{T}^R = (T_0^R, \dots, T_n^R)$  denote the optimal solution of *R-d-MR* for  $\mathbf{X} = \mathbf{X}^P$ .

Using the results in Roundy (1985) (partly summarized in Section IV.5.3.1), the retailers can be divided into three disjoint and collectively exhaustive subsets, namely  $G$ ,  $L$  and  $E$  such that we have the following:



- $i \in G$  for which  $T_i^R > T_0^R$  and  $T_i^R = \sqrt{\frac{2S_i^P}{(h_i+h_0)D_i}}$ ,
- $i \in L$  for which  $T_i^R < T_0^R$  and  $T_i^R = \sqrt{\frac{2S_i^P}{h_i D_i}}$ , and
- $i \in E$  for which  $T_i^R = T_0^R$  and  $T_i^R = \sqrt{\frac{2[S_0^P + \sum_{i \in E} S_i^P]}{\sum_{i \in E} (h_i+h_0)D_i + \sum_{i \in L} h_0 D_i}}$ .

Then, it is straightforward to show that  $T_i^R$  is an increasing function of  $S_i^P$ , for all  $i \in \mathcal{I} \cup \{0\}$ . Noting that, by definition,  $S_i^{\min} < S_i^P < S_i^{\max}$  for all  $i \in \mathcal{I} \cup \{0\}$ , we have

$$T_i^{R \min} < T_i^R < T_i^{R \max}, \quad \forall i \in \mathcal{I} \cup \{0\}. \quad (4.23)$$

Also, for all  $i \in \mathcal{I} \cup \{0\}$ , the corresponding power-of-two reorder interval  $T_i^P$  satisfies

$$\frac{1}{\sqrt{2}} T_i^R \leq T_i^P = T_b 2^{v_i} \leq \sqrt{2} T_i^R, \quad v_i \in \mathbb{Z}. \quad (4.24)$$

Using (4.24) and (4.23), we obtain

$$\frac{1}{\sqrt{2}} T_i^{R \min} < \frac{1}{\sqrt{2}} T_i^R \leq T_i^P \leq \sqrt{2} T_i^R < \sqrt{2} T_i^{R \max}, \quad \forall i \in \mathcal{I} \cup \{0\}. \quad (4.25)$$

Now, let us first compare  $\sqrt{2} T_i^{R \min}$  with  $T_i^P$ .

- If  $\sqrt{2} T_i^{R \min} < T_i^P$ , then  $T_i^{P \min} < T_i^P$  since, by (4.24), we have  $T_i^{R \min}/\sqrt{2} \leq T_i^{P \min} \leq \sqrt{2} T_i^{R \min}$ .
- If  $T_i^P \leq \sqrt{2} T_i^{R \min}$ , then  $T_i^{P \min} = T_i^P$  since, by (4.25), we have  $T_i^{R \min}/\sqrt{2} < T_i^P$  and there is only one power-of-two value between  $T_i^{R \min}/\sqrt{2}$  and  $\sqrt{2} T_i^{R \min}$ .

Similarly, let us compare  $T_i^{R \max}/\sqrt{2}$  with  $T_i^P$ .

- If  $T_i^P < T_i^{R \max}/\sqrt{2}$ , then  $T_i^P < T_i^{P \max}$  since, by (4.24), we have  $T_i^{R \max}/\sqrt{2} \leq T_i^{P \max} \leq \sqrt{2} T_i^{R \max}$ .
- If  $T_i^{R \max}/\sqrt{2} \leq T_i^P$ , then  $T_i^P = T_i^{P \max}$  since, by (4.25), we have  $T_i^P < (\sqrt{2}) T_i^{R \max}$  and there is only one power-of-two value between  $T_i^{R \max}/\sqrt{2}$  and  $\sqrt{2} T_i^{R \max}$ .

Therefore, we conclude that  $T_i^{P \min} \leq T_i^P \leq T_i^{P \max}$  for all  $i \in \mathcal{I} \cup \{0\}$ , and this completes the proof. ■

## IV.6. Algorithmic Approaches for *PII-d-MR*

In this section, we present algorithmic approaches for the solution of the integrated location-inventory problem in three-stage distribution systems with continuous facility location setting. We emphasize that these approaches do not assume any specific form of the distance function. First, in Section IV.6.1, we address the issue of computing a solution of *R-d-MR*. Second, in Section IV.6.2, we develop methods for computing a solution of *PII-d-MR* that do not necessarily rely on the optimal solution of *R-d-MR*. In the remainder of this section, considering Remark 1,

- the best relaxed or power-of-two reorder intervals corresponding to a fixed DC location (i.e., fixed vector of setup costs) are obtained using the approach in Roundy (1985), and
- the best DC location corresponding to a fixed vector of reorder intervals is obtained by using the appropriate technique that depends on the distance function, as described before in Section II.1.1,

without specific references to these methods.

### IV.6.1. Solving *R-d-MR*

Although the proposed algorithmic approaches for *PII-d-MR* do not rely on the optimal solution of *R-d-MR*, the solution of this problem is still important because it provides a realistic lower bound, and, hence, a useful benchmark, for our purposes. In order to solve *R-d-MR*, we rely on two approaches. Namely, the L-BFGS-B Algorithm and the *IterativeRelaxed* Algorithm.

#### IV.6.1.1. The L-BFGS-B Algorithm

One can utilize the L-BFGS-B algorithm (Byrd et al., 1995; Zhu et al., 1997) provided in the NEOS server<sup>2</sup> (Czyzyk et al., 1998; Dolan, 2001; Gropp and More, 1997) for solving  $R$ - $d$ - $MR$ . The L-BFGS-B algorithm is for large-scale optimization problems with simple bounds on the variables and does not require any structure in the objective function. It is a limited-memory quasi-Newton algorithm for bound-constrained or unconstrained optimization. In this algorithm, the step-length is determined at each iteration by a line-search routine that enforces a sufficient decrease condition and a curvature condition. When the squared Euclidean are employed, the L-BFGS-B algorithm guarantees the optimal relaxed solution  $(\mathbf{X}^*, \mathbf{T}^*)$  due to Theorem 4. However, if the  $\ell_p$  distances are employed, then the objective function of  $R$ - $d$ - $MR$  is no longer convex, and, hence, the L-BFGS-B algorithm does not guarantee  $(\mathbf{X}^*, \mathbf{T}^*)$ . The L-BFGS-B algorithm can still be employed to find the local optimum employing a multi-start framework; but, this is computationally expensive. We take advantage of the L-BFGS-B algorithm to obtain benchmarks for our computational tests in Section IV.7. However, we note that its computational times are excessive for problems with large number of retailers. Hence, the second approach discussed below, which can also be implemented in a multi-start framework, presents an effective alternative for computing a heuristic solution for  $R$ - $d$ - $MR$ .

#### IV.6.1.2. The *IterativeRelaxed* Algorithm

The *IterativeRelaxed* algorithm starts with a random DC location  $\mathbf{X}^r$  in the convex hull and computes the corresponding best relaxed reorder intervals (with the given  $\mathbf{X}^r$ ), denoted by  $\mathbf{T}^r$ , followed by computing the corresponding new best location (con-

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<sup>2</sup><http://www-neos.mcs.anl.gov>

sidering the computed  $\mathbf{T}^r$ ), iteratively for a preset number of iterations leading to a heuristic solution for  $R$ - $d$ - $MR$ . Based on our computational results, its average solution quality is comparable to the L-BFGS-B algorithm whereas its run-time is much shorter. Due to these advantages, the solution obtained from the *IterativeRelaxed* algorithm is also utilized as a warm-start for our proposed approaches for solving  $PII$ - $d$ - $MR$ .

#### IV.6.2. Solving $PII$ - $d$ - $MR$

We propose three specific approaches for computing the solution of  $PII$ - $d$ - $MR$  effectively.

##### IV.6.2.1. The *Perturb* Algorithm

Based on our discussion in Section IV.5.3.2, the best power-of-two reorder intervals for our problem are not guaranteed by solving the underlying SWMR and Weber problems iteratively, even if we could start with the optimal solution of  $R$ - $d$ - $MR$ ,  $(\mathbf{X}^*, \mathbf{T}^*)$ . This is because these iterations generally result in premature termination. To alleviate this situation, we utilize the observation provided before concluding Section IV.5.3.2. More specifically, for a given DC location  $\mathbf{X}$ , it is clear that the setup costs,  $S_i(\mathbf{X})$ , are unique and so are the corresponding best relaxed reorder intervals. Then, for a given DC location  $\mathbf{X}$ , the best relaxed reorder intervals can be computed and transformed to power-of-two reorder intervals,  $\mathbf{T}$ . We observe that, due to this rounding approach, there exist other locations that lead to the same power-of-two reorder intervals as  $\mathbf{T}$ . That is, for a fixed  $\mathbf{T}$ , there is a region of possible locations (a subset of point coordinates in the convex hull of retailers and the supplier) that imply the same  $\mathbf{T}$ . Once a  $\mathbf{T}$  is fixed, the corresponding location problem has well-defined “weights” and its solution gives the best location in the region for  $\mathbf{T}$ . Fixing this

best location as the new  $\mathbf{X}$  and calculating the new setup costs  $S_i(\mathbf{X})$ , we can obtain the new best relaxed reorder intervals, and then the new best power-of-two reorder intervals.

There are now two possibilities. If the new  $\mathbf{T}$  is different than the previous one, then we are in a new region of locations leading to the new  $\mathbf{T}$ . We find the new best location, with the new weights, which is different than the previous location. On the other hand, if the new  $\mathbf{T}$  is the same as the previous one, then we have not left the region of locations associated with the previous iteration (the corresponding best location will not change). In this case, there are again two possibilities. Either a local optimum is found or we have not been able to move out of the region of locations leading to the same  $\mathbf{T}$  (called premature termination earlier). The *Perturb* algorithm provides a remedy to this latter situation via a perturbation of the current location coordinates in a direction that improves the objective function value thereby either avoiding premature termination or verifying that a local optimum is reached (if there is no improving direction).

The *Perturb* algorithm is outlined in Display 3. The algorithm parameters (step size,  $\lambda$ , and sweeping angle increment,  $\delta$ ) facilitate the perturbation process. After  $\lambda$  and  $\delta$  are initialized, the iterations begin with choosing a random location  $\mathbf{X}^{prev}$  in the convex hull. Given  $\mathbf{X}^{prev}$ , we compute the corresponding best power-of-two reorder intervals  $\mathbf{T}^{prev}$ . Next, we compute the best location  $\mathbf{X}^{next}$  given  $\mathbf{T}^{prev}$ . We iterate one step further to obtain the new best power-of-two reorder intervals  $\mathbf{T}^{next}$  for given location  $\mathbf{X}^{next}$ . After obtaining  $\mathbf{T}^{prev}$  and  $\mathbf{T}^{next}$ , we compare these values. They may or may not be the same. If  $\mathbf{T}^{prev} \neq \mathbf{T}^{next}$ , we update the value of  $\mathbf{T}^{prev}$  with  $\mathbf{T}^{next}$  and continue with the iterative procedure at Step 4. If  $\mathbf{T}^{prev} = \mathbf{T}^{next}$ , there are two possibilities: a premature termination in the iterative procedure or the best power-of-two reorder intervals are found. Thus, we check whether the objective function can be

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**DISPLAY 3** The *Perturb* Algorithm.
 

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**0:** Initialize step size  $\lambda$  and angular increment  $\delta$ .  
**1:** Pick a location  $\mathbf{X}^{prev}$  in the convex hull randomly.  
**2:** Calculate the modified setup costs  $S_i(\mathbf{X}^{prev})$ ,  $\forall i \in \mathcal{I} \cup \{0\}$ .  
 Obtain the best power-of-two reorder intervals  $\mathbf{T}^{prev}$  corresponding to  $\mathbf{X}^{prev}$ .  
**3:** Find the best facility location  $\mathbf{X}^{next}$  corresponding to  $\mathbf{T}^{prev}$ .  
**4:** Calculate the modified setup costs  $S_i(\mathbf{X}^{next})$ ,  $\forall i \in \mathcal{I} \cup \{0\}$ .  
 Obtain the corresponding best power-of-two reorder intervals  $\mathbf{T}^{next}$ .  
**5:** Compare  $\mathbf{T}^{prev}$  and  $\mathbf{T}^{next}$  :  
   **if**  $\mathbf{T}^{prev} \neq \mathbf{T}^{next}$  **then**  
      $\mathbf{T}^{prev} = \mathbf{T}^{next}$ .  
     Go to Step 4.  
   **else**  
      $\theta = 0$ .  
     **while**  $\theta < 2\pi$  **do**  
        $\mathbf{X}^{perturb} = \mathbf{X}^{next} + \lambda(\cos \theta, \sin \theta)$ .  
        $\mathbf{T}^{perturb} = \arg \min_{\mathbf{T}} Z(\mathbf{X}^{perturb}, \mathbf{T})$ .  
       **if**  $Z(\mathbf{X}^{perturb}, \mathbf{T}^{perturb}) < Z(\mathbf{X}^{next}, \mathbf{T}^{next})$  **then**  
          $\mathbf{T}^{prev} = \mathbf{T}^{perturb}$ .  
         Go to Step 4.  
       **else**  
          $\theta = \theta + \delta$ .  
       **end if**  
     **end while**  
   **end if**  
 Terminate with  $\mathbf{X}^{next}$  and  $\mathbf{T}^{next}$  as the solution.

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improved further by perturbing the facility location  $\mathbf{X}^{next}$ . An improvement direction, if there is one, is found by sweeping on a circle around  $\mathbf{X}^{next}$ . More specifically, we obtain  $\mathbf{X}^{perturb}$  by modifying the existing  $\mathbf{X}^{next}$  in the direction  $\theta$  as much as the step size  $\lambda$ . Considering location  $\mathbf{X}^{perturb}$ , we compute for the corresponding best power-of-two policy  $\mathbf{T}^{perturb}$ . At this point, if the perturbed solution improves the existing solution, we update  $\mathbf{T}^{prev}$  with  $\mathbf{T}^{perturb}$  and go back to the Step 4 of the algorithm. Otherwise, using  $\delta$ , we modify the direction  $\theta$ , which is varied from  $0^\circ$

to  $360^\circ$ . At the end of the full circle sweep, if there does not exist an improving direction, we terminate with  $\mathbf{X}^{next}$  and  $\mathbf{T}^{next}$  as the solution. The *Perturb* algorithm produces high quality solutions, i.e., within 6% of the lower bound  $Z(\mathbf{X}^*, \mathbf{T}^*)$ , as evidenced by our numerical results in Section IV.7 and may find the optimum solution under the provision that its parameters are selected carefully. If the step size is too big, the algorithm may overshoot the optimum or fall short moving away a local optimum. However, a relatively small step size that moves the current location to another location in a different (improving) region of  $\mathbf{T}$  is a better choice. On the other hand, if the angular increment is too large, the approach may skip an improving direction, and if it is too small, the runtime may increase.

#### IV.6.2.2. The *Circle* Algorithm

Theorem 6 suggests that one can compute the upper and lower bounds,  $T_i^{P \min}$  and  $T_i^{P \max}$ , respectively, on the values of  $T_i^P$  using  $S_i^{\min}$  and  $S_i^{\max}$  for all  $i \in \mathcal{I} \cup \{0\}$ . The corresponding ranges,  $[T_i^{P \min}, T_i^{P \max}]$ , dictate the values of possible power-of-two reorder intervals for each  $i \in \mathcal{I} \cup \{0\}$ , and if they offer a reasonable number of possibilities then an enumeration algorithm can be utilized to find the best power-of-two reorder intervals  $T_i^P$ ,  $i \in \mathcal{I} \cup \{0\}$  along with the implied  $\mathbf{X}^P$ . Although it becomes immediately clear that this approach is computationally prohibitive for geographically realistic instances with large distances—where  $S_i^{\max}$  and, hence,  $T_i^{P \max}$  values are large—the concept of bounding the power-of-two reorder intervals is useful for developing the *Circle* algorithm. The *Circle* algorithm successively considers portions (discs), specified by circles, of the convex hull  $\Omega$  and determines the best power-of-two reorder intervals for that part of  $\Omega$ . The algorithm uses the *RestrictedEnumeration* procedure, discussed later, as a subroutine to determine the possible power-of-two reorder intervals implied by a circle of location coordinates.

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**DISPLAY 4** The *Circle* Algorithm.

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- 0:** Initialize the circle radius  $R$ ,  $IterNo=0$  and set  $MaxIter$ .  
**1:** Pick a location  $\mathbf{X}^{prev}$  in  $\Omega$  randomly (center of the circle).  
**2:** For all  $i \in \mathcal{I} \cup \{0\}$ , calculate

$$S_i^L = K_i + p_i + r_i^d \begin{cases} d(\mathbf{X}^{prev}, \mathbf{P}_i) - R, & \text{if } d(\mathbf{X}^{prev}, \mathbf{P}_i) \geq R. \\ 0, & \text{otherwise.} \end{cases}$$

$$S_i^U = K_i + p_i + r_i^d(d(\mathbf{X}^{prev}, \mathbf{P}_i) + R).$$

- 3:** Using the setup cost vectors  $\mathbf{S}^L = (S_0^L, \dots, S_n^L)$  and  $\mathbf{S}^U = (S_0^U, \dots, S_n^U)$  perform the *RestrictedEnumeration* procedure to obtain the solution  $(\mathbf{X}^{next}, \mathbf{T}^{next})$ .  
**4:** If the  $\mathbf{X}^{next}$  returned by the *RestrictedEnumeration* procedure is in the current circle, **then** save the solution in Elite set.  $IterNo++$ . If  $IterNo < MaxIter$ , go to Step 1, otherwise, go to Step 5.  
**Else**  $\mathbf{X}^{prev} = \mathbf{X}^{next}$ .  $IterNo++$ . If  $IterNo < MaxIter$ , go to Step 2, otherwise, go to Step 5.  
**5:** Return the solution with the lowest cost from the Elite set.
- 

We provide the details of the *Circle* algorithm in Display 4. After we initialize the radius of the circle  $R$ , we pick a location  $\mathbf{X}^{prev}$  randomly in  $\Omega$  and let  $\mathbf{X}^{prev}$  represent the center of the current circle. For each  $i \in \mathcal{I} \cup \{0\}$ , we determine the smallest and largest setup costs (denoted by  $S_i^L$  and  $S_i^U$ , respectively, and computed as explained in Display 5) considering the closest and the farthest points on the circle from the existing location  $\mathbf{P}_i$ . Note that if  $\mathbf{P}_i$  falls into the circle, its  $S_i^L$  is zero. Employing the *RestrictedEnumeration* procedure (in Display 5), which is initialized with  $\mathbf{S}^L$  and  $\mathbf{S}^U$ , we find the best vector of power-of-two reorder intervals ( $\mathbf{T}^{next}$ ) and a corresponding facility location  $\mathbf{X}^{next}$ . If  $\mathbf{X}^{next}$  falls into the circle, we place the solution  $(\mathbf{X}^{next}, \mathbf{T}^{next})$  in the Elite set and return to Step 1. Otherwise, we update  $\mathbf{X}^{prev}$  with  $\mathbf{X}^{next}$ , and return to Step 2. The iterations continue in this fashion until





retailers. The possible set combinations for this example are, starting from rightmost range in Figure 10,  $\{\bar{L}, \bar{L}, \bar{L}\}$ ,  $\{\bar{L}, \bar{E}, \bar{L}\}$ ,  $\{\bar{L}, \bar{G}, \bar{L}\}$ ,  $\{\bar{E}, \bar{G}, \bar{L}\}$ ,  $\{\bar{E}, \bar{E}, \bar{L}\}$ ,  $\{\bar{G}, \bar{G}, \bar{L}\}$ ,  $\{\bar{G}, \bar{G}, \bar{E}\}$ ,  $\{\bar{G}, \bar{G}, \bar{G}\}$ .

For each set combination, we determine the feasible vectors of power-of-two reorder intervals using  $\mathbf{T}^L$  and  $\mathbf{T}^U$ . For instance, consider the set combination  $\{\bar{E}, \bar{E}, \bar{L}\}$ . Then, retailer 1 and 2 are in set  $\bar{E}$ , and retailer 3 is in set  $\bar{L}$ . Hence,  $T_1 = T_2 = T_0$ , and  $T_3 < T_0$ . Considering these restrictions, along with  $T_i^L \leq T_i \leq T_i^U, i = 1, 2, 3$ , leads to two feasible vectors of power-of-two reorder intervals,  $(2, 2, 2, 0.5)$  and  $(2, 2, 2, 1)$ , and these vectors are added to the restricted power-of-two search space. Similarly, consider set combination  $\{\bar{L}, \bar{L}, \bar{L}\}$  which indicates that  $T_0$  is greater than  $T_1, T_2$ , and  $T_3$ . However, since  $T_i^L \leq T_i \leq T_i^U, i = 1, 2, 3$ , there is no such feasible solution. Hence, not every set combination contributes to the restricted power-of-two search space. After forming the restricted power-of-two search space in this manner, we compute the best facility location ( $\mathbf{X}'$ ) corresponding to each vector of power-of-two reorder intervals in this space, and subsequently determine the best vector of power-of-two reorder intervals ( $\mathbf{T}'$ ) for each of these possible locations. We finally return the solution with the lowest total cost.

The *Circle* algorithm also produces quality solutions as evidenced in Section IV.7. We note that if the circle radius  $R$  is large enough to cover the set  $\mathcal{I} \cup \{0\}$ , then the *RestrictedEnumeration* procedure takes into account all of the feasible power-of-two reorder intervals implied by  $\mathbf{T}^{P \min} = (T_0^{P \min}, \dots, T_n^{P \min})$  and  $\mathbf{T}^{P \max} = (T_0^{P \max}, \dots, T_n^{P \max})$  without performing a total enumeration, and, hence, finds the optimal solution in one iteration. However, this iteration may take an excessive amount of time. More specifically, if  $R$  is sufficiently large then all of the retailers are in set  $B$  (defined in Display 5). Then, each retailer has at least one range

in Figure 10 containing all three sets,  $\bar{G}$ ,  $\bar{E}$ , and  $\bar{L}$ , simultaneously. If the ranges containing all three sets coincide at least once for all retailers, then the number possible set combinations is  $3^n$  which, in turn, implies an exponential worst-case run time. That is, the larger the radius, the more time consuming the enumeration procedure is, especially for systems with a large number of retailers. On the other hand, if the radius  $R$  is zero, then the *Circle* algorithm becomes identical to the initial stages (prior to the first perturbation) of the *Perturb* algorithm.

#### IV.6.2.3. A Local Search Algorithm

A local search algorithm, starting with a given initial feasible solution, finds a local optima by iteratively generating feasible solutions with at least the same objective value of the current solution under a given neighborhood function. We define the neighborhood of an existing solution using the discrete neighborhood of the reorder intervals. More specifically, let  $(\mathbf{X}^k, \mathbf{T}^k)$  be an existing solution where  $\mathbf{T}^k = \{T_i : 2^{v_i}T_b, \forall i \in \mathcal{I} \cup \{0\}, v_i \in \mathbb{Z}\}$ . The neighborhood of  $\mathbf{T}^k$ ,  $\mathcal{N}(\mathbf{T}^k)$ , is obtained by modifying each of the reorder intervals in  $\mathbf{T}^k$ , one at a time, to its closest discrete neighbors, i.e.,  $T_j = 2^{m_j}T_b$  is modified as either  $2^{m_j-1}T_b$  or  $2^{m_j+1}T_b$ . Therefore, in a given neighborhood, there are  $2n+1$  vectors of power-of-two reorder intervals. Furthermore, this neighborhood is also constrained using Theorem 6, i.e., the reorder intervals that are outside the bounds in Theorem 6 are not considered. We compute the best DC location corresponding to each vector of power-of-two reorder intervals in  $\mathcal{N}(\mathbf{T}^k)$ . We pick the best overall solution in the existing neighborhood and compare it against the current best solution. If it is better than the current best solution, we accept it as the new best solution and the initial solution of the next iteration. The algorithm terminates when no improving solution exists in the neighborhood of the current best solution in which case a local minimum is found.

## IV.7. Numerical Results

The purpose of the numerical experiments is twofold. First, we present the computational results demonstrating that our algorithms produce good quality solutions when compared with the lower bound  $Z(\mathbf{X}^*, \mathbf{T}^*)$ , i.e., within 6% of the lower bound, in Section IV.7.1. Second, we discuss the potential benefit of *PII-d-MR* in practical decision-making situations by comparing the integrated model with benchmark models in Section IV.7.2.

### IV.7.1. Results Regarding the Performance of Algorithms

The computational results reported here were obtained considering the squared Euclidean distances, i.e., the only case where  $Z(\mathbf{X}^*, \mathbf{T}^*)$  can be computed exactly for benchmarking purposes. Under the  $\ell_p$  distances, although the objective function of *PII-d-MR* is no longer convex, our algorithms are still applicable. The results presented below offer promise that these algorithms may lead to acceptable solutions under the  $\ell_p$  distances as well—especially when they are employed in a multi-start framework with different initial solutions.

The algorithms were implemented using C++ and run on a Pentium IV 3.2Ghz machine with 1 GB memory considering numerous data sets. We report detailed results based on four data groups that consist of 5, 20, 35, and 50 retailers. In each group, we have 500 problem instances, generated randomly using the uniform distributions in Table 10, resulting in a total of 2,000 of problem instances. We note that computational results were also obtained for additional 20,480 problem instances with other demand and cost parameter ranges<sup>3</sup>, and our fundamental conclusions are

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<sup>3</sup>For the additional 20,480 instances, the demand and cost parameters were generated using the approach in Chen et al. (2001) whereas the existing locations were generated as indicated in Table 10. In these instances, the demand and cost parameters of the retailers were varied only slightly,

the same for all of the 22,480 problem instances examined. We do not report the solution times here since they are very small, less than 1 second for the majority of instances.

TABLE 10. Parameter Values for Testing the Performance of Proposed Algorithms

Parameters	Range
$D_i$	U[630,770]
$K_0$	U[720,880]
$K_i$	U[135,165]
$h_0$	U[1.8,2.2]
$h'_i$	U[6.3,7.7]
$p_0$	U[765,935]
$p_i$	U[180,120]
$r_0$	U[1.8,2.2]
$r_i^d$	U[1.35,1.65]
$\mathbf{P}_i$	U[0,100] $\times$ U[0,100]
$ \mathcal{I} $	5, 10, 35, 50

Our main findings regarding the performance of our algorithms, including the minimum, average, and maximum percentage gaps in the objective function value when compared to the lower bound  $Z(\mathbf{X}^*, \mathbf{T}^*)$ , are summarized in Table IV.7.1. For each problem instance, the gap is defined as the percentage difference

$$\text{Gap (\%)} = \frac{Z_{\text{algorithm}} - Z_{L\text{-}BFGS\text{-}B}}{Z_{L\text{-}BFGS\text{-}B}} \times 100,$$

where  $Z_{\text{algorithm}}$  is the cost of the solution suggested by the algorithm of interest and  $Z_{L\text{-}BFGS\text{-}B}$  is the optimal solution of  $R$ - $d$ - $MR$  obtained using the L-BFGS-B algorithm, i.e.,  $Z_{L\text{-}BFGS\text{-}B} = Z(\mathbf{X}^*, \mathbf{T}^*)$ .

Recall that the ***IterativeRelaxed*** algorithm provides a heuristic solution for  $R$ - $d$ - $MR$ . We employ this algorithm in a multi-start framework (with 50 re-starts) and take advantage of the resulting solution as a warm-start for our algorithms de-

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i.e., not as significantly as in the other 2000 instances.

TABLE 11. Summary of Percentage Gaps for Problem Instances Generated Using the Data in Table 10

$ \mathcal{I} $	<i>IterativeRelaxed</i>			<i>Perturb</i>			<i>Circle</i>			<i>Local Search</i>		
	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max
5	0.00 (97)	0.31	2.94	0.02	1.69	4.14	0.02	1.81	5.61	0.02	1.72	4.46
20	0.00 (27)	0.50	5.67	0.57	1.63	3.26	0.57	1.67	3.83	0.57	1.66	3.27
35	0.00 (10)	0.48	5.37	0.79	1.84	2.91	0.79	1.92	4.35	0.79	1.88	2.96
50	0.00 (7)	0.49	5.22	0.85	1.93	2.89	0.85	2.06	3.73	0.91	1.98	3.17

veloped for solving *PII-d-MR*. For this reason, first we discuss our findings regarding the performance of the *IterativeRelaxed* algorithm by providing a comparison of the solution quality of this algorithm with the solution of *R-d-MR* obtained using the L-BFGS-B algorithm on NEOS. The first portion of Table IV.7.1 summarizes these findings<sup>4</sup> where the average gaps are less than 0.50% and the maximum gaps do not exceed 5.67%. We note that the L-BFGS-B algorithm is quite slow compared to the *IterativeRelaxed* algorithm which provides good heuristic solutions for *R-d-MR* in a fast manner.

The second portion of Table IV.7.1 displays our findings regarding the performance of the ***Perturb* algorithm**. Observe that the average gaps are less than 1.93% and the maximum gaps are less than 4.14%. Although these findings are based on the case where the solution of the *IterativeRelaxed* algorithm was utilized as a warm-start, we also obtained similar average *and* maximum gaps for the additional 20,480 problem instances where the *Perturb* algorithm was initiated considering a random DC location, i.e., the average gaps were always less than 2% and the maximum gaps were always less than 6%. This observation provides strong evidence that the *Perturb* algorithm is capable of effectively producing high quality solutions regardless of the

<sup>4</sup>The information in parentheses is the number of instances for which the *IterativeRelaxed* algorithm finds  $(\mathbf{X}^*, \mathbf{T}^*)$ .

initial DC location. As we have noted earlier, the solution quality and run time of the *Perturb* algorithm depends on the step size  $\lambda$  and the sweeping angle  $\delta$ . Hence, we tested the impact of different  $\lambda$  and  $\delta$  values, and the best results in terms of the solution quality and run time were obtained with  $\lambda = 0.1$  and  $\delta = 5^\circ$  for all data groups.

The results on the performance of the **Circle Algorithm** in Table IV.7.1 are also based on the case where the solution of the *IterativeRelaxed* algorithm was used as a warm-start. These results suggest that the average percentage gaps are less than 2.06% and the maximum gaps are less than 5.61%. We note that, although the maximum gaps associated with the *Circle* Algorithm were sometimes larger than 6% for the additional 20,480 problem instances examined, they are always less than 6.831%. We also note that when a random initial location was used as the center of the circle, interestingly, the average gaps were slightly better. However, in terms of the maximum gaps, the warm-start improved on random initialization. In order to find the best radius  $R$  in terms of both the solution quality and run time, we also experimented with different radius values. A radius that is 10% of  $d_{max} = \max_{i,j \in \mathcal{I} \cup \{0\}} \{d(\mathbf{P}_i, \mathbf{P}_j)\}$  seems to perform better for all data groups.

Finally, the last portion of Table IV.7.1 displays our findings regarding the performance of the **Local Search algorithm** where the average gaps are less than 1.98% and the maximum gaps are less than 4.46%. Again, the solution of the *IterativeRelaxed* algorithm was utilized as a warm-start for obtaining these findings. Since the *IterativeRelaxed* algorithm operates on neighborhood function of power-of-two reorder intervals, we also initialized this algorithm with a random a feasible (power-of-two) vector and observed that this approach consistently performed worse. However, although maximum gaps with random initialization are high (potentially more than 6%), the average gaps are generally only slightly larger than those in Table

#### IV.7.1.

In summary, these findings indicate that the proposed algorithms are capable of effectively producing high quality solutions.

#### IV.7.2. Results Regarding the Practical Use of *PII-d-MR*

For the purpose of illustrating the potential benefit of (*PII-d-MR*) in practical decision-making situations, we discuss three benchmark models, and report additional numerical results where the distances are modeled using the squared Euclidean distance norm. We note that similar numerical results can be obtained for the case where the distances are modeled using the  $\ell_p$  distance norm as well.

##### IV.7.2.1. Measuring the Value of Integrated Decision-Making

The first two benchmark models, called BM1 and BM2, build on the idea that location and inventory decisions have been made *sequentially* in traditional applications where, typically, facility location decisions precede inventory decisions. We call this approach the sequential framework. We utilize BM1 and BM2 in order to quantify the cost advantage of *PII-d-MR* relative to the sequential framework with the purpose of measuring the practical value of *PII-d-MR*.

In BM1, we apply the typical sequential framework in our problem setting where the DC location decision is committed first. The DC location decision can be addressed by minimizing the annual transportation cost, via solving the problem  $\min_{\mathbf{X} \in \mathbb{R}^2} \{\sum_{i=0}^n c_i D_i d(\mathbf{X}, \mathbf{P}_i)\}$  where  $c_i$ ,  $i \in \mathcal{I} \cup \{0\}$ , denotes the per-unit-per-mile cost between the DC and existing location  $\mathbf{P}_i$ . During our discussion in Section IV.5.1, we have demonstrated that the per-unit-per-mile costs, in fact, depend on the inventory policy parameters. However, within the sequential framework, due to the lack of a better approach, we have to rely on crude estimates of  $c_i$ ,  $i \in \mathcal{I} \cup \{0\}$ , that may not ac-



curately consider the impact of inventory policy parameters on per-unit per-mile costs. One alternative is simply to focus on minimizing the total weighted distance to address the DC location decision, by solving the problem  $\min_{\mathbf{X} \in \mathbb{R}^2} \{\sum_{i=0}^n D_i d(\mathbf{X}, \mathbf{P}_i)\}$ , with the hope that the resulting solution will eventually minimize distribution related expenses. We use this alternative in BM1 for illustrative purposes but note that one can pick any set of crude estimates of  $c_i, i = 0, \dots, n$ . The best DC location is then given by (4.18), and this solution has a clear impact on the setup costs associated with inventory decisions (due to distance-based transportation costs which translate into DC-location-dependent replenishment costs) as explained in our discussion leading to (4.16). Next, inventory decisions are addressed by solving the corresponding SWMR problem using the approach in Roundy (1985). Obviously, BM1<sup>5</sup> may perform very poorly when compared to *PII-d-MR* as it relies on the idea of decoupling the location and inventory decisions which have been demonstrated to be coupled in Section IV.5.1.

BM2 is similar to BM1 except that in BM2 we attempt to develop realistic estimates of  $c_i, i \in \mathcal{I} \cup \{0\}$ , before committing the DC location decision. For this purpose, recalling the discussion leading to and following (4.17), one can imagine to

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<sup>5</sup>When the distances are modeled using the  $\ell_p$  distance norm, in BM1 we solve both  $\min_{\mathbf{X} \in \mathbb{R}^2} \{\sum_{i=0}^n D_i d(\mathbf{X}, \mathbf{P}_i)\}$  and  $\min_{\mathbf{X} \in \mathbb{R}^2} \{\sum_{i=1}^n D_i d(\mathbf{X}, \mathbf{P}_i)\}$  using (2.3) and (2.4) and obtain two alternative DC locations. The former problem leads to  $\mathbf{P}_0$  as the DC location due to the majority theorem. Next, we proceed with solving the corresponding inventory problems associated with these two DC locations, and adopt the approach that results in the lowest average annual total cost as BM1.

solve the *inventory problem* given by

$$\text{Min} \quad \sum_{i \in \mathcal{I} \cup \{0\}} \frac{p_i + K_i}{T_i} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_0 D_i \max\{T_0, T_i\} + \sum_{i \in \mathcal{I}} \frac{1}{2} h_i D_i T_i.$$

subject to

$$T_i = 2^{v_i} T_b \quad \text{and} \quad v_i \in \mathbb{Z}, \quad \text{for} \quad i = 0, \dots, n.$$

$$\mathbf{T} \in \mathbb{R}_+^{n+1}, \quad \mathbf{X} \in \mathbb{R}^2.$$

Then, the resulting  $T_i$ ,  $i \in \mathcal{I} \cup \{0\}$ , values can be utilized to obtain the implied per-unit-per-mile costs,  $r_i^d/Q_i$ ,  $i \in \mathcal{I} \cup \{0\}$ . In BM2, we consider the case where the facility location decision is addressed by solving the problem  $\min_{\mathbf{X} \in \mathbb{R}^2} \{\sum_{i=0}^n c_i D_i d(\mathbf{X}, \mathbf{P}_i)\}$  after replacing  $c_i$ ,  $i = 0, \dots, n$ , with these implied-per-unit-per-mile costs. As in BM1, once the DC location decision is committed, the inventory decisions in BM2 are addressed by solving the corresponding SWMR problem using the approach in Roundy (1985).

A numerical comparison of the cost of *PII-d-MR* with those of BM1 and BM2 can then be utilized to measure the value of the integrated framework we suggest relative to the sequential one described above. For this purpose, we analyze the  $2 \times 3^9 = 39,366$  problem instances obtained via a factorial design of the demand and cost parameters in Table 12 where the existing locations are

- Case 1:  $\mathbf{P}_0 = (0, 0)$ ,  $\mathbf{P}_1 = (10, 0)$ ,  $\mathbf{P}_2 = (10, 10)$  in 19,683 problem instances, and
- Case 2:  $\mathbf{P}_0 = (0, 0)$ ,  $\mathbf{P}_1 = (100, 0)$ ,  $\mathbf{P}_2 = (100, 100)$  in the remaining 19,683 problem instances.

We measure the value of the integrated framework by computing

$$\text{Percentage gain by } PII-d-MR \text{ (\%)} = \frac{Z_{(BM\cdot)} - Z_{(PII-d-MR)}}{Z_{(BM\cdot)}} \times 100,$$

for each problem instance where  $Z_{(BM.)}$  is the cost associated with the solution of the benchmark model of interest and  $Z_{(PII-d-MR)}$  is the cost of the solution of *PII-d-MR* obtained using the *Perturb* algorithm. Obviously, in all instances, the integrated framework is preferred over the sequential one.

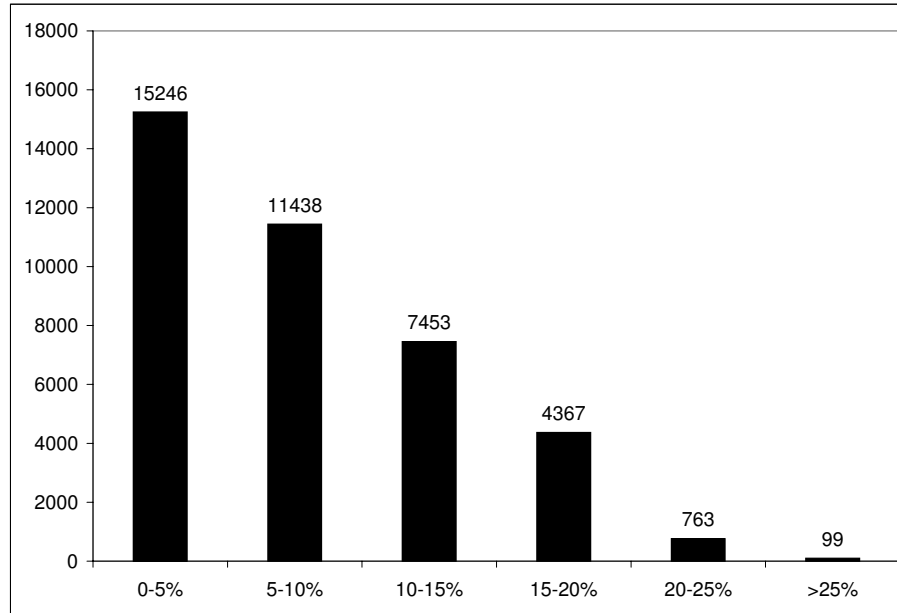
TABLE 12. Parameters of Benchmark Experiments,  $i \in \mathcal{I}$

Parameters	$D_i$	$K_0$	$K_i$	$h_0$	$h'_i$	$p_0$	$p_i$	$r_0$	$r'_i$
Low (L)	350	400	75	1	5.5	425	100	1	0.75
Medium (M)	700	800	150	2	7	850	200	2	1.5
High (H)	1400	1600	300	4	10	1700	400	4	3

In Figure 11, we present the distribution of the percentage gains obtained by *PII-d-MR* over BM1 and observe that the graph is skewed towards lower percentage gains with the median gain of 7.1%. Hence, employing BM1 (or BM2) rather than *PII-d-MR* for a quick-and-dirty analysis may seem to be reasonable at a first glance. However, in 24,120 problem instances the gain with joint optimization is at least 5% compared to the cost of BM1. In particular, the average and maximum percentage gains are 7.7%, and 29.6%, respectively. These percentages when converted to dollar amounts can result in substantial savings justifying the value of integrated framework. One important observation is that, although the maximum gain in Case 1 (29.6%) is higher than the maximum gain in Case 2 (26.2%), the average gain in Case 2 (9.2%) is higher than that of Case 1 (6.3%). This observation indicates that the integrated framework has more a pronounced impact, on the average, in Case 2 where the existing facilities are farther apart from each other.

We have similar observations associated with BM2 for which average and maximum gains are 4.0% and 29.4%, respectively. In Case 1, the average and maximum gains over BM2 (4.9% and 29.4%, respectively). In Case 2, the average and maximum gains over BM2 are 3.1% and 14.2%, respectively. In both cases, these gains are less

FIGURE 11. The Distribution of Percentage Gains by Joint Optimization Over Sequential Decision-Making Using BM1.



than those over BM1, and one can argue that BM2 performs better than BM1, in particular, in Case 2. This is because BM2 adopts a more sophisticated approach within the sequential framework by attempting to develop reasonable estimates of the per-unit-per-mile costs. Also, the impact of having reasonable estimates of per-unit-per-mile costs is more pronounced when the existing facilities are farther apart from each other, as in Case 2. Nevertheless, BM2 cannot be replaced by *PII-d-MR* without taking over a full scale analysis that compares BM2 and *PII-d-MR* because our results for BM2 provide clear evidence that the gains over BM2 are at least 5% for a large number of problem instances, i.e., for a total of 12,214 problem instances.

In summary, there is substantial value associated with the integrated framework we suggest; but, it may not be always feasible to implement it due to various practical constraints, such as lack of demand and cost data associated with the inventory

decisions. However, whenever there is opportunity, one should give serious consideration to adopting this framework. Next, we proceed with an illustrative practical application where there is, in fact, opportunity to implement *PII-d-MR*.

### IV.7.3. A Practical Application Setting

Let us recall the VMI setting described at the beginning of the chapter. and illustrate the use of *PII-d-MR* for a comparison of the two options, O1 and O2. Clearly, the cost of O2 can be evaluated by solving *PII-d-MR* whereas the cost of O1 can be evaluated by solving<sup>6</sup> the problem

$$\min_{T_i > 0, \in \mathcal{I} \cup \{0\}} \left\{ \sum_{i=1}^n \left[ \frac{p_0 + r_0 d(\mathbf{P}_0, \mathbf{P}_i) + K_i}{T_i} + \frac{1}{2} h'_i D_i T_i \right] \right\},$$

whose solution is simply given by

$$T_i = \sqrt{\frac{2(p_0 + r_0 d(\mathbf{P}_0, \mathbf{P}_i) + K_i)}{h'_i}}, \quad \forall i \in I.$$

A numerical comparison of the cost of O2, evaluated by using the solution of *PII-d-MR*, with the cost of O1, evaluated by using the solution of the above problem, can then be utilized to measure the amount the supplier is willing to pay to establish a DC. That is, the amount the supplier is willing to pay to establish a DC can be estimated by

$$\text{Savings due to O2 (\%)} = \max \left\{ 0, \frac{Z_{(O1)} - Z_{(O2)}}{Z_{(O1)}} \times 100 \right\},$$

where  $Z_{(O1)}$  and  $Z_{(O2)}$  are the average annual total costs of O1 and O2, respectively.

Clearly,  $Z_{(O2)}$  is not always superior to  $Z_{(O1)}$  as O2 is not always preferable over O1,

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<sup>6</sup>Since we consider the case where the supplier replenishes on a lot-for-lot basis, there is no cost advantage of coordinating replenishments *for the purpose of evaluating the cost of O1*. Hence, the cost of O1 can be evaluated without considering power-of-two policies.

i.e., retailer shipments directly from the supplier may make more practical sense than retailer shipments via a DC, especially when the retailers are not far away from the supplier.

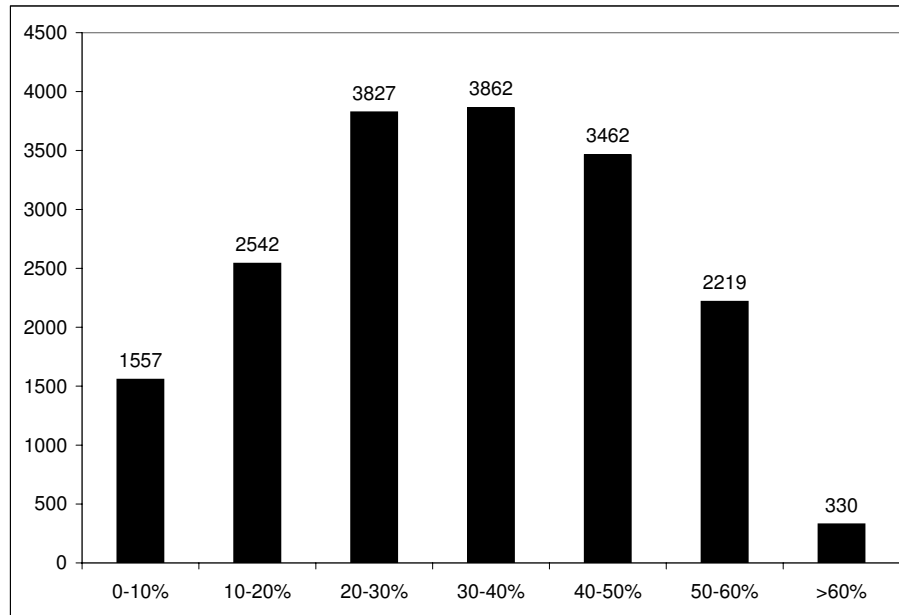
For the problem instances discussed in Section IV.7.2.1, we have numerical results illustrating the savings in Case 1 and Case 2. In particular, for Case 1, out of 19,683, in only 8735 problem instances O2 is preferable over O1, leading to average and maximum savings of 14.4% and 48.5%, respectively. However, for Case 2, out of 19,683, in 17,799 problem instances O2 is preferable over O1, leading to average and maximum savings of 32.3% and 63.6%, respectively. Recalling that Case 2 corresponds to a situation where the retailers are farther apart from the supplier, these findings are not surprising. In any case, there are substantial savings associated with O2 which cannot be evaluated and realized without using *PII-d-MR*. In Figure 12, we present the distribution of the percentage savings due to O2 for Case 2.

## IV.8. Summary and Conclusion

In this chapter, we consider a three-stage distribution network—consisting of a single supplier at a given location, a single intermediate DC whose location is to be determined, and multiple retailers at given locations—where the supplier is interested in cost saving opportunities associated with *coordinated replenishments* via the *to-be-established* DC. We propose an integrated location-inventory model that explicitly considers the quantity-based, quantity- and distance-based, and just distance-based transportation costs.

For problems *PII-Q* and *PII-Qd*, the location and inventory problems can be decomposed. Problem *PII-d* require joint optimization of location and inventory decisions since the objective functions of these problems are not separable with respect

FIGURE 12. The Distribution of Percentage Savings Due to Establishing a Warehouse



to the corresponding decision variables. For  $PII-d-SR$ , we present closed form solution for the location and inventory decision variables. Furthermore, we show that  $PII-d-MR$  is an extension of the SWMR problem to explicitly consider the DC location decision and DC-location-dependent replenishment costs. We also discuss that  $PII-d-MR$  is a practical generalization of the Weber problem to explicitly consider the inventory decisions and costs. We examine important characteristics of  $PII-d-MR$  that relate to the SWMR and Weber problems, and we build on these characteristics for developing solution algorithms that do not assume any specific form of the distance function.

Considering the squared Euclidean distances, we report computational results demonstrating the efficient and effective performance our algorithms using warm-start. We reiterate that we do not have any significant differences in the final solu-

tion quality when our algorithms are implemented using random initialization. We also emphasize that, under the  $\ell_p$  distances, although the convexity of the objective function vanishes, our algorithms can be implemented in a multi-start framework and potentially lead to acceptable solutions. Finally, we provide numerical results investigating the practical value of *PII-d-MR*. We conclude that substantial cost savings are realizable by the integration of location and inventory decisions, and, one should give serious consideration to this approach for logistical coordination.

As we have noted earlier, our focus in this chapter is on coordinated replenishments considering the case of direct deliveries between the successive stages of the underlying network. Incorporation of vehicle routing and truck capacity considerations in *PII-d-MR* remains a noteworthy generalization. Furthermore, we propose the generalizations of *PII-d-MR* to consider multiple DCs in dynamic and stochastic demand settings. These generalizations lead to challenging research problems aimed at extending both the existing location theory and inventory theory. We emphasize that the multi-DC generalization can be formulated as a set-partitioning problem. Then, our algorithms provide convenient methods of evaluating the cost performance of each partition and, hence, they are potentially beneficial in developing an effective solution methodology for the multi-DC problem.



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**DISPLAY 5** The *RestrictedEnumeration* Procedure.
 

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**0:** Input setup cost vectors  $\mathbf{S}^L = (S_0^L, \dots, S_n^L)$  and  $\mathbf{S}^U = (S_0^U, \dots, S_n^U)$ .

**1:** Calculate the best vectors of power-of-two reorder intervals,  $\mathbf{T}^L$  and  $\mathbf{T}^U$ , corresponding to  $\mathbf{S}^L$  and  $\mathbf{S}^U$ , respectively. This process requires the breakpoints  $\tau_i^L$ ,  $\tau_i^L$ ,  $\tau_i^U$ , and  $\tau_i^U$  for  $i \in \mathcal{I}$  (where  $\tau_i^L$  and  $\tau_i^U$  can be computed using (4.19) and  $\tau_i^L$  and  $\tau_i^U$  can be computed using (4.20)).

**2:** Sort the breakpoints in non-decreasing order.

**3:** Let  $A = \{i \in I : \tau_i^L < \tau_i^U \leq \tau_i^L < \tau_i^U\}$  and  $B = \{i \in I : \tau_i^L < \tau_i^L < \tau_i^U < \tau_i^U\}$ . Each retailer belongs to either in  $A$  or  $B$ . Also, let  $\bar{G} = \{i \in \mathcal{I} : T_0 < \tau_i^U\}$ ,  $\bar{E} = \{i \in \mathcal{I} : \tau_i^L \leq T_0 \leq \tau_i^U\}$ , and  $\bar{L} = \{i \in \mathcal{I} : \tau_i^U < T_0\}$ . Over each of the  $4n + 1$  ranges of  $T_0$  resulting from the breakpoints, determine into which one(s) of the sets  $\bar{G}$ ,  $\bar{E}$  and  $\bar{L}$  each retailer falls as follows:

- Consider  $i \in A$ . Depending on the value of  $T_0$ ,  $i \in A$  may belong to the following sets: For  $T_0 > \tau_i^U$ ,  $i \in \bar{L}$ ; for  $\tau_i^U \geq T_0 > \tau_i^L$ ,  $i \in \bar{E}$  or  $\bar{L}$ ; for  $\tau_i^L \geq T_0 > \tau_i^U$ ,  $i \in \bar{E}$ ;  
for  $\tau_i^U \geq T_0 > \tau_i^L$ ,  $i \in \bar{E}$  or  $\bar{G}$ ; for  $\tau_i^L \geq T_0$ ,  $i \in \bar{G}$ .
- Consider  $i \in B$ . Depending on the value of  $T_0$ ,  $i \in B$  may belong to the following sets: For  $T_0 > \tau_i^U$ ,  $i \in \bar{L}$ ; for  $\tau_i^U \geq T_0 > \tau_i^L$ ,  $i \in \bar{E}$  or  $\bar{L}$ ; for  $\tau_i^U \geq T_0 > \tau_i^L$ ,  $i \in \bar{G}$ ,  $\bar{E}$ , or  $\bar{L}$ ;  
for  $\tau_i^L \geq T_0 > \tau_i^L$ ,  $i \in \bar{E}$  or  $\bar{G}$ ; for  $\tau_i^L \geq T_0$ ,  $i \in \bar{G}$ .

**4:** Over each of the  $4n + 1$  ranges of  $T_0$  resulting from the breakpoints, determine the set combinations associated each retailer.

**5:** For each set combination and the corresponding possible  $T_0$  values (implied by that set combination), determine the feasible vectors of power-of-two reorder intervals using  $\mathbf{T}^L$  and  $\mathbf{T}^U$ , and add them to the restricted power-of-two search space.

**6:** For each vector in the restricted power-of-two search space find the corresponding best DC location  $\mathbf{X}'$ .

**7:** For each  $\mathbf{X}'$  thus obtained, determine the best  $\mathbf{T}'$ .

**8:** Return the  $(\mathbf{X}', \mathbf{T}')$  with the lowest cost.

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## CHAPTER V

FIXED CHARGE FACILITY LOCATION PROBLEMS WITH INVENTORY  
CONSIDERATIONS

Starting with this chapter, we examine the impact of inventory decisions in the context of discrete facility location models. In discrete facility location models, instead of calculating the coordinates of the new facility location, we evaluate a list of candidate facilities to determine which, and how many, facilities to select in two-stage and three-stage settings. In this chapter, we analyze fixed charge facility location problems arising in the context of two-stage distribution systems. In the next chapter, we analyze production distribution system design problems arising in the context of three-stage distribution systems.

The classical fixed charge facility location problem (FCFLP) (see Section II.1.2) consists of two stages. In the first stage, there are geographically dispersed established retailers. In the second stage, there is a set of potential facilities- henceforth called distribution centers (DCs)- with fixed operational costs. The objective is to choose a subset to open from the set of potential DCs, and then to assign the retailers to the open DCs while minimizing the sum of the fixed location and the unit-based transportation costs for satisfying the demand.

In this chapter, we develop models to analyze the impact on the classical FCFLP of inventory holding at the retailers. In particular, we consider establishing a number of DCs from a candidate set to serve geographically dispersed retailers with deterministic stationary demand. In this setting, each retailer operates under the assumption of the EOQ model. That is, retailers hold inventory to meet the deterministic stationary demand. Hence, we explicitly account for the inventory holding and replenishment costs at the retailers. We also account for the transportation costs associated

with direct shipments between the retailers and their respective DCs. We model the transportation costs as a function of distance as in (3.5). Similar to the models in Chapters III and IV, the facility location and inventory problems are decomposable, and, hence, the corresponding FCFLP is trivial for transportation cost functions (3.3) and (3.4). As a result, we focus on the distance-based transportation costs given by (3.5). This type of transportation cost function is a generalization of the per mile per unit transportation costs that are utilized in the classical FCFLP.

We assume that each DC operates as a cross-dock facility with a facility-specific fixed operational cost. That is, the DCs are not inventory keeping points, and, hence, they are replenished on an as-needed basis by an external supplier incurring a sunk cost. Therefore, there is no need to explicitly model the external suppliers. We relax this assumption while considering production/distribution system network design problems in Chapter VI. We note that due to the existence of facility-specific fixed operational costs, there is a significant trade-off between the tactical and strategic decisions.

In this setting, the goal is to minimize the total cost in the system, including not only the sum of the transportation costs between the DCs and the retailers and the fixed operational costs of selected DCs, but, also, the inventory replenishment and holding costs at the retailers by determining

- the number and locations of DCs,
- the assignment of each selected DC to a retailer, and
- the inventory decisions of each retailer.

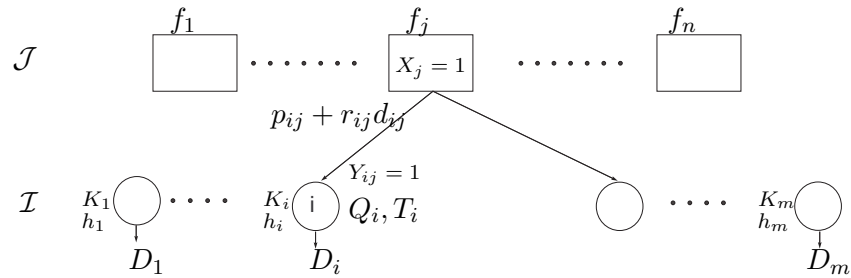
We emphasize, once again, that our focus is on improving the distribution system design by considering the total operational cost of logistics, rather than just the

facility location or inventory costs. For this purpose, we first formulate and solve the integrated problem, considering the location, assignment, and inventory decisions simultaneously. Next, we quantify the impact of integrated decision-making by comparing the results with the solution obtained from a benchmark. As before, the benchmark uses a sequential decision-making framework. We estimate the savings from integrated decision-making and identify those parameters that influence the amount of savings.

We observe that the capacity limitations of the DCs impact the difficulty and applicability of the FCFLP of interest. In this chapter, therefore, we analyze both uncapacitated and capacitated problems with different types of practical problem considerations.

We note that one possible application of the models in this chapter can be found in the vendor selection literature. There are two basic decisions that need to be made in the vendor selection process. A firm must decide which vendors it should work with, and it must determine the appropriate annual order quantity for each selected vendor. We refer to these two decisions together as the vendor selection problem. Weber et al. (1991) review seventy-four articles related to the vendor selection problem that have appeared since 1966. Despite the economic importance and inherent complexity of the vendor selection process, surprisingly little research has been devoted to developing quantitative models to address the problem. Several researchers use linear programming (Anthony and Buffa, 1989; Pan, 1989) and goal programming (Buffa and Jackson, 1983), while some others propose mixed integer programming (Chaudhry et al., 1993; Gaballa, 1974; Rosenthal et al., 1995) and mixed integer nonlinear programming approaches (Ghodsypour and O'Brien, 2001). Current and Weber (1994) demonstrate that vendor selection problems may be formulated within the mathematical constructs of discrete facility location models. As a consequence,

FIGURE 13. Fixed Charge Facility Location with Inventory Considerations



the extensive literature devoted to formulating and solving discrete facility location problems may be utilized for specific vendor selection problems. In particular, the FCFLP with inventory considerations, captures important practical characteristics of the vendor selection problem and provides a foundation for further analytical work generalizing the existing literature in the area. More specifically, the solution of the FCFLP with inventory considerations reveals, not only the selected vendors (DCs) and the annual order quantities for each selected vendor, but also the replenishment plan for each retailer.

The remainder of the chapter is organized as follows. In the next section, we introduce the notation and the general model. In Sections V.2 and V.3, we consider the corresponding uncapacitated and capacitated problems, respectively. For both cases, we discuss the structural properties of the formulation and provide efficient solution approaches. The numerical results clearly indicate the importance of integrated decision-making and help us draw valuable insights regarding the factors influencing the costs. Finally, in Section V.4, we summarize our findings and conclude this chapter by discussing the potential impact of this work.

### V.1. General Model and Notation

Recalling that we are considering a two-stage distribution system as in Figure 13, we introduce the following notation:

- $\mathcal{I}$  set of retailers,  $\mathcal{I} = \{1, \dots, m\}$ .
- $\mathcal{J}$  set of potential DCs,  $\mathcal{J} = \{1, \dots, n\}$ .
- $D_i$  annual demand rate faced by retailer  $i$ ,  $\forall i \in \mathcal{I}$ .
- $h'_i$  inventory holding cost rate for each unit of inventory at retailer  $i$ ,  $\forall i \in \mathcal{I}$ .
- $K_i$  fixed ordering cost of retailer  $i$ ,  $\forall i \in \mathcal{I}$ .
- $d_{ij}$  distance between retailer  $i$ ,  $\forall i \in \mathcal{I}$ , and DC  $j$ ,  $\forall j \in \mathcal{J}$ .
- $p_{ij}$  fixed dispatch cost per replenishment to retailer  $i$ ,  $\forall i \in \mathcal{I}$ ,  
from DC  $j$ ,  $\forall j \in \mathcal{J}$ .
- $r_{ij}$  variable mileage cost per replenishment to retailer  $i$ ,  $\forall i \in \mathcal{I}$ ,  
from DC  $j$ ,  $\forall j \in \mathcal{J}$ .
- $f_j$  fixed cost of selecting DC  $j$ ,  $\forall j \in \mathcal{J}$ .

We have three sets of decision variables. The first set of decision variables is associated with selecting the DCs. For each DC  $j$ ,  $j \in \mathcal{J}$ ,

$$X_j = \begin{cases} 1, & \text{if DC } j \text{ is selected,} \\ 0, & \text{otherwise.} \end{cases}$$

The second set of decision variables pertains to the assignment of retailers to DCs. For retailer  $i$ ,  $i \in \mathcal{I}$ , and DC  $j$ ,  $j \in \mathcal{J}$ ,

$$Y_{ij} = \begin{cases} 1, & \text{if retailer } i \text{ is assigned to DC } j, \\ 0, & \text{otherwise.} \end{cases}$$

By letting assignment variables have only binary values, we ensure that each retailer

will receive shipments from only one DC. This property is also known as *single-sourcing* in the location literature. For the uncapacitated FCFLP, even if the assignment variables are allowed to be fractional, due to the lack of capacity restrictions, the assignment variables naturally assume integer values. For the capacitated FCFLP, single-sourcing is a restrictive assumption resulting in a more difficult optimization problem.

Finally, the third set of decision variables relates to inventory decisions of the retailers. For each retailer  $i$ ,  $i \in \mathcal{I}$ , we define

$Q_i$  order quantity of retailer  $i$ , and

$T_i$  reorder interval of retailer  $i$ .

We also define  $\mathbf{Q}$  as the vector of the order quantities and  $\mathbf{T}$  as the vector of the reorder intervals. Clearly,  $T_i = Q_i/D_i$ ,  $\forall i \in \mathcal{I}$ . Hence, for the rest of the analysis, we use  $\mathbf{Q}$  to represent inventory decisions, keeping in mind that the corresponding reorder intervals can be easily obtained from the order quantities.

Then, the FCFLP with inventory considerations can be formulated as the following mixed integer nonlinear program denoted by *PIII*

$$\text{Min} \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left\{ \frac{(p_{ij} + r_{ij} d_{ij}) D_i Y_{ij}}{Q_i} \right\} + \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2} \right\} \quad (PIII)$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad \forall i \in \mathcal{I}. \quad (5.1)$$

$$Y_{ij} \leq X_j, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.2)$$

$$X_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (5.3)$$

$$Y_{ij} \in \{0, 1\}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.4)$$

$$Q_i \geq 0, \quad \forall i \in \mathcal{I}. \quad (5.5)$$

The objective function of *PIII* minimizes the total costs: the fixed operating cost of opening the DCs, the transportation cost from DCs to retailers, and the inventory replenishment and holding cost at the retailers. Constraints (5.1) ensure that the demand of each retailer is satisfied. Constraints (5.2) establish that each retailer will be assigned to an open (selected) DC. Constraints (5.3) and (5.4) ensure integrality, whereas (5.5) ensure nonnegativity.

## V.2. Uncapacitated Case

In this section, we present the structural properties of *PIII*. Next, we discuss solution approaches for determining location, assignment, and inventory decisions simultaneously and sequentially. Finally, we provide numerical results that show the effectiveness of the solution approaches. Through numerical tests, we also offer evidence about the factors that influence the effectiveness and impact of simultaneous decision-making.

### V.2.1. Structural Properties

The objective function of *PIII* can be reorganized as follows:



$$\sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2} + \frac{D_i}{Q_i} \sum_{j \in \mathcal{J}} (p_{ij} + r_{ij} d_{ij}) Y_{ij} \right\}. \quad (5.6)$$

For any  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ , let  $Y_{ij}$  be known. Then, it is easy to observe that the second part of (5.6) is convex in  $\mathbf{Q}$  due to the similarity of its corresponding terms for each  $i \in \mathcal{I}$  to the EOQ model in Section II.2.1. Hence, the order quantity of each retailer  $i \in \mathcal{I}$  can be calculated as

$$Q_i = \sqrt{\frac{2[K_i + \sum_{j \in \mathcal{J}} (p_{ij} + r_{ij} d_{ij}) Y_{ij}] D_i}{h'_i}}. \quad (5.7)$$

Substituting (5.7) in (5.6), we obtain:

$$\begin{aligned} & \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sqrt{2[K_i + \sum_{j \in \mathcal{J}} (p_{ij} + r_{ij} d_{ij}) Y_{ij}] D_i h'_i} \\ &= \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sqrt{A_i + \sum_{j \in \mathcal{J}} B_{ij} Y_{ij}}, \end{aligned} \quad (5.8)$$

where  $A_i = 2K_i D_i h'_i$  and  $B_{ij} = 2(p_{ij} + r_{ij} d_{ij}) D_i h'_i$  for  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ .

With these simplifications, *PIII* can be stated as determining the open DCs and DC-retailer assignments so that (5.8) is minimized while satisfying the constraints (5.1), (5.2), (5.3), and (5.4).

**PROPOSITION 1** *In the optimal solution of PIII, for retailer  $i$ ,  $i \in \mathcal{I}$ ,  $Y_{i,j^*} = 1$  where  $j^* = \arg \min_{j \in \mathcal{J}'} \{p_{ij} + r_{ij} d_{ij}\}$  and  $\mathcal{J}' = \{j \in \mathcal{J} : X_j = 1\}$ .*

**Proof:**

For some retailer  $i \in \mathcal{I}$ , let  $j^*$  be the open DC with the lowest transportation cost,  $p_{ij^*} + r_{ij^*} d_{ij^*}$ , among all the open DCs. Then, the cost of this assignment is given as

$$C^* = \sqrt{2(K_i + p_{ij^*} + r_{ij^*} d_{ij^*}) D_i h'_i}. \quad (5.9)$$

Now, assume that in the optimal assignment, retailer  $i$  is assigned to open DC  $j^o$ .

The cost of this assignment is

$$C^o = \sqrt{2(K_i + p_{ij^o} + r_{ij^o}d_{ij^o})D_i h'_i}. \quad (5.10)$$

Furthermore, since  $j^o$  is the optimally selected facility,

$$\sqrt{2(K_i + p_{ij^o} + r_{ij^o}d_{ij^o})D_i h'_i} \leq \sqrt{2(K_i + p_{ij^*} + r_{ij^*}d_{ij^*})D_i h'_i}. \quad (5.11)$$

However, for  $p_{ij^*} + r_{ij^*}d_{ij^*} \leq p_{ij^o} + r_{ij^o}d_{ij^o}$ ,

$$\sqrt{2(K_i + p_{ij^*} + r_{ij^*}d_{ij^*})D_i h'_i} \leq \sqrt{2(K_i + p_{ij^o} + r_{ij^o}d_{ij^o})D_i h'_i}. \quad (5.12)$$

Using (5.11) and (5.12), we conclude that  $j^o = j^*$ . ■

The following theorem states an important structural result and simplifies the solution approach.

**THEOREM 7** *The objective function (5.8) of PIII is equivalent to the following equation:*

$$\sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} Y_{ij}, \quad (5.13)$$

where  $c_{ij} = \sqrt{2(K_i + p_{ij} + r_{ij}d_{ij})D_i h'_i}$ , for  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ .

**Proof:** For any retailer  $i^o \in \mathcal{I}$  and any potential DC  $j^o \in \mathcal{J}$ , if  $Y_{i^o j^o} = 1$ , the total assignment cost denoted by  $TAC_{i^o j^o}$  is given as the sum of the average annual transportation, inventory holding, and replenishment costs:

$$TAC_{i^o j^o} = \frac{K_{i^o} D_{i^o}}{Q_{i^o}} + \frac{h'_{i^o} Q_{i^o}}{2} + \frac{D_{i^o} (p_{i^o j^o} + r_{i^o j^o} d_{i^o j^o})}{Q_{i^o}},$$

since  $Y_{i^o j} = 0$  for  $j \in \mathcal{J} \setminus \{j^o\}$ . The order quantity for this retailer that optimizes  $TAC_{i^o j^o}$  is

$$Q_{i^o}^* = \sqrt{\frac{2(K_{i^o} + p_{i^o j^o} + r_{i^o j^o} d_{i^o j^o})D_{i^o}}{h'_{i^o}}}.$$

Hence, the optimal  $TAC_{i^o j^o}$ , denoted by  $c_{i^o j^o}$ , for retailer  $i^o$  and DC  $j^o$  is equal to

$$c_{i^o j^o} = \sqrt{2(K_{i^o} + p_{i^o j^o} + r_{i^o j^o} d_{i^o j^o}) D_{i^o} h'_{i^o}}.$$

Calculating  $c_{ij}$  for all retailers  $i \in \mathcal{I}$  and all DCs  $j \in \mathcal{J}$  in the same manner completes the proof. ■

**COROLLARY 2** *PIII reduces to the **classical** uncapacitated FCFLP.*

### V.2.2. Solution Methodology

Due to Corollary 2, *PIII* can be solved using the existing techniques developed for the classical uncapacitated FCFLP (see Section II.1.2). We proceed with a discussion of the details of a Lagrangian relaxation-based heuristic that we implement for the following formulation of our problem:

$$\text{Min} \quad \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sqrt{A_i + \sum_{j \in \mathcal{J}} B_{ij} Y_{ij}} \quad (5.8)$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad \forall i \in \mathcal{I}. \quad (5.1)$$

$$Y_{ij} \leq X_j, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.2)$$

$$X_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (5.3)$$

$$Y_{ij} \in \{0, 1\}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.4)$$

This heuristic is an iterative procedure that utilizes a Lagrangian relaxation of the original problem obtained by excluding the complicating constraints and incorporating them into the objective function via Lagrange multipliers that represent the penalty coefficients for violating the relaxed constraint.

The approach has been efficiently applied to classical uncapacitated integer pro-

gramming problems (see Geoffrion (1974); Shapiro (1979); Fisher (1981, 1985)). For this formulation, the following Lagrangian decompositions can be stated:

- (1) **Relaxation with respect to constraints (5.1):** For this relaxation, we utilize Lagrangian multipliers  $\mathbf{u} = \{i \in \mathcal{I} : u_i\}$  associated with the retailers. The Lagrangian problem is then

$$\begin{aligned} \text{(LR1)} \quad \max_{\mathbf{u}} L(\mathbf{u}) = \min_{\mathbf{X}, \mathbf{Y}} \quad & \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sqrt{A_i + \sum_{j \in \mathcal{J}} B_{ij} Y_{ij}} - \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} u_i Y_{ij} \\ & + \sum_{i \in \mathcal{I}} u_i \end{aligned} \quad (5.14)$$

subject to (5.2), (5.3), and (5.4).

- (2) **Relaxation with respect to constraints (5.2):** For this relaxation, we utilize Lagrangian multipliers  $\mathbf{v} = \{i \in \mathcal{I}, j \in \mathcal{J} : v_{ij}\}$  associated with the assignment of DCs to retailers. The Lagrangian problem is then

$$\begin{aligned} \text{(LR2)} \quad \max_{\mathbf{v}} L(\mathbf{v}) = \min_{\mathbf{X}, \mathbf{Y}} \quad & \sum_{j \in \mathcal{J}} (f_j - \sum_{i \in \mathcal{I}} v_{ij}) X_j + \sum_{i \in \mathcal{I}} \sqrt{A_i + \sum_{j \in \mathcal{J}} B_{ij} Y_{ij}} \\ & + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} v_{ij} Y_{ij} \end{aligned} \quad (5.15)$$

subject to (5.1), (5.3), and (5.4).

- (3) **Relaxation with respect to constraints (5.1) and (5.2):** Using Lagrange multipliers  $\mathbf{u}$  and  $\mathbf{v}$ , the Lagrangian problem is given as

$$\begin{aligned} \text{(LR3)} \quad \max_{\mathbf{u}, \mathbf{v}} L(\mathbf{u}, \mathbf{v}) = \min_{\mathbf{X}, \mathbf{Y}} \quad & \sum_{j \in \mathcal{J}} (f_j - \sum_{i \in \mathcal{I}} v_{ij}) X_j + \sum_{i \in \mathcal{I}} \sqrt{A_i + \sum_{j \in \mathcal{J}} B_{ij} Y_{ij}} \\ & + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} (v_{ij} - u_i) Y_{ij} + \sum_{i \in \mathcal{I}} u_i \end{aligned} \quad (5.16)$$

subject to (5.3) and (5.4).

In our implementation, we utilize (LR2) for its ease in solving  $L(\mathbf{v})$  for a given  $\mathbf{v}$  to obtain a lower bound, based on the following steps:

- 0:** Initialize  $Z^{LB} = \infty$ ,  $Z^{UB} = 0$ ,  $k = 0$ ,  $v_{ij} = 0$  for  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ ,  $\lambda = 2$ ,  $\epsilon_1 = 0.00001$ , and  $\epsilon_2 = 0.001$ .
- 1:** For given  $\mathbf{v}$ , solve (LR2) for a lower bound, i.e.,  $Z^{LB} = L(\mathbf{v})$ . If  $Z^{LB}$  is not improved in 30 iterations, halve  $\lambda$ .
- 2:** Obtain an upper bound using the solution of (LR2). Update  $Z^{UB}$  if necessary.
- 3:** Update  $\mathbf{v}^k$  using subgradient optimization. Increase iteration number, i.e.,  $k \leftarrow k + 1$ .
- 4:** Terminate if

[1] iteration number reaches a predetermined value, i.e.  $k > 300$ ,

[2]  $\lambda < \epsilon_1$ , or,

[3]  $Z^{UB} - Z^{LB} < \epsilon_2$ .

Otherwise, go back to Step 1.

Next, we explain how to obtain lower and upper bounds and how to update the Lagrange multipliers using subgradient optimization.

### V.2.3. Solving the Relaxed Problem

For fixed values of Lagrange multipliers  $\mathbf{v}$ , we want to minimize (5.15) of formulation (LR2) in order to obtain a lower bound for *PIII*. By relaxing the assignment constraints, we remove the link between DC selection/location  $X_j$  and DC-retailer assignment  $Y_{ij}$  variables for all  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ . Hence, in order to minimize  $L(\mathbf{v})$ , we treat the  $\mathbf{X}$  and  $\mathbf{Y}$  variables separately.

Consider DC selection/location variables  $\mathbf{X}$ . There is only one term in the cost function (5.15) with  $\mathbf{X}$ :

$$\sum_{j \in \mathcal{J}} (f_j - \sum_{i \in \mathcal{I}} v_{ij}) X_j. \quad (5.17)$$

To minimize (5.17), for all  $j \in \mathcal{J}$ , we set

$$X_j = \begin{cases} 1, & \text{if } f_j - \sum_{i \in \mathcal{I}} v_{ij} < 0, \\ 0, & \text{otherwise.} \end{cases} \quad (5.18)$$

Next, consider the assignment variables. Using Proposition 1, we state that each retailer is optimally assigned to the DC with the lowest operating costs. For each retailer  $i \in \mathcal{I}$ , find the DC  $j^* \in \mathcal{J}$  such that  $j_i^* = \arg \min_{j \in \mathcal{J}} \{\sqrt{A_i + B_{ij}} + v_{ij}\}$ . Then, we set assignment variables for  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$  as

$$Y_{ij} = \begin{cases} 1, & \text{if } j = j_i^*, \\ 0, & \text{otherwise.} \end{cases} \quad (5.19)$$

For any values of Lagrange multipliers,  $\mathbf{v}$ , evaluating (5.15) using location and assignment variables determined by (5.18) and (5.19) provides a *lower bound* ( $Z^{LB}$ ) on the objective function value of *PIII*.

#### V.2.4. Obtaining an Upper Bound

The solution obtained from the relaxed problem (LR2) is likely to violate the constraint (5.2) since, otherwise, the solution obtained would be feasible, and, hence, optimal for *PIII*.

To construct a feasible solution using the relaxed solution, we keep a set of open facilities,  $\mathcal{J}' \subset \mathcal{J}$ . Next, using Proposition 1, we assign each retailer to the open DC with the lowest operating cost. That is, for each retailer  $i \in \mathcal{I}$ , determine the open

DC  $j_i \in \mathcal{J}'$  such that

$$j_i = \arg \min_{j \in \mathcal{J}'} \sqrt{A_i + B_{ij}}.$$

Then, we set the assignment variables for each  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$  as follows:

$$Y_{ij} = \begin{cases} 1, & \text{if } j = j_i, \\ 0, & \text{otherwise.} \end{cases} \quad (5.20)$$

The upper bound  $Z$  is calculated by evaluating (5.8) using  $\mathbf{X}$  and  $\mathbf{Y}$  given in (5.18) and (5.20), respectively. If  $Z < Z^{UB}$ , then update the best upper bound  $Z^{UB}$  by setting it equal to  $Z$ .

### V.2.5. Subgradient Optimization

One of the challenging issues related to the Lagrangian relaxation heuristic is the computation of a good set of Lagrange multipliers. In general, this is known to be a difficult task (Gavish, 1978). In practice, a good, but not necessarily optimal, set of multipliers are calculated by using either subgradient optimization or various multiplier adjustment methods (Bazaraa and Goode, 1979).

In our implementation, we utilize subgradient optimization to update the Lagrange multipliers (see Held et al. (1974)). Given multiplier vector  $\mathbf{v}^k$  for the  $k^{\text{th}}$  iteration of the Lagrangian heuristic, the next set of multipliers  $\mathbf{v}^{k+1}$  are calculated using the following rule:

$$v_{ij}^{k+1} = \max\{0, v_{ij}^k + t^k(Y_{ij}^k - X_j^k)\}, \quad (5.21)$$

where  $\mathbf{X}^k$  and  $\mathbf{Y}^k$  are the optimal solution to  $L(\mathbf{v}^k)$ , and  $t^k$  is a positive scalar step size. We use the following step size that is frequently used in the literature:

$$t^k = \frac{\lambda(Z^{UB} - Z^{LB})}{\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} (Y_{ij}^k - X_j^k)^2},$$

where  $\lambda$  is a scalar satisfying  $0 \leq \lambda \leq 2$ .

### V.2.6. Numerical Results

In this section, we present computational results related to the *PIII*. The algorithms were implemented using C++ and run on a Pentium IV 3.2Ghz machine with 1 GB memory on numerous data sets. In particular, we conducted two computational experiments. The first experiment focuses on the performance of the Lagrangian heuristic over a variety of data sets. The second experiment is aimed at quantifying the benefit of integrated decision-making via comparison of the results obtained from joint optimization and a sequential benchmark model. In the second experiment, via a factorial design, we also identify the parameters that influence the implication of integrated decision-making the most.

#### V.2.6.1. Experiment 1: Performance of Lagrangian Relaxation

In this first experiment, our goal is to show the performance of the Lagrangian relaxation heuristic in solving *PIII*. For this purpose, we report results from 9 different data sets where both the number of retailers and the number of potential DCs have three alternatives. Each data set consists of either 25, 50, or 100 retailers and 10, 20, or 30 potential DCs. In each group, we have 100 problem instances, generated randomly using the uniform distributions given in Table 13, resulting in a total of 900 problem instances.

Our main findings regarding the performance of the Lagrangian relaxation heuristic, including the minimum, average, and maximum percentage gaps in the objective function value when compared to the lower bound, are summarized in Table 14. For



TABLE 13. Parameter Values for Experiment 1.

Parameters	Range
$D_i$	U[350, 1400]
$K_i$	U[75, 300]
$h'_i$	U[5, 10]
$p_{ij}$	U[425, 1700]
$r_{ij}$	U[180, 120]
$d_{ij}$	U[1, 150]
$f_j$	U[100000,150000]

each problem instance, the gap is defined as the percentage difference

$$\text{Gap (\%)} = \frac{Z^{UB} - Z^{LB}}{Z^{LB}} \times 100,$$

where  $Z^{UB}$  is the cost of the best feasible solution suggested by the Lagrangian relaxation heuristic, and  $Z^{LB}$  is the cost of the lower bound obtained by solving  $L(\mathbf{v})$ .

TABLE 14. Performance of Lagrangian Relaxation Heuristic.

Data set	$ \mathcal{I} $	$ \mathcal{J} $	Gap(%)			Duration (s.)		
			Min	Ave	Max	Min	Ave	Max
1	25	10	0.00	0.04	0.85	0.56	2.94	4.92
2	25	20	0.00	0.00	0.21	3.14	7.99	10.00
3	25	30	0.00	0.01	0.19	1.56	11.04	12.28
4	50	10	0.00	0.04	0.60	2.56	8.73	9.45
5	50	20	0.00	0.02	0.33	15.47	15.74	19.89
6	50	30	0.00	0.03	0.15	21.98	22.53	26.69
7	100	10	0.00	0.06	0.58	6.48	16.86	17.83
8	100	20	0.00	0.04	0.41	21.98	30.83	37.77
9	100	30	0.00	0.05	0.33	43.33	44.01	49.16

The first portion of Table 14 provides minimum, average, and maximum percentage gaps between upper and lower bounds for the different data sets. In all of the data sets, we observe that the Lagrangian heuristic is very effective in solving

*PIII* with average gaps less of than 0.1%. Furthermore, the maximum gaps tend to decrease as the number of potential DCs is increased while the number of retailers remains constant.

The second portion of Table 14 reports the duration of the heuristic for the different data sets. The heuristic duration tends to increase as the number of retailers and the number of potential DCs increase. This is expected since the problem is NP-hard. Increasing problem size lengthens the runtimes of the heuristics.

#### V.2.6.2. Experiment 2: Impact of Integrated Decision-Making

We proceed with reporting the results for Experiment 2, which illustrates the value of the integrated framework over the sequential framework that is commonly utilized in practice. For this purpose, we first provide a detailed description and formulation of the benchmark model. Next, we analyze results from two computational tests comparing the costs using the benchmark model and the *PIII*.

#### Benchmark Model (*BMIII*)

Similar to the benchmark models in Chapter IV, the benchmark model (*BMIII*) builds on the idea that location and inventory decisions are made *sequentially* and that facility location decisions precede inventory decisions. For this purpose, in *BMIII*, we first solve the following location-allocation problem regarding the selection of the DCs and the assignment of the DCs to the retailers:

$$\text{Min } \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} Y_{ij} \quad (\text{BMIII-}LocAlloc)$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad \forall i \in \mathcal{I}. \quad (5.22)$$

$$Y_{ij} \leq X_j, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.23)$$

$$X_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (5.24)$$

$$0 \leq Y_{ij} \leq 1, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.25)$$

As explained in IV, due to the lack of a better approach, we have to rely on crude estimates of  $c_{ij}$ ,  $i \in \mathcal{I} \cup \{0\}$ , which may not accurately consider the impact of the inventory policy parameters on per-unit per-mile costs. We let  $c_{ij} = \alpha d_{ij} D_i$ , where  $\alpha$  is the per-unit per-mile transportation cost which is set to 1 in our computational tests.

This formulation is a typical formulation of the uncapacitated FCFLP<sup>1</sup> and in our computational tests, we solve this problem *exactly* using CPLEX 9.0<sup>2</sup>.

Let  $\mathbf{X}^{BM}$  and  $\mathbf{Y}^{BM}$  be the solution of the *BMIII-LocAlloc*. Next, given the assignment variables  $Y_{ij}^{BM}$  for all  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ , the inventory decisions of the retailers are determined by solving the following problem:

$$\text{Min} \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i + \sum_{j \in \mathcal{J}} \frac{(p_{ij} + r_{ij} d_{ij}) D_i Y_{ij}^{BM}}{Q_i} \right\}. \quad (BMIII-Inv)$$

The optimal order quantity of retailer  $i \in \mathcal{I}$  obtained from the solution of the *BMIII-Inv* is

$$Q_i^{BM} = \sqrt{\frac{2[K_i + \sum_{j \in \mathcal{J}} (p_{ij} + r_{ij} d_{ij} Y_{ij}^{BM})] D_i}{h'_i}}.$$

The cost of *BMIII* ( $Z^{BM}$ ) is calculated using the decision variables  $(\mathbf{X}^{BM}, \mathbf{Y}^{BM}, \mathbf{Q}^{BM})$ ,

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<sup>1</sup>Although the assignment variables are modeled as a fraction of the demand of the retailer that is served by a DC, since the facilities are uncapacitated, they will assume integer values.

<sup>2</sup>CPLEX is a trademark of ILOG, Inc.

obtained from the solution of the *BMIII-LocAlloc* and the *BMIII-Inv*, in (5.8). We then measure the value of the integrated framework by computing

$$\text{The percentage gain by the } PIII \text{ over } BMIII (\%) = \frac{Z^{BM} - Z^{PIII}}{Z^{BM}} \times 100,$$

where  $Z^{PIII}$  is the cost of the *PIII* obtained by the Lagrangian relaxation heuristic.

### Test 1: Impact of $|\mathcal{I}|$ and $|\mathcal{J}|$

We first compare the performance of the *PIII* and the *BMIII* for 900 instances. The main goal of this experiment is to measure the impact of integrated decision-making for different configurations of distribution systems, i.e., for different numbers of retailers and potential DCs.

TABLE 15. Performance of *PIII* Compared to the Sequential Benchmark Model.

Data set	$ \mathcal{I} $	$ \mathcal{J} $	Gain(%)			Open DCs		
			Min	Ave	Max	%-Less	Ave	Max
1	25	10	10.74	20.39	30.19	89	1.56	4
2	25	20	13.76	22.85	32.57	78	1.32	4
3	25	30	11.93	24.39	31.45	81	1.47	4
4	50	10	2.38	20.88	25.71	100	2.21	4
5	50	20	2.86	22.44	28.58	92	2.01	5
6	50	30	4.89	23.82	30.26	94	2.00	4
7	100	10	17.45	21.23	24.70	99	2.73	5
8	100	20	8.48	23.25	26.99	98	3.23	6
9	100	30	15.31	24.45	28.07	94	2.85	6

We report the minimum, average, and maximum percentage gains by *PIII* over the *BMIII* in Table 15. We also compare the number of open DCs in both approaches and report the results. For each instance, the difference in the number of open DCs is calculated as  $\sum_{j \in \mathcal{J}} X_j^{BM} - \sum_{j \in \mathcal{J}} X_j^{PIII}$  where  $\mathbf{X}^{PIII}$  represents the open DC locations obtained from the solution of the *PIII*. In Table 15, we present the average

and maximum differences for the *PIII* and the *BMIII*. Furthermore, we also present the percentage of instances where the *PIII* has fewer open DCs than the *BMIII* under ‘%-Less’ column.

For all of the data sets in Table 15, we observe that significant savings are obtained over the *BMIII* when using the *PIII*, with average gains of more than 20% and maximum gains up to 32.57%. One interesting observation is that, for a given number of retailers, the average gains increase as the number of potential DCs increase. For instance, from data set 1 to data set 2, the average gain increases to 22.85% from 20.39%. Similarly, from data set 2 to data set 3, the average gain increases to 24.39%. A similar pattern is observed between data sets 4, 5, 6 and data sets 7, 8, 9 as well. In other words, when the number of potential DCs increase for a fixed number of retailers, the *PIII* is more efficient than the *BMIII* in determining open DCs and assignments of DCs to retailers, which translates to an increasing gap between the costs of the *PIII* and *BMIII*.

Another interesting observation from Table 15 relates to the difference in the number of open DCs in the solutions of the *BMIII* and *PIII*. In many of the instances, the number of open DCs in *PIII* is less than those in *BMIII* which is not surprising. On average, this difference is at least one DC for data sets with smaller networks (data sets 1, 2, and 3) and can be as high as three DCs for data sets with larger networks (data sets 8 and 9). By explicit consideration of the transportation cost and the impact of inventory decisions on location decisions, *PIII* generally requires fewer number of open DCs. For instance, for data set 4, in all of the instances, the number of open DCs with *PIII* are less than with the *BMIII*. Hence, *PIII* evaluates the trade-off among the fixed facility costs and the operating costs, including inventory and transportation, more efficiently than the *BMIII*.

## Test 2: Impact of Problem Parameters

In this experiment, we measure the impact of the problem parameters on integrated decision-making. For this purpose, by considering the high and low ranges in Table 16, we first generate  $2^7 = 128$  experimental settings using a factorial design. For each of the 128 settings, we generate 100 instances where the values of the seven critical parameters are generated randomly, using the uniform distributions dictated by the setting. We also note that the high and low ranges in Table 16 include the ranges in Table 13.

TABLE 16. Factorial Design

Parameters	Low (L)	High (H)
$D_i$	U[315, 385]	U[1260, 1540]
$K_i$	U[67.5, 82.5]	U[270, 330]
$h'_i$	U[4.5, 5.5]	U[9, 11]e
$p_{ij}$	U[425.5, 522.5]	U[1530, 1870]
$r_{ij}$	U[0.675, 0.825]	U[2.7, 3.3]
$d_{ij}$	U[30, 45]	U[120, 180]
$f_j$	U[45000, 55000]	U[180000, 220000]

In Tables 25, 26, 27, and 28, we present the factorial experiment settings and the results for each setting including the minimum, average, and maximum percentage gains for 100 instances with *PIII* over the *BMIII*. We also report the average and maximum differences in the number of open DCs between these two approaches.

We next summarize the other key points of our numerical study and establish the trade-offs among the problem parameters to determine under which settings the integrated decision-making has a higher impact.

- In Table 17, we provide the factorial designs that result in the 10 highest average percentage gains. The highest average gain is 19.01%, and the maximum gain

among these factorial designs is 33.44%. An important observation is that the ten highest average percentage gains are attained when demand and distance are drawn from the uniform distribution at high levels. Hence, demand and distance appear to be the most influential parameters. This result is not surprising considering the fact that the cost of the *BMIII* increases with demand and distance, and the cost of *PIII* increases with the square root of demand and distance.

TABLE 17. Factorial Designs with the 10 Highest Average Gains for *PIII*

F. D. No.	Levels of Parameters							Gain (%)			Open DCs		
	$D_i$	$K_i$	$h'_i$	$p_{ij}$	$r_{ij}$	$d_{ij}$	$f_j$	Min	Ave	Max	%-Less	Ave	Max
67	H	L	L	L	L	H	L	6.26	19.01	26.88	100	3.10	5
68	H	L	L	L	L	H	H	1.46	18.86	33.44	100	0.80	2
72	H	L	L	L	H	H	H	4.08	15.19	28.84	100	0.80	2
71	H	L	L	L	H	H	L	1.38	14.44	21.17	100	3.10	5
84	H	L	H	L	L	H	H	2.95	14.42	27.52	100	0.80	2
83	H	L	H	L	L	H	L	1.25	14.27	20.71	100	3.10	5
75	H	L	L	H	L	H	L	1.04	12.66	18.10	100	3.10	5
76	H	L	L	H	L	H	H	1.26	12.52	24.38	100	0.80	2
88	H	L	H	L	H	H	H	1.36	11.27	22.97	100	0.80	2
80	H	L	L	H	H	H	H	1.10	11.22	22.62	100	0.80	2

Another significant observation from Table 17 is related to the difference between *PIII* and the *BMIII* in the number of open DCs. The number of open DCs in the solution of *PIII* is always less than in the *BMIII* for all instances of these factorial designs. The average difference can be up to 3 DCs and the maximum difference can be as high as 5 DCs. High demand and distance levels seem to force the benchmark model to open more (unnecessary) DCs than *PIII*.

- In Table 18, we provide the factorial designs that result in the 10 lowest average percentage gains. Similar to the results in Table 17, demand and distance

appear to be the most influential parameters. The lowest average gains are attained when demand and distance are drawn from uniform distributions at low levels. The 10 lowest average gains range between 0.25% and 0.41%. The maximum gains corresponding to these factorial designs go up to 3.15% which is quite significant considering the dollar values associated with these percentage savings.

TABLE 18. Factorial Designs with the 10 Lowest Average Gains for *PIII*

F. D.		Levels of Parameters						Gain (%)			Open DCs		
No.	$D_i$	$K_i$	$h'_i$	$p_{ij}$	$r_{ij}$	$d_{ij}$	$f_j$	Min	Ave	Max	%-Less	Ave	Max
54	L	H	H	L	H	L	H	0.00	0.25	2.08	0	0.00	0
38	L	H	L	L	L	L	H	0.00	0.30	2.51	0	0.00	0
62	L	H	H	H	H	L	H	0.00	0.35	2.20	0	0.00	0
46	L	H	L	H	H	L	H	0.00	0.35	2.46	0	0.00	0
22	L	L	H	L	H	L	H	0.00	0.36	3.01	0	0.00	0
30	L	L	H	H	H	L	H	0.00	0.37	2.79	0	0.00	0
50	L	H	H	L	L	L	H	0.00	0.37	2.76	0	0.00	0
58	L	H	H	H	L	L	H	0.00	0.40	2.43	0	0.00	0
42	L	H	L	H	L	L	H	0.00	0.40	2.69	0	0.00	0
34	L	H	L	L	L	L	H	0.00	0.41	3.15	0	0.00	0

As observed in Table 18, another important parameter is the fixed cost of opening DCs. For the factorial designs with the 10 lowest average gains, the fixed location costs are drawn from a uniform distribution at high levels. In all of the factorial designs in Table 18, the number of open DCs in the solutions of *PIII* and the *BMIII* are the same. Hence, the average and maximum difference for the number of open DCs is zero for all of the instances in these ten factorial designs. High levels of fixed location costs, coupled with low levels of demand and distance, force both *PIII* and the *BMIII* to open fewer DCs, frequently, only one DC.



- As the above two points demonstrate, the most influential parameters influencing the impact of integrated decision-making are demand and distance. Increasing either one, while the other parameters are kept the same, increases the average cost savings.
- Another important influential parameter that affects integrated decision-making is fixed location costs ( $f_j, \forall j \in \mathcal{J}$ ). In general, increasing fixed location costs, while holding other parameters the same, decreases the average gap between the costs of *PIII* and the *BMIII*. Increased fixed costs force both models to open fewer DCs, and hence, the average costs savings are more limited. There is an exception to this observation. For factorial designs with high demand and distance levels, increasing the fixed location costs may not impact the average gap between *PIII* and the *BMIII*, or it may even increase it. This is due to the classical trade-off between fixed location costs and *implied* transportation costs. If the implied transportation costs are higher than the highest levels of the fixed transportation costs, *PIII* and the *BMIII* may open a different number of DCs.
- The impact of other parameters on integrated decision-making can be summarized as follows:
  - Increasing the variable transportation cost or the fixed transportation cost, while keeping all the other parameters the same, decreases the average gain. Increasing either the variable or the fixed transportation cost impacts the trade-off between the fixed location costs and transportation costs in *PIII* and force *PIII* to open more DC locations to decrease the total transportation cost. Hence, the total cost of *PIII* increases. Since a change in the variable or the fixed transportation costs does not impact the *BMIII*, the average gap between

$PMIII$  and the  $BMIII$  decreases.

- Increasing the holding cost when all of the other parameters are kept the same, decreases the average gain.

### V.3. Capacitated Case

In this section, we introduce capacity restrictions to the generic (uncapacitated)  $PIII$  model and analyze the impact of capacity on integrated decision-making. For this purpose, we first discuss the potential capacity restrictions that can be posed on the  $PIII$  and express these restrictions mathematically. We then examine their influence on the structural properties of the  $PIII$  and its solution approaches. Finally, via computational tests, we measure the value of integrated decision-making under these capacity restrictions.

#### V.3.1. Definition of Capacity

In many of the real life problems related to distribution system design, there are several operational constraints that further complicate the solution procedures. Some of these restrictions, such as production (for plants) or throughput capacities (for DCs), are considered in the context of extensions of the classical FCFLP in the previous literature. However, in the  $PIII$ , by considering inventory decisions simultaneously with facility location decisions, we create a venue for defining other capacity restrictions such as storage space and dispatch size and number.

For the  $PIII$ , we define the following capacity restrictions:

**Throughput Capacity** Throughput capacity for a DC (likewise, production capacity for a plant) represents the maximum allowed throughput through that DC; the term also means the total annual demand assigned to a DC should be less

than the annual throughput capacity of that DC, i.e.,

$$\sum_{i \in \mathcal{I}} D_i Y_{ij} \leq P_j X_j, \quad \forall j \in \mathcal{J}, \quad (5.26)$$

where  $P_j$  is the annual throughput capacity of DC  $j$ ,  $j \in \mathcal{J}$ .

The addition of this constraint to the formulation of the FCFLP complicates the solution, see Section II.1.2. Several exact and heuristic methods are developed for the capacitated FCFLP, including Lagrangian relaxations and Bender's decomposition (Daskin, 1995). We include the throughput capacity constraint in the formulation of the *PIII* and refer to this new problem as the *PIII-PC*. We analyze the structural properties of the *PIII-PC* and discuss a solution approach in Section V.3.2.

**Storage Capacity** Storage capacity refers to the physical capacity of a DC. Generally, storage capacity is different from the throughput capacity due to inventory turnover, i.e., the rate at which inventory is depleted/replenished. The relation between inventory turnover and storage capacity is not taken into account explicitly in the context of the classical FCFLP in which operational inventory considerations are omitted. In the context of the *PIII*, however, we can formulate the relationship between inventory turnover and storage space by considering

$$\sum_{i \in \mathcal{I}} Q_i Y_{ij} \leq S_j X_j, \quad j \in \mathcal{J}. \quad (5.27)$$

where  $S_j$  is the physical storage capacity of DC  $j$ ,  $j \in \mathcal{J}$ . This constraint states that the storage capacity at DC  $j$  should be large enough to accommodate the inventory that all of the retailers assigned to DC  $j$  order together. Hence, the constraint ensures that the space limitation is never violated. Even in the worst case, storage capacity can handle the simultaneous arrival of orders from the

all of the retailers assigned to this DC. The right hand side of (5.27) not only limits the storage availability but also ensures that we consider only open DCs.

**Number of Dispatches** In certain distribution systems, the number of dispatches from a DC can not exceed a certain number due to restrictions regarding the loading/unloading capabilities of the facility. However, this practical constraint is generally overlooked in the classical FCFLP simply because the number of dispatches out of a facility is dictated by the size of shipments. In the context of the *PIII*, we can easily incorporate a restriction on the number of dispatches as follows:

$$\sum_{i \in \mathcal{I}} \frac{D_i}{Q_i} Y_{ij} \leq R_j X_j, \quad j \in \mathcal{J}, \quad (5.28)$$

where  $R_j$  is the maximum number of dispatches from DC  $j$ ,  $j \in \mathcal{J}$ , annually.

**Truck Capacity** In many distribution systems, one of the main goals is to achieve efficient truck utilization to minimize transportation costs by reducing the number of required trips and/or the number of trucks used for each shipment. Again, in order to achieve this goal, the location model must take inventory decisions into account as in the *PIII*. The number of trucks (or number of trips) required by retailer  $i$ ,  $i \in \mathcal{I}$ , for a shipment size of  $Q_i$  is given by  $\left\lceil \frac{Q_i}{C_T} \right\rceil$  where  $C_T$  is the truck/cargo capacity. Hence, the objective function of the *PIII* for considering truck capacities is:

$$\text{Min} \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left\{ \frac{(p_{ij} + r_{ij} d_{ij}) \left\lceil \frac{Q_i}{C_T} \right\rceil D_i Y_{ij}}{Q_i} \right\} + \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h_i Q_i \right\}.$$

We call the extension of the *PIII* with the above objective function subject to constraints (5.1), (5.2), (5.3), (5.4), and (5.5) the *PIII-TC*. In this problem, if there is not a limit on the number of available trucks, the truck capacity is not

a hard constraint, and, hence, is referred to as an *installable* truck capacity. If the number of trucks at hand, denoted by  $N_T$  is also limited, we can incorporate the following constraint

$$\sum_{i \in \mathcal{I}} \left\lceil \frac{Q_i}{C_T} \right\rceil \leq N_T, \quad (5.29)$$

into the *PIII-TC*.

### V.3.2. Structural Properties and Analysis

In this section, we analyze the *PIII* with different capacity constraints. In particular, we consider

- *PIII-PC*: *PIII* with limited throughput capacities at the potential DCs,
- *PIII-SC*: *PIII* with limited storage capacity at the potential DCs,
- *PIII-DC*: *PIII* with a limited number of annual dispatches from the potential DCs, and
- *PIII-TC*: *PIII* with installable truck capacity.

#### V.3.2.1. Analysis of the *PIII-PC*

The *PIII-PC* is the most naive extension of the *PIII*. The relationship between the *PIII* and the *PIII-PC* is analogous to the relationship between the classical uncapacitated and capacitated FCFLP. Using the notation defined in Section V.1, the formulation of the *PIII-PC* is as follows:

$$\text{Min} \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left\{ \frac{(p_{ij} + r_{ij} d_{ij}) D_i Y_{ij}}{Q_i} \right\} + \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\} \quad (\text{PIII-PC})$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad \forall i \in \mathcal{I}. \quad (5.30)$$

$$Y_{ij} \leq X_j, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.31)$$

$$\sum_{i \in \mathcal{I}} D_i Y_{ij} \leq P_j X_j, \quad \forall j \in \mathcal{J}. \quad (5.32)$$

$$X_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (5.33)$$

$$Y_{ij} \in \{0, 1\}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.34)$$

$$Q_i \geq 0, \quad \forall i \in \mathcal{I}. \quad (5.35)$$

The above formulation is identical to the uncapacitated *PIII* defined in Section V.2 except that we have now included a throughput capacity constraint (5.32). Note that constraint (5.31) is not needed any more since the capacity constraint (5.32) ensures that retailer  $i$ ,  $i \in \mathcal{I}$ , is not assigned to a potential DC  $j$ ,  $j \in \mathcal{J}$ , if that DC is not open. However, considering this constraint strengthens the linear programming relaxation of the classical capacitated FCFLP (Daskin, 1995). As we will show next, the *PIII-PC* can be converted to the classical capacitated FCFLP, and, hence, inclusion of constraint (5.31) is useful in computing the solution of the *PIII-PC*.

Since the objective functions of the *PIII* and the *PIII-PC* are the same, we can eliminate the order quantity from the objective function of the *PIII-PC* via structural properties in Section V.2.1, and we can use the following modified objective function for the *PIII-PC*.

$$\sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sqrt{A_i + \sum_{j \in \mathcal{J}} B_{ij} Y_{ij}}, \quad (5.36)$$

where  $A_i = 2K_i D_i h'_i$  and  $B_{ij} = 2(p_{ij} + r_{ij} d_{ij}) D_i h'_i$  for  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ . Then,

Theorem 7 is applicable for the *PIII-PC*. As a corollary, the *PIII-PC* can be solved using the techniques developed for the single-source capacitated FCFLP, see Section 2.9. We present the computational results regarding the *PIII-PC* in Section V.3.3.

### V.3.2.2. Analysis of the *PIII-SC*

Using the notation defined in Section V.1, the formulation of the *PIII-SC* is as follows:

$$\text{Min } \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left\{ \frac{(p_{ij} + r_{ij} d_{ij}) D_i Y_{ij}}{Q_i} \right\} + \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h_i Q_i \right\} \quad (\text{PIII-SC})$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad \forall i \in \mathcal{I}. \quad (5.37)$$

$$Y_{ij} \leq X_j, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.38)$$

$$\sum_{i \in \mathcal{I}} Q_i Y_{ij} \leq S_j X_j, \quad \forall j \in \mathcal{J}. \quad (5.39)$$

$$X_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (5.40)$$

$$Y_{ij} \in \{0, 1\}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.41)$$

$$Q_i \geq 0, \quad \forall i \in \mathcal{I}. \quad (5.42)$$

The novelty of this formulation is the explicit consideration of storage capacity related to physical DC size in association with inventory decisions.

As we mentioned earlier, in this formulation, the storage capacity constraint is written with a conservative worst case scenario in mind. Here, we shall not account for the possibility that orders can be phased so that they coincide; therefore, it will be never necessary to have the maximum order quantity of each retailer at the same time. The following example demonstrates the value of this conservative estimate as well as the impact of considering inventory turnover while defining capacity requirements

clearly. Hence, it shows the importance of integrating location, assignment, and inventory decisions.

**EXAMPLE 3** Consider a particular open DC  $j^*$  and the set of retailers that are assigned to this DC  $\mathcal{J}^* = \{1_{j^*}, 2_{j^*}, 3_{j^*}, 4_{j^*}\}$  with annual demands  $D_{1_{j^*}} = 3000$ ,  $D_{2_{j^*}} = 1200$ ,  $D_{3_{j^*}} = 600$ , and  $D_{4_{j^*}} = 200$ . Retailer  $1_{j^*}$  orders monthly;  $2_{j^*}$  orders every two months;  $3_{j^*}$  orders quarterly; and  $4_{j^*}$  orders semi-annually. Accordingly, the annual ordering schedule for DC  $j^*$  is given in Table 19. The last row in Table 19 shows

TABLE 19. The Annual Ordering Schedule for DC  $j^*$

$\mathcal{J}^*$	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	$D_{i_{j^*}}$
$1_{j^*}$	250	250	250	250	250	250	250	250	250	250	250	250	3000
$2_{j^*}$	200		200		200		200		200		200		1200
$3_{j^*}$	150			150			150			150			600
$4_{j^*}$	100						100						200
Total	700	250	450	400	450	250	700	250	450	400	450	250	5000

the monthly quantity that leaves DC  $j^*$ , and the last column summarizes the demand of each retailer in  $\mathcal{J}^*$ . The maximum monthly amount served by this DC is equal to the sum of the order quantities of retailers in  $\mathcal{J}^*$ , and hence, the storage space of DC  $j^*$  should be at least 700. On the other hand, the annual throughput of this DC (the cumulative annual order quantity) is 5000. As a conclusion, with a storage capacity of 700, we can serve a DC with a throughput requirement of 5000. Hence, it is not correct to model throughput capacity as storage capacity.

Next, we discuss the structural properties of the *PIII-SC* leading to an efficient solution approach for the problem. For the sake of notational clarity and ease of



explanation, we represent the problem in the following compact form:

$$\text{Min}_{\mathbf{X}, \mathbf{Y}, \mathbf{Q}} \quad F(\mathbf{X}) + G(\mathbf{Y}, \mathbf{Q}) \quad (\text{P-SC})$$

subject to

$$\mathbf{H}(\mathbf{X}, \mathbf{Y}, \mathbf{Q}) \leq \mathbf{0}, \quad (5.43)$$

$$\mathbf{M}(\mathbf{Y}) = \mathbf{0}, \quad (5.44)$$

$$\mathbf{N}(\mathbf{X}, \mathbf{Y}) \leq \mathbf{0}, \quad (5.45)$$

$$\mathbf{X} \in \{0, 1\}^n, \quad \mathbf{Y} \in \{0, 1\}^{mn}, \quad \text{and} \quad \mathbf{Q} \in \mathbb{R}_+^m, \quad (5.46)$$

where

$$F(\mathbf{X}) = \sum_{j \in \mathcal{J}} f_j X_j,$$

$$G(\mathbf{Y}, \mathbf{Q}) = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left\{ \frac{(p_{ij} + r_{ij} d_{ij}) D_i Y_{ij}}{Q_i} \right\} + \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\},$$

and  $\mathbf{H}(\cdot)$ ,  $\mathbf{M}(\cdot)$ , and  $\mathbf{N}(\cdot)$  are vector functions with components:

$$H_j(X_j, \mathbf{Y}, \mathbf{Q}) = \sum_{i \in \mathcal{I}} Q_i Y_{ij} - S_j X_j, \quad j \in \mathcal{J}; \quad (5.47)$$

$$M_i(\mathbf{Y}) = \sum_{j \in \mathcal{J}} Y_{ij} - 1, \quad i \in \mathcal{I}; \quad (5.48)$$

$$N_{ij}(X_j, Y_{ij}) = Y_{ij} - X_j, \quad i \in \mathcal{I}, \quad j \in \mathcal{J}. \quad (5.49)$$

This problem is a difficult mixed integer nonlinear programming (MINLP) problem, and it is NP-hard. For this problem, it is easy to see now that the discrete variables  $\mathbf{X}$  and  $\mathbf{Y}$  are the complicating variables in the sense that we obtain a multi-retailer capacitated EOQ problem when the discrete variables are temporarily held fixed. The key idea that enables (P-SC) to be viewed as a problem in the  $\mathbf{X}$  and  $\mathbf{Y}$  space is the concept of *projection*, sometimes known also as *partitioning*. The

projection of (P-SC) onto  $\mathbf{X}$ - $\mathbf{Y}$  is

$$v(\mathbf{X}, \mathbf{Y}) = \min_{\mathbf{Q}} G(\mathbf{Y}, \mathbf{Q}) \quad (\text{Sub-P-SC})$$

subject to

$$\mathbf{H}(\mathbf{X}, \mathbf{Y}, \mathbf{Q}) \leq \mathbf{0}, \quad (5.50)$$

$$\mathbf{Q} \in \mathbb{R}_+^m. \quad (5.51)$$

Then, the overall problem can be rewritten as

$$\text{Min}_{\mathbf{X}, \mathbf{Y}} \quad F(\mathbf{X}) + v(\mathbf{X}, \mathbf{Y}) \quad (\text{Master-P-SC})$$

subject to

$$\mathbf{M}(\mathbf{Y}) = \mathbf{0}, \quad (5.52)$$

$$\mathbf{N}(\mathbf{X}, \mathbf{Y}) \leq \mathbf{0}, \quad (5.53)$$

$$\mathbf{X} \in \{0, 1\}^n \cap \Gamma, \text{ and } \mathbf{Y} \in \{0, 1\}^{mn} \cap \Gamma, \quad (5.54)$$

where  $\Gamma = \{\mathbf{X}, \mathbf{Y} : \exists \mathbf{Q} \in \mathbb{R}_+^m \text{ and } H(\mathbf{X}, \mathbf{Y}, \mathbf{Q}) \leq 0\}$ , i.e., set  $\Gamma$  consists of values of  $\mathbf{X}$  and  $\mathbf{Y}$  such that the problem (Sub-P-SC) is feasible.

Benders (1962) was one of the first to appreciate the importance of problem (Sub-P-SC) as a route to solving (P-SC) in classical mixed integer programming. Benders (1962) showed that the master problem and the subproblem can be solved independently with information being communicated between them. The integer solution is passed from the master problem to the subproblem, and the subproblem generates cuts for the master problem. Geoffrion (1972) developed a framework, called ‘‘Generalized Bender’s Decomposition’’ (GBD), to implement Bender’s decomposition for nonlinear and possibly mixed integer problems.

The basic idea in the GBD is the generation, at each iteration, of an upper bound and a lower bound on the sought solution of the original MINLP problem.

The upper bound results from the solution of the subproblem while the lower bound results from the solution of the master problem. The solution of the subproblem not only provides an upper bound but also the Lagrange multipliers associated with the constraints removed from the original problem. The master problem is derived via nonlinear duality theory (Geoffrion, 1971) through manipulations to the original problem. For GBD, it is shown that, as the iterations proceed, the sequence of the lower bounds is non-decreasing, and the sequence of updated upper bounds is non-increasing. Hence, the sequences converge in a finite number of iterations (Geoffrion, 1972).

GBD is used for solving many MINLP problems (Floudas, 1995). For solving (P-SC), we utilize the GBD due to the special structure of our problem. That is, when the location and assignment variables are known, the remaining problem (Sub-P-SC) is a multi-retailer capacitated EOQ problem for each open DC. On the other hand, when the  $\mathbf{Q} = \{Q_1, \dots, Q_m\}$  values are known, the remaining problem is a capacitated FCFLP. Next, we will discuss each component of the GBD and its application to (P-SC) in detail. In particular, we will discuss

- the subproblem with feasible and infeasible cases,
- the derivation of the master problem, and
- the overall GBD procedure.

### **The Subproblem**

Given  $\mathbf{X}$  and  $\mathbf{Y}$ , the problem reduces to finding a solution to a multi-retailer EOQ problem with storage capacity for each open DC  $j$ ,  $j \in \mathcal{J}$ . That is, the subproblem

at any iteration  $k$

$$v(\mathbf{X}^k, \mathbf{Y}^k) = \min_{\mathbf{Q}} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \frac{(p_{ij} + r_{ij}d_{ij})D_i Y_{ij}^k}{Q_i} + \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}$$

subject to

$$\sum_{i \in \mathcal{I}} Q_i Y_{ij}^k \leq S_j X_j^k, \quad \forall j \in \mathcal{J}, \quad (5.55)$$

$$Q_i \geq 0, \quad \forall i \in \mathcal{I}, \quad (5.56)$$

is decomposable for each  $j \in \mathcal{J}$ , since each retailer can only be assigned to one DC. That is, for each  $j \in \mathcal{J}$ , there is a set of retailers  $\mathcal{I}_j$  that is served by DC  $j$ , i.e.,  $\mathcal{I}_j = \{i \in \mathcal{I} : Y_{ij} = 1\}$ ,  $\forall j \in \mathcal{J}$ . Then, the solution of the subproblem is obtained by solving the following problem for all  $j \in \mathcal{J}$ :

$$v_j(X_j^k, \mathbf{Y}^k) = \min_{\mathbf{Q}} \sum_{i \in \mathcal{I}_j} \left\{ \frac{(p_{ij} + r_{ij}d_{ij})D_i Y_{ij}^k}{Q_i} + \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}$$

subject to

$$\sum_{i \in \mathcal{I}_j} Q_i Y_{ij}^k \leq S_j X_j^k, \quad \forall j \in \mathcal{J}. \quad (5.57)$$

$$Q_i \geq 0, \quad \forall i \in \mathcal{I}_j. \quad (5.58)$$

In the above, although displaying  $Y_{ij}^k$  explicitly,  $i \in \mathcal{I}_j$  and  $j \in \mathcal{J}$ , is unnecessary, keeping it in the formulation is useful for generating the cuts later.

For each  $j \in \mathcal{J}$ , the above problem  $v_j(X_j^k, \mathbf{Y}^k)$  is solved following a similar approach in Hadley and Whitin (1963)[p. 54-55]. In particular, for a fixed  $j \in \mathcal{J}$ , we first ignore constraints (5.57), i.e., we minimize the terms of the objective function for  $i \in \mathcal{I}_j$  separately to determine the order quantity  $Q_i$ . This yields

$$Q_i^* = \sqrt{\frac{2D_i[K_i + \sum_{j \in \mathcal{J}} (p_{ij} + r_{ij}d_{ij})Y_{ij}^k]}{h'_i}}. \quad (5.59)$$

For each  $j \in \mathcal{J}$ , if  $Q_i^*$ ,  $i \in \mathcal{I}_j$ , satisfy constraints (5.57), then these  $Q_i^*$  are optimal, and the capacity constraints (5.57) are not active. On the other hand, if for some  $j \in \mathcal{J}$  these  $Q_i^*$  do not satisfy constraints (5.57), then the constraint is active, and  $Q_i^*$  of (5.59) is not optimal. To find the optimal  $Q_i^*$ ,  $\forall i \in \mathcal{I}_j$ , the Lagrange multiplier technique is used.

For DCs with violated capacity constraints,  $j \in \mathcal{J}$ , we form the following Lagrange function:

$$\mathfrak{L}_j = \sum_{i \in \mathcal{I}_j} \left\{ \frac{(p_{ij} + r_{ij}d_{ij})D_i Y_{ij}^k}{Q_i} + \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\} + \mu_j \left( \sum_{i \in \mathcal{I}_j} Q_i Y_{ij}^k - S_j X_j^k \right), \quad (5.60)$$

where  $\mu_j$  is the Lagrange multiplier, and  $\mu_j \geq 0$ ,  $j \in \mathcal{J}$ . For each  $j \in \mathcal{J}$ , the optimal  $Q_i$ , for  $i \in \mathcal{I}_j$ , is obtained from the well known KKT conditions (Bertsekas, 2004):

$$\frac{\partial \mathfrak{L}_j}{\partial Q_i} = 0 = \frac{-(p_{ij} + r_{ij}d_{ij})D_i}{Q_i^2} - \frac{K_i D_i}{Q_i^2} + \frac{1}{2} h'_i + \mu_j, \quad i \in \mathcal{I}_j. \quad (5.61)$$

$$\frac{\partial \mathfrak{L}_j}{\partial \mu_j} = 0 = \sum_{i \in \mathcal{I}_j} Q_i - s_j. \quad (5.62)$$

$$Q_i \geq 0, \quad i \in \mathcal{I}_j. \quad (5.63)$$

$$\mu_j \geq 0. \quad (5.64)$$

This equation system has a unique and, hence, optimal solution:

$$Q_i^{**} = \sqrt{\frac{2D_i[K_i + p_{ij} + r_{ij}d_{ij}]}{h'_i + 2\mu_j^*}}, \quad i \in \mathcal{I}_j, \quad (5.65)$$

where  $\mu_j^*$  is the value of  $\mu_j$  such that  $Q_i^{**}$  of (5.65) satisfy (5.62). For  $j \in \mathcal{J}$ , the equation

$$W_j(\mu_j) = \sum_{i \in \mathcal{I}_j} \sqrt{\frac{A_{ij}}{B_i + \mu_j}} - s_j = 0$$

is satisfied by a *unique*  $\mu_j^* > 0$  since

- $W_j(\mu_j)$  is a monotone decreasing function of  $\mu_j$  for  $A_{ij} = D_i[K_i + p_{ij} + r_{ij}d_{ij}] \geq 0$

and  $B_i = \frac{1}{2}h'_i \geq 0$ , and

- as  $\mu_j \rightarrow \infty$ ,  $W_j(\mu_j) \rightarrow -s_j$ .

In our algorithm, we find such  $\mu_j^* > 0$  using a bisection algorithm.

Using this solution, we form the following cut:

$$\begin{aligned}
\mathfrak{L}(\mathbf{X}^k, \mathbf{Y}^k, \mathbf{Q}^k, \mu^k) &= \sum_{j \in \mathcal{J}} \mathfrak{L}_j \\
&= \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \frac{(p_{ij} + r_{ij}d_{ij})D_i Y_{ij}^k}{Q_i^k} + \sum_{i \in \mathcal{I}} \frac{K_i D_i}{Q_i^k} + \frac{1}{2}h'_i Q_i^k \\
&\quad + \sum_{j \in \mathcal{J}} \mu_j^k \left( \sum_{i \in \mathcal{I}} Q_i^k Y_{ij}^k - S_j X_j^k \right) \\
&= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \alpha_{ij}^k Y_{ij}^k + \sum_{j \in \mathcal{J}} \beta_j^k X_j^k + \gamma^k, \tag{5.66}
\end{aligned}$$

where

$$Q_i^k = Q_i^{**}, \quad i \in \mathcal{I}. \tag{5.67}$$

$$\mu_j^k = \mu_j^*, \quad j \in \mathcal{J}. \tag{5.68}$$

$$\alpha_{ij} = \frac{(p_{ij} + r_{ij}d_{ij})D_i}{Q_i^k} + \mu_j^k S_j, \quad i \in \mathcal{I} \text{ and } j \in \mathcal{J}. \tag{5.69}$$

$$\beta_j = -\mu_j^k S_j, \quad j \in \mathcal{J}. \tag{5.70}$$

$$\gamma = \sum_{i \in \mathcal{I}} \frac{K_i D_i}{Q_i^k} + \frac{1}{2}h'_i Q_i^k. \tag{5.71}$$

If the capacity constraint (5.57) is not tight,  $\mu_j = 0$ , and, consequently,  $\beta_j = 0$ .

Next, we describe the master problem.

### The Master Problem

The desired master problem is obtained by invoking the key ideas of the GBD including partitioning (or projection of) problem (P-SC) onto  $\mathbf{X}$ - $\mathbf{Y}$ , the dual representation of (Sub-P-SC), and the dual representation of the feasible region of (Sub-P-SC)  $\Gamma$ , (Geoffrion, 1971). By introducing a new variable  $\eta$ , we formulate the master

problem as follows:

$$\text{Min}_{\mathbf{X}, \mathbf{Y}, \eta} \quad \sum_{j \in \mathcal{J}} f_j X_j + \eta \quad (\text{Master-P-SC})$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad i \in \mathcal{I}. \quad (5.72)$$

$$Y_{ij} \leq X_j, \quad i \in \mathcal{I} \text{ and } j \in \mathcal{J}. \quad (5.73)$$

$$\eta \geq \min_{\mathbf{Q}} \left\{ \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \frac{(p_{ij} + r_{ij} d_{ij}) D_i Y_{ij}}{Q_i} + \sum_{i \in \mathcal{I}} \left( \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right) \right. \\ \left. \sum_{j \in \mathcal{J}} \mu_j \left( \sum_{i \in \mathcal{I}} Q_i Y_{ij} - S_j X_j \right) \right\}, \quad \forall \mu \geq 0. \quad (5.74)$$

This formulation of the master problem is equivalent to the original formulation of the *PIII-SC*. It involves, however, an infinite number of constraints, and hence, we need to consider a relaxation of the master problem by dropping a number of constraints. In particular, we add constraints (5.74) as needed. This reduced formulation is called the relaxed master problem (RM-P-SC):

$$\text{Min}_{\mathbf{X}, \mathbf{Y}, \eta} \quad \sum_{j \in \mathcal{J}} f_j X_j + \eta \quad (\text{RM-P-SC})$$

subject to

$$(5.72), (5.73),$$

$$\eta \geq \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \alpha_{ij}^k Y_{ij} + \sum_{j \in \mathcal{J}} \beta_j^k X_j + \gamma^k, \quad \text{for } k = 1, \dots, K. \quad (5.75)$$

In this formulation, the cuts (5.75) follow the same form as in (5.66). Furthermore, they are added at each iteration as explained earlier. In (5.75),  $K$  refers to the number of cuts generated using the subproblem. (RM-P-SC) is still a difficult mixed integer

programming formulation. We solve (RM-P-SC) at each iteration of the GBD using CPLEX 9.0.

Next, we describe the overall procedure of the GBD for (P-SC).

### The Overall GBD Procedure

The overall procedure of the GBD follows closely from the initial framework introduced by Geoffrion (1972). The framework developed for (P-SC) is represented in the following steps:

- 1:** Set  $UB = \infty$  and  $bestUB = \infty$ . Set the counter  $k = 0$ . Select convergence tolerance  $\epsilon = 0.001$ . Solve (RM-P-SC). Let  $(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\eta})$  be an optimal solution of (RM-P-SC). Set  $LB = \sum_{j \in \mathcal{J}} f_j \hat{X}_j + \hat{\eta}$ .
- 2:** Given  $(\hat{\mathbf{X}}, \hat{\mathbf{Y}})$ , solve (Sub-P-SC). Let  $UB = v(\hat{\mathbf{X}}, \hat{\mathbf{Y}})$ . Update  $bestUB$  with  $\min\{bestUB, UB\}$ . If  $bestUB - LB \leq \epsilon$ , then terminate. Otherwise, set  $k = k + 1$ , and  $(\alpha^k, \beta^k, \gamma^k) = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ . Go to Step 3.
- 3:** Solve (RM-P-SC). Update  $LB$ . If  $bestUB - LB \leq \epsilon$ , then terminate. Otherwise, update  $(\hat{\mathbf{X}}, \hat{\mathbf{Y}})$  and return to Step 2.

The algorithm iterates between Step 2 and Step 3. The termination conditions of the algorithm imply that the solution is  $\epsilon$ -optimal for the original problem (P-SC). Note that the  $LB$  values form a non-decreasing series and the  $bestUB$  values form a non-increasing series. However, in our computational experience, we observe that it is possible that the same cut is generated more than once. Then, the values of  $LB$  and  $bestUB$  are not updated any further, and the  $\epsilon$ -convergence is not achieved. In those cases, we terminate the algorithm if the same cut is generated again. We report our computational results regarding the performance of the GBD procedure in Section V.3.3.



### V.3.2.3. Analysis of the *PIII-DC*

In this section, we analyze the generalization of the *PIII* with limited number of annual dispatches from the potential DCs. Using the notation defined in Section V.1, the formulation of this problem is as follows:

$$\text{Min } \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left\{ \frac{(p_{ij} + r_{ij} d_{ij}) D_i Y_{ij}}{Q_i} \right\} + \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h_i Q_i \right\} \quad (\text{PIII-DC})$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad \forall i \in \mathcal{I}. \quad (5.76)$$

$$Y_{ij} \leq X_j, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.77)$$

$$\sum_{i \in \mathcal{I}} \frac{D_i}{Q_i} Y_{ij} \leq R_j X_j, \quad \forall j \in \mathcal{J}. \quad (5.78)$$

$$X_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (5.79)$$

$$Y_{ij} \in \{0, 1\}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.80)$$

$$Q_i \geq 0, \quad \forall i \in \mathcal{I}. \quad (5.81)$$

The *PIII-DC* is challenging to solve because of the nonlinearities both in the objective function and constraint (5.78). In this regard, it has similarities with the *PIII-SC*.

One way to approach the *PIII-DC* is to ignore constraint (5.78), and solve the problem using a Lagrangian Relaxation Heuristic. If the solution satisfies the limit on the number of dispatches, the solution is optimal for the *PIII-DC*. On the other hand, if the solution does not satisfy the limit, one can modify this solution to obtain a feasible solution for the *PIII-DC*. This may be a good heuristic approach to solve the problem the *PIII-DC*; however, this may not guarantee an optimal solution.

Therefore, since the structural properties of the *PIII-DC* are similar to those of the *PIII-SC*, we develop a GBD algorithm for the *PIII-DC*. This algorithm follows the same steps as the GBD for the *PIII-SC* except the solution of the subproblem. We discuss the formulation and solution of the subproblem next as well as the formulations of the master problem (Master-P-DC) and the relaxed master problem (RM-P-DC).

### The Subproblem

Given  $\mathbf{X}$  and  $\mathbf{Y}$ , the problem reduces to finding a solution to a multi-retailer EOQ problem with dispatch limitations for each open DC  $j$ ,  $j \in \mathcal{J}$ . That is, the subproblem at any iteration  $k$

$$v(\mathbf{X}^k, \mathbf{Y}^k) = \min_{\mathbf{Q}} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \frac{(p_{ij} + r_{ij}d_{ij})D_i Y_{ij}^k}{Q_i} + \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}$$

subject to

$$\sum_{i \in \mathcal{I}} \frac{D_i}{Q_i} Y_{ij}^k \leq R_j X_j^k, \quad \forall j \in \mathcal{J}, \quad (5.82)$$

$$Q_i \geq 0, \quad \forall i \in \mathcal{I}, \quad (5.83)$$

is decomposable for each  $j \in \mathcal{J}$ , since each retailer can only be assigned to one DC as in Section V.3.2.2. Let  $\mathcal{I}_j = \{i \in \mathcal{I} : Y_{ij} = 1\}$ ,  $\forall j \in \mathcal{J}$ . Then, the solution of the subproblem is obtained by solving the following problem for all  $j \in \mathcal{J}$ :

$$v_j(X_j^k, \mathbf{Y}^k) = \min_{\mathbf{Q}} \sum_{i \in \mathcal{I}_j} \left\{ \frac{(p_{ij} + r_{ij}d_{ij})D_i Y_{ij}^k}{Q_i} + \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\}$$

subject to

$$\sum_{i \in \mathcal{I}_j} \frac{D_i}{Q_i} Y_{ij}^k \leq R_j X_j^k. \quad (5.84)$$

$$Q_i \geq 0, \quad \forall i \in \mathcal{I}_j. \quad (5.85)$$

For each  $j \in \mathcal{J}$ , the problem  $v_j(X_j^k, \mathbf{Y}^k)$  is solved via the approach in Hadley and

Whitin (1963)[p. 56-57]. To find the optimal  $Q_i^*$ ,  $\forall i \in \mathcal{I}_j$ , the Lagrange multiplier technique is used. In particular, for each  $j \in \mathcal{J}$ , we form the following Lagrange function:

$$\mathfrak{L}_j = \sum_{i \in \mathcal{I}_j} \left\{ \frac{(p_{ij} + r_{ij}d_{ij})D_i Y_{ij}^k}{Q_i} + \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right\} + \mu_j \left( \sum_{i \in \mathcal{I}_j} \frac{D_i}{Q_i} Y_{ij}^k - R_j X_j^k \right), \quad (5.86)$$

where  $\mu_j$  is the Lagrange multiplier, and  $\mu_j \geq 0$ ,  $j \in \mathcal{J}$ . For all  $j \in \mathcal{J}$ , the optimal  $\mu_j^*$  and  $Q_i^*$ , for  $i \in \mathcal{I}_j$ , are obtained from the well known KKT conditions (Bertsekas, 2004):

$$\frac{\partial \mathfrak{L}_j}{\partial \mu_j} = 0 = \sum_{i \in \mathcal{I}_j} \frac{D_i}{Q_i} - R_j. \quad (5.87)$$

$$\frac{\partial \mathfrak{L}_j}{\partial Q_i} = 0 = \frac{-(p_{ij} + r_{ij}d_{ij})D_i}{Q_i^2} - \frac{K_i D_i}{Q_i^2} + \frac{1}{2} h'_i - \frac{\mu_j D_i}{Q_i^2}, \quad i \in \mathcal{I}_j. \quad (5.88)$$

$$Q_i \geq 0, \quad i \in \mathcal{I}_j. \quad (5.89)$$

$$\mu_j \geq 0. \quad (5.90)$$

From (5.88), the optimal order quantity is given as

$$Q_i^* = \sqrt{\frac{2D_i[p_{ij} + r_{ij}d_{ij} + K_i + \mu_j^*]}{h_i}}, \quad i \in \mathcal{I}_j, j \in \mathcal{J}. \quad (5.91)$$

By substituting (5.91) in (5.87) for  $j \in \mathcal{J}$ , we obtain

$$W_j(\mu_j^*) = \sum_{i \in \mathcal{I}_j} \frac{\sqrt{D_i h_i}}{\sqrt{K_i + p_{ij} + r_{ij}d_{ij} + \mu_j^*}} - R_j.$$

There is a unique  $\mu_j^* \geq 0$  such that  $W_j(\mu_j^*) = 0$  since

- $W_j(\mu_j^*)$  is a decreasing function of  $\mu_j^*$ , and
- as  $\mu_j^* \rightarrow \infty$ ,  $W_j(\mu_j^*) \rightarrow -R_j$ .

This unique  $\mu_j^*$  is found using a bisection algorithm.

Using the solution of the subproblem, we form the following cut:

$$\begin{aligned}
\mathfrak{L}(\mathbf{X}^k, \mathbf{Y}^k, \mathbf{Q}^k, \mu^k) &= \sum_{j \in \mathcal{J}} \mathfrak{L}_j \\
&= \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \frac{(p_{ij} + r_{ij}d_{ij})D_i Y_{ij}^k}{Q_i^k} + \sum_{i \in \mathcal{I}} \frac{K_i D_i}{Q_i^k} + \frac{1}{2} h'_i Q_i^k \\
&\quad + \sum_{j \in \mathcal{J}} \mu_j^k \left( \sum_{i \in \mathcal{I}} \frac{D_i}{Q_i^k} Y_{ij}^k - R_j X_j^k \right) \\
&= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \alpha_{ij}^k Y_{ij}^k + \sum_{j \in \mathcal{J}} \beta_j^k X_j^k + \gamma^k, \tag{5.92}
\end{aligned}$$

where

$$Q_i^k = Q_i^*, \quad i \in \mathcal{I}. \tag{5.93}$$

$$\mu_j^k = \mu_j^*, \quad j \in \mathcal{J}. \tag{5.94}$$

$$\alpha_{ij} = \frac{(p_{ij} + r_{ij}d_{ij})D_i}{Q_i^k} + \mu_j^k \frac{D_i}{Q_i^k}, \quad i \in \mathcal{I} \text{ and } j \in \mathcal{J}. \tag{5.95}$$

$$\beta_j = -\mu_j^k R_j, \quad j \in \mathcal{J}. \tag{5.96}$$

$$\gamma = \sum_{i \in \mathcal{I}} \frac{K_i D_i}{Q_i^k} + \frac{1}{2} h'_i Q_i^k. \tag{5.97}$$

If the dispatch capacity constraint (5.84) is not tight,  $\mu_j^k = 0$ , and, consequently,  $\beta_j = 0$ .

### The Master Problem

After discussing the subproblem and its solution, we are ready to present the formulation of the master problem for the *PIII-DC*. Again, by introducing a new

variable  $\eta$ , we formulate the master problem as follows:

$$\text{Min}_{\mathbf{x}, \mathbf{y}, \eta} \quad \sum_{j \in \mathcal{J}} f_j X_j + \eta \quad (\text{Master-P-DC})$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad i \in \mathcal{I}. \quad (5.98)$$

$$Y_{ij} \leq X_j, \quad i \in \mathcal{I} \text{ and } j \in \mathcal{J}. \quad (5.99)$$

$$\eta \geq \min_{\mathbf{Q}} \left\{ \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \frac{(p_{ij} + r_{ij} d_{ij}) D_i Y_{ij}}{Q_i} + \sum_{i \in \mathcal{I}} \left( \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i \right) \right. \\ \left. \sum_{j \in \mathcal{J}} \mu_j \left( \sum_{i \in \mathcal{I}} \frac{D_i}{Q_i} Y_{ij} - R_j X_j \right) \right\}, \quad \forall \mu \geq 0. \quad (5.100)$$

This formulation of the master problem is equivalent to the original formulation of the *PIII-DC*. It involves, however, an infinite number of constraints, and hence, we consider a relaxation of the master problem where we drop constraints (5.100) and add a number of them as needed. This reduced formulation is called the relaxed master problem (RM-P-DC) and is given by

$$\text{Min}_{\mathbf{x}, \mathbf{y}, \eta} \quad \sum_{j \in \mathcal{J}} f_j X_j + \eta \quad (\text{RM-P-SC})$$

subject to

$$(5.98), (5.99),$$

$$\eta \geq \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \alpha_{ij}^k Y_{ij} + \sum_{j \in \mathcal{J}} \beta_j^k X_j + \gamma^k, \quad \text{for } k = 1, \dots, K. \quad (5.101)$$

In this formulation, the cuts (5.101) follow the same form as in (5.92). Furthermore, they are added at each iteration as explained earlier. In (5.101),  $K$  refers to the number of cuts generated for the subproblems. (RM-P-DC) is still a difficult mixed

integer programming formulation. We solve (RM-P-DC) at each iteration of the GBD algorithm using CPLEX 9.0.

The overall procedure of the GBD is the same as in Section V.3.2.2. We present the computational results regarding the performance of the algorithm in Section V.3.3.

#### V.3.2.4. Analysis of *PIII-TC*

In this section, we analyze a generalization of the *PIII* with installable truck capacity. Using the notation defined in Section V.1, the formulation of the *PIII-TC* is as follows:

$$\text{Min } \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left\{ \frac{(p_{ij} + r_{ij} d_{ij}) \left\lceil \frac{Q_i}{C_T} \right\rceil D_i Y_{ij}}{Q_i} \right\} + \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h_i Q_i \right\}$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad \forall i \in \mathcal{I}. \quad (5.102)$$

$$Y_{ij} \leq X_j, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.103)$$

$$X_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (5.104)$$

$$Y_{ij} \in \{0, 1\}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (5.105)$$

$$Q_i \geq 0, \quad \forall i \in \mathcal{I}. \quad (5.106)$$

Observe that in this case the nonlinearity is in only the objective function. As we will show next, we can eliminate this nonlinearity by converting the *PIII-TC* to the classical uncapacitated FCFLP, and, hence, the *PIII-TC* can be solved via the techniques developed for the uncapacitated FCFLP (see Section II.1.2).

First, note that, if  $\mathbf{X}$  and  $\mathbf{Y}$  are known, then the remaining problem is a multi-retailer EOQ-model with a generalized replenishment cost structure. Let  $\mathcal{I}_j = \{i \in$

$\mathcal{I} : Y_{ij} = 1\}$  for a given  $j \in \mathcal{J}$ . Then, for each  $j \in \mathcal{J}$  and each  $i \in \mathcal{I}_j$ , we solve the following EOQ problem with a generalized replenishment cost structure:

$$g(Q_i) = \frac{(K_i + (p_{ij} + r_{ij}d_{ij}) \left\lceil \frac{Q_i}{C_T} \right\rceil) D_i}{Q_i} + \frac{1}{2} h'_i Q_i. \quad (5.107)$$

For a single-retailer EOQ model with similar generalized replenishment costs, Toptal et al. (2003) provide an algorithm to find the optimal order quantity. Since our problem is decomposable for each retailer, we can utilize the algorithm by Toptal et al. (2003) to find the optimal order quantity for each retailer. Here, we reiterate the algorithm for our problem in (5.107) for the sake of completeness.

**Algorithm for finding  $Q^*$**

For retailer  $i \in \mathcal{I}_j$  and  $j \in \mathcal{J}$ :

- 1: Compute  $Q_i^{EOQ} = \sqrt{2K_i D_i / h'_i}$ .
- 2: Let  $N$  denote the integer such that  $NC_T < Q_i^{EOQ} \leq (N+1)C_T$ . Compute

$$Q_i^{N+1} = \sqrt{\frac{2D_i [K_i + (N+1)(p_{ij} + r_{ij}d_{ij})]}{h'_i}}.$$

If  $Q_i^{N+1} \geq (N+1)C_T$ , then go to Step 3. Otherwise, go to Step 4.

- 3:  $Q_i^* = \arg \min\{g(NC_T), g((N+1)C_T)\}$ . Stop.

- 4:  $Q_i^* = \arg \min\{g(NC_T), g(Q_i^{N+1})\}$ . Stop.

Observe that the optimal  $Q_i^*$  generated by this algorithm is given by

$$\arg \min\{g(Q_i^{N+1}), g(NC_T), g((N+1)C_T)\}.$$

Hence, for each  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ , we can obtain the order quantities of the

retailers via this algorithm. Let

$$c_{ij} = \frac{(K_i + (p_{ij} + r_{ij}d_{ij}) \lceil \frac{Q_i^*}{C_T} \rceil) D_i}{Q_i^*} + \frac{1}{2} h'_i Q_i^*.$$

Then, we can restate the *PIII-TC* as follows:

$$\text{Min}_{\mathbf{x}, \mathbf{y}} \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} Y_{ij}$$

subject to

$$(5.102), (5.103), (5.104), \text{ and } (5.105),$$

and this formulation is equivalent to the formulation of the uncapacitated FCFLP.

Hence, the following corollary summarizes the desired result.

**COROLLARY 3** *The PIII-TC can be solved using the techniques developed for the uncapacitated FCFLP.*

**REMARK 3** *If the number of trucks at hand is also limited, then the above approach is no longer sufficient. The solution approaches for that problem remain a challenging problem for future research.*

### V.3.3. Numerical Results

In this section, we present numerical results for the problems *PIII-PC*, *PIII-SC*, and *DC*. Again, the computational results for the *PIII-PC* illustrate the impact of integrated decision-making. For the *PIII-PC*, we first describe the benchmark model that represents the typical practice. Next, we provide a numerical comparison the *PIII-PC* and the benchmark model. We not only quantify the benefits obtained from integrated decision-making, but we also identify the problem parameters that contribute the most to these benefits.



The computational results for the *PIII-SC* and the *PIII-DC* show the performance of the GBD algorithm with these problems. Since these models are only applicable when location and inventory decisions are made simultaneously, there is no fair benchmark with which to compare these models to measure the value of integrated decision-making.

All of the numerical results are obtained through algorithms implemented using C++ and run on a Pentium IV 3.2Ghz machine with 1 GB memory.

### V.3.3.1. Numerical Results Regarding the *PIII-PC*

In this section, we first introduce a benchmark model that is comparable to *PIII-PC*. We describe the benchmark model, its assumptions, formulation and solution in detail. Next, we perform two different types of computational experiments. In the first experiment, we investigate the impact of the network structure and the problem size on the integrated decision-making. In the second experiment, we perform a factorial design to identify the influential parameters affecting the impact of integrated decision-making.

#### Benchmark Model (*BMIII-PC*)

Similar to the benchmark model in Section V.2.6, the benchmark model (*BMIII-PC*) builds on the idea that location and inventory decisions are made *sequentially* and that facility location decisions precede inventory decisions. For this purpose, in the *BMIII-PC*, we first solve the following capacitated single-source FCFLP regarding the selection of DCs and assignment of DCs to the retailers:

$$\text{Min } \sum_{j \in \mathcal{J}} f_j X_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} Y_{ij} \quad (\text{BMIII-PC-Loc})$$

subject to

$$\begin{aligned}
\sum_{j \in \mathcal{J}} Y_{ij} &= 1, & \forall i \in \mathcal{I}. \\
Y_{ij} &\leq X_j, & \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \\
\sum_{i \in \mathcal{I}} D_i Y_{ij} &\leq P_j X_j, & \forall j \in \mathcal{J}. \\
X_j &\in \{0, 1\}, & \forall j \in \mathcal{J}. \\
Y_{ij} &\in \{0, 1\}, & \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}.
\end{aligned}$$

Similarly to Section V.2.6,  $c_{ij}$  represents an estimate for per-unit per-mile transportation cost. This formulation is the typical formulation of the single source capacitated FCFLP and can be solved via several different methods developed for the FCFLP (see Section II.1.2). In our computational tests, we solve this problem exactly using CPLEX 9.0 to obtain  $\mathbf{X}^{BM}$  and  $\mathbf{Y}^{BM}$ .

Next, given the assignment variables  $Y_{ij}^{BM}$  for all  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ , the inventory decisions of retailers are determined by solving the following problem:

$$\min_{\mathbf{Q}} \sum_{i \in \mathcal{I}} \left\{ \frac{K_i D_i}{Q_i} + \frac{1}{2} h'_i Q_i + \sum_{j \in \mathcal{J}} \frac{(p_{ij} + r_{ij} d_{ij}) D_i Y_{ij}^{BM}}{Q_i} \right\}. \quad (BMIII-PC-Inv)$$

The optimal order quantity of retailer  $i \in \mathcal{I}$  obtained from the solution of the *BMIII-PC-Inv* is

$$Q_i^{BM} = \sqrt{\frac{2[K_i + \sum_{j \in \mathcal{J}} (p_{ij} + r_{ij} d_{ij}) D_i Y_{ij}^{BM}]}{h'_i}} D_i, \quad i \in \mathcal{I}.$$

The cost of *BMIII-PC* ( $Z^{BM}$ ) is calculated using the decision variables ( $\mathbf{X}^{BM}$ ,  $\mathbf{Y}^{BM}$ ,  $\mathbf{Q}^{BM}$ ), obtained from the solutions of the *BMIII-Loc* and the *BMIII-Inv*, in (5.8). We then measure the value of the integrated framework by computing

$$\text{The percentage gain of the } PIII-PC \text{ over the } BMIII-PC (\%) = \frac{Z^{BM} - Z^{PIII-PC}}{Z^{PIII-PC}} \times 100,$$

where  $Z^{PIII-PC}$  is the cost of the *PIII-PC*.

### Test 1: Impact of $|\mathcal{I}|$ and $|\mathcal{J}|$

We first compare the performance of the *PIII-PC* and the *BMIII-PC* for 900 instances using data given in Table 13. As before, the main goal of this experiment is to measure the impact of integrated decision-making for different configurations of the distribution systems, i.e., for different numbers of retailers and potential DCs, under throughput capacity constraints.

We generate the throughput capacity randomly using the uniform distribution  $U[L_{cap}, U_{cap}]$  for each DC  $j \in \mathcal{J}$  where  $L_{cap} = 4 * \frac{\sum_{i \in \mathcal{I}} D_i}{|\mathcal{I}|}$  and  $U_{cap} = 16 * \frac{\sum_{i \in \mathcal{I}} D_i}{|\mathcal{I}|}$ . With this capacity restriction, each DC, on average, will be capable of serving a random number of retailers between 4 and 16.

We report the minimum, average, and maximum percentage gain of the *PIII-PC* over the *BMIII-PC* in Table 20 for different configurations of the distribution systems. We also compare the number of open DCs in both approaches and report the results. For each instance, the difference in the number of open DCs is calculated as  $\sum_{j \in \mathcal{J}} X_j^{BM} - \sum_{j \in \mathcal{J}} X_j^{PIII-PC}$  where  $\mathbf{X}^{PIII-PC}$  is obtained from the solution of the *PIII-PC*. In Table 20, we also present the minimum, average, and maximum difference in the number of open DCs for the *PIII-PC* and the *BMIII-PC*.

In all of the data sets in Table 20, we observe that significant gains are obtained with the *PIII-PC* over the *BMIII-PC*, with average gains more than 34.85% and maximum gains up to 61.42%. One interesting observation is that, for a given number of retailers, the average gain increases as the number of potential DCs increases. Similarly to the findings in Section V.2.6, integrated decision-making adds gains above sequential decision-making by the increment of the number of potential DC locations.

TABLE 20. Performance of the *PIII-PC* Compared to the Sequential Benchmark Model *BMIII-PC*.

Data set	$ \mathcal{I} $	$ \mathcal{J} $	Gain(%)			Open DCs		
			Min	Ave	Max	Min	Ave	Max
1	25	10	4.37	34.85	60.17	0	2.14	4
2	25	20	3.36	37.40	58.21	0	2.60	5
3	25	30	5.39	37.48	60.63	0	2.78	5
4	50	10	6.60	39.52	61.42	2	3.84	6
5	50	20	7.10	39.96	58.79	0	4.49	8
6	50	30	5.09	40.86	61.07	1	4.69	8
7	100	10	8.34	39.65	59.64	4	5.65	9
8	100	20	2.89	40.38	60.31	4	7.52	10
9	100	30	5.25	41.26	59.35	1	7.40	11

Similarly, for a given number of potential DCs, the average gain increases as the number of retailers increases. Hence, as the distribution network gets larger, it is more beneficial to consider the *PIII-PC*.

From the results in Table 20, we also observe that using the *PIII-PC* reduces the number of open DCs significantly. In the uncapacitated case *PIII*, there are instances where the number of open DCs in the *PIII* is larger than the number in the *BMIII* (see Section V.2.6). For the *PIII-PC*, this is no longer the case. For all of the instances in all data sets, the number of open DCs in the *PIII-PC* is less than the *BMIII-PC*. On average, this difference is at least two DCs for data sets with smaller networks (data sets 1, 2, and 3) and can be as high as seven DCs for data sets with larger networks (data sets 8 and 9). Due to the explicit consideration of transportation costs and the impact of inventory decisions on location decisions, the *PIII-PC* requires fewer number of open DCs.

## Test 2: Impact of Problem Parameters

In this experiment, to measure the impact of the problem parameters on in-

tegrated decision-making, we use the factorial design setting introduced in Section V.2.6 while comparing the *PIII-PC* and the *BMIII-PC*. The throughput capacity is generated in the same way as in Test 1. In Tables 29, 30, 31, and 32, we present the factorial experiment settings and the results for each setting including the minimum, average, and maximum percentage gains for 100 instances and the average and maximum differences in the number of open DCs for the two approaches.

The results we obtain follow the key findings for *PIII* as in Section V.2.6. Furthermore, due to the impact of the throughput capacity, the savings are more pronounced. We next summarize the other key points of our numerical study and establish the trade-offs among the problem parameters to determine under which settings the integrated decision-making has a higher impact.

- In Table 21, we provide the settings that result in 10 highest average percentage gains. The highest average gain is 42.57%, and maximum gain among these factorial designs is 63.61%. One of our main observations is that the ten highest average percentage gains are attained when demand and distance is obtained from uniform distributions at high levels. Hence, demand and distance appear to be the most influential parameters as in the comparison of the *BMIII* and the *PIII*.

Another important observation from Table 21 is related to the difference in the number of open DCs between the *PIII-PC* and the *BMIII-PC*. The average difference in the number of DCs can be up to 6 DCs and the maximum difference can go as high as 8 DCs. High demand and distance levels force the *BMIII-PC* to open more (unnecessary) DCs than the *PIII-PC*. This difference is more distinct due to the impact of throughput capacity.

- In Table 22, we provide the factorial design settings that result in 10 lowest av-

TABLE 21. Factorial Designs with the 10 Highest Average Gains for *PIII-PC*

F. D. No.	Levels of Parameters							Gain (%)			Open DCs		
	$D_i$	$K_i$	$h'_i$	$p_{ij}$	$r_{ij}$	$d_{ij}$	$f_j$	Min	Ave	Max	Min	Ave	Max
68	H	L	L	L	L	H	H	11.18	42.57	55.72	1	3.03	4
72	H	L	L	L	H	H	H	1.04	37.14	48.91	1	3.02	5
84	H	L	H	L	L	H	H	6.73	36.21	47.12	1	3.04	5
76	H	L	L	H	L	H	H	6.82	32.83	43.14	1	3.04	5
88	H	L	H	L	H	H	H	5.32	31.38	50.90	1	3.07	5
80	H	L	L	H	H	H	H	3.76	31.08	61.59	1	3.05	5
67	H	L	L	L	L	H	L	22.53	30.78	37.77	4	6.08	8
100	H	H	L	L	L	H	H	3.03	30.45	63.61	1	3.05	5
3	L	L	L	L	L	H	L	3.03	30.45	63.61	1	3.05	5
92	H	L	H	H	L	H	H	0.97	26.74	36.09	1	3.05	5

erage percentage gains. Similar to the results in Table 21, demand and distance appear as the most influential parameters. The lowest average gains occur when demand and distance are drawn from uniform distributions at low levels. The ten lowest average gains range between 0.51% and 1.18%. The maximum gains corresponding to these settings go up to 11.07% which is significant.

TABLE 22. Factorial Designs with the 10 Lowest Average Gains for *PIII-PC*

F. D. No.	Levels of Parameters							Gain (%)			Open DCs		
	$D_i$	$K_i$	$h'_i$	$p_{ij}$	$r_{ij}$	$d_{ij}$	$f_j$	Min	Ave	Max	Min	Ave	Max
62	L	H	H	H	H	L	H	0.00	0.51	5.25	0	0.00	0
58	L	H	H	H	L	L	H	0.00	0.52	2.80	0	0.00	0
54	L	H	H	L	H	L	H	0.00	0.62	3.38	0	0.00	0
50	L	H	H	L	L	L	H	0.00	0.76	6.39	0	0.00	0
46	L	H	L	H	H	L	H	0.00	0.81	11.07	0	0.00	0
34	L	H	L	L	L	L	H	0.00	0.84	4.42	0	0.00	0
22	L	L	H	L	H	L	H	0.00	0.94	4.72	0	0.00	0
14	L	L	L	H	H	L	H	0.00	0.94	5.81	0	0.00	0
18	L	L	H	L	L	L	H	0.00	1.13	9.96	0	0.00	0
6	L	L	L	L	H	L	H	0.00	1.18	6.02	0	0.00	0

As observed in Table 22, another important parameter is the fixed cost of open-

ing DCs. For the factorial designs with the ten lowest average gains, the fixed location costs are drawn from a uniform distribution at high levels. In all of the factorial designs in Table 22, the number of open DCs in the solutions of the *PIII-PC* and the *BMIII-PC* are the same. Hence, the average and maximum difference for the number of open DCs are zero for all of the instances in these ten factorial designs. High levels of fixed location costs, coupled with low levels of demand and distance, force both the *PIII-PC* and the *BM-III-PC* to open fewer DCs, often, only one DC.

- The impact of other parameters on integrated decision-making is similar to our earlier comparison of the *PIII* and the *BMIII*.

### V.3.3.2. Numerical Results Regarding the *PIII-SC*

In this section, we present computational results regarding the performance of the GBD algorithm. We test the *PIII-SC* for 900 instances generated using the data given in Table 13. For these instances the storage capacity  $S_j$  for each potential DC  $j \in \mathcal{J}$  is generated randomly using the uniform distribution  $U[SL_{cap}, SU_{cap}]$  for each DC  $j \in \mathcal{J}$  where  $L_{cap} = \frac{\sum_{i \in \mathcal{I}} D_i}{|\mathcal{I}|}$  and  $U_{cap} = 4 * \frac{\sum_{i \in \mathcal{I}} D_i}{|\mathcal{I}|}$ . With this storage capacity restriction, each DC, on average, will be capable of serving a random number of retailers from 1 to 4.

The main goal of this experiment is to

- demonstrate the quality of the solution generated by the GBD,
- present the time and number of iterations required to achieve that result, and
- compare the performance of the GBD with a generic mixed integer nonlinear programming solver.

To demonstrate the performance of the GBD, we record the percentage gap between the best upper and lower bounds returned by the algorithm, the number of cuts generated, and the duration of the algorithm. For each problem instance, the gap is defined as the percentage difference

$$\text{Gap(UB-LB) (\% )} = \frac{Z_{PIII-SC}^{UB} - Z_{PIII-SC}^{LB}}{Z_{PIII-SC}^{LB}} \times 100,$$

where  $Z_{PIII-SC}^{UB}$  is the cost of the best feasible solution suggested by the GBD and  $Z_{PIII-SC}^{LB}$  is the lower bound on the optimal solution of the *PIII-SC*.

Furthermore, for comparing the solution provided by the GBD algorithm with the generic methods, we utilize the MINLP algorithm on the NEOS server<sup>3</sup> (Czyzyk et al., 1998; Dolan, 2001; Gropp and More, 1997) for solving the *PIII-SC*. The MINLP implements a branch-and-bound algorithm searching a tree whose nodes correspond to continuous nonlinearly constrained optimization problems. The continuous problems are solved using filterSQP, a Sequential Quadratic Programming solver which is suitable for solving large nonlinearly constrained problems. For each problem instance, we define the gap between the solution returned by the MINLP and the lower bound on the optimal solution of the *PIII-SC* as the percentage difference

$$\text{Gap(Neos-LB) (\% )} = \frac{Z_{MINLP} - Z_{PIII-SC}^{LB}}{Z_{PIII-SC}^{LB}} \times 100.$$

In Table 23, for each data set, we provide the minimum, average, and maximum percentage gaps, number of cuts, and duration of the 100 instances. The first portion of Table 23 summarizes the findings regarding the percentage gaps. The average gap between the upper bound and lower bound varies between 1.05% and 2.79%. The maximum gap can go as high as 9% for larger data sets 8 and 9. On the other hand,

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<sup>3</sup><http://www-neos.mcs.anl.gov>



TABLE 23. Results of the GBD for *PIII-SC*

DS	GAP (UB-LB)(%)			GAP(Neos-LB)(%)			Cuts			Duration (s.)		
	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max
1	0.01	2.38	7.28	0.08	8.83	27.98	3	9.17	17	0.16	1.29	7.16
2	0.01	2.79	7.38	0.10	8.92	29.91	3	10.17	18	0.31	3.62	13.53
3	0.05	2.49	7.01	0.32	10.47	28.63	3	10.90	19	0.67	6.76	25.16
4	0.00	2.27	7.53	0.00	7.08	20.28	5	12.86	25	0.36	8.39	34.36
5	0.00	1.87	7.09	0.31	9.39	23.65	4	12.65	30	1.94	23.85	90.09
6	0.00	1.92	7.19	0.25	9.99	26.91	4	13.00	27	2.20	37.52	179.37
7	0.00	1.16	6.93	0.05	9.24	34.88	5	15.82	39	2.06	77.49	19675.57
8	0.00	1.05	8.91	-	-	-	5	15.14	39	8.59	500.70	19663.30
9	0.00	1.77	9.16	-	-	-	5	13.94	72	9.69	741.59	21062.50

the generic MINLP algorithm has average gaps around 9%, and maximum gaps up to 34.88%. Furthermore, the generic MINLP cannot solve instances in data set 8 and 9. On the other hand, the GBD is quite effective in solving the *PIII-SC*.

In terms of speed, for smaller distribution systems (data sets 1, 2, 3, and 4), the GBD algorithm is quite fast with an average duration of less than 10 seconds and a maximum duration up to 34.36 seconds. For data sets 8 and 9, the duration is higher with average times of 500.70 and 741.59 seconds, respectively. The maximum duration for these data sets can be as high as six hours although the number of cuts added only up to seventy-two. Hence, the main driver behind this duration of the GBD method is the solution of the relaxed master problem which is currently solved by CPLEX. As a future research direction, an alternative, and possibly faster, solution approach can be developed to improve the current GBD implementation. Although the maximum durations for data sets 8 and 9 are higher, NEOS could not provide a solution for these data sets. Note that we do not report and compare the duration of the solution provided by NEOS for two reasons. First, it is not possible to obtain the duration of the algorithms in NEOS. Secondly, even if it were possible, it would not be fair to compare these durations since they are run on different computer configurations.

TABLE 24. Results of the GBD for *PIII-DC*

DS	GAP (UB-LB)(%)			GAP(Neos-LB)(%)			Cuts			Duration (s.)		
	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max
1	0.00	0.15	3.48	0.00	0.46	7.80	1	2.18	8	0.05	0.16	1.20
2	0.00	0.33	5.84	0.00	0.54	5.84	1	2.17	8	0.06	0.27	2.81
3	0.00	0.69	6.37	0.00	1.06	9.08	1	2.09	9	0.09	0.39	3.30
4	0.00	0.25	7.33	0.00	0.28	7.33	1	2.87	11	0.06	0.89	6.70
5	0.00	0.61	8.16	0.00	0.69	8.16	1	2.80	12	0.09	2.55	48.77
6	0.00	0.46	8.99	0.00	0.50	8.99	1	3.16	10	0.16	4.11	28.92
7	0.00	0.42	5.29	0.00	0.43	5.30	1	3.35	15	0.09	3.99	65.50
8	0.00	0.50	6.43	0.00	0.50	6.43	1	3.47	18	0.20	18.41	425.20
9	0.00	0.51	6.72	0.00	0.51	6.72	1	3.63	12	0.34	438.84	17598.50

### V.3.3.3. Numerical Results Regarding the *PIII-DC*

In this section, we present computational results regarding the performance of the GBD algorithm for the *PIII-DC*. Again, we report results for 900 instances generated using the data given in Table 13. For these instances, the dispatch capacity  $R_j$  for each potential DC  $j \in \mathcal{J}$  is generated randomly using the uniform distribution  $U[10, 50]$  for each DC  $j \in \mathcal{J}$ . With this dispatch capacity restriction, each DC, on average, will be capable of filling at least 10 and at most 50 orders per year.

With this experiment, which is similar to the experiment for the *PIII-SC*, our aim is to demonstrate the quality of the solution generated by the GBD by presenting the percentage gaps between the upper bound and the lower bound, the duration of the algorithm, and the number of iterations required to achieve the result. We also compare the performance of the GBD with generic mixed-integer nonlinear programming solvers.

In Table 24, for each data set, we provide the minimum, average, and maximum percentage gaps, number of cuts, and duration of the 100 instances. The first portion of Table 24 summarizes the findings regarding the percentage gap. The average gap between the upper bound and lower bound varies between 0.15% and 0.69%, which shows the effectiveness of the GBD in solving the *PIII-DC*. However, the maximum

gap can be as high as 9%. For the *PIII-DC*, the performance of the generic MINLP algorithm is quite close to the performance of the GBD. For data instances in data sets 5 to 9, the best feasible solutions obtained by GBD are the same, or slightly better than, the solutions obtained by the MINLP algorithm.

In terms of the duration of the algorithm, the GBD algorithm is quite fast for data sets 1 to 7 with an average duration of less than 5 seconds and a maximum duration of up to 65.50 seconds. For data set 8, the duration is slightly higher with an average and a maximum time of 18.41 and 425.20 seconds, respectively. Data set 9 is the only data set that takes significant time with an average and maximum time of 438.84 seconds and approximately 5 hours, respectively. The average and maximum number of cuts for all of the data sets are less than 4 and 20 cuts, respectively. Hence, the main driver behind the duration of the algorithm is the solution of the relaxed master problem, which we are currently solving with CPLEX. As a future research direction, an alternative and possibly, faster solution -probably an heuristic- approach can be developed to improve the current GBD implementation for the *PIII-SC* and *PIII-DC*.

#### **V.4. Summary and Conclusions**

In this chapter, we generalize the classical FCFLP problem to consider inventory decisions at the retailers. In particular, we consider establishing a number of DCs from a candidate set to serve geographically dispersed retailers with stationary and deterministic demand. Each retailer operates under the assumptions of the EOQ model. In order to exploit the complex trade-offs in distribution system design, our goal is to minimize the total costs in the system including inventory holding and ordering costs, direct transportation costs, and fixed facility (DC) location costs.

We classify this problem with respect to the capacity restriction. When the potential DCs do not have a capacity restriction, the problem  $P_{III}$  is a generalization of the uncapacitated FCFLP. For the  $P_{III}$ , we show that the problem can be converted to an equivalent uncapacitated FCFLP, and, hence, can be solved via the techniques developed for the FCFLP. To quantify the benefits from integrated decision-making, we compare the solution of the  $P_{III}$  with a benchmark model where the facility location decisions precede the inventory decisions. From our computational experiments, we conclude that there are, on average, 25% to 30% savings with the integrated approach over the benchmark model. Furthermore, the demand of retailers, distance between the retailers and DCs, and fixed costs of the DCs emerge as the parameters that have the most influence on the savings. For the problem instances with higher demands and farther distances, it is more beneficial to utilize the integrated location-inventory model in order to efficiently exploit the trade-offs in the distribution system, and, hence, realize more cost savings. For problem instances with lower demands and closer distances, the fixed costs of the DCs appear as an influential parameter due to the classical trade-off among the fixed facility location costs and the *implied* transportation costs.

For the purpose of studying capacity restricted extensions of the  $P_{III}$ , we first discuss capacity considerations in real-life distribution systems. We consider four different types of capacity restrictions: throughput capacity restrictions, storage capacity restrictions, and dispatch capacity restrictions at the DCs, and the truck/cargo capacity restrictions on the transportation links. We call the corresponding problems the  $P_{III-PC}$ ,  $P_{III-SC}$ ,  $P_{III-DC}$ , and  $P_{III-TC}$ , respectively. Under these different capacity restrictions, we revise the model  $P_{III}$  and its solution approaches.

We show that the  $P_{III-PC}$  can be converted to the classical capacitated FCFLP, and, hence, can be solved via the techniques developed for the capacitated FCFLP.

We compare the *PIII-PC* with a benchmark that relies on sequential decision-making to quantify the benefits associated with integrated decision-making. In our computational tests, we observe that due to the impact of capacity restriction, the savings with the *PIII-PC* rise to, on average, 65%-70% savings over the benchmark model. As in the *PIII*, the demand of retailers, distance between the retailers and DCs, and fixed costs of the DCs emerge as the most influential parameters affecting savings.

The *PIII-SC*, *PIII* with storage capacities, and the *PIII-DC*, *PIII* with dispatch capacities, include the order quantities of retailers in capacity constraints which make the problems more challenging. For these problems, we develop GBD-based algorithms by exploiting the structure of the problems. Compared to the generic MINLP solver, the algorithms provide good quality solutions with short durations.

The *PIII-TC* is the *PIII* with installable truck capacities. In the *PIII-TC*, we take into account the truck/cargo costs in the objective function. We show that the *PIII-TC* can be converted to an equivalent uncapacitated FCFLP problem and, hence, can be solved via the techniques developed for the uncapacitated FCFLP.

In conclusion, the contributions of this chapter can be summarized as follows:

- developing integrated location and inventory models generalizing the uncapacitated and capacitated FCFLP,
- analyzing these models and developing efficient solution approaches,
- identifying the conditions where integrated decision-making is beneficial, and
- quantifying the benefits from integrated decision-making.

In future research, the results of this chapter can be extended in many ways. One noteworthy extension would be to consider a three-stage distribution system with inventory considerations at both the DC and retailer levels. The coordination

of inventory issues raised in the three-stage continuous facility location problem in Chapter IV are also relevant for this problem setting. We analyze this particular problem in the next chapter. Other areas for future research might include

- improving the solution mechanism developed for the *PIII-SC* and the *PIII-DC*,
- hard truck capacities for the *PIII-TC*,
- including several capacity considerations in the same model, for instance, considering dispatch and storage capacities simultaneously, and
- the impact of uncertainty of demand on the models, solution approaches, and results.

TABLE 25. Results of Experiments with Factorial Design Settings for *PIII*: Part 1

F. D.		RANGES of PARAMETERS												GAIN (%)			OPEN DCs		
No.	$D_i$	$K_i$		$h'_i$		$p_{ij}$		$r_{ij}$		$d_{ij}$		$f_j$		Min	Ave	Max	Ave	Max	
1	315	385	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	30	45	45000	55000	0	0.93	3.50	0.00	0
2	315	385	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	0.56	4.15	0.00	0
3	315	385	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	120	180	45000	55000	0.79	10.83	21.98	0.80	2
4	315	385	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	120	180	180000	220000	0.00	1.80	7.18	0.00	0
5	315	385	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	30	45	45000	55000	0.00	0.70	3.04	0.00	0
6	315	385	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	0.45	3.58	0.00	0
7	315	385	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	120	180	45000	55000	1.27	8.59	17.66	0.80	2
8	315	385	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	120	180	180000	220000	0.00	1.29	5.90	0.00	0
9	315	385	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	30	45	45000	55000	0.00	0.80	2.96	0.00	0
10	315	385	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	30	45	180000	220000	0.00	0.45	3.41	0.00	0
11	315	385	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	120	180	45000	55000	0.33	7.40	14.90	0.80	2
12	315	385	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	120	180	180000	220000	0.00	1.39	5.14	0.00	0
13	315	385	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	30	45	45000	55000	0.00	0.71	2.81	0.00	0
14	315	385	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.41	3.20	0.00	0
15	315	385	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	120	180	45000	55000	0.11	6.30	13.35	0.80	2
16	315	385	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	120	180	180000	220000	0.00	1.15	4.65	0.00	0
17	315	385	67.5	82.5	9	11	427.5	522.5	0.675	0.825	30	45	45000	55000	0.00	0.79	3.09	0.00	0
18	315	385	67.5	82.5	9	11	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	0.47	3.62	0.00	0
19	315	385	67.5	82.5	9	11	427.5	522.5	0.675	0.825	120	180	45000	55000	0.72	8.50	17.07	0.80	2
20	315	385	67.5	82.5	9	11	427.5	522.5	0.675	0.825	120	180	180000	220000	0.00	1.44	5.77	0.00	0
21	315	385	67.5	82.5	9	11	427.5	522.5	2.7	3.3	30	45	45000	55000	0.00	0.56	2.64	0.00	0
22	315	385	67.5	82.5	9	11	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	0.36	3.01	0.00	0
23	315	385	67.5	82.5	9	11	427.5	522.5	2.7	3.3	120	180	45000	55000	0.42	6.09	13.09	0.80	2
24	315	385	67.5	82.5	9	11	427.5	522.5	2.7	3.3	120	180	180000	220000	0.00	0.94	4.55	0.00	0
25	315	385	67.5	82.5	9	11	1530	1870	0.675	0.825	30	45	45000	55000	0.00	0.75	2.71	0.00	0
26	315	385	67.5	82.5	9	11	1530	1870	0.675	0.825	30	45	180000	220000	0.00	0.41	3.01	0.00	0
27	315	385	67.5	82.5	9	11	1530	1870	0.675	0.825	120	180	45000	55000	0.24	5.60	11.30	0.80	2
28	315	385	67.5	82.5	9	11	1530	1870	0.675	0.825	120	180	180000	220000	0.00	1.14	4.08	0.00	0
29	315	385	67.5	82.5	9	11	1530	1870	2.7	3.3	30	45	45000	55000	0.00	0.66	2.56	0.00	0
30	315	385	67.5	82.5	9	11	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.37	2.79	0.00	0
31	315	385	67.5	82.5	9	11	1530	1870	2.7	3.3	120	180	45000	55000	0.76	4.68	9.88	0.80	2
32	315	385	67.5	82.5	9	11	1530	1870	2.7	3.3	120	180	180000	220000	0.00	0.89	3.58	0.00	0

TABLE 26. Results of Experiments with Factorial Design Settings for *PIII*: Part 2

F. D.		RANGES of PARAMETERS												GAIN (%)			OPEN DCs		
No.	$D_i$	$K_i$			$h'_i$		$p_{ij}$		$r_{ij}$		$d_{ij}$		$f_j$		Min	Ave	Max	Ave	Max
33	315	385	270	330	4.5	5.5	427.5	522.5	0.675	0.825	30	45	45000	55000	0.00	0.71	2.78	0.00	0
34	315	385	270	330	4.5	5.5	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	0.41	3.15	0.00	0
35	315	385	270	330	4.5	5.5	427.5	522.5	0.675	0.825	120	180	45000	55000	0.34	6.20	12.92	0.80	2
36	315	385	270	330	4.5	5.5	427.5	522.5	0.675	0.825	120	180	180000	220000	0.00	1.13	4.52	0.00	0
37	315	385	270	330	4.5	5.5	427.5	522.5	2.7	3.3	30	45	45000	55000	0.00	0.48	2.34	0.00	0
38	315	385	270	330	4.5	5.5	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	0.30	2.51	0.00	0
39	315	385	270	330	4.5	5.5	427.5	522.5	2.7	3.3	120	180	45000	55000	0.62	4.15	9.34	0.80	2
40	315	385	270	330	4.5	5.5	427.5	522.5	2.7	3.3	120	180	180000	220000	0.00	0.65	3.38	0.00	0
41	315	385	270	330	4.5	5.5	1530	1870	0.675	0.825	30	45	45000	55000	0.00	0.72	2.53	0.00	0
42	315	385	270	330	4.5	5.5	1530	1870	0.675	0.825	30	45	180000	220000	0.00	0.40	2.69	0.00	0
43	315	385	270	330	4.5	5.5	1530	1870	0.675	0.825	120	180	45000	55000	0.39	4.13	8.45	0.80	2
44	315	385	270	330	4.5	5.5	1530	1870	0.675	0.825	120	180	180000	220000	0.00	0.94	3.55	0.00	0
45	315	385	270	330	4.5	5.5	1530	1870	2.7	3.3	30	45	45000	55000	0.00	0.63	2.38	0.00	0
46	315	385	270	330	4.5	5.5	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.35	2.46	0.00	0
47	315	385	270	330	4.5	5.5	1530	1870	2.7	3.3	120	180	45000	55000	0.13	3.39	7.15	0.80	2
48	315	385	270	330	4.5	5.5	1530	1870	2.7	3.3	120	180	180000	220000	0.00	0.69	3.06	0.00	0
49	315	385	270	330	9	11	427.5	522.5	0.675	0.825	30	45	45000	55000	0.00	0.66	2.55	0.00	0
50	315	385	270	330	9	11	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	0.37	2.76	0.00	0
51	315	385	270	330	9	11	427.5	522.5	0.675	0.825	120	180	45000	55000	0.71	4.54	9.57	0.80	2
52	315	385	270	330	9	11	427.5	522.5	0.675	0.825	120	180	180000	220000	0.00	0.88	3.55	0.00	0
53	315	385	270	330	9	11	427.5	522.5	2.7	3.3	30	45	45000	55000	0.00	0.43	2.11	0.00	0
54	315	385	270	330	9	11	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	0.25	2.08	0.00	0
55	315	385	270	330	9	11	427.5	522.5	2.7	3.3	120	180	45000	55000	0.37	3.01	6.36	0.80	2
56	315	385	270	330	9	11	427.5	522.5	2.7	3.3	120	180	180000	220000	0.00	0.45	2.44	0.00	0
57	315	385	270	330	9	11	1530	1870	0.675	0.825	30	45	45000	55000	0.00	0.72	2.40	0.00	0
58	315	385	270	330	9	11	1530	1870	0.675	0.825	30	45	180000	220000	0.00	0.40	2.43	0.00	0
59	315	385	270	330	9	11	1530	1870	0.675	0.825	120	180	45000	55000	0.44	3.33	6.26	0.80	2
60	315	385	270	330	9	11	1530	1870	0.675	0.825	120	180	180000	220000	0.00	0.80	3.13	0.00	0
61	315	385	270	330	9	11	1530	1870	2.7	3.3	30	45	45000	55000	0.00	0.63	2.25	0.00	0
62	315	385	270	330	9	11	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.35	2.20	0.00	0
63	315	385	270	330	9	11	1530	1870	2.7	3.3	120	180	45000	55000	1.80	2.66	5.07	0.80	2
64	315	385	270	330	9	11	1530	1870	2.7	3.3	120	180	180000	220000	0.00	0.55	2.64	0.00	0



TABLE 27. Results of Experiments with Factorial Design Settings for *PIII*: Part 3

F. D. No.	RANGES of PARAMETERS														GAIN (%)			OPEN DCs	
	$D_i$		$K_i$		$h'_i$		$p_{ij}$		$r_{ij}$		$d_{ij}$		$f_j$		Min	Ave	Max	Ave	Max
65	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	30	45	45000	55000	0.06	7.08	14.32	0.80	2
66	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	1.36	4.96	0.00	0
67	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	120	180	45000	55000	6.26	19.01	26.88	3.10	5
68	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	120	180	180000	220000	1.46	18.86	33.44	0.80	2
69	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	30	45	45000	55000	0.34	6.20	12.92	0.80	2
70	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	1.13	4.52	0.00	0
71	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	120	180	45000	55000	1.38	14.44	21.17	3.10	5
72	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	120	180	180000	220000	4.08	15.19	28.84	0.80	2
73	1260	1540	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	30	45	45000	55000	0.74	4.45	8.84	0.80	2
74	1260	1540	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	30	45	180000	220000	0.00	1.03	3.70	0.00	0
75	1260	1540	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	120	180	45000	55000	1.04	12.66	18.10	3.10	5
76	1260	1540	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	120	180	180000	220000	1.26	12.52	24.38	0.80	2
77	1260	1540	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	30	45	45000	55000	0.39	4.13	8.45	0.80	2
78	1260	1540	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.94	3.55	0.00	0
79	1260	1540	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	120	180	45000	55000	4.40	11.07	15.99	3.10	5
80	1260	1540	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	120	180	180000	220000	1.10	11.22	22.62	0.80	2
81	1260	1540	67.5	82.5	9	11	427.5	522.5	0.675	0.825	30	45	45000	55000	0.11	5.36	10.84	0.80	2
82	1260	1540	67.5	82.5	9	11	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	1.11	4.01	0.00	0
83	1260	1540	67.5	82.5	9	11	427.5	522.5	0.675	0.825	120	180	45000	55000	1.25	14.27	20.71	3.10	5
84	1260	1540	67.5	82.5	9	11	427.5	522.5	0.675	0.825	120	180	180000	220000	2.95	14.42	27.52	0.80	2
85	1260	1540	67.5	82.5	9	11	427.5	522.5	2.7	3.3	30	45	45000	55000	0.71	4.54	9.57	0.80	2
86	1260	1540	67.5	82.5	9	11	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	0.88	3.55	0.00	0
87	1260	1540	67.5	82.5	9	11	427.5	522.5	2.7	3.3	120	180	45000	55000	3.62	10.43	15.35	3.10	5
88	1260	1540	67.5	82.5	9	11	427.5	522.5	2.7	3.3	120	180	180000	220000	1.36	11.27	22.97	0.80	2
89	1260	1540	67.5	82.5	9	11	1530	1870	0.675	0.825	30	45	45000	55000	0.29	3.50	6.62	0.80	2
90	1260	1540	67.5	82.5	9	11	1530	1870	0.675	0.825	30	45	180000	220000	0.00	0.89	3.28	0.00	0
91	1260	1540	67.5	82.5	9	11	1530	1870	0.675	0.825	120	180	45000	55000	2.57	9.55	13.66	3.10	5
92	1260	1540	67.5	82.5	9	11	1530	1870	0.675	0.825	120	180	180000	220000	0.33	9.70	19.28	0.80	2
93	1260	1540	67.5	82.5	9	11	1530	1870	2.7	3.3	30	45	45000	55000	0.44	3.33	6.26	0.80	2
94	1260	1540	67.5	82.5	9	11	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.80	3.13	0.00	0
95	1260	1540	67.5	82.5	9	11	1530	1870	2.7	3.3	120	180	45000	55000	0.75	7.96	11.62	3.10	5
96	1260	1540	67.5	82.5	9	11	1530	1870	2.7	3.3	120	180	180000	220000	0.97	8.80	17.60	0.80	2

TABLE 28. Results of Experiments with Factorial Design Settings for *PIII*: Part 4

F. D.		RANGES of PARAMETERS												GAIN (%)			OPEN DCs		
No.	$D_i$	$K_i$		$h'_i$		$p_{ij}$		$r_{ij}$		$d_{ij}$		$f_j$		Min	Ave	Max	Ave	Max	
97	1260	1540	270	330	4.5	5.5	427.5	522.5	0.675	0.825	30	45	45000	55000	0.30	3.96	8.11	0.80	2
98	1260	1540	270	330	4.5	5.5	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	0.93	3.50	0.00	0
99	1260	1540	270	330	4.5	5.5	427.5	522.5	0.675	0.825	120	180	45000	55000	4.13	10.72	15.48	3.10	5
100	1260	1540	270	330	4.5	5.5	427.5	522.5	0.675	0.825	120	180	180000	220000	0.79	10.83	21.98	0.80	2
101	1260	1540	270	330	4.5	5.5	427.5	522.5	2.7	3.3	30	45	45000	55000	0.08	3.31	6.94	0.80	2
102	1260	1540	270	330	4.5	5.5	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	0.70	3.04	0.00	0
103	1260	1540	270	330	4.5	5.5	427.5	522.5	2.7	3.3	120	180	45000	55000	3.55	7.11	10.58	3.10	5
104	1260	1540	270	330	4.5	5.5	427.5	522.5	2.7	3.3	120	180	180000	220000	1.27	8.59	17.66	0.80	2
105	1260	1540	270	330	4.5	5.5	1530	1870	0.675	0.825	30	45	45000	55000	0.41	2.93	4.97	0.80	2
106	1260	1540	270	330	4.5	5.5	1530	1870	0.675	0.825	30	45	180000	220000	0.00	0.80	2.96	0.00	0
107	1260	1540	270	330	4.5	5.5	1530	1870	0.675	0.825	120	180	45000	55000	3.66	7.15	10.33	3.10	5
108	1260	1540	270	330	4.5	5.5	1530	1870	0.675	0.825	120	180	180000	220000	0.33	7.40	14.90	0.80	2
109	1260	1540	270	330	4.5	5.5	1530	1870	2.7	3.3	30	45	45000	55000	1.61	2.71	4.62	0.80	2
110	1260	1540	270	330	4.5	5.5	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.71	2.81	0.00	0
111	1260	1540	270	330	4.5	5.5	1530	1870	2.7	3.3	120	180	45000	55000	2.73	5.67	8.25	3.10	5
112	1260	1540	270	330	4.5	5.5	1530	1870	2.7	3.3	120	180	180000	220000	0.11	6.30	13.35	0.80	2
113	1260	1540	270	330	9	11	427.5	522.5	0.675	0.825	30	45	45000	55000	0.40	3.21	6.01	0.80	2
114	1260	1540	270	330	9	11	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	0.79	3.09	0.00	0
115	1260	1540	270	330	9	11	427.5	522.5	0.675	0.825	120	180	45000	55000	0.62	7.73	11.25	3.10	5
116	1260	1540	270	330	9	11	427.5	522.5	0.675	0.825	120	180	180000	220000	0.72	8.50	17.07	0.80	2
117	1260	1540	270	330	9	11	427.5	522.5	2.7	3.3	30	45	45000	55000	1.72	2.62	4.93	0.80	2
118	1260	1540	270	330	9	11	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	0.56	2.64	0.00	0
119	1260	1540	270	330	9	11	427.5	522.5	2.7	3.3	120	180	45000	55000	1.87	4.51	7.23	2.91	5
120	1260	1540	270	330	9	11	427.5	522.5	2.7	3.3	120	180	180000	220000	0.42	6.09	13.09	0.80	2
121	1260	1540	270	330	9	11	1530	1870	0.675	0.825	30	45	45000	55000	1.15	2.41	3.87	0.25	2
122	1260	1540	270	330	9	11	1530	1870	0.675	0.825	30	45	180000	220000	0.00	0.75	2.71	0.00	0
123	1260	1540	270	330	9	11	1530	1870	0.675	0.825	120	180	45000	55000	2.57	5.35	8.08	2.59	4
124	1260	1540	270	330	9	11	1530	1870	0.675	0.825	120	180	180000	220000	0.24	5.60	11.30	0.80	2
125	1260	1540	270	330	9	11	1530	1870	2.7	3.3	30	45	45000	55000	1.00	2.11	3.55	0.28	2
126	1260	1540	270	330	9	11	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.66	2.56	0.00	0
127	1260	1540	270	330	9	11	1530	1870	2.7	3.3	120	180	45000	55000	1.39	3.99	6.22	2.45	4
128	1260	1540	270	330	9	11	1530	1870	2.7	3.3	120	180	180000	220000	0.76	4.68	9.88	0.80	2

TABLE 29. Results of Experiments with Factorial Design Settings for *PIII-PC*: Part 1

F. D. No.	RANGES of PARAMETERS														GAIN (%)			OPEN DCs	
	$D_i$		$K_i$		$h'_i$		$p_{ij}$		$r_{ij}$		$d_{ij}$		$f_j$		Min	Ave	Max	Ave	Max
1	315	385	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	30	45	45000	55000	2.592079	11.50	23.96	0.79	2
2	315	385	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	1.23	6.59	0	0
3	315	385	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	120	180	45000	55000	3.03	30.45	63.61	3.05	5
4	315	385	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	120	180	180000	220000	9.97	26.58	43.70	0.78	2
5	315	385	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	30	45	45000	55000	0.77	10.31	22.54	0.80	2
6	315	385	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	1.18	6.02	0.00	0
7	315	385	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	120	180	45000	55000	15.97	24.89	34.20	2.95	5
8	315	385	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	120	180	180000	220000	2.13	22.94	40.92	0.79	2
9	315	385	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	30	45	45000	55000	0.74	7.19	15.83	0.78	2
10	315	385	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	30	45	180000	220000	0.00	1.19	19.56	0.00	0
11	315	385	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	120	180	45000	55000	13.11	21.24	29.34	3.01	5
12	315	385	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	120	180	180000	220000	2.23	19.60	36.37	0.79	2
13	315	385	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	30	45	45000	55000	0.44	6.78	15.24	0.79	2
14	315	385	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.94	5.81	0.00	0
15	315	385	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	120	180	45000	55000	11.52	19.46	29.96	3.02	5
16	315	385	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	120	180	180000	220000	0.02	18.25	34.36	0.81	2
17	315	385	67.5	82.5	9	11	427.5	522.5	0.675	0.825	30	45	45000	55000	0.06	9.20	18.95	0.80	2
18	315	385	67.5	82.5	9	11	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	1.13	9.96	0.00	0
19	315	385	67.5	82.5	9	11	427.5	522.5	0.675	0.825	120	180	45000	55000	15.97	25.01	50.92	3.05	5
20	315	385	67.5	82.5	9	11	427.5	522.5	0.675	0.825	120	180	180000	220000	4.70	21.83	37.91	0.76	2
21	315	385	67.5	82.5	9	11	427.5	522.5	2.7	3.3	30	45	45000	55000	0.14	7.67	15.65	0.76	2
22	315	385	67.5	82.5	9	11	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	0.94	4.72	0.00	0
23	315	385	67.5	82.5	9	11	427.5	522.5	2.7	3.3	120	180	45000	55000	12.02	19.26	27.05	3.02	5
24	315	385	67.5	82.5	9	11	427.5	522.5	2.7	3.3	120	180	180000	220000	0.77	18.38	34.70	0.80	2
25	315	385	67.5	82.5	9	11	1530	1870	0.675	0.825	30	45	45000	55000	0.05	5.55	12.21	0.78	2
26	315	385	67.5	82.5	9	11	1530	1870	0.675	0.825	30	45	180000	220000	0.00	1.55	51.10	0.01	1
27	315	385	67.5	82.5	9	11	1530	1870	0.675	0.825	120	180	45000	55000	10.20	16.83	29.94	3.06	5
28	315	385	67.5	82.5	9	11	1530	1870	0.675	0.825	120	180	180000	220000	4.28	15.79	30.42	0.80	2
29	315	385	67.5	82.5	9	11	1530	1870	2.7	3.3	30	45	45000	55000	0.73	5.35	15.42	0.78	2
30	315	385	67.5	82.5	9	11	1530	1870	2.7	3.3	30	45	180000	220000	0.00	1.67	58.71	0.01	1
31	315	385	67.5	82.5	9	11	1530	1870	2.7	3.3	120	180	45000	55000	8.78	14.71	24.05	3.01	5
32	315	385	67.5	82.5	9	11	1530	1870	2.7	3.3	120	180	180000	220000	2.34	13.92	28.32	0.80	2

TABLE 30. Results of Experiments with Factorial Design Settings for *PIII-PC*: Part 2

F. D. No.	RANGES of PARAMETERS														GAIN (%)			OPEN DCs	
	$D_i$		$K_i$		$h'_i$		$p_{ij}$		$r_{ij}$		$d_{ij}$		$f_j$		Min	Ave	Max	Ave	Max
33	315	385	270	330	4.5	5.5	427.5	522.5	0.675	0.825	30	45	45000	55000	0.21	6.49	14.68	0.81	2
34	315	385	270	330	4.5	5.5	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	0.84	4.42	0.00	0
35	315	385	270	330	4.5	5.5	427.5	522.5	0.675	0.825	120	180	45000	55000	11.91	18.61	24.25	3.02	4
36	315	385	270	330	4.5	5.5	427.5	522.5	0.675	0.825	120	180	180000	220000	8.16	17.88	31.82	0.77	2
37	315	385	270	330	4.5	5.5	427.5	522.5	2.7	3.3	30	45	45000	55000	0.63	5.67	12.86	0.78	2
38	315	385	270	330	4.5	5.5	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	1.32	35.86	0.01	1
39	315	385	270	330	4.5	5.5	427.5	522.5	2.7	3.3	120	180	45000	55000	8.70	15.10	55.76	3.01	5
40	315	385	270	330	4.5	5.5	427.5	522.5	2.7	3.3	120	180	180000	220000	3.20	14.30	28.37	0.79	2
41	315	385	270	330	4.5	5.5	1530	1870	0.675	0.825	30	45	45000	55000	0.23	4.90	16.95	0.79	2
42	315	385	270	330	4.5	5.5	1530	1870	0.675	0.825	30	45	180000	220000	0.00	1.47	48.54	0.01	1
43	315	385	270	330	4.5	5.5	1530	1870	0.675	0.825	120	180	45000	55000	6.55	13.04	30.93	3.05	5
44	315	385	270	330	4.5	5.5	1530	1870	0.675	0.825	120	180	180000	220000	0.27	11.92	24.75	0.80	2
45	315	385	270	330	4.5	5.5	1530	1870	2.7	3.3	30	45	45000	55000	0.28	4.31	19.80	0.77	2
46	315	385	270	330	4.5	5.5	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.81	11.07	0.00	0
47	315	385	270	330	4.5	5.5	1530	1870	2.7	3.3	120	180	45000	55000	6.29	11.46	25.57	3.09	5
48	315	385	270	330	4.5	5.5	1530	1870	2.7	3.3	120	180	180000	220000	1.28	11.02	22.62	0.79	2
49	315	385	270	330	9	11	427.5	522.5	0.675	0.825	30	45	45000	55000	0.60	5.15	11.21	0.82	2
50	315	385	270	330	9	11	427.5	522.5	0.675	0.825	30	45	180000	220000	0.00	0.76	6.39	0.00	0
51	315	385	270	330	9	11	427.5	522.5	0.675	0.825	120	180	45000	55000	8.47	14.40	24.75	3.02	5
52	315	385	270	330	9	11	427.5	522.5	0.675	0.825	120	180	180000	220000	1.94	13.51	25.79	0.78	2
53	315	385	270	330	9	11	427.5	522.5	2.7	3.3	30	45	45000	55000	1.03	4.50	11.28	0.80	2
54	315	385	270	330	9	11	427.5	522.5	2.7	3.3	30	45	180000	220000	0.00	0.62	3.38	0.00	0
55	315	385	270	330	9	11	427.5	522.5	2.7	3.3	120	180	45000	55000	5.79	11.19	38.97	3.01	5
56	315	385	270	330	9	11	427.5	522.5	2.7	3.3	120	180	180000	220000	1.75	10.46	22.31	0.81	2
57	315	385	270	330	9	11	1530	1870	0.675	0.825	30	45	45000	55000	0.13	7.64	57.11	0.77	2
58	315	385	270	330	9	11	1530	1870	0.675	0.825	30	45	180000	220000	0.00	0.52	2.80	0.00	0
59	315	385	270	330	9	11	1530	1870	0.675	0.825	120	180	45000	55000	5.27	11.66	54.59	3.07	5
60	315	385	270	330	9	11	1530	1870	0.675	0.825	120	180	180000	220000	0.26	8.92	19.63	0.78	2
61	315	385	270	330	9	11	1530	1870	2.7	3.3	30	45	45000	55000	0.11	6.57	47.93	0.81	2
62	315	385	270	330	9	11	1530	1870	2.7	3.3	30	45	180000	220000	0.00	0.51	5.25	0.00	0
63	315	385	270	330	9	11	1530	1870	2.7	3.3	120	180	45000	55000	2.44	9.27	58.06	3.00	5
64	315	385	270	330	9	11	1530	1870	2.7	3.3	120	180	180000	220000	2.25	8.36	17.54	0.79	2

TABLE 31. Results of Experiments with Factorial Design Settings for *PIII-PC*: Part 3

F. D. No.	RANGES of PARAMETERS											GAIN (%)			OPEN DCs				
	$D_i$	$K_i$	$h'_i$		$p_{ij}$		$r_{ij}$		$d_{ij}$		$f_j$	Min	Ave	Max	Ave	Max			
65	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	30	45	45000	55000	13.15	20.71	28.39	3.10	5
66	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	30	45	180000	220000	1.40	18.94	35.47	0.78	2
67	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	120	180	45000	55000	22.53	30.78	37.77	6.08	8
68	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	0.675	0.825	120	180	180000	220000	11.18	42.57	55.72	3.03	4
69	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	30	45	45000	55000	11.91	18.61	24.25	3.02	4
70	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	30	45	180000	220000	8.16	17.88	31.82	0.77	2
71	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	120	180	45000	55000	17.29	26.10	61.43	6.13	8
72	1260	1540	67.5	82.5	4.5	5.5	427.5	522.5	2.7	3.3	120	180	180000	220000	1.04	37.14	48.91	3.02	5
73	1260	1540	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	30	45	45000	55000	7.74	13.79	52.15	3.02	5
74	1260	1540	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	30	45	180000	220000	0.90	12.35	25.36	0.80	2
75	1260	1540	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	120	180	45000	55000	16.16	21.80	36.03	6.08	8
76	1260	1540	67.5	82.5	4.5	5.5	1530	1870	0.675	0.825	120	180	180000	220000	6.82	32.83	43.14	3.04	5
77	1260	1540	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	30	45	45000	55000	6.55	13.04	30.93	3.05	5
78	1260	1540	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	30	45	180000	220000	0.27	11.92	24.75	0.80	2
79	1260	1540	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	120	180	45000	55000	13.19	19.91	28.36	6.12	8
80	1260	1540	67.5	82.5	4.5	5.5	1530	1870	2.7	3.3	120	180	180000	220000	3.76	31.08	61.59	3.05	5
81	1260	1540	67.5	82.5	9	11	427.5	522.5	0.675	0.825	30	45	45000	55000	9.74	16.57	47.09	3.15	5
82	1260	1540	67.5	82.5	9	11	427.5	522.5	0.675	0.825	30	45	180000	220000	3.69	15.15	29.56	0.79	2
83	1260	1540	67.5	82.5	9	11	427.5	522.5	0.675	0.825	120	180	45000	55000	17.19	24.60	30.94	6.13	8
84	1260	1540	67.5	82.5	9	11	427.5	522.5	0.675	0.825	120	180	180000	220000	6.73	36.21	47.12	3.04	5
85	1260	1540	67.5	82.5	9	11	427.5	522.5	2.7	3.3	30	45	45000	55000	8.47	14.40	24.75	3.02	5
86	1260	1540	67.5	82.5	9	11	427.5	522.5	2.7	3.3	30	45	180000	220000	1.94	13.51	25.79	0.78	2
87	1260	1540	67.5	82.5	9	11	427.5	522.5	2.7	3.3	120	180	45000	55000	12.37	20.26	41.34	6.14	8
88	1260	1540	67.5	82.5	9	11	427.5	522.5	2.7	3.3	120	180	180000	220000	5.32	31.38	50.90	3.07	5
89	1260	1540	67.5	82.5	9	11	1530	1870	0.675	0.825	30	45	45000	55000	5.04	12.78	60.97	3.05	5
90	1260	1540	67.5	82.5	9	11	1530	1870	0.675	0.825	30	45	180000	220000	0.26	9.33	20.23	0.80	2
91	1260	1540	67.5	82.5	9	11	1530	1870	0.675	0.825	120	180	45000	55000	11.43	18.73	57.07	6.09	8
92	1260	1540	67.5	82.5	9	11	1530	1870	0.675	0.825	120	180	180000	220000	0.97	26.74	36.09	3.05	5
93	1260	1540	67.5	82.5	9	11	1530	1870	2.7	3.3	30	45	45000	55000	5.27	11.70	54.59	3.07	5
94	1260	1540	67.5	82.5	9	11	1530	1870	2.7	3.3	30	45	180000	220000	0.26	8.92	19.63	0.78	2
95	1260	1540	67.5	82.5	9	11	1530	1870	2.7	3.3	120	180	45000	55000	9.58	16.54	59.08	6.12	8
96	1260	1540	67.5	82.5	9	11	1530	1870	2.7	3.3	120	180	180000	220000	15.76	24.89	33.69	3.03	5

TABLE 32. Results of Experiments with Factorial Design Settings for *PIII-PC*: Part 4

F. D.		RANGES of PARAMETERS												GAIN (%)			OPEN DCs		
No.	$D_i$	$K_i$			$h'_i$		$p_{ij}$		$r_{ij}$		$d_{ij}$		$f_j$		Min	Ave	Max	Ave	Max
97	1260	1540	270	330	4.5	5.5	427.5	522.5	0.675	0.825	30	45	45000	55000	7.07	12.84	28.44	3.13	5
98	1260	1540	270	330	4.5	5.5	427.5	522.5	0.675	0.825	30	45	180000	220000	2.59	11.50	23.96	0.79	2
99	1260	1540	270	330	4.5	5.5	427.5	522.5	0.675	0.825	120	180	45000	55000	12.79	19.85	59.96	6.12	8
100	1260	1540	270	330	4.5	5.5	427.5	522.5	0.675	0.825	120	180	180000	220000	3.03	30.45	63.61	3.05	5
101	1260	1540	270	330	4.5	5.5	427.5	522.5	2.7	3.3	30	45	45000	55000	6.06	11.91	57.36	3.08	5
102	1260	1540	270	330	4.5	5.5	427.5	522.5	2.7	3.3	30	45	180000	220000	0.77	10.31	22.54	0.80	2
103	1260	1540	270	330	4.5	5.5	427.5	522.5	2.7	3.3	120	180	45000	55000	9.92	19.07	57.71	6.00	8
104	1260	1540	270	330	4.5	5.5	427.5	522.5	2.7	3.3	120	180	180000	220000	15.97	24.89	34.20	2.95	5
105	1260	1540	270	330	4.5	5.5	1530	1870	0.675	0.825	30	45	45000	55000	4.87	14.68	52.78	2.84	5
106	1260	1540	270	330	4.5	5.5	1530	1870	0.675	0.825	30	45	180000	220000	0.74	7.19	15.83	0.78	2
107	1260	1540	270	330	4.5	5.5	1530	1870	0.675	0.825	120	180	45000	55000	8.52	19.52	59.47	5.93	8
108	1260	1540	270	330	4.5	5.5	1530	1870	0.675	0.825	120	180	180000	220000	13.11	21.24	29.34	3.01	5
109	1260	1540	270	330	4.5	5.5	1530	1870	2.7	3.3	30	45	45000	55000	3.66	14.66	65.80	2.92	5
110	1260	1540	270	330	4.5	5.5	1530	1870	2.7	3.3	30	45	180000	220000	0.44	6.78	15.24	0.79	2
111	1260	1540	270	330	4.5	5.5	1530	1870	2.7	3.3	120	180	45000	55000	6.78	18.99	62.16	5.94	8
112	1260	1540	270	330	4.5	5.5	1530	1870	2.7	3.3	120	180	180000	220000	11.52	19.46	29.95	3.02	5
113	1260	1540	270	330	9	11	427.5	522.5	0.675	0.825	30	45	45000	55000	4.35	10.78	56.63	3.02	5
114	1260	1540	270	330	9	11	427.5	522.5	0.675	0.825	30	45	180000	220000	0.06	9.20	18.95	0.80	2
115	1260	1540	270	330	9	11	427.5	522.5	0.675	0.825	120	180	45000	55000	9.30	16.30	66.54	6.09	8
116	1260	1540	270	330	9	11	427.5	522.5	0.675	0.825	120	180	180000	220000	15.97	25.01	50.92	3.05	5
117	1260	1540	270	330	9	11	427.5	522.5	2.7	3.3	30	45	45000	55000	4.10	10.22	63.14	3.05	5
118	1260	1540	270	330	9	11	427.5	522.5	2.7	3.3	30	45	180000	220000	0.14	7.67	15.65	0.76	2
119	1260	1540	270	330	9	11	427.5	522.5	2.7	3.3	120	180	45000	55000	7.30	17.97	57.23	5.61	8
120	1260	1540	270	330	9	11	427.5	522.5	2.7	3.3	120	180	180000	220000	12.02	19.26	27.05	3.02	5
121	1260	1540	270	330	9	11	1530	1870	0.675	0.825	30	45	45000	55000	3.48	18.49	64.38	2.42	5
122	1260	1540	270	330	9	11	1530	1870	0.675	0.825	30	45	180000	220000	0.05	5.55	12.21	0.78	2
123	1260	1540	270	330	9	11	1530	1870	0.675	0.825	120	180	45000	55000	6.50	24.85	65.10	5.60	8
124	1260	1540	270	330	9	11	1530	1870	0.675	0.825	120	180	180000	220000	10.20	16.83	29.94	3.06	5
125	1260	1540	270	330	9	11	1530	1870	2.7	3.3	30	45	45000	55000	0.75	19.25	63.29	2.51	5
126	1260	1540	270	330	9	11	1530	1870	2.7	3.3	30	45	180000	220000	0.73	5.35	15.42	0.78	2
127	1260	1540	270	330	9	11	1530	1870	2.7	3.3	120	180	45000	55000	5.20	20.74	65.88	5.51	8
128	1260	1540	270	330	9	11	1530	1870	2.7	3.3	120	180	180000	220000	8.78	14.71	24.05	3.01	5

## CHAPTER VI

PRODUCTION DISTRIBUTION SYSTEM DESIGN PROBLEMS WITH  
INVENTORY CONSIDERATIONS

Production-distribution system design (PDS) problems are aimed at addressing strategic and tactical decisions regarding the design and operation of supply chains. A typical PDS model simultaneously considers the decisions regarding plant locations, distribution center (DC) locations, plant-DC and DC-retailer assignments as well as product flows from plants to retailers through DCs. The objective of the PDS problem is to minimize the variable distribution and transportation costs as well as the fixed costs of opening, equipping and managing plants and DCs.

Several companies including Elkem Silicone (Ulstein et al., 2006), GE Plastics (Tyagi et al., 2004), the Kellogg Company (Brown et al., 2001), Frito-Lay Inc. (Erlebacher and Meller, 2000), Digital Equipment Corporation (Arntzen et al., 1995), Ault Foods Limited (Pooley, 1994), Libbey-Owens-Ford (Martin et al., 1993), and DowBrands Inc. (Robinson et al., 1993) have achieved substantial cost savings through the optimization of production-distribution systems. Review papers on PDS (Erengüç et al., 1999; Geoffrion and Powers, 1995; Goetschalckx et al., 2002; Sarmiento and Nagi, 1999) also summarize the benefits and challenges of integrating the overall decision process in the whole supply chain. In particular, all of the previous studies emphasize the need for practical analytical models and efficient solution methods to aid decision-making. With this motivation, in this chapter, we consider an integrated location and inventory problem in the context of PDS. The models in this chapter generalize the previous PDS literature by taking inventory decisions into account and modeling the interaction between inventory and location decisions explicitly through transportation costs. These models also generalize the problem set-

tings in three-stage continuous facility location models in Chapter IV and two-stage discrete facility location models in Chapter V as explained next.

We consider a three-stage PDS problem. In the first stage, there are retailers (customers) with stationary and deterministic demand at established locations. The second and third stages consist of candidate locations of DCs and capacitated suppliers, respectively. Opening (selecting) each DC or supplier results in a facility-specific fixed operational cost. This problem is a generalization of the two-stage discrete facility location problems discussed in Chapter V, since we now consider multiple capacitated suppliers and decide on the number and the location of the suppliers as well as the assignment of DCs to the suppliers.

Another generalization of the models in Chapter V is related to inventory considerations. In this chapter, as opposed to Chapter V, inventory is kept at both the first and the second stages. Each retailer replenishes its inventory from a particular (established) DC at the second stage via *direct shipments*, and each selected DC replenishes its inventory from a particular *capacitated* supplier located at the third stage via *direct shipments*. Since the inventory is kept at both the retailer and DC levels, the issue of coordinating inventory decisions raised in Chapter IV is also relevant for this problem setting. We assume that the inventory systems of each selected DC (once established) and the associated retailers are operated under the assumptions of the classical single-warehouse multi-retailer (SWMR) problem studied by Roundy (1985). Also, as in Roundy (1985), the replenishment lead times are assumed to be negligible.

In designing the distribution system, we pose single-sourcing restrictions between the selected suppliers and the DCs as well as between the selected DCs and the retailers. In other words, each retailer can be served by only one DC, and each DC can be served by only one supplier. Contrarily, each supplier serves a set of DCs



and each DC serves a set of retailers. Although single-sourcing constraints make the location problem much harder to solve due to the capacity restrictions at the suppliers, they ease the control of the inventory decisions throughout the supply chain since the set of retailers assigned to different DCs are disjoint. Under this setting, the problem is the simultaneous optimization of

- the number and location of DCs,
- the number and location of suppliers,
- the assignment of each retailer to a selected DC,
- the assignment of each selected DC to a selected supplier, and
- the inventory decisions of each retailer and each selected DC.

with the objective of minimizing *the total costs in the system*, including (i) the inventory replenishment and holding costs of the retailers and the DCs, (ii) the transportation costs between the DCs and the retailers and between the suppliers and the DCs, and (iii) the fixed operational costs of the selected DCs and suppliers.

We note that this problem is a generalization of the work by Teo and Shu (2004) who consider a two-stage network design problem with inventory decisions and address the issue of coordinating replenishment activities between the warehouses (DCs) and the retailers. We generalize their setting to consider a three-stage distribution system by modeling the number and location of capacitated suppliers as well as the impact of trip distances and frequencies on the transportation costs. In order to capture this impact properly, we model the transportation costs as in (3.5).

The main contributions of this chapter are as follows:

- We create a formal model that provides an integrated view of strategic facility location decisions and operational transportation and inventory decisions.

- We show that the PDSD problem with inventory considerations can be modeled as a set partitioning problem.
- We develop efficient construction and improvement heuristics to find near optimal solutions for the problem.
- By comparing the cost of the PDSD problem with inventory considerations with a benchmark obtained through a sequential framework, we quantify the value of integrated decision making. Through this comparison, we also identify the important problem parameters that contribute the most to this value.

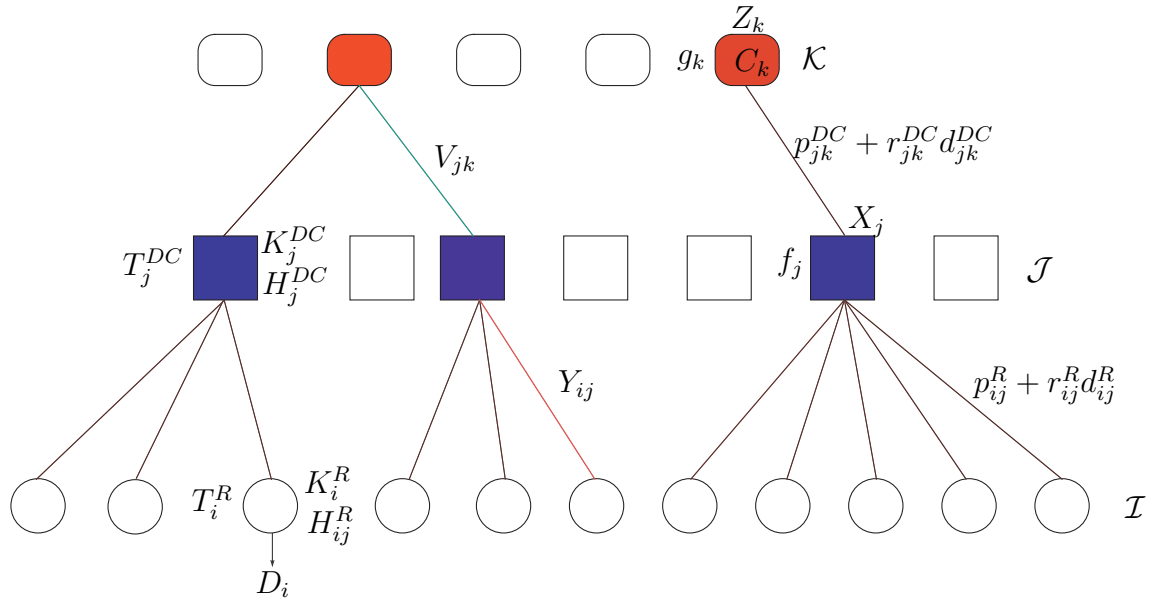
The remainder of the chapter is organized as follows. In the next section, we introduce the notation and our PDSD model with inventory considerations. In Section VI.1.1, we present a set partitioning formulation of our model. Next, in Section VI.2, we discuss the heuristic solution approaches to this problem. In Section VI.3, we describe the details of the benchmark model. In Section VI.4, we present the numerical results regarding the performance of the solution approaches and the value of integrated decision-making. Finally, in Section VI.5, we summarize our findings and conclude the chapter by discussing the potential impact of this work.

## **VI.1. General Model and Notation**

In this section, we present a nonlinear integer programming formulation for the PDSD problem with inventory considerations that is described in the previous section and illustrated in Figure 14.

To model the PDSD problem with inventory considerations, we define the following notation:

FIGURE 14. The Problem Setting for PDSB Problems with Inventory Considerations



$\mathcal{I}$  set of retailers,  $\mathcal{I} = \{1, \dots, m\}$ .

$\mathcal{J}$  set of candidate DC locations,  $\mathcal{J} = \{1, \dots, n\}$ .

$\mathcal{K}$  set of candidate supplier locations,  $\mathcal{K} = \{1, \dots, K\}$ .

$D_i$  deterministic and stationary demand rate faced by retailer  $i$ ,  $\forall i \in \mathcal{I}$ .

$P_i^R$  location of retailer  $i$ ,  $\forall i \in \mathcal{I}$ ,  $P_i^R = (a_i^R, b_i^R)$ .

$P_j^{DC}$  location of DC  $j$ ,  $\forall j \in \mathcal{J}$ ,  $P_j^{DC} = (a_j^{DC}, b_j^{DC})$ .

$P_k^S$  location of supplier  $k$ ,  $\forall k \in \mathcal{K}$ ,  $P_k^S = (a_k^S, b_k^S)$ .

- $d_{jk}^{DC}$  Euclidean distance between DC  $j$  and supplier  $k$ ,  
 $d_{jk}^{DC} = \sqrt{(a_j^{DC} - a_k^S)^2 + (b_j^{DC} - b_k^S)^2}$ ,  $j \in \mathcal{J}$  and  $k \in \mathcal{K}$ .
- $d_{ij}^R$  Euclidean distance between retailer  $i$  and DC  $j$ ,  
 $d_{ij}^R = \sqrt{(a_i^R - a_j^{DC})^2 + (b_i^R - b_j^{DC})^2}$ ,  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ .
- $W_k$  annual throughput capacity at supplier  $k$ ,  $k \in \mathcal{K}$ .
- $f_j$  fixed (annual) cost of selecting candidate DC  $j$ ,  $j \in \mathcal{J}$ .
- $g_k$  fixed (annual) cost of selecting candidate supplier  $k$ ,  $k \in \mathcal{K}$ .
- $K_j^{DC}$  fixed ordering cost of DC  $j$ ,  $j \in \mathcal{J}$ .
- $K_i^R$  fixed ordering cost of retailer  $i$ ,  $i \in \mathcal{I}$ .
- $h_i^R$  inventory holding cost rate for each unit of inventory at retailer  $i$ ,  $i \in \mathcal{I}$ .
- $h_j^{DC}$  inventory holding cost rate for each unit of inventory at DC  $j$ ,  $j \in \mathcal{J}$ .  
 $h_i^R \geq h_j^{DC}$ ,  $\forall i \in \mathcal{I}$  and  $\forall j \in \mathcal{J}$ .
- $H_j^{DC}$  echelon holding cost rate at the DC,  $H_j^{DC} = h_j^{DC}$ ,  $j \in \mathcal{J}$ .
- $H_{ij}^R$  echelon holding cost rate at retailer  $i$ ,  $H_{ij}^R = h_i^R - H_j^{DC}$ ,  $\forall i \in \mathcal{I}$ .
- $T_b$  fixed base period (set a priori).
- $p_{jk}^{DC}$  fixed cost of transportation (releasing a shipment) to DC  $j$   
from supplier  $k$ ,  $j \in \mathcal{J}$  and  $k \in \mathcal{K}$ .
- $p_{ij}^R$  fixed cost of transportation (releasing a shipment) to retailer  $i$   
from DC  $j$ ,  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ .
- $r_{jk}^{DC}$  per mile transportation cost to DC  $j$  from supplier  $k$ ,  $j \in \mathcal{J}$  and  $k \in \mathcal{K}$ .
- $r_{ij}^R$  per mile transportation cost to retailer  $i$  from DC  $j$ ,  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ .

Next, we define the decision variables. We have six sets of decision variables. The first set of decision variables relates to selecting suppliers. For each supplier  $k$ ,

$k \in \mathcal{K}$ ,

$$Z_k = \begin{cases} 1, & \text{if supplier } k \text{ is selected,} \\ 0, & \text{otherwise.} \end{cases}$$

Second, for each DC  $j$ ,  $j \in \mathcal{J}$ , we define

$$X_j = \begin{cases} 1, & \text{if DC } j \text{ is selected,} \\ 0, & \text{otherwise.} \end{cases}$$

The third set of decision variables relates to the assignment of retailers to DCs. For retailer  $i$ ,  $i \in \mathcal{I}$ , and DC  $j$ ,  $j \in \mathcal{J}$ ,

$$Y_{ij} = \begin{cases} 1, & \text{if retailer } i \text{ is assigned to DC } j, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we have a set of decision variables for assigning the suppliers to the DCs. For DC  $j$ ,  $j \in \mathcal{J}$ , and supplier  $k$ ,  $k \in \mathcal{K}$ ,

$$V_{jk} = \begin{cases} 1, & \text{if DC } j \text{ is assigned to supplier } k, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the last two sets of decision variables relate to the inventory policy parameters at the selected DCs and their retailers. For  $j \in \mathcal{J}$ ,  $T_j^{DC}$  represents the reorder interval of DC  $j$ , and for  $i \in \mathcal{I}$ ,  $T_i^R$  represents the reorder interval of retailer  $i$ .

Before introducing the mathematical model, we recall and summarize the underlying modeling assumptions:

- Each selected supplier  $k \in \mathcal{K}$  serves a set of DCs, but each DC  $j \in \mathcal{J}$  is served by only one supplier.
- Each selected DC  $j$ ,  $j \in \mathcal{J}$ , serves a set of retailers, but each retailer  $i$ ,  $i \in \mathcal{I}$ , is served by only one DC. In other words, there is single-sourcing between the

suppliers and the DCs and between the DCs and the retailers. Hence, the sets of retailers assigned to different DCs are disjoint.

- Each candidate supplier has a facility-specific annual throughput capacity, and the annual demand satisfied by a selected supplier cannot exceed this capacity.
- Deliveries between the suppliers and the DCs and between the DCs and the retailers are direct and instantaneous. The sizes and frequencies of the deliveries are determined by the inventory policy parameters. Each delivery between two facilities results in a transportation cost. This cost consists of a fixed (loading/unloading) cost and a variable shipment cost that depends on the distance between the two facilities.
- Both DCs and retailers are inventory holding points. From an inventory modeling viewpoint, the inventories of each selected DC  $j \in \mathcal{J}$  and the corresponding set of retailers are operated under the assumptions of the single warehouse multi-retailer (SWMR) lot sizing problem (see Section II.2.2). That is, given a *fixed* DC  $j \in \mathcal{J}$ , its supplier  $k \in \mathcal{K}$ , and a set of retailers assigned to this DC, i.e.,  $\mathcal{I}_j = \{i \in \mathcal{I} : Y_{ij} = 1\}$  and  $\mathcal{I}_j \subset \mathcal{I}$ ,
  - DC  $j$ 's inventory is replenished at successive reorder intervals of  $T_j^{DC}$  incurring  $p_{jk}^{DC} + r_{jk}^{DC} d_{jk}^{DC} + K_j^{DC}$ , which represents the total costs of transportation and ordering per replenishment.
  - Retailer  $i \in \mathcal{I}_j$  is replenished at successive reorder intervals of  $T_i^R$  incurring  $p_{ij}^R + r_{ij}^R d_{ij}^R + K_i^R$ ,  $i \in \mathcal{I}_j$ , which represents the total costs of transportation and ordering per retailer replenishment.
  - Echelon holding costs accumulate at rate  $H_j^{DC}$  at DC  $j$  over  $T_j^{DC}$  and at rate  $H_{ij}^R$ , at retailer  $i$  over  $T_i^R$ .

- The reorder intervals,  $T_j^{DC}$  and  $T_i^R$ ,  $i \in \mathcal{I}_j$ , are chosen as power-of-two multiples of a fixed base period,  $T_b$ .

The model can now be formulated as the following integer nonlinear program:

$$\begin{aligned} \text{Min} \quad & \sum_{j \in \mathcal{J}} f_j X_j + \sum_{j \in \mathcal{J}} g_k Z_k + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \frac{(p_{ij}^R + r_{ij}^R d_{ij}^R) Y_{ij}}{T_i^R} + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \frac{(p_{jk}^{DC} + r_{jk}^{DC} d_{jk}^{DC}) V_{jk}}{T_j^{DC}} \\ & + \sum_{j \in \mathcal{J}} \left\{ \frac{K_j^{DC}}{T_j^{DC}} + \sum_{i \in \mathcal{I}_j} \frac{K_i^R}{T_i^R} + \sum_{i \in \mathcal{I}_j} \frac{1}{2} H_{ij}^R D_i T_i^R + \sum_{i \in \mathcal{I}_j} \frac{1}{2} H_i^{DC} D_i \max\{T_i^R, T_j^{DC}\} \right\} X_j \end{aligned}$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad \forall i \in \mathcal{I}. \quad (6.1)$$

$$\sum_{k \in \mathcal{K}} V_{jk} = 1, \quad \forall j \in \mathcal{J}. \quad (6.2)$$

$$Y_{ij} \leq X_j, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (6.3)$$

$$V_{jk} \leq Z_k, \quad \forall j \in \mathcal{J} \text{ and } \forall k \in \mathcal{K}. \quad (6.4)$$

$$V_{jk} \leq X_j, \quad \forall j \in \mathcal{J} \text{ and } \forall k \in \mathcal{K}. \quad (6.5)$$

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} D_i Y_{ij} V_{jk} \leq C_k Z_k, \quad \forall k \in \mathcal{K}. \quad (6.6)$$

$$T_j^{DC} = \begin{cases} 2^{\nu_j} T_b, \nu_j \in \mathbb{Z}, & \text{if } X_j = 1, \\ \text{N.A.}, & \text{O.W.} \end{cases} \quad \forall j \in \mathcal{J}. \quad (6.7)$$

$$T_i^R = 2^{v_i} T_b \quad \text{and} \quad v_i \in \mathbb{Z}, \quad \forall i \in \mathcal{I}. \quad (6.8)$$

$$X_j \in \{0, 1\}, Y_{ij} \in \{0, 1\}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (6.9)$$

$$Z_k \in \{0, 1\}, V_{jk} \in \{0, 1\}, \quad \forall j \in \mathcal{J} \text{ and } \forall k \in \mathcal{K}. \quad (6.10)$$

The objective function minimizes the total cost of designing and operating the distribution system, that is the sum of (i) the fixed cost of locating the DCs, (ii) the fixed cost of locating the suppliers, (iii) the transportation cost from the selected

suppliers to the selected DCs, (iv) the transportation cost from the selected DCs to the retailers, and (v) the annual ordering and holding costs at the DCs and the retailers, respectively.

Constraints (6.1) ensure that each retailer is served by exactly one DC. Constraints (6.2) stipulate that each DC is assigned to exactly one supplier. These sets of constraints, together with the integrality constraints (6.9) and (6.10), state that there is single-sourcing between the first and second stage and between the second and third stage. Constraints (6.3), (6.4), (6.5) are the assignment constraints that ensure that only selected facilities (DCs and suppliers) are utilized in the distribution system design. In particular, constraints (6.3) state that the retailers can be assigned only to the selected candidate DCs. Constraints (6.4) ensure that a DC can be assigned only to a selected supplier. Constraints (6.5) ensure that a supplier can be assigned only to a selected candidate DC. Constraints (6.6) ensure capacity restrictions at the suppliers. Constraints (6.7) and (6.8) state the power-of-two restrictions on the reorder intervals of the selected DCs and the retailers. Finally, constraints (6.9) and (6.10) are the standard integrality constraints.

Note that this formulation is complicated, not only due to the nonlinearities in the objective function, but also due to the nonlinearities in the constraints (6.6). We may eliminate the nonlinearity in the capacity constraint by introducing a new variable that keeps track of the retailers assigned to a supplier. In particular, for retailer  $i, i \in \mathcal{I}$ , DC  $j, j \in \mathcal{J}$ , and supplier  $k, k \in \mathcal{K}$ , define

$$W_{ijk} = \begin{cases} 1, & \text{if retailer } i \text{ is assigned to supplier } k \text{ through DC } j, \\ 0, & \text{otherwise.} \end{cases}$$



Then, we can restate constraint (6.6) as follows:

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} D_i W_{ijk} \leq C_k Z_k, \quad \forall k \in \mathcal{K}. \quad (6.11)$$

To establish the relation of  $W_{ijk}$  with the other assignment variables  $V_{jk}$  and  $Y_{ij}$ , we introduce the following constraints:

$$W_{ijk} > V_{jk} + Y_{ij} - 2, \quad i \in \mathcal{I}, \quad j \in \mathcal{J}, \quad k \in \mathcal{K}. \quad (6.12)$$

$$2W_{ijk} \leq V_{jk} + Y_{ij}, \quad i \in \mathcal{I}, \quad j \in \mathcal{J}, \quad k \in \mathcal{K}. \quad (6.13)$$

Now, by replacing constraints (6.6) with constraints (6.11), (6.12), and (6.13), we eliminate the nonlinearity in the constraints at the expense of additional variables and constraints. However, the nonlinearity in the objective function that is due to inventory considerations still complicates the problem. Providing a solution to this formulation even in its current form is challenging, and, hence, we formulate the problem as a set partitioning formulation. In the next section, we discuss the set partitioning formulation. The set partitioning formulation leads to efficient heuristic approaches that we explain in detail in Section VI.2.

### VI.1.1. Set Partitioning Model

The set partitioning problem is widely used for modeling difficult combinatorial optimization problems, including vehicle routing, crew scheduling, network design, graph partitioning, graph coloring, etc. (Balas and Carrera, 1996; Ceria et al., 1997). Several different exact and heuristic methods are proposed to solve the set partitioning problem (Caprara et al., 1998).

To formulate our problem as a set partitioning problem, we define an additional decision variable. Let  $j$  be a particular DC in  $\mathcal{J}$ ,  $k$  be a particular supplier in  $\mathcal{K}$ , and

$S$  be a subset of retailers in  $\mathcal{I}$ . Then, the set partitioning decision variable is

$$x_{jkS} = \begin{cases} 1, & \text{if supplier } k \text{ is selected to serve DC } j, \text{ and } j \text{ is used to serve} \\ & \text{retailers in } S \text{ and no one else,} \\ 0, & \text{otherwise.} \end{cases}$$

Our PDS problem with inventory considerations then reduces to finding a minimum cost partition of the set of retailers into subsets  $S_{jk}$ ,  $j \in \mathcal{J}$  and  $k \in \mathcal{K}$ , where  $S_{jk}$  denotes the subset of retailers served by DC  $j$  that is supplied by supplier  $k$ . Note that  $S_{jk}$  may be empty if (i) the DC  $j$  or supplier  $k$  is not selected, and (ii) the DC  $j$  is not assigned to supplier  $k$ .

Next, we discuss the total cost associated with a given partition. In particular, let  $c_{jkS}$  denote the cost of serving the retailers in  $S \subset \mathcal{I}$  using DC  $j \in \mathcal{J}$  which is assigned to supplier  $k \in \mathcal{K}$ . It consists of the following cost components:

- DC specific fixed location cost,  $f_j$ .
- Systemwide inventory replenishment, inventory holding, and transportation costs, denoted by  $IT(j, k, S)$ . This cost is estimated by solving the following single-warehouse multi-retailer (SWMR) lot-sizing problem:

$$IT(j, k, S) \equiv \min \left\{ \sum_{i \in S} \frac{(p_{ij}^R + r_{ij}^R d_{ij}^R)}{T_i^R} + \frac{(p_{jk}^{DC} + r_{jk}^{DC} d_{jk}^{DC})}{T_j^{DC}} + \frac{K_j^{DC}}{T_j^{DC}} + \sum_{i \in S} \frac{K_i^R}{T_i^R} + \sum_{i \in S} \frac{1}{2} H_{ij}^R D_i T_i^R + \sum_{i \in S} \frac{1}{2} H_i^{DC} D_i \max\{T_i^R, T_j^{DC}\} \right\}. \quad (6.14)$$

subject to

$$T_j^{DC} = 2^{\nu_j} T_b \quad \text{and} \quad \nu_j \in \mathbb{Z}. \quad (6.15)$$

$$T_i^R = 2^{v_i} T_b \quad \text{and} \quad v_i \in \mathbb{Z}, \quad \forall i \in \mathcal{S}. \quad (6.16)$$

$$T_j^{DC} \in \mathbb{R}_+ \quad \text{and} \quad T_i^R \in \mathbb{R}_+, \quad \forall i \in \mathcal{S}. \quad (6.17)$$

Note that this SWMR lot-sizing problem takes the interaction of the inventory decision variables and the transportation costs into account. In particular, the transportation cost between supplier  $k$  and DC  $j$  has an explicit impact on the reorder interval of the DC,  $T_j^{DC}$  by influencing the total cost per replenishment from the supplier. Similarly, the transportation cost between DC  $j$  and the retailers in  $S$  influences the reorder interval of the retailers.

Hence, we let  $c_{jkS} = f_j + IT(j, k, S)$ , for all  $j \in \mathcal{J}$ , for all  $k \in \mathcal{K}$ , and for all  $S \subset \mathcal{I}$ . Now, the PDS with inventory considerations can be formulated as a set partitioning problem in the following way:

$$\text{Min} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \sum_{S \subset \mathcal{I}} c_{jkS} x_{jkS} + \sum_{k \in \mathcal{K}} g_k Z_k \quad (PIV-SP)$$

subject to

$$\sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \sum_{S \subset \mathcal{I}; i \in S} x_{jkS} = 1, \quad \forall i \in \mathcal{I}. \quad (6.18)$$

$$\sum_{S \subset \mathcal{I}; i \in S} x_{jkS} \leq M Z_k, \quad \forall j \in \mathcal{J} \text{ and } \forall k \in \mathcal{K}. \quad (6.19)$$

$$\sum_{i \in S; S \subset \mathcal{I}} \sum_{j \in \mathcal{J}} D_i x_{jkS} \leq C_k Z_k, \quad \forall k \in \mathcal{K}. \quad (6.20)$$

$$x_{jkS} \in \{0, 1\}, \quad \forall j \in \mathcal{J}, \forall k \in \mathcal{K}, \text{ and } \forall S \subset \mathcal{I}. \quad (6.21)$$

$$Z_k \in \{0, 1\}, \quad \forall k \in \mathcal{K}. \quad (6.22)$$

This formulation is different than the classical set partitioning formulation due to the selection of suppliers and the assignment of suppliers to the partitioned sets. The objective function minimizes the total cost of serving the sets of retailers and selecting the suppliers. Constraint (6.18) ensures that each retailer  $i \in \mathcal{I}$  is assigned to a set that is served by a single DC and a single supplier. Constraint (6.19) satisfies that we can assign partitions to only open suppliers, where  $M$  is a big number. Constraint (6.20) ensures that the capacity requirement at each selected supplier is satisfied. Constraints (6.21) and (6.22) are the integrality constraints.

Formulation *PIV-SP* has no nonlinearity in its objective function or constraints. However, there are a large number of variables ( $|J| \times |K| \times O(2^{|R|})$ ), and  $c_{jkS}$  can be obtained only by solving a convex programming problem. Hence, it is not easy to solve this formulation effectively via exact optimization methods. In this chapter, we develop heuristic approaches for the solution of *PIV*. Our heuristic approaches rely on the ideas examined in Chapter IV and utilize the results of *PII-d-MR* for solving a simpler version of the problem under consideration, as will be clear in the next section. Furthermore, the efficiency and short runtime of the heuristic approaches make it possible for solving larger- possibly, real-life size- networks. One of the most preferred approaches for solving the set partitioning problem is branch-and-price. Branch-and-price is a generalization of branch-and-bound in which the nodes are processed by solving LP relaxations via column generation (see Barnhart et al. (1994)). However, in order to initialize a branch-and-price solution, we need a set of good solutions, i.e. initial columns to solve. Heuristic approaches are highly effective at providing such initial solutions for a branch-and-price implementation. For the reasons mentioned above, we focus on heuristic approaches and leave the implementation of branch-and-price to future research on this problem.

## VI.2. Heuristic Approaches

In this section, we first describe the solution representation and evaluation of heuristic approaches. We begin with a construction heuristic which provides a feasible solution to *PIV*. The feasible solution obtained from the construction heuristic is later used as the initial input in two improvement heuristics.

### VI.2.1. Solution Representation and Evaluation

As we observe from the set partitioning formulation *PIV-SP*, any feasible solution can be represented as mutually exclusive subsets of retailers. Let  $\mathcal{S}$  denote the set of mutually exclusive subsets of retailers. Each  $S \in \mathcal{S}$  is served by a unique DC  $j$ ,  $j \in \mathcal{J}$ . The set of open DCs,  $\mathcal{J}' \subset \mathcal{J}$ , is defined by these unique DCs. In other words, if DC  $j$  is serving a subset of retailers  $S$ , then  $j \in \mathcal{J}'$ . Similarly, a set of open suppliers,  $\mathcal{K}' \subset \mathcal{K}$ , is determined if they are serving any set of retailers.

In general, the quality of a feasible solution is represented by the value of the objective function, or the cost it implies. To evaluate the cost, we use the objective function of the set partitioning formulation *PIV-SP*. For this purpose, we let

$$OBJ(\mathcal{K}', \mathcal{J}', \mathcal{S}) = \sum_{k \in \mathcal{K}'} g_k + \sum_{j \in \mathcal{J}'} f_j + \sum_{j \in \mathcal{J}'} \sum_{k \in \mathcal{K}'} \sum_{S \in \mathcal{S}} IT(j, k, S) x_{jks}$$

where  $x_{jks}$  is the binary variable denoting whether subset  $S$  is assigned to supplier  $k$  through DC  $j$ .

### VI.2.2. A Construction Heuristic

Based on our solution representation described above, the main objective of the construction heuristic for *PIV* is to partition the set of retailers into mutually exclusive subsets and then determine a feasible DC-supplier assignment for each set.

The construction heuristic utilizes a greedy approach. First, we initialize:

- the set of unserved retailers (*unservedRet*) as the set of retailers  $\mathcal{I}$ ,
- the set of available suppliers (*aS*) as the set of suppliers  $\mathcal{K}$ ,
- the set of available DCs (*aDC*) as the set of DCs  $\mathcal{J}$ ,
- the set of open suppliers (*oS*), the set of open DCs (*oDC*), and the set of subsets of retailers ( $\mathcal{S}$ ) to be empty.

Next, we sort the set of available suppliers in decreasing order according to the ratio of their capacity to fixed facility opening cost. In other words, we start by selecting the suppliers with the highest capacity and the lowest fixed facility opening cost. Starting with supplier  $k_1$  at the beginning of the sorted list, we construct a new (empty) subset of retailers  $S$  to be served by this supplier. First, among the available DCs, we determine the closest DC  $j_1$  to this supplier. Next, we sort the unserved retailers with respect to their proximity to this selected DC. We greedily add retailers to subset  $S$  in the determined order as long as they satisfy the capacity constraint of supplier  $k_1$ . If a retailer is added to  $S$ , it is removed from the set of unserved retailers. We mark  $k_1$  and  $j_1$  as open supplier and DC, respectively, and remove them from the sets of available suppliers and DCs. These steps are repeated until all of the retailers are served by a DC and a supplier. Finally, the construction heuristic returns the objective function value of the solution. This procedure is shown in Display 6.

### VI.2.3. Link-based Improvement Heuristic

An improvement heuristic modifies the initial feasible solution using a neighborhood function to search for a better solution. A neighborhood function modifies the key attributes in order to generate neighboring solutions in a heuristic search framework.

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**DISPLAY 6** The Construction Heuristic.

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**0:** Initialize:  $unservedRet \leftarrow \mathcal{I}$ ,  $oS \leftarrow \emptyset$ ,  $aS \leftarrow \mathcal{K}$ ,  $oDC \leftarrow \emptyset$ ,  $aDC \leftarrow \mathcal{J}$ ,  $\mathcal{S} \leftarrow \emptyset$ .

**1:** Sort  $aS$  according to the ratio  $C_k/g_k$ ,  $k \in \mathcal{K}$ , in a decreasing order.

**2:** Determine  $\mathcal{S}$ ,  $oS$ , and  $oDC$ :

**while**  $|unservedRet| > 0$  **do**

$k_1 \leftarrow aS[1]$ .

$S \leftarrow \emptyset$ .

$remainingCap = C_{k_1}$ .

$j_1 = \arg \min_{j \in aDC} \{(p_{j,k_1}^{DC} + r_{j,k_1}^{DC} d_{j,k_1}^{DC})\}$ .

Sort  $unservedRet$  according to  $\{(p_{i,j_1}^R + r_{i,j_1}^R d_{i,j_1}^R)\}$  in an increasing order.

**for**  $i \in unservedRet$  **do**

**if**  $D_i < remainingCap$  **then**

$remainingCap \leftarrow remainingCap - D_i$ .

Insert  $i$  to  $S$ , remove  $i$  from  $unservedRet$ .

**end if**

**end for**

Insert  $S$  to  $\mathcal{S}$ . Insert  $k_1$  to  $oS$ , and remove  $k_1$  from  $aS$ . Insert  $j_1$  to  $oDC$ , and remove  $j_1$  from  $aDC$ .

**end while**

**3:** Return the cost  $\sum_{k \in oS} g_k + \sum_{j \in oDC} f_j + \sum_{j \in oDC} \sum_{k \in oS} \sum_{S \in \mathcal{S}} IT(j, k, S)$ .

---

For our problem, we can list the attributes of our solution as mutually exclusive subsets of retailers and DC-supplier pairs (links) that serve these subsets.

In developing the link-based improvement heuristic, we first secure the subsets of retailers from the initial feasible solution obtained using the construction heuristic. Next, we construct a set of potential DC-supplier links to be used while serving these subsets. Note that, it is possible that the set of potential DC-supplier links may contain all the possible DC-supplier links. However, searching through such an

extensive set for each change in the solution can be quite time-intensive. Hence, we confine ourselves to a small subset of all of the possible DC-supplier links.

In our solution approach, we define the neighborhood of a solution through a combination of simple neighborhood functions. In particular, we utilize the *link exchange neighborhood*, *move neighborhood*, *exchange neighborhood*, and *new set construction* functions. The link exchange neighborhood helps us modify the set of potential links available to serve the subsets of retailers. The move and exchange neighborhoods help us modify the contents of each subset of retailers for intensification purposes. Finally, the *new set construction* function helps us construct new subsets of retailers for diversification purposes. Combining several different simple neighborhood functions helps us search the solution more thoroughly and efficiently. We explain these components in detail below. Before concluding this section, we present an overall framework of the link-based improvement heuristic.

### VI.2.3.1. Link Exchange Neighborhood Search

Link-exchange neighborhood search is one of the most important components of the link-based improvement heuristic since it impacts the set of potential links directly and the solution quality indirectly. Before discussing the details of link exchange neighborhood search, let us explain how the initial set of potential links is constructed.

As mentioned before, the set of potential links  $\mathcal{L}$  is a subset of all of the possible DC-supplier links. Primarily, it is set to contain 10% of all of the possible DC-supplier links. Hence, the size of  $\mathcal{L}$  is fixed. Let us denote this size as  $LSize$ . At the start of the overall algorithm, we initialize  $\mathcal{L}$  using the subsets obtained from the construction heuristic. For each subset, we find the best DC-supplier pair and add it to the set  $\mathcal{L}$  until  $LSize$  is reached. Once  $\mathcal{L}$  is established, it is utilized to determine the best DC-supplier pairs for subsets of retailers throughout the algorithm.



Link exchange neighborhood search helps us modifying the contents of  $\mathcal{L}$ . In Display 7, we provide the steps of link exchange neighborhood search. In link exchange neighborhood search, for each link  $l_1 \in \mathcal{L}$ , we first find the a subset of retailers  $S^*$  that is served *feasibly* by this link at a minimum cost ( $z^{best}$ ). The cost  $z(S, l_1)$  includes the cost of establishing the DC-supplier pair implied by link  $l_1$  as well as the total inventory and transportation costs implied by the assignment  $(S, l_1)$ . Using  $S^*$ , we search through the remaining set of all potential links ( $\mathcal{P}\mathcal{L} \setminus \mathcal{L}$ ) to check if there exists another link  $l_2 \in \mathcal{P}\mathcal{L} \setminus \mathcal{L}$  such that  $z(S^*, l_2) < z^{best}$ . If we find such a link, we update  $z^{best}$  and update the potential exchange link  $l_1^{best}$  with  $l_2$ . After searching through all of the links in  $\mathcal{P}\mathcal{L} \setminus \mathcal{L}$ , if an  $l_1^{best}$  exists, we swap  $l_1$  with  $l_1^{best}$  and modify the contents of  $\mathcal{P}\mathcal{L}$  and  $\mathcal{L}$ . Finally, once the exchange of all of the links in  $\mathcal{L}$  is complete, we return the modified set  $\mathcal{L}$ .

### VI.2.3.2. Move Neighborhood Search

Move neighborhood search modifies the content of a subset  $S$  by moving a retailer in  $S$  to another subset in the set of the subsets  $\mathcal{S}$ . The steps of move neighborhood search are given in Display 8.

In the beginning of the search, we initialize the best cost  $z^{best}$  with the best cost obtained so far in the overall procedure that we denote as  $z^{CH}$ . Furthermore, we initialize the control cost  $z^0$  as infinity. Starting with a subset  $S_1 \in \mathcal{S}$ , we move each retailer  $i$  in  $S_1$  to another subset  $S_2 \in \mathcal{S}$ . After each move, we calculate the objective value of the new solution,  $z^{move}$ . Note that, before calculating the objective value, we need to determine the DC-supplier pairs that will be serving the newly constructed subsets  $S_1$  and  $S_2$ . The best DC-supplier pairs are determined via

- a search through the potential link set,  $\mathcal{L}$ , in the link-based improvement heuris-

---

**DISPLAY 7** The Link Exchange Neighborhood Search.

---

```

for  $\forall l_1 \in \mathcal{L}$  do
   $S^* \leftarrow \arg \min_{S \in \mathcal{S}} \{z(S, l_1)\}$ .
   $z^{best} \leftarrow \min_{S \in \mathcal{S}} \{z(S, l_1)\}$ .
  for  $\forall l_2 \in \mathcal{P}\mathcal{L} \setminus \mathcal{L}$  do
    Calculate  $z(S^*, l_2)$ .
    if  $z(S^*, l_2) < z^{best}$  then
       $z^{best} \leftarrow z(S^*, l_2)$ .
       $l_1^{best} \leftarrow l_2$ .
       $improved \leftarrow 1$ .
    end if
  end for
  if  $improved = 1$  then
     $\mathcal{L} \leftarrow \mathcal{L} \setminus \{l_1\}$ .  $\mathcal{L} \leftarrow \mathcal{L} \cup \{l_1^{best}\}$ .
     $\mathcal{P}\mathcal{L} \leftarrow \mathcal{P}\mathcal{L} \setminus \{l_1^{best}\}$ .  $\mathcal{P}\mathcal{L} \leftarrow \mathcal{P}\mathcal{L} \cup \{l_1\}$ .
  end if
end for
Return  $\mathcal{L}$ .

```

---

tic, and

- solving *PII-d-MR* for each subset and supplier combination in the *PII-d-MR*-based improvement heuristic.

If, with this move, we can improve  $z^{best}$ , we modify  $\mathcal{S}$  to include the new updated  $S_1$  and  $S_2$ . We update the value of  $z^{best}$ , and start the move neighborhood search over taking into account the new modified  $\mathcal{S}$ . For this purpose, the move neighborhood searches for the “*first-best-move*.”

We continue in this manner until we can no longer improve  $z^{best}$ . At that point, we return  $z^{best}$  and the associated  $\mathcal{S}$  as the new solution.

---

**DISPLAY 8** The Move Neighborhood.
 

---

**0:** Set  $z^{best} \leftarrow z^{CH}$ ,  $z^0 \leftarrow \infty$ .  
**while**  $z^{best} < z^0$  **do**  
    $z^0 \leftarrow z^{best}$ .  
   **for**  $\forall S_1 \in \mathcal{S}$  **do**  
     **for**  $\forall S_2 \in \mathcal{S}$  **do**  
       **if**  $S_1 \neq S_2$  and  $|S_1| > 1$  **then**  
         **for**  $\forall i \in S_1$  **do**  
            $S_1 \leftarrow S_1 \setminus \{i\}$ .  $S_2 \leftarrow S_2 \cup \{i\}$ . Calculate  $z^{move}$ .  
           **if**  $z^{move} < z^{best}$  **then**  
              $z^{best} \leftarrow z^{move}$ . Update  $\mathcal{S}$ . Start over.  
           **end if**  
         **end for**  
       **end if**  
     **end for**  
   **end for**  
**end while**  
 Return  $z^{best}$ .

---

**VI.2.3.3. Exchange Neighborhood Search**

The exchange neighborhood search modifies the contents of two subsets by swapping retailers among subsets. The steps of the exchange neighborhood search are given in Display 9.

In the beginning of the search, we initialize the best cost  $z^{best}$  with the best cost  $z^{CH}$  obtained so far in the overall procedure. We also initialize the control cost  $z^0$  as infinity. Starting with two subsets  $S_1 \in \mathcal{S}$  and  $S_2 \in \mathcal{S}$ , such that  $S_1 \neq S_2$ , we swap each retailer  $i$  in  $S_1$  with another retailer  $j$  in subset  $S_2 \in \mathcal{S}$ . After each exchange,

---

**DISPLAY 9** The Exchange Neighborhood.
 

---

**0:** Set  $z^{best} \leftarrow z^{CH}$ ,  $z^0 \leftarrow \infty$ .  
**while**  $z^{best} < z^0$  **do**  
    $z^0 \leftarrow z^{best}$ .  
   **for**  $\forall S_1 \in \mathcal{S}$  **do**  
     **for**  $\forall S_2 \in \mathcal{S}$  **do**  
       **if**  $S_1 \neq S_2$  **then**  
         **for**  $\forall i \in S_1$  **do**  
           **for**  $\forall j \in S_2$  **do**  
              $S_1 \leftarrow S_1 \setminus \{i\}$ .  $S_1 \leftarrow S_1 \cup \{j\}$ .  $S_2 \leftarrow S_2 \setminus \{j\}$ .  $S_2 \leftarrow S_2 \cup \{i\}$ .  
             Calculate  $z^{exchange}$ .  
             **if**  $z^{exchange} < z^{best}$  **then**  
                $z^{best} \leftarrow z^{exchange}$ . Update  $\mathcal{S}$ . Start over.  
             **end if**  
           **end for**  
         **end for**  
       **end for**  
     **end for**  
   **end while**  
   Return  $z^{best}$ .

---

we calculate the objective value of the new solution,  $z^{exchange}$ . Similar to the step in move neighborhood search, before calculating  $z^{exchange}$ , we need to determine the DC-supplier pairs that will serve the newly constructed subsets  $S_1$  and  $S_2$ , either via a search through the potential link set,  $\mathcal{L}$ , (in link-based improvement heuristic) or via solving *PII-d-MR* for each subset and supplier combination (in the *PII-d-MR*-based improvement heuristic).

If, with this exchange, we can improve  $z^{best}$ , we modify  $\mathcal{S}$  to include the new

updated  $S_1$  and  $S_2$ . We update the value of  $z^{best}$  and start the exchange neighborhood search again over the new modified  $\mathcal{S}$ . For this purpose, the exchange neighborhood searches for the “*first-best-exchange*.”

We continue in this manner until we can no longer improve  $z^{best}$ . At that point, we return  $z^{best}$  and the associated  $\mathcal{S}$  as the new solution.

#### VI.2.3.4. New Set Construction

In both link-based and *PII-d-MR*-based improvement, we use a *new set construction* module to modify the contents of  $\mathcal{S}$ . In particular, via new set construction, we increase the number of subsets in  $\mathcal{S}$  by one. While generating this new subset, we utilize the distances among retailers.

First, we initialize *smallestSize* to the smallest  $|S|$  such that  $S \in \mathcal{S}$ . Let  $S^*$  be the new set to be constructed. Let  $S_1$  be the first set in  $\mathcal{S}$ . We first find the two most distant retailers in  $S_1$ . We remove one of those retailers, say  $i$ , from  $S_1$  randomly and add  $i$  to the new set  $S^*$ . Afterwards, we loop through all the subsets to determine the retailers closest to retailer  $i$ . We include these retailers in the new subset  $S^*$  and eliminate them from their original subsets until the size of  $S^*$  is at least as large as *smallestSize*. Then, we add  $S^*$  to  $\mathcal{S}$ . Finally, we determine the objective value  $z^{newSet}$  associated with the new  $\mathcal{S}$  and return  $z^{newSet}$ . As in the move and exchange neighborhood search algorithms, before determining  $z^{newSet}$ , we first obtain DC-supplier pairs for each subset in  $\mathcal{S}$  according to the type (link-based or *PII-d-MR*-based) of the improvement heuristic. Display 10 summarizes this procedure.

#### VI.2.3.5. Overall Algorithm

In Display 11, we present the overall procedure for the link-based improvement heuristic. The link-based improvement heuristic starts with the initial feasible solution

---

**DISPLAY 10** The New Set Construction.

---

**0:** Set  $smallestSize = \min_{S \in \mathcal{S}} \{|S|\}$ . Construct  $S^*$ .  
**1:** Pick  $S_1 \in \mathcal{S}$  Find  $(i^*, j^*) = \arg \max_{i \in S_1, j \in S_i} d_{ij}$ .  
**2:**  $S_1 \leftarrow S_1 \setminus \{i^*\}$ .  $S^* \leftarrow S^* \cup \{i^*\}$ .  
**while**  $|S^*| < smallestSize$  **do**  
    **for**  $\forall S \in \mathcal{S}$  **do**  
        Find  $j^* = \arg \min_{j \in S} \{d_{i^*, j}\}$ .  
         $S \leftarrow S \setminus \{j^*\}$ .  $S^* \leftarrow S^* \cup \{j^*\}$ .  
    **end for**  
**end while**  
Calculate and return  $z^{newSet}$ .

---

that was obtained from the construction heuristic. We initialize the set of subsets of retailers  $\mathcal{S}$  and the best objective value ( $z^{best}$ ) from the solution of the construction heuristic. We also initialize the set of potential links  $\mathcal{L}$  as explained in Section VI.2.3.1. We set the initial cost  $z^0$  to infinity.

At the initialization, if  $\mathcal{S}$  contains more than one subset of retailers, we can perform a subset content improvement (move and exchange) neighborhood search. Otherwise, we can check to see if we can generate new subsets by invoking the *new subset construction* function described in Section VI.2.3.4. If generating new subsets proves beneficial and reduces the overall cost, we update the set of subsets of retailers with the new set obtained from the new subset construction function.

In either case, the subset's content can be improved through move and exchange neighborhood searches that are described in Sections VI.2.3.2 and VI.2.3.3, respectively. With the link-based heuristic, whenever the set content is changed, new DC-supplier assignments for each subset are determined via a search through the set of potential links. After move and exchange neighborhood searches, we modify the

set of potential links to check whether or not there are other DC-supplier links that return a better cost for the current  $\mathcal{S}$ . We modify  $\mathcal{L}$  using the link exchange function described in Section VI.2.3.1. Next, we check whether the number of subsets can be increased by invoking the new subset construction module. We continue in this manner by updating  $\mathcal{S}$  and  $\mathcal{L}$  through a series of operations until the cost stops improving. when that happens, we return the best cost  $z^{best}$  as the solution to the link-based improvement heuristic.

---

**DISPLAY 11** The Link-based Improvement Heuristic.

---

**0:** Obtain  $\mathcal{S}$  from the initial solution. Set  $z^{best} \leftarrow z^{CH}$ ,  $z^0 \leftarrow \infty$ . Initialize  $\mathcal{L}$ .

**1:**

**while**  $z^{best} < z^0$  **do**

$z^0 \leftarrow z^{best}$ .

Perform *Move*. If  $z^0 < z^{best}$ , set  $z^{best} \leftarrow z^{move}$ , and update  $\mathcal{S}$ .

Perform *Exchange*. If  $z^0 < z^{best}$ , set  $z^{best} \leftarrow z^{exc}$ , and update  $\mathcal{S}$ .

Perform *Link Exchange*. If  $z^0 < z^{best}$ , set  $z^{best} \leftarrow z^{link}$ , and update  $\mathcal{S}$  and  $\mathcal{L}$ .

Perform *New Set Construction*. If  $z^0 < z^{best}$ , set  $z^{best} \leftarrow z^{nsc}$ , and update  $\mathcal{S}$ .

**end while**

Return  $z^{best}$ .

---

#### VI.2.4. *PII-d-MR*-based Improvement Heuristic

As with the link-based improvement heuristic, our *PII-d-MR*-based improvement heuristic utilizes the simple neighborhood functions move and exchange neighborhoods as well as the new set construction module. The most important difference between the link-based and *PII-d-MR*-based improvement heuristics is the way the subsets of retailers are assigned to a DC and a supplier. In the next section, we describe how to make use of the continuous three stage joint location inventory problem

examined in Chapter IV when determining such assignments. Finally, Section VI.2.4.2 describes the overall procedure for the *PII-d-MR*-based improvement heuristic.

#### VI.2.4.1. Using the *PII-d-MR*

Recall that the *PII-d-MR* considers an integrated location and inventory problem in a three stage continuous facility location setting where the transportation costs are a function of distance. The solution of the *PII-d-MR* determines the location of the central DC given the locations of the supplier and the retailers as well as the inventory policy parameters of the central DC and the retailers.

Given a subset of retailers with known locations, we can determine the best central DC location and supplier to serve this subset by solving a *PII-d-MR* problem for each potential supplier that has enough capacity to serve this subset. In other words, by solving the *PII-d-MR* at most  $|\mathcal{K}|$  times, we obtain a continuous DC location associated with each candidate supplier as well as the corresponding cost. We pick the lowest cost solution to determine which DC and supplier to assign to this subset of retailers. However, since we are restricted to candidate DC locations, rather than a continuous DC location, we determine the three best candidate DC locations that are closest to the continuous DC location with the lowest cost. We evaluate each one of these candidate DC locations and their associated supplier by estimating the cost of assigning it to the current subset of retailers. In other words, we compare the cost of three DC-supplier pairs for the current subset of retailers. We assign the subset of retailers to the one with the lowest total cost. Before we move on to a new subset of retailers, we update the remaining capacity of the selected supplier. We continue in this manner until all of the subsets are served by a DC-supplier pair.

We invoke this procedure to determine the DC-supplier assignments whenever a subset's content is changed via move or exchange operations or a new subset is



created.

#### **VI.2.4.2. Overall Algorithm**

In this section, we describe the overall procedure for the *PII-d-MR*-based improvement heuristic. As we mentioned before, the main difference between the *PII-d-MR*-based and the link-based improvement heuristics is the way the subsets of retailers are assigned to the corresponding DCs and suppliers. However, the overall procedure is quite similar to the overall procedure of the link-based improvement heuristic.

The overall algorithm is initialized by the solution obtained from the construction heuristic. Then, if the set of subsets contains more than one subset, we employ move, exchange, and new set construction routines as long as the objective function continues to improve. If the set of subsets contains only one subset, then the new set construction function is invoked to check whether it would be beneficial to generate new subsets of retailers. If new subsets are created, we again employ move, exchange, and new set construction routines as long as the objective function keeps improving. Otherwise, we return the objective value of the construction heuristic.

We present the computational results for the performance of the *PII-d-MR*-based heuristic in Section VI.4.

### **VI.3. Benchmark Model**

In this section, we describe a benchmark model for comparing the solution of the integrated location and inventory model and evaluating the effectiveness of the integrated model *PIV*. The benchmark model (*BMIV*) follows the typical sequential framework where location decisions precede inventory decisions. Specifically, we first solve a two-stage PDS problem without inventory considerations to determine the

supplier and DC locations, the assignments of the selected suppliers to the open DCs, and the assignments of the open DCs to the retailers. Then, given these location and assignment decisions, we determine the inventory policy parameters at each open DC and their assigned retailers. In the next two subsections, we explain the details of the PDSD and the inventory components of the benchmark model.

### VI.3.1. Modeling *BMIV-PDSD*

Ignoring the inventory decisions initially, the problem *PIV* reduces to a two-level PDSD problem where a number of DCs and capacitated suppliers are to be located with respect to the retailer locations while minimizing the total cost in the system. The total cost includes the fixed cost of locating DCs and suppliers as well the transportation costs from the selected suppliers to the open DCs and from the open DCs to the retailers. These transportation costs are unit-based transportation costs and ignore the impact of inventory decisions.

Using the notation defined in Section VI.1, we formulate this PDSD as the following integer program:

$$\text{Min } \sum_{j \in \mathcal{J}} f_j X_j + \sum_{j \in \mathcal{J}} g_k Z_k + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} u_{ijk} D_i W_{ijk} \quad (BMIV-PDSD).$$

subject to

$$\sum_{j \in \mathcal{J}} Y_{ij} = 1, \quad \forall i \in \mathcal{I}. \quad (6.23)$$

$$Y_{ij} \leq X_j, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (6.24)$$

$$W_{ijk} \leq Z_k, \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J} \text{ and } \forall k \in \mathcal{K}. \quad (6.25)$$

$$W_{ijk} \leq X_j, \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J} \text{ and } \forall k \in \mathcal{K}. \quad (6.26)$$

$$\sum_{k \in \mathcal{K}} W_{ijk} = Y_{ij}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (6.27)$$

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} D_i W_{ijk} \leq C_k Z_k, \quad \forall k \in \mathcal{K}. \quad (6.28)$$

$$X_j \in \{0, 1\}, Y_{ij} \in \{0, 1\}, \quad \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}. \quad (6.29)$$

$$Z_k \in \{0, 1\}, W_{ijk} \in \{0, 1\}, \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J} \text{ and } \forall k \in \mathcal{K}. \quad (6.30)$$

In this formulation, the objective function minimizes the sum of the fixed cost of locating the DCs, the fixed cost of locating the suppliers, and the total transportation costs from the selected suppliers to the retailers through open DCs. In the objective function,  $u_{ijk}$  denotes the per-unit transportation cost from supplier  $k$  to retailer  $i$  through DC  $j$ , for all  $i \in \mathcal{I}$ ,  $j \in \mathcal{J}$ ,  $k \in \mathcal{K}$ . As we noted earlier in Chapters IV and V, we need to use an estimate for  $u_{ijk}$  due to lack of better information and we let  $u_{ijk} = \text{fracr}_{ij}^R + r_{jk}^{DC} 2 * 0.1\%$  for all  $i \in \mathcal{I}$ ,  $j \in \mathcal{J}$ ,  $k \in \mathcal{K}$ .

Constraints (6.23) ensure that each retailer is served by exactly one DC. Constraints (6.24), (6.25), (6.26) are the assignment constraints that ensure only selected facilities (DCs and suppliers) are utilized in the distribution system design. Constraints (6.24) ensure that a retailer can only be assigned to a selected DC. Constraints (6.25) ensure that a DC can only be assigned to a selected supplier. Constraints (6.26) ensure that a supplier can only be assigned to a selected candidate DC. Constraints (6.27) stipulate that if a link between a customer and a DC exists, it can be served

by only one selected supplier. In other words, these constraints ensure that a selected DC can be only assigned to only one selected supplier. Constraints (6.28) ensure the capacity restrictions at the suppliers. Finally, constraints (6.29) and (6.30) are the standard integrality constraints.

We solve *BMIV-PDSD* to optimality using CPLEX 9.0 and determine the optimal values of  $X_j$ ,  $Z_k$ , and  $W_{ijk}$  for all  $i \in \mathcal{I}$ ,  $j \in \mathcal{J}$ ,  $k \in \mathcal{K}$ .

### VI.3.2. Modeling *BMIV-Inv*

Once  $W_{ijk}$  is determined by solving *BMIV-PDSD* for all  $i \in \mathcal{I}$ ,  $j \in \mathcal{J}$ ,  $k \in \mathcal{K}$ , we use this information to form subsets of retailers that are served by a single DC and a single supplier. Then, for each such subset  $S$ , the inventory decisions of each retailer in that set and the DC serving that set are addressed by solving the corresponding SWMR problem using the approach in Roundy (1985).

Let  $\mathcal{S}$  be the set of subsets of retailers. Then, for  $S \in \mathcal{S}$ , there is a unique DC-supplier  $(j_S, k_S)$  pair that serves subset  $S$  such that  $j_S \in \mathcal{J}$  and  $k_S \in \mathcal{K}$ . Then, we formulate the inventory problem as follows:

$$\begin{aligned} \text{Min} \quad & \sum_{S \in \mathcal{S}} \left\{ \sum_{i \in S} \frac{(p_{i,j_S}^R + r_{i,j_S}^R d_{i,j_S}^R)}{T_i^R} + \frac{(p_{j_S,k_S}^{DC} + r_{j_S,k_S}^{DC} d_{j_S,k_S}^{DC})}{T_{j_S}^{DC}} + \frac{K_{j_S}^{DC}}{T_{j_S}^{DC}} + \sum_{i \in S} \frac{K_i^R}{T_i^R} \right. \\ & \left. + \sum_{i \in S} \frac{1}{2} H_{i,j_S}^R D_i T_i^R + \sum_{i \in S} \frac{1}{2} H_i^{DC} D_i \max\{T_i^R, T_{j_S}^{DC}\} \right\} \quad (BMIV-Inv) \end{aligned}$$

subject to

$$T_{jS}^{DC} = 2^{\nu_{jS}} T_b \quad \text{and} \quad \nu_{jS} \in \mathbb{Z}, \quad \forall j_S \in \mathcal{J}, S \in \mathcal{S}. \quad (6.31)$$

$$T_i^R = 2^{v_i} T_b \quad \text{and} \quad v_i \in \mathbb{Z}, \quad \forall i \in S, S \in \mathcal{S}. \quad (6.32)$$

$$T_{jS}^{DC} \in \mathbb{R}_+ \quad \text{and} \quad T_i^R \in \mathbb{R}_+, \quad \forall i \in S, \forall j_S \in \mathcal{J}, S \in \mathcal{S}. \quad (6.33)$$

This formulation can be decomposed for each subset  $S \in \mathcal{S}$ , and the inventory decisions for each retailer  $i \in S$ , and DC  $j_S$ ,  $j_S \in \mathcal{J}$ ,  $S \in \mathcal{S}$  can be obtained by solving a corresponding SWMR lot-sizing problem using the approach in Roundy (1985).

After the location, assignment, and inventory decisions are determined through solutions of the *BMIV-PDSD* and *BMIV-Inv*, we evaluate the cost of the benchmark model using the objective function of the original formulation. We compare this cost with the cost of the original formulation solved by the heuristic approaches to determine the value of integrated decision-making. We report our results in the next section.

#### VI.4. Numerical Results

In this section, we present numerical results to demonstrate the performance of the heuristic solution approaches. In Section VI.4.1, we explain the test data. In Section VI.4.2, we present results detailing the performance of the improvement heuristics. In Section VI.4.3, we compare the results of the benchmark heuristics with the solution approaches to determine the value of integrated decision-making.

All of the numerical results are obtained with algorithms implemented using C++ and run on a Pentium IV 3.2Ghz machine with 1 GB memory.

### VI.4.1. Experiment Data

To test the solution approaches, we generate 8 different data sets where the number of retailers, number of potential DCs, and the number of potential suppliers have two alternatives. Each data set consists of either 50 or 100 retailers, 20 or 40 potential DCs, and 5 or 10 potential suppliers. In each group, we have 50 problem instances, generated randomly using the uniform distributions given in Table 33, resulting in a total of 400 problem instances. To generate the supplier capacities, we use the average demand in the data instance. In particular, we generate the supplier capacity randomly using the uniform distribution  $U[L_{cap}, U_{cap}]$  for each supplier  $k \in \mathcal{K}$  where  $L_{cap} = 25 * \frac{\sum_{i \in \mathcal{I}} D_i}{|\mathcal{I}|} * \frac{|\mathcal{I}|}{50} = 0.5 * \sum_{i \in \mathcal{I}} D_i$  and  $U_{cap} = 50 * \frac{\sum_{i \in \mathcal{I}} D_i}{|\mathcal{I}|} * \frac{|\mathcal{I}|}{50} = \sum_{i \in \mathcal{I}} D_i$ . With this capacity restriction, on average, each supplier is capable of serving a random number of retailers between 25 and 50 if there are 50 retailers and between 50 and 100 if there are 100 retailers in the system.

In order to test the performance of the solution algorithms, we generate an extreme case where the fixed costs of facilities are 10% of the value that is suggested by the original data while the rest of the data is kept the same. In the original data set, the location costs for both DCs and suppliers are higher than the other operational costs. In the second data set, the operating costs dominate over location costs. We report results for both data sets in the following sections.

### VI.4.2. Performance of Improvement Heuristics

In this section, we report computational results illustrating the performance and duration of the improvement heuristics. We define the performance of an improvement heuristic as “improvement over the construction heuristic”, and we measure it as

$$\text{Improvement Over Construction (\%)} = \frac{Z(C) - Z(H.)}{Z(C)} \times 100,$$

TABLE 33. Parameter Values for Numerical Experiments.

Parameters	Range
$D_i$	U[350, 1400]
$f_j$	U[100000,150000]
$g_k$	U[200000,300000]
$C_k$	U[0.5 $\sum_{i \in \mathcal{I}} D_i$ , $\sum_{i \in \mathcal{I}} D_i$ ]
$K_i^R$	U[75, 300]
$K_j^{DC}$	U[400,1600]
$h_i^R$	U[5, 10]
$h_j^{DC}$	U[1, 4]
$p_{ij}^R$	U[100, 400]
$p_{ij}^{DC}$	U[425, 1700]
$r_{ij}^R$	U[0.75, 3]
$r_{jk}^{DC}$	U[1, 4]
$\mathbf{P}_i^R$	U[0,150] $\times$ U[0,150]
$\mathbf{P}_j^{DC}$	U[0,150] $\times$ U[0,150]
$\mathbf{P}_k^S$	U[0,150] $\times$ U[0,150]

where  $Z(C)$  is the objective value of the construction heuristic, and  $Z(H.)$  is the objective value of the appropriate improvement heuristic.

In Table 34, we present the minimum, average, and maximum percentage improvement over construction heuristic (C1) for both link-based (H1) and *PII-d-MR*-based (H2) improvement heuristics for eight data sets containing 400 instances. We also present the minimum average and maximum durations for construction, link-based improvement and *PII-d-MR*-based improvement heuristics.

According to the results in Table 35, with the link-based improvement heuristic, the average improvement gap ranges from 3.75% (data set 1) to 5.37% (data set 7). Not surprisingly, the link-based improvement heuristic performs better as  $|\mathcal{J}| \times |\mathcal{K}|$  increases since the size of the potential link set gets larger. A similar effect is observed for the maximum improvement gap. The maximum improvement gap ranges from 17.31% (data set 3) to 29.55% (data set 4).

Although the link-based improvement heuristic performs quite well, the *PII-d-*

TABLE 34. Comparison of Improvement Heuristics.

DATA	Improvement Over Construction					DURATIONS (in sec.)		
				H1	H2	C1	H1	H2
DataSet1	$ \mathcal{I} $	50	Min	0.00%	0.00%	0.00	0.00	0.28
	$ \mathcal{J} $	20	Ave	3.75%	23.19%	0.00	1.69	44.50
	$ \mathcal{K} $	5	Max	18.48%	52.04%	0.02	3.98	88.11
DataSet2	$ \mathcal{I} $	50	Min	0.00%	0.00%	0.00	0.02	0.86
	$ \mathcal{J} $	20	Ave	4.60%	22.73%	0.00	2.66	94.60
	$ \mathcal{K} $	10	Max	23.47%	50.93%	0.02	9.36	151.53
DataSet3	$ \mathcal{I} $	50	Min	0.00%	2.82%	0.00	0.73	23.08
	$ \mathcal{J} $	40	Ave	3.96%	25.92%	0.01	4.61	47.58
	$ \mathcal{K} $	5	Max	17.31%	46.62%	0.02	14.13	84.47
DataSet4	$ \mathcal{I} $	50	Min	0.00%	0.00%	0.00	0.02	0.59
	$ \mathcal{J} $	40	Ave	5.36%	22.96%	0.01	6.20	92.29
	$ \mathcal{K} $	10	Max	29.55%	50.47%	0.02	23.67	229.92
DataSet5	$ \mathcal{I} $	100	Min	0.00%	0.00%	0.00	0.02	0.92
	$ \mathcal{J} $	20	Ave	3.82%	17.95%	0.01	9.05	223.00
	$ \mathcal{K} $	5	Max	21.51%	37.98%	0.02	44.02	517.08
DataSet6	$ \mathcal{I} $	100	Min	0.00%	0.00%	0.00	1.92	2.11
	$ \mathcal{J} $	20	Ave	4.14%	16.20%	0.01	20.54	242.13
	$ \mathcal{K} $	10	Max	21.33%	38.79%	0.02	107.13	494.33
DataSet7	$ \mathcal{I} $	100	Min	0.00%	0.00%	0.00	0.02	0.94
	$ \mathcal{J} $	40	Ave	5.37%	16.56%	0.01	25.69	203.80
	$ \mathcal{K} $	5	Max	22.88%	33.48%	0.02	150.95	487.25
DataSet8	$ \mathcal{I} $	100	Min	0.00%	0.00%	0.00	0.03	1.63
	$ \mathcal{J} $	40	Ave	4.68%	17.77%	0.01	44.08	374.57
	$ \mathcal{K} $	10	Max	26.48%	36.71%	0.02	238.84	637.61

*MR*-based improvement heuristic performs much better. With the *PII-d-MR*-based improvement heuristic, instead of searching through a limited set of DC-supplier links, we approximate the location of the DC and determine the best DC-supplier link through the use of *PII-d-MR*. This simple component change has a great impact on the improvements. With the *PII-d-MR*-based improvement heuristic, the average improvement ranges from 16.20% (data set 6) to 25.92% (data set 3) which is almost equivalent to the maximum improvement gap with the link-based improvement heuristic. Furthermore, the maximum improvement gap with the *PII-d-MR*-based



improvement heuristic ranges from 33.48% (data set 7) to 52.04% (data set 1).

In terms of duration, all of the heuristics run quite rapidly. The duration of the construction heuristic is almost insignificant. Even for the instances with maximum duration, the run time is under 0.2 seconds. The link-based improvement heuristic runs quite fast as well. The average runtime of the link-based improvement heuristic is less than ten seconds for smaller networks and up to 45 seconds for larger ones. The maximum runtime of the link-based improvement heuristic does not exceed four minutes. Even though the *PII-d-MR*-based improvement heuristic is the most time-intensive heuristic, it still runs, on average, under five minutes with the runtime going up to ten minutes in the largest instances.

In Table 35, we present the comparative performances of the solution approaches for data with lower fixed location costs. For this data, the performance of the link-based improvement heuristic and the *PII-d-MR*-based heuristic are almost equivalent in terms of improvement over the construction heuristic. With the link-based improvement heuristic, the average improvement gap ranges from 1.10% to 1.79%, whereas the *PII-d-MR*-based heuristic performs slightly better with the average improvement ranging from 1.43% to 2.18%. Both improvement heuristics improve over the construction heuristic by as much as 21% which is quite significant. In terms of the runtime of the heuristics, the *PII-d-MR*-based heuristic is the most time intensive heuristic with the average duration ranging from 45 seconds to six minutes.

### VI.4.3. Value of Integrated Decision-Making

In this section, we present computational results to demonstrate the value of integrated decision-making over sequential decision-making in terms of location and

TABLE 35. Comparison of Improvement Heuristics for Data with Low Fixed Location Costs.

DATA		Improvement Over Construction				DURATIONS (in sec.)		
				H1	H2	C1	H1	H2
DataSet1	$\mathcal{I}$	50	Min	0.00%	0.00%	0.00	0.00	0.50
	$\mathcal{J}$	20	Ave	1.10%	1.64%	0.00	1.94	43.26
	$\mathcal{K}$	5	Max	5.80%	6.45%	0.02	6.44	80.61
DataSet2	$\mathcal{I}$	50	Min	0.00%	0.23%	0.00	0.02	37.89
	$\mathcal{J}$	20	Ave	1.76%	2.13%	0.00	5.05	88.57
	$\mathcal{K}$	10	Max	10.53%	11.82%	0.02	44.41	147.25
DataSet3	$\mathcal{I}$	50	Min	0.17%	0.00%	0.00	0.72	25.70
	$\mathcal{J}$	40	Ave	1.43%	2.18%	0.01	5.14	42.20
	$\mathcal{K}$	5	Max	11.18%	11.87%	0.02	14.20	64.89
DataSet4	$\mathcal{I}$	50	Min	0.00%	0.00%	0.00	0.02	0.55
	$\mathcal{J}$	40	Ave	1.67%	2.10%	0.01	11.67	87.41
	$\mathcal{K}$	10	Max	11.90%	13.49%	0.02	37.80	159.16
DataSet5	$\mathcal{I}$	100	Min	0.00%	0.00%	0.00	0.00	0.83
	$\mathcal{J}$	20	Ave	1.16%	1.45%	0.01	10.19	205.42
	$\mathcal{K}$	5	Max	21.03%	21.06%	0.02	35.20	378.30
DataSet6	$\mathcal{I}$	100	Min	0.04%	0.15%	0.00	3.11	1.69
	$\mathcal{J}$	20	Ave	1.79%	1.84%	0.01	23.67	214.05
	$\mathcal{K}$	10	Max	20.94%	21.33%	0.02	89.19	376.73
DataSet7	$\mathcal{I}$	100	Min	0.08%	0.18%	0.02	4.58	80.79
	$\mathcal{J}$	40	Ave	1.34%	1.43%	0.02	35.55	189.50
	$\mathcal{K}$	5	Max	8.43%	10.84%	0.03	118.47	373.17
DataSet8	$\mathcal{I}$	100	Min	0.20%	0.00%	0.02	6.38	177.82
	$\mathcal{J}$	40	Ave	1.69%	1.77%	0.02	59.72	376.56
	$\mathcal{K}$	10	Max	13.87%	13.90%	0.03	271.52	693.92

inventory decisions. We measure the value of integrated decision-making as

$$\text{Percentage gain over benchmark(\%)} = \frac{Z(BIV) - Z(H.)}{Z(BMIV)} \times 100,$$

where  $Z(BMIV)$  is the objective value of  $PIV$  evaluated with the decision variables obtained through the solution of  $BMIV$ , and  $Z(H.)$  is the objective value of the appropriate solution approach for  $PIV$ .

In Table 36, we report the minimum, average, and maximum runtimes for  $BMIV$

TABLE 36. Comparison of Benchmark Model with Solution Approaches.

		Duration	Gain (%)		
		(s.)	C1	H1	CH2
DataSet1	Min	1.50	24.09%	24.11%	31.84%
	Ave	13.91	42.09%	44.35%	55.87%
	Max	22.95	64.97%	64.97%	73.65%
DataSet2	Min	0.63	19.23%	21.46%	34.33%
	Ave	5.79	41.72%	44.57%	55.14%
	Max	13.59	67.40%	67.40%	73.15%
DataSet3	Min	18.95	24.56%	26.87%	36.80%
	Ave	53.14	40.66%	43.04%	56.22%
	Max	90.27	57.80%	58.26%	72.10%
DataSet4	Min	3.11	20.07%	20.07%	36.16%
	Ave	23.06	41.28%	44.52%	55.26%
	Max	63.30	71.03%	71.03%	72.12%
DataSet5	Min	0.80	36.86%	37.11%	47.72%
	Ave	2.21	51.03%	52.96%	60.01%
	Max	5.50	67.50%	67.50%	71.23%
DataSet6	Min	4.38	39.69%	46.08%	46.49%
	Ave	12.87	53.64%	55.60%	61.18%
	Max	53.83	61.99%	67.78%	73.56%
DataSet7	Min	1.86	37.68%	41.88%	45.31%
	Ave	10.78	51.33%	54.02%	59.57%
	Max	27.06	71.30%	71.30%	72.60%
DataSet8	Min	6.27	42.27%	46.45%	52.49%
	Ave	56.11	54.77%	56.98%	62.99%
	Max	160.55	75.74%	75.74%	75.74%

and the value of integrated decision-making using the construction, the link-based improvement, and the *PII-d-MR* improvement heuristics. All of the solution procedures, when compared to the *BMIV*, perform extremely well with average gains ranging from 40.66% to 54.77% for the construction heuristic, from 43.04% to 56.08% for the link-based improvement heuristic, and from 55.14% to 62.99% for the *PII-d-MR*-based improvement heuristic. Recall that, with the original data, the improvement over construction heuristics is quite significant for both the link-based and the *PII-d-MR*-based improvement heuristics, with the *PII-d-MR*-based improvement heuristic being

slightly better.

TABLE 37. Comparison of Benchmark Model with Solution Approaches for Data with Low Fixed Location Costs.

		Duration (s.)	Gain (%)		
			C1	H1	CH2
DataSet1	Min	0.58	17.19%	17.81%	18.09%
	Ave	0.60	20.23%	21.11%	21.54%
	Max	1.00	23.98%	24.99%	25.78%
DataSet2	Min	0.56	9.43%	10.16%	10.28%
	Ave	0.64	24.50%	25.83%	26.11%
	Max	1.22	33.09%	38.24%	39.12%
DataSet3	Min	1.72	19.20%	20.18%	20.55%
	Ave	1.82	24.26%	25.34%	25.91%
	Max	2.13	27.45%	32.56%	33.08%
DataSet4	Min	1.27	18.87%	20.68%	21.12%
	Ave	1.29	28.52%	29.73%	30.04%
	Max	1.56	37.29%	37.52%	38.65%
DataSet5	Min	0.72	4.28%	4.63%	4.65%
	Ave	0.77	13.62%	14.63%	14.88%
	Max	1.05	26.61%	30.77%	30.80%
DataSet6	Min	1.58	9.52%	10.33%	10.31%
	Ave	1.62	16.48%	17.98%	18.02%
	Max	1.95	25.50%	35.49%	35.81%
DataSet7	Min	1.77	11.48%	12.95%	12.57%
	Ave	1.82	17.86%	18.96%	19.03%
	Max	2.06	26.69%	27.39%	28.77%
DataSet8	Min	3.59	10.32%	11.64%	11.56%
	Ave	3.71	20.74%	22.09%	22.14%
	Max	4.05	27.83%	29.61%	29.78%

In Table 37, we report the results for the *BMIV* and the value of integrated decision-making using the data with lower fixed facility location costs. The average gain over the benchmark model is almost the same for all of the solution approaches, ranging from 13.62% to 30.04%. In this case, the savings from all of the solution approaches are less than the savings obtained with the original data. However, these gains are still significant, considering their monetary value. One important observa-

tion is that the runtime of the benchmark model *BMIV* is significantly faster using the data with lower fixed facility location costs because it is easier to solve the *BMIV-PDSD* problem in this situation.

## VI.5. Summary and Conclusions

In this chapter, we analyze a three-stage PDSD problem with inventory considerations where the first stage consists of retailers and the second and third stages consist of candidate locations of DCs and capacitated suppliers, respectively. Furthermore, we consider inventory decisions at the retailer and the DC levels. Each retailer replenishes its inventory from a specific established DC at the second stage via *direct shipments*, and each selected DC replenishes its inventory from a specific capacitated supplier located at the third stage via *direct shipments*. This problem is a generalization of the problems considered in the other chapters. In particular,

- it generalizes the three-stage continuous location problems (discussed in Chapter IV) to a discrete setting with multiple DCs and multiple suppliers;
- it generalizes the two-stage discrete location problems (discussed in Chapter V) due to the consideration of multiple capacitated suppliers and additional inventory decisions at the DC level.

After presenting two different formulations for *PIV*, we developed efficient heuristic solution approaches using advanced neighborhood search algorithms, i.e., the link-based improvement heuristic and the *PII-d-MR*-based improvement heuristic. The link-based heuristic utilizes an efficient search among potential DC-supplier pairs (links) to determine which subset of retailers is served by which link. On the other hand, the *PII-d-MR*-based improvement heuristic utilizes the algorithm developed for *PII-d-MR* to determine the DC-supplier assignments for each retailer set. Based on

our computational results, the *PII-d-MR*-based improvement heuristic finds better quality solutions (i.e., with lower cost) at the expense of duration. Nevertheless, both the link-based and the *PII-d-MR*-based improvement heuristics provide significant cost savings over the benchmark model.

As we have noted earlier, our focus in this chapter is on the development of efficient heuristic approaches. As a future research, these heuristics can be modified in several different ways. One way to modify these heuristics is to consider complete local search in move and exchange neighborhoods instead of “first-best-improvements.” Another modification is due to consideration of a parallel neighborhood search, i.e., considering move and exchange neighborhood search simultaneously instead of considering them sequentially. Finally, another noteworthy extension of this research would be the development of exact optimization algorithms for *PIV*.

## CHAPTER VII

## CONTRIBUTIONS AND CONCLUSIONS

Inbound and outbound distribution system design and redesign has become a major challenge for companies as they simultaneously try to reduce logistics costs and improve customer service in today's increasingly competitive business environment. Recent SCM initiatives, including third party logistics (3PL) and vendor managed inventory (VMI), are aimed at reducing total logistics costs by forcing geographically dispersed suppliers, manufacturers (plants), DCs, and retailers to unite as cross-functional teams. Hence, the *systems approach* is key for successful cost effectiveness across an entire supply chain network. With this motivation, the models and algorithms in this dissertation are aimed to provide efficient solutions to allow managers to benefit from these recent SCM trends by utilizing integrated decision-making.

This dissertation contributes to the SCM literature by investigating the impact of joint optimization of facility location and inventory decisions. In particular, this research makes an attempt to fill this gap in the literature by addressing the integration of facility location and inventory decisions for two-stage and three-stage distribution systems in continuous and discrete facility location models and by providing insights into

- the impact of *explicit* transportation costs on integrated decisions,
- the impact of *different* transportation cost functions on integrated decisions in the context of continuous facility location problems of interest,
- the value of integrated decision-making in different supply chain settings, and
- the performance of solution methods that jointly optimize facility location and inventory decisions.

Chapters III and IV considered continuous facility location problems in two-stage and three-stage distribution systems, respectively. In Chapter III, we analyzed the integration of facility location and inventory decisions in a continuous facility location problem setting for a two-stage supply chain. We showed that the underlying location problem in this model is similar to the classical Weber problem and the underlying inventory problem is a multi-retailer EOQ problem. In this chapter, we discussed three main transportation cost structures that arise in the context of real-life applications. Considering these different transportation cost structures, we developed and analyzed three different integrated location and inventory models, namely, the  $PI-Q$ ,  $PI-Qd$ , and  $PI-d$  in a continuous facility location problem setting for a two-stage supply chain.

The analysis of the  $PI-Q$  and the  $PI-Qd$  are trivial primarily due to the simplifications in the average annual cost function in these models. However, analysis of the  $PI-d$  is not trivial. We analyzed this problem under two different distance norms, namely Squared Euclidean and Euclidean distances. For the  $PI-d-SE$ , which is a convex nonlinear optimization problem, we devised an iterative algorithm that converges to the optimal central DC location and the parameters of the inventory policies of the retailers. The  $PI-d-E$  is also a nonlinear optimization problem. However, the objective function of the  $PI-d-E$  problem is neither convex nor concave in location and inventory decisions. For the  $PI-d-E$ , we proved that the existing retail locations are local minima for the central DC location, and we propose an effective search algorithm.

In Chapter IV, we analyzed the integration of facility location and inventory decisions in a continuous facility location problem setting for a three-stage supply chain. In this problem, in addition to the complexities in Chapter III, the central DC is an inventory holding location and its inventory replenishment must be coordi-



nated with the retailers' replenishments. Hence, this problem has similarities to the classical Weber problem and the single-warehouse, multi-retailer (SWMR) lot-sizing problem. We analyzed this problem under three main transportation cost structures that were introduced in Chapter III, namely, the *PII-Q*, *PII-Qd*, and *PII-d*. Similar to the analysis in Chapter III, for the *PII-Q* and *PII-Qd*, the location and inventory problems can be decomposed, but the *PII-d* requires joint optimization of location and inventory decisions since the objective function is not separable with respect to the corresponding decision variables.

We analyzed the *PII-d* with single and multiple retailers, the *PII-d-SR* and *PII-d-MR*, respectively. For the *PII-d-SR*, we presented a closed form solution for the location and inventory decision variables. Furthermore, we showed that the *PII-d-MR* is an extension of the SWMR problem that explicitly considers the DC location decision and the DC-location-dependent replenishment costs. We also discussed the fact that the *PII-d-MR* is a practical generalization of the Weber problem that explicitly considers inventory decisions and costs. We examined important characteristics of the *PII-d-MR* that relate to the SWMR and Weber problems, and we built on these characteristics for developing solution algorithms that do not assume any specific form of the distance function. We provided numerical results that demonstrate the efficient and effective performance of our algorithms and investigate the practical value of the *PII-d-MR*. We concluded that substantial cost savings can be realized by the integration of location and inventory decisions in the continuous facility location setting and that one should give serious consideration to this approach for logistical coordination.

Chapters V and VI considered discrete facility location problems in two-stage and three-stage distribution systems, respectively. In Chapter V, we analyzed the integration of facility location and inventory decisions in a discrete facility location

problem setting for a two-stage supply chain. In particular, this model generalized the classical FCFLP problem in two-stages to consider inventory decisions at the retailers where it is modeled as a multi-retailer EOQ problem. We classified this problem with respect to capacity restriction. When the potential DCs do not have a capacity restriction, the problem  $P_{III}$  is a generalization of the uncapacitated FCFLP. For the  $P_{III}$ , we showed that the problem can be converted to an equivalent uncapacitated FCFLP, and, hence, can be solved via the techniques developed for the FCFLP.

For the capacity restricted  $P_{III}$ , we first expanded on the definition of capacity in distribution systems. We considered four different types of capacity restrictions: throughput capacity restrictions, storage capacity restrictions, dispatch capacity restrictions at the DCs, and the truck/cargo capacity restrictions on the transportation links. Under these different capacity restrictions, we revised the model  $P_{III}$ , to the  $P_{III-PC}$  with throughput capacity, the  $P_{III-SC}$  with storage capacity, the  $P_{III-DC}$  with dispatch capacity, and the  $P_{III-TC}$  with truck capacity. For the  $P_{III-PC}$  and  $P_{III-TC}$ , we showed that the models can be converted to the classical *capacitated* FCFLP and, hence, can be solved via the techniques developed for the *capacitated* FCFLP. The  $P_{III-SC}$  and the  $P_{III-DC}$  include the order quantities of retailers in defining the capacity restrictions, which makes these problems more challenging. For these problems, we developed generalized Benders decomposition based algorithms by exploiting the structure of the problem. Compared to the generic MINLP solver, the algorithms provide good quality solutions in short durations.

In Chapter V, for problems  $P_{III}$  and  $P_{III-PC}$ , we identified the conditions where integrated decision-making is beneficial and quantified the benefits from integrated decision-making by comparing the solutions of  $P_{III}$  and  $P_{III-PC}$  with respective benchmark models.

Finally, in Chapter VI, we analyzed the integration of facility location and inven-

tory decisions in a discrete facility location problem setting for a three-stage supply chain. This chapter includes the most comprehensive model by generalizing the production distribution system design problems with inventory decisions. The problem *PIV* in this chapter generalizes the problem settings in three-stage continuous facility location models in Chapter IV and two-stage discrete facility location models in Chapter V, since multiple capacitated suppliers and the assignment of DCs to the suppliers are considered. For the *PIV*, we provided two formulations including a non-linear integer programming formulation and set-partitioning problem. We developed highly efficient solution methodologies that involve a greedy construction heuristic, a combined neighborhood of single-moves and single-exchanges in a local search improvement procedure. The algorithms perform extremely well, both in terms of solution duration and gap over the benchmark model that solves *PIV* in a sequential manner.

The models presented in Chapters III through VI are state-of-the-art models aimed at investigating the integration of location and inventory decisions. The models and analysis, apart from the practical contributions, make significant theoretical contributions in modeling and algorithmic development. The computational results also shed light on the importance of integrating location and inventory decisions in different supply chain settings. We established that substantial cost savings can be achieved in certain contexts and with certain parameters. Hence, this research will advance the practical knowledge on supply chain design at a time when such knowledge is crucial in the real world for optimizing supply chain performance.

Building on these models and analysis, research in this dissertation can be extended in several directions in the future. To mention a few, one may consider

- stochastic demand at the retailers,

- vehicle routing instead of direct shipments from the DC to the retailers,
- multi-period decision-making where the demand and cost parameters may change periodically with respect to changes in economy, and
- supply chain coordination and contracting in decentralized systems

in joint inventory-location models.

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## VITA

Burcu Barış Keskin received a Bachelor of Science degree in Industrial Engineering from Boğaziçi University, Turkey, in 2001. In July 2001, she joined the Industrial Engineering graduate program at Boğaziçi University. She conducted research in the areas of supply chain management and inventory theory under the guidance of Professor Taner Bilgiç. In July 2002, she was granted a Master of Science degree in Industrial Engineering with a thesis titled “Integrated Lot-sizing for a Single Supplier Multi-retailer Supply Chain Under Asymmetric Information.”

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