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## NON-NORMALITY OF CONTINUED FRACTION PARTIAL QUOTIENTS MODULO *q*

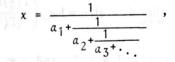
by

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It is well known that almost all real numbers (in the sense of Lebesgue measure) are normal to base q where  $q \ge 2$  is any integer base. More precisely, if  $a \in \{0, 1, 2, \dots, q-1\}$  and  $d_n^*(a,q)$  denotes the number of occurences of the digit a amongst the first n digits in the "decimal" expansion base q of a real number x, with 0 < x < 1, then for almost all  $x \in (0,1)$ ,

 $\lim_{n\to\infty} d_n^*(a,q)/n = q^{-1}.$ 

It seems natural to wonder whether this normality property still holds for other representations for real numbers; in particular for the digits occurring in the regular continnued fraction expansion of 0 < x < 1. Recall that each such x has a unique expansion in the form



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where the positive integer digits  $a_{i}$  are known as the partial quotients. Unfortunately, for almost all  $x \in (0,1)$  the partial quotients form an unbounded set of integers. It is there fore necessary to define what is meant by normality in such a case. One possible way is to consider the partial quotients modulo q and ask for the asymptotic frequency of occurrences of digits congruent to  $a \mod q$ , if this exists. These asymptotic frequencies can in fact be deduced rather easily from the following general theorem of Khinchine [1, p.95]:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(a_{k})=\sum_{r=1}^{\infty}\frac{f(r)\ln\{1+\frac{1}{r(r+2)}\}}{\ln 2},$$

where  $a_k$  denotes the kth partial quotient of x, for  $1 \le k \le n$ .

As far as we are aware, however, our particular quiestion has not explicitly been considered. Let  $d_n(a,q)$  denote the number of occurences amongst the first *n* continued fraction partial quotients that are congruent to a mod q. For convenience let us denote the case a = 0 by  $d_n(q,q)$ . It turns out, perhaps surprisingly, that the asymptotic frequencies  $\lim_{n\to\infty} d_n(a,q)/n$  are not the same for each a and that in fact these frequencies form a strictly decreasing sequence of values as a increases from 1 to q. These values can be expressed exactly in terms of the gamma function as follows:

**THEOREM.** Let  $a \in \{1, 2, \dots, q\}$ . Then for almost all  $x \in (0, 1)$ 

$$\lim_{n\to\infty} d_n(a,q)/n = \log_2\left(\frac{\Gamma(\frac{a}{q})\Gamma(\frac{a+2}{q})}{\{\Gamma(\frac{a+1}{q})\}^2}\right).$$

**Proof.** For fixed q > 2 and  $a \in \{0, 1, 2, \dots, q-1\}$  define

$$\delta_{a,q}(r) = \begin{cases} 1 , r \equiv a \mod q \\ 0 , \text{ otherwise.} \end{cases}$$

Then  $\frac{1}{n}\sum_{k=1}^{n} \delta_{a,q}(a_k)$  represents the density of partial quotients which are congruent to a modg in the first *n* elements of the continued fraction. Using Khinchine's theorem and our previous notation (replacing the case a = 0 by a = q)

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \int_{a,q} (a_k) = \lim_{n \to \infty} d_n(a,q)/n = \frac{1}{\ln 2} \sum_{r \equiv a \mod q} \ln\left(1 + \frac{1}{r(r+2)}\right)$$
$$= \log_2 \prod_{n=0}^{\infty} \left(1 + \frac{1}{(nq+a)(nq+a+2)}\right) = \log_2 \prod_{n=0}^{\infty} \frac{(nq+a+1)^2}{(nq+a)(nq+a+2)}$$
$$= \log_2 \prod_{n=0}^{\infty} \frac{(n + \frac{a+1}{q})^2}{(n + \frac{a}{q})(n + \frac{a+2}{q})}.$$

We can now evaluate the infinite product using Theorem 5 of Rainville [2, p.14]. Then simplifying using the identity  $\Gamma(t+1) = t\Gamma(t), t \in \mathbb{R}$ , gives the result.

In particular we deduce that the asymptotic frequency of even partial quotients is  $\log_2 \frac{4}{\pi} = 0.348504...$ , and of odd partial quotients is  $\log_2 \frac{\pi}{2} = 0.651496...$  The tables below gives the asymptotic frequencies for q = 3 and q = 4.

a	=	1	0.546954				0.5
			0.274052	a	=	2	0.239203
durch given non to a				a	=	3	0.151496
				a	=	4	0.109300

$$q = 3$$

q = 4

It is not hard to show that as  $q \rightarrow \infty$ 

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{(nq+a)(nq+a+2)}\right) \to 1,$$

and thus for fixed k < q, as  $q \rightarrow \infty$ 

 $\lim_{n\to\infty} d_n(k,q)/n \to \log_2(1+\frac{1}{k(k+2)}).$ 

Of course  $\log_2(1 + \frac{1}{k(k+2)})$  is the familiar expression for the asymptotic frequency of the proportion of partial quotient having the value k (see [1]). In particular for every q,  $\lim_{n \to \infty} d_n(1,q) > \log_2 \frac{4}{3} = 0.415037...$ 

## REFERENCES

- A.Ya. Khinchine, Continued Fractions. P.Noordhoff Ltd. (1963).
- [2] E. Rainville, Special Functions, Chelsea Publ. Co. (1960).

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