UNIVERSIDAD NACIONAL DE COLOMBIA

Fuzzy Linear Programming with Interval Type-2 fuzzy constraints

Programación Lineal Difusa con restricciones Difusas Tipo-2 de Intervalo

Juan Carlos Figueroa-García

Universidad Nacional de Colombia
Facultad de Ingeniería, Departamento de Ingeniería Industrial y de Sistemas
Bogotá, Colombia
2014

Fuzzy Linear Programming with Interval Type-2 fuzzy constraints

Programación Lineal Difusa con restricciones Difusas Tipo-2 de Intervalo

Juan Carlos Figueroa-García

Thesis presented as a partial requirement for obtaining the degree of:
Ph.D. en Ingeniería - Industria y Organizaciones

Advisor:
Ph.D. Germán Jairo Hernández-Pérez

Research line:
Operations Research - Fuzzy sets Theory
Research Group:
Algorithms and Combinatorics (ALGOS)

Universidad Nacional de Colombia
Facultad de Ingeniería, Departamento de Ingeniería Industrial y de Sistemas
Bogotá, Colombia
2014

Dedicatory

This work is specially dedicated to my mother María Irene García-Bautista, my father Antonio Figueroa-Barón, and my sister Mónica Figueroa-García, for their invaluable support through all these years, from the very beginning to this final step...

Nothing was possible without them...

Aknowledgements

I would like to thank to everyone who made possible this work in the Universidad Distrital Francisco José de Caldas and the Universidad Nacional in Bogotá - Colombia. A special thank to everybody in LAMIC, ALGOS and MMAI research groups, Martín Caberio & Vladik Kreinovich at UTEP - USA, Dharmaraja Selvamuthu, Aparna Mehra & Suresh Chandra at IIT Delhi - India, Yuri Chalco-Cano & Heriberto Román-Flores at Universidad de Tarapacá - Chile, and Roman Neruda at Charles University in Prague - Czech Republic, for their invaluable help and hospitality, and the reviewers of this thesis for their invaluable comments that helped me to improve it a lot.

Cooperation visits

The following visits and research stays helped me to improve this work:

- ELAVIO (Escuela Latinoamericana de Verano en Investigación de Operaciones). February 5-11, 2012 - Bento Ǫonzalves, Brasil.
- Czech Academy of Sciences (Invited talks and cooperation). April 21-28, 2013 - Prague, Czech Republic.
- Universidad de Tarapacá (Invited talks and cooperation). September 1-7, 2013 - Arica, Chile.
- University of Texas at El Paso (Invited talks and cooperation). October 1 to November 3, 2013 - Texas, USA.
- Indian Institute for Technology (IIT-UN bilateral cooperation). November 4 to December 6, 2013 - New Delhi, India.

Abstract

Fuzzy Linear Programming (FLP) is an extension of linear programming that allows us to deal with inexact coefficients like the ones that appear in real decision-making problems where human estimations of those coefficients involve uncertainty.

For the past 40 years there has been a vibrant research activity on methods to solve Fuzzy Linear Programming (FLP) problems with uncertain coefficients, especially its constraints. Some real decision-making problems need to represent multiple human expert estimates with no consensus among their perceptions or opinions, so there is a need for finding solutions to this kind of practical issues. In this thesis, both a general model and a method for solving FLP problems whose constraints are represented as Type-2 fuzzy numbers based on the perceptions and opinions of multiple experts, are proposed.

Keywords: Fuzzy set theory, Operations Research, Linear programming, Optimization, Type-2 fuzzy sets.

Abstract

La Programación Lineal Difusa (FLP) es una extensión de programación lineal que nos permite manejar coeficientes inexactos como aquellos que aparecen en situaciones reales de toma de decisiones donde dichos parámetros incluyen incertidumbre proveniente del razonamiento humano.

En los últimos 40 años ha habido un vibrante movimiento en investigación alrededor de métodos para resolver problemas FLP con coeficientes inciertos, especialmente sus restricciones. Algunos problemas reales de toma de decisiones necesitan múltiples estimaciones dadas por múltiples expertos, los cuales no necesariamente están de acuerdo en sus opiniones y/o percepciones, por lo que se necesita encontrar soluciones a dichos problemas. En esta Tesis Doctoral, se propone tanto un modelo como un método para la resolución de problemas FLP cuyas restricciones son números difusos Tipo-2 provenientes de múltiples opiniones y percepciones de múltiples expertos.

Palabras clave: Teoría de conjuntos difusos, Investigación de operaciones, Programación lineal, Optimización, Conjuntos diifusos Tipo-2.

Contents

Aknowledgements **vii**

Abstract **ix**

I Preliminaries **3**

1 Introduction and Motivation **5**

2 The initial proposal **7**
2.1 Problem Identification 7
2.1.1 Objectives 9
2.2 Antecedents and rationale 9
2.2.1 From Type-1 to Interval Type-2 LPs 10
2.3 The Linear Programming (LP) problem 11
2.3.1 LP methods and algorithms 11
2.4 Basics on Fuzzy sets theory 12
2.4.1 Computing with words 12
2.4.2 Imprecision and fuzzy sets 12
2.4.3 Crisp, Type-1 and Type-2 fuzzy sets concepts 13
2.5 The Fuzzy Linear Programming model (FLP) 19

II The Proposal **23**

3 The IT2FLP model **25**
3.1 Uncertain constraints 25
3.2 The IT2FLP model 29
3.3 Methodology for solving an IT2FLP 30
3.4 Feasibility of an IT2FLP 31
3.5 Convexity of an IT2FLP 32
3.6 Existence of an optimal solution of an IT2FLP 32

4 The concept of optimal solution under fuzzy uncertainty 35
4.1 What is a fuzzy optimal solution? 35
4.1.1 Fuzzy optimal solution 36
4.1.2 Fuzzy global optimal solution 37
4.1.3 Operation points 38
4.1.4 Application example 39
4.1.5 Computing the set of optimal solutions $\tilde{z}$ 40

5 Solution procedure of an IT2FLP 43
5.1 First method 43
5.2 Second method 44
5.2.1 Selecting α 46
5.3 Optimization software 47

6 Application examples 49
6.1 Production Planning example 49
6.1.1 FLP results 54
6.1.2 IT2FLP results 54
6.1.3 Discussion of the results 56
6.2 Simulated Data example 56
6.2.1 Discussion of the results 58
6.3 Behavior of the Zimmermann's method into $\tilde{z}$ 58
6.3.1 Discussion of the results 59

7 Concluding Remarks 63
7.1 Contribution of the Thesis 64
7.1.1 Publications: 64
7.1.2 Awards 65

Appendix A 65

Part I

Preliminaries

Chapter 1

Introduction and Motivation

During the last six decades, optimization and heuristic techniques became an important tool for decision makers, usually managers or engineers. Uncertainty in decision making increases the complexity of the problem because the solution method have to deal with: i) an appropriate representation of the uncertainty, ii) the effects on the behavior of the system due to the possible fluctuations introduced by the uncertainty and iii) a strategy or method to find a decision or solution that balances adequately the goals of the decision making problem with the uncertainty of achieving them.

The two most popular frameworks used to represent uncertainty are probability theory (random variables) and fuzzy logic (fuzzy sets) alongside appropriate methods to analyze and solve decision-making problems. While probability theory is based on measuring repeated events to compute its frequency and infer the frequency of occurrence of known events in order to have an idea of the behavior of a random variable, fuzzy sets measure the perception of concepts and words regarding a random variable.

Probability theory originated on early 1700's with Bernoulli's works on probable events, and started to be developed to measure randomness of the occurrence of known events which are defined over a well known variable a.k.a *Probabilistic Uncertainty*. Its application over different scenarios is wide, worth to mention statistical inference, experimental design, multivariate analysis, and stochastic optimization.

As fast as Lofti A. Zadeh created fuzzy sets in 60's, its use in computing with words (as Type-n fuzzy uncertainty) was proposed, where his main focus were classical fuzzy sets (a.k.a Type-1 fuzzy sets). Type-1 fuzzy sets measure numerical uncertainty regarding a word or concept defined over a variable a.k.a *Fuzzy Uncertainty*. The analysis of non-probabilistic uncertainty is a recent area that offers an appropriate starting point to cover these problems, visualized in a fuzzy sets environment in the sense that no perfect measurement exists. From late 70's, decision making has been influenced by Zadeh's results, so Linear programming (FLP) has applied its definitions to cover imprecision on their parameters, having successful applications in the last four decades.

This way, both approaches are considered itself as theories, where their most popular optimization applications are stochastic and fuzzy programming including stochastic linear programming and fuzzy linear programming. Many problems not only include randomness and/or numerical uncertainty but fuzzy, such as the one coming from imprecision of numerical measures and human perception about its meaning (usually expressed by words and concepts).

On the other hand, many LP cases have lost numerical data or they have unreliable data, so the parameters of the model have to be defined using experts as information source. Perceptions of different experts about the parameters of an LP include two kinds of uncertainty: numerical and linguistic (words and concepts), so LPs in those conditions cannot be fully covered by classical fuzzy sets, or statistical approaches.

Uncertainty related to words and Type-2 fuzzy sets for words are called to be important fields of study in the solution of complex problems. This work shows a way to use Type-2 fuzzy sets to cover uncertainty coming from perceptions of multiple experts alongside numerical imprecision over the constraints of an LP problem, in order to solve LPs that include linguistic uncertainty.

This Doctoral thesis is divided into two main parts: a first one that presents preliminary concepts and the main aspects of the initial proposal, and a second part that shows the results of the work, including two optimization methods for solving LPs with Interval Type-2 fuzzy constraints, two application examples alongside its respective explanations, and the obtained publications of this thesis. Finally, this report shows some concluding remarks, and the bibliography used in the work.

Chapter 2

The initial proposal

2.1 Problem Identification

Many decision making problems has no either historical data or information, so the knowledge of the experts is an alternative for solving the problem. LP models are popular, so its use in cases where only information from experts is available, is an interesting field to be covered.
A common issue presented in many applications is related to the perception of different experts about its parameters. This leads to misspecification increasing its uncertainty level. So, those opinions should be considered by the analyst when trying to find an optimal solution. Figure 2.1 shows how different opinions can lead to define a Type-2 fuzzy set in decision making scenarios.

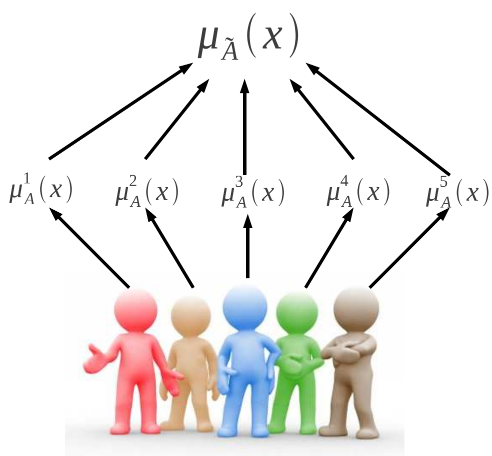

Figure 2.1: Different experts' opinions

The relatively recent use of Type-2 fuzzy sets has opened an important window for modeling engineering problems under uncertainty. Its applicability to LP, decision making and related problems seems to be wide. Furthermore, the design of optimization methods for

LP problems involving linguistic uncertainty is an opened problem, where Type-2 fuzzy sets appeared as useful uncertainty-based information measures.
It seems that the natural course in fuzzy optimization is going to uncertain fuzzy sets from crisp sets. More specifically, the research problem is related with the following aspects:

- There is no an optimization method for handling the experts' opinions and perceptions of different variables of the system.
- Different kind of uncertainties can be presented when an LP model is defined, so the use of linguistic uncertainty arises as a new field to be covered.
- The absence of decision making techniques for solving discord fuzzy LP problems in the sense that different experts can define different fuzzy sets to represent his individual perception about imprecise variables of the system.

Then, the main question of the problem becomes:

How to involve higher uncertainty levels to LP problems?

This problem can be divided in two main aspects: a first one related to mathematical definitions about the model itself and a second one related to optimization methods that potentially should be used to find solutions to the problem.
Some additional questions complements the problem:

1. What analytical and geometrical issues has the use of Interval Type-2 fuzzy sets on the constraints of the LP model?
2. Can the classical optimization methods be extended to LP models with Interval Type-2 fuzzy constraints?
3. What mathematical and application issues should be kept in mind for defining a linear programming problem in this framework?

This work focuses on provide basic mathematical definitions about linguistic uncertain LP models and the design of an optimization method for this kind of problems. As always, some selected theories should be taken as a basis for extending the problem by means of Type-2 fuzzy sets and solve it.

Remark 1 *An important limitation of the use of fuzzy sets in some applications is related to the definition of their shapes. This problem is not addressed in this work since LP models are not intended to represent the dynamic behavior of a system but reach optimal solutions over predefined systems. This way, we assume that the experts' definitions of the shapes of fuzzy sets lead to feasible problems like in classical LP models.*

2.1.1 Objectives

General Objective :

Define an optimization model that allows to solve LP problems where its constraints includes linear Interval Type-2 fuzzy sets.

Specific Objectives

- Formally define the concept of a solution of an Interval Type-2 fuzzy LP problem.
- Develop a formal framework to solve LP problems with Interval Type-2 constraints.
- Evaluate the potential techniques that can find solutions to Interval Type-2 fuzzy LP problems, including an extension of the Zimmermann's method.
- Develop an optimization method for a linear programming model with IT2FS constraints.
- Validate the developed method through both simulated and study cases related to industrial engineering sciences like logistics and production planning.

2.2 Antecedents and rationale

Optimization, heuristics and statistical techniques have had an important role in decision making during the last sixty years, being useful for engineers, economists, staticians, managers and people that make decisions as best as possible according to their goals.
Uncertainty is an important aspect of information which is defined in different ways, depending on the problem and its characteristics. For our purposes we refer to uncertainty in two ways: uncertainty from randomness that is defined as probabilistic measures where there is no discussion about its definition, so we can say that probabilities are *Precise*. On the other hand, uncertainty coming from the perception of an expert about a concept related to a variable which can be measured by a *Concept or Word* with *Imprecision*.
Standard optimization techniques assume that the parameters of the system are uncertainty-free and precise, which is not true in many real applications. Stochastic programming deals with optimization problems where its parameters are random and comes from probabilistic measures. On the other hand, the analysis of non-probabilistic uncertainty coming from imprecision about a variable $x \in X$ offers an appropriate starting point to cover problems related to imprecise values of $x \in X$. This imprecision may appear in the parameters of an LP model, so its potential applications are wide. LP models are among the most used techniques for decision making under uncertainty due to its applicability, efficiency, and capability to fit into hybrid concepts.

2.2.1 From Type-1 to Interval Type-2 LPs

As we pointed out before, a Type-2 fuzzy set can deal with multiple experts' opinion. This way, Type-1 Fuzzy LP *(T1FLP)* deals with imprecision coming from a single expert, and Interval Type-2 Fuzzy LP *(IT2FLP)* deals with imprecision coming from multiple experts which leads to have uncertainty over the linguistic label of a set A.

The use of Type-2 fuzzy sets in LP instead of Type-1 fuzzy sets allows us to handle higher uncertainty levels which come from typical scenarios where the problem is being defined by multiple experts, and they are not in agreement of using a single fuzzy set for representing their perceptions about the problem.

The classical models proposed by Zimmermann, Verdegay, Rommelfanger, and Ramík among others are based on the idea of satisfying one or more fuzzy objectives regarding a set of fuzzy constraints, where all sets are defined by a single expert. The presented work is intended to extend the results of the soft constraints method to an IT2FLP problem defined by multiple experts, using a similar reasoning done by classical authors.

The use of fuzzy sets in LP problems has been treated by different approaches, all of them trying to deal with imprecision on the parameters of a LP model through a linguistic variable that defines a fuzzy set in order to measure the perception of an expert about a parameter. Uncertainty related to linguistic variables (words) is given by different sources, so Type-2 fuzzy sets for words is a tool that deals with the opinion and perception of multiple experts for solving complex problems.

We can summarize optimization techniques as follows:

Deterministic	$\rightarrow$	Simplex, Interior point methods
Probabilistic	$\rightarrow$	Stochastic processes, Stochastic optimization
Imprecise-valued	$\rightarrow$	Fuzzy Optimization, Fuzzy LP
Linguistic	$\rightarrow$	Type-2 fuzzy sets

Now, the problem of having higher uncertainty levels in fuzzy optimization is an open problem which we propose to be handled through Type-2 fuzzy sets, as a way to represent the knowledge of the experts about the problem.

Fuzzy Linear Programming (FLP) is an approach which deals with imprecision on the parameters of an LP model using fuzzy sets. This model still faces with some open problems such as non-polynomial complexity[1], integer problems, etc, and our proposal which is related to higher uncertainty levels.

[1]Although the simplex method proposed by Dantzig [10] has polynomial resolution time in practice, Klee and Minty [11] demonstrated that there is a family of instances where the simplex method spends an exponential solving time. This gave a formal framework to later develop *Interior Point* methods.

2.3 The Linear Programming (LP) problem

The mathematical representation of the LP model is as follows:

$$\begin{aligned} &\underset{x}{\text{Max}}\ z = c'x + c_0 \\ &s.t. \\ &Ax \leqslant b \\ &x \geqslant 0 \end{aligned} \tag{2.1}$$

where $(x, c) \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

According to Mora [12], Bazaraa [13], Dantzig [10], and Dantizg & Thapa [14] this model assumes that all its parameters are deterministic. This means it has neither uncertainty nor imprecision on A, b, c and c_0. In other words, the parameters of a classical LP model are considered as constants.

A crisp LP problem $\{c'x \mid Ax \leqslant b, x \geqslant 0\}$ is the problem of solving n inequalities of the system $Ax \leqslant b$ achieving the best value of given a goal $z = c'x$, where a solution is a vector $x \in \mathbb{R}$ which satisfies all the constraints of the problem $Ax \leqslant b$, $\{c'x \mid Ax \leqslant b, x \geqslant 0\}$. An optimal solution x^* is an extreme point with coordinates $(x_1, x_2, \cdots, x_n)$ for which there is no improvement of z, that is $z^* = \sup\{c'x \mid Ax = b, x \geqslant 0\}$, so we also have that $z(x^*) \geqslant z(x)\, \forall\, x \in \mathbb{R}$.

From another point of view, the LP problem is based on the idea of composing a feasible region which is simply the intersection among all constraints (defined as inequalities), and then find the best value of the goal function $z = c'x$. Now, a solution is then any point $x \in \mathbb{R}^n$ enclosed into $\cap_{i \in \mathbb{R}^m} Ax \leqslant b_i$, and its optimal solution is simply $\sup\limits_{z=c'x} \{\cap_{i \in \mathbb{R}^m} Ax \leqslant b_i\}$.

2.3.1 LP methods and algorithms

The LP model has well known optimization methods, and some extensions of the LP general model shown in (3.11) have well defined solutions. The well known simplex method proposed by Dantzig [10], and Dantizg & Thapa [14] and some interior point methods (see Karmarkar [15] and [16], Mehrotra [17], Mehrotra & Ye [18], Mizuno et.al [19] and Roos et.al [19]) are widely applied in LP problems due to their efficiency and robustness.

In this work, we are not focused on the complexity of the algorithm used to achieve a solution, so we apply classical simplex routines over GAMS® and MatLab®. Perhaps the selection of an efficient algorithm is an important computational aspect in optimization, our goal is to find a solution of the problem before designing specialized algorithms.

2.4 Basics on Fuzzy sets theory

2.4.1 Computing with words

As fuzzy sets were introduced by Lofti A. Zadeh in 1965, they became an alternative for solving uncertain problems. The term *"Computing with Words"* was coined by Zadeh [20], [21] and Jerry Mendel [22], [23], [24], [25] to refer to the way of how to represent words, ideas and concepts for solving problems. This modeling framework tries to compute words as inputs of a computing machine in order to obtain a word as an output while handling linguistic uncertainty through Type-2 fuzzy sets (or higher-order fuzzy sets).
Although this framework is still not fully usable in optimization problems, its concepts, definitions of linguistic uncertainty, and uncertain fuzzy sets are compatibles with optimization. The aim and scope of this work is to define a general model for LP problems with parameters treated as Interval Type-2 Fuzzy Sets (IT2FS) and design an optimization method. The main idea is to deal with the perception and opinion of multiple experts about the parameters of an LP model under linguistic uncertainty as a decision making tool.

2.4.2 Imprecision and fuzzy sets

Classical uncertainty is defined by probabilistic concepts and all their measures are based on experimental evidence. This means that both probabilistic and stochastic approaches assume that all observations are realizations of a well known process, and the underlying uncertainty is considered as uncontrollable noise.
On the other hand, some problems have imprecise information about X. An alternative way to handle this problem is by asking an expert for its own perception of the variable, so his/her opinion may be a starting point to represent it.
As usual, the expert does not know the truth about X and different experts can have different perceptions. Next sections are intended to explain how the concepts of both imprecision and/or uncertainty can be applied in optimization.

Imprecision

The concept of imprecision is intimately related with uncertainty and the information needed to define a specific variable. According to Klir [26], [27] and Wang and Klir [28], uncertainty involved in any problem-solution situation can be given by some information deficiency, so this concept is related to the information perceived by the analyst.
Another point of view is given by the mathematical body of evidence of X. X can be defined as a constant, but it also can be defined by any function that represents imprecision over the universe of discourse, so X can be represented through different measures: Fuzzy, Rough

Sets, etc.

Uncertainty

According to Mendel [29], [22] and Klir & Yuan [26], the following sources of uncertainty can be found in the quantitative analysis:

Uncertainty: Attribute of information, it is related to imprecision, vagueness and ambiguity measures of a granular variable X, for instance, an interval valued variable $x \in [x_1, x_2]$.

- **i)** *Fuzziness*: Lack of definite or sharp distinctions.
- **ii)** *Ambiguity*: One to many relationships.
  - **a)** *Nonspecifity*: Two or more alternatives are left unspecified.
  - **b)** *Discord*: Disagreement in choosing among several alternatives.

Roughly speaking, uncertainty represents higher levels of information deficiency, and their measures are more complex than classical fuzzy sets. Some recent approaches used to measure this kind of uncertainty are Type-2 fuzzy sets, Intuitionistic fuzzy sets and interval-valued sets, which are defined in next sections.

2.4.3 Crisp, Type-1 and Type-2 fuzzy sets concepts

First, some basic suppositions and concepts are presented before define the focus of this work.

Membership function

The concept of *membership* function is a generalization of the *indicator* function used in classical sets theory. In this work, the indicator function is applied as membership function without distinction. The definition of a membership function $\mu_A(\cdot)$ is:

$$\mu_A(\cdot) \, : \, X \rightarrow [0, 1] \tag{2.2}$$

Remark 2 *In this work we define the Universal Set* $(x \in U)$ *as the space of real numbers, so we have that* $(x \in \mathbb{R})$.

Singleton and Crisp sets

A set S is called singleton $\{S\}$ if has a single element $x \in \mathbb{R}$. In the real numbers $\mathbb{R}$, S is a constant.

$$S := \{x : x = S\} \tag{2.3}$$

This implies that $\mu_S(x) = 1$, and $\mu_S(\cdot) = 0$ for every $x \notin S$

$$S : X \to \{0, 1\}$$

A set S is called *Crisp* if it is composed by an interval of elements that either belong or not to S, that is:

$$\mu_S(\cdot) = \begin{cases} 1 & \text{for } x \in S \\ 0 & \text{for } x \notin S \end{cases} \tag{2.4}$$

where S is an interval of elements $[\check{x}, \hat{x}]$, $x, \check{x}, \hat{x} \in \mathbb{R}$.
Note that the membership function of a crisp set is defined as

$$\mu_S(\cdot) \,:\, X \to \{0, 1\} \tag{2.5}$$

Formally, the LP problem uses *Singleton* namely *Constant*, and *Crisp* namely *Deterministic* or *Boolean* sets, so classical optimization LP models and algorithms are based on classical sets operations.

Type-1 fuzzy sets

A fuzzy set is a generalization of a *Crisp* or Boolean set. It is defined on an universe of discourse X and is characterized by a *Membership Function* namely $\mu_A(x)$ that takes values in the interval [0,1]. A fuzzy set A may be represented as a set of ordered pairs of a generic element x and its grade of membership function, $\mu_A(x)$, i.e.,

$$A = \{(x, \mu_A(x)) \,|\, x \in X\} \tag{2.6}$$

Now, A is contained into a family of fuzzy sets $\mathcal{F} = \{A_1, A_2, \cdots, A_m\}$, each one with a membership function $\{\mu_{A_1}(x),\, \mu_{A_2}(x); \cdots, \mu_{A_m}(x)\}$ where $\mu_A(\cdot)$ is a measure of belonging of x regarding the fuzzy set A.
Here, A is a *Linguistic Label* that defines the sense of the fuzzy set through the concept A, and it is the way how an expert perceives X and the shape of A_i. These concepts are explained as follows.

Definition 1 (Membership Function $\mu_A(x)$) *The Membership Function of a set A called $\mu_A(x)$ is a function which represents the degree of similarity of an element $x \in X$ to the fuzzy set A. It takes values in the interval [0,1], that is:*

$$\mu_A(x) : X \to [0, 1] \tag{2.7}$$

Formally, $\mathcal{F}$ is the set of all possible membership functions that $x \in X$ could take. This leads to an infinite number of fuzzy sets, which can be obtained as follows:

$$\mathcal{F} = [0, 1]^X$$

Henceforth we do not make any distinction between a membership function $\mu_A(\cdot)$ and its linguistic label A when refer to a membership function, and we will refer to a Type-1 fuzzy set by using $\mu_A(x)$ or A.
In this approach, $\mu_A(x)$ is defined either by mathematical evidence or by an expert who establish his perception about X by means of A and its membership function $\mu_A(x)$. This flexibility becomes fuzzy sets into an alternative tool for solving uncertain problems which comes from imprecision. Some basic definitions of fuzzy sets are:
The *height* of A, $h(A)$, of a fuzzy set is the largest membership degree obtained by any element in A, this is

$$h(A) = \sup_{x \in X} A(x)$$

Now, the *Normalization Law* of a probability function sets that

$$\int_{-\infty}^{\infty} f(x;\theta)dx = 1$$

while a *Normalized* fuzzy must fulfill

$$\max_{x \in X} \{\mu_A(x)\} = 1$$

Remark 3 *There is the possibility of having non-normalized fuzz sets, this means* $\max_{x \in X} \{\mu_A(x)\} \neq 1$. *In this work we only consider normalized fuzzy sets due to its interpretability and mathematical properties. This implies that* $h(A) = 1$.

The *Support* of A, $supp(A)$, is composed by all the elements of X that have nonzero membership in A, this means

$$supp(A) = \{x \mid \mu_A(x) > 0\} \; \forall \; x \in X \tag{2.8}$$

The *Core* of a fuzzy set $core(A)$ is the crisp set that contains all the elements of X that have $h(A)$ as membership degree, this is $h(A) = 1$

$$core(A) = \{x \mid \mu_A(x) = 1\} \; \forall \; x \in X \tag{2.9}$$

The α-*cut* of $\mu_A(x)$ namely ${}^\alpha A$ represents the interval of all values of x which has a membership degree equal or greatest than α, this means:

$${}^\alpha A = \{x \mid \mu_A(x) \geqslant \alpha\} \; \forall \; x \in X \tag{2.10}$$

where the interval of values which satisfies ${}^\alpha A$ is defined by

$${}^\alpha A \in \left[\inf_x {}^\alpha\mu_A(x), \sup_x {}^\alpha\mu_A(x)\right] \tag{2.11}$$

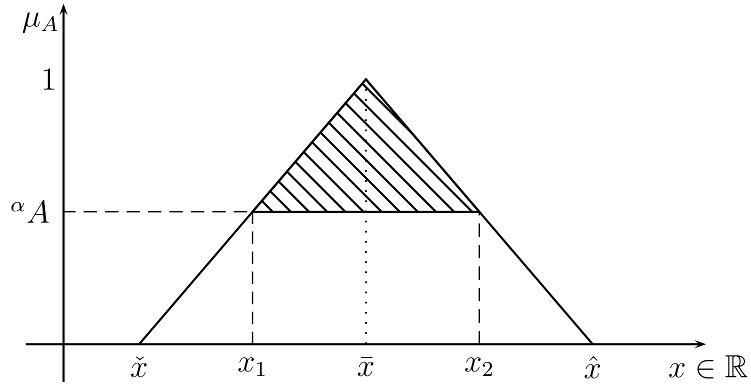

Figure 2.2: Type-1 Fuzzy set A

For simplicity we denote $x_1 = \inf_x {}^{\alpha}\mu_A(x)$ and $x_2 = \sup_x {}^{\alpha}\mu_A(x)$. A graphical display of a triangular fuzzy set is given in Figure 2.2.
Note that *supp*(A) is the α-*cut* made over A for $\alpha = 0$ and *core*(A) is the α-*cut* made over A for $\alpha = 1$. This is equivalent to say *supp*$(A) = {}^{0}A$ and *core*$(A) = {}^{1}A$.
Here, A is a Type-1 fuzzy set, its universe of discourse is the set of all values $x \in \mathbb{R}$, the *support* of A, *supp*(A) is the interval $x \in [\check{x}, \hat{x}]$ and μ_A is a triangular function with parameters $\check{x}, \bar{x}$ and $\hat{x}$. α is the degree of membership that an specific value x has regarding A and the dashed region is an α-*cut* done over A.
A desirable property of a fuzzy set is the *convexity*, defined as follows

Theorem 1 (Klir [26]) *A fuzzy set A on $\mathbb{R}$ is convex iff*

$$A(\lambda x_1 + (1 - \lambda)x_2) \geqslant \min[A(x_1), A(x_2)]$$

for all $x_1, x_2 \in \mathbb{R}$ and all $\lambda \in [0, 1]$, where min *denotes the minimum operator.*

This property is especially useful in optimization because non-convex fuzzy sets have non-continuous $^{\alpha}A$ which implies that we should face the problem of having multiple intervals that fulfill $^{\alpha}A$. To avoid this problem, we provide the following remark

Remark 4 *For simplicity purposes, all the fuzzy sets used in this work are convex, often known as α-convex sets. The set shown in Figure 2.2 is an example of an α-convex set.*

Type-2 fuzzy sets

Now, a Type-2 fuzzy set is a collection of infinite Type-1 fuzzy sets into a single fuzzy set. It is defined by two membership functions: The first one defines the degree of membership of the universe of discourse Ω and the second one weights each one of the first Type-1 fuzzy sets.

According to Jerry Mendel in his book *"Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions"*, (see [29]), [30], [22], [31], [32], [33], [34], [35], [36], [37], [38], [39] some basic definitions of Type-2 fuzzy sets are given next:

Definition 2 (Type-2 fuzzy set) *A Type-2 fuzzy set, $\tilde{A}$, is described as the following ordered pairs:*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \,|\, x \in X\} \tag{2.12}$$

where $\mu_{\tilde{A}}(x))$ is composed by an infinite amount of Type-1 fuzzy sets in two ways: Primary fuzzy sets J_x weighted by Secondary fuzzy sets $f_x(u)$. In other words

$$\tilde{A} = \{((x, u), J_x, f_x(u)) \mid x \in X; u \in [0, 1]\} \tag{2.13}$$

And finally we can get the following compact representation of $\tilde{A}$

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} f_x(u)/(x, u) = \int_{x \in X} \left[\int_{u \in J_x} f_x(u)/u \right] \Big/ x, \tag{2.14}$$

where x is the primary variable, J_x is the primary membership function associated to x, u is the secondary variable and $\int_{u \in J_x} f_x(u)/u$ is the secondary membership function.

Uncertainty about $\tilde{A}$ is conveyed by the union of all of the primary memberships, called the *Footprint Of Uncertainty* of $\tilde{A}$ [FOU($\tilde{A}$)], i.e.

$$\mathrm{FOU}(\tilde{A}) = \bigcup_{x \in X} J_x \tag{2.15}$$

Therefore, the FOU includes all the embedded J_x weighted by the secondary membership function $f_x(u)/u$. These Type-2 fuzzy sets are known as *Generalized* Type-2 fuzzy sets, *(T2FS)*, since $f_x(u)/u$ is a Type-1 membership function. Now, an *Interval Type-2* fuzzy set *(IT2FS)* is a simplification of a T2FS in the sense that its secondary membership function is assumed to be 1, as follows

Definition 3 (Interval Type-2 fuzzy set) *An Interval Type-2 fuzzy set, $\tilde{A}$, is described as:*

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} 1/(x, u) = \int_{x \in X} \left[\int_{u \in J_x} 1/u \right] \Big/ x, \tag{2.16}$$

The main difference between T2 FS and IT2 FS lies in $f_x(u)/u$, while a T2 FS uses any form of Type-1 membership functions, an IT2 FS uses 1 as a unique weight for each J_x, being an interval fuzzy set.
A FOU is bounded by two membership functions: An *Upper* membership function *(UMF)* $\bar{\mu}_{\tilde{A}}(x)$ and a *Lower* membership function *(LMF)* $\underline{\mu}_{\tilde{A}}(x)$.

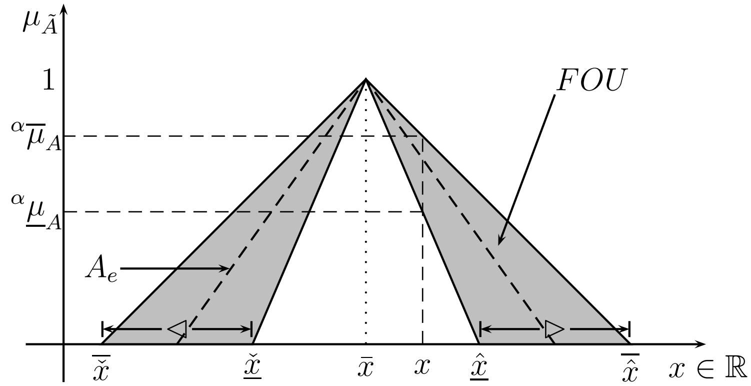

Figure 2.3: Interval Type-2 Fuzzy set $\tilde{A}$ with Uncertain $\triangleleft = \triangleright$

For discrete universes of discourse $X = \{x_1, x_2, \cdots, x_N\}$ and discrete J_x, an *Embedded T1 FS*, A_e has N elements, one each from $J_{x_1}, J_{x_2}, \cdots, J_{x_N}$, namely $u_1, u_2, \cdots, u_N$, e.g.

$$A_e = \sum_{i=1}^{N} u_i/x_i \quad u_i \in J_{x_i} \subseteq [0,1] \tag{2.17}$$

A graphical representation of these concepts is given in Figure 2.3
Here, $\tilde{a}$ is a Type-2 fuzzy set, the universe of discourse is the set $x \in X$, the *support* of $\tilde{A}$, *supp*$(\breve{A})$ is the interval $x \in [\overline{\breve{x}}, \overline{\hat{x}}]$ and $\mu_{\tilde{A}}$ is a linear function with parameters $\overline{\breve{x}}, \overline{\hat{x}}, \underline{\breve{x}}, \underline{\hat{x}}$ and $\bar{x}$. $^{\alpha}\,\bar{\mu}_A(x)$ is the degree of membership an specific value x has regarding the upper fuzzy set $\bar{A}$ and $^{\alpha}\underline{\mu}_A(x)$ is the degree of membership an specific value x has regarding the lower fuzzy set $\underline{A}$. FOU is the *Footprint of Uncertainty* of the Type-2 fuzzy set and A_e is a Type-1 fuzzy set embedded in its FOU.
An infinite number of embedded A_e Type-1 fuzzy sets are in the FOU, each one representing the knowledge of an expert about the universe of discourse or his perception about it. This *"knowledgeability"* of an expert is weighted by a T2FS through the secondary membership function, so an IT2FS do a uniform weighting of all A_e contained into its FOU. Figure 2.3 shows an IT2FS and its bounds where the opinion of all experts about X through a is given by an uncertain word $\tilde{A}$ with multiple perceptions about the word A embedded into FOU.
Now, as a fuzzy set involves more uncertainty sources, it becomes more complex (mathematically speaking) and the search of defuzzified measures is also more complex. This leads us to think that the search of an optimal value is also a complex problem and optimization under linguistic uncertainty should be a more complex problem than either a Type-1 fuzzy or crisp problem.
Another example of a Type-2 fuzzy set is given by its vertical slice
Note that each x has a primary membership function J_x weighted by a secondary membership function.
Therefore, in this work we refer to Type-1 fuzzy sets when we refer to imprecision about a

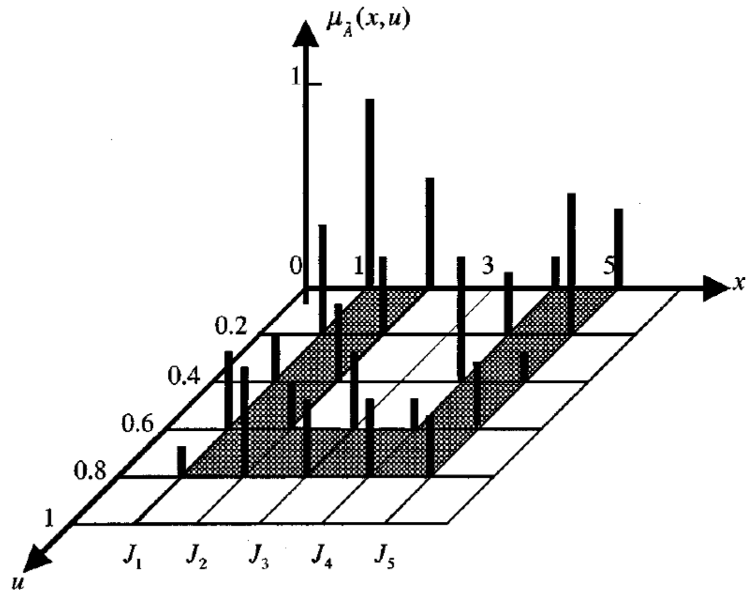

Figure 2.4: Vertical slice of a Type-2 fuzzy set

variable X and Type-2 fuzzy sets when we refer to an uncertain variable X. An uncertain Type-1 fuzzy set can be represented by a Type-2 fuzzy set, in the sense that there exist the possibility of having fuzziness and/or ambiguity about the choice of a Type-1 fuzzy set.

2.5 The Fuzzy Linear Programming model (FLP)

The fuzzy optimization field is treated by Klir & Yuan [26], Klir & Folger [27], Y. J. Lai & C. Hwang [40], Timothy J. Ross [41], J. Kacprzyk & S. A. Orlovski [42], Zimmermann [43, 44] who propose two methods by using RHS parameters treated as Type-1 fuzzy sets called *Soft Constraints*. Tanaka & Asai [45] and Tanaka, Okuda & Asai [46] propose fuzzy solutions for FLP problems, among others.

This work focuses on the *Soft Constraints* model introduced by Zimmermann [43, 44] due to its popularity. Its simplicity and interpretability becomes into an appropriate basic model to be extended to a Type-2 fuzzy environment. The mathematical representation of this model is

$$\begin{aligned} \underset{x}{\text{Max}}\ z &= c'x + c_0 \\ s.t.& \\ Ax &\lesssim B \\ x &\geqslant 0 \end{aligned} \tag{2.18}$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$, $A \in \mathbb{R}^{n \times m}$. B denotes the set of fuzzy constraints, where every of their elements B_i is defined by two parameters $\check{b}_i$ and $\hat{b}_i$ (see Figure 2.5).

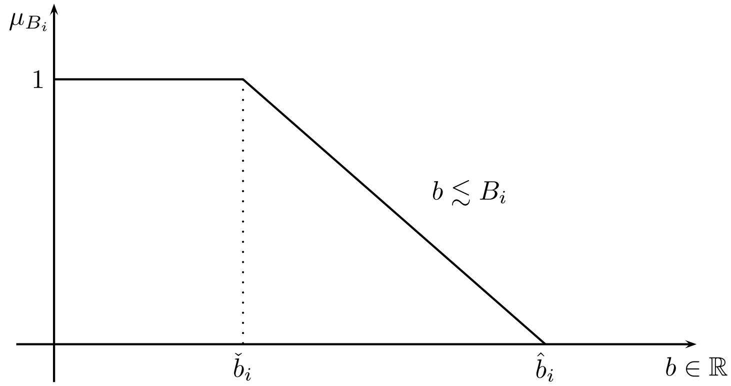

Figure 2.5: Fuzzy set B_i

This approach assumes that the Right Hand Side (RHS) parameters of an LP problems as Type-1 fuzzy sets. To do so, Zimmermann designed an algorithm that solves this kind of problems through an α-cut approach, which consists on defining a set of solutions $\tilde{z}(x^*)$ to find a joint-optimal $\alpha - cut$ for $\tilde{z}(x^*)$ and $\tilde{b}$. The method is summarized next:

Algorithm 1

1. Calculate an inferior bound called *Z Minimum* ($\check{z}$) by using $\check{b}$ as a frontier of the model.

2. Calculate a superior bound called *Z Maximum* ($\hat{z}$) by using $\hat{b}$ as a frontier of the model.

3. Define a Fuzzy Set $\tilde{z}(x^*)$ with bounds $\check{z}$ and $\hat{z}$ and trapezoidal membership function. This set represents the degree that any feasible solution has regarding the optimization objective.

4. Create an auxiliary variable α and solve the following LP model

$$\begin{gathered} \text{Max } \{\alpha\} \\ s.t. \\ c'x + c_0 - \alpha(\hat{z} - \check{z}) = \check{z} \\ Ax + \alpha(\hat{b} - \check{b}) \leqslant \hat{b} \\ x \geqslant 0 \end{gathered} \tag{2.19}$$

This method uses α as a decision variable that finds the max intersection among all fuzzy constraints. This means that α operates as an overall satisfaction degree of all fuzzy constraints, so also it represents a defuzzification degree that returns a fuzzy measure that reaches a crisp optimal solution of the global problem.

Some conceptual papers about fuzzy and possibilistic optimization are given by Lodwick & Jamison [47], Lodwick & Bachman [48], M. Inuiguchi & M. Sakawa [49, 50], Masahiro Inuiguchi & Jaroslav Ramík [51], and Figueroa & López [52].

Other important advances of theory of Fuzzy Optimization are given by Jaroslav Ramík [53] who find solutions of FLP with fuzzy objective functions, Masahiro Inuiguchi [54] define some necessity measures for optimization problems, Heinrich Rommelfanger [55] did an important discussion about multicriteria problems with mixed variables, M. Kurano, M. Yasuda, J. Nakagami & Y. Yoshida [56] describe an optimization method for fuzzy perceptive transition matrices by using average mesures, Tomoe Entani & Hideo Tanaka [57] use interval-valued probability to find optimal solutions analytic hierarchy process (AHP) problems, Hiroaki Ishii, Yung Lung Lee & Kuang Yih Yeh [58] use fuzzy measures to make optimal selection of candidates for facility location, João Paulo Carvalho & José Tomé [59] use fuzzy logic systems to optimize causal rule bases by using fuzzy boolean nets, C.A. Silva, J.M.C. Sousa & T.A. Runkler [60] addressed a Logistic problem by using fuzzy weighted aggregation, obtaining optimal solutions and N. Mahdavi-Amiri & S.H. Nasseri [61] design a dual simplex method for LP problems with trapezoidal membership functions.

Multicriteria Linear Fractional Programming *(MOLP)* problem is treated by Choo [62] and Kornbluth [63], and a fuzzy approach called Multiobjective Fuzzy Linear Fractional Programming *(MOFLP)* was designed by Lee and Tcha [64] to solve multiple linear inequalities. Yan-Kuen, Sy-Ming & Yu-Chih [65] propose a method based on the max-Archimedean composition by means of max-Archimedean t-norm composition. Ghodousiana & Khorram [66] proposed a fuzzy operator constructed by the convex combination of two known operators, max-min and max-average compositions. B.B. Pal, B.N. Moitra & U. Maulik [67] solve fuzzy multiobjective linear fractional programming *(FMOLFP)* problems by using goal programming, and Masatoshi Sakawa & Ichiro Nishizaki [68] solve two-level fuzzy fractional problems.

Lertworasirikul, Fang, Nuttle & Joines [69] reported the design and implementation of a fuzzy DEA (Data Envelopment Analysis) BCC model based in $\alpha - cut$ defined on *Possibility* and *Credibility* measurements. This model is called FDEA (Fuzzy DEA) and proposes an alternative means to measure the efficiency with imprecise data, later Lertworasirikul, Fang & Joines [70] propose a possibilistic approach of the above model. T. León, V. Liern, J.L. Ruiz & I. Sirvent [71] define a fuzzy set of efficient DMU's *(Decision Making Units)* and solve a fuzzy optimization model by using symmetric triangular fuzzy sets. Peijun Guo & Hideo Tanaka [72] propose a fuzzy DEA-Regression Analysis *(RA)* model called *DEARA*, under the base that DEA is a reference method while RA is an average method. S. Saati and A.d Memariani [73] propose a methodology to handle the classical fuzzy DEA model by using a common set of weights *(CSW)* and an auxiliary variable (γ).

G.R. Jahanshahloo, M. Soleimani-damaneh & E. Nasrabadi [74] propose a methodology for evaluating and ranking of the DMUs of the model by using concepts of fuzzy inputs-outputs. They compare their results with the proposal of Peijun Guo & Hideo Tanaka [72]. Angelov

[75] proposed a method for intuitionistic FLP, and Dubey et.al. [76] used interval fuzzy sets in LP problems. An application of IT2FS to the DEA model was reported by Qin et.al [77] where they applied Type reduction of the IT2 parameters of the DEA problem before solving as an LP problem. Those applications are based on the concept of Type-reduction of a T2FS before optimizing it, so its results has no implicit fuzzy uncertainty.

Part II

The Proposal

Chapter 3

The IT2FLP model

In this section, we propose a mathematical programming model to solve an LP problem involving Type-2 fuzzy constraints. Some concepts about uncertain constraints and the meaning of optimality under fuzzy uncertainty are presented to understand the proposed results.

3.1 Uncertain constraints

There are many ways to define the *"knowledgeability"* of an expert, so an infinite number of A_e fuzzy sets can be seen into $FOU(\tilde{A})$. Each A_e is a representation of either the the knowledge of an expert about A or his perception about it. When multiple experts are defining a constraint, linguistic issues and multiple opinions about the same word A do appear, which is an uncertainty source itself.
Now we have defined what a Type-2 fuzzy set is, then an uncertain constraint can be defined as follows

Definition 4 (IT2FS Constraint - Figueroa [78]) *Consider a set of constraints of an FLP problem defined as an IT2FS called $\tilde{b}$ defined on the closed interval $\tilde{b}_i \in [\,\underline{b}_i, \overline{b}_i\,]$, $\{\underline{b}_i, \overline{b}_i\} \in \mathbb{R}$ and $i \in \mathbb{N}_m$. The membership function which represents $\tilde{b}_i$ is:*

$$\tilde{b}_i = \int_{b_i \in \mathbb{R}} \left[\int_{u \in J_{b_i}} 1/u \right] \Big/ \ b_i, \quad i \in \mathbb{N}_m, \ J_{b_i} \subseteq [0,1] \tag{3.1}$$

Note that $\tilde{b}$ is bounded by both *Lower* and *Upper* primary membership functions, namely $\underline{\mu}_{\tilde{b}}(x)$ with parameters $\underline{\check{b}}$ and $\underline{\hat{b}}$ and $\bar{\mu}_{\tilde{b}}(x)$ with parameters $\overline{\check{b}}$ and $\overline{\hat{b}}$. Now, the *(FOU)* of the set $\tilde{b}$ can be composed by two distances called $\triangle$ and ∇, defined as follows.

Definition 5 (Figueroa [78]) *Consider an Interval Type-2 FLP problem* (IT2FLP) *with restrictions in the form $\leqslant$. Then $\triangle$ is defined as the distance between $\underline{\check{b}}$ and $\overline{\check{b}}$, $\triangle = \overline{\check{b}} - \underline{\check{b}}$ and ∇ is defined as the distance between $\underline{\hat{b}}$ and $\overline{\hat{b}}$, $\nabla = \overline{\hat{b}} - \underline{\hat{b}}$.*

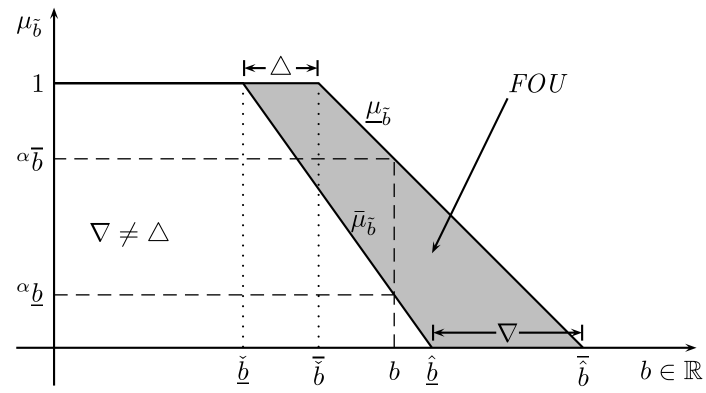

Figure 3.1: IT2FS constraint with joint uncertain $\triangle$ & ∇.

A graphical representation of $\tilde{b}_i$ is shown in Figure 3.1
In Figure 3.1, $\tilde{b}$ is an IT2FS with linear membership functions $\underline{\mu}_{\tilde{b}}$ and $\bar{\mu}_{\tilde{b}}$. A particular value b has an interval of infinite membership degrees $u \in J_b$, as follows

$$J_b \in \left[{}^{\alpha}\bar{\mu}_{\tilde{b}},\, {}^{\alpha}\underline{\mu}_{\tilde{b}}\right] \;\; \forall \; b \in \mathbb{R} \tag{3.2}$$

where J_b is the set of all possible membership degrees (u) associate to $b \in \mathbb{R}$. ${}^{\alpha}\bar{\mu}_{\tilde{b}}$ is the α-cut made over the upper membership function of $\tilde{b}$, and ${}^{\alpha}\underline{\mu}_{\tilde{b}}$ is the α-cut made over the upper membership function of $\tilde{b}$, where the α-cut of a fuzzy set b is defined as ${}^{\alpha}b = \{x \,|\, \mu_b(x) \geqslant \alpha\}$. Now, the FOU of $\tilde{b}$ can be composed by the union of all values of u, as defined as follows

Definition 6 (FOU of $\tilde{b}$) *As defined in* (3.2), *it is possible to compose the footprint of uncertainty of $\tilde{b}, u \in J_b$ as follows:*

$$\mathrm{FOU}(\tilde{b}) = \bigcup_{b \in \mathbb{R}} \left[{}^{\alpha}\bar{\mu}_{\tilde{b}},\, {}^{\alpha}\underline{\mu}_{\tilde{b}}\right] \;\; \forall \; b \in \tilde{b},\, u \in J_b,\, \alpha \in [0,1] \tag{3.3}$$

Remark 5 *Definition 6 presents an L-R Type-2 fuzzy set as the union of all possible L-R Type-1 fuzzy sets into its FOU. Definition 4 defines an uncertain constraint as a monotonic decreasing Type-2 fuzzy set which represents the statement "Approximately less or equal than b_i". In this way, we refer to an uncertain constraint as the IT2FS defined in Definitions 5 and 6 with a membership function as displayed in Figure 3.1.*

Alternatively, we can decompose $\tilde{b}$ into *α-cuts*, as shown in Figure 3.2.
In this Figure, $\tilde{b}$ is an IT2FS composed by two piece-wise linear membership functions $\underline{\mu}_{\tilde{b}}$ and $\bar{\mu}_{\tilde{b}}$. A particular value b projects an interval of infinite membership degrees $u \in J_b$, as follows

$$J_b \in \left[{}^{\alpha_1}\bar{b},\, {}^{\alpha_2}\underline{b}\right] \;\; \forall \; b \in \mathbb{R} \tag{3.4}$$

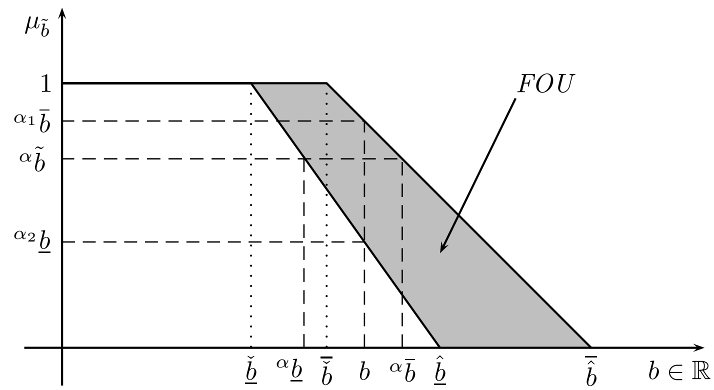

Figure 3.2: IT2FS constraint $Ax \precsim \tilde{b}$

where J_b is the set of all possible membership degrees associated to $b \in \mathbb{R}$, and the interval obtained by a particular membership degree namely $\alpha \in [0,1]$ is

$$^{\alpha}\tilde{b} \to \left[{}^{\alpha}\bar{b},\, {}^{\alpha}\underline{b}\right] \;\forall\; \alpha \in [0,1],\, u \in J_b, \tag{3.5}$$

In the case of an IT2FS, we have that $u = 1$, so all values of $u \in J_b$ are 1, which is an interval itself. Note that also $^{\alpha}\tilde{b}$, and subsequently $^{\alpha}\bar{b}$ and $^{\alpha}\underline{b}$ lead to intervals.
We can use the concept of an α-*cut* of a classical fuzzy set, extended to an IT2FS in order to use interval optimization methods for finding a solution of an IT2FLP. Then, the α-*cut* of an IT2FS $\tilde{b}$ namely $^{\alpha}\tilde{b}$ can be expressed as:

$$^{\alpha}\tilde{b} = \{x \,|\, \mu_{\tilde{b}}(x,u) \geqslant \alpha\} \;\forall u \in J_b \subseteq [0,1] \tag{3.6}$$

Now, $^{\alpha}\tilde{b}$ is bounded by the α-*cut* of their membership functions:

$$^{\alpha}\bar{b} = \{x \,|\, \mu_{\bar{b}}(x,u) \geqslant \alpha\} \;\forall u \in J_b \subseteq [0,1] \tag{3.7}$$

$$^{\alpha}\underline{b} = \{x \,|\, \mu_{\underline{b}}(x,u) \geqslant \alpha\} \;\forall u \in J_b \subseteq [0,1] \tag{3.8}$$

Since for an IT2FS $u = 1$ for every J_b, then the α-*cut* of an IT2 fuzzy constraint is equivalent to:

$$^{\alpha}\tilde{b} = \{x \,|\, \mu_{\tilde{b}}(x) \geqslant \alpha\} \;\forall u \in J_b \tag{3.9}$$

In this way, the crisp bounds of the α-*cut* of an IT2 fuzzy constraint called α^c-*cut*, is defined as follows:

$$^{\alpha^c}\tilde{b} = \{x \,|\, \mu_{\tilde{b}}(x) = \alpha\} \;\forall u \in J_b \tag{3.10}$$

This leads to obtain the following boundaries of $^{\alpha^c}\tilde{b}$:

$$^{\alpha^c}\mu_{\underline{b}}(x) = \inf_{b}\,\{x \,|\, \mu_{\tilde{b}}(x) = \alpha\} \;\forall u \in J_b \tag{3.11}$$

$$^{\alpha^c}\mu_{\bar{b}}(x) = \sup_{b}\,\{x \,|\, \mu_{\tilde{b}}(x) = \alpha\} \;\forall u \in J_b \tag{3.12}$$

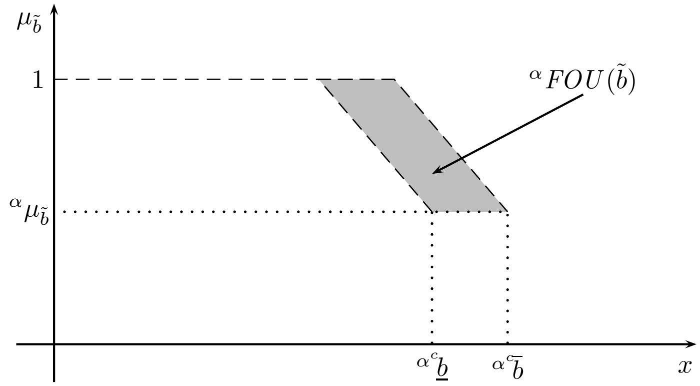

Figure 3.3: α-*cuts* $^{\alpha}\tilde{b}$ and $^{\alpha^c}\tilde{b}$ of a constraint $\tilde{b}$.

For simplicity, we rename (3.11) and (3.12) as $^{\alpha^c}\underline{b}$ and $^{\alpha^c}\overline{b}$ respectively. Figure 3.3 presents a graphical representation of $^{\alpha^c}\underline{b}$ and $^{\alpha^c}\overline{b}$.

Note that an α-*cut* made over the FOU of an IT2FS contains an infinite amount of embedded α-*cuts*, hence we only use the α^c-*cut* at a level α as defuzzification measure. In the following sections, some definitions about LP problems with IT2FS constraints are provided together with a method for finding optimal solutions in terms of $x \in \mathbb{R}$ regarding z and $\tilde{b}$.

The problem of having a type-2 fuzzy constrained problem cannot solved in a closed form, so there is a need for finding an appropriate solution. Some interesting ideas about the concept of an optimal solution in terms of the decision variables $x \in \mathbb{R}$ given uncertain constraints $\tilde{b}$, can be discussed. A first way would be to use Type-reduction to all IT2FS based on centroid methods, and afterwards solve the resultant interval-valued optimization problem. However, this is not recommendable because the centroid of an IT2FS constraint is usually outside its FOU. Another easy way is by using the *Center of FOU* which is simply to use the center of ∇ and $\triangle$ as extreme points of a fuzzy set embedded into the FOU of $\tilde{b}$, and then apply the Zimmermann's method. This method can be used in cases where the analyst has no a defuzzification criteria.

We have based our results in the Bellman-Zadeh fuzzy decision making principle, so the idea is to find a maximum intersection value between all constraints and Z. To do so, we need to provide some definitions of LP problems with IT2FS constraints in order to design a method for finding an optimal solution in terms of $x \in \mathbb{R}$ regarding z and $\tilde{b}$.

3.2 The IT2FLP model

Given the concept of an IT2FS constraint and the definition of an FLP (see Figueroa and Hernández [1, 2]), an uncertain constrained FLP model (IT2FLP) can be defined as follows:

$$\begin{aligned} \max_{x \in X} \; & z = c'x + c_0 \\ & s.t. \\ & Ax \precsim \tilde{b} \\ & x \geqslant 0 \end{aligned} \tag{3.13}$$

where $x, c \in \mathbb{R}^m$, $c_0 \in \mathbb{R}$, $A \in \mathbb{R}^{n \times m}$. $\tilde{b}$ is an IT2FS vector defined by two primary membership functions $\underline{\mu}_b$ and $\bar{\mu}_b$. $\precsim$ is a Type-2 fuzzy partial order.
The binary relation $\precsim$ for classical fuzzy sets has been proposed and investigated by Ramík and Řimánek [79]. Let A and B be two fuzzy numbers. Then $A \precsim B$ if and only if $\sup {}^\alpha A \leqslant \sup {}^\alpha B$ and $\inf {}^\alpha A \leqslant \inf {}^\alpha B$ for each $\alpha \in [0,1]$, where ${}^\alpha A$ and ${}^\alpha B$ are α-cuts of A and B respectively, and ${}^\alpha A := [\inf {}^\alpha A, \sup {}^\alpha A]$ and ${}^\alpha B := [\inf {}^\alpha B, \sup {}^\alpha B]$. This binary relation satisfies the axioms of a partial order relation on $\mathcal{F}(\mathbb{R})$ and is called *fuzzy max order* (see Figueroa-García, Chalco-Cano & Román-Flores [80]). It is possible to extend the fuzzy max order to the case of $Ax \precsim \tilde{b}$, as follows:

Theorem 2 *Let $\tilde{b} \in \mathbb{R}_+$ be an IT2FS constraint as defined in Definition 4, $f(x)$ be a real function defined over b, $f(x) \in b$, then the binary relation $\precsim$, $f(x) \precsim \tilde{b}$ holds if only if $f(x) \leqslant {}^\alpha\tilde{b}, \alpha \in [0,1]$, this is*

$$f(x) \leqslant \inf {}^\alpha\tilde{b},\ \alpha \in [0,1]. \tag{3.14}$$

where ${}^\alpha\tilde{b}$ is the α-cut of $\tilde{b}$, and $f(x)$ is a real function over $b \in \mathbb{R}^+$.

Proof 1 *From a crisp point of view, $f(x) \precsim \tilde{b}$ implies that every α-cut done over $\tilde{B}$ can be decomposed into the crisp points $\inf {}^\alpha\tilde{b} := {}^{\alpha^c}\underline{b}$, $\sup {}^\alpha\tilde{b} := {}^{\alpha^c}\bar{b}$, so ${}^{\alpha^c}\tilde{b} := [\inf {}^\alpha\tilde{b}, \sup {}^\alpha\tilde{b}] := [\underline{b}, \bar{b}]$. This implies that*

$$f(x) \leqslant {}^{\alpha^c}\tilde{b},\ \alpha \in [0,1], \tag{3.15}$$

thus, if the condition (3.15) holds then the following condition

$$f(x) \leqslant {}^{\alpha^c}\underline{b},\ \alpha \in [0,1] \tag{3.16}$$

is enough to proof that $f(x) \precsim \tilde{b}$, since $f(x) \leqslant {}^{\alpha^c}\underline{b} \leqslant {}^{\alpha^c}\bar{b}$, $\alpha \in [0,1]$.

Corollary 1 *Let B a Type-1 fuzzy set as shown in Figure 2.5 embedded into the FOU of $\tilde{b}$ as defined in (2.17). The binary relation $\lesssim$, $f(x) \lesssim B$ holds if only if $f(x) \leqslant {}^\alpha B$, $\alpha \in [0,1]$. This is equivalent to the following result*

$$f(x) \leqslant {}^\alpha B,\ \alpha \in [0,1]. \tag{3.17}$$

where ${}^\alpha B$ is the α-cut of B, and $f(x)$ is a real function over $b \in \mathbb{R}^+$.

Proof 2 *Follows directly from proof of Theorem 2.*

Two possible partial orders $\precsim$ and $\succsim$ can be used depending on the problem. We only use linear membership functions since we solve LP models using classical algorithms, which means less complexity. The membership function of $\precsim$ (see Figure 3.1) is:

$$\underline{\mu}_{\tilde{b}}(x; \underline{\check{b}}, \underline{\hat{b}}) = \begin{cases} 1, & x \leqslant \underline{\check{b}} \\ \dfrac{\underline{\hat{b}} - x}{\underline{\hat{b}} - \underline{\check{b}}}, & \underline{\check{b}} \leqslant x \leqslant \underline{\hat{b}} \\ 0, & x \geqslant \underline{\hat{b}} \end{cases} \tag{3.18}$$

and its upper membership function (see Figure 2.2) is:

$$\bar{\mu}_{\tilde{b}}(x; \bar{\check{b}}, \bar{\hat{b}}) = \begin{cases} 1, & x \leqslant \bar{\check{b}} \\ \dfrac{\bar{\hat{b}} - x}{\bar{\hat{b}} - \bar{\check{b}}}, & \bar{\check{b}} \leqslant x \leqslant \bar{\hat{b}} \\ 0, & x \geqslant \bar{\hat{b}} \end{cases} \tag{3.19}$$

3.3 Methodology for solving an IT2FLP

A first approach for solving IT2FS problems is by reducing its complexity into a simpler form in order to use well known algorithms. In this case, we propose the following three-step methodology:

1. Compute a fuzzy set of optimal solutions namely $\tilde{z}$
2. Apply a Type-reduction strategy to find a single fuzzy set Z embedded into the FOU of $\tilde{z}$
3. Apply the Zimmermann's soft constraints method to find a crisp solution

This allows us to see the above problem as the problem of finding a vector of solutions $x \in \mathbb{R}^m$ such that:

$$\underset{x \in \mathbb{R}^n}{\text{Max}}\ \alpha \left\{ \bigcap_{i=1}^{m} \{\, \tilde{b}_i,\ b_i \} \bigcap \tilde{z} \right\} \tag{3.20}$$

where α is the α-cut made over all fuzzy constraints $\tilde{b}_i$ and $\tilde{z}$, defined as follows

$$\mu_{\tilde{z}}(\tilde{b})(z) = \sup_{z = c'x^* + c_0} \min_i \left\{ \mu_{\tilde{b}_i}(x^*) \,|\, x^* \in \mathbb{R}^m \right\} \tag{3.21}$$

Given $\mu_{\tilde{z}}$, the problem becomes in how to find the maximal intersection point between $\tilde{z}$ and $\tilde{b}$, where α is an auxiliary variable. In practice, the problem is solved by x^*, so α allows us

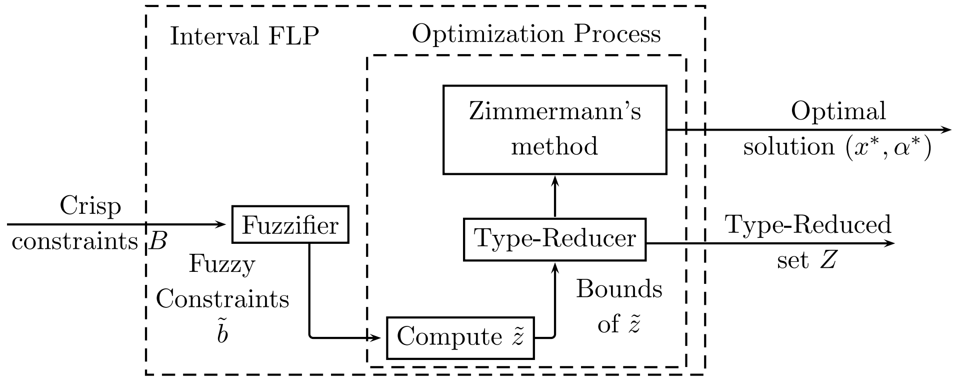

Figure 3.4: IT2FLP proposed methodology

to find x^* according to (3.20). The proposed methodology for using α over $\tilde{z}$ and $\tilde{b}$ to find x^* is presented in Figure 3.4.

Figure 3.4 shows the three basic steps of this approach: fuzzification, fuzzy optimization, and defuzzification. The main idea is to compute the fuzzy set $\tilde{Z}$ before applying a Type-reduction method for Type-2 fuzzy sets, to finally obtain a crisp solution using the Zimmermann's soft constraints method. Now, there are two important conditions which ensures that an IT2FLP has an optimal solution in some point of $supp(\tilde{b})$: *feasibility* and *convexity* which are described next

3.4 Feasibility of an IT2FLP

An important condition to be satisfied by any LP problem before being optimized is that it should be *feasible*. The extension of this concept to an IT2FLP leads us to think that the problem should be feasible at every element of the halfspace generated by $b \in supp(\tilde{b})$ (see Niewiadomski [81, 82]). In other words:

Proposition 3 (Feasibility of the IT2FLP) *An IT2FLP is feasible iff the system*

$$A(x_{ij}) \leqslant \bar{\hat{b}}_i \quad \forall \ i \in \mathbb{N}_m \tag{3.22}$$

is feasible itself.

This means that an IT2FLP is feasible only if the broadest value of $supp(\tilde{b})$ is feasible, i.e. the boundary provided by $\bar{\hat{b}}$. It is clear that if a solution in this point exists, then every value of $b \leqslant \bar{\hat{b}}$ is feasible as well, since they are contained into the convex hull defined by $\bar{\hat{b}}$ (see Wolsey [14], and Papadimitriou and Steiglitz [83]).

3.5 Convexity of an IT2FLP

Another important condition to be satisfied by any LP model is *convexity*. In an LP problem, convexity means that the halfspace generated by all $A(x_{ij}) \leqslant b$ should be continuous and compact. This implies that every set b should not be empty (non-null).
An IT2FLP has to guarantee two convexity conditions: a first one regarding $b \in supp(\tilde{b})$ and a second one regarding $\mu_{\tilde{b}}$. This leads us to the following proposition:

Proposition 4 (Convexity of an IT2FLP) *An IT2FLP is said to be convex iff*

$$A(x_{ij}) \precsim \tilde{b} \quad \forall \;\; i \in \mathbb{N}_m \tag{3.23}$$

is a non-null halfspace, and $\tilde{b}$ is composed by convex $\overline{\mu}_{\tilde{b}}$ and $\underline{\mu}_{\tilde{b}}$ membership functions.

According to Kearfott and Kreinovich [84], global optimization is only possible for convex objective functions, so the Proposition 4 agrees with this. As $\tilde{z}$ is a function of $\tilde{b}$, then we need to guarantee that $\mu_{\tilde{b}}$ be convex to ensure that $\tilde{z}$ be convex as well.
This definition of uncertain constraints is a starting point for designing optimization methods which deal with linear IT2FS, so its use in different scenarios is an open problem for decision making sciences.

3.6 Existence of an optimal solution of an IT2FLP

Apart from feasibility and convexity of an IT2FLP, the existence of an optimal defuzzification value α which is a basic feasible solution, should be proven. To do so, we have to show that at least one of the embedded sets into the FOU of $\tilde{b}$ is an optimal solution of the system, defined as the following FLP

$$\begin{gathered}
\text{Max } \{\alpha\} \\
s.t. \\
c'x + c_0 - \alpha(\hat{z} - \check{z}) = \check{z} \\
Ax + \alpha(\hat{b} - \check{b}) \leqslant \hat{b} \\
x \geqslant 0
\end{gathered}$$

Lemma 1 *Assume that IT2FLP has a non-degenerate optimal solution, i.e., there exists an optimal basic feasible solution α for which there exists a basis matrix A_B such that the problem*

$$A_B \alpha = [\check{z}, \hat{b}],\ x_N = 0,\ \alpha \in [0, 1] \tag{3.24}$$

where x_N is a vector of non-basis variables.
Then there exist dual vectors $y \in \mathbb{R}_m$ and $s \in \mathbb{R}_n$, which can be partitioned conformally with x such that

$$A'_B y = c_B,\ s_B = 0,\ N'y + s_N = c_N,\ s_N \in [0, 1]$$

Proof 3 *Let $\mathcal{P} = \{p \in \mathbb{B}_n : A_B p_B + N p_N = 0, \, p_N \in [0,1]\}$ where p_b is a basis vector, p_N is a non-basis vector, and N is a non-basis matrix. Then, since $\alpha > 0$, we have that $p \in \mathcal{P}$ is a feasible direction i.e., which is feasible to LP for $\beta \in [0,1]$. Now, given that α is optimal for $c = 1$, then $c'p = p \geqslant 0$ holds for $p \in \mathcal{P}$. Hence,*

$$0 \leqslant p = c_B' p_B + c_N' p_N = (c_N - (N A_B^{-1})' c_B)' p_N$$

where c_B is the vector of basis costs, and c_N is a vector of non-basis costs.
It is clear that $[c_B, c_N] = [\check{z}, \hat{b}]$, so for A_B, we can define y and s from (3.24). It remains to show that $s_N \geqslant 0$ holds, and the above equation gives

$$0 \leqslant p = s_N' p_N$$

Keeping in mind that p_N is an arbitrary vector, $p = S_N' p_N$ implies that $s_N \geqslant 0$, and the proof is complete.

This lemma shows us that if $p = s_N' p_N \geqslant 0$, then $s_N' p_N$ is a feasible (non-optimal) direction of the dual problem which is (by duality) the optimal direction of the primal problem, this is $s_N' p_N = \alpha$ where $s_N' = \alpha$ and $p_N = c = 1$.
From another point of view, $\hat{b} \in \triangle$, $\check{b} \in \nabla$ (see Definition 5) and Z is a Type-1 fuzzy set enclosed into the FOU of $\tilde{z}$. This leads us to the following result:

$$\begin{array}{rcc} \check{z} = f_1^{-1}(\hat{b}) & \Leftrightarrow & \nabla_z = f_1^{-1}(\triangle_b) \\ \hat{z} = f_2^{-1}(\check{b}) & \Leftrightarrow & \triangle_z = f_2^{-1}(\nabla_b) \\ \alpha^* = g_1(c, x^*), b = g_2(A, x^*) & \Leftrightarrow & x^* \to \alpha^*, b \end{array}$$

Now, if $\hat{b} \in \triangle_b$ and $\check{b} \in \nabla_b$ are feasible points (see Proposition 3) and μ_B is a set of convex membership functions, then $\check{z} = f_1^{-1}(\hat{b})$ and $\hat{z} = f_2^{-1}(\check{b})$ should be feasible with convex μ_Z. Note that the shapes of μ_B and μ_Z are opposed by definition, so the Bellman-Zadeh fuzzy decision making principle (see Bellman and Zadeh [85]) leads us to see that $(Z \bigcap B_1 \bigcap \cdots \bigcap B_n)$ is non-null.
Every T1FLP is solved using the auxiliary variable α, so every T1FLP enclosed into $\triangle_b$ and ∇_b has an associated α. We can see that if $\check{z} = f_1^{-1}(\hat{b})$ and $\hat{z} = f_2^{-1}(\check{b})$ are feasible points with linear μ_Z, then $0 < (\mu_Z \bigcap \mu_{B_1} \bigcap \cdots \bigcap \mu_{B_n}) < 1$. Therefore $\alpha < 1$ is a feasible variable, so $\alpha(\hat{z} - \check{z})$ and $\alpha(\hat{b} - \check{b})$ are feasible as well, which is enough to show the existence of a solution of the T1FLP through α.

Chapter 4

The concept of optimal solution under fuzzy uncertainty

In this chapter, some additional aspects are pointed out regarding IT2FLPs. A first aspect is related to the concept of an optimal solution under fuzziness, a second aspect is about the computation of the set of optimal solutions called $\tilde{z}$, and the third one regards to the behavior of the Algorithm 1 (Zimmermann's soft constraints method) inside $\tilde{z}$. All aspects are explained next.

4.1 What is a fuzzy optimal solution?

In crisp LP models, the concept of a feasible solution is based on the idea of having a convex halfspace where any element contained into it, is feasible (or possible). In our case, this concept should be extended to halfspace which has no crisp boundaries, it has fuzzy boundaries related to a membership degree.

As shown in Section 3.4, the boundary of the vector B, $\hat{b}$, generates a halfspace namely $h(\cdot)$ generated by the set of all the values of x contained into the support of B_i, $x \in supp(B_i)$. This way, a fuzzy constrained LP is feasible only if the polyhedron (or polytope) generated by $h(\cdot)$ is a non-trivial set, that is:

$$\mathcal{P} = \{x \,|\, h(\cdot) \leqslant \hat{b}\}, \tag{4.1}$$

where $\mathcal{P}$ is a non-trivial set of solutions (polyhedron or polytope) of a crisp LP model. Here, $\mathcal{P}$ is a convex set of solutions of all possible solutions bounded by $\hat{b}$.

Therefore, any feasible solution has an associated **membership degree**, which leads us to define the following

Definition 7 *Let $x' \in \mathcal{P}$ any feasible solution of $\sum a_{ij}x'_j = b'_i$ where $b'_i \in h(\cdot)$. Then, the linear combination $\sum a_{ij}x'_j$ belongs to B_i with a membership degree $\mu_{B_i}(x')$.*

This means that every feasible solution x' can be projected into each B_i, with a membership degree $\mu_{B_i}(x')$, which basically is equivalent to say that in a fuzzy environment, we have not only a feasible solution x', we have its membership degree $\mu_{B_i}(x')$ as well. Figure 4.1 shows what a fuzzy feasible solution is.

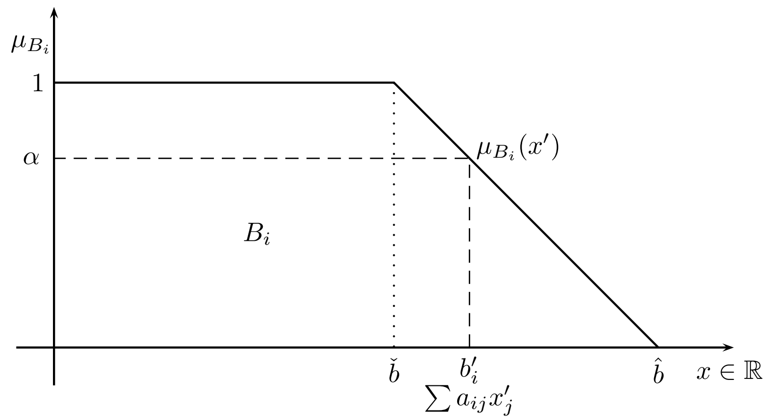

Figure 4.1: Fuzzy feasible solution x' projected over B_i

4.1.1 Fuzzy optimal solution

The concept of *optimal solution* of a fuzzy constrained LP is close to the optimality concept in LP. While in a crisp LP we have that an optimal solution is a vector x^* for which the function $z = c'x$ is certainly maximal, in a fuzzy constrained LP we have a set of optimal solutions (Z) which is a function of ${}^{\alpha}B$. This leads us to the following definition

Definition 8 *A fuzzy optimal solution is defined as a vector x^* for which $\exists\, x : \max\{{}^{\alpha}Z = c'x \mid Ax \leqslant {}^{\alpha}B, x \geqslant 0, \alpha \in [0,1]\}$, so ${}^{\alpha}Z(x^*) \geqslant {}^{\alpha}Z(x)\, \forall\, x \in \mathbb{B}$. $\mathbb{B} \subseteq \mathbb{R}^+$ is the set of all feasible values of x, and ${}^{\alpha}Z(x^*)$ is the optimal objective value of $c'x^*$ given α.*

Hence, a fuzzy optimal solution is a vector x^* which fulfills all constraints and obtains a maximal value of the goal $z'^* = c'x^*$, at a membership degree α'^*. Now, as we are using ${}^{\alpha}B$ as a crisp value, then the value of each ${}^{\alpha}Z(x^*)$ is a *crisp global optimal* solution.

In Figure 4.2, a particular value of z' comes from an LP model, so what we have is an optimal solution $z' = c'x^*$ given a particular value α, projected into ${}^{\alpha}B$. Note that between $\check{z}$ and $\hat{z}$ there is an infinite amount of possible optimal solutions (see Algorithm 2.5).

Also note that as more α values are used, more x^* and ${}^{\alpha}Z(x^*)$ values exists. Indeed, there is an infinite amount of possible optimal solutions that can be computed, each one leading to a global optimal value given α.

Now, the concept of optimal solution in a fuzzy environment is a fuzzy set itself, which is a function of the parameters of the problem (and their membership functions). This leads to think about the concept what a *fuzzy global optimal* solution means.

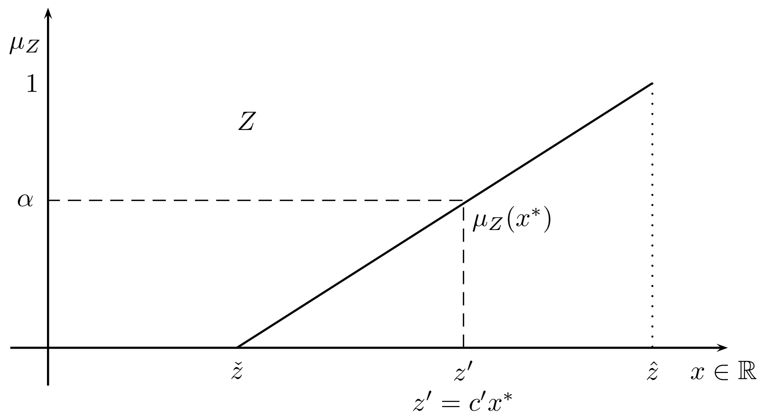

Figure 4.2: Fuzzy optimal solution $z' = c'x^*$ projected over Z

4.1.2 Fuzzy global optimal solution

The importance of having a *global optimal solution* in LP problems is large, since it is a key point for implementations and development of new algorithms. In many applications, decision makers ask for a single solution because they want to have a single operation point which returns the best possible results.

This is useful when the analyst asks for a solution *to be implemented*, but in practice there is no reliability that optimal results can be applied. What sometimes happen is that the optimal solution cannot be implemented, so there is a need for having choices to be applied. This makes sense to the fuzzy approach, since it obtains a set of optimal solutions that the analyst can use to compare the obtained results in practice to a set of possible choices. This allows to see how close (or far) are its results from the best possible solution.

In general, there is a need for having a relationship between **theoretical optimal** and what it is obtained in **practical applications**. Moreover, there is a need for clarify the sense of having an fuzzy optimal decision and what it means in practice. To do so, we have to take a look about concepts of global optimal solutions and its extension to a fuzzy environment.

The existence of a global optimal solution has become one of the most important issues when solving optimization problems, so when involving fuzzy uncertainty to the analysis, we need to think not only in crisp results but in membership degrees.

Then, it is clear that the search of a global optimal solution in a fuzzy environment should include a search of a uncertainty (imprecision) degree in which a decision has to be optimal, or at least plausible. To do so, we have to find a fuzzy global solution keeping in mind the following key aspects:

- How to find crisp optimal solutions

- A fuzzy decision making criteria

- How to find an optimal solution which fulfills fuzzy decision making criteria and still crisp optimal

The Bellman-Zadeh fuzzy decision making principle (see [85]) (its fuzzy LP version is shown in 3.6), where the main idea is to solve a $\max - \min$ decision making problem given known fuzzy sets. This leads to find a single optimal intersection point between fuzzy constraints and the goal (see (2.19) in Algorithm 2.5), and consequently a solution x^* located at ${}^{\alpha}B$ and ${}^{\alpha}Z$.
This way, we can infer that a *fuzzy global solution* can be considered as a solution which is **crisp global optimal given a fuzzy decision making criteria**. As usual, find a crisp point which fulfils both requirements is expensive, so there is a need for using well known optimization methods able to handle the shapes of all fuzzy sets while computing optimum solutions.
Various authors have proven the existence of boundaries for optimal values of a given goal function when solving fuzzy optimization problems, Ramík [86], Fiedler et.al [87], and Zimmermann [44, 43] have defined well known methods for finding the boundaries of the goal function ($\check{z}$ and $\hat{z}$ in this case).
This leads us to think in the following situation: **what is the usefulness of the solutions between $\check{z}$ and $\hat{z}$?**. This question leads us to think in all those points as alternatives to decision making in practical applications.

4.1.3 Operation points

From a practical point of view, there is no any certainty of reaching the optimal solution. When having a single optimal solution, the analyst should set the system in terms of that referring point in order to get its best performance.
If the analyst has choices or *operation points*, then decision making can be enriched because the analyst can use those points when setting the system, so basically if the system does not reach the expected results, the analyst can compare its current performance to a set of possible choices and see how good the performance of the system is. Then, an operation point is defined as follows:

Definition 9 (Operation point) *An operation point is a set of observed values of b namely b' contained into the boundaries of B, $b' \in [\check{b}, \hat{b}]$ which leads to the optimal solution x^*, and z'.*

An operation point is then an observed performance of the system which obeys to given running conditions. What in real implementations decision maker observes, is what an operation point is itself, so as many of running conditions can be applied to the system, as many of operation points the analyst can compare to make a decision.
To do so, we propose the following rank index for comparing an operation point (what was measured in real world) of the system against the optimal results after fuzzy decision making.

Definition 10 (Ranking a solution) *Let b' a set of observed constraints $b' \in [\check{b}, \hat{b}]$, Z be the set of optimal solutions provided by any fuzzy decision making method, $z' \in Z$ be the optimal solution of the LP problem given b', α^* its optimal uncertainty degree, z^* the optimal solution of the fuzzy problem given α^*, and α' the membership degree of z' into Z. Then the relative degree of fulfilment ($Df_{z'}$) of z' compared to z^* is:*

$$Df_{z'} = \frac{z' - z^*}{\hat{z} - \check{z}} \tag{4.2}$$

which is equivalent to

$$Df_{z'} = \alpha' - \alpha^* \tag{4.3}$$

It is clear that $Df_{z'} \in [-1, 1]$, so its interpretation is as follows: if $Df_{z'} > 0$ then the observed values of b' lead to improve the expected results; if $Df_{z'} < 0$ then observed values of b' did not reach the expected results, and if $Df_{z'} = 0$, then the values of b' have reached the expected results.
A comprehensive graph is provided in Figure 4.3

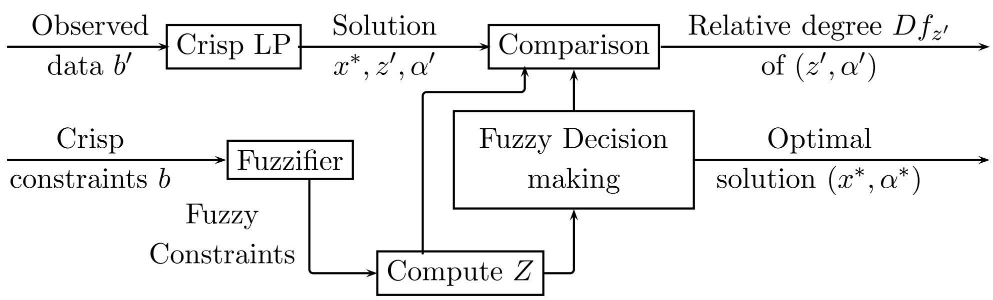

Figure 4.3: Comparing observed values of b

Therefore, we can see that *every* operation point (b') observed in reality leads to x^*, z', α', having its own $Df_{z'}$. This allows us to compare the performance of the observed system and take actions to improve it. Moreover, $Df_{z'}$ allows us to compare different operation points at different stages of the system in order to make an appropriate decision.

4.1.4 Application example

The application example is taken from Klir and Folger [27], example 15.8 at page 413. Assume that a company makes two products. Product P_1 has a \$0.4 per unit profit and product P_2 has a \$0.3 per unit profit. Product P_1 requires twice as many labor hours as each product P_2. The total available labor hours are at least 500 hours per day, and may be possible extended to 600 hours per day, due to some special arrangements for overtime work. The supply of material is at least sufficient for 400 units of both products P_1 and P_2, per day, but may be possible extended to 500 units per day according to previous experience.

The problem is, how many units of products P_1 and P_2 should be made per day to maximize the total profit? The main problem can be expressed as follows

$$\begin{aligned} \underset{x}{\text{Max}}\ z = 0.4x_1 + 0.3x_2 & \quad \text{(profit)} \\ \text{s.t.} & \\ x_1 + x_2 \lesssim B_1 & \quad \text{(material)} \\ 2x_1 + x_2 \lesssim B_2 & \quad \text{(labor hours)} \\ x_1, x_2 \geqslant 0 & \end{aligned}$$

Then we have $\check{b} = [400; 500]$ and $\hat{b} = [500; 600]$. Using the Algorithm 2.5 (Zimmermann's method). the obtained results are $\check{z} = 130, \hat{z} = 160, z^* = 145, \alpha^* = 0.5, x_1^* = 100$ and $x_2^* = 350$.

Now suppose that the analyst did an experiment to try to set the system, and after all their attempts, the available labor hours and material were 530 and 415 units respectively, $b' = [415; 530]$. Then, the obtained results for this operation point were $z' = 136, \alpha' = 0.2, x_1^* = 115$ and $x_2^* = 300$. The relative degree of fulfilment of the current operation point is

$$Df_{z'} = \frac{136 - 145}{160 - 130} = -0.3 \tag{4.4}$$

This means that this current operation point has not been reached the expected results (since $Df_{z'} < 0$), so the analyst has to take actions to improve the system's performance.

Now suppose that the analyst has taken more actions to improve the availability of their resources. After some negotiations and improvements, it can increased the available labor hours and material to 560 and 465 units respectively, so $b' = [465; 560]$. Then, the obtained results for this *operation point* were $z' = 149, \alpha' = 0.6333, x_1^* = 95$ and $x_2^* = 370$. The relative degree of fulfilment of the current operation point is

$$Df_{z'} = \frac{149 - 145}{160 - 130} = 0.1333 \tag{4.5}$$

At this point, the current operation point of the system has overtaken its expected performance (since $Df_{z'} > 0$), so the analyst has taken actions which have improved the system's performance.

4.1.5 Computing the set of optimal solutions $\tilde{z}$

The fuzzy set of optimal solutions can be obtained by applying the Zadeh's extension principle and the computation of the optimal bounds of the problem. This means that $\tilde{z}$ is a function of the optimal values of the problem $z^* = c(x^*)$ regarding the bounds of $\tilde{b}$.

Now, $\mu_{\tilde{z}}$ is a linear function of x^*, $z^* = c(x^*)$ and $\tilde{b}_i$, then we can define its bounds as function of the bounds of $\tilde{b}_i$, as defined in Definition 5. This way, we can extend this property to the IT2FLP by using (3.21), and convex optimization tools to compose $\tilde{z}$ through $\tilde{b}_i(Ax)$ and $z^* = c(x^*)$.

Note that $\tilde{b}_i$ is a fuzzy partial order in the form $\precsim$ or $\succsim$ (see Theorem 2) with well defined parameters $\underline{\check{b}}, \bar{\check{b}}, \underline{\hat{b}}$ and $\bar{\hat{b}}$. Mathematically speaking we can compute the bounds of the global optimization problem by using each one of these parameters. To do so, the following relations are defined

Theorem 5 (Bounds of $\tilde{z}$) *Let (6.12) be an optimization problem where $\tilde{b}$ is composed by $\underline{\mu}_{\tilde{b}}$ with parameters $\underline{\check{b}}$ and $\underline{\hat{b}}$, and $\bar{\mu}_{\tilde{b}}$ parametrized by $\bar{\check{b}}$ and $\bar{\hat{b}}$ (see Appendix). Then the fuzzy set $\mu_{\tilde{z}}$ is composed by $\underline{\mu}_{\tilde{z}}$ and $\bar{\mu}_{\tilde{z}}$ in the following way.*

$$\underline{\mu}_{\tilde{z}}(z; \underline{\check{z}}, \underline{\hat{z}}) = \begin{cases} 0, & z \leqslant \underline{\check{z}} \\ \dfrac{z - \underline{\check{z}}}{\underline{\hat{z}} - \underline{\check{z}}}, & \underline{\check{z}} \leqslant z \leqslant \underline{\hat{z}} \\ 1, & z \geqslant \underline{\hat{z}} \end{cases} \tag{4.6}$$

$$\bar{\mu}_{\tilde{z}}(z; \bar{\check{z}}, \bar{\hat{z}}) = \begin{cases} 0, & z \leqslant \bar{\check{z}} \\ \dfrac{z - \bar{\check{z}}}{\bar{\hat{z}} - \bar{\check{z}}}, & \bar{\check{z}} \leqslant z \leqslant \overline{\hat{z}} \\ 1, & z \geqslant \bar{\hat{z}} \end{cases} \tag{4.7}$$

where $\underline{\check{z}}, \underline{\hat{z}}, \bar{\check{z}}$, and $\bar{\hat{z}}$ are crisp parameters computed from $\tilde{b}$.

Proof 4 *Firstly, we can assume the following:*

$$\underline{\check{b}} \geqslant \underline{\hat{b}} \tag{4.8}$$

$$\bar{\check{b}} \geqslant \bar{\hat{b}} \tag{4.9}$$

$$\underline{\check{b}} \leqslant \bar{\check{b}} \tag{4.10}$$

$$\underline{\hat{b}} \leqslant \bar{\hat{b}} \tag{4.11}$$

Now, replacing b by each of the above parameters in the following crisp LP problem:

$$\begin{gathered} \max z = c(x) + c_0 \\ \text{s.t.} \\ Ax \leqslant b \\ x \geqslant 0 \end{gathered} \tag{4.12}$$

we can obtain the following optimal results as functions of z^*:

$$\underline{\check{b}} \rightarrow \hat{\check{z}} \tag{4.13}$$

$$\underline{\hat{b}} \rightarrow \hat{\hat{z}} \tag{4.14}$$

$$\bar{\check{b}} \rightarrow \underline{\check{z}} \tag{4.15}$$

$$\bar{\hat{b}} \rightarrow \underline{\hat{z}} \tag{4.16}$$

Then, the computation of each bound of $\mu_{\tilde{b}}(Ax)$ leads to a corresponding bound of $\mu_{\tilde{z}}$, and by mathematical induction is clear that:

$$\underline{\check{z}} \geqslant \underline{\hat{z}} \tag{4.17}$$

$$\bar{\check{z}} \geqslant \bar{\hat{z}} \tag{4.18}$$

$$\underline{\check{z}} \leqslant \bar{\check{z}} \tag{4.19}$$

$$\underline{\hat{z}} \leqslant \bar{\hat{z}} \tag{4.20}$$

so we can compose $\mu_{\tilde{z}}$ by its primary membership functions $\underline{\mu}_{\tilde{z}}$ and $\bar{\mu}_{\tilde{z}}$ in the following way:

$$\underline{\mu}_{\tilde{z}}(z; \underline{\check{z}}, \underline{\hat{z}}) = \begin{cases} 0, & z \leqslant \underline{\check{z}} \\ \dfrac{z - \underline{\check{z}}}{\underline{\hat{z}} - \underline{\check{z}}}, & \underline{\check{z}} \leqslant z \leqslant \underline{\hat{z}} \\ 1, & z \geqslant \underline{\hat{z}} \end{cases}$$

and its upper membership function is:

$$\bar{\mu}_{\tilde{z}}(z; \bar{\check{z}}, \bar{\hat{z}}) = \begin{cases} 0, & z \leqslant \bar{\check{z}} \\ \dfrac{z - \bar{\check{z}}}{\bar{\hat{z}} - \bar{\check{z}}}, & \bar{\check{z}} \leqslant z \leqslant \overline{\hat{z}} \\ 1, & z \geqslant \bar{\hat{z}} \end{cases}$$

which concludes the proof.

On the other hand, an interesting question arises from the analysis of $\mu_{\tilde{z}}$: Is there exists the possibility of having a linear combination of $\tilde{b}_i$ that reaches out a solution outside $\mu_{\tilde{z}}$? To answer that, we define the following corollary

Corollary 2 (FOU of $\tilde{z}$) *The FOU of $\tilde{z}$ is defined as follows*

$$\mathrm{FOU}(\tilde{z}) = \bigcup_{z \in Z} \left[\underline{\mu}_{\tilde{z}}(z^*), \overline{\mu}_{\tilde{z}}(z^*)\right] \tag{4.21}$$

$$\mathrm{FOU}(\tilde{z}) = \bigcup_{z \in Z} J_{z^*} \tag{4.22}$$

where J_{z^*} is composed by (4.6) *and* (4.7).

Proof 5 *Straightforward to the reader.*

Corollary 2 defines that all possible Type-1 fuzzy sets Z are embedded into $FOU(\tilde{z})$. Definition 4, convex $\bar{\mu}_{\tilde{b}}$ and $\underline{\mu}_{\tilde{b}}$ alongside equations (4.8) to (4.11) allow us to think that there is no choice to have a linear fuzzy set outside ∇ and $\triangle$, so the question has a natural answer: No. This is not possible if we consider that $\mu_{\tilde{b}}$ is composed by linear primary membership functions and $\mu_{\tilde{z}}$ is also a linear fuzzy set, so there is no any possibility to define a linear combination of $\tilde{b}_i$ that reaches a value outside $\mu_{\tilde{z}}$.

Chapter 5

Solution procedure of an IT2FLP

Until now our main problem is how to deal with interval fuzzy sets, since most of fuzzy optimization methods were designed for Type-1 fuzzy sets, and what we have is an interval of infinite choices. This way, our proposal is based on finding two endpoints enclosed into $\triangle$ and ∇ (see Figure 2.2), and use these points as the parameters of a single fuzzy set, suitable to be optimized using the Algorithm 1.

5.1 First method

Figueroa [78, 88], Figueroa [3], and Figueroa et.al [4] have proposed a method to find an optimal fuzzy set embedded into the FOU of the problem using $\triangle$, ∇ as auxiliary variables weighted by $c^{\triangle}$ and c^{∇} and the Zimmermann's method. A description of the algorithm is presented next.

Algorithm 2

1. Compute an optimal inferior boundary called Z *minimum* ($\check{z}$) by using $\underline{\check{b}} + \triangle$ as a frontier of the model, where $\triangle$ (see Definition 5) is an auxiliary set of variables weighted by $c^{\triangle}$ which represents the lower uncertainty interval subject to the following statement:

$$\triangle \leqslant \bar{\check{b}} - \underline{\check{b}} \tag{5.1}$$

   To do so, $\triangle^{*}$ is obtained solving the following LP problem

$$\begin{aligned} \underset{x,\triangle}{\operatorname{Max}}\ z = c'x + c_0 - c^{\triangle}{}'\triangle \\ s.t. \\ Ax - \triangle \leqslant \underline{\check{b}} \\ \triangle \leqslant \bar{\check{b}} - \underline{\check{b}} \\ x \geqslant 0 \end{aligned} \tag{5.2}$$

2. Compute an optimal superior boundary called *Z maximum* ($\hat{z}$) by using $\bar{\hat{b}} + \nabla$ as a frontier of the model, where ∇ (see Definition 5) is an auxiliary set of variables weighted by c^{∇} which represents the upper uncertainty interval subject to the following statement:
$$\nabla \leqslant \bar{\hat{b}} - \underline{\hat{b}} \tag{5.3}$$
To do so, ∇^* is obtained solving the following LP problem
$$\begin{aligned} \underset{x,\nabla}{\text{Max}}\ & z = c'x + c_0 - c^{\nabla\,\prime}\nabla \\ & s.t. \\ & Ax - \nabla \leqslant \underline{\hat{b}} \\ & \nabla \leqslant \bar{\hat{b}} - \underline{\hat{b}} \\ & x \geqslant 0 \end{aligned} \tag{5.4}$$

3. Find the final solution using the third and subsequent steps of the Algorithm 1 using the following values of $\check{b}$ and $\hat{b}$
$$\check{b} = \underline{\check{b}} + \triangle^* \tag{5.5}$$
$$\hat{b} = \underline{\hat{b}} + \nabla^* \tag{5.6}$$

Remark 6 (About $c^{\triangle}$ and c^{∇}) *In Algorithm 2, we have defined $c^{\triangle}$ and c^{∇} as the weights of $\triangle$ and ∇. In other words, $c_i^{\triangle}$ and c_i^{∇} are the unitary cost associated to increase each resource $\underline{\check{b}}_i$ and $\underline{\hat{b}}_i$ respectively.*

Remark 7 (Max − Min **objectives)** *The proposed algorithm was designed for maximization problems, so equations* (5.5) *and* (5.6) *apply to a* Max *goal. For a* min *goal, equations* (5.2)*,* (5.4)*,* (5.5) *and* (5.6) *have to be changed.*

Therefore, $\triangle$ and ∇ are auxiliary variables that operate as Type-reducers[1], where $\triangle_i^*$ and ∇_i^* become $\check{b}_i$ and $\hat{b}_i$ as the inputs of the Zimmermann's method which returns $\check{z}^*$, $\hat{z}^*$ and α^* (see Section 2.5).

5.2 Second method

Figueroa and Hernández [5] have designed a method for solving IT2FLP problems using Interval Linear Programming (ILP). This way, the main idea of this proposal is to pre-defuzzify all constraints $\tilde{b}$ using an *α-cut* to obtain an interval optimization problem, and then

[1] A Type-reduction strategy regards to a method for finding a single fuzzy set embedded into the FOU of a Type-2 fuzzy set.

solve this new problem using bounded LP methods such as the simplex method, Karmarkar interior point methods, Khachiyan, etc. Now, the Bellman-Zadeh fuzzy decision making principle is based on the idea of obtaining a maximum intersection among fuzzy goals and constraints for making a decision (see Bellman and Zadeh [85]), in the sense that we propose to pre-defuzzify the problem before find a solution using crisp and interval optimization methods, so our approach is a crisp-interval feasible operation point given a value of α-*cut*. Some facts about this proposal are:

1. The complexity of $\tilde{b}$ can be reduced using α-*cuts*, which lead to an ILP problem.
2. The solution provided by the proposed approach is an operation point x^* which is optimal in the sense of a crisp LP problem, not in the Bellman-Zadeh decision making principle.
3. An optimal solution is understood as an extreme point for which we have $z^* = \inf\{c'x \mid Ax = {}^{\alpha}\tilde{b}, x \geqslant 0\}$. It can be computed using any optimization method. As there is a single optimal value for z, any choice of an optimization algorithm should obtain the same value z^*.

Then the idea is to reduce the complexity of an IT2FLP using α-cuts to solve the problem as an interval-valued one. Although there are different ways to solve the same problem, we propose the methodology shown in Figure 5.1 (see Figueroa and Hernández [5]).

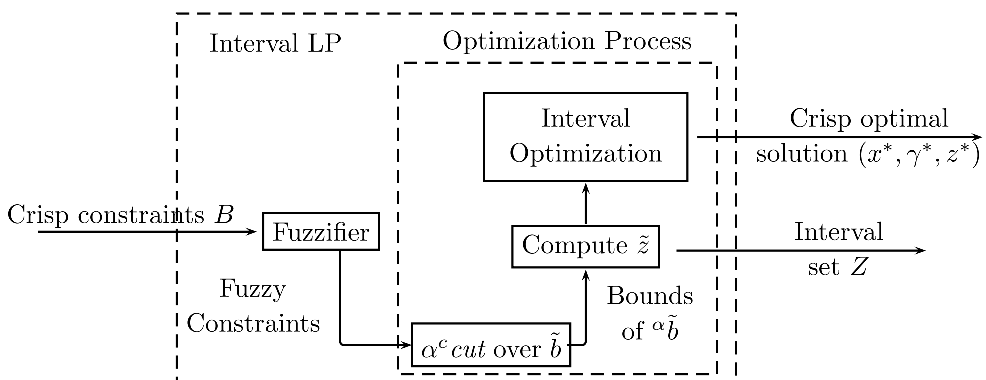

Figure 5.1: Interval optimization methodology

This methodology introduces a pre-defuzzification α-*cut* for finding a solution of the resultant ILP model by means of classical linear optimization methods. The procedure is summarized as follows.

Algorithm 3

1. Select a pre-defuzzification level named α for all fuzzy constraints.

2. Compute the $\alpha^c - cut$ for all fuzzy constraints, ${}^{\alpha^c}\tilde{b}_i$. This generates an interval in the form $[{}^{\alpha^c}\underline{b}_i, {}^{\alpha^c}\overline{b}_i]$ (see (3.11) and (3.12)).

3. Set a new variable γ_i:
$$ {}^{\alpha^c}\underline{b}_i \leqslant \gamma_i \leqslant {}^{\alpha^c}\overline{b}_i \tag{5.7}$$

4. Solve the problem as an ILP model in the form:
$$\begin{aligned} \underset{x,\gamma}{\text{Max}}\ z &= c'x + c_0 - c^{\gamma}{}'\gamma \\ & s.t. \\ Ax - \gamma &\leqslant {}^{\alpha^c}\underline{b} \\ {}^{\alpha^c}\underline{b} \leqslant \gamma &\leqslant {}^{\alpha^c}\overline{b} \\ x &\geqslant 0 \end{aligned} \tag{5.8}$$
where $x, c, c^{\gamma} \in \mathbb{R}^n$; $b, \gamma \in \mathbb{R}^m$; $c_0 \in \mathbb{R}$, and $A \in \mathbb{R}^{n\times m}$.

5. Return x^*, γ^* and z^*.

Remark 8 (About c^{γ}) *The cost c^{γ} defined in our method is the unitary cost of increasing an specific resource. Different choices of c^{γ} could lead to different results, so we recommend to analyze the shadow prices of the LP model using ${}^{\alpha^c}\underline{b}$ before defining c^{γ}.*

Remark 9 (On computing $\tilde{z}$) *The set $\tilde{z}$ is an interval set of optimal solutions defined by two boundaries $\underline{z}$ and $\bar{z}$ which comes from the following equations:*
$$ {}^{\alpha^c}\underline{b} \to \underline{z} \tag{5.9}$$
$$ {}^{\alpha^c}\overline{b} \to \bar{z} \tag{5.10}$$
This means that $\underline{z}$ and $\bar{z}$ comes from the LP model shown in (5.8) *using ${}^{\alpha^c}\underline{b}$ and ${}^{\alpha^c}\overline{b}$ respectively.*

5.2.1 Selecting α

It is clear that the selection of α is an entirely expert-based issue. The values of x^* and γ^* depend on the selection of α, so the analyst can use one of the following four main choices of α (for a maximization goal):

Pessimistic α: This scenario is provided by selecting $\alpha = 1 \to {}^{1}\tilde{b}$. This means that the analyst has a pessimistic perception about uncertainty on $\tilde{b}$ since he/she thinks that the system should have the minimum availability of resources, more than likely.

Optimistic α: In this case we have $\alpha = 0 \rightarrow {}^{0}\tilde{b}$. This means that the analyst thinks that the system should have the minimum availability of their resources, with a low possibility of occurrence.

Uniform α: This option uses $0 \leqslant \alpha^c \leqslant 1 \rightarrow {}^{\alpha^c}\tilde{b}$. The analyst selects a single defuzzification degree α for all constraints, following their perception about the system.

Non-uniform α: This choice uses $0 \leqslant \alpha_i^c \leqslant 1 \rightarrow {}^{\alpha^c}\tilde{b}_i$. The analyst selects a different defuzzification degree α_i for each constraint $\tilde{b}_i$.

In this way, the analyst can use any of the proposed choices to pre-defuzzify all constraints $\tilde{b}$ depending on the necessities of the system and/or their perception about its behavior.

5.3 Optimization software

Most of the reported FLP applications and models are based on convex optimization techniques. This means that all the solutions of FLP models can be obtained by using classical optimization principles, so the main problem is how to compose the set of all possible optimal solutions, and a satisfaction degree regarding all fuzzy constraints.
However, Rommelfanger [89, 90, 91] has designed some specialized computer routines for several FLP models: Soft constraints, fuzzy objective coeficients and fuzzy piecewise technological coeficients. Although those routines are available for T1FLP, we tend to use convex optimization software as GAMS® and MatLab® due to their flexibility and capabilities to handle large problems.

Chapter 6

Application examples

6.1 Production Planning example

An important activity in Production Planning is the optimization of the production quantities of "j" products. LP models are efficient at solving the problem in terms of the capacities of the system as well as the demand of each product. Time units are widely used to express the capacities of the system. However, different units can be used to measure capacities such as energy, materials, space, money units, etc. All of them are included in a general model where these capacities and demands are expressed as restrictions of an LP model and its solution is found through the optimal mix of products regarding an objective function.

In this example, multiple experts have defined the demands of the system using their expertise and perceptions about the markets. If each expert defines demands using the same concept, we obtain the FOU of an IT2FS, so we can compose an IT2FS around the same demand. This way, in a case where different experts define the demands in different ways over the same linguistic label, it is possible to bypass from a fuzzy model to an Type-2 one by using IT2FSs.

Fuzzy production planning problems have been addressed by Peidro et.al [92, 93], Mula et.al [94], Lee et.al [95, 96], Chanas et.al [97], Ángeles et.al [98], and Gen et.al [99], so we address a Mixed Production Planning problem (MPP) using a production, inventory and backorder strategy under a utility maximization objective. This example has been taken from Figueroa et.al [4]. For simplicity, we only consider resources capacity, workforce availability and constrained demands as defined as follows:

$$\text{Max}\, z = \sum_{j=1}^{n}\sum_{k=1}^{K} Sp_{jk}\,[x^r_{jk} + x^o_{jk} + x^s_{jk}] -$$

$$[Cp^r_{jk}\, x^r_{jk} + Cp^o_{jk}\, x^o_{jk} + Cp^s_{jk}\, x^s_{jk} + h_{jk}\, q_{jk} + o_{jk}\, s_{jk}] \tag{6.1}$$

$$s.t.$$

$$\sum_{j=1}^{n} ts_{ijk}\, x^r_{jk} \leqslant Ac^r_{ik} \quad \forall\; i \in \mathbb{N}_m; k \in \mathbb{N}_K \tag{6.2}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ts_{ijk}\, Op_i\, x^r_{jk} \leqslant Wc^r_k \quad \forall\; k \in \mathbb{N}_K \tag{6.3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} e_{jkl}\,[x^r_{jk} + x^o_{jk}] \leqslant Ae_{kl} \quad \forall\; k \in \mathbb{N}_K; l \in \mathbb{N}_L \tag{6.4}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} rm_{jkr}\,[x^r_{jk} + x^o_{jk}] \leqslant Am_{kr} \;\; \forall\; k \in \mathbb{N}_K; r \in \mathbb{N}_R \tag{6.5}$$

$$\sum_{j=1}^{n} ts_{ijk}\, x^o_{jk} \leqslant Ac^o_{ik} \quad \forall\; i \in \mathbb{N}_m; k \in \mathbb{N}_K \tag{6.6}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ts_{ijk}\, Op_i\, x^o_{jk} \leqslant Wc^o_k \quad \forall\; k \in \mathbb{N}_K \tag{6.7}$$

$$x^s_{jk} \leqslant As_{jk} \quad \forall\; j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{6.8}$$

$$\sum_{j=1}^{n} a_j\, q_{jk} \leqslant A_k \quad \forall\; k \in \mathbb{N}_K \tag{6.9}$$

$$x^r_{jk} + x^o_{jk} + x^s_{jk} + q_{j,k-1} + s_{jk} - q_{jk} - s_{j,k-1} \leqslant d^{(+)}_{jk} \quad \forall\; j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{6.10}$$

$$x^r_{jk} + x^o_{jk} + x^s_{jk} + q_{j,k-1} + s_{jk} - q_{jk} - s_{j,k-1} \geqslant d^{(-)}_{jk} \quad \forall\; j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{6.11}$$

$$x^r_{jk}; x^o_{jk}; x^s_{jk}; q_{jk}, s_{jk} \geqslant 0 \tag{6.12}$$

Index sets:
$\mathbb{N}_m$ is the set of all "i" resources, $i \in \mathbb{N}_m$, $\mathbb{N}_m = 1, 2, \cdots, m$.
$\mathbb{N}_n$ is the set of all "j" products, $j \in \mathbb{N}_n$, $\mathbb{N}_n = 1, 2, \cdots, n$.
$\mathbb{N}_K$ is the set of all "k" periods, $k \in \mathbb{N}_K$, $\mathbb{N}_K = 1, 2, \cdots, K$.
$\mathbb{N}_L$ is the set of all "l" energy units, $l \in \mathbb{N}_L$, $\mathbb{N}_L = 1, 2, \cdots, L$.
$\mathbb{N}_R$ is the set of all "r" raw materials, $r \in \mathbb{N}_R$, $\mathbb{N}_R = 1, 2, \cdots, R$.

Decision variables:
x^r_{jk}: Amount of product "j" to be manufactured in regular time per period "k".
x^o_{jk}: Amount of product "j" to be manufactured in overtime per period "k".

x^s_{jk}: Amount of product "j" to be manufactured by outsourcing per period "k".
q_{jk}: Inventory amount of product "j" to be held per period "k".
s_{jk}: Shortage amount of product "j" to be supported in the period "k".

Parameters:
Sp_{jk} = Unitary selling price of product "j" in the period "k".
Cp^r_{jk} = Unitary production cost for the Regular time of product "j" in the period "k".
Cp^o_{jk} = Unitary production cost for overtime of product "j" in the period "k".
Cp^s_{jk} = Unitary outsourcing cost for the product "j" in the period "k".
ts_{ijk} = Unitary standard production time that product "j" uses from the "i" resource in the period "k".
Op_i = Amount of workers needed to operate resource "i".
e_{jkl} = Amount of the "l" energy units used to manufacture product "j", in the period "k".
rm_{jkl} = Amount of the "r" raw material units used to manufacture product "j", in the period "k".
a_j = Amount of space (in area or volume units) used to hold an unit of product "j".
$d^{(-)}_{jk}$ = Minimum demand of product "j" in the period "k".
$d^{(+)}_{jk}$ = Maximum (Potential) demand of product "j" in the period "k".
Ac^r_{ik} = Available capacity on regular time of resource "i" in the period "k", expressed in hours.
Wc^r_k = Available capacity on regular time of workforce in the period "k", expressed in hours.
Ac^o_{ik} = Available capacity on overtime of resource "i" in the period "k", expressed in hours.
Wc^o_k = Available capacity on overtime of workforce in the period "k", expressed in hours.
Ae_{kl} = Availability of the energy type "l" in the period "k", expressed in energy units.
Am_{kr} = Availability of the raw material type "r" in the period "k", expressed as units as Kg, Lt, etc.
As_{jk} = Available outsourced units of product "j" in the period "k".
A_k = Available space units (in area or volume units) in the period "k".

The example is composed by 4 products manufactured in 3 stations for 4 periods on the base of the crisp model shown in (6.1). It is solved by the methods presented in Algorithm 1 and Algorithm 2 with a brief discussion about the results of both MPP models. For the sake of understanding, we refer to the Zimmermann's classical fuzzy method as the *Soft* MPP, and the IT2FS approach as the *Uncertain* MPP.
It is possible to get multiple opinions and estimates coming from different experts of the demands of the system, leading to uncertain demands denoted by $\tilde{D}_{jk}$. Therefore, an IT2FS approach is a suitable way to deal with those opinions and estimates of that variable. Hence, the IT2FLP model for the MPP problem should be solved in two stages before using the Zimmermann's method: a first MPP model in terms of (5.2) and a second one in terms of (5.4). A graphical representation of an uncertaind demand $\tilde{D}_{jk}$ is shown in Figure 6.1.

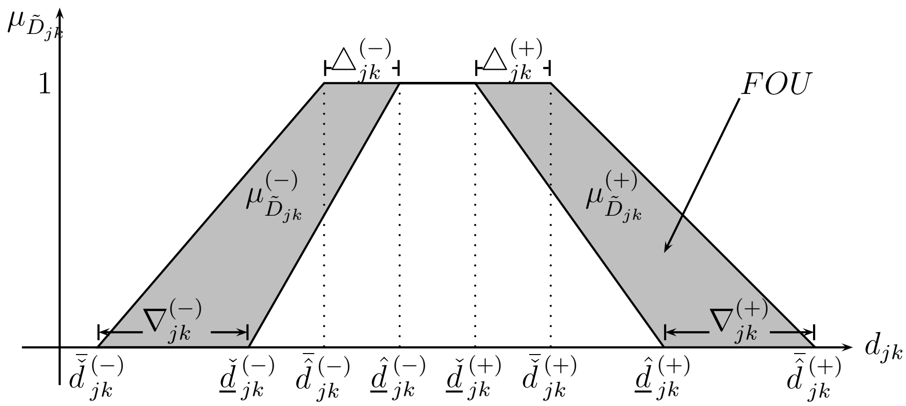

Figure 6.1: IT2FS for d_{jk}, $\tilde{D}_{jk}$.

Thus, the MPP in terms of (5.2), Tables 1 and 2 is presented next

$$\text{Max}\, z = \sum_{j=1}^{n}\sum_{k=1}^{K} Sp_{jk}\,[x^r_{jk} + x^o_{jk} + x^s_{jk}] -$$

$$[Cp^r_{jk}\,x^r_{jk} + Cp^o_{jk}\,x^o_{jk} + Cp^s_{jk}\,x^s_{jk} + h_{jk}\,q_{jk} + o_{jk}\,s_{jk}] \tag{6.13}$$

$$s.t.$$

$$\sum_{j=1}^{n} ts_{ijk}\,x^r_{jk} \leqslant Ac^r_{ik} \quad \forall\; i \in \mathbb{N}_m; k \in \mathbb{N}_K \tag{6.14}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ts_{ijk}\,Op_i\,x^r_{jk} \leqslant Wc^r_k \quad \forall\; k \in \mathbb{N}_K \tag{6.15}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} e_{jkl}\,[x^r_{jk} + x^o_{jk}] \leqslant Ae_{kl} \quad \forall\; k \in \mathbb{N}_K; l \in \mathbb{N}_L \tag{6.16}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} rm_{jkr}\,[x^r_{jk} + x^o_{jk}] \leqslant Am_{kr} \;\; \forall\; k \in \mathbb{N}_K; r \in \mathbb{N}_R \tag{6.17}$$

$$\sum_{j=1}^{n} ts_{ijk}\,x^o_{jk} \leqslant Ac^o_{ik} \quad \forall\; i \in \mathbb{N}_m; k \in \mathbb{N}_K \tag{6.18}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ts_{ijk}\,Op_i\,x^o_{jk} \leqslant Wc^o_k \quad \forall\; k \in \mathbb{N}_K \tag{6.19}$$

$$x^s_{jk} \leqslant As_{jk} \quad \forall\; j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{6.20}$$

$$\sum_{j=1}^{n} a_j\,q_{jk} \leqslant A_k \quad \forall\; k \in \mathbb{N}_K \tag{6.21}$$

$$x^r_{jk} + x^o_{jk} + x^s_{jk} + q_{j,k-1} + s_{jk} - q_{jk} - s_{j,k-1} - \triangle^{(+)}_{jk} \leqslant 0 \quad \forall j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{6.22}$$

$$x^r_{jk} + x^o_{jk} + x^s_{jk} + q_{j,k-1} + s_{jk} - q_{jk} - s_{j,k-1} - \triangle^{(-)}_{jk} \geqslant 0 \quad \forall j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{6.23}$$

$$\triangle_{jk}^{(-)} \leqslant \underline{\hat{d}}_{jk}^{(-)} - \bar{\bar{d}}_{jk}^{(-)} \tag{6.24}$$

$$\triangle_{jk}^{(+)} \leqslant \underline{\check{d}}_{jk}^{(+)} - \bar{\bar{d}}_{jk}^{(+)} \tag{6.25}$$

$$x_{jk}^{r}; x_{jk}^{o}; x_{jk}^{s}; q_{jk}, s_{jk} \geqslant 0$$

Here, $\triangle_{jk}^{(-)}$ is related to $\tilde{D}_{jk}^{(-)}$, $\triangle_{jk}^{(+)}$ is related to $\tilde{D}_{jk}^{(+)}$, $C_{jk}^{\triangle(-)}$ and $C_{jk}^{\triangle(+)}$ are the unitary cost of uncertainty associated with increasing an unit of $\triangle_{jk}^{(-)}$ or $\triangle_{jk}^{(+)}$, and (5.2) as defined before. Now, the uncertain MPP model in terms of (5.4) is:

$$\text{Max } z = \sum_{j=1}^{n}\sum_{k=1}^{K} Sp_{jk}\,[x_{jk}^{r} + x_{jk}^{o} + x_{jk}^{s}] -$$

$$[Cp_{jk}^{r}\,x_{jk}^{r} + Cp_{jk}^{o}\,x_{jk}^{o} + Cp_{jk}^{s}\,x_{jk}^{s} + h_{jk}\,q_{jk} + o_{jk}\,s_{jk}] \tag{6.26}$$

$$s.t.$$

$$\sum_{j=1}^{n} ts_{ijk}\,x_{jk}^{r} \leqslant Ac_{ik}^{r} \quad \forall\ i \in \mathbb{N}_m; k \in \mathbb{N}_K \tag{6.27}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ts_{ijk}\,Op_i\,x_{jk}^{r} \leqslant Wc_{k}^{r} \quad \forall\ k \in \mathbb{N}_K \tag{6.28}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} e_{jkl}\,[x_{jk}^{r} + x_{jk}^{o}] \leqslant Ae_{kl} \quad \forall\ k \in \mathbb{N}_K; l \in \mathbb{N}_L \tag{6.29}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} rm_{jkr}\,[x_{jk}^{r} + x_{jk}^{o}] \leqslant Am_{kr} \ \ \forall\ k \in \mathbb{N}_K; r \in \mathbb{N}_R \tag{6.30}$$

$$\sum_{j=1}^{n} ts_{ijk}\,x_{jk}^{o} \leqslant Ac_{ik}^{o} \quad \forall\ i \in \mathbb{N}_m; k \in \mathbb{N}_K \tag{6.31}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ts_{ijk}\,Op_i\,x_{jk}^{o} \leqslant Wc_{k}^{o} \quad \forall\ k \in \mathbb{N}_K \tag{6.32}$$

$$x_{jk}^{s} \leqslant As_{jk} \quad \forall\ j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{6.33}$$

$$\sum_{j=1}^{n} a_j\,q_{jk} \leqslant A_k \quad \forall\ k \in \mathbb{N}_K \tag{6.34}$$

$$x_{jk}^{r} + x_{jk}^{o} + x_{jk}^{s} + q_{j,k-1} + s_{jk} - q_{jk} - s_{j,k-1} - \nabla_{jk}^{(+)} \leqslant 0 \quad \forall j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{6.35}$$

$$x_{jk}^{r} + x_{jk}^{o} + x_{jk}^{s} + q_{j,k-1} + s_{jk} - q_{jk} - s_{j,k-1} - \nabla_{jk}^{(-)} \geqslant 0 \quad \forall j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{6.36}$$

$$\nabla_{jk}^{(-)} \leqslant \underline{\check{d}}_{jk}^{(-)} - \bar{\check{d}}_{jk}^{(-)} \tag{6.37}$$

$$\nabla_{jk}^{(+)} \leqslant \bar{\hat{d}}_{jk}^{(+)} - \underline{\hat{d}}_{jk}^{(+)} \tag{6.38}$$

$$x_{jk}^{r}; x_{jk}^{o}; x_{jk}^{s}; q_{jk}, s_{jk} \geqslant 0$$

Here, $\nabla_{jk}^{(-)}$ is related to $\tilde{D}_{jk}^{(-)}$, $\nabla_{jk}^{(+)}$ is related to $\tilde{D}_{jk}^{(+)}$, $C_{jk}^{\nabla(-)}$ and $C_{jk}^{\nabla(+)}$ are the unitary uncertainty costs associated with increasing an unit of $\nabla_{jk}^{(-)}$ or $\nabla_{jk}^{(+)}$, respectively. All data used in this example is shown in Tables 1, 2, 3, and 4 (see Appendix).

6.1.1 FLP results

Solving the problem as a soft MPP using the results of Algorithm 1, (3.12) and the parameters shown in Table 3, we found that $\check{z}$=15'223.900 is obtained using $\check{d}_{jk}^{(+)}$ and $\hat{z}$=15'282.940 is obtained through $\hat{d}_{jk}^{(+)}$. The optimal satisfaction degree is α_1=0.5 and the maximal utility of the system is 15'229.750. A detailed report of the solution is shown in Table 6.1 and its graphical representation is shown next.

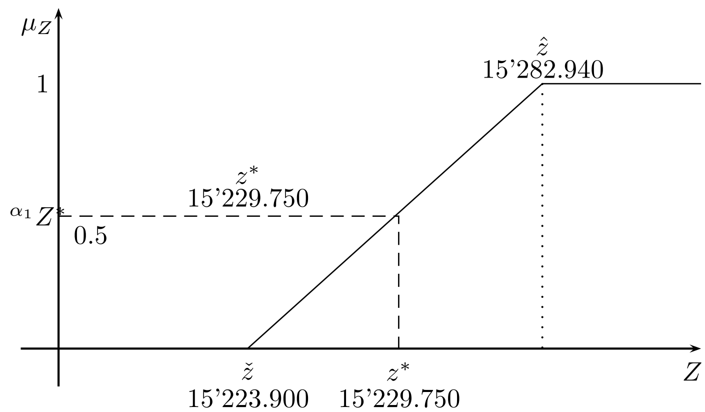

Figure 6.2: Obtained solution Z^* for soft demands

6.1.2 IT2FLP results

By using the Algorithm 2, Propositions 3 and 4, and the parameters shown in Table 4, an uncertain problem is solved using the simplex algorithm. The optimal satisfaction degree is α_1=0,7076 and the optimal values of z computed by the Algorithm 3 through key points of $\tilde{D}_{jk}$ (see Figure 6.3) are:

$$
\begin{aligned}
\underline{\check{d}}_{jk}^{(+)} &\to \underline{\check{z}} = 15'208.730\\
\bar{\check{d}}_{jk}^{(+)} &\to \bar{\check{z}} = 15'247.030\\
\underline{\hat{d}}_{jk}^{(+)} &\to \underline{\hat{z}} = 15'252.880\\
\bar{\hat{d}}_{jk}^{(+)} &\to \bar{\hat{z}} = 15'311.860
\end{aligned}
$$

Each of the above values of $\tilde{z}$ are computed by applying the simplex algorithm to its respective value of $\tilde{D}_{jk}$. On the other hand, $\check{z}^*$=15'229.250 and $\hat{z}^*$=15'281.160 are obtained after computing (5.1) and (5.3) which become (5.5) and (5.6). This means that $\triangle_{jk}^{(+)*}$ and $\nabla_{jk}^{(+)*}$

Table 6.1: Summary report of both soft and uncertain problems

Soft demands problem				Uncertain demands problem				
j,k	x^{r*}_{jk}	I^*_{jk}	S^*_{jk}	x^{r*}_{jk}	I^*_{jk}	S^*_{jk}	$\triangle^{(+)*}_{jk}$	$\nabla^{(+)*}_{jk}$
1,1	11700	2688	0	11700	2490.32	0	320	440
1,2	3200	0	0	3200	0	0	0	0
1,3	2400	0	0	2400	0	0	0	0
1,4	3400	0	0	3400	0	0	0	0
2,1	17216.66	13426.17	0	17216.66	13110.24	0	241	364
2,2	6350	3250	0	6350	3250	0	0	0
2,3	4200	2450	0	4200	2450	0	0	0
2,4	2600	0	0	2600	0	0	0	0
3,1	18313.64	26833.31	0	18313.64	26409.6	0	117	338
3,2	4500	0	0	4500	0	0	0	0
3,3	0	0	0	0	0	0	0	0
3,4	10636.37	1295.87	0	10636.37	1186.53	0	144	312
4,1	16926.55	37281.37	0	16926.55	36704.55	0	277	351
4,2	5174.96	0	0	5174.96	0	0	0	0
4,3	1900	0	0	1900	0	0	0	0
4,4	7031.83	2585.7	0	7031.83	2347.79	0	254	205

operate as location points in the FOU of $\tilde{D}_{jk}$, which is useful to bypass from an uncertain problem to a soft one before using the Zimmermann's method, so this can be interpreted as a Type-reduction method. The maximal utility of the system is 15'265.981 through an α-cut of 0,7076. Table 6.1 shows a detailed report of the IT2FLP solution; and Figure 6.3 shows a graph of $\tilde{z}$.

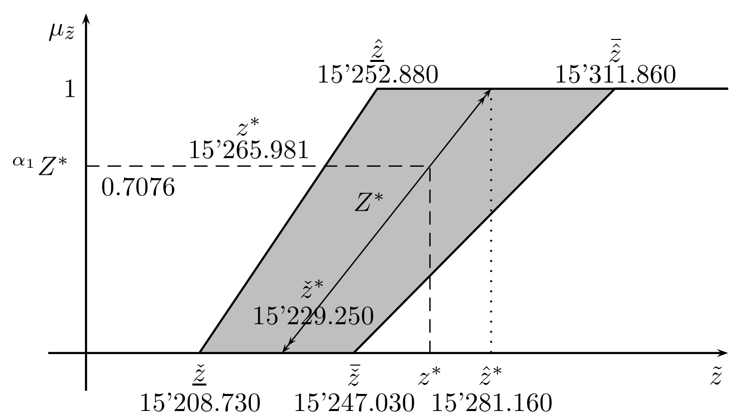

Figure 6.3: Solution Z^* embedded in the FOU of $\tilde{z}$.

In the interval FLP approach, $C^{\triangle(+)}_{jk}$ and $C^{\nabla(+)}_{jk}$ work as incremental costs to bypass from the certain feasible region to an uncertain region. These uncertainty costs are supported by the system in order to increase its demand, understood as the opportunity cost of covering the market.

As shown in Table 6.1, the model yields a crisp optimal solution for both soft and uncertain demands. Note that the production levels x_{jk}^{r*} are the same, only varying the inventory levels I_{jk}^{*}. This indicates that both models are consistent on their results, going from fuzzy to crisp measures obtaining an optimal solution, and collecting the opinion of the experts in this case represented by non-probabilistic uncertainty related to the demands of the market.

6.1.3 Discussion of the results

As the problem involves higher uncertainty measures, decision making processes become complex. The FLP model reaches a solution based on a single perception about the demands of the market instead of the IT2FLP model that involves an infinite amount of perceptions about the same concept.

Both the FLP approach and the IT2FLP one reach an α_1 degree, the difference between the two lies in their modeling process. The final results are similar, this means that the production levels x_{jk}^{r*} are the same and the optimal profits are similar, though at different satisfaction degrees.

The optimal satisfaction degree α^* operates as a location point for planning the demands that the system should cover in terms of its production strategy. Notice that the IT2FLP approach yielded a higher satisfaction degree with better profits than the FS approach. This means that the FLP approach tries to cover less demands than the IT2FLP approach with less profits, and the IT2FLP model tries to achieve an equilibrium point among demands, profits and uncertainty.

The IT2FLP model returns a higher value of z^* than the FLP approach, even using $C_{jk}^{\triangle(+)}$ and $C_{jk}^{\nabla(+)}$ as uncertainty costs. This means that the analyst can make better decisions, even by paying for some uncertainty costs. The possibility of having multiple experts might become an opportunity for decision making achieving positive results since the use of multiple choices and opinions can lead to better results and profits.

6.2 Simulated Data example

The following application data is used to explain how the Algorithm 3 works. All necessary data to solve the problem is shown as follows.

$$A = \begin{bmatrix} 5 & 4 & 3 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 4 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix} ; \; \check{\underline{b}} = \begin{bmatrix} 25 \\ 35 \\ 20 \\ 30 \\ 20 \end{bmatrix} ; \; \hat{\underline{b}} = \begin{bmatrix} 36 \\ 52 \\ 33 \\ 48 \\ 40 \end{bmatrix} \; \check{\overline{b}} = \begin{bmatrix} 30 \\ 40 \\ 28 \\ 38 \\ 29 \end{bmatrix}$$

$$\bar{\hat{b}} = \begin{bmatrix} 48 \\ 55 \\ 38 \\ 51 \\ 49 \end{bmatrix} ; \; c = \begin{bmatrix} 11 \\ 10 \\ 11 \end{bmatrix} ; \; c^{\gamma} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

The Algorithm 3 uses a predefuzzification degree α per approach. Firstly, the *uniform* approach is solved using an $\alpha = 0,75$ which leads to the following values:

$${}^{0,75}\underline{b} = \begin{bmatrix} 33.25 \\ 47.75 \\ 29.75 \\ 43.5 \\ 35 \end{bmatrix} ; \; {}^{0,75}\bar{b} = \begin{bmatrix} 43.5 \\ 51.25 \\ 35.5 \\ 47.75 \\ 44 \end{bmatrix}$$

Now, the selected values of α_i for the *non-uniform* approach, and the resultant values of ${}^{\alpha^c}\tilde{b}_i$ are as follows:

$$\alpha_i = \begin{bmatrix} 0.5 \\ 0.25 \\ 0.6 \\ 0.85 \\ 0.4 \end{bmatrix} ; \; {}^{\alpha_i}\underline{b} = \begin{bmatrix} 30.5 \\ 39.25 \\ 27.8 \\ 45.3 \\ 28 \end{bmatrix} ; \; {}^{\alpha_i}\bar{b} = \begin{bmatrix} 39 \\ 43,75 \\ 34 \\ 49,05 \\ 37 \end{bmatrix}$$

The boundaries of $\tilde{z}$ are computed as proposed by Figueroa [6]. The resultant fuzzy set is shown in Figure 6.4.

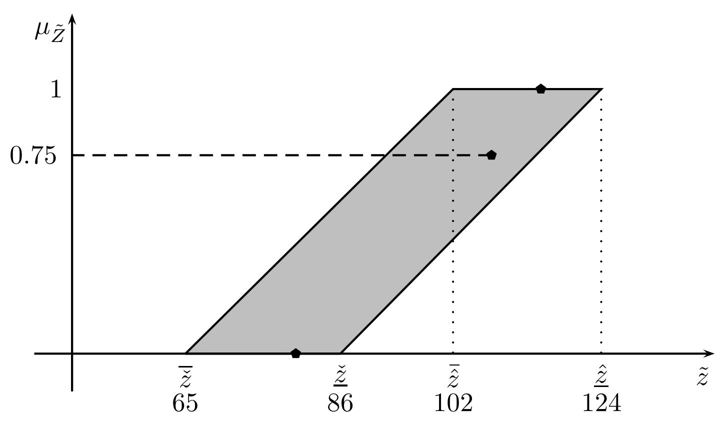

Figure 6.4: Set of optimal solutions $\tilde{Z}$

In this figure we show the set of all possible optimal solutions $\tilde{Z}$ and the solutions for $\alpha = 0, \alpha = 0.75$ and $\alpha = 1$. A detailed report of the obtained solution is shown next.

Table 6.2: Optimization results for $\alpha = 0, 0.75, 1$ and α_i

α	x_1^*	x_2^*	x_3^*	γ_1^*	γ_2^*	γ_3^*	$\gamma_4^* = \gamma_5^*$	z^*
0	0,25	5	3	5	0	8	0	80
0.75	7,09	0	4,18	12	0	5	0	115
1	6,14	0	4,27	10,2	0	5,8	0	106,5
α_i	3,88	1,63	4,38	8,5	0	6,2	0	99,7

6.2.1 Discussion of the results

All methods have obtained an optimal solution. Note that both z^* and x^* have different results because γ and c^γ change the boundaries of the problem as a function of α^c. This means that the decision maker has to analyze the implications of using any of the presented choices.

At this point, an optimal solution of the problem is obtained in terms of x^* and γ^*. For the sake of understanding, the proposed method uses a predefuzzification degree α before using any optimization algorithm, so γ operates as an auxiliary variable that reaches a defuzzified value of each constraint.

The values of c^γ are additional degrees of freedom that the analyst should keep in mind before applying our proposal. In this example, those costs has to be paid by the system, so the method achieves a solution where the system has to increase only selected constraints to improve z, even by paying for γ.

There are some interesting reasons for: the method selects the constraints that increase the objective function, accomplishing (6.12) instead of a simpler reasoning of treating all constraints in the same way.

Note that there is an infinite amount of possible choices of x_{ij} that can solve the problem, so we point out that our approach is based on the idea of a selection done over the possible set of choices through α^c.

6.3 Behavior of the Zimmermann's method into $\tilde{z}$

Even when the Zimmermann's method has a linear behavior (if considered as a single model), the existence of infinite sets B into the FOU of $\tilde{b}$ permits that infinite choices of $\hat{b} \in \triangle$ and $\underline{b} \in \nabla$ can be selected to perform an IT2FLP. Figueroa and Hernández [7] has shown that the Zimmermann's method has no a linear behavior for different selections of $\hat{b}$ and $\check{b}$. Anyways, any selection is inside $\tilde{z}$ as shown in last section.

The experimental evidence has been obtained by computing optimal solutions for some selected combinations of fuzzy sets using two key points $\check{b}$ and $\hat{b}$ enclosed into $\triangle$ and ∇ respectively. The selected combinations of parameters for computing α^* are:

Pessimistic approach: This approach uses $\mu_{\underline{b}}$, namely $\check{\underline{b}}$ and $\hat{\underline{b}}$, based on the idea of having the minimum possible availability of resources.

Optimistic approach: This approach uses $\mu_{\bar{b}}$, namely $\check{\bar{b}}$ and $\hat{\bar{b}}$, based on the idea of having the maximum possible availability of resources.

min-max approach: This approach uses $\check{\underline{b}}$ and $\hat{\bar{b}}$, based on the idea of having extreme availability of resources.

max-min approach: This approach uses $\check{\bar{b}}$ and $\hat{\underline{b}}$, based on the idea of having extreme availability of resources.

Incremental approach: This approach divides $\triangle$ and ∇ into a set of proportional values from $\check{\underline{b}}$ to $\check{\bar{b}}$ and $\hat{\underline{b}}$ to $\hat{\bar{b}}$ using a value $\delta \in [0, 1]$, as follows:

$$\check{b} = \check{\underline{b}} + \triangle * \delta \tag{6.39}$$

$$\hat{b} = \hat{\underline{b}} + \nabla * \delta \tag{6.40}$$

Note that $\delta = 0$ and $\delta = 1$ equals to the pessimistic and optimistic approach.
A maximization problem of four variables and five fuzzy constraints using the following data is solved to show the behavior of the Algorithm 1 regarding the selected combinations of $\hat{b} \in \triangle$ and $\check{b} \in \nabla$:

$$A = \begin{bmatrix} 2 & 3 & 3 & 2 \\ 3 & 1 & 2 & 3 \\ 4 & 5 & 2 & 6 \\ 3 & 2 & 4 & 4 \\ 3 & 6 & 4 & 5 \end{bmatrix} ; \check{\underline{b}} = \begin{bmatrix} 20 \\ 24 \\ 18 \\ 20 \\ 32 \end{bmatrix} ; \hat{\underline{b}} = \begin{bmatrix} 24 \\ 30 \\ 23 \\ 26 \\ 40 \end{bmatrix}$$

$$\check{\bar{b}} = \begin{bmatrix} 22 \\ 30 \\ 20 \\ 24 \\ 40 \end{bmatrix} ; \hat{\bar{b}} = \begin{bmatrix} 22 \\ 30 \\ 20 \\ 24 \\ 40 \end{bmatrix} ; c = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 5 \end{bmatrix}$$

Now, we have selected 9 equally distributed values of δ, $\delta = \{0.1, 0.2, \cdots, 0.9\}$ for which we computed the mixed approach. Note that each of the resultant problems have a value of $\check{z}$, $\hat{z}$ and $z^* = f(\alpha^*, \check{z}, \hat{z})$. The results are summarized in Table 6.3, and using the results of the Theorem 5, we compute $\tilde{z}$ which is displayed in Figure 6.5.

6.3.1 Discussion of the results

The largest and smallest values of α were achieved by the $\delta = 0.4$ and max $-$ min approach respectively. This means that the FOU of $\tilde{b}$ contains extreme points of α. This also suggests

Table 6.3: Optimization results for different values of α.

Value	α^*	z^*	$\check{z}$	$\hat{z}$
Pessimistic	0.5	25,125	22	28,25
Optimistic	0.5149	29,377	25	33,5
$\min - \max$	0.5194	27,973	22	33,5
$\max - \min$	0.5	26,625	25	28,25
$\delta = 0.1$	0.5	25,650	22,3	29
$\delta = 0.2$	0.5072	26,124	22,6	29,5474
$\delta = 0.3$	0.515	26,586	22,9	30,0579
$\delta = 0.4$	0.5208	27,037	23,2	30,5684
$\delta = 0.5$	0.5189	27,433	23,5	31,0789
$\delta = 0.6$	0.5171	27,828	23,8	31,5895
$\delta = 0.7$	0.5154	28,223	24,1	32,1
$\delta = 0.8$	0.5138	28,619	24,4	32,6105
$\delta = 0.9$	0.5138	29,003	24,7	33,075

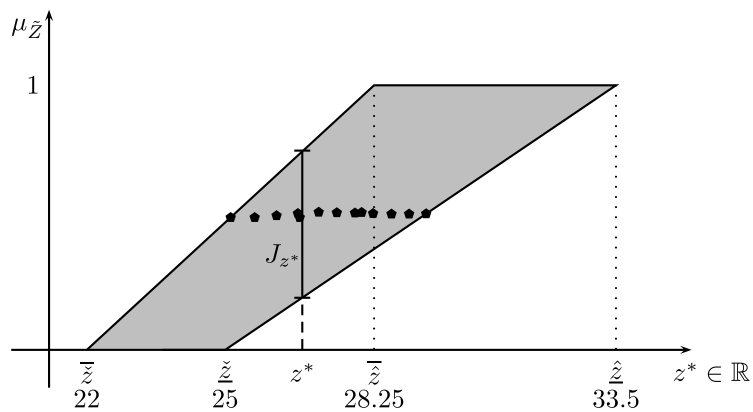

Figure 6.5: Boundaries of $\tilde{z}$

that extreme points ($\max - \min$ and $\min - \max$) do not lead to larger values of α.
At a first glance, $\check{z}, z^*, \hat{z}$ and α^* seem not to be proportional to δ. To do so, we compute the variational rate, namely θ as

$$^{\theta}\alpha_i = \alpha_i - \alpha_{i-1} \ \forall i \in \delta \tag{6.41}$$

$$^{\theta}z_i^* = z_i^* - z_{i-1}^* \ \forall i \in \delta \tag{6.42}$$

$$^{\theta}\check{z}_i = \check{z}_i - \check{z}_{i-1} \ \forall i \in \delta \tag{6.43}$$

$$^{\theta}\hat{z}_i = \hat{z}_i - \hat{z}_{i-1} \ \forall i \in \delta \tag{6.44}$$

The obtained results are shown in Table 6.4
Note that the behavior of α, z^* and $\hat{z}$ has no linear increments, so we can see that even when $\check{z}$ is linearly incremented, the remaining results has no a proportional variation rate. This leads us to think that the soft constraints method has no a linear behavior, even when it is an LP problem itself.

Table 6.4: Variational rate θ for $\alpha, z^*, \check{z}$ and $\hat{z}$.

Value	${}^{\theta}\alpha^*$	${}^{\theta}z^*$	${}^{\theta}\check{z}$	${}^{\theta}\hat{z}$
$\delta = 0.1$	0	0.525	0.3	0.750
$\delta = 0.2$	0.0072	0.474	0.3	0.547
$\delta = 0.3$	0.0078	0.463	0.3	0.511
$\delta = 0.4$	0.0058	0.451	0.3	0.511
$\delta = 0.5$	-0.0019	0.395	0.3	0.511
$\delta = 0.6$	-0.0018	0.395	0.3	0.511
$\delta = 0.7$	-0.0017	0.395	0.3	0.511
$\delta = 0.8$	-0.0016	0.395	0.3	0.511
$\delta = 0.9$	0	0.385	0.3	0.465
$\delta = 1$	0.0011	0.374	0.3	0.425

On the other hand, the interaction between $z(x^*)$ and B is not proportional to δ, so the soft constraints method leads to nonlinear results.

There is no any α less than 0.5, so it seems that α has minimum and maximum boundaries. It is clear that $\alpha \leqslant 0.5$, so the soft constraints method fits the Bellman-Zadeh fuzzy decision making principle through LP methods to achieve nonlinear results.

The shape of $\tilde{b}$ affects the solution and the behavior of α as well. Different configurations of $\mu_{\underline{b}}$ and $\mu_{\bar{b}}$ lead to different values of α^*, so its behavior is a function of the FOU of $\tilde{b}$. The effect of having multiples shapes into the FOU of $\tilde{b}$ is to have nonlinear increments of α, z^*, $\check{z}$ and $\hat{z}$.

Also note that if we compute linear increments on $\mu_{\underline{b}}$ instead of proportional increments, their results should be different from using $\mu_{\bar{b}}$, due to their shapes.

Finally, note that any value $z \in \mathbb{R}$ (except $\bar{\check{z}}$ and $\underline{\hat{z}}$) in the IT2FLP approach has no a single membership value as pointed out before. In this case, every $z \in \mathbb{R}$ has an interval set of membership degrees, namely $u \in J_{z^*} \subseteq [0, 1]$, similarly as shown in Figures 6.5 and 3.1.

Chapter 7

Concluding Remarks

The presented thesis final report shows a mathematical programming model (see model (6.12)) for LP problems whose constraints involve linguistic uncertainty coming from multiple expert perceptions and opinions (see Figueroa & Hernández [1, 2]), and Section 5 presents two methods to solve an IT2FLP problem which are applied to different examples.

An extension of the Zimmermann soft constraints method [43, 44] (equivalent to Verdegay's model [100], see Section 2.5) to an IT2FSs environment is presented and solved using LP optimization methods. Some theoretical and geometrical considerations has been presented in Section 3.1, Definitions 4, 5, and equation (3.4), in order to provide better information to decision makers.

Two application examples (see Chapter 6) have been solved using the proposed methods, a production planning in Section 6.1 and a simulated example in Section 6.2. Some practical considerations about the behavior of the Zimmermann's soft constraints method regarding Type-2 fuzzy constraints are provided in Section 6.3 to improve decision making.

The concept of fuzzy optimal solution regarding IT2FLP has been presented and discussed in Definition 7 (see Chapter 4), and the existence of an optimal solution is proven in Section 3.6. Feasibility condition is defined in Section 3.4, and convexity conditions are defined in Section 3.5, which provide robustness to the proposal. Some considerations about the practical implications of our proposal are glimpsed in Section 4.1.3, and some recommendations regarding its application are provided in Section 6.3.

7.1 Contribution of the Thesis

7.1.1 Publications:

The following publications compose the contribution of this Thesis to the academic community:

[1] Figueroa-García, J.C., Hernández, G.: A method for solving Linear Programming models with Interval Type-2 fuzzy constraints. *Pesquisa Operacional* **34**(1) (2014) 1-17.

[2] Figueroa-García, J.C., Hernández, G.: Linear Programming with Interval Type-2 fuzzy constraints. In: *Constraint Programming and Decision Making - Studies in Computational Intelligence*. Volume 539. Springer Verlag (2014) 19-34.

[8] Figueroa-García, J.C., Hernández, G.: A Multiple Means Transportation Model with Type-2 Fuzzy Uncertainty. In: *Supply Chain Management Under Fuzziness - Studies in Fuzziness and Soft Computing*. Volume 313. Springer Verlag (2014).

[7] Figueroa-García, J.C., Hernández, G.: Behavior of the soft constraints method applied to Interval Type-2 Fuzzy linear programming problems. *Lecture Notes in Computer Science* **7996** (1) (2013) 101-109.

[5] Figueroa-García, J.C., Hernández, G.: Solving linear programming problems with Interval Type-2 fuzzy constraints using interval optimization. In: 32th Annual NAFIPS conference. Volume 32., IFSA-IEEE (2013) 1-6.

[9] Figueroa-García, J.C., Hernández, G.: A note on "Solving Fuzzy Linear Programming problems with Interval Type-2 RHS". In: 32th Annual NAFIPS conference. Volume 32., IFSA-IEEE (2013) 1-6.

[6] Figueroa-García, J.C., Hernández, G.: Computing optimal solutions of a linear programming problem with Interval Type-2 fuzzy constraints. *Lecture Notes in Computer Science* **7208** (2012) 567-576.

[3] Figueroa-García, J.C., Hernández, G.: Interval Type-2 fuzzy linear programming: Uncertain constraints. In: IEEE Symposium Series on Computational Intelligence, IEEE (2011) 1-6.

[4] Figueroa et.al: Multi-period mixed production planning with uncertain demands: Fuzzy and Interval fuzzy sets approach. *Fuzzy Sets and Systems* **206** (2012) 21-38.

7.1.2 Awards

The following awards and grants are also a component of the results of this Thesis:

- IFSA (International Fuzzy Sets Association) *Young Talent* award. Given at FEDCSIS 2012 conference for the paper *"An approximation method for Type Reduction of an Interval Type-2 fuzzy set based on α-cuts"* (see Figueroa-García [101]).
- NAFIPS (North American Fuzzy Information Processing Society) *Best Interval Session Student paper* award. Given at NAFIPS 2013 conference for the paper *"Solving linear programming problems with Interval Type-2 fuzzy constraints using interval optimization"* (see Figueroa-García and Hernández [5]).
- IFORS (Institute for Operational Research) grant to attend ELAVIO (Escuela Latinoamericana de Verano en Investigación de Operaciones). February 5-11, 2012 - Bento Ģonzalves, Brasil.

Appendix A

This appendix shows Tables 1, 2, 3, and 4 which are used in the first application example.

Table 1: Capacities AC_{ik} and available labor hours WC_k.

k	AC_{1k}	AC_{2k}	AC_{3k}	WC_k
1	6105	5280	3960	22200
2	5610	6435	5610	19200
3	4455	5775	5940	16800
4	5362.5	5610	6270	23800

Table 2: Unitary standard production time, ts_{ijk}
$ts_{ijk}\ \forall\ k = 1, 2, 3, 4.$

i, j	ts_{i1k}	ts_{i2k}	ts_{i3k}	ts_{i4k}
ts_{1jk}	0.15	0.25	0.25	0.15
ts_{2jk}	0.25	0.1	0.12	0.09
ts_{3jk}	0.12	0.28	0.21	0.34

Table 3: Production, inventory, backorder costs and soft demands.

Unitary costs.							Soft demands.			
j,k	Cp^r_{jk}	Sp_{jk}	h_{jk}	o_{jk}	$C^{\triangle(+)}_{jk}$	$C^{\nabla(+)}_{jk}$	$\check{d}^{(-)}_{jk}$	$\hat{d}^{(-)}_{jk}$	$\check{d}^{(+)}_{jk}$	$\hat{d}^{(+)}_{jk}$
1,1	225	350	12	156.25	12	14	3700	3900	8775	9249
1,2	165	300	14	168.75	12	14	3200	3400	7650	8128
1,3	160	280	13	150	12	14	2400	2700	6075	6834
1,4	105	220	15	143.75	12	14	3400	3500	7875	8107
2,1	205	320	12	143.75	18	20	2650	2800	6300	6657
2,2	175	300	14	156.25	18	20	3100	3400	7650	8390
2,3	160	270	13	137.5	18	20	1750	1900	4275	4641
2,4	225	330	15	131.25	18	20	2600	2800	6300	6785
3,1	53	190	12	171.25	20	18	1950	2100	4725	5088
3,2	74	220	14	182.5	20	18	3700	3900	8775	9249
3,3	103	250	13	183.75	20	18	2450	2600	5850	6208
3,4	138	320	15	227.5	10	18	4000	4100	9225	9456
4,1	160	350	12	237.5	10	10	2650	2800	6300	6657
4,2	90	270	14	225	12	10	3300	3400	7650	7882
4,3	65	210	13	181.25	12	10	1900	2100	4725	5222
4,4	50	240	15	237.5	12	10	2400	2500	5625	5859

Table 4: Uncertain demands.

j,k	$\underline{\check{d}}^{(-)}_{jk}$	$\bar{\check{d}}^{(-)}_{jk}$	$\underline{\hat{d}}^{(-)}_{jk}$	$\hat{d}^{(-)}_{jk}$	$\underline{\check{d}}^{(+)}_{jk}$	$\bar{\check{d}}^{(+)}_{jk}$	$\underline{\hat{d}}^{(+)}_{jk}$	$\hat{d}^{(+)}_{jk}$
1,1	3700	4070	3900	4290	8661	8981	9039	9479
1,2	3200	3520	3400	3740	7594	7821	8002	8274
1,3	2400	2640	2700	2970	5899	6287	6654	6943
1,4	3400	3740	3500	3850	7805	8083	7909	8340
2,1	2650	2915	2800	3080	6201	6442	6415	6779
2,2	3100	3410	3400	3740	7580	7818	8292	8525
2,3	1750	1925	1900	2090	4077	4501	4510	4785
2,4	2600	2860	2800	3080	6201	6422	6633	6848
3,1	1950	2145	2100	2310	4664	4781	4951	5289
3,2	3700	4070	3900	4290	8714	8994	9026	9372
3,3	2450	2695	2600	2860	5631	5994	6097	6405
3,4	4000	4400	4100	4510	9169	9313	9299	9611
4,1	2650	2915	2800	3080	6187	6464	6478	6829
4,2	3300	3630	3400	3740	7513	7725	7797	8064
4,3	1900	2090	2100	2310	4603	4909	5020	5395
4,4	2400	2640	2500	2750	5516	5770	5784	5989

Self References

[1] Figueroa-García, J.C., Hernández, G.: A method for solving Linear Programming models with Interval Type-2 fuzzy constraints. Pesquisa Operacional **34**(1) (2014) 1–17

[2] Figueroa-García, J.C., Hernández, G.: Linear Programming with Interval Type-2 fuzzy constraints. In: Constraint Programming and Decision Making - Studies in Computational Intelligence. Volume 539. Springer Verlag (2014) 19–34

[3] Figueroa, J.C.: Interval Type-2 fuzzy linear programming: Uncertain constraints. In: IEEE Symposium Series on Computational Intelligence, IEEE (2011) 1–6

[4] Figueroa-García, J.C., Kalenatic, D., Lopez-Bello, C.A.: Multi-period mixed production planning with uncertain demands: Fuzzy and interval fuzzy sets approach. Fuzzy Sets and Systems **206** (2012) 21–38

[5] Figueroa-García, J.C., Hernández, G.: Solving linear programming problems with interval type-2 fuzzy constraints using interval optimization. In: 32th Annual NAFIPS conference. Volume 32., IFSA-IEEE (2013) 1–6

[6] Figueroa, J.C., Hernández, G.: Computing optimal solutions of a linear programming problem with interval type-2 fuzzy constraints. Lecture Notes in Computer Science **7208** (2012) 567–576

[7] Figueroa-García, J.C., Hernández, G.: Behavior of the soft constraints method applied to interval type-2 fuzzy linear programming problems. Lecture Notes in Computer Science **7996**(1) (2013) 101–109

[8] Figueroa-García, J.C., Hernández, G.: A Multiple Means Transportation Model with Type-2 Fuzzy Uncertainty. In: Supply Chain Management Under Fuzziness - Studies in Fuzziness and Soft Computing. Volume 313. Springer Verlag (2014)

[9] Figueroa-García, J.C., Hernández, G.: A note on "solving fuzzy linear programming problems with interval type-2 rhs". In: 32th Annual NAFIPS conference. Volume 32., IFSA-IEEE (2013) 1–6

Bibliography

[10] Dantzig, G.: Linear Programming and Extensions. Princeton University Press (1963)

[11] Klee, V., Minty, G.: How good is the simplex algorithm? In Shisha, O., ed.: Inequalities III, Academic Press (1972) 159–175

[12] Mora, H.M.: Optimización no lineal y dinámica. Universidad Nacional de Colombia (2001)

[13] Bazaraa: Linear Programming and Networks Flow. John Wiley and Sons (1998)

[14] Dantzig, G.B., Thapa, M.N.: Linear Programming: Introduction. Springer-Verlag (1995)

[15] Karmarkar, N.: A new polynomial time algorithm for linear programming. Combinatorica **4**(4) (1984) 373–395

[16] Adler, I., Karmarkar, N., Resende, M.G., Veiga, G.: An implementation of karmarkar's algorithm for linear programming. Mathematical Programming **44**(3) (1989) 297–335

[17] Mehrotra, S.: On the implementation of a primal-dual interior point method. SIAM Journal on Optimization **2**(4) (1992) 575–601

[18] Mehrotra, S., Ye, Y.: Finding an interior point in the optimal face of linear programs. Mathematical Programming **62**(3) (1993) 497–515

[19] S. Mizuno, M.J.T., Ye, Y.: On adaptive-step primal-dual interior-point algorithms for linear programming. Mathematics of Operations Research **18**(4) (1993) 964–981

[20] Zadeh, L.A.: Fuzzy logic = computing with words. IEEE Transactions on Fuzzy Systems **4**(1) (1996) 103–111

[21] Zadeh, L.A.: From computing with numbers to computing with words – from manipulation of measures to manipulation of perceptions. IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications **4**(1) (1999) 105–119

[22] Mendel, J.: Fuzzy sets for words: a new beginning. In: The IEEE International Conference on Fuzzy Systems. (2003) 37–42

[23] Mendel, J.M.: The perceptual computer: an architecture for computing with words. Proceedings of Modeling With Words Workshop – FUZZ-IEEE 2001 (2001)

[24] Mendel, J.M.: An architecture for making judgments using computing with words. Int. J. Appl. Math. Comput. Sci. **12**(3) (2002) 325–335

[25] Mendel, J.M.: Computing with words and its relationships with fuzzistics. Information Sciences **177**(1) (2007) 998–1006

[26] Klir, G.J., Yuan, B.: Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prentice Hall (1995)

[27] Klir, G.J., Folger, T.A.: Fuzzy Sets, Uncertainty and Information. Prentice Hall (1992)

[28] Wang, Z., Klir, G.J.: Generalized Measure Theory. Springer-Verlag (2009)

[29] Mendel, J.: Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions. Prentice Hall (1994)

[30] Mendel, J.M.: Type-2 Fuzzy Sets: Some Questions and Answers. IEEE coNNectionS. A publication of the IEEE Neural Networks Society. (8) (2003) 10–13

[31] Mendel, J.M., John, R.I., Liu, F.: Interval type-2 fuzzy logic systems made simple. IEEE Transactions on Fuzzy Systems **14**(6) (2006) 808–821

[32] Mendel, J.M., John, R.I.: Type-2 fuzzy sets made simple. IEEE Transactions on Fuzzy Systems **10**(2) (2002) 117–127

[33] Liang, Q., Mendel, J.M.: Interval type-2 fuzzy logic systems: Theory and design. IEEE Transactions on Fuzzy Systems **8**(5) (2000) 535–550

[34] Karnik, N.N., Mendel, J.M.: Operations on type-2 fuzzy sets. Fuzzy Sets and Systems **122** (2001) 327–348

[35] Karnik, N.N., Mendel, J.M., Liang, Q.: Type-2 fuzzy logic systems. Fuzzy Sets and Systems **17**(10) (1999) 643–658

[36] Mendel, J.M., Liu, F.: Super-exponential convergence of the Karnik-Mendel algorithms for computing the centroid of an interval type-2 fuzzy set. IEEE Transactions on Fuzzy Systems **15**(2) (2007) 309–320

[37] Melgarejo, M.A.: Implementing Interval Type-2 Fuzzy processors. IEEE Computational Intelligence Magazine **2**(1) (2007) 63–71

[38] Melgarejo, M.A.: A Fast Recursive Method to compute the Generalized Centroid of an Interval Type-2 Fuzzy Set. In: Annual Meeting of the North American Fuzzy Information Processing Society (NAFIPS), IEEE (2007) 190–194

[39] Melgarejo, M., Peńa, C.A., Sanchez, E.: A genetic-fuzzy system approach to control a model of the hiv infection dynamics. In IEEE, ed.: 2006 International Conference on Fuzzy Systems, IEEE (2006) 2323–2330

[40] Lai, Y.J., Hwang, C.: Fuzzy Mathematical Programming. Springer Verlag (1992)

[41] Ross, T.J.: Fuzzy Logic with Engineering Applications. Mc Graw Hill (1995)

[42] Kacprzyk, J., Orlovski, S.A.: Optimization Models using Fuzzy sets and Possibility Theory. Kluwer Academic Press (1987)

[43] Zimmermann, H.J.: Fuzzy programming and Linear Programming with several objective functions. Fuzzy Sets and Systems **1**(1) (1978) 45–55

[44] Zimmermann, H.J., Fullér, R.: Fuzzy Reasoning for solving fuzzy Mathematical Programming Problems. Fuzzy Sets and Systems **60**(1) (1993) 121–133

[45] Tanaka, H., Asai, K.: Fuzzy Solution in Fuzzy Linear Programming Problems. IEEE Transactions on Systems, Man and Cybernetics. **14** (1984) 325–328

[46] Tanaka, H., Asai, K., Okuda, T.: On Fuzzy Mathematical Programming. Journal of Cybernetics. **3** (1974) 37–46

[47] Lodwick, W.A., Jamison, K.D.: Theoretical and semantic distinctions of fuzzy, possibilistic, and mixed fuzzy/possibilistic optimization. Fuzzy Sets and Systems **158**(1) (2007) 1861–1872

[48] Lodwick, W.A., Bachman, K.A.: Solving Large-Scale Fuzzy and Possibilistic Optmization Problems. Fuzzy Optimization and Decision Making **4** (2005) 257–78

[49] Inuiguchi, M., Sakawa, M.: A possibilistic linear program is equivalent to a stochastic linear program in a special case. Fuzzy Sets and Systems **76**(1) (1995) 309–317

[50] Inuiguchi, M., Sakawa, M.: Possible and necessary optimality tests in possibilistic linear programming problems. Fuzzy Sets and Systems **67** (1994) 29–46

[51] Inuiguchi, M., Ramík, J.: Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. Fuzzy Sets and Systems **111** (2000) 3–28

[52] Figueroa, J.C., López, C.A.: Linear programming with fuzzy joint parameters: A cumulative membership function approach. In: 2008 Annual Meeting of the IEEE North American Fuzzy Information Processing Society (NAFIPS). (2008)

[53] Ramík, J.: Optimal solutions in optimization problem with objective function depending on fuzzy parameters. Fuzzy Sets and Systems **158**(17) (2007) 1873–1881

[54] Inuiguchi, M.: Necessity measure optimization in linear programming problems with fuzzy polytopes. Fuzzy Sets and Systems **158**(17) (2007) 1882–1891

[55] Rommelfanger, H.: A general concept for solving linear multicriteria programming problems with crisp, fuzzy or stochastic values. Fuzzy Sets and Systems **158**(17) (2007) 1892–1904

[56] Kurano, M., Yasuda, M., Nakagami, J., Yoshida, Y.: Fuzzy optimality relation for perceptive MDPs: the average case. Fuzzy Sets and Systems **158**(17) (2007) 1905–1912

[57] Entani, T., Tanaka, H.: Interval estimations of global weights in AHP by upper approximation. Fuzzy Sets and Systems **158**(17) (2007) 1913–1921

[58] Ishii, H., Lee, Y.L., Yeh, K.Y.: Fuzzy facility location problem with preference of candidate sites. Fuzzy Sets and Systems **158**(17) (2007) 1922–1930

[59] Carvalho, J.P., Tomé, J.: Qualitative optimization of fuzzy causal rule bases using fuzzy boolean nets. Fuzzy Sets and Systems **158**(17) (2007) 1931–1946

[60] Silva, C., Sousa, J., Runkler, T.: Optimization of logistic systems using fuzzy weighted aggregation. Fuzzy Sets and Systems **158**(17) (2007) 1947–1960

[61] Mahdavi-Amiri, N., Nasseri, S.: Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables. Fuzzy Sets and Systems **158**(17) (2007) 1961–1978

[62] Choo, E.U.: Multicriteria Linear Fractional Programming. Doctoral thesis, University of British Columbia (1980)

[63] Kornbluth, J.S.H., Steuer, R.S.: Multiple objective linear fractional programming. Management Science **27**(1) (1981) 1024–1039

[64] Lee, B.I., Tcha, D.W.: An iterative procedure for fuzzy programming with linear fractional objectives. Computers ind. Engineering **16**(1) (1989) 269–275

[65] Yan-Kuen, W., Sy-Ming, G., Yu-Chih, L.J.: Optimizing the linear fractional programming problem with max-archimedean t-norm fuzzy relational equation constraints. In IEEE, ed.: The 2007 IEEE International Conference on Fuzzy Systems, IEEE (2007) 1–6

[66] Ghodousiana, A., Khorram, E.: Solving a linear programming problem with the convex combination of the max-min and the max-average fuzzy relation equations. Applied Mathematics and Computation **180**(1) (2006) 411–418

[67] Pal, B., Moitra, B., Maulik, U.: A goal programming procedure for fuzzy multiobjective linear fractional programming problem. Fuzzy Sets and Systems. **139**(1) (2003) 395–405

[68] Sakawa, M., Nishizaki, I.: Interactive fuzzy programming for two-level linear fractional programming problems. Fuzzy Sets and Systems. **119**(1) (2001) 31–40

[69] Lertworasirikul, S., Fang, S.C., Nuttle, H.L.W., Joines, J.A.: Fuzzy bcc model for data envelopment analysis. Fuzzy Optimization and Decision Making **2**(4) (Dec 2003) 337–359

[70] Lertworasirikul, S., Fang, S.C., Joines, J.A.: Fuzzy data envelopment analysis (DEA): a possibility approach. Fuzzy Sets and Systems. **139**(1) (2003) 379–394

[71] León, T., Liern, V., Ruiz, J., Sirvent, I.: A fuzzy mathematical programming approach to the assessment of efficiency with dea models. Fuzzy Sets and Systems. **139**(1) (2003) 407–419

[72] Guo, P., Tanaka, H.: Fuzzy DEA: a perceptual evaluation method. Fuzzy Sets and Systems. **119**(1) (2001) 149–160

[73] Saati, S., Memariani, A.: Reducing weight flexibility in fuzzy dea. Applied Mathematics and Computation. **161**(1) (2005) 611–622

[74] Jahanshahloo, G., Soleimani-damaneh, M., Nasrabadi, E.: Measure of efficiency in dea with fuzzy input–output levels: a methodology for assessing, ranking and imposing of weights restrictions. Applied Mathematics and Computation. **156**(1) (2004) 175–187

[75] Angelov, P.P.: Optimization in an intuitionistic fuzzy environment. Fuzzy Sets and Systems **86**(3) (1997) 299–306

[76] Dubey, D., Chandra, S., Mehra, A.: Fuzzy linear programming under interval uncertainty based on IFS representation. Fuzzy Sets and Systems **188**(1) (2012) 68–87

[77] Qin, R., Liu, Y., Liu, Z., Wang, G.: Modeling fuzzy dea with type-2 fuzzy variable coefficients. Lecture Notes in Computer Science **5552**(2) (2009) 25–34

[78] Figueroa, J.C.: Solving fuzzy linear programming problems with interval type-2 RHS. In: 2009 Conference on Systems, Man and Cybernetics, IEEE (2009)

[79] Ramík, J., Řimánek, J.: Inequality relation between fuzzy numbers and its use in fuzzy optimization. Fuzzy Sets and Systems **16** (1985) 123–138

[80] Figueroa-García, J.C., Chalco-Cano, Y., Román-Flores, H.: Distance measures for interval type-2 fuzzy numbers. Discrete Applied Mathematics **To appear**(1) (2015)

[81] Niewiadomski, A.: On Type-2 fuzzy logic and linguistic summarization of databases. Bulletin of the Section of Logic **38**(3) (2009) 215–227

[82] Niewiadomski, A.: Imprecision measures for Type-2 fuzzy sets: Applications to linguistic summarization of databases. Lecture Notes in Computer Science **5097**(1) (2009) 285–294

[83] Cooper, C.A., Seiford, L.M., Tone, K.: Introduction to Data Envelopment Analysis and Its Uses: With DEA-Solver Software and References. Springer (2005)

[84] Kearfott, R.B., Kreinovich, V.: Beyond convex? global optimization is feasible only for convex objective functions: A theorem. Journal of Global Optimization **33**(4) (2005) 617–624

[85] Bellman, R.E., Zadeh, L.A.: Decision-making in a fuzzy environment. Management Science **17**(1) (1970) 141–164

[86] Ramík, J.: Soft computing: overview and recent developments in fuzzy optimization. Technical report, Institute for Research and Applications of Fuzzy Modeling (2001)

[87] Fiedler, M., Nedoma, J., Ramík, J., Rohn, J., Zimmermann, K.: Linear Optimization Problems with Inexact Data. Springer-Verlag (2006)

[88] Figueroa, J.C.: Linear programming with interval type-2 fuzzy right hand side parameters. In: 2008 Annual Meeting of the IEEE North American Fuzzy Information Processing Society (NAFIPS). (2008)

[89] Rommelfanger, H.: Entscheiden bei UnschSrfe - Fuzzy Decision Support-Systeme 2nd ed. Springer-Verlag, Berlin/Heidelberg (1994)

[90] Rommelfanger, H. In: FULPAL - An interactive method for solving multiobjective fuzzy linear programming problems. Reidel, Dordrecht (1990) 279–299

[91] Rommelfanger, H. In: FULP - A PC-supported procedure for solving multicriteria linear programming problems with fuzzy data. Springer-Verlag (1991) 154–167

[92] Peidro, D., Mula, J., Poler, R., Verdegay, J.: Fuzzy optimization for supply chain planning under supply, demand and process uncertainties. Fuzzy Sets and Systems **160** (2009) 2640–2657

[93] Peidro, D., Mula, J., Poler, R.: Supply chain planning under uncertainty: a fuzzy linear programming approach. 2007 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE) **1** (2007) 1–6

[94] Mula, J., Poler, R., García, J.: MRP with flexible constraints: a fuzzy mathematical programming approach. Fuzzy Sets and Systems **157** (2006) 74–97

[95] Lee, Y., Kim, S., Moon, C.: Production-distribution planning in supply chain using a hybrid approach. Production Planning & Control **13** (2002) 35–46

[96] Lee, Y., Kim, S.: Production-distribution planning in supply chain considering capacity constraints. Computers & Industrial Engineering **43** (2002) 169–190

[97] Chanas, S., Delgado, M., Verdegay, J., Vila, M.: Interval and fuzzy extensions of classical transportation problems. Transportation Planning and Technology **17** (1993) 202–218

[98] Ángeles Gil, M., López-Díaz, M., Ralescu, D.A.: Overview on the development of fuzzy random variables. Fuzzy Sets and Systems **157**(16) (2007) 2546–2557

[99] Gen, M., Tsujimura, Y., Ida, K.: Method for solving multiobjective aggregate production planning problem with fuzzy parameters. Computers & Industrial Engineering **23** (1992) 117–120

[100] Delgado, M., Verdegay, J.L., Vila, M.: A general model for fuzzy linear programming. Fuzzy Sets and Systems **29**(1) (1989) 21–29

[101] Figueroa-García, J.C.: An approximation method for type reduction of an interval Type-2 fuzzy set based on α-cuts. In IEEE, ed.: Proceedings of FEDCSIS 2012, IEEE (2012) 1–6