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NON-NORMALITY OF CONTINUED FRACTION PARTIAL QUOTIENTS MODULO q

by

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It is well known that almost all real numbers (in the sense of Lebesgue measure) are normal to base q where $q \geqslant 2$ is any integer base. More precisely, if $a \in \{0,1,2,...,q-1\}$ and $d_n^*(a,q)$ denotes the number of occurences of the digit a amongst the first n digits in the "decimal" expansion base q of a real number x, with $0 < x < 1$, then for almost all $x \in (0,1)$,

$$\lim_{n\to\infty} d_n^*(a,q)/n = q^{-1}.$$

It seems natural to wonder whether this normality property still holds for other representations for real numbers; in particular for the digits occurring in the regular continued fraction expansion of $0 < x < 1$. Recall that each such x has a unique expansion in the form

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where the positive integer digits a_i are known as the partial quotients. Unfortunately, for almost all $x \in (0,1)$ the partial quotients form an unbounded set of integers. It is therefore necessary to define what is meant by normality in such a case. One possible way is to consider the partial quotients modulo q and ask for the asymptotic frequency of occurrences of digits congruent to a mod q, if this exists. These asymptotic frequencies can in fact be deduced rather easily from the following general theorem of Khinchine [1, p.95]:

THEOREM. *Let c, δ represent positive constants and suppose* $0 \leqslant f(r) < cr^{\frac{1}{2}-\delta}$, $r = 1,2,3,\ldots$ *Then for almost all* $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(a_k) = \sum_{r=1}^{\infty} \frac{f(r)\ln\{1+\frac{1}{r(r+2)}\}}{\ln 2},$$

where a_k *denotes the kth partial quotient of* x, *for* $1 \leqslant k \leqslant n$.

As far as we are aware, however, our particular quiestion has not explicitly been considered. Let $d_n(a,q)$ denote the number of occurences amongst the first n continued fraction partial quotients that are congruent to a mod q. For convenience let us denote the case $a = 0$ by $d_n(q,q)$. It turns out, perhaps surprisingly, that the asymptotic frequencies $\lim_{n\to\infty} d_n(a,q)/n$ are not the same for each a and that in fact these frequencies form a strictly decreasing sequence of values as a increases from 1 to q. These values can be expressed exactly in terms of the gamma function as follows:

THEOREM. *Let* $a \in \{1,2,\ldots,q\}$. *Then for almost all* $x \in (0,1)$

$$\lim_{n\to\infty} d_n(a,q)/n = \log_2\left(\frac{\Gamma(\frac{a}{q})\Gamma(\frac{a+2}{q})}{\{\Gamma(\frac{a+1}{q})\}^2}\right).$$

Proof. For fixed $q > 2$ and $a \in \{0,1,2,\ldots,q-1\}$ define

$$\delta_{a,q}(r) = \begin{cases} 1, & r \equiv a \bmod q \\ 0, & \text{otherwise.} \end{cases}$$

Then $\frac{1}{n}\sum_{k=1}^{n} \delta_{a,q}(a_k)$ represents the density of partial quotients which are congruent to a mod$q$ in the first n elements of the continued fraction. Using Khinchine's theorem and our previous notation (replacing the case $a = 0$ by $a = q$)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \delta_{a,q}(a_k) = \lim_{n\to\infty} d_n(a,q)/n = \frac{1}{\ln 2} \sum_{r\equiv a \bmod q} \ln\left(1 + \frac{1}{r(r+2)}\right)$$

$$= \log_2 \prod_{n=0}^{\infty}\left(1 + \frac{1}{(nq+a)(nq+a+2)}\right) = \log_2 \prod_{n=0}^{\infty} \frac{(nq+a+1)^2}{(nq+a)(nq+a+2)}$$

$$= \log_2 \prod_{n=0}^{\infty} \frac{(n+\frac{a+1}{q})^2}{(n+\frac{a}{q})(n+\frac{a+2}{q})}.$$

We can now evaluate the infinite product using Theorem 5 of Rainville [2, p.14]. Then simplifying using the identity $\Gamma(t+1) = t\Gamma(t)$, $t \in \mathbb{R}$, gives the result.

In particular we deduce that the asymptotic frequency of even partial quotients is $\log_2\frac{4}{\pi} = 0.348504\ldots$, and of odd partial quotients is $\log_2\frac{\pi}{2} = 0.651496\ldots$ The tables below gives the asymptotic frequencies for $q = 3$ and $q = 4$.

$a = 1$	0.546954...
$a = 2$	0.274052...
$a = 3$	0.178992...

$q = 3$

$a = 1$	0.5
$a = 2$	0.239203...
$a = 3$	0.151496...
$a = 4$	0.109300...

$q = 4$

It is not hard to show that as $q \to \infty$,

$$\prod_{n=1}^{\infty}\left(1 + \frac{1}{(nq+a)(nq+a+2)}\right) \to 1,$$

and thus for fixed $k < q$, as $q \to \infty$

$$\lim_{n\to\infty} d_n(k,q)/n \to \log_2\left(1+\frac{1}{k(k+2)}\right).$$

Of course $\log_2(1+\frac{1}{k(k+2)})$ is the familiar expression for the asymptotic frequency of the proportion of partial quotient having the value k (see [1]). In particular for every q, $\lim_{n\to\infty} d_n(1,q) > \log_2\frac{4}{3} = 0.415037\ldots$

REFERENCES

[1] A.Ya. Khinchine, *Continued Fractions*. P.Noordhoff Ltd. (1963).

[2] E. Rainville, Special Functions, Chelsea Publ. Co. (1960).

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