N-Koszul algebras, Calabi-Yau algebras and skew PBW extensions

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Álgebras N-Koszul, álgebras Calabi-Yau y extensiones PBW torcidas

Abstract: In the schematic approach to non-commutative algebraic geometry arises some important classes of non-commutative algebras like Koszul algebras, Artin-Schelter regular algebras, Calabi-Yau algebras, and closely related with them, the skew PBW extensions. There exist some relations between these algebras and the skew PBW extensions. We give conditions to guarantee that skew PBW extensions over fields are nonhomogeneous Koszul or Koszul algebras. We also show that a constant skew PBW extension of a field is a PBW deformation of its homogeneous version. We define graded skew PBW extensions, study some properties of these algebras and showed that if R is a PBW algebra then a graded skew PBW extension of R is a PBW algebra, and therefore, a Koszul algebra. As a generalization of the above results, we prove that every graded skew PBW extension of a finitely presented Koszul algebra is Koszul. Artin-Schelter regularity and the skew Calabi-Yau condition are studied for graded skew PBW extensions. We prove that every graded quasi-commutative skew PBW extension of an Artin-Schelter regular algebra is an Artin-Schelter regular algebra and, more general, graded skew PBW extensions of a finitely presented Auslander-regular algebra, are Artin-Schelter regular algebras. As a consequence, every graded quasi-commutative skew PBW extension of a finitely presented skew Calabi-Yau algebra is skew Calabi-Yau, and graded skew PBW extensions of a finitely presented Auslander-regular algebra are skew Calabi-Yau. Since graded quasi-commutative skew PBW extensions with coefficients in a finitely presented skew Calabi-Yau algebra are skew Calabi-Yau, the Nakayama automorphism exists for these extensions. With this in mind, we give a description of Nakayama automorphism for these non-commutative algebras using the Nakayama automorphism of the ring of the coefficients.

Resumen: En el enfoque esquemático de la geometría algebraica no conmutativa surgen algunas clases importantes de álgebras no conmutativas como álgebras de Koszul, álgebras Artin-Schelter regulares, álgebras Calabi-Yau y, estrechamente relacionadas con estas, las extensiones PBW torcidas. Existen algunas relaciones entre estas álgebras y las extensiones PBW torcidas. Nosotros damos condiciones para garantizar cuáles extensiones PBW torcidas de un cuerpo son álgebras no homogéneas de Koszul o álgebras de Koszul. También, mostramos que una extensión PBW torcida constante de un cuerpo es una deformación PBW de su versión homogénea. Definimos las extensiones PBW torcidas graduadas, estudiamos algunas propiedades de estas álgebras y mostramos que si R es un álgebra PBW, entonces cada extensión PBW torcida graduada de R es un álgebra PBW, y por lo tanto un álgebra de Koszul. Como una generalización de los resultados anteriores, se demuestra que cada extensión PBW torcida graduada de un álgebra de Koszul finitamente presentada, es un álgebra de Koszul. La regularidad de Artin-Schelter y la condición de Calabi-Yau torcida se estudian para las extensiones PBW torcidas graduadas. Se demuestra que cada extensión PBW torcida cuasi-conmutativa graduada de un álgebra Artin-Schelter regular es un álgebra Artin-Schelter regular, y más general, extensiones PBW torcidas graduadas de un álgebra finitamente presentada Auslander-regular, son álgebras Artin-Schelter regulares. Como consecuencia, cada extensión PBW torcida cuasi-conmutativa graduada de un álgebra Calabi-Yau torcida finitamente presentada, es Calabi-Yau torcida, y las extensiones PBW torcidas graduadas de un álgebra Auslander-regular finitamente presentada son álgebras Calabi-Yau torcidas. Dado que las extensiones PBW torcidas cuasi-conmutativas graduadas con coeficientes en un álgebra Calabi-Yau torcida finitamente presentada, son Calabi-Yau torcidas, existe el automorfismo de Nakayama para estas extensiones. Con esto en mente, damos una descripción del automorphism de Nakayama para estas álgebras no conmutativas, usando el automorphism de Nakayama del anillo de coeficientes.

Keywords: Skew PBW extensions, Koszul algebras, Artin-Schelter regular algebras, Calabi-Yau algebras.

Palabras clave: Extensiones PBW torcidas, álgebras de Koszul, álgebras Artin-Schelter regulares, álgebras Calabi-Yau.

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Dedicated to

My mom, my daughters and my sons

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Introduction

In the schematic approach to non-commutative algebraic geometry arises some important classes of non-commutative algebras like Koszul algebras, Artin-Schelter regular algebras, Calabi-Yau algebras (see for example [3], [4], [5], [14], [26], [38], [76]), and closely related with them, the skew PBW extensions (see [44], [46]). Koszul algebras were introduced by Priddy in [63], later, Berger in [10] introduces a generalization of Koszul algebras, which are then called generalized Koszul algebras or N-Koszul algebras, for $N \geq 2$ (2-Koszul algebras coincide with Koszul algebras). Li in [50] defined the notions of generalized Koszul modules and Koszul algebras in a similar way to the classical case. Phan in [59] and [60] defined Koszul algebras for augmented algebras and R-augmented algebras. Regular algebras were defined by Artin and Schelter in [3] and they are now known in the literature as Artin-Schelter regular algebras. Calabi-Yau algebras were defined by Ginzburg in [26], and as a generalization of them, were defined the skew (also named twisted) Calabi-Yau algebras. These algebras and the relations between them, have been studied recently in several papers (see also [84]):

(i) Berger and Marconnet in Proposition 5.2 of [12] show that if $B = T(V)/\langle R\rangle$ is a connected graded $\mathbb{K}$-algebra ($\mathbb{K}$ a field) such that the space V of generators is concentrated in degree 1, the space R of relations lives in degrees ≥ 2, the global dimension d of B is 2 or 3, and that B is an Artin-Schelter regular algebra (the polynomial growth imposed by Artin and Schelter is often removed and in fact, it is not necessary), then B is N-Koszul if $d = 3$, and Koszul if $d = 2$.

(ii) Berger and Taillefer in Proposition 4.3 of [14] show that if B is a connected $\mathbb{N}$-graded Calabi-Yau algebra then B is an Artin-Schelter regular algebra. In Proposition 5.4 they prove that if B is an Artin-Schelter regular $\mathbb{C}$-algebra of global dimension 3 (with polynomial growth), then B is Calabi-Yau if and only if B is of type $\mathtt{A}$ in the classification of Artin and Schelter given in [3].

(iii) Let $\mathbb{K}$ be a field of characteristic zero, V be an n-dimensional space with $n \geq 1$, w be a non-zero homogeneous potential of V of degree $N+1$ with $N \geq 2$, and $B = B(w)$ be the potential algebra defined by w (so that the space of generators of B is V). Berger and Solotar in Theorem 2.6 of [13] prove that if the space of relations R (i.e. the subspace of $V^{\otimes N}$ generated by the relations $\partial_x(w)$, $x \in X$) of B is n-dimensional, then B is 3-Calabi-Yau if and only if B is N-Koszul of global dimension 3 and $dim R_{N+1} = 1$, where $R_{N+1} = (R \otimes V) \cap (V \otimes R) \subseteq V^{\otimes(N+1)}$.

(iv) Let $\mathbb{K}$ be a field of characteristic zero, let V be an n-dimensional space with

$n \geq 1$, let $X = \{x_1, \ldots, x_n\}$ be a fix basis of V. The subspace of tensor algebra $T(V)$ generated by the commutators is denoted by $[T(V), T(V))]$. The elements of the vector space $Pot(V) = T(V)/[T(V), T(V)]$ are called potentials of V or potentials in the variables $x_1, \ldots, x_n$. For any potential w of V, let $I(\partial_x(w) : x \in X)$ denote the two-sided ideal generated by all the cyclic partial derivatives of w. We say that the associative $\mathbb{K}$-algebra $B = B(w) = T(V)/I(\partial_x(w) : x \in X)$ is derived from the potential w, or that it is the potential algebra defined from w. Bocklandt, Schedler and Wemyss in [16], Theorem 6.8 proved that if B is an algebra defined by a skew potential w and B is N-Koszul, then B is skew d-Calabi-Yau if and only if a certain complex defined from w (a bimodule version of a complex previously considered by Dubois-Violette, [20]) is exact. Berger and Solotar in [13] give a necessary and sufficient condition for that the homogeneous potentials w of degree $N+1$ in n variables is 3-Calabi-Yau, when the algebra B defined by the potential w is N-Koszul of global dimension 3: Let w be a non-zero homogeneous potential of V of degree $N+1$ with $N \geq 2$, let $B = B(w)$ be the potential algebra defined by w. Assume that B is N-Koszul. Then B is 3-Calabi-Yau if and only the Hilbert series of the graded algebra B is given by $h_B(t) = (1-nt+nt^N-t^{N+1})^{-1}$ ([13], Theorem 2.7). As an application, they study skew polynomial algebras over non-commutative quadrics algebras: For any $n \geq 2$, let R be a non-degenerate non-commutative quadric in n variables $x_1, \ldots, x_n$ of degree 1, let z be an extra variable of degree 1, let B be an algebra defined by a non-zero cubic potential w in the variables $x_1, \ldots, x_n, z$. Assume that the graded algebra B is isomorphic to a skew polynomial algebra $R[z; \sigma, \delta]$ over R in the variable z, defined by a 0-degree homogeneous automorphism σ of R and a 1-degree homogeneous σ-derivation δ of R. Then B is Koszul and 3-Calabi-Yau ([13], Proposition 4.1).

(v) Reyes, Rogalski and Zhang in Lemma 1.2 of [75] show that if B is a connected graded algebra, then B is graded skew Calabi-Yau if and only if B is Artin-Schelter regular.

(vi) In [90], Artin-Schelter regular algebras of dimension 5 generated by two generators of degree 1 with three generating relations of degree 4 are classified under some generic condition. There are nine types such Artin-Schelter regular algebras in this classification list. Among them, the algebras $\mathbf{D}$ and $\mathbf{G}$ are given by iterated Ore extensions (see [90], Section 5.2). The algebra $\mathbf{D}$ is skew Calabi-Yau with the Nakayama automorphism ν given by $\nu(x) = p^{-3}q^4x$; $\nu(y) = p^3q^{-4}y$. $\mathbf{D}$ is Calabi-Yau if and only if p, q satisfy the system of equations (see [51], Theorem 4.3):

$$\begin{cases} p^3 = q^4, \\ 2p^4 - p^2q + q^2 = 0. \end{cases}$$

The algebra $\mathbf{G}$ is skew Calabi-Yau with the Nakayama automorphism ν given by $\nu(x) = gx$; $\nu(y) = g^{-1}y$. $\mathbf{D}$ is Calabi-Yau if and only if $g = 1$.

(vii) Zhou and Lu in [94] study and classify Artin-Schelter regular algebras of dimension five with two generators under an additional $\mathbb{Z}^2$-grading by Hilbert driven Gröbner basis computations. All the algebras obtained there, are strongly noetherian, Auslander regular, and Cohen-Macaulay.

(viii) Let $\mathbb{K}$ be a field, let n be an even natural number ≥ 2, and let B be the associative

$\mathbb{K}$-algebra defined by generators $x_1, \ldots, x_n$ subject to the single relation

$$\sum_{1 \leq i \leq \frac{n}{2}} [x_i, x_{i+\frac{n}{2}}] = \nu + \lambda,$$

where the bracket stands for the commutator, ν is a linear combination of the x_i's, and $\lambda \in \mathbb{K}$. Then the filtered algebra B is Koszul. Furthermore B is 2-Calabi-Yau if and only if $\nu = 0$ (see [11], Theorem 6.4).

(ix) For an Artin-Schelter Gorenstein algebra B, the homological determinant, denoted hdet, is a homomorphism from the graded automorphism group $GrAut(B)$ of B to the multiplicative group $\mathbb{K} \setminus \{0\}$, generalizing the usual determinant of a matrix. For the precise definition, we refer to [37]. Let R be a Koszul Artin-Schelter regular algebra of global dimension d, with Nakayama automorphism ν. Let $B = R[x_1, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ be an iterated skew polynomial ring (see [27], page 23-24), with σ_i graded. B is Calabi-Yau if and only if $\sigma_1\sigma_2 \cdots \sigma_n = \nu$ and $(\text{hdet}\sigma_i) = 1$ for all i (see [95], Theorem 4.6).

Other authors have studied some properties of algebras constructed from Koszul Artin-Schelter regular algebras. For example He, Van Oystaeyen and Zhang in [33] show that for a Koszul Artin-Schelter regular algebra R with Nakayama automorphism ν, the Yoneda Ext-algebra of the skew polynomial algebra $R[z; \nu]$ is a trivial extension of a Frobenius algebra. Then they prove that $R[z; \nu]$ is Calabi-Yau and hence each Koszul Artin-Schelter regular algebra is a subalgebra of a Koszul Calabi-Yau algebra. A super potential $\breve{w}$ is also constructed so that the Calabi-Yau algebra $R[z; \nu]$ is isomorphic to the derivation quotient of $\breve{w}$. The Calabi-Yau property of a skew polynomial algebra with coefficients in a PBW-deformation of a Koszul Artin-Schelter regular algebra is also discussed.

Suppose $\sigma : R \to R$ is a graded algebra automorphism and $\delta : R(-1) \to R$ is a graded σ-derivation. If $B := R[z; \sigma, \delta]$ is the associated Ore extension, then B is a skew PBW extension. In this case we have $B = R[z; \sigma, \delta] = \sigma(R)\langle x \rangle$ (see [24], Example 5). Some properties are preserved by Ore extensions, for example:

- If R is a connected graded algebra then B is a connected graded algebra.
- If R is homologically smooth, then so is B (see [51], Proposition 3.1).
- R is Koszul if and only if B is Koszul (see [61], Corollary 1.3).
- Let $R = \mathbb{K}\langle x_1, \ldots, x_n \rangle / \langle f \rangle$ where $f = (x_1, \ldots, x_n)M(x_1, \ldots, x_n)^t$ and M is an $n \times n$ matrix. Then R is Calabi-Yau of dimension 2 if and only if M is invertible and anti-symmetric (see [32], Corollary 1). Let δ be a graded derivation of the free algebra $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ of degree 1. If $\delta(f) = 0$, then δ induces a graded derivation $\overline{\delta}$ on R. Let $B = R[z; \overline{\delta}]$ be the Ore extension of R defined by the graded derivation $\overline{\delta}$. Then B is a graded Calabi-Yau algebra of dimension 3 (see [34], Proposition 1.3).
- If R is a skew Calabi-Yau projective K-algebra of dimension d with Nakayama automorphism ν, then B is skew Calabi-Yau of dimension $d + 1$ and the Nakayama automorphism ν' of B satisfies that $\nu'_{|R} = \sigma^{-1}\nu$ and $\nu'(z) = uz + b$, with $u, b \in R$ and u invertible (see [51], Theorem 3.3).

- Let R be a Koszul Artin-Schelter regular algebra of global dimension d with the Nakayama automorphism ν. Then $B = R[z;\nu]$ is a Calabi-Yau algebra of dimension $d+1$ (see [33], Theorem 3.3).

- Let R be a skew Calabi-Yau algebra of dimension d with Nakayama automorphism ν, then $R[x;\nu]$ and $R[x^{\pm 1};\nu]$ are Calabi-Yau algebras of dimension $d+1$ (see [28], Theorema 1.1 and Remark 5.1). Furthermore, if $R[x;\nu]$ is Calabi-Yau then $R[x^{\pm 1};\nu]$ is Calabi-Yau (see [28], Corollary 5.5).

In the current literature there are not explicit relations between Artin-Schelter regular algebras, N-Koszul algebras or Calabi-Yau algebras with the skew PBW extensions defined in [24], and recently studied in many papers from a homological and constructive approach (see [1], [2], [23], [25], [43], [44], [45], [67], [68], [69], [70], [72], [73], [74], [89]).

In the present monograph we want to analyze the skew PBW extensions from the non-commutative algebraic-geometric point of view induced by Koszulity, Artin-Schelter regularity and the skew Calabi-Yau condition.

In Chapter 1 we give some definitions and elementary properties of skew PBW extensions. We also define some special subclasses of skew PBW extensions and classify most of the known examples according to these subclasses. The new results of this chapter are in Subsection 1.2.2 where the skew PBW extensions are classified in a very important subclasses. In Chapter 2 we present some properties of skew PBW extensions with the filtration and graduation defined in [45]. Later, we define a more general graduation of skew PBW extensions, give some examples and study some general properties of graded skew PBW extensions. The main results of the second chapter are Theorem 2.2.1, Corollary 2.3.14 and Theorem 2.3.22. The graded skew PBW extensions defined here are one of the main tools of the present thesis. In Chapter 3 we study the Koszulity for skew PBW extensions. We study the Koszul property for skew PBW extensions over fields and the Koszul property for graded skew PBW extensions. The new results here are Corollary 3.1.14, Theorem 3.1.15, Example 3.1.17, Example 3.1.18, Proposition 3.1.26, Theorem 3.2.5, Corollary 3.2.7, Example 3.2.8, Remark 3.2.9, and Theorem 3.2.17. In Chapter 4 we investigate the Artin-Schelter regular property and the skew Calabi-Yau property for graded skew PBW extensions, and we present a description of the Nakayama automorphism of graded quasi-commutative skew PBW extensions over finitely presented skew Calabi-Yau algebras, taking into account the Nakayama automorphism of the ring of coefficients. The results in this final chapter are Theorem 4.1.2, Theorem 4.1.3, Theorem 4.2.8, Example 4.2.9, Example 4.2.10, Theorem 4.3.3, Example 4.3.4 and Example 4.3.5.

CHAPTER 1

Skew PBW extensions and preliminaries

In this chapter we recall some definitions and elementary properties of skew PBW extensions; in addition, we will introduce some sub-classes of them: constant, pre-commutative and semi-commutative. Examples of these sub-classes are presented. The new results in this chapter are in Subsection 1.2.2. For more details and to check other recent properties related to skew PBW extensions, see [1], [2], [23], [25], [43], [44], [45], [67], [68], [69], [70], [72], [73], [74], [89].

1.1 Definition and basic properties

Definition 1.1.1. Let R and A be rings. We say that A is a *skew PBW extension of* R if the following conditions hold:

(i) $R \subseteq A$;

(ii) there exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the set of standard monomials

$$\mathrm{Mon}(A) := \{x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1.1.1}$$

(iv) For $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n. \tag{1.1.2}$$

Under these conditions we will write $A = \sigma(R)\langle x_1, \dots, x_n \rangle$.

Remark 1.1.2. Skew PBW extensions are a generalization of PBW extensions. *PBW* extensions were defined by Bell and Goodearl in [9]. Let R and A be rings. It is said that A is a *Poincaré-Birkhoff-Witt extension* of R, noted PBW, if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left R-free module with basis

$$Mon(A) := Mon\{x_1, \dots, x_n\} := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

In this case it is also said that A *is a ring of a left polynomial type over* R with respect to $\{x_1, \dots, x_n\}$ and $Mon(A)$ is the set of standard monomials of A. Moreover, $x_1^0 \cdots x_n^0 := 1 \in Mon(A)$.

(iii) $x_i r - r x_i \in R$, for each $r \in R$ and $1 \leq i \leq n$.

(iv) $x_i x_j - x_j x_i \in R + Rx_1 + \cdots + Rx_n$, for any $1 \leq i, j \leq n$.

Remark 1.1.3 ([24], Remark 2)**.** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension.

(i) Since Mon(A) is a left R-basis of A, the elements $c_{i,r}$ and $c_{i,j}$ in Definition 1.1.1 are unique.

(ii) If $r = 0$, then $c_{i,0} = 0$. In Definition 1.1.1 (iv), $c_{i,i} = 1$. This follows from $x_i^2 - c_{i,i}x_i^2 = s_0 + s_1x_1 + \cdots + s_nx_n$, with $s_j \in R$, which implies $1 - c_{i,i} = 0 = s_j$, for $0 \leq j \leq n$.

(iii) Let $i < j$. By (1.1.2) there exist elements $c_{j,i}, c_{i,j} \in R$ such that $x_ix_j - c_{j,i}x_jx_i \in R + Rx_1 + \cdots + Rx_n$ and $x_jx_i - c_{i,j}x_ix_j \in R + Rx_1 + \cdots + Rx_n$, and hence $1 = c_{j,i}c_{i,j}$, that is, for each $1 \leq i < j \leq n$, $c_{i,j}$ has a left inverse and $c_{j,i}$ has a right inverse. In general, the elements $c_{i,j}$ are not two-sided invertible. For instance, $x_1x_2 = c_{2,1}x_2x_1 + p = c_{21}(c_{1,2}x_1x_2 + q) + p$, where $p, q \in R + Rx_1 + \cdots + Rx_n$, so $1 = c_{2,1}c_{1,2}$, since x_1x_2 is a basic element of Mon(A). Now, $x_2x_1 = c_{1,2}x_1x_2 + q = c_{1,2}(c_{2,1}x_2x_1 + p) + q$, but we cannot conclude that $c_{1,2}c_{2,1} = 1$ because x_2x_1 is not a basic element of Mon(A).

(iv) Each element $f \in A \setminus \{0\}$ has a unique representation as $f = c_1X_1 + \cdots + c_tX_t$, with $c_i \in R \setminus \{0\}$ and $X_i \in$ Mon(A) for $1 \leq i \leq t$.

If B is a ring and σ is a ring endomorphism $\sigma : B \to B$, a σ-derivation $\delta : B \to B$ satisfies by definition $\delta(r + s) = \delta(r) + \delta(s)$, and $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in B$.

Proposition 1.1.4 ([24], Proposition 3)**.** *Let A be a skew PBW extension of R. For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r)x_i + \delta_i(r), \qquad r \in R. \tag{1.1.3}$$

Proof. For every $1 \leq i \leq n$ and each $r \in R$ we have elements $c_{i,r}, r_i \in R$ such that $x_ir = c_{i,r}x_i + r_i$; since $Mon(A)$ is an R-basis of A, $c_{i,r}$ and r_i are unique for r, so we define $\sigma_i, \delta_i : R \to R$ by $\sigma_i(r) := c_{i,r}$, $\delta_i(r) := r_i$. We can check that σ_i is a ring endomorphism and δ_i is a σ_i-derivation of R, i.e., $\delta_i(r+r') = \delta_i(r)+\delta_i(r')$ and $\delta_i(rr') = \sigma_i(r)\delta_i(r')+\delta_i(r)r'$, for any $r, r' \in R$. Moreover, by the Definition 1.1.1-(iii), $c_{i,r} \neq 0$ for $r \neq 0$. This means that σ_i is injective. □

The notation $\sigma(R)\langle x_1, \dots, x_n\rangle$ and the name of the skew PBW extensions are due to Proposition 1.1.4.

Definition 1.1.5. Let A be a skew PBW extension of R, $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$, where σ_i and δ_i $(1 \leq i \leq n)$ are as in Proposition 1.1.4.

(a) A is called *pre-commutative* if the conditions (iv) in Definition 1.1.1 are replaced by: For any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in Rx_1 + \cdots + Rx_n. \tag{1.1.4}$$

(b) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 1.1.1 are replaced by

(iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r = c_{i,r} x_i; \tag{1.1.5}$$

(iv') for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \tag{1.1.6}$$

(c) A is called *bijective*, if σ_i is bijective for each $\sigma_i \in \Sigma$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

(d) If $\sigma_i = \mathrm{id}_R$ for every $\sigma_i \in \Sigma$, we say that A is a skew PBW extension of *derivation type*.

(e) If $\delta_i = 0$ for every $\delta_i \in \Delta$, we say that A is a skew PBW extension of *endomorphism type*.

(f) Any element r of R such that $\sigma_i(r) = r$ and $\delta_i(r) = 0$ for all $1 \leq i \leq n$, will be called a *constant*. A is called *constant* if every element of R is constant.

(g) A is called *semi-commutative* if A is quasi-commutative and constant.

Definition 1.1.6 ([24], Definition 6)**.** Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 1.1.4.

(i) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) For $X = x^\alpha \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$. The symbol $\succeq$ will denote a total order defined on $\mathrm{Mon}(A)$. For an element $x^\alpha \in \mathrm{Mon}(A)$, $\exp(x^\alpha) := \alpha$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. If $f = c_1 X_1 + \cdots + c_t X_t \in A$, $c_i \in R \setminus \{0\}$, with $X_1 \succ \cdots \succ X_t$, then $\mathrm{lm(f)} := X_1$ is the *leading monomial* of f, $\mathrm{lc}(f) := c_1$ is the *leading coefficient* of f, $\mathrm{lt}(f) := c_1 X_1$ is the *leading term* of f, $\exp(f) := \exp(X_1)$ is the *order* of f, and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. For a detailed description of monomial orders in skew PBW extensions, see [24], Section 3.

(iii) If f is an element as in Remark 1.1.3 (iv), then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.

Skew PBW extensions can be characterized as the following proposition shows.

Proposition 1.1.7 ([24], Theorem 7)**.** *Let A and R be rings that satisfy the conditions* (i) *and* (ii) *of Definition 1.1.1. A is a skew PBW extension of R if and only if the following conditions are satisfied:*

(a) *for each $x^\alpha \in \mathrm{Mon}(A)$ and all $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$ such that*
$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}, \tag{1.1.7}$$
where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.

(b) *For each $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that*
$$x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{1.1.8}$$
where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

Proof. $\Rightarrow$) We divide the proof of (a) in three steps.

Step 1. For each $1 \leq i \leq n$, $0 \neq r \in R$ and for every $k \in \mathbb{N}$,
$$x_i^k r = r_k x_i^k + p_{k,r},$$
where $r_k := \sigma_i^k(r) \in R - \{0\}$, $p_{k,r} \in A$ and $p_{k,r} = 0$ or $\deg(p_{k,r}) < k$. Moreover, if r is left invertible, then r_k is left invertible. In fact, we will prove this by induction on k: for $k = 0$ we have $r_0 = \sigma_i^0(r) = r$ and $p_{0,r} = 0$; for $k = 1$ we have $x_i r = \sigma_i(r)x_i + \delta_i(r)$, so $r_1 := \sigma_i(r) \neq 0$ and $p_{1,r} = \delta_i(r)$, with $\delta_i(r) = 0$ or $\deg(\delta_i(r)) = 0 < 1$ (if r is left invertible, then $\sigma_i(r)$ is left invertible). By induction we have $x_i^{k+1} r = x_i x_i^k r = x_i(r_k x_i^k + p_{k,r})$, where $r_k = \sigma_i^k(r) \in R - \{0\}$, $p_{k,r} \in A$, $p_{k,r} = 0$ or $\deg(p_{k,r}) < k$ (if r is left invertible, then r_k is left invertible). So, $x_i^{k+1} r = (x_i r_k)x_i^k + x_i p_{k,r} = (\sigma_i(r_k)x_i + \delta_i(r_k))x_i^k + x_i p_{k,r}$ $= \sigma_i(r_k)x_i^{k+1} + \delta_i(r_k)x_i^k + x_i p_{k,r}$. Note that $r_{k+1} := \sigma_i(r_k) = \sigma_i(\sigma_i^k(r)) = \sigma_i^{k+1}(r) \neq 0$ since $r_k \neq 0$ and σ_i is injective; moreover, $p_{k+1,r} := \delta_i(r_k)x_i^k + x_i p_{k,r} = 0$ or $\deg(p_{k+1,r}) < k+1$ since $p_{k,r} = 0$ or $\deg(x_i p_{k,r}) \leq k < k+1$ (if r_k is left invertible, then $\sigma_i(r_k)$ is left invertible).

Step 2. We complete the proof by induction on the number of variables involved in x^α. For one variable only, the proof is the content of the step 1. Then, $x^\alpha r = x_1^{\alpha_1} \cdots x_n^{\alpha_n} r = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}(x_n^{\alpha_n} r) = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}(r_{\alpha_n} x_n^{\alpha_n} + p_{\alpha_n,r})$, with $r_{\alpha_n} = \sigma_n^{\alpha_n}(r) \neq 0$ and $p_{\alpha_n,r} \in A$, $p_{\alpha_n,r} = 0$ or $\deg(p_{\alpha_n,r}) < \alpha_n$ (if r is left invertible, then r_{α_n} is left invertible). So by induction
$$\begin{aligned} x^\alpha r &= (x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} r_{\alpha_n})x_n^{\alpha_n} + x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} p_{\alpha_n,r} \\ &= (r_\alpha x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} + q_{\alpha,r_{\alpha_n}})x_n^{\alpha_n} + x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} p_{\alpha_n,r} \\ &= r_\alpha x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} + p_{\alpha,r} \\ &= r_\alpha x^\alpha + p_{\alpha,r}, \end{aligned}$$
where $r_\alpha = \sigma_1^{\alpha_1} \cdots \sigma_{n-1}^{\alpha_{n-1}}(r_{\alpha_n}) = (\sigma_1^{\alpha_1} \cdots \sigma_{n-1}^{\alpha_{n-1}} \sigma_n^{\alpha_n})(r) = \sigma^\alpha(r) \neq 0$ (by induction and since $r_{\alpha_n} \neq 0$), $q_{\alpha,r_{\alpha_n}} \in A$ and $p_{\alpha,r} := q_{\alpha,r_{\alpha_n}} x_n^{\alpha_n} + x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} p_{\alpha_n,r} \in A$ (if r_{α_n} is left invertible, then r_α is left invertible); note that $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha| = \alpha_1 + \cdots + \alpha_n$ since $p_{\alpha_n,r} = 0$ or $\deg(p_{\alpha_n,r}) < \alpha_n$, and, $q_{\alpha,r_{\alpha_n}} = 0$ or $\deg(q_{\alpha,r_{\alpha_n}}) < \alpha_1 + \cdots + \alpha_{n-1}$.

Step 3. Since $Mon(A)$ is R-basis for A, then r_α and $p_{\alpha,r}$ are unique.

Now we will consider the proof of (b). We divide the proof also in three steps.

Step 3.1. We will prove first that for $i < j$ and $k, m \geq 0$

$$x_j^k x_i^m = c_{k,m} x_i^m x_j^k + p_{k,m},$$

with $c_{k,m} \in R$ left invertible, $p_{k,m} \in A$, $p_{k,m} = 0$ or $\deg(p_{k,m}) < k + m$. For this, we will use double induction, on k and on m. For $k = 0$ we have $c_{0,m} := 1$, $p_{0,m} := 0$.

$k = 1$: For $i < j$ and $m \geq 0$ we will prove by induction on m that

$$x_j x_i^m = c_{1,m} x_i^m x_j + p_{1,m},$$

with $c_{1,m} \in R$ left invertible, $p_{1,m} \in A$, $p_{1,m} = 0$ or $\deg(p_{1,m}) < 1+m$. For $m = 0$, $c_{1,0} = 1$, $p_{1,0} = 0$. Let $m = 1$, then $x_j x_i = c_{i,j} x_i x_j + p_{1,1}$, with $c_{i,j} \in R$ left invertible (see Definition 1.1.3-(iv)), $p_{1,1} \in A$, $p_{1,1} = 0$ or $\deg(p_{1,1}) \leq 1 < 1+1$. Now we use the induction hypothesis, so $x_j x_i^{m+1} = x_j x_i^m x_i = (c_{1,m} x_i^m x_j + p_{1,m}) x_i$, with $c_{1,m} \in R$ left invertible, $p_{1,m} \in A$, $p_{1,m} = 0$ or $\deg(p_{1,m}) < 1 + m$. Then, $x_j x_i^{m+1} = c_{1,m} x_i^m x_j x_i + p_{1,m} x_i = c_{1,m} x_i^m (c_{i,j} x_i x_j + p_{1,1}) + p_{1,m} x_i = c_{1,m} x_i^m c_{i,j} x_i x_j + c_{1,m} x_i^m p_{1,1} + p_{1,m} x_i = c_{1,m}(r_m x_i^m + p_{m,c_{i,j}}) x_i x_j + c_{1,m} x_i^m p_{1,1} + p_{1,m} x_i$, where $r_m \in R$ is left invertible since $c_{i,j}$ is left invertible (part (a)); moreover $p_{m,c_{i,j}} \in A$, $p_{m,c_{i,j}} = 0$ or $\deg(p_{m,c_{i,j}}) < m$. Hence, $x_j x_i^{m+1} = c_{1,m+1} x_i^{m+1} x_j + p_{1,m+1}$, with $c_{1,m+1} := c_{1,m} r_m \in R$ left invertible, $p_{1,m+1} := c_{1,m} p_{m,c_{i,j}} x_i x_j + c_{1,m} x_i^m p_{1,1} + p_{1,m} x_i \in A$, $p_{1,m+1} = 0$ or $\deg(p_{1,m+1}) \leq m + 1 < m + 2$. This completes the proof for $k = 1$.

$k+1$: $x_j^{k+1} x_i^m = x_j x_j^k x_i^m = x_j(c_{k,m} x_i^m x_j^k + p_{k,m})$, with $c_{k,m} \in R$ left invertible, $p_{k,m} \in A$, $p_{k,m} = 0$ or $\deg(p_{k,m}) < k+m$. Thus, $x_j^{k+1} x_i^m = (x_j c_{k,m}) x_i^m x_j^k + x_j p_{k,m} = (r_1 x_j + p_{1,c_{k,m}}) x_i^m x_j^k + x_j p_{k,m}$, with $r_1 \in R$ left invertible, $p_{1,c_{k,m}} = 0$ or $\deg(p_{1,c_{k,m}}) < 1$; then, $x_j^{k+1} x_i^m = r_1 x_j x_i^m x_j^k + p_{1,c_{k,m}} x_i^m x_j^k + x_j p_{k,m} = r_1(c_{1,m} x_i^m x_j + p_{1,m}) x_j^k + p_{1,c_{k,m}} x_i^m x_j^k + x_i p_{k,m}$, by induction $c_{1,m} \in R$ is left invertible, $p_{1,m} \in A$, $p_{1,m} = 0$ or $\deg(p_{1,m}) < 1 + m$, hence $x_j^{k+1} x_i^m = c_{k+1,m} x_i^m x_j^{k+1} + p_{k+1,m}$, with $c_{k+1,m} := r_1 c_{1,m} \in R$ left invertible, $p_{k+1,m} := r_1 p_{1,m} x_j^k + p_{1,c_{k,m}} x_i^m x_j^k + x_j p_{k,m} \in A$, $p_{k+1,m} = 0$ or $\deg(p_{k+1,m}) \leq k + m < k + 1 + m$. This complete the step 1.

Step 3.2. The proof is done by induction on the number of variables involved in x^α or x^β. If x^α and x^β include only one variable, then we apply the step 1. So by induction we assume that (1.1.8) is true when the number of variables of x^α or x^β is $\leq n - 1$.

Then,

$$\begin{aligned}
x^\alpha x^\beta =& x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_1^{\beta_1} \cdots x_n^{\beta_n} = (x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_1^{\beta_1}) x_2^{\beta_2} \cdots x_n^{\beta_n} \\
=& (c_1 x_1^{\alpha_1+\beta_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} + p_1)(x_2^{\beta_2} \cdots x_n^{\beta_n}), c_1 \in R \text{ left invertible}, \\
& p_1 \in A, p_1 = 0 \text{ or } \deg(p_1) < \alpha_1 + \cdots + \alpha_n + \beta_1 \leq |\alpha + \beta| \\
=& c_1(x_1^{\alpha_1+\beta_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n})(x_2^{\beta_2} \cdots x_n^{\beta_n}) + p_1 x_2^{\beta_2} \cdots x_n^{\beta_n} \\
=& c_1(c_2 x_1^{\alpha_1+\beta_1} x_2^{\alpha_2+\beta_2} \cdots x_n^{\alpha_n+\beta_n} + p_2) + p_1 x_2^{\beta_2} \cdots x_n^{\beta_n}, \\
& c_2 \in R \text{ left invertible}, p_2 \in A, p_2 = 0 \text{ or} \\
& \deg(p_2) < \alpha_1 + \beta_1 + \alpha_2 + \cdots + \alpha_n + \beta_2 + \cdots + \beta_n = |\alpha + \beta| \\
=& c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta},
\end{aligned}$$

with $c_{\alpha,\beta} := c_1c_2 \in R$, left invertible, $p_{\alpha,\beta} := c_1p_2 + p_1x_2^{\beta_2}\cdots x_n^{\beta_n} \in A$, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha+\beta|$.

Step 3.3. Since $Mon(A)$ is R-basis for A, then $c_{\alpha,r}$ and $p_{\alpha,r}$ are unique.

$\Leftarrow$) The condition (iii) of the Definition 1.1.1 is a particular case of (1.1.7), and the condition (iv) is a particular case of (1.1.8), thus (a) and (b) implies that A is a skew BBW extension.

□

Remark 1.1.8 ([24], Remark 8)**.** (i) A left inverse of $c_{\alpha,\beta}$ will be denoted by $c'_{\alpha,\beta}$. We observe that if $\alpha = 0$ or $\beta = 0$, then $c_{\alpha,\beta} = 1$ and hence $c'_{\alpha,\beta} = 1$.

(ii) Let $\theta, \gamma, \beta \in \mathbb{N}^n$ and $c \in R$, then we have the following identities:

$$\sigma^\theta(c_{\gamma,\beta})c_{\theta,\gamma+\beta} = c_{\theta,\gamma}c_{\theta+\gamma,\beta},$$
$$\sigma^\theta(\sigma^\gamma(c))c_{\theta,\gamma} = c_{\theta,\gamma}\sigma^{\theta+\gamma}(c).$$

In fact, since $x^\theta(x^\gamma x^\beta) = (x^\theta x^\gamma)x^\beta$, then

$$x^\theta(c_{\gamma,\beta}x^{\gamma+\beta} + p_{\gamma,\beta}) = (c_{\theta,\gamma}x^{\theta+\gamma} + p_{\theta,\gamma})x^\beta,$$
$$\sigma^\theta(c_{\gamma,\beta})c_{\theta,\gamma+\beta}x^{\theta+\gamma+\beta} + p = c_{\theta,\gamma}c_{\theta+\gamma,\beta}x^{\theta+\gamma+\beta} + q,$$

with $p = 0$ or $\deg(p) < |\theta+\gamma+\beta|$, and, $q = 0$ or $\deg(q) < |\theta+\gamma+\beta|$. From this we get the first identity. For the second, $x^\theta(x^\gamma c) = (x^\theta x^\gamma)c$, and hence

$$x^\theta(\sigma^\gamma(c)x^\gamma + p_{\gamma,c}) = (c_{\theta,\gamma}x^{\theta+\gamma} + p_{\theta,\gamma})c,$$
$$\sigma^\theta(\sigma^\gamma(c))c_{\theta,\gamma}x^{\theta+\gamma} + p = c_{\theta,\gamma}\sigma^{\theta+\gamma}(c)x^{\theta+\gamma} + q,$$

with $p = 0$ or $\deg(p) < |\theta+\gamma|$, and, $q = 0$ or $\deg(q) < |\theta+\gamma|$. This proves the second identity.

(iii) We observe if A is quasi-commutative, then from the proof of Proposition 1.1.7 we conclude that $p_{\alpha,r} = 0$ and $p_{\alpha,\beta} = 0$ for every $0 \neq r \in R$ and every $\alpha, \beta \in \mathbb{N}^n$. On the other hand, we note that the evaluation function at 0, i.e., $A \to R$, $f \in A \mapsto f(0) \in R$, is a ring surjective homomorphism with kernel $\langle x_1, \ldots, x_n\rangle$ the two-sided ideal generated by $x_1, \ldots, x_n$. Thus, $A/\langle x_1, \ldots, x_n\rangle \cong R$.

(iv) From the proof of Proposition 1.1.7 we also get that if A is bijective, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}^n$.

A natural and useful result that we will use later is the following property.

Proposition 1.1.9 ([45], Proposition 1.7)**.** *Let A be a bijective skew PBW extension of a ring R. Then, A_R is free with basis $Mon(A)$.*

Proof. First, note that A_R is a module where the product $f \cdot r$ is defined by the multiplication in A: $f \cdot r := fr$, $f \in A$, $r \in R$. We prove next that $Mon(A)$ is a system of generators of A. Let $f \in A$, then f is a finite sum of terms like rx^α, with $r \in R$ and $x^\alpha \in Mon(A)$, so it is enough to prove that each of these terms is a right linear R-combination of elements of $Mon(A)$. From Proposition 1.1.7, $rx^{\alpha} = x^{\alpha}\sigma^{-\alpha}(r) - p_{\alpha,\sigma^{-\alpha}(r)}$, with $\deg(p_{\alpha,\sigma^{-\alpha}(r)}) < |\alpha|$ if $p_{\alpha,\sigma^{-\alpha}(r)} \neq 0$, so by induction on $|\alpha|$ we get the result.

Now we will show that $Mon(A)$ is linearly independent: let $x^{\alpha_1}r_1 + \cdots x^{\alpha_t}r_t = 0$, with $x^{\alpha_1} \succ \cdots \succ x^{\alpha_t}$ for the total order $\succeq$ on $Mon(A)$ defined in the previous remark, then $\sigma^{\alpha_1}(r_1)x^{\alpha_1} + p_{\alpha_1,r_1} + \cdots + \sigma^{\alpha_t}(r_t)x^{\alpha_t} + p_{\alpha_t,r_t} = 0$, with $\deg(p_{\alpha_i,r_i}) < |\alpha_i|$ if $p_{\alpha_i,r_i} \neq 0$, $1 \leq i \leq t$; hence, $\sigma^{\alpha_1}(r_1) = 0$ and from this $r_1 = 0$. By induction on t we obtain the result. □

1.2 Examples and classification

From now on, and if not stated otherwise, we fix the following notation: $\mathbb{K}$ is a field, K is a commutative ring with unity, all algebras are $\mathbb{K}$-algebras, vector spaces are $\mathbb{K}$-vector spaces and $\otimes$ is $\otimes_{\mathbb{K}}$. In [24], [45] and [65] are presented a considerable number of examples of quasi-commutative, bijective, or skew PBW extensions of derivation type and endomorphism type. These examples include algebras of interest for modern mathematical physicists such group rings of polycyclic-by-finite groups, Ore algebras, operator algebras, diffusion algebras, some quantum groups, quadratic algebras in three variables, and 3-dimensional skew polynomial rings. In this section we present some of these examples and others that are important in the remainder of this paper. We also classify these examples according to Definition 1.1.5. The examples of this section and their classification can also be found in [88].

1.2.1 General examples

In [45] are presented some particular examples of skew PBW extensions. These examples are classified as: PBW extensions, Ore extensions of bijective type, operator algebras, diffusion algebras, quantum algebras, quadratic algebras in three variables, etc. Here we present a list of those examples and some other examples, including a brief description of them.

Example 1.2.1. *Classical polynomial ring.* $A = R[t_1, \ldots, t_n]$ is the classical polynomial ring , so $t_ir - rt_i = 0$ and $t_it_j - t_jt_i = 0$, for any $r \in R$ and $1 \leq i, j \leq n$. The R-free basis is Mon(A).

Example 1.2.2. *Ore extensions of bijective type.* The ring $A = R[x; \sigma, \delta]$ is the noncommutative polynomial ring with product defined by $xr = \sigma(r)x + \delta(r)$, where $\sigma : R \to R$ is an injective endomorphism of R and δ is a σ-derivation of R, i.e., $\delta(r + r') = \delta(r) + \delta(r')$ and $\delta(rr') = \sigma(r)\delta(r') + \delta(r)r'$, for any r, $r' \in R$. The R-free basis is $\{x^l | l \geq 0\}$. We have $R[x; \sigma, \delta] \cong \sigma(R)\langle x\rangle$. More generally we can consider the *iterated Ore extension* $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ where σ_i, δ_i are defined on $R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}]$, i.e.,

$$\sigma_i, \delta_i : R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}] \to R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}].$$

The iterated Ore extension is of *bijective type* if the following conditions hold:

1. for $1 \leq i \leq n$, σ_i is bijective;

2. for every $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r)$, $\delta_i(r) \in R$;

3. for $i < j$, $\sigma_j(x_i) = cx_i + d$, with $c, d \in R$ and c has a left inverse;

4. for $i < j$, $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_n$,

then $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ is a bijective skew PBW extension. Under these conditions we have $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n] \cong \sigma(R)\langle x_1,\ldots,x_n\rangle$. A particular case are the *iterated skew polynomial ring* (see [27], page 23-24), i.e., the iterated Ore extension $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ such that the following conditions hold: $\sigma_i(x_j) = x_j$, $j < i$; $\delta_i(x_j) = 0$, $j < i$; $\sigma_i\sigma_j = \sigma_j\sigma_i$, $1 \leq i,j \leq n$; $\delta_i\delta_j = \delta_j\delta_i$, $1 \leq i,j \leq n$; $\sigma_i\delta_j = \delta_j\sigma_i$, $1 \leq i \neq j \leq n$; for $1 \leq i \leq n$, σ_i is bijective. From these rules we get that $x_ix_j = x_jx_i, 1 \leq i,j \leq n$, $\sigma_i(R) \subseteq R$ and $\delta_i(R) \subseteq R$, $1 \leq i \leq n$.

Example 1.2.3. *Weyl algebra and extended Weyl algebra.* The Weyl algebra $A_n(\mathbb{K}) = \mathbb{K}[t_1,\ldots,t_n][x_1,\partial/\partial t_1]\cdots[x_n,\partial/\partial t_n]$ is an Ore extension. Note that, $x_ip = px_i + \partial p/\partial t_i$, $x_ix_j - x_jx_i = 0$, for any $p \in \mathbb{K}[t_1,\ldots,t_n]$ and $1 \leq i,j \leq n$. So, $A_n(\mathbb{K}) \cong \sigma(\mathbb{K}[t_1,\ldots,t_n])\langle x_1\ldots,x_n\rangle$. Let $\mathbb{K}(t_1,\ldots,t_n)$ the field of fractions of $\mathbb{K}[t_1,\ldots,t_n]$, then extended Weyl algebra $B_n(\mathbb{K}) = \mathbb{K}(t_1,\ldots,t_n)[x_1,\partial/\partial t_1]\cdots[x_n,\partial/\partial t_n]$ is also a skew PBW extension.

Example 1.2.4. *Jordan plane.* The Jordan plane A is the free algebra generated by x, y and relation $yx = xy + x^2$, so $A = \mathbb{K}\langle x,y\rangle/\langle yx - xy - x^2\rangle \cong \sigma(\mathbb{K}[x])\langle y\rangle$.

Example 1.2.5. *Particular Sklyanin algebra.* The Sklyanin algebra (Example 1.14, [76]) is the algebra $S = \mathbb{K}\langle x,y,z\rangle/\langle ayx + bxy + cz^2, axz + bzx + cy^2, azy + byz + cx^2\rangle$, where $a,b,c \in \mathbb{K}$. If $c = 0$ and $a,b \neq 0$ then in S: $yx = -\frac{b}{a}xy$; $zx = -\frac{a}{b}xz$ and $zy = -\frac{b}{a}yz$, therefore $S \cong \sigma(\mathbb{K})\langle x,y,z\rangle$ is a skew PBW extension of $\mathbb{K}$, and we call this algebra a *particular Sklyanin algebra*.

Example 1.2.6. *Universal enveloping algebra of a Lie algebra.* Let $\mathcal{G}$ be a finite dimensional Lie algebra over K with basis $\{x_1,\ldots,x_n\}$; the universal enveloping algebra of $\mathcal{G}$, $\mathcal{U}(\mathcal{G})$, is a skew PBW extension of K. In this case, $x_ir - rx_i = 0$ and $x_ix_j - x_jx_i = [x_i,x_j] \in \mathcal{G} = K + Kx_1 + \cdots + Kx_n$, for any $r \in K$ and $1 \leq i,j \leq n$.

Example 1.2.7. *Tensor product.* Let $\mathcal{G}$ be a finite dimensional Lie algebra over K with basis $\{x_1,\ldots,x_n\}$ and $\mathcal{U}(\mathcal{G})$ the universal enveloping algebra of $\mathcal{G}$; let R be a K-algebra containing K. The tensor product $A := R \otimes_K \mathcal{U}(\mathcal{G})$ is a skew PBW extension of R with R-base $1 \otimes W = \{z_1^{\sigma_1}\cdots z_n^{\sigma_n} | \alpha = (\sigma_1,\ldots,\sigma_n) \in \mathbb{N}^n\}$, with $z_i := 1 \otimes x_i$; $(r \otimes 1)(1 \otimes x_i) - (1 \otimes x_i)(r \otimes 1) = 0$, for all $r \in R$ and $z_iz_j - z_jz_i \in Rz_1 + \cdots + Rz_n$.

Example 1.2.8. *Crossed product.* Let $\mathcal{G}$ be a finite dimensional Lie algebra over K with basis $\{x_1,\ldots,x_n\}$ and $\mathcal{U}(\mathcal{G})$ the universal enveloping algebra of $\mathcal{G}$; let R be a K-algebra containing K. $R * \mathcal{U}(\mathcal{G})$ is called crossed product of R if satisfies the following conditions: $R \subset R * \mathcal{U}(\mathcal{G})$; there exist an injective homomorphism of K-modules $\mathcal{G} \hookrightarrow R * \mathcal{U}(\mathcal{G})$, $x \mapsto \bar{x}$ such that $\bar{x}r - r\bar{x} \in R$ for all $r \in R$, $\bar{x}\bar{y} - \bar{y}\bar{x} \in \overline{[x,y]} + R$ for all $x, y \in \mathcal{G}$; $R * \mathcal{U}(\mathcal{G})$ is a free left R-module with the standard monomials over $\{\bar{x}_1,\ldots,\bar{x}_n\}$ as a base. So $R * \mathcal{U}(\mathcal{G})$ is a skew PBW extension of R.

Example 1.2.9. *Algebra of q-differential operators.* Let $q, h \in \mathbb{K}$, $q \neq 0$; consider $\mathbb{K}[y][x;\sigma,\delta]$, $\sigma(y) := qy$ and $\delta(y) := h$. By definition of skew polynomial ring we have

$xy = \sigma(y)x + \delta(y) = qyx + h$, and hence $xy - qyx = h$. Therefore, the algebra of q-differential operators $D_{q,h}[x, y] \cong \sigma(\mathbb{K}[y])\langle x\rangle$ is a skew PBW extension of $\mathbb{K}[y]$.

Example 1.2.10. *Algebra of shift operators.* Let $h \in \mathbb{K}$, the algebra of shift operators is defined by $S_h := \mathbb{K}[t][x_h; \sigma_h]$, where $\sigma_h(p(t)) := p(t - h)$. Notice that $x_h t = (t - h)x_h$ and for $p(t) \in \mathbb{K}[t]$ we have $x_h p(t) = p(t - ih)x_h^i$. Thus, $S_h \cong \sigma(\mathbb{K}[t])\langle x_h\rangle$ is a skew PBW extension of $\mathbb{K}[t]$.

Example 1.2.11. *Mixed algebra.* Let $h \in \mathbb{K}$, the algebra mixed algebra D_h is defined by $D_h := \mathbb{K}[t][x; \frac{d}{dt}][x_h; \sigma_h]$, where σ_h is as in Example 1.2.10 and $\sigma_h(x) = x$. Then, $D_h \cong \sigma(\mathbb{K}[t])\langle x, x_h\rangle$ is a skew PBW extension of $\mathbb{K}[t]$.

Example 1.2.12. *Algebra of discrete linear systems.* This algebra is defined by $D := \mathbb{K}[t_1, \dots, t_n][x_1; \sigma_1] \cdots [x_n; \sigma_n]$, where $\sigma_i(p(t_1, \dots, t_n)) := p(t_1, \dots, t_{i-1}, t_i + 1, t_{i+1}, \dots, t_n)$, $\sigma_i(x_i) = x_i$, $1 \le i \le n$. So, $D \cong \sigma(\mathbb{K}[t_1, \dots, t_n])\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of $\mathbb{K}[t_1, \dots, t_n]$.

Example 1.2.13. *Linear partial differential operators.* Algebra of linear partial differential operators. The *nth* Weyl algebra $A_n(\mathbb{K})$ over $\mathbb{K}$ coincides with the algebra of linear partial differential operators with polynomial coefficients $\mathbb{K}[t_1, \dots, t_n]$. As we have seen, the generators of $A_n(\mathbb{K})$ satisfy the following relations $t_i t_j = t_j t_i$, $\partial_i \partial_j = \partial_j \partial_i, 1 \le i < j \le n$; $\partial_j t_i = t_i \partial_j + \delta_{ij}, 1 \le i, j \le n$, where δ_{ij} is the Kronecker symbol. Let $\mathbb{K}(t_1, \dots, t_n)$ be the field of rational functions in n variables. Then the algebra of linear partial differential operators with rational function coefficients is the algebra $B_n(\mathbb{K}) = \mathbb{K}(t_1, \dots, t_n)[\partial_1, \dots, \partial_n]$, where the generators satisfy the relations above.

Example 1.2.14. *Linear partial shift operators.* It is the algebra with polynomial coefficients (respectively with rational coefficients), $\mathbb{K}[t_1, \dots, t_n][E_1, \dots, E_m]$ (respectively $\mathbb{K}(t_1, \dots, t_n)[E_1, \dots, E_m]$, $n \ge m$), subject to the relations: $t_j t_i = t_i t_j, 1 \le i < j \le n$; $E_i t_i = (t_i + 1)E_i = t_i E_i + E_i$, $1 \le i \le m$; $E_j t_i = t_i E_j$, $i \ne j$; $E_j E_i = E_i E_j$, $1 \le i < j \le m$.

Example 1.2.15. *Algebra of linear partial difference operators.* It is the algebra with polynomial coefficients (respectively rational coefficients), $\mathbb{K}[t_1, \dots, t_n][\Delta_1, \dots, \Delta_m]$ (respectively $\mathbb{K}(t_1, \dots, t_n)[\Delta_1, \dots, \Delta_m]$), $n \ge m$, subject to the relations: $t_j t_i = t_i t_j$, $1 \le i < j \le n$; $\Delta_i t_i = (t_i + 1)\Delta_i + 1 = t_i \Delta_i + \Delta_i + 1$, $1 \le i \le m$; $\Delta_j t_i = t_i \Delta_j$, $i \ne j$; $\Delta_j \Delta_i = \Delta_i \Delta_j$, $1 \le i < j \le m$.

Example 1.2.16. *Algebra of linear partial q-dilation operators.* For a fixed $q \in \mathbb{K} - \{0\}$, the algebra of linear partial q-dilation operators with polynomial coefficients (respectively rational coefficients) is $\mathbb{K}[t_1, \dots, t_n][H_1^{(q)}, \dots, H_m^{(q)}]$ (respectively $\mathbb{K}(t_1, \dots, t_n)[H_1^{(q)}, \dots, H_m^{(q)}]$), $n \ge m$, subject to the relations: $t_j t_i = t_i t_j$, $1 \le i < j \le n$; $H_i^{(q)} t_i = q t_i H_i^{(q)}$, $1 \le i \le m$; $H_j^{(q)} t_i = t_i H_j^{(q)}$, $i \ne j$; $H_j^{(q)} H_i^{(q)} = H_i^{(q)} H_j^{(q)}$, $1 \le i < j \le m$.

Example 1.2.17. *Algebra of linear partial q-differential operators.* For a fixed $q \in \mathbb{K} - \{0\}$, the algebra of linear partial q-differential operators with polynomial coefficients, respectively with rational coefficients is $\mathbb{K}[t_1, \dots, t_n][D_1^{(q)}, \dots, D_m^{(q)}]$, respectively the ring $\mathbb{K}(t_1, \dots, t_n)[D_1^{(q)}, \dots, D_m^{(q)}]$, $n \ge m$, subject to the relations: $t_j t_i = t_i t_j$, $1 \le i < j \le n$; $D_i^{(q)} t_i = q t_i D_i^{(q)} + 1$, $1 \le i \le m$; $D_j^{(q)} t_i = t_i D_j^{(q)}$, $i \ne j$; $D_j^{(q)} D_i^{(q)} = D_i^{(q)} D_j^{(q)}$, $1 \le i < j \le m$.

Note that if $n = m$, then this operator algebra coincides with the additive analogue $A_n(q_1, \ldots, q_n)$ of the Weyl algebra $A_n(q)$.

Example 1.2.18. *Diffusion algebra.* The diffusion algebra $\mathcal{D}$ is generated by $\{D_i, x_i | 1 \leq i \leq n\}$ over $\mathbb{K}$ with relations $x_ix_j = x_jx_i$, $x_iD_j = D_jx_i$, $1 \leq i,j \leq n$; $\quad c_{i,j}D_iD_j - c_{j,i}D_jD_i = x_jD_i - x_iD_j$, $i < j$, $c_{i,j}$, $c_{j,i} \in \mathbb{K}^*$. Thus, $A \cong \sigma(\mathbb{K}[x_1, \ldots, x_n])\langle D_1, \ldots, D_n\rangle$ is a bijective non quasi-commutative skew PBW extension of $\mathbb{K}[x_1, \ldots, x_n]$. Observe that $\mathcal{D}$ is not a PBW extension neither an iterated Ore extension of bijective type.

Example 1.2.19. *Additive analogue of the Weyl algebra.* The algebra $A_n(q_1, \ldots, q_n)$ is generated by the variables $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ subject to the relations:

$$
\begin{aligned}
x_jx_i &= x_ix_j, && 1 \leq i,j \leq n, && (1.2.1)\\
y_jy_i &= y_iy_j, && 1 \leq i,j \leq n, && (1.2.2)\\
y_ix_j &= x_jy_i, && i \neq j, && (1.2.3)\\
y_ix_i &= q_ix_iy_i + 1, && 1 \leq i \leq n, && (1.2.4)
\end{aligned}
$$

where $q_i \in \mathbb{K} \setminus \{0\}$. From the relations above we have

$$A_n(q_1, \ldots, q_n) \cong \sigma(\mathbb{K})\langle x_1, \ldots, x_n; y_1, \ldots, y_n\rangle \cong \sigma(\mathbb{K}[x_1, \ldots, x_n])\langle y_1, \ldots, y_n\rangle.$$

Example 1.2.20. *Multiplicative analogue of the Weyl algebra.* This algebra is denoted by $\mathcal{O}_n(\lambda_{ji})$ and is generated by $x_1, \ldots, x_n$ subject to the relations: $x_jx_i = \lambda_{ji}x_ix_j$, $1 \leq i < j \leq n$, $\lambda_{ji} \in \mathbb{K}\backslash\{0\}$. Thus $\mathcal{O}_n(\lambda_{ji}) \cong \sigma(\mathbb{K})\langle x_1, \ldots, x_n\rangle \cong \sigma(\mathbb{K}[x_1])\langle x_2, \ldots, x_n\rangle$. If $n = 2$, this algebra is called the *quantum plane*.

Example 1.2.21. *Quantum algebra $\mathcal{U}'(\mathfrak{so}(3, \mathbb{K}))$.* It is the algebra generated by I_1, I_2, I_3 subject to relations $I_2I_1 - qI_1I_2 = -q^{1/2}I_3$; $\quad I_3I_1 - q^{-1}I_1I_3 = q^{-1/2}I_2$; $\quad I_3I_2 - qI_2I_3 = -q^{1/2}I_1$, where $q \in \mathbb{K} - \{0\}$. In this way $\mathcal{U}'(\mathfrak{so}(3, \mathbb{K})) \cong \sigma(\mathbb{K})\langle I_1, I_2, I_3\rangle$.

Example 1.2.22. *Dispin algebra.* Dispin algebra $\mathcal{U}(osp(1, 2))$ it is generated by x, y, z over K satisfying the relations $yz - zy = z$, $zx + xz = y$, $xy - yx = x$. Thus, $\mathcal{U}(osp(1, 2)) \cong \sigma(K)\langle x, y, z\rangle$.

Example 1.2.23. *Woronowicz algebra.* Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2, \mathbb{K}))$ is generated by x, y, z subject to the relations $xz - \nu^4zx = (1 + \nu^2)x$; $\quad xy - \nu^2yx = \nu z$; $\quad zy - \nu^4yz = (1 + \nu^2)y$, where $\nu \in \mathbb{K} - \{0\}$ is not a root of unity. We have $\mathcal{W}_\nu(\mathfrak{sl}(2, \mathbb{K})) \cong \sigma(\mathbb{K})\langle x, y, z\rangle$.

Example 1.2.24. *Complex algebra.* The complex algebra is noted by $V_q(\mathfrak{sl}_3(\mathbb{C}))$. Let q be a complex number such that $q^8 \neq 1$. The complex algebra is generated by e_{12}, e_{13}, e_{23}, f_{12}, f_{13}, f_{23}, k_1, k_2, l_1, l_2 with the following relations:

$$
\begin{aligned}
e_{13}e_{12} &= q^{-2}e_{12}e_{13}, & f_{13}f_{12} &= q^{-2}f_{12}f_{13},\\
e_{23}e_{12} &= q^{2}e_{12}e_{23} - qe_{13}, & f_{23}f_{12} &= q^{2}f_{12}f_{23} - qf_{13},\\
e_{23}e_{13} &= q^{-2}e_{13}e_{23}, & f_{23}f_{13} &= q^{-2}f_{13}f_{23},
\end{aligned}
$$

$$
\begin{aligned}
e_{12}f_{12} &= f_{12}e_{12} + \frac{k_1^2 - l_1^2}{q^2 - q^{-2}}, & e_{12}k_1 &= q^{-2}k_1e_{12}, & k_1f_{12} &= q^{-2}f_{12}k_1,\\
e_{12}f_{13} &= f_{13}e_{12} + qf_{23}k_1^2, & e_{12}k_2 &= qk_2e_{12}, & k_2f_{12} &= qf_{12}k_2,\\
e_{12}f_{23} &= f_{23}e_{12}, & e_{13}k_1 &= q^{-1}k_1e_{13}, & k_1f_{13} &= q^{-1}f_{13}k_1,
\end{aligned}
$$

$$
\begin{array}{lll}
e_{13}f_{12} = f_{12}e_{13} - q^{-1}l_1^2e_{23}, & e_{13}k_2 = q^{-1}k_2e_{13}, & k_2f_{13} = q^{-1}f_{13}k_2, \\
e_{13}f_{13} = f_{13}e_{13} - \dfrac{k_1^2k_2^2 - l_1^2l_2^2}{q^2 - q^{-2}}, & e_{23}k_1 = qk_1e_{23}, & k_1f_{23} = qf_{23}k_1, \\
e_{13}f_{23} = f_{23}e_{13} + qk_2^2e_{12}, & e_{23}k_2 = q^{-2}k_2e_{23}, & k_2f_{23} = q^{-2}f_{23}k_2,
\end{array}
$$

$$
\begin{array}{lll}
e_{23}f_{12} = f_{12}e_{23}, & e_{12}l_1 = q^2l_1e_{12}, & l_1f_2 = q^2f_{12}l_1, \\
e_{23}f_{13} = f_{13}e_{23} - q^{-1}f_{12}l_2^2, & e_{12}l_2 = q^{-1}l_2e_{12}, & l_2f_{12} = q^{-1}f_{12}l_2, \\
e_{23}f_{23} = f_{23}e_{23} + \dfrac{k_2^2 - l_2^2}{q^2 - q^{-2}}, & e_{13}l_1 = ql_1e_{13}, & l_1f_{13} = qf_{13}l_1, \\
e_{13}l_2 = ql_2e_{13}, & l_2f_{13} = qf_{13}l_2, & e_{23}l_1 = q^{-1}l_1e_{23},
\end{array}
$$

$$
\begin{array}{lll}
l_1f_{23} = q^{-1}f_{23}l_1, & e_{23}l_2 = q^2l_2e_{23}, & l_2f_{23} = q^2f_{23}l_2, \\
l_1k_1 = k_1l_1, & l_2k_1 = k_1l_2, & k_2k_1 = k_1k_2, \\
l_1k_2 = k_2l_1, & l_2k_2 = k_2l_2, & l_2l_1 = l_1l_2.
\end{array}
$$

This algebra is a skew PBW extension of the commutative polynomial ring $\mathbb{C}[l_1, l_2, k_1, k_2]$, $V_q(\mathfrak{sl}_3(\mathbb{C})) \cong \sigma(\mathbb{C}[l_1, l_2, k_1, k_2])\langle e_{12}, e_{13}, e_{23}, f_{12}, f_{13}, f_{23}\rangle$.

Example 1.2.25. *Algebra* $\mathbf{U}$. Let $\mathbf{U}$ be the algebra generated over the field $\mathbb{K} = \mathbb{C}$ by the set of variables x_i, y_i, z_i, $1 \leq i \leq n$ subject to the relations:

$$
\begin{array}{ll}
x_jx_i = x_ix_j, \quad y_jy_i = y_iy_j, \quad z_jz_i = z_iz_j, & 1 \leq i < j \leq n, \\
x_jy_i = q^{-\delta_{ij}}y_ix_j, \quad z_jx_i = q^{-\delta_{ij}}x_iz_j, & 1 \leq i, j \leq n, \\
z_jy_i = y_iz_j, & i \neq j, \\
z_iy_i - q^2y_iz_i = -q^2x_i^2, & 1 \leq i \leq n,
\end{array}
$$

where $q \in \mathbb{K} - \{0\}$. We can see that $\mathbf{U}$ is a is a bijective skew PBW extension of $\mathbb{K}[x_1, \dots, x_n]$, i.e., $\mathbf{U} \cong \sigma(\mathbb{K}[x_1, \dots, x_n])\langle y_1, \dots, y_n; z_1, \dots, z_n\rangle$.

Example 1.2.26. *Manin algebra.* The coordinate algebra of the quantum matrix space $M_q(2)$ is also known as Manin algebra of 2×2 quantum matrices. By definition, $\mathcal{O}_2(M_n(\mathbb{K}))$, also denoted $\mathcal{O}(M_q(2))$, is the coordinate algebra of the quantum matrix space $M_q(2)$, it is the algebra generated by the variables x, y, u, v satisfying the relations $xu = qux$; $yu = q^{-1}uy$; $vu = uv$, and $xv = qvx$; $vy = qyv$; $yx - xy = -(q - q^{-1})uv$, where $q \in \mathbb{K}\backslash\{0\}$. Thus, $\mathcal{O}(M_q(2)) \cong \sigma(\mathbb{K}[u])\langle x, y, v\rangle$. It is not possible to consider $\mathcal{O}(M_q(2))$ as a skew PBW extension of $\mathbb{K}$. This algebra can be generalized to n variables, $\mathcal{O}_q(M_n(\mathbb{K}))$, and coincides with the coordinate algebra of the quantum group $SL_q(2)$.

Example 1.2.27. *Algebra of quantum matrices* $\mathcal{O}_q(M_n(\mathbb{K}))$. This algebra of order n coincides with the coordinate algebra of the quantum group $SL_q(2)$ and is generated by $\mathbb{K}$

and the variables x_{ij}, $1 \leq i, j \leq n$, subject to

$$\begin{aligned}
&x_{im}x_{ik} = q^{-1}x_{ik}x_{im}, && 1 \leq k < m \leq n,\\
&x_{jk}x_{ik} = q^{-1}x_{ik}x_{jk}, && 1 \leq i < j \leq n,\\
&x_{im}x_{jk} = x_{jk}x_{im}, && 1 \leq i < j, k < m \leq n,\\
&x_{jm}x_{im} = q^{-1}x_{im}x_{jm}, && 1 \leq i < j \leq n,\\
&x_{jm}x_{jk} = q^{-1}x_{jk}x_{jm}, && 1 \leq k < m \leq n,\\
&x_{ik}x_{jm} - x_{jm}x_{ik} = (q - q^{-1})x_{im}x_{jk}, && 1 \leq i < j, k < m \leq n.
\end{aligned}$$

From these relations we can see that $\mathcal{O}_q(M_n(\mathbb{K})) \cong \sigma(\mathbb{K}[x_{im}, x_{jk}])\langle x_{ik}, x_{jm}\rangle$, for $1 \leq i < j, k < m \leq n$. If $n = 2$, and by the identification $x_{11} := y, x_{12} := u, x_{21} := v$ y $x_{22} := x$, we obtain $\mathcal{O}_q(M_2(\mathbb{K}))$.

Example 1.2.28. *q-Heisenberg algebra.* The algebra $H_n(q)$ is generated by the set of variables $x_1, \dots, x_n$, $y_1, \dots, y_n$, $z_1, \dots, z_n$ subject to the relations:

$$x_jx_i = x_ix_j, \qquad z_jz_i = z_iz_j, \qquad y_jy_i = y_iy_j, \quad 1 \leq i, j \leq n, \tag{1.2.5}$$

$$z_jy_i = y_iz_j, \qquad z_jx_i = x_iz_j, \qquad y_jx_i = x_iy_j, \quad i \neq j, \tag{1.2.6}$$

$$z_iy_i = qy_iz_i, \qquad z_ix_i = q^{-1}x_iz_i + y_i, \qquad y_ix_i = qx_iy_i, \ 1 \leq i \leq n, \tag{1.2.7}$$

with $q \in \mathbb{K} \setminus \{0\}$. Then $H_n(q) \cong \sigma(\mathbb{K}[y_1, \dots, y_n])\langle x_1, \dots, x_n; z_1, \dots, z_n\rangle$. Note that $H_n(q)$ is isomorphic to the iterated Ore extension

$$\mathbb{K}[x_1, \dots, x_n][y_1; \sigma_1] \cdots [y_n; \sigma_n][z_1; \theta_1, \delta_1] \cdots [z_n; \theta_n, \delta_n]$$

on the commutative polynomial ring $\mathbb{K}[x_1, \dots, x_n]$:

$$\begin{gathered}
\theta_j(z_i) := z_i, \ \delta_j(z_i) := 0, \sigma_j(y_i) := y_i, \ 1 \leq i < j \leq n,\\
\theta_j(y_i) := y_i, \ \delta_j(y_i) := 0, \theta_j(x_i) := x_i, \ \delta_j(x_i) := 0, \sigma_j(x_i) := x_i, \ i \neq j,\\
\theta_i(y_i) := qy_i, \ \delta_i(y_i) := 0, \theta_i(x_i) := q^{-1}x_i, \ \delta_i(x_i) := y_i, \sigma_i(x_i) := qx_i, \ 1 \leq i \leq n,
\end{gathered}$$

Since $\delta_i(x_i) = y_i \notin \mathbb{K}[x_1, \dots, x_n]$, then $H_n(q)$ is not a skew PBW extension of $\mathbb{K}[x_1, \dots, x_n]$, however, with respect to $\mathbb{K}$, $H_n(q)$ satisfies the conditions of (iii), and hence, $H_n(q)$ is a bijective skew PBW extension of $\mathbb{K}$:

$$H_n(q) = \sigma(\mathbb{K})\langle x_1, \dots, x_n; y_1, \dots, y_n; z_1, \dots, z_n\rangle.$$

Example 1.2.29. *Quantum enveloping algebra of* $\mathfrak{sl}(2, \mathbb{K})$. $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{K}))$ is defined as the algebra generated by the variables x, y, z, z^{-1} with relations

$$zz^{-1} = z^{-1}z = 1, \tag{1.2.8}$$

$$xz = q^{-2}zx, \quad yz = q^2zy, \quad xz^{-1} = q^2z^{-1}x, \quad yz^{-1} = q^{-2}z^{-1}y, \tag{1.2.9}$$

$$xy - yx = \frac{z - z^{-1}}{q - q^{-1}}, \quad q \neq 1, -1 \tag{1.2.10}$$

From these relations we can see that $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{K})) = \sigma(\mathbb{K}[\boldsymbol{z^{\pm 1}}])\langle x, y\rangle$.

Example 1.2.30. *Hayashi's algebra* $W_q(J)$. The quantized algebra $W_q(J)$ is defined as the algebra generated by the variables $x_i, y_i, z_i, 1 \leq i \leq n$, $i \in J$, where $\mid J \mid = n$, and relations (1.2.5)-(1.2.7), replacing $z_i x_i = q^{-1} x_i z_i + y_i$ by

$$(z_i x_i - q x_i z_i) y_i = 1 = y_i (z_i x_i - q x_i z_i), \quad i = 1, \ldots, n, \quad q \in \mathbb{K} - \{0\}. \tag{1.2.11}$$

We note that Hayashi's algebra $W_q(J)$ is a skew PBW extension of the Laurent polynomial ring $\mathbb{K}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$. Indeed, if we consider

$$x_i y_j^{-1} = y_j^{-1} x_i, \; z_i y_j^{-1} = y_j^{-1} z_i, \; y_j y_j^{-1} = y_j^{-1} y_j = 1, \; z_i x_i = q x_i z_i + y_i^{-1}, \tag{1.2.12}$$

for $1 \leq i, j \leq n$, then $W_q(J) \cong \sigma(\mathbb{K}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}])\langle x_1, \ldots, x_n; z_1, \ldots, z_n \rangle$.

Example 1.2.31. *Multi-parameter quantum affine* n*-spaces.* Let $n \geq 1$ and $\mathbf{q}$ be a matrix $(q_{ij})_{n \times n}$ with entries in $\mathbb{K}$, where $q_{ii} = 1$ and $q_{ij} q_{ji} = 1$ for all $1 \leq i, j \leq n$. Then multi-parameter quantum affine n-space $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ is defined to be $\mathbb{K}-$algebra generated by $x_1, \ldots, x_n$ with the relations $x_j x_i = q_{ij} x_i x_j$ for all $1 \leq i, j \leq n$.

Example 1.2.32. *The algebra of differential operators* $D_q(S_q)$ *on a quantum space* S_q. Let $q = [q_{ij}]$ be a matrix with entries in K^* such that $q_{ii} = 1 = q_{ij} q_{ji}$ for all $1 \leq i, j \leq n$. The K-algebra S_q is generated by x_i, $1 \leq i \leq n$, subject to the relations $x_i x_j = q_{ij} x_j x_i$. The algebra S_q is regarded as the algebra of functions on a quantum space. Note that if K is a field, then S_q is the multi-parameter quantum affine n-space $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ of Example 1.2.31. The algebra $D_q(S_q)$ of q-differential operators on S_q is defined by $\partial_i x_j - q_{ij} x_j \partial_i = \delta_{ij}$ for all i, j. The relations between ∂_i, are given by $\partial_i \partial_j = q_{ij} \partial_j \partial_i$, for all i, j. Thus, $D_q(S_q) \cong \sigma(\sigma(K)\langle x_1, \ldots, x_n \rangle)\langle \partial_1, \ldots, \partial_n \rangle$. More exactly, $D_q(S_q)$ is a bijective skew PBW extension of a quasicommutative bijective skew PBW extension of K.

Example 1.2.33. *Witten's deformation of* $\mathcal{U}(\mathfrak{sl}(2, \mathbb{K}))$ E. Witten introduced and studied a 7- parameter deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2, \mathbb{K}))$ depending on a 7-tuple of parameters $\xi = (\xi_1, ..., \xi_7)$ and subject to relations $xz - \xi_1 zx = \xi_2 x$; $\quad zy - \xi_3 yz = \xi_4 y$; $\quad yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z$. The resulting algebra is denoted by $W(\xi)$. Assuming that $\xi_1 \xi_3 \xi_5 \neq 0$ (see [40]) we get that $W(\xi) \cong \sigma(\sigma(K[x])\langle z \rangle)\langle y \rangle$.

Example 1.2.34. *Quantum Weyl algebra of Maltsiniotis* $A_n^{q,\lambda}$. Let $q = [q_{ij}]$ be a matrix over K such that $q_{ij} q_{ji} = 1$ and $q_{ii} = 1$ for all $1 \leq i, j \leq n$. Fix an element $\lambda := (\lambda_1, \ldots, \lambda_n)$ of $(K^*)^n$. By definition, this algebra is generated by x_i, y_j, $1 \leq i, j \leq n$ subject to the relations: For any $1 \leq i < j \leq n$, $x_i x_j = \lambda_i q_{ij} x_j x_i$; $\quad y_i y_j = q_{ij} y_j y_i$; $\quad x_i y_j = q_{ji} y_j x_i$; $y_i x_j = \lambda_i^{-1} q_{ji} x_j y_i$. For any $1 \leq i \leq n$, $x_i y_i - \lambda_i y_i x_i = 1 + \sum_{1 \leq j < i} (\lambda_j - 1) y_j x_j$. From the relations above we have that $A_n^{q,\lambda}$ is isomorphic to a bijective skew PBW extension

$$\sigma(\sigma(\cdots(\sigma(\sigma(K)\langle x_1, y_1 \rangle)\langle x_2, y_2 \rangle)\cdots)\langle x_{n-1}, y_{n-1} \rangle)\langle x_n, y_n \rangle.$$

Example 1.2.35. *Quantum Weyl algebra* $A_n(q, p_{i,j})$. The ring $A_n(q, p_{i,j})$ arising as $A_n(R)$ for the "standard" multiparameter Hecke symmetry. This ring can be viewed as a quantization of the usual Weyl algebra $A_n(\mathbb{K})$. By definition, $A_n(q, p_{i,j})$ is the ring generated over $\mathbb{K}$ by the variables x_i, ∂_j with $i, j = 1, \ldots, n$ and subject to relations $x_i x_j = p_{ij} q x_j x_i$, for all $i < j$; $\quad \partial_i \partial_j = p_{ij} q^{-1} \partial_j \partial_i$, for all $i < j$; $\quad \partial_i x_j = p_{ij}^{-1} q x_j \partial_i$; for all $i \neq j$; $\partial_i x_i = 1 + q^2 x_i \partial_i + (q^2 - 1) \sum_{i<j} x_j \partial_j$, for all i. When $q = 1$ and each $p_{ij} = 1$, these relations give the usual Weyl algebra $A_n(\mathbb{K})$. From relations above we have that $A_n(q, p_{i,j})$ is a bijective skew PBW extension,

$$A_n(q, p_{i,j}) \cong \sigma(\sigma(\cdots\sigma(\sigma(\mathbb{K})\langle x_n, \partial_n \rangle)\langle x_{n-1}, \partial_{n-1} \rangle)\cdots)\langle x_2, \partial_2 \rangle)\langle x_1, \partial_1 \rangle.$$

Example 1.2.36. *Multiparameter quantized Weyl algebra* $A_n^{Q,\Gamma}(\mathbb{K})$. Let $Q := [q_1, \dots, q_n]$ be a vector in $\mathbb{K}^n$ with no zero components, and let $\Gamma = [\gamma_{ij}]$ be a multiplicatively antisymmetric $n \times n$ matrix over $\mathbb{K}$. The multiparameter Weyl algebra $A_n^{Q,\Gamma}(\mathbb{K})$ is the algebra generated by $\mathbb{K}$ and the indeterminates $y_1, \dots, y_n$, $x_1, \dots, x_n$ subject to the relations: $y_iy_j = \gamma_{ij}y_jy_i$, $1 \leq i, j \leq n$; $x_ix_j = q_i\gamma_{ij}x_jx_i$, $1 \leq i < j \leq n$; $x_iy_j = \gamma_{ji}y_jx_i$, $1 \leq i < j \leq n$; $x_iy_j = q_j\gamma_{ji}y_jx_i$, $1 \leq j < i \leq n$; $x_jy_j = q_jy_jx_j + 1 + \sum -l < j(q_l - 1)y_lx_l$, $1 \leq j \leq n$.
From relations above we have that $A_n^{Q,\Gamma}(\mathbb{K})$ is isomorphic to a bijective skew PBW extension,

$$A_n^{Q,\Gamma}(\mathbb{K}) \cong \sigma(\sigma(\cdots \sigma(\sigma(\mathbb{K})\langle x_1, y_1\rangle)\langle x_2, y_2\rangle) \cdots)\langle x_{n-1}, y_{n-1}\rangle)\langle x_n, y_n\rangle.$$

Example 1.2.37. *Quantum symplectic space* $\mathcal{O}_q(\mathfrak{sp}(\mathbb{K}^{2n}))$. For every nonzero element $q \in \mathbb{K}$, one defines this quantum algebra $\mathcal{O}_q(\mathfrak{sp}(\mathbb{K}^{2n}))$ as the algebra generated by $\mathbb{K}$ and the variables $y_1, \dots, y_n$, $x_1, \dots, x_n$, subject to the relations: $y_jx_i = q^{-1}x_iy_j$, $y_jy_i = qy_iy_j$, $1 \leq i < j \leq n$; $x_jx_i = q^{-1}x_ix_j$, $x_jy_i = qy_ix_j$, $1 \leq i < j \leq n$; $x_iy_i - q^2y_ix_i = (q^2 - 1)\sum_{l=1}^{i-1} q^{i-l}y_lx_l$, $1 \leq i \leq n$. From relations above we have that $\mathcal{O}_q(\mathfrak{sp}(\mathbb{K}^{2n}))$ is isomorphic to a bijective skew PBW extension,

$$\mathcal{O}_q(\mathfrak{sp}(\mathbb{K}^{2n})) \cong \sigma(\sigma(\cdots \sigma(\sigma(\mathbb{K})\langle x_1, y_1\rangle)\langle x_2, y_2\rangle) \cdots)\langle x_{n-1}, y_{n-1}\rangle)\langle x_n, y_n\rangle.$$

Example 1.2.38. *Quadratic algebras.* Quadratic algebras in 3 variables are considered as a class of G-algebras in 3 variables with relations homogeneous of degree 2. More exactly, *a quadratic algebra in 3 variables* $\mathcal{A}$ is an algebra generated by x, y, z subject to the relations

$$\begin{aligned}
yx &= xy + a_1xz + a_2y^2 + a_3yz + \xi_1z^2,\\
zx &= xz + \xi_2y^2 + a_5yz + a_6z^2,\\
zy &= yz + a_4z^2.
\end{aligned}$$

If $a_1 = a_4 = 0$ we obtain the relations

$$\begin{aligned}
yx &= xy + a_2y^2 + a_3yz + \xi_1z^2,\\
zx &= xz + \xi_2y^2 + a_5yz + a_6z^2,\\
zy &= yz.
\end{aligned}$$

One can check that $\mathcal{A} \cong \sigma(\mathbb{K}[y, z])\langle x\rangle$. If $a_5 = a_3 = 0$, which implies $a_2 = a_6 = 0$ and thus there is a family of algebras with relations

$$\begin{aligned}
yx &= xy + a_1xz + \xi_1z^2,\\
zx &= xz,\\
zy &= yz + a_4z^2.
\end{aligned}$$

These algebras are skew PBW extensions of the form $\sigma(\mathbb{K}[x, z])\langle y\rangle$.

Example 1.2.39. *Quantum Weyl algebra* $A_2(J_{a,b})$. It is the algebra generated by the variables $x_1, x_2, \partial_1, \partial_2$, with relations (depending on parameters $a, b \in \mathbb{K}$)

$$\begin{aligned}
x_1x_2 &= x_2x_1 + ax_1^2, & \partial_2\partial_1 &= \partial_1\partial_2 + b\partial_2^2\\
\partial_1x_1 &= 1 + x_1\partial_1 + ax_1\partial_2, & \partial_1x_2 &= -ax_1\partial_1 - abx_1\partial_2 + x_2\partial_1 + bx_2\partial_2\\
\partial_2x_1 &= x_1\partial_2, & \partial_2x_2 &= 1 - bx_1\partial_2 + x_2\partial_2.
\end{aligned}$$

If we consider the skew PBW extension of $\mathbb{K}[x_1, \partial_2]$, from the relations above we obtain the isomorphism $A_2(J_{a,b}) \cong \sigma(\mathbb{K}[x_1, \partial_2])\langle x_2, \partial_1\rangle$. In the particular case $a = b = 0$, $A_2(J_{0,0}) \cong A_2(\mathbb{K})$, is the usual Weyl algebra. Note that $A_2(J_{a,a}) \cong A_2(\mathbb{K})$ for all $a \in \mathbb{K}$.

Example 1.2.40 ([80], Chapter 12)**.** *Homogenized enveloping algebra.* Let $\mathcal{G}$ be a finite dimensional Lie algebra over $\mathbb{K}$ with basis $\{x_1, \dots, x_n\}$ and $\mathcal{U}(\mathcal{G})$ its enveloping algebra. The *homogenized enveloping algebra* of $\mathcal{G}$ is $\mathcal{A}(\mathcal{G}) := T(\mathcal{G} \oplus \mathbb{K}z)/\langle R\rangle$, where $T(\mathcal{G} \oplus \mathbb{K}z)$ is the tensor algebra, z is a new variable, and R is spanned by $\{z \otimes x - x \otimes z \mid x \in \mathcal{G}\} \cup \{x \otimes y - y \otimes x - [x, y] \otimes z \mid x, y \in \mathcal{G}\}$. From the PBW theorem for $\mathcal{G} \otimes \mathbb{K}(z)$, considered as a Lie algebra over $\mathbb{K}(z)$, we get that $\mathcal{A}(\mathcal{G})$ is a skew PBW extension of $\mathbb{K}[z]$.

1.2.2 Classification and some other examples

In Table 1.1 we classify Examples 1.2.1 - 1.2.40 of skew PBW extensions as constant, bijective, pre-commutative, quasi-commutative and semi-commutative, according to the Definition 1.1.5. This classification is new and it is very useful for the whole thesis.

Remark 1.2.41. In Example 1.2.28, the q-Heisenberg algebra $H_n(q)$ can be seen as skew PBW extension of $\mathbb{K}$ or as skew PBW extension of $\mathbb{K}[y_1, \dots, y_n]$. $H_n(q)$ as skew PBW extension of $\mathbb{K}$ is pre-commutative, but $H_n(q)$ as skew PBW extension of $\mathbb{K}[y_1, \dots, y_n]$ is not pre-commutative because $z_i x_i = q^{-1} x_i z_i + y_i,\ \ 1 \le i \le n$. In Table 1.1 we classify q-Heisenberg algebra as a skew PBW extension of $\mathbb{K}$.

1.2.3 3-dimensional skew polynomial algebra

It is the algebra $\mathcal{A}$ generated by the variables x, y, z restricted to relations

$$yz - \alpha zy = \lambda, \quad zx - \beta xz = \mu, \quad xy - \gamma yx = \nu, \tag{1.2.13}$$

such that

1. $\lambda, \mu, \nu \in \mathbb{K} + \mathbb{K}x + \mathbb{K}y + \mathbb{K}z$, and $\alpha, \beta, \gamma \in \mathbb{K}^*$;
2. Standard monomials $\{x^i y^j z^l \mid i, j, l \ge 0\}$ are a $\mathbb{K}$-basis of the algebra.

Thus, $\mathcal{A} \cong (K)\langle x, y, z\rangle$ are skew PBW extensions of $\mathbb{K}$. There are fifteen 3-dimensional skew polynomial algebras not isomorphic (see [77], Theorem C.4.3.1), i.e. $\mathcal{A}$ is one of the following algebras:

(a) if $|\{\alpha, \beta, \gamma\}| = 3$, then $\mathcal{A}$ is defined by

$$yz - \alpha zy = 0, \quad zx - \beta xz = 0, \quad xy - \gamma yx = 0. \tag{1.2.14}$$

(b) if $|\{\alpha, \beta, \gamma\}| = 2$ y $\beta \ne \alpha = \gamma = 1$, $\mathcal{A}$ is one of the following algebras:

  (i) $yz - zy = z, \quad zx - \beta xz = y, \quad xy - yx = x;$

  (ii) $yz - zy = z, \quad zx - \beta xz = b, \quad xy - yx = x;$

(iii) $yz - zy = 0, \quad zx - \beta xz = y, \quad xy - yx = 0;$
(iv) $yz - zy = 0, \quad zx - \beta xz = b, \quad xy - yx = 0;$
(v) $yz - zy = az, \quad zx - \beta xz = 0, \quad xy - yx = x;$
(vi) $yz - zy = z, \quad zx - \beta xz = 0, \quad xy - yx = 0.$

Here a, b are any elements $\mathbb{K}$. All nonzero values of b give isomorphic algebras.

(c) If $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma \neq 1$, then $\mathcal{A}$ is one of the following algebras:

(i) $yz - \alpha zy = 0, \quad zx - \beta xz = y + b, \quad xy - \alpha yx = 0;$
(ii) $yz - \alpha zy = 0, \quad zx - \beta xz = b, \quad xy - \alpha yx = 0.$

In this case b is an arbitrary element of $\mathbb{K}$. Again, nonzero values of b give isomorphic algebras.

(d) If $\alpha = \beta = \gamma \neq 1$, then $\mathcal{A}$ is the algebra

$$yz - \alpha zy = a_1x + b_1, \quad zx - \alpha xz = a_2y + b_2, \quad xy - \alpha yx = a_3z + b_3.$$

If $a_i = 0, i = 1, 2, 3$, all nonzero values of b_i give isomorphic algebras.

(e) If $\alpha = \beta = \gamma = 1$, $\mathcal{A}$ is isomorphic to one of the following algebras

(i) $yz - zy = x, \quad zx - xz = y, \quad xy - yx = z;$
(ii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = z;$
(iii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = b;$
(iv) $yz - zy = -y, \quad zx - xz = x + y, \quad xy - yx = 0;$
(v) $yz - zy = az, \quad zx - xz = z, \quad xy - yx = 0;$

Parameters $a, b \in \mathbb{K}$ are arbitrary and all nonzero values of b generate isomorphic algebras.

Note that the fifteen 3-dimensional skew polynomial algebras are constant and bijective. Therefore, in Table 1.2 we classify the fifteen 3-dimensional skew polynomial algebras as pre-commutative, quasi-commutative and semi-commutative according to Definition 1.1.5.

1.2.4 Sridharan enveloping algebra of 3-dimensional Lie algebra $\mathcal{G}$

Let $\mathcal{G}$ be a finite dimensional Lie algebra, and let $f \in Z^2(\mathcal{G}, \mathbb{K})$ be an arbitrary $2-$*cocycle*, that is, $f : \mathcal{G} \times \mathcal{G} \to \mathbb{K}$ such that $f(x, x) = 0$ and

$$f(x, [y, z]) + f(z, [x, y]) + f(y, [z, x]) = 0$$

for all $x, y, z \in \mathcal{G}$. The Sridharan enveloping algebra of $\mathcal{G}$ is defined to be the associative algebra $\mathcal{U}_f(\mathcal{G}) = T(\mathcal{G})/I$, where $T(\mathcal{G})$ is the tensor algebra of $\mathcal{G}$ and I is the two-sided ideal of $T(\mathcal{G})$ generated by the elements

$$(x \otimes y) - (y \otimes x) - [x, y] - f(x, y), \text{ for all } x, y \in \mathcal{G}.$$

Note that if $f = 0$ then $\mathcal{U}_f(\mathcal{G}) = \mathcal{U}_0(\mathcal{G}) = \mathcal{U}(\mathcal{G})$. For $x \in \mathcal{G}$, we still denote by x its image in $\mathcal{U}_f(\mathcal{G})$. $\mathcal{U}_f(\mathcal{G})$ is a filtered algebra with the associated graded algebra $gr(\mathcal{U}_f(\mathcal{G}))$ being a polynomial algebra. Let $\mathbb{K}$ be a field algebraically closed with characteristic zero. If $\mathcal{G}$ is a Lie algebra of dimension three, then the Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ for $f \in Z^2(\mathcal{G}, \mathbb{K})$ is isomorphic to one of the ten types of associative algebras with the commuting relations listed in Table 1.3, defined by three generators x, y, z and where $\alpha \in \mathbb{K}\backslash\{0\}$ (see [58], Theorem 1.3). Therefore the Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ is a skew PBW extension of $\mathbb{K}$, i.e. $\mathcal{U}_f(\mathcal{G}) \cong \sigma(\mathbb{K})\langle x, y, z\rangle$.

According to the commutation relations (Table 1.3), in Table 1.4 we classify the algebra $\mathcal{U}_f(\mathcal{G})$.

Classification of Examples 1.2.1 - 1.2.40						
Example	*Skew PBW extension*	*C*	*B*	*P*	*QC*	*SC*
1.2.1	Classical polynomial ring $R[x_1,\dots,x_n]$	✓	✓	✓	✓	✓
1.2.2	Ore extensions of bijective type $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$	⋆	✓	✓	⋆	⋆
1.2.3	Weyl algebra $A_n(\mathbb{K})$	⋆	✓	✓	⋆	⋆
1.2.3	Extended Weyl algebra $A_n(\mathbb{K})$	⋆	✓	✓	⋆	⋆
1.2.4	Jordan plane	⋆	✓	✓	⋆	⋆
1.2.5	Particular Sklyanin algebra	✓	✓	✓	✓	✓
1.2.6	Universal enveloping algebra of a Lie algebra $\mathcal{U}(\mathcal{G})$	✓	✓	✓	⋆	⋆
1.2.7	Tensor product $R\otimes_K \mathcal{U}(\mathcal{G})$	✓	✓	✓	⋆	⋆
1.2.8	Crossed product $R*\mathcal{U}(\mathcal{G})$	⋆	✓	⋆	⋆	⋆
1.2.9	Algebra of q-differential operators $D_{q,h}[x,y]$	⋆	✓	✓	⋆	⋆
1.2.10	Algebra of shift operators S_h	⋆	✓	✓	✓	⋆
1.2.11	Mixed algebra D_h	⋆	✓	⋆	⋆	⋆
1.2.12	Algebra of discrete linear systems $\mathbb{K}[t_1,\dots,t_n][x_1;\sigma_1]\cdots[x_n;\sigma_n]$	⋆	✓	✓	✓	⋆
1.2.13	Linear partial differential operators $\mathbb{K}[t_1,\dots,t_n][\partial_1,\dots,\partial_n]$	⋆	✓	✓	⋆	⋆
1.2.13	Linear partial differential operators $\mathbb{K}(t_1,\dots,t_n)[\partial_1,\dots,\partial_n]$	⋆	✓	✓	⋆	⋆
1.2.14	Linear partial shift operators $\mathbb{K}[t_1,\dots,t_n][E_1,\dots,E_m]$	⋆	✓	✓	✓	⋆
1.2.14	Linear partial shift operators $\mathbb{K}(t_1,\dots,t_n)[E_1,\dots,E_m]$	⋆	✓	✓	✓	⋆
1.2.15	Linear partial difference operators $\mathbb{K}[t_1,\dots,t_n][\Delta_1,\dots,\Delta_m]$	⋆	✓	✓	⋆	⋆
1.2.15	Linear partial difference operators $\mathbb{K}(t_1,\dots,t_n)[\Delta_1,\dots,\Delta_m]$	⋆	✓	✓	⋆	⋆
1.2.16	Linear partial q-dilation operators $\mathbb{K}[t_1,\dots,t_n][H_1^{(q)},\dots,H_m^{(q)}]$	⋆	✓	✓	✓	⋆
1.2.16	Linear partial q-dilation operators $\mathbb{K}(t_1,\dots,t_n)[H_1^{(q)},\dots,H_m^{(q)}]$	⋆	✓	✓	✓	⋆
1.2.17	Linear partial q-differential operators $\mathbb{K}[t_1,\dots,t_n][D_1^{(q)},\dots,D_m^{(q)}]$	⋆	✓	✓	⋆	⋆
1.2.17	Linear partial q-differential operators $\mathbb{K}(t_1,\dots,t_n)[D_1^{(q)},\dots,D_m^{(q)}]$	⋆	✓	✓	⋆	⋆
1.2.18	Diffusion algebra: $\mathcal{D}$	✓	✓	✓	⋆	⋆
1.2.19	Additive analogue of the Weyl algebra $A_n(q_1,\dots,q_n)$	✓	✓	⋆	⋆	⋆
1.2.20	Multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$	✓	✓	✓	✓	✓
1.2.21	Quantum algebra $\mathcal{U}'(\mathfrak{so}(3,\mathbb{K}))$	✓	✓	✓	⋆	⋆
1.2.22	Dispin algebra $\mathcal{U}(osp(1,2))$	✓	✓	✓	⋆	⋆
1.2.23	Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2,\mathbb{K}))$	✓	✓	✓	⋆	⋆
1.2.24	Complex algebra $V_q(\mathfrak{sl}_3(\mathbb{C}))$	⋆	✓	⋆	⋆	⋆
1.2.25	Algebra $\mathbf{U}$	⋆	✓	⋆	⋆	⋆
1.2.26	Manin algebra $M_q(2)$, $\mathcal{O}(M_q(2))$	⋆	✓	✓	⋆	⋆
1.2.27	Coordinate algebra of the quantum group $SL_q(2)$	⋆	✓	⋆	⋆	⋆
1.2.28	q-Heisenberg algebra $\mathbf{H}_n(q)$	✓	✓	✓	⋆	⋆
1.2.29	Quantum enveloping algebra of $\mathfrak{sl}(2,\mathbb{K})$, $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{K}))$	⋆	✓	⋆	⋆	⋆
1.2.30	Hayashi's algebra $W_q(J)$	⋆	✓	⋆	⋆	⋆
1.2.31	Multi-parameter quantum affine $n-$spaces $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$	✓	✓	✓	✓	✓
1.2.32	The algebra of differential operators $D_q(S_q)$ on a quantum space S_q	⋆	✓	⋆	⋆	⋆
1.2.33	Witten's deformation of $\mathcal{U}(\mathfrak{sl}(2,\mathbb{K}))$, $W(\xi)$	⋆	✓	⋆	⋆	⋆
1.2.34	Quantum Weyl algebra of Maltsiniotis $A_n^{q,\lambda}$	⋆	✓	⋆	⋆	⋆
1.2.35	Quantum Weyl algebra $A_n(q,p_{i,j})$	⋆	✓	⋆	⋆	⋆
1.2.36	Multiparameter quantized Weyl algebra $A_n^{Q,\Gamma}(\mathbb{K})$	⋆	✓	⋆	⋆	⋆
1.2.37	Quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}(\mathbb{K}^{2n}))$	⋆	✓	⋆	⋆	⋆
1.2.38	Quadratic algebras	⋆	✓	⋆	⋆	⋆
1.2.39	Quantum Weyl algebra $A_2(J_{a,b})$	⋆	✓	⋆	⋆	⋆
1.2.40	Homogenized enveloping algebra $\mathcal{A}(\mathcal{G})$	✓	✓	✓	⋆	⋆

Table 1.1: Classification of Examples 1.2.1 - 1.2.40.

C: Constant B: Bijective P: Pre-commutative QC: Quasi-commutative SC: Semi-commutative
$\star$: Negation ✓: Affirmation

Cardinal	*3-dimensional skew polynomial algebra*	P	QC	SC
$\lvert\{\alpha,\beta,\gamma\}\rvert = 3$	$yz-\alpha zy=0,\quad zx-\beta xz=0,\quad xy-\gamma yx=0$	✓	✓	✓
$\lvert\{\alpha,\beta,\gamma\}\rvert = 2$, $\beta\neq\alpha=\gamma=1$	$yz-zy=z,\quad zx-\beta xz=y,\quad xy-yx=x$	✓	⋆	⋆
	$yz-zy=z,\quad zx-\beta xz=b,\quad xy-yx=x$	⋆	⋆	⋆
	$yz-zy=0,\quad zx-\beta xz=y,\quad xy-yx=0$	✓	⋆	⋆
	$yz-zy=0,\quad zx-\beta xz=b,\quad xy-yx=0$	⋆	⋆	⋆
	$yz-zy=az,\quad zx-\beta xz=0,\quad xy-yx=x$	✓	⋆	⋆
	$yz-zy=z,\quad zx-\beta xz=0,\quad xy-yx=0$	✓	⋆	⋆
$\lvert\{\alpha,\beta,\gamma\}\rvert = 2$, $\beta\neq\alpha=\gamma\neq 1$	$yz-\alpha zy=0,\quad zx-\beta xz=y+b,\quad xy-\alpha yx=0$	⋆	⋆	⋆
	$yz-\alpha zy=0,\quad zx-\beta xz=b,\quad xy-\alpha yx=0$	⋆	⋆	⋆
$\alpha=\beta=\gamma\neq 1$	$yz-\alpha zy=a_1x+b_1,\quad zx-\alpha xz=a_2y+b_2,\quad xy-\alpha yx=a_3z+b_3$	⋆	⋆	⋆
$\alpha=\beta=\gamma=1$	$yz-zy=x,\quad zx-xz=y,\quad xy-yx=z$	✓	⋆	⋆
	$yz-zy=0,\quad zx-xz=0,\quad xy-yx=z$	✓	⋆	⋆
	$yz-zy=0,\quad zx-xz=0,\quad xy-yx=b$	⋆	⋆	⋆
	$yz-zy=-y,\quad zx-xz=x+y,\quad xy-yx=0$	✓	⋆	⋆
	$yz-zy=az,\quad zx-xz=x,\quad xy-yx=0$	✓	⋆	⋆

Table 1.2: Classification of 3-dimensional skew polynomial algebras.

P: Pre-commutative QC: Quasi-commutative SC: Semi-commutative
⋆: Negation ✓: Affirmation

Type	$[x,y]$	$[y,z]$	$[z,x]$
1	0	0	0
2	0	x	0
3	x	0	0
4	0	αy	$-x$
5	0	y	$-(x+y)$
6	z	$-2y$	$-2x$
7	1	0	0
8	1	x	0
9	x	1	0
10	1	y	x

Table 1.3: Sridharan enveloping algebras of a 3-dimensional Lie algebra.

Sridharan enveloping algebra of 3-dimensional Lie algebra $\mathcal{G}$								
Type	$[x,y]$	$[y,z]$	$[z,x]$	C	B	P	QC	SC
1	0	0	0	✓	✓	✓	✓	✓
2	0	x	0	✓	✓	✓	⋆	⋆
3	x	0	0	✓	✓	✓	⋆	⋆
4	0	αy	$-x$	✓	✓	✓	⋆	⋆
5	0	y	$-(x+y)$	✓	✓	✓	⋆	⋆
6	z	$-2y$	$-2x$	✓	✓	✓	⋆	⋆
7	1	0	0	✓	✓	⋆	⋆	⋆
8	1	x	0	✓	✓	⋆	⋆	⋆
9	x	1	0	✓	✓	⋆	⋆	⋆
10	1	y	x	✓	✓	⋆	⋆	⋆

Table 1.4: Classification of the Sridharan enveloping algebras of a 3-dimensional Lie algebra.

C: Constant B: Bijective P: Pre-commutative QC: Quasi-commutative SC: Semi-commutative
⋆: Negation ✓: Affirmation

CHAPTER 2

Graded skew PBW extensions

Lezama and Reyes in [45] defined a filtration for skew PBW extensions. The associate graduation to this filtration is a quasi-commutative skew PBW extension. Some properties of skew PBW extensions are deduced from this filtration and the associated graduation (see for example [45], [64], [66], [46]). There are skew PBW extensions which are graded but not quasi-commutative. In the first section we present some properties of skew PBW extensions with the standard filtration and graduation defined in [45]. In the second section, we define a more general graduation of skew PBW extensions, give some examples and study some specific properties of graded skew PBW extensions. In the third section, we study more general properties of graded skew PBW extensions. The main new results of the present chapter are Theorem 2.2.1, Corollary 2.3.14 and Theorem 2.3.22.

2.1 Standard filtration for skew PBW extensions and its associated graduation

Recall that a *filtered ring* is a ring B with a family $F(B) = \{F_n(B) \mid n \in \mathbb{Z}\}$ of subgroups of the additive group of B where we have the ascending chain $\cdots \subset F_{n-1}(B) \subset F_n(B) \subset \cdots$ such that $1 \in F_0(B)$ and $F_n(B)F_m(B) \subseteq F_{n+m}(B)$ for all $n, m \in \mathbb{Z}$. From a filtered ring B it is possible to construct its associated graded ring $Gr(B)$ taking $Gr(B)_n := F_n(B)/F_{n-1}(B)$. The following proposition establishes that one can construct a quasi-commutative skew PBW extension from a given skew PBW extension of a ring R.

Proposition 2.1.1 ([45], Proposition 2.1)**.** *Let A be a skew PBW extension of R. Then, there exists a quasi-commutative skew PBW extension A^σ of R in n variables $z_1, \dots, z_n$ defined by the relations $z_i r = c_{i,r} z_i$, $z_j z_i = c_{i,j} z_i z_j$, for $1 \leq i \leq n$, where $c_{i,r}, c_{i,j}$ are the same constants that define A. Moreover, if A is bijective then A^σ is also bijective.*

Proof. We consider n variables $z_1, \dots, z_n$ and the set of standard monomials

$$\mathcal{M} := \{z_1^{\alpha_1} \cdots z_n^{\alpha_n} | \alpha_i \in \mathbb{N}^n, 1 \leq i \leq n\},$$

let A^σ be the free R-module with basis $\mathcal{M}$ (i.e., A and A^σ are isomorphic as R-modules); we define the product in A^σ by the distributive law and the rules

$$rz^\alpha sz^\beta := r\sigma^\alpha(s)c_{\alpha,\beta}z^{\alpha+\beta},$$

where the σ's and the constants c's are as in Proposition 1.1.7. The identities of Remark 1.1.8 show that this product is associative, moreover $R \subseteq A^\sigma$ since for $r \in R$, $r = rz_1^0 \cdots z_n^0$. Thus, A^σ is a quasi-commutative skew PBW extension of R, and also, each element f^σ of A^σ corresponds to a unique element $f \in A$, replacing the variables x's by the variables z's. The last assertion of the proposition is trivial. □

The next proposition describes the standard filtration of a skew PBW extensions and its corresponding graduation.

Theorem 2.1.2. *Let A be an arbitrary skew PBW extension of R. Then, A is a filtered ring with increasing filtration given by*

$$F_m(A) := \begin{cases} R & \text{if } \ m = 0 \\ \{f \in A \mid \deg(f) \leq m\} \cup \{0\} & \text{if } \ m \geq 1 \end{cases} \tag{2.1.1}$$

and the corresponding graded ring $Gr(A)$ is isomorphic to A^σ.

Proof. See [45], Theorem 2.2. □

Proposition 2.1.3 establishes the relation between skew PBW extensions and iterated Ore extension in the sense of Proposition 1.1.4.

Proposition 2.1.3. *Let A be a quasi-commutative skew PBW extension of a ring R.*

(i) *A is isomorphic to an iterated Ore extension of endomorphism type $R[z_1; \theta_1] \cdots [z_n; \theta_n]$, where $\theta_1 = \sigma_1$;*

$$\theta_j : R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}] \to R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(z_i) = c_{i,j}z_i$ ($c_{i,j} \in R$ as in (1.1.2)), $1 \leq i < j \leq n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$.

(ii) *If A is bijective, then each θ_i in* (i) *is bijective.*

Proof. See [45], Theorem 2.3. □

Proposition 2.1.4 ([45], Corollary 2.4)**.** *Let A be a bijective skew PBW extension of R. If R is a left (right) noetherian ring then A is also a left (right) noetherian ring.*

Proof. According to Theorem 2.1.2, $Gr(A)$ is a quasi-commutative skew PBW extension, and by the hypothesis, $Gr(A)$ is also bijective. By Proposition 2.1.3, $Gr(A)$ is isomorphic to an iterated Ore extension $R[z_1; \theta_1] \cdots [z_n; \theta_n]$ such that each θ_i is bijective, $1 \leq i \leq n$. This implies that $Gr(A)$ is a left (right) noetherian ring, and hence, A is left (right) noetherian (see [55], Theorem 1.6.9). □

In the second section of the present chapter we will need some homological properties of skew PBW extensions that we will recall next. Let B be a ring and let M be a B-module. Let us denote $\mathrm{pd}_B(M)$ the projective dimension of M and $\mathrm{injdim}_B(M)$ the injective dimension of M. Let $\mathrm{lgld}(B)$ ($\mathrm{rgld}(B)$) be the left (right) global dimension of B. Do not always the left and right global dimensions of B are equal. However, if the ring B is noetherian the equality holds. If B is noetherian and if $\mathrm{injdim}(_BB) < \infty$ and $\mathrm{injdim}(B_B) < \infty$, then $\mathrm{injdim}(_BB) = \mathrm{injdim}(B_B)$. We say that B has *finite global dimension* (resp. finite injective dimension) if the left and right global dimensions of B are finite and equal (resp. the modules $_BB$ and B_B have finite injective dimensions which are equal). In such case we denote these numbers by $\mathrm{gld}(B)$ (resp. $\mathrm{injdim}(B)$).

Proposition 2.1.5. *Let $R \subseteq B$ be rings such that B_R ($_RB$) is faithfully flat and $_RB$ (B_R) is projective. Then,*

$$\mathrm{lgld}(R) \leq \mathrm{lgld}(B), \text{ if } \mathrm{lgld}(R) < \infty,$$
$$\mathrm{rgld}(R) \leq \mathrm{rgld}(B), \text{ if } \mathrm{lgld}(R) < \infty.$$

Proof. See [55], Theorem 7.2.6. □

Proposition 2.1.6. *Let B be a filtered ring, then*

$$\mathrm{lgld}(B) \leq \mathrm{lgld}(Gr(B)),\ \mathrm{rgld}(B) \leq \mathrm{rgld}(Gr(B)).$$

Proof. See [55], Corollary 7.6.18. □

Proposition 2.1.7. *Let R be a ring and σ an automorphism of R. Then,*

$$\mathrm{lgld}(R[x;\sigma]) = \mathrm{lgld}(R) + 1,\ \mathrm{rgld}(R[x;\sigma]) = \mathrm{rgld}(R) + 1.$$

Proof. See [55], Theorem 7.5.3. □

Proposition 2.1.8 ([45], Theorem 4.2)**.** *Let A be a bijective skew PBW extension of a ring R. Then,*

$$\mathrm{lgld}(R) \leq \mathrm{lgld}(A) \leq \mathrm{lgld}(R) + n, \text{ if } \mathrm{lgld}(R) < \infty,$$
$$\mathrm{rgld}(R) \leq \mathrm{rgld}(A) \leq \mathrm{rgld}(R) + n, \text{ if } \mathrm{rgld}(R) < \infty.$$

If A is quasi-commutative, then

$$\mathrm{lgld}(A) = \mathrm{lgld}(R) + n,\ \mathrm{rgld}(A) = \mathrm{rgld}(R) + n.$$

In particular, if R is semisimple, then $\mathrm{lgld}(A) = n = \mathrm{rgld}(A)$.

Proof. Since A is a filtered ring, then by Proposition 2.1.6

$$\mathrm{lgld}(A) \leq \mathrm{lgld}(Gr(A)),\ \mathrm{rgld}(A) \leq \mathrm{rgld}(Gr(A));$$

according to Theorem 2.1.2 and Proposition 2.1.3, $Gr(A)$ is isomorphic to an iterated Ore extension of automorphism type. From Proposition 2.1.7 we get the left inequalities. By Proposition 1.1.9, A_R is free, and hence projective and faithfully flat (by definition ${}_RA$ is free, and hence, projective and faithfully flat). From Proposition 2.1.5 we get the left inequalities. If A is quasi-commutative, Proposition 2.1.3 and Proposition 2.1.7 give the equalities. Finally, if R is semisimple, then $\operatorname{lgld}(R) = 0 = \operatorname{lgld}(R)$. □

Let B be a finitely generated algebra with finite generating set $\{b_1, \dots, b_n\}$. Let V be a finite dimensional subspace of B. V is called a *finite dimensional generating* subspace for B if we can express every element of B as a linear combination of monomials formed by elements of V. An example is the case where V is the subspace of B spanned by the generators $\{b_1, \dots, b_n\}$. If we set $V^0 := \mathbb{K}$ and V^s the subspace spanned by monomials of the form $b_{i_1}^{l_1} \cdots b_{i_m}^{l_m}$, $b_{i_j} \in \{b_1, \dots, b_n\}$ and $\sum_{j=1}^m l_j = s$, we have $B_m = \sum_{s=0}^m V^s$ and $B = \bigcup_{m=0}^\infty B_m$. Define $d_V(m) := \dim_{\mathbb{K}}(B_m)$. Gelfand-Kirillov dimension is a measure of the rate of growth of the algebra in terms of any generating set. More exactly: The *Gelfand-Kirillov dimension* (GK-dimension) of B is defined by

$$\operatorname{GKdim}(B) := \overline{\lim}\Big(\frac{\log d_V(m)}{\log m}\Big),$$

for a finite dimensional generating subspace V of B. The Gelfand-Kirillov dimension of the algebra B is independent of the choice of V. GK-dimension coincides with the Krull dimension in the commutative case. Algebras with GK-dimension zero are precisely those finite dimensional. The *Gelfand-Kirillov dimension of a module* M over an algebra B is:

$$\operatorname{GKdim}_B(M) = \sup_{V, M_0} \overline{\lim}_{n\to\infty} \log_n \dim_{\mathbb{K}} V^n M_0$$

where the supremum is taken over all finite-dimensional subspaces $V \subset B$ and $M_0 \subset M$.

We recall that a filtration $\{F_i(B)\}_{i\in\mathbb{Z}}$ of an algebra B is said to be *finite* if each $F_i(B)$ is a finite dimensional subspace.

Proposition 2.1.9 ([39], Proposition 6.6)**.** *Let B be an algebra with a finite filtration $\{F_i(B)\}_{i\in\mathbb{Z}}$ such that $Gr(B)$ is finitely generated. Then*

$$\operatorname{GKdim}(Gr(B)) = \operatorname{GKdim}(B).$$

Lemma 2.1.10 ([36], Lemma 2.2)**.** *Let B be an algebra with a finite dimensional generating subspace V, σ a $\mathbb{K}$-automorphism of B and δ a σ-derivation. If $\sigma(V) \subseteq V$, then*

$$\operatorname{GKdim}(B[x; \sigma, \delta]) = \operatorname{GKdim}(B) + 1.$$

Proposition 2.1.11 ([64], Theorem 14)**.** *Let R be an algebra with a finite dimensional generating subspace V and let $A = \sigma(R)\langle x_1 \dots, x_n\rangle$ be a bijective skew PBW extension of R. If $\sigma_n(V) \subseteq V$, then GKdim$(A) =$ GKdim$(R) + n$.*

Proof. From Theorem 2.1.2 it is clear that A is an algebra with a finite filtration. Let X be the $\mathbb{K}$-linear subspace of A spanned by $1, x_1, \dots, x_n$. Then VX is a finite dimensional generating subspace of $Gr(A) \cong A^\sigma$ and hence Proposition 2.1.9 implies $\operatorname{GKdim}(A) =$

GKdim($Gr(A)$). Now, from (2.1.1) and Proposition 2.1.3 we have that the ring A^σ is isomorphic to the Ore extension of automorphism type $R[x_1;\sigma_1]\cdots[x_n;\sigma_n]$ say. Note that $R[x_1;\sigma_1]\cdots[x_{n-1};\sigma_{n-1}]$ is an algebra and the automorphism σ_n of $R[x_1;\sigma_1]\cdots[x_{n-1};\sigma_{n-1}]$ given by $\sigma_n(r)=c_{n,r}$ and $\sigma_n(x_i)=c_{i,n}x_i$ for $r\in R, 1\le i<n$ is a $\mathbb{K}$-automorphism. If X' is the $\mathbb{K}$-linear subspace of A spanned by $1,x_1,\dots,x_{n-1}$, then VX' a finite dimensional generating subspace of $R[x_1;\sigma_1]\cdots[x_{n-1};\sigma_{n-1}]$, and from the assumption $\sigma_n(V)\subseteq V$ follows that $\sigma_n(VX')\subseteq VX'$. Lemma 2.1.10 guarantees $\mathrm{GKdim}(A)=\mathrm{GKdim}(Gr(A))=\mathrm{GKdim}(R)+n$. ☐

2.2 Graded skew PBW extensions and examples

In this section we define for skew PBW extensions a graduation more general than the graduation of Theorem 2.1.2. We also give some examples of these special classes of graded algebras. The results presented here, and the next section, represent the main tools developed in the present thesis, they will be applied in the next two chapters which contain the central results of the monograph. Many of these results have been mainly published in [83], and others are in [85].

A graded algebra $B=\bigoplus_{p\geq 0}B_p$ is called *connected* if $B_0=\mathbb{K}$. In [76], Definition 1.4, the concept of finitely graded algebra was presented. It is said that an algebra B is *finitely graded* if the following conditions hold:

(i) B is $\mathbb{N}$-graded (positively graded): $B=\bigoplus_{j\geq 0}B_j$,

(ii) B is connected.

(iii) B is *finitely generated* as algebra, i.e., there is a finite set of elements $x_1,\dots,x_n\in B$ such that the set $\{x_{i_1}x_{i_2}\cdots x_{i_m}\mid 1\le i_j\le n, m\geq 1\}\cup\{1\}$ spans B as a vector space.

We recall that an algebra B is called *augmented* when is given a morphism of algebras $\varepsilon: B\to\mathbb{K}$, called the *augmentation map*. In particular $\varepsilon(1_B)=1_{\mathbb{K}}$. If B is augmented, then B is canonically isomorphic, as a vector space, to $\mathbb{K}\cdot 1_B\oplus\ker(\varepsilon)$. The ideal $\ker(\varepsilon)$ is called the *augmentation ideal*. A $\mathbb{N}$-graded and connected algebra B is augmented, the augmentation is given by the projection $\varepsilon: B\to B_0=\mathbb{K}$. A graded algebra $B=\bigoplus_{j\in\mathbb{Z}}B_j$ is called *locally finite* if $dim_{\mathbb{K}}B_j<\infty$, for all $j\in\mathbb{Z}$. A graded B-module $M=\bigoplus_{j\in\mathbb{Z}}M_j$ is called *locally finite* if $dim_{\mathbb{K}}M_j<\infty$, for all $j\in\mathbb{Z}$. We say that the graded B-module M is *generated in degree* s if $M=B\cdot M_s$. M is *concentrated* in degree m if $M=M_m$. For any integer l, $M(l)$ is a graded B-module whose degree i component is $M(l)_i=M_{i+l}$.

The free associative algebra (tensor algebra) L in n generators $x_1,\dots,x_n$ is the ring $L:=\mathbb{K}\langle x_1,\dots,x_n\rangle$, whose underlying vector space is the set of all words in the variables x_i, that is, expressions $x_{i_1}x_{i_2}\dots\ x_{i_m}$ for some $m\geq 1$, where $1\le i_j\le n$ for all j. The length of a word $x_{i_1}x_{i_2}\dots x_{i_m}$ is m. We include among the words a symbol 1, which we think of as the empty word, and which has length 0. The product of two words is concatenation, and this operation is extended linearly to define an associative product on all elements. Note that L is $\mathbb{N}$-graded with graduation given by $L:=\bigoplus_{j\geq 0}L_j$ where $L_0=\mathbb{K}$ and L_j spanned by all words of length j in the alphabet $\{x_1,\dots,x_n\}$, for $j>0$; L is connected, the augmentation of L is given by the natural projection $\varepsilon:\mathbb{K}\langle x_1,\dots,x_n\rangle\to L_0=\mathbb{K}$

and the augmentation ideal is given by $L_+ =: \bigoplus_{j>0} L_j$. Consider the natural filtration $F_i(L) = \{\bigoplus L_j \mid j \leq i\}$ of L. Let P be a subspace of $F_2(L) := \mathbb{K} \bigoplus L_1 \bigoplus L_2$, the algebra $L/\langle P\rangle$ is called nonhomogeneous *quadratic algebra*. $L/\langle P\rangle$ is called *quadratic algebra* if P is a subspace of L_2, where $\langle P\rangle$ the two-sided ideal of L generated by P.

Let $I \subseteq \sum_{n\geq 2} L_n$ be a finitely generated homogeneous ideal of $\mathbb{K}\langle x_1, \dots, x_n\rangle$ and let $R = \mathbb{K}\langle x_1, \dots, x_n\rangle/I$, which is a connected graded algebra generated in degree 1. Suppose $\sigma : R \to R$ is a graded algebra automorphism and $\delta : R(-1) \to R$ is a graded σ-derivation (i.e. a degree $+1$ graded σ-derivation δ of R). Let $B := R[x; \sigma, \delta]$ be the associated *graded Ore extension* of R; that is, $B = \bigoplus_{n\geq 0} Rx^n$ as an R-module, and for $r \in R$, $xr = \sigma(r)x + \delta(r)$. We consider x to have degree 1 in B, and under this grading B is a connected graded algebra which is generated in degree 1 (see [19] and [61]). We introduce the definition of graded skew PBW extensions following [19].

Theorem 2.2.1. *Let $R = \bigoplus_{m\geq 0} R_m$ be a $\mathbb{N}$-graded algebra and let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of R satisfying the following two conditions:*

(i) *σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation for all $1 \leq i \leq n$, where σ_i and δ_i are as in Proposition 1.1.4.*

(ii) *$x_j x_i - c_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in (1.1.2) and $c_{i,j} \in R_0$.*

For $p \geq 0$, let A_p the $\mathbb{K}$-space generated by the set

$$\Big\{r_t x^\alpha \mid t + |\alpha| = p,\ r_t \in R_t \text{ and } x^\alpha \in \mathrm{Mon}(A)\Big\}.$$

Then A is a $\mathbb{N}$-graded algebra with graduation

$$A = \bigoplus_{p\geq 0} A_p. \tag{2.2.1}$$

Proof. It is clear that $1 = x_1^0 \cdots x_n^0 \in A_0$. Let $f \in A \setminus \{0\}$, then by Remark 1.1.3-(iv), f has a unique representation as $f = r_1 X_1 + \cdots + r_s X_s$, with $r_i \in R \setminus \{0\}$ and $X_i := x_1^{\alpha_{i_1}} \cdots x_n^{\alpha_{i_n}} \in \mathrm{Mon}(A)$ for $1 \leq i \leq s$. Let $r_i = r_{i_{q_1}} + \cdots + r_{i_{q_m}}$ the unique representation of r_i in homogeneous elements of R. Then $f = (r_{1_{q_1}} + \cdots + r_{1_{q_m}}) x_1^{\alpha_{1_1}} \cdots x_n^{\alpha_{1_n}} + \cdots + (r_{s_{q_1}} + \cdots + r_{s_{q_u}}) x_1^{\alpha_{s_1}} \cdots x_n^{\alpha_{s_n}} = r_{1_{q_1}} x_1^{\alpha_{1_1}} \cdots x_n^{\alpha_{1_n}} + \cdots + r_{1_{q_m}} x_1^{\alpha_{1_1}} \cdots x_n^{\alpha_{1_n}} + \cdots + r_{s_{q_1}} x_1^{\alpha_{s_1}} \cdots x_n^{\alpha_{s_n}} + \cdots + r_{s_{q_u}} x_1^{\alpha_{s_1}} \cdots x_n^{\alpha_{s_n}}$ is the unique representation of f in homogeneous elements of A. Therefore A is a direct sum of the family $\{A_p\}_{p\geq 0}$ of subspaces of A.

Now, let $x \in A_p A_q$. Without loss of generality we can assume that $x = (r_t x^\alpha)(r_s x^\beta)$ with $r_t \in R_t$, $r_s \in R_s$, $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, $t + |\alpha| = p$ and $s + |\beta| = q$. By Proposition 1.1.7-(a), we have that for r_s and x^α there exist unique elements $r_{s_\alpha} := \sigma^\alpha(r_s) \in R \setminus \{0\}$ and $p_{\alpha,r_s} \in A$ such that $x = r_t(r_{s_\alpha} x^\alpha + p_{\alpha,r_s})x^\beta = r_t r_{s_\alpha} x^\alpha x^\beta + r_t p_{\alpha,r_s} x^\beta$, where $p_{\alpha,r_s} = 0$ or $\deg(p_{\alpha,r_s}) < |\alpha|$ if $p_{\alpha,r_s} \neq 0$. Now, by Proposition 1.1.7-(b), we have that for x^α, x^β there exists unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x = r_t r_{s_\alpha}(c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}) + r_t p_{\alpha,r_s} x^\beta = r_t r_{s_\alpha} c_{\alpha,\beta} x^{\alpha+\beta} + r_t r_{s_\alpha} p_{\alpha,\beta} + r_t p_{\alpha,r_s} x^\beta$, where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$. We note that:

1. Since σ_i is graded for $1 \leq i \leq n$, then $\sigma_i^{\alpha_i}$ is graded and therefore σ^α is graded. Then $r_{s_\alpha} := \sigma^\alpha(r_s) \in R_s$ and $\delta_i^{\alpha_i}(r_s) \in R_{s+\alpha_i}$, for $1 \leq i \leq n$ and $\alpha_i \geq 0$.

2. If $W[\delta_i^{\nu}\sigma_i^{\alpha_i-\nu}]$ represents the sum of the possible words that can be constructed with the alphabet composed of ν times the symbol δ_i and $\alpha_i-\nu$ times the symbol σ_i, then $x_i^{\alpha_i}r_s=\sum_{\nu=0}^{\alpha_i}W[\delta_i^{\nu}\sigma_i^{\alpha_i-\nu}](r_s)x_i^{\alpha_i-\nu}\in A_{s+\alpha_i}$, since each summand in the above expression is in $A_{s+\alpha_i}$.

3. From condition (ii), we have that for $1\leq i<j\leq n$, $x_jx_i=c_{i,j}x_ix_j+r_{0_{ij}}+r_{1_{ij}}x_1+\cdots+r_{n_{ij}}x_n\in A_2$. Then, for $1\leq i<j<k\leq n$, we have that

$$\begin{aligned}
&x_k(x_jx_i)=x_k(c_{i,j}x_ix_j+r_{0_{ij}}+r_{1_{ij}}x_1+\cdots+r_{n_{ij}}x_n)\\
&=(\sigma_k(c_{i,j})x_kx_ix_j+\delta_k(c_{i,j})x_ix_j)+(\sigma_k(r_{0_{ij}})x_k+\delta_k(r_{0_{ij}}))\\
&\quad+(\sigma_k(r_{1_{ij}})x_kx_1+\delta_k(r_{1_{ij}})x_1)+\cdots+(\sigma_k(r_{n_{ij}})x_kx_n+\delta_k(r_{n_{ij}})x_n)\\
&=\sigma_k(c_{i,j})[c_{i,k}x_ix_k+r_{0_{ik}}+r_{1_{ik}}x_1+\cdots+r_{n_{ik}}x_n]x_j+\delta_k(c_{i,j})x_ix_j+\sigma_k(r_{0_{ij}})x_k\\
&\quad+\delta_k(r_{0_{ij}})+\sigma_k(r_{1_{ij}})[c_{1,k}x_1x_k+r_{0_{1k}}+r_{1_{1k}}x_1+\cdots+r_{n_{1k}}x_n]\\
&\quad+\delta_k(r_{1_{ij}})x_1+\cdots+\sigma_k(r_{n_{ij}})x_kx_n+\delta_k(r_{n_{ij}})x_n\\
&=\sigma_k(c_{i,j})c_{i,k}x_i[c_{j,k}x_jx_k+r_{0_{jk}}+r_{1_{jk}}x_1+\cdots+r_{n_{jk}}x_n]+\sigma_k(c_{i,j})r_{0_{ik}}x_j\\
&\quad+\sigma_k(c_{i,j})r_{1_{ik}}x_1x_j+\cdots+\sigma_k(c_{i,j})r_{n_{ik}}x_nx_j+\delta_k(c_{i,j})x_ix_j+\sigma_k(r_{0_{ij}})x_k\\
&\quad+\delta_k(r_{0_{ij}})+\sigma_k(r_{1_{ij}})[c_{1,k}x_1x_k+r_{0_{1k}}+r_{1_{1k}}x_1+\cdots+r_{n_{1k}}x_n]\\
&\quad+\delta_k(r_{1_{ij}})x_1+\cdots+\sigma_k(r_{n_{ij}})x_kx_n+\delta_k(r_{n_{ij}})x_n\\
&=\sigma_k(c_{i,j})c_{i,k}\sigma_i(c_{j,k})x_ix_jx_k+\sigma_k(c_{i,j})c_{i,k}\delta_i(c_{j,k})x_jx_k+\sigma_k(c_{i,j})c_{i,k}\sigma(c_{j,k})\sigma_i(r_{0_{ij}})x_i\\
&\quad+\sigma_k(c_{i,j})c_{i,k}\delta_i(r_{0_{ij}})+\sigma_k(c_{i,j})c_{i,k}\sigma_i(r_{1_{jk}})x_ix_1+\sigma_k(c_{i,j})c_{i,k}\delta_i(r_{1_{jk}})x_1+\cdots\\
&\quad+\sigma_k(c_{i,j})c_{i,k}\sigma_i(r_{n_{jk}})x_ix_n+\sigma_k(c_{i,j})c_{i,k}\delta_i(r_{n_{jk}})x_n+\sigma_k(c_{i,j})r_{0_{ik}}x_j\\
&\quad+\sigma_k(c_{i,j})r_{1_{ik}}x_1x_j+\cdots+\sigma_k(c_{i,j})r_{n_{ik}}x_nx_j+\delta_k(c_{i,j})x_ix_j+\sigma_k(r_{0_{ij}})x_k+\delta_k(r_{0_{ij}})\\
&\quad+\sigma_k(r_{1_{ij}})c_{1,k}x_1x_k+\sigma_k(r_{1_{ij}})r_{0_{1k}}+\sigma_k(r_{1_{ij}})r_{1_{1k}}x_1+\cdots+\sigma_k(r_{1_{ij}})r_{n_{1k}}x_n\\
&\quad+\delta_k(r_{1_{ij}})x_1+\cdots+\sigma_k(r_{n_{ij}})x_kx_n+\delta_k(r_{n_{ij}})x_n\\
&=\sigma_k(c_{i,j})c_{i,k}\sigma_i(c_{j,k})x_ix_jx_k+\sigma_k(c_{i,j})c_{i,k}\delta_i(c_{j,k})x_jx_k+\sigma_k(c_{i,j})c_{i,k}\sigma(c_{j,k})\sigma_i(r_{0_{ij}})x_i\\
&\quad+\sigma_k(c_{i,j})c_{i,k}\delta_i(r_{0_{ij}})+\sigma_k(c_{i,j})c_{i,k}\sigma_i(r_{1_{jk}})c_{1,i}x_1x_i+\sigma_k(c_{i,j})c_{i,k}\sigma_i(r_{1_{jk}})r_{0_{1i}}\\
&\quad+\sigma_k(c_{i,j})c_{i,k}\sigma_i(r_{1_{jk}})r_{1_{1i}}x_1+\cdots+\sigma_k(c_{i,j})c_{i,k}\sigma_i(r_{1_{jk}})r_{n_{1i}}x_n\\
&\quad+\sigma_k(c_{i,j})c_{i,k}\delta_i(r_{1_{jk}})x_1+\cdots+\sigma_k(c_{i,j})c_{i,k}\sigma_i(r_{n_{jk}})x_ix_n+\sigma_k(c_{i,j})c_{i,k}\delta_i(r_{n_{jk}})x_n\\
&\quad+\sigma_k(c_{i,j})r_{0_{ik}}x_j+\sigma_k(c_{i,j})r_{1_{ik}}x_1x_j+\cdots+\sigma_k(c_{i,j})r_{n_{ik}}c_{j,n}x_jx_n+\sigma_k(c_{i,j})r_{n_{ik}}\delta_n(x_j)\\
&\quad+\sigma_k(c_{i,j})r_{n_{ik}}r_{0_{jn}}+\sigma_k(c_{i,j})r_{n_{ik}}r_{1_{jn}}x_1+\cdots+\sigma_k(c_{i,j})r_{n_{ik}}r_{n_{jn}}x_n+\delta_k(c_{i,j})x_ix_j\\
&\quad+\sigma_k(r_{0_{ij}})x_k+\delta_k(r_{0_{ij}})+\sigma_k(r_{1_{ij}})c_{1,k}x_1x_k+\sigma_k(r_{1_{ij}})r_{0_{1k}}+\sigma_k(r_{1_{ij}})r_{1_{1k}}x_1+\cdots\\
&\quad+\sigma_k(r_{1_{ij}})r_{n_{1k}}x_n+\delta_k(r_{1_{ij}})x_1+\cdots+\sigma_k(r_{n_{ij}})x_kx_n+\delta_k(r_{n_{ij}})x_n.
\end{aligned}$$

Since all summands in the above sum have the form rx, where r is an homogeneous element of R, $x\in\mathrm{Mon}(A)$ and $rx\in A_3$, we have that $x_kx_jx_i\in A_3$. Following this procedure we get in general that $x_{i_1}x_{i_2}\cdots x_{i_m}\in A_m$ for $1\leq i_k\leq n$, $1\leq k\leq m$, $m\geq 1$.

4. In a similar way and following the proof of Proposition 1.1.7, we obtain that $p_{\alpha,r_s}\in A_{|\alpha|+s}$ and $p_{\alpha,\beta}\in A_{|\alpha|+|\beta|}$. Then $r_tr_{s_\alpha}c_{\alpha,\beta}x^{\alpha+\beta}\in A_{t+s+|\alpha|+|\beta|}$, $r_tr_{s_\alpha}p_{\alpha,\beta}\in A_{t+s+|\alpha|+|\beta|}$ and $r_tp_{\alpha,r_s}x^{\beta}\in A_{t+|\alpha|+s+|\beta|}$, i.e., $x\in A_{p+q}$.

□

Definition 2.2.2. Let $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ be a bijective skew PBW extension of a $\mathbb{N}$-graded algebra $R=\bigoplus_{m\geq 0}R_m$. We say that A is a *graded skew PBW extension* if A satisfies the conditions (i) and (ii) in Theorem 2.2.1.

In [93], Definition 1.3, James J. Zhang and Jun Zhang introduced the concept of double

Ore extensions. Let R be an algebra and B be another algebra, containing R as a subring. We say B is a *right double extension* of R if the following conditions hold:

(a) B is generated by R and two new variables x_1 and x_2.

(b) $\{x_1, x_2\}$ satisfies a relation $x_2x_1 = p_{12}x_1x_2 + p_{11}x_1^2 + \tau_1x_1 + \tau_2x_2 + \tau_0$, where $p_{12}, p_{11} \in \mathbb{K}$ and $\tau_1, \tau_2, \tau_0 \in R$.

(c) As a left R-module, $B = \sum\limits_{\alpha_1,\alpha_2\geq 0} Rx_1^{\alpha_1}x_2^{\alpha_2}$ and it is a left free R-module with $\{x_1^{\alpha_1}x_2^{\alpha_2} \mid \alpha_1 \geq 0, \alpha_2 \geq 0\}$.

(d) $x_1R + x_2R \subseteq Rx_1 + Rx_2 + R$.

We remark that every graded skew PBW extension $A = \sigma(R)\langle x_1, x_2\rangle$ of a connected algebra R is a right double extension. In this case, $p_{11} = 0$. On the other hand, if B is a right double extension of R with $p_{12} \neq 0$ and $p_{11} = 0$, then B is a (not necessarily graded) skew PBW extension of R.

Remark 2.2.3. Let A be a skew PBW extension of an algebra R. Note that if we endow R with the trivial graduation, $Gr(A)$ (as in Theorem 2.1.2) is a graded skew PBW extension (as in Definition 2.2.2). The reciprocal is false, since there are graded skew PBW extensions that are not quasi-commutative (see Example 2.2.7) and by Theorem 2.1.2 and Proposition 2.1.1 $Gr(A)$ is a quasi-commutative skew PBW extension. So, the graduation of Definition 2.2.2 generalizes the graduation of Theorem 2.1.2.

Proposition 2.2.4. *Quasi-commutative skew PBW extensions with the trivial graduation of R are graded skew PBW extensions. If we assume that R has a different graduation to the trivial graduation, then A is a graded skew PBW extension if and only if σ_i is graded and $c_{i,j} \in R_0$, for $1 \leq i, j \leq n$.*

Proof. Let $R = R_0$ and $r \in R = R_0$. From (1.1.5) we have that $x_ir = c_{i,r}x_i = \sigma_i(r)x_i$. So, $\sigma_i(r) = c_{i,r} \in R = R_0$ and $\delta_i = 0$, for $1 \leq i \leq n$. Therefore σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation for all $1 \leq i \leq n$. On the other hand, from (1.1.6) we have that $x_jx_i - c_{i,j}x_ix_j = 0 \in R_2 + R_1x_1 + \cdots + R_1x_n$ and $c_{i,j} \in R = R_0$. If R has a nontrivial graduation, then we get the result from relations (1.1.5), (1.1.6) and Definition 2.2.2. $\square$

Example 2.2.5. We present some examples of graded quasi-commutative skew PBW extensions.

1. The particular Sklyanin algebra S (Example 1.2.5) is a graded quasi-commutative skew PBW extension of $\mathbb{K}$.

2. The algebra of linear partial q-dilation operators $\mathbb{K}[t_1, \dots, t_n][H_1^{(q)}, \dots, H_m^{(q)}]$ (Example 1.2.16) is a graded quasi-commutative skew PBW extension of $\mathbb{K}[t_1, \dots, t_n]$, where $\mathbb{K}[t_1, \dots, t_n]$ is endowed with usual graduation.

3. The multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji}) \cong \sigma(\mathbb{K})\langle x_1, \dots, x_n\rangle \cong \sigma(\mathbb{K}[x_1])\langle x_2, \dots, x_n\rangle$ (Example 1.2.20) is a graded quasi-commutative skew PBW extension of $\mathbb{K}[x_1]$, where $\mathbb{K}[x_1]$ is endowed with usual graduation.

4. The multi-parameter quantum affine n-space $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ (Example 1.2.31).

Remark 2.2.6. The algebra of shift operators $S_h := \mathbb{K}[t][x_h; \sigma_h] \cong \sigma(\mathbb{K}[t])\langle x_h \rangle$ (Example 1.2.10) is a quasi-commutative skew PBW extension of $R := \mathbb{K}[t]$. S_h is a graded quasi-commutative skew PBW extension if $\mathbb{K}[t]$ is endowed with trivial graduation. But if $h \neq 0$ and $\mathbb{K}[t]$ is endowed with the usual graduation, i.e. $R_0 = \mathbb{K}$, R_1 is the subspace generated by t, R_2 is the subspace generated by t^2, etc., then S_h is not a graded skew PBW extension, since $\sigma_h(t) = t - h \notin R_1$, i.e, σ_h is not graded.

Examples 2.2.7. Next, we present specific examples of graded skew PBW extensions of the classical polynomial ring R with coefficients in $\mathbb{K}$, which are not quasi-commutative and where R has the usual graduation.

1. The Jordan plane $A = \mathbb{K}\langle x, y\rangle/\langle yx - xy - x^2\rangle \cong \sigma(\mathbb{K}[x])\langle y\rangle$ (Example 1.2.4).
2. The homogenized enveloping algebra. $\mathcal{A}(\mathcal{G}) \cong \sigma(\mathbb{K}[z])\langle x_1, \dots, x_n\rangle$ (Example 1.2.40).
3. The Diffusion algebra. $A \cong \sigma(\mathbb{K}[x_1, \dots, x_n])\langle D_1, \dots, D_n\rangle$ (Example 1.2.18).
4. The algebra $U \cong \sigma(\mathbb{K}[x_1, \dots, x_n])\langle y_1, \dots, y_n; z_1, \dots, z_n\rangle$ (Example 1.2.25).
5. Manin algebra. $\mathcal{O}(M_q(2)) \cong \sigma(\mathbb{K}[u])\langle x, y, v\rangle$ (Example 1.2.26).
6. Algebra of quantum matrices. $\mathcal{O}_q(M_n(\mathbb{K})) \cong \sigma(\mathbb{K}[x_{im}, x_{jk}])\langle x_{ik}, x_{jm}\rangle$, for $1 \leq i < j, k < m \leq n$ (Example 1.2.27).
7. Quadratic algebras in three variables. If $a_1 = a_4 = 0$ then the quadratic algebra is a graded skew PBW extension of $R = \mathbb{K}[y, z]$, and if $a_5 = a_3 = 0$ then quadratic algebras are graded skew PBW extensions of $R = \mathbb{K}[x, z]$ (Example 1.2.38).

2.3 Some properties

In this section we present some properties of graded skew PBW extensions.

Proposition 2.3.1. *Let B be a connected $\mathbb{N}$-graded algebra. B is finitely generated as $\mathbb{K}$-algebra if and only if $B = \mathbb{K}\langle x_1, \dots, x_m\rangle/I$, where I is a proper homogeneous two-sided ideal of $\mathbb{K}\langle x_1, \dots, x_m\rangle$. Moreover, for every $n \in \mathbb{N}$, $\dim_{\mathbb{K}} B_n < \infty$, i.e., B is locally finite.*

Proof. $\Leftarrow$): As the free algebra $L := \mathbb{K}\langle x_1, \dots, x_m\rangle$ is $\mathbb{N}$-graded and I is homogeneous, i.e., graded, then L/I es $\mathbb{N}$-graded with graduation given by $(L/I)_n := (L_n + I)/I$. Note that L/I is connected since $(L/I)_0 = \mathbb{K}$. Moreover, L/I is finitely generated as algebra by the elements $\overline{x_i} := x_i + I$, $1 \leq i \leq m$. Observe that L_n is finitely generated as vector space, whence, $(L/I)_n$ is also finitely generated as vector space, i.e., $\dim_{\mathbb{K}}((L/I)_n) < \infty$.
$\Rightarrow$): Let $a_1, \dots, a_m \in B$ be a finite collection of elements that generate B as $\mathbb{K}$-algebra; by the universal property of the free algebra $\mathbb{K}\langle x_1, \dots, x_m\rangle$, there exists a $\mathbb{K}$-algebra homomorphism $f : \mathbb{K}\langle x_1, \dots, x_m\rangle \to B$ with $f(x_i) := a_i$, $1 \leq i \leq m$; it is clear that f is surjective. Let $I := \ker(f)$, then I is a proper two-sided ideal of $\mathbb{K}\langle x_1, \dots, x_m\rangle$ and

$$B \cong \mathbb{K}\langle x_1, \dots, x_m\rangle/I. \tag{2.3.1}$$

Since B is $\mathbb{N}$-graded, we can assume that every a_i is homogeneous, $a_i \in B_{d_i}$ for some $d_i \geq 1$, moreover, at least one of generators is of degree 1. We define a new graduation for $L = \mathbb{K}\langle x_1, \dots, x_m\rangle$: we put weights d_i to the variables x_i and we set $L'_n := {}_{\mathbb{K}}\langle x_{i_1} \cdots x_{i_m} \mid \sum_{j=1}^m d_{i_j} = n\rangle$, $n \in \mathbb{N}$. This implies that f is graded, and from this we obtain that I is homogeneous. In fact, let $X_1 + \cdots + X_t \in I$, where $X_l \in L'_{n_l}$, $1 \leq l \leq t$, so $f(X_1) + \cdots + f(X_t) = 0$, and hence, $f(X_l) = 0$ for every l, i.e., $X_l \in I$. Finally, note that under the isomorphism $\widetilde{f}$ in (2.3.1), $\widetilde{f}((L'_n + I)/I) = B_n$, so $\dim_{\mathbb{K}}(B_n) < \infty$. □

Let B be a finitely graded algebra; it is said that B is *finitely presented* if the two-sided ideal I of relations in Proposition 2.3.1 is finitely generated.

Remark 2.3.2. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a graded skew PBW extension. Then, we immediately have the following properties:

(i) A is a $\mathbb{N}$-graded algebra and $A_0 = R_0$.

(ii) R is connected if and only if A is connected.

(iii) If R is finitely generated then A is finitely generated. Indeed, as $\mathrm{Mon}(A) = \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ is R-base for A, and R is finitely generated as algebra, then there is a finite set of elements $t_1, \dots, t_s \in R$ such that the set $\{t_{i_1}t_{i_2} \cdots t_{i_m} | 1 \leq i_j \leq s, m \geq 1\} \cup \{1\}$ spans R as a vector space. Then there is a finite set of elements $t_1, \dots, t_s, x_1, \dots, x_n \in A$ such that the set $\{t_{i_1}t_{i_2} \cdots t_{i_m} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 1 \leq i_j \leq s, m \geq 1, \alpha_1, \dots, \alpha_n \in \mathbb{N}\}$ spans A as a vector space. So, if R is generated in degree 1 then A is generated in degree 1.

(iv) For (i), (ii) and (iii) above we have that if R is a finitely graded algebra then A is a finitely graded algebra.

(v) If R is locally finite, A as algebra is a locally finite. Indeed, $\dim_{\mathbb{K}} A_0 = \dim_{\mathbb{K}} R_0$, $\dim_{\mathbb{K}} A_1 = \dim_{\mathbb{K}} R_1 + n$; let $\mathcal{B}_t$ be a (finite) base of R_t, $t \geq 0$, then for a fixed $p \geq 2$ the set $\{r_t x^\alpha \mid t + |\alpha| = p,\ r_t \in B_t \text{ and } x^\alpha \in \mathrm{Mon}(A)\}$ is a finite base for A_p.

(vi) A as R-module is locally finite.

(vii) If A is quasi-commutative and R is concentrate in degree 0, then $A_0 = R$.

(viii) If R is a quadratic algebra then A is a quadratic algebra.

(ix) If R is finitely presented then A is finitely presented. Indeed, by Proposition 2.3.1, $R = \mathbb{K}\langle t_1, \dots, t_m\rangle/I$ where
$$I = \langle r_1, \dots, r_s\rangle \tag{2.3.2}$$
is a two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m\rangle$ generated by a finite set $r_1, \dots, r_s$ of homogeneous polynomials in $\mathbb{K}\langle t_1, \dots, t_m\rangle$. Then $A = \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle/J$ where
$$J = \langle r_1, \dots, r_s,\ f_{hk},\ g_{ji} \mid 1 \leq i, j, h \leq n,\ 1 \leq k \leq m\rangle \tag{2.3.3}$$
is the two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle$ generated by a finite set of homogeneous elements $r_1, \dots, r_s$, f_{hk}, g_{ji} where $r_1, \dots, r_s$ are as in (2.3.2);
$$f_{hk} := x_h t_k - \sigma_h(t_k)x_h - \delta_h(t_k) \tag{2.3.4}$$

with σ_h and δ_h as in Proposition 1.1.4;

$$g_{ji} := x_j x_i - c_{i,j} x_i x_j - (r_{0_{j,i}} + r_{1_{j,i}} x_1 + \cdots + r_{n_{j,i}} x_n) \tag{2.3.5}$$

as in (1.1.2) of Definition 1.1.1.

Let $B = R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ be an iterated Ore extension. Then B is called a *graded iterated Ore extension* if $x_1, \ldots, x_n$ have degree 1 in B,

$$\sigma_i : R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}] \to R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}]$$

is a graded algebra automorphism and

$$\delta_i : R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}](-1) \to R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}]$$

is a graded σ_i-derivation, $2 \leq i \leq n$.

Remark 2.3.3. The class of graded iterated Ore extensions $\subsetneq$ class of graded skew PBW extensions. For example, the homogenized enveloping algebra $\mathcal{A}(\mathcal{G})$ and the Diffusion algebra are graded skew PBW extensions but these are not iterated Ore extensions. Therefore, the definition of graded skew PBW extensions is more general than the definition of graded Ore extensions.

An algebra is called *noetherian* if it is right and left noetherian.

Proposition 2.3.4. *A graded algebra B is right (left) noetherian if and only if it is* graded right (left) noetherian*, which means that every graded right (left) ideal is finitely generated.*

Proof. See [41], Proposition 1.4. ☐

Proposition 2.3.5. *Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a graded skew PBW extension.*

(i) *If R is a graded left (right) noetherian algebra, then every graded skew PBW extension A of R is graded left (right) noetherian.*

(ii) *If A is quasi-commutative, then A is isomorphic to a graded iterated Ore extension of endomorphism type $R[z_1; \theta_1] \cdots [z_n; \theta_n]$, where θ_i is bijective, for each i; $\theta_1 = \sigma_1$;*

$$\theta_j : R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}] \to R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(z_i) = c_{i,j} z_i$ ($c_{i,j} \in R_0$ as in (1.1.2)), $1 \leq i < j \leq n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$.

Proof. (i) Since A is bijective, then by Proposition 2.1.4 we have that A is a left (right) noetherian algebra. As A is graded then by Proposition 2.3.4, A is graded left (right) noetherian.

(ii) By Proposition 2.1.3 we have that A is isomorphic to an iterated Ore extension of endomorphism type $R[z_1; \theta_1] \cdots [z_n; \theta_n]$, where θ_i is bijective; $\theta_1 = \sigma_1$;

$$\theta_j : R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}] \to R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(z_i) = c_{i,j} z_i$ ($c_{i,j} \in R$ as in (1.1.2)), $1 \leq i < j \leq n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$. Since A is graded then σ_i is graded and $c_{i,j} \in R_0$. Moreover, since $\theta_i(r) = \sigma_i(r)$, then θ_i is a graded automorphism for each i. Note that z_i has graded 1 in A, for all i. Thus, $A \cong R[z_1; \theta_1] \cdots [z_n; \theta_n]$ is a graded iterated Ore extension. ☐

Let M be a B-module. The *grade number* of M is $j_B(M) := \min\{p \mid Ext_B^p(M, B) \neq 0\}$ or ∞ if no such p exists. Notice that $j_B(0) = +\infty$. When B is noetherian, $j_B(M) \leq pd_B(M)$, and if furthermore $\text{injdim}(B) = q < \infty$, we have $j_B(M) \leq q$ for all non-zero finitely generated B-module M (see [41]).

Definition 2.3.6. Let B be a filtered ring with filtration $\{F_n(B)\}_{n\in\mathbb{Z}}$. The *Rees ring* associated to B is a graded ring defined by

$$\tilde{B} := \bigoplus_{n\in\mathbb{Z}} F_n(B).$$

The filtration $\{F_n(B)\}_{n\in\mathbb{Z}}$ is *left (right) Zariskian*, and B is called a *left (right) Zariski ring*, if $F_{-1}(B) \subseteq Rad(F_0(B))$ and the associated Rees ring $\tilde{B}$ is left (right) noetherian.

Li in [47] shows the following result regarding Zariski's rings.

Proposition 2.3.7. *Let B be a $\mathbb{N}$-filtered ring such that $Gr(B)$ is left (right) noetherian. Then, B is left (right) Zariskian.*

Definition 2.3.8 ([41], Definition 2.1)**.** Let B be a noetherian ring.

(i) An B-module M satisfies the *Auslander-condition* if $\forall p \geq 0$, $j_B(N) \geq p$ for all B-submodules N of $Ext_B^p(M, B)$.

(ii) The ring B is said to be *Auslander-Gorenstein* of dimension q if $\text{injdim}(B) = q < \infty$, and every left or right finitely generated B-module satisfies the Auslander-condition.

(iii) The ring B is said to be *Auslander-regular* of dimension q if $\text{gld}(B) = q < \infty$ and every left or right finitely generated B-module satisfies the Auslander-condition.

(iv) Let B be an algebra; B is Cohen-Macaulay if $\text{GKdim}(B) = j_B(M) + \text{GKdim}_B(M)$ for every non-zero noetherian B-module M.

Proposition 2.3.9 ([15], Theorem 3.9)**.** *Let B be a left and right Zariski ring. If its associated graded ring $Gr(B)$ is Auslander-Gorenstein (respectively Auslander-regular), then B is Auslander-Gorenstein (respectively Auslander-regular).*

Proposition 2.3.10 ([21], Theorem 4.2)**.** *If B is Auslander-Gorenstein (Auslander-regular), then the Ore extension $B[x; \sigma, \delta]$, with σ bijective, is also Auslander-Gorenstein (Auslander-regular).*

Proposition 2.3.11 ([46], Lemma 2.8)**.** *If A is a bijective skew PBW extension of a noetherian ring R, then A is a left and right Zariski ring.*

Proof. Since A is $\mathbb{N}$-filtered, $0 = F_{-1}(A) \subseteq Rad(F_0(A)) = Rad(R)$. By Proposition 2.1.3, $Gr(A)$ is isomorphic to an iterated Ore extension $R[z_1; \theta_1] \cdots [z_n; \theta_n]$, with θ_i bijective, $1 \leq i \leq n$. Whence $Gr(A)$ is noetherian. Proposition 2.3.7 says that A is a left and right Zariski ring. $\square$

Proposition 2.3.12 ([46], Theorem 2.9)**.** *Let A be a bijective skew PBW extension of a ring R such that R is Auslander-Gorenstein (respectively Auslander-regular), then A is Auslander-Gorenstein (respectively Auslander-regular).*

Proof. According to Theorem 2.1.2 $Gr(A)$ is a quasi-commutative bijective skew PBW extension. By Proposition 2.1.3, $Gr(A)$ is isomorphic to an iterated Ore extension

$$R[z_1; \theta_1] \cdots [z_n; \theta_n],$$

with θ_i is bijective, $1 \leq i \leq n$. Proposition 2.3.10 says that $Gr(A)$ is Auslander-Gorenstein (respectively Auslander-regular). From Proposition 2.3.11, A is a left and right Zariski ring, so by Proposition 2.3.9, A is Auslander-Gorenstein (respectively Auslander-regular). □

A graded ring B has finite graded injective dimension q if ${}_BB$ and B_B are both of injective dimension q in the category of graded B-modules. Then, we write $\mathrm{grinjdim}(B) = q$. If M and N are graded B-modules, we use $\underline{Hom}^d_B(M, N)$ to denote the set of all B-module homomorphisms $h : M \to N$ such that $h(M_i) \subseteq N_{i+d}$. We set $\underline{Hom}_B(M, N) = \bigoplus_{d \in \mathbb{Z}} Hom^d_B(M, N)$, and we denote the corresponding derived functors by $\underline{Ext}^i_B(M, N)$. Given any graded B-module M, for the graded case, the *grade number* ($\underline{j}$-number) of M is $\underline{j}_B(M) = \min\{p \mid \underline{Ext}^p_B(M, B) \neq 0\}$ or ∞ if no such p exists. In particular, if $M = 0$, then $\underline{j}_B(M) = 0$. For finitely graded algebras, we have two additional remarks: Let B be a finitely graded algebra and let M, N be $\mathbb{Z}$-graded B-modules. Then there is a natural inclusion $\underline{Hom}_B(M, N) \hookrightarrow Hom_B(M, N)$. If M is an B-module finitely generated, then $\underline{Hom}_B(M, N) \cong Hom_B(M, N)$ and $\underline{Ext}^i_B(M, N) \cong Ext^i_B(M, N)$.

For the case of graded modules, in Definition 2.3.8, one can define the notion of a *graded-Auslander-Gorenstein* ring, or *graded-Auslander-regular* ring.

Proposition 2.3.13. *The noetherian graded ring B is Auslander-Gorenstein (resp. regular) if and only if B is graded Auslander-Gorenstein (resp. graded Auslander-regular).*

Proof. See [41], Theorem 3.1. □

Corollary 2.3.14. *Let A be a graded skew PBW extension of an Auslander-Gorenstein (respectively Auslander-regular) algebra R, then A is graded Auslander-Gorenstein (respectively graded Auslander-regular).*

Proof. As A is bijective, then from Proposition 2.3.12 we have that A is Auslander-Gorenstein (respectively Auslander-regular). Since A is graded then by Proposition 2.3.13 we have that A is graded Auslander-Gorenstein (resp. graded Auslander-regular). □

Proposition 2.3.15. *Let B be a $\mathbb{N}$-graded noetherian ring and let $grgld(B)$ and $grinjdim(B)$ be a graded global dimension and graded injective dimension of B respectively. Then $gld(B)$ (resp. $injdim(B)$) is finite if and only if $grgld(B)$ (resp. $grinjdim(B)$) is finite, in which case these two numbers are equal.*

Proof. See [41], Lemma 3.3. □

Corollary 2.3.16. *Let A be a graded skew PBW extension of a noetherian algebra R. If $gld(R)$ is finite, then $grgld(A)$ is finite.*

Proof. From Remark 2.3.2-(i) A is a $\mathbb{N}$-graded algebra. Now, by Proposition 2.3.5-(i) A is a graded noetherian algebra. As $gld(R)$ is finite, by Proposition 2.1.8 we have that $gld(A)$ is finite. Then by 2.3.15 we have that $grgld(A)$ is finite. □

When B is graded, one can define a *graded Cohen-Macaulay* property by taking $M \neq 0$ as a graded finitely generated B-module.

Remark 2.3.17. Let B be an algebra.

(i) If B is a graded right (left) noetherian algebra and B_0 is finite dimensional, then B is locally finite ([82], page 1).

(ii) ([82], Theorem 2.4) Every connected graded left (right) noetherian algebra with finite global dimension has finite GK-dimension.

Proposition 2.3.18. *Let A be a graded skew PBW extension of a connected left (right) noetherian algebra R. Then*

(i) *A is locally finite.*

(ii) *If R has finite global dimension then A has finite GK-dimension.*

Proof. From Proposition 2.3.5-(i), A is a graded left (right) noetherian algebra. Since R is connected then by Remark 2.3.2-(ii) we have that A is connected, i.e., $A_0 = \mathbb{K}$.

(i) Since $\mathbb{K}$ is finite dimensional then from Remark 2.3.17-(i), A is locally finite.

(ii) If R has finite global dimension then by Corollary 2.3.16, A has finite graded global dimension. Then by Remark 2.3.17-(ii) we have that A has finite GK-dimension.

□

Proposition 2.3.19 ([46], Proposition 3.5)**.** *Let B be a left and right Zariski ring with finite filtration and such that $Gr(B)$ is Auslander-Gorenstein. If $Gr(B)$ is Cohen-Macaulay, then B is Cohen-Macaulay.*

Proof. Let M be a noetherian B-module, then

$$\begin{aligned} GKdim(B) = GKdim(Gr(B)) &= GKdim(_{Gr(B)}Gr(M)) + j_{Gr(B)}(M) \\ &= GKdim(_BM) + j_B(M). \end{aligned}$$

Therefore B is Cohen-Macaulay. □

Proposition 2.3.20. *Suppose that R is Auslander-regular (Auslander-Gorenstein) and Cohen-Macaulay ring. Let $B = R[x; \sigma, \delta]$ be an Ore extension with σ bijective. If $R = \bigoplus_{i\geq 0} R_i$ is a connected graded algebra such that $\sigma(R_i) \subseteq R_i$ for each $i \geq 0$, i.e. σ is graded. Then B is Cohen-Macaulay.*

Proof. See [42], Lemma-(ii), page 184. □

Proposition 2.3.21 ([46], Theorem 3.9)**.** *Let A be a bijective skew PBW extension of a ring R such that R is Auslander-Gorenstein, Cohen-Macaulay, and $R = \bigoplus_{i\geq 0} R_i$ is a connected graded algebra such that $\sigma_j(R_i) \subseteq R_i$ for each $i \geq 0$ and $1 \leq j \leq n$, then A is Cohen-Macaulay.*

Proof. From Theorem 2.1.2 we have that A is an algebra with a finite filtration and $Gr(A)$ is a quasi-commutative skew PBW extensions, and by the hypothesis, $Gr(A)$ is also bijective. By Proposition 2.1.3, $Gr(A)$ is isomorphic to an iterated Ore extension $R[z_1;\theta_1]\cdots[z_n;\theta_n]$ such that each θ_i is bijective, $1 \leq i \leq n$. Proposition 2.3.10 says that $Gr(A)$ is Auslander-Gorenstein. From Proposition 2.3.11, A is a left and right Zariski ring, and by Proposition 2.3.20 $Gr(A)$ is Cohen-Macaulay, so by Proposition 2.3.19 A is Cohen-Macaulay. □

Theorem 2.3.22. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a graded skew PBW extension of a connected algebra R. If R is graded Auslander-Gorenstein and graded Cohen-Macaulay then A is graded Cohen-Macaulay.*

Proof. Since A is bijective, R is a $\mathbb{N}$-graded algebra, connected and each σ_i is graded, i.e., $\sigma_i(R_m) \subseteq R_m$ for each $m \geq 0$ and $1 \leq i \leq n$, then by Proposition 2.3.21 we have that A is graded Cohen-Macaulay. □

CHAPTER 3

Koszulity for skew PBW extensions

Koszul algebras were first introduced by Priddy in [63] under the name of homogeneous Koszul algebras and have revealed important applications in algebraic geometry, Lie theory, quantum groups, algebraic topology and combinatorics. The structure and history of Koszul algebras are detailed in [62]. Backelin and Fröberg in [7] show several equivalent definitions of Koszul algebras. Later they emerged some general notions of Koszul algebras (or Koszul rings), for example in [8], a *Koszul ring* is a positively graded ring $B = \oplus_{j\geq 0} B_j$ such that B_0 is semisimple and B_0, considered as a graded left B-module, admits a graded projective resolution

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \twoheadrightarrow B_0$$

such that P_i is generated by its degree i component, i.e., $P_i = B(P_i)_i$. Many interesting algebras with similar properties to a Koszul algebra do not satisfy that B_0 is semisimple, there do already exist several generalized Koszul theories where the degree 0 part B_0 of a graded algebra B is not required to be semisimple, see [30], [49], [53], [91]. Let B be a graded algebra, Li in [49] develops a generalized Koszul theory by assuming that B_0 is self-injective instead of semisimple and generalize many classical results. Each Koszul ring B defined by Woodcock in [91] is supposed to satisfy that B is both a left projective B_0-module and a right projective B_0-module. This requirement is too strong. In [49] define generalized Koszul modules and Koszul algebras in a similar way to the classical case, that is, a graded B-module M is *Koszul* if M has a linear projective resolution, and B is a Koszul algebra if B_0 viewed as a graded B-module is Koszul. More precisely (see [49], Definition 2.4), a locally finite graded B-module M generated in degree 0 (we say $M = \oplus_{i\in\mathbb{Z}} M_i$ is generated in degree s if $M = B \cdot M_s$) is called a Koszul module if it has a (minimal) projective resolution

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

such that P_i is generated in degree i for all $i \geq 0$.

Let B be a generalized Koszul algebra over a finite dimensional algebra S. He, Van Oystaeyen and Zhang in [35] construct a bimodule Koszul resolution of B when the projective dimension of S_B equals 2. Using this they prove a Poincaré-Birkhoff-Witt (PBW) type theorem for a deformation of a generalized Koszul algebra. When the projective dimension of S_B is greater than 2, they construct bimodule Koszul resolutions for generalized smash product algebras obtained from braiding between finite dimensional algebras and Koszul algebras, and then prove the PBW type theorem. The results obtained can be applied to standard Koszul Artin-Schelter Gorenstein algebras in the sense of Minamoto and Mori in [57].

On the other hand, Koszul algebras attracted the interest of those studying noncommutative algebraic geometry because some of the Artin-Schelter regular algebras (introduced in [3]) are Koszul. Koszul algebras must always be quadratic. There have been some attempts to generalize Koszulity to connected graded algebras with non quadratic relations. Hoping to capture more of the Artin-Schelter regular algebras, Berger introduced N-Koszul algebras in [10]. These algebras are N-homogeneous, that is, their ideal of relations may be generated by degree N homogeneous elements, where N is an integer ≥ 2. The 2-Koszul algebras are exactly the Koszul algebras. The term N-Koszul as used by Berger, is different than the sense of the term seen in [62]. Green, Marcos, Martinez and Zhang in [29] study d-Koszul algebras: Let $B = B_0 + B_1 + B_2 + \cdots$ be a graded K-algebra generated in degrees 0 and 1 where K is a commutative noetherian ring. Assume that B_0 is a finitely generated semisimple K-algebra, B_1 is a finitely generated K-module and that

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to B_0 \to 0,$$

is a minimal graded B-projective resolution of B_0. B is a *d-Koszul algebra* if, for each $i \geq 0$, P_i can be generated in exactly one degree, $\delta(i)$, and

$$\delta(i) = \begin{cases} \frac{i}{2}d, & \text{if } i \text{ is even;} \\ \frac{i-1}{2}d + 1, & \text{if } i \text{ is odd.} \end{cases}$$

By assumption that B is generated in degrees 0 and 1, B is a quotient of the tensor algebra $T_{B_0}(B) = B_0 + (B_1 \otimes_{B_0} B_1) + (\otimes^3_{B_0} B_1) + \cdots$. If $B = T_{B_0}(B_1)/I$ is a d-Koszul algebra, then I is finitely generated and can be generated by elements in $\otimes^d_{B_0} B_1$ since P_2 can be generated in degree d. Furthermore, the finiteness assumptions on B_0 and B_1 and that K is noetherian imply that each P_n is finitely generated. We note that if $d = 2$, then B is a Koszul algebra since B_0 has linear projective resolution. For $d \geq 3$, B is not a Koszul algebra (see [29], page 147).

Cassidy and Shelton introduced $\mathcal{K}_2$ algebras in [19], this class of algebras contains all the N-Koszul algebras and the Koszul algebras, but also admits algebras whose ideals of relations are generated by homogeneous elements in different degrees (see [60]).

Since Priddy in [63] defined homogeneous Koszul algebras and Koszul algebras, from now on in this thesis Koszul algebras, defined by Priddy, will be called nonhomogeneous Koszul and the homogeneous Koszul algebras will be simply called Koszul. In the first section, we study the nonhomogeneous Koszul property and the Koszul property for skew PBW extensions over fields. In the second section, we study the Koszul property for graded skew PBW extensions using PBW algebras and the Backelin's criterion in terms of distributivity of lattices. The main results are Corollary 3.1.14, Theorem 3.1.15, Example 3.1.17, Example 3.1.18, Proposition 3.1.26, Theorem 3.2.5, Corollary 3.2.7, Example 3.2.8, Remark 3.2.9, and Theorem 3.2.17.

3.1 Koszulity for skew PBW extensions over fields

Some authors have studied nonhomogeneous Koszul algebras. For example, nonhomogeneous Koszul algebras are defined in [62], analogous to Definition 3.1.9 below: Let $L = \mathbb{K}\langle x_1, \ldots, x_n\rangle$ the free associative algebra (tensor algebra) in n generators $x_1, \ldots, x_n$, let $P \subseteq \mathbb{K} \bigoplus L_1 \bigoplus L_2$ be a subspace of $F_2(L)$ and $B = L/\langle P\rangle$, where $\langle P\rangle$ is the two-sided ideal of L generated by P; let $B^{(0)} = L/\langle R\rangle$, where R is obtained by taking homogeneous part of P (i.e., $R = \pi(P)$, where $\pi : L \twoheadrightarrow L_2$ is the projection onto the quadratic part of the free algebra L). B is said to be nonhomogeneous Koszul if $B^{(0)}$ is Koszul (see [62], page 140). In [52] nonhomogeneous Koszul algebras are defined as follows: Let V a graded vector space and a degree homogeneous subspace $P \subseteq V \bigoplus V^{\otimes 2}$, the algebra $B = T(V)/\langle P\rangle$ is called nonhomogeneous Koszul if $P \cap V = \{0\}$, $\{P \otimes V + V \otimes P\} \cap V^{\otimes 2} \subseteq P \cap V^{\otimes 2}$ and $T(V)/\langle \pi(P)\rangle$ is Koszul, where $\pi : T(V) \twoheadrightarrow V^{\otimes 2}$ is the projection onto the quadratic part of the tensor algebra. Berger in [10] defined the notion of N-Koszul algebra, for $N \geq 2$. If $N = 2$, the notion of Koszul algebra is obtained. In this section we study the nonhomogeneous Koszul property and Koszul property for skew PBW extensions over fields, taking int account the definitions given in [63]. The results of this section were published in [88].

Let B be an algebra; if B is finitely generated and $\mathbb{N}$-graded, then B is generated by a finite set of homogeneous elements and from this we can conclude that a positively graded algebra B is finitely generated if and only if there is a degree preserving surjective ring homomorphism $\mathbb{K}\langle x_1, \ldots, x_n\rangle \to B$ for some free algebra $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ with some weighting of the variables, and thus $B \cong \mathbb{K}\langle x_1, \ldots, x_n\rangle/I$ for some homogeneous ideal I.

3.1.1 Pre-Koszul algebras

We present a definition of nonhomogeneous pre-Koszul and pre-Koszul algebras, analogous to the definition given by Priddy in [63].

Definition 3.1.1. Let I be a proper two sided ideal of $L = \mathbb{K}\langle x_1, \ldots, x_n\rangle$ and let $B := L/I$.

(i) B is said to be a *nonhomogeneous pre-Koszul* algebra if I is a two sided ideal generated by elements of the form

$$\sum_{i=1}^{n} c_i x_i + \sum_{1 \leq j,k \leq n} c_{j,k} x_j x_k, \text{ where } c_i \text{ and } c_{j,k} \text{ are in } \mathbb{K}. \tag{3.1.1}$$

(ii) A nonhomogeneous pre-Koszul algebra is said to be *pre-Koszul* if $c_i = 0$, for $1 \leq i \leq n$ in (3.1.1).

Presentations of special types of skew PBW extensions are given in the following remark.

Remark 3.1.2. Let $A = \sigma(\mathbb{K})\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of $\mathbb{K}$.

1. We note that $A = \mathbb{K}\langle x_1, \ldots, x_n\rangle/I$, where I is the two sided ideal generated by elements as in *(iii)* and *(iv)* of the Definition 1.1.1, i.e., elements of the form

$$c_r + x_i r - c_{i,r} x_i, \quad r_0 + r_1 x_1 + \cdots + r_n x_n + x_j x_i - c_{i,j} x_i x_j, \tag{3.1.2}$$

where $r \neq 0$, c_r, $c_{i,r} \neq 0$, r_0, $r_1, \dots, r_n$, $c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$.

2. If A is pre-commutative, then $A = \mathbb{K}\langle x_1, \dots, x_n\rangle/I$, where I is the two sided ideal generated by elements of the form

$$c_r + x_i r - c_{i,r}x_i, \quad r_1x_1 + \cdots + r_nx_n + x_jx_i - c_{i,j}x_ix_j, \tag{3.1.3}$$

where $r \neq 0$, c_r, $c_{i,r} \neq 0$, $r_1, \dots, r_n$, $c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$.

3. If A is constant, then $A = \mathbb{K}\langle x_1, \dots, x_n\rangle/I$, where I is the two sided ideal generated by elements of the form

$$r_0 + r_1x_1 + \cdots + r_nx_n + x_jx_i - c_{i,j}x_ix_j, \tag{3.1.4}$$

where $r_0, r_1, \dots, r_n$, $c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$.

4. If A is quasi-commutative then $A = \mathbb{K}\langle x_1, \dots, x_n\rangle/I$, where I is the two sided ideal generated by elements as in (iii') and (iv') of the Definition 1.1.5, i.e., elements of the form

$$x_ir - c_{i,r}x_i, \quad x_jx_i - c_{i,j}x_ix_j \tag{3.1.5}$$

where $r \neq 0$, $c_{i,r} \neq 0$, $c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$.

5. If A is semi-commutative then $A = \mathbb{K}\langle x_1, \dots, x_n\rangle/I$, where I is the two sided ideal generated by elements of the form

$$x_jx_i - c_{i,j}x_ix_j \tag{3.1.6}$$

where $c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$.

If otherwise is not assumed, in this section all skew PBW extensions are algebras and extensions of $\mathbb{K}$ (i.e., $R = \mathbb{K}$ in Definition 1.1.1), so $A = \sigma(\mathbb{K})\langle x_1 \dots, x_n\rangle$ is necessarily a constant skew PBW extension.

Proposition 3.1.3. *Let A be a skew PBW extension. If A is pre-commutative then A is nonhomogeneous pre-Koszul.*

Proof. From (3.1.3) and (3.1.4) we have that $A = \mathbb{K}\langle x_1, \dots, x_n\rangle/I$, where I is the two sided ideal generated by elements of the form

$$r_1x_1 + \cdots + r_nx_n + x_jx_i - c_{i,j}x_ix_j. \tag{3.1.7}$$

Then we conclude that A is nonhomogeneous pre-Koszul. □

Proposition 3.1.4. *Let A be a skew PBW extension. If A is semi-commutative then A is pre-Koszul.*

Proof. Since semi-commutative skew PBW extensions are quasi-commutative, and quasi-commutative skew PBW extensions of $\mathbb{K}$ are pre-commutative, then from (3.1.6) and Proposition 3.1.3 we get that A is pre-Koszul. □

Let B be a nonhomogeneous pre-Koszul algebra. One can truncate the relations in (3.1.1) leaving only their homogeneous quadratic parts. Let $B^{(0)}$ be the obtained algebra. Then $B^{(0)}$ is called the *associated pre-Koszul algebra* of B. Note that B is homogeneous if and only if $B^{(0)} \cong B$ as algebras.

Proposition 3.1.5. *Let A be a pre-commutative skew PBW extension, then A^σ is the associated pre-Koszul algebra of A.*

Proof. From Proposition 3.1.3 we have that A is a nonhomogeneous pre-Koszul algebra. By (3.1.3), $A = \mathbb{K}\langle x_1, \dots, x_n\rangle / I$ where I is the two sided ideal generated by elements of the form $c_r + x_i r - c_{i,r} x_i$, $\quad r_1 x_1 + \cdots + r_n x_n + x_j x_i - c_{i,j} x_i x_j$, with $r \neq 0$, c_r, $c_{i,r} \neq 0$, $r_1, \dots, r_n$, $c_{i,j} \neq 0$ elements in $\mathbb{K}$, $1 \leq i, j \leq n$. Since A is constant then $A = \mathbb{K}\langle x_1, \dots, x_n\rangle / I$, where I is the two sided ideal generated by elements of the form $r_1 x_1 + \cdots + r_n x_n + x_j x_i - c_{i,j} x_i x_j$. Then $A^{(0)} = \mathbb{K}\langle x_1, \dots, x_n\rangle / J$, where J is the two sided ideal generated by elements of the form $x_j x_i - c_{i,j} x_i x_j$, where $c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$. Now, by Proposition 2.1.1 there exists a quasi-commutative skew PBW extension A^σ of $\mathbb{K}$ in n variables $z_1, \dots, z_n$ defined by the relations $z_i r = c_{i,r} z_i$, $z_j z_i = c_{i,j} z_i z_j$, for $1 \leq i \leq n$, where $c_{i,r}, c_{i,j}$ are the same constants that define A. Since A is constant, then A^σ is defined by the relations $z_j z_i = c_{i,j} z_i z_j$, with $c_{i,j} \in \mathbb{K} \setminus \{0\}$, $1 \leq i, j \leq n$. Without loss of generality, we can assume that $x_i = z_i$, for $1 \leq i \leq n$. Then,

$$A^\sigma = \mathbb{K}\langle x_1, \dots, x_n\rangle / \langle x_j x_i - c_{i,j} x_i x_j \mid c_{i,j} \in \mathbb{K} \setminus \{0\}, \quad 1 \leq i, j \leq n\rangle = A^{(0)}.$$

□

3.1.2 Koszul algebras and skew PBW extensions

We start this subsection with some homological constructions that will be used in the definitions and results presented below. If B is an augmented algebra with augmentation ideal B_+, then for every left B-module M we have the following resolution by free B-modules, called the *normalized bar-resolution* (see [62]).

$$\begin{aligned}&\widetilde{Bar}_\bullet(B, M) :\\ &\cdots \to B \otimes B_+^{\otimes i+1} \otimes M \to B \otimes B_+^{\otimes i} \otimes M \to \cdots \to B \otimes B_+ \otimes M \to B \otimes M \to 0,\end{aligned} \quad (3.1.8)$$

where

$$\widetilde{Bar}_i(B, M) := B \otimes B_+^{\otimes i} \otimes M \quad (3.1.9)$$

and the differential is given by

$$\begin{aligned}\partial(a_0 \otimes \cdots \otimes a_i \otimes m) := \sum_{s=1}^{i} (-1)^s a_0 \otimes \cdots \otimes a_{s-1} a_s \otimes \cdots \otimes a_i \otimes m \\ + (-1)^{i+1} a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i m.\end{aligned} \quad (3.1.10)$$

We can view $\widetilde{Bar}_i(B, M)$ as a B-module by letting B act on the left. Since $B_+^{\otimes i} \otimes M$ is a B-module, then in particular, this is a $\mathbb{K}$-space, therefore it has a base. Then, tensoring with B, we obtain that each $B \otimes B_+^{\otimes i} \otimes M$ in (3.1.9) is a free B-module. Since we have the canonical homomorphism $B \otimes M \to M$, defined by $b \otimes m \mapsto bm$, $\widetilde{Bar}_{\bullet}(B, M)$ is a free resolution of M. Then,

$$Ext^i_B(M, N) = H^i(Hom_B(\widetilde{Bar}_{\bullet}(B, M), N)). \tag{3.1.11}$$

In fact, the *cobar-complex* is

$$COB^{\bullet}(B, M, N) := Hom_B(\widetilde{Bar}_{\bullet}(B, M), N), \tag{3.1.12}$$

so, $Ext^i_B(M, N)$ can be computed as the *i-th* cohomology of the cobar-complex. Note that

$$COB^i(B, M, N) = Hom_B(B \otimes B_+^{\otimes i} \otimes M, N) \cong Hom_{\mathbb{K}}(B_+^{\otimes i} \otimes M, N). \tag{3.1.13}$$

There is a similar construction of (3.1.8) for non-augmented algebra B, with B_+ replace by B everywhere, called the *non-normalized bar-resolution*.

Let B be a graded algebra and M, N two graded B-modules. The *graded cobar-complex* is:

$$Cob^{\bullet}(B, M, N) := \underline{Hom}_B(\widetilde{Bar}_{\bullet}(B, M), N)) \cong \underline{Hom}_{\mathbb{K}}(B_+^{\otimes \bullet} \otimes M, N). \tag{3.1.14}$$

This identification is compatible with internal gradings.
As in (3.1.11), we can compute these spaces using the graded cobar-complex:

$$\underline{Ext}^i_B(M, N) = H^i(Cob^{\bullet}(B, M, N)). \tag{3.1.15}$$

For an algebra B, a right B-module R, and a left B-module L, we denote by $Tor^B_i(R, L)$ the derived functor of the tensor product over B, so that $Tor^B_0(R, L) = R \otimes_B L$. If B is a graded algebra and the modules R and L are graded then the spaces $Tor^B_i(R, L)$ acquire the corresponding internal graduation induced by the grading of

$$Tor^B_i(R, L) = \bigoplus_{j \in \mathbb{Z}} Tor^B_{i,j}(R, L). \tag{3.1.16}$$

So, $Tor^B_{0,j}(R, L) = (R \otimes_B L)_j$ is spanned by elements $x \otimes y$ where $x \in R_s$ and $y \in L_{j-s}$.

The *bar-complex* is defined as

$$Bar_{\bullet}(R, B, L) := R \otimes_B \widetilde{Bar}_{\bullet}(B, L). \tag{3.1.17}$$

We can compute the spaces $Tor^B_i(R, L)$ as homology of the bar-complex:

$$Tor^B_i(R, L) = H_i(Bar_{\bullet}(R, B, L)) = H_i(R \otimes_B \widetilde{Bar}_{\bullet}(B, L)). \tag{3.1.18}$$

For a vector space V, let $V^{\vee} := Hom_{\mathbb{K}}(V, \mathbb{K})$. There is a duality between bar and cobar-complexes

$$COB^{\bullet}(B, M, R^{\vee}) = Bar_{\bullet}(R, B, M)^{\vee}, \tag{3.1.19}$$

where the left-hand side we use the natural structure of a left B-module on $R^{\vee}$. This leads to the corresponding duality between homology:

$$Ext^i_B(M, R^\vee) = Tor^B_i(R, M)^\vee \tag{3.1.20}$$

In the graded case we have a similar duality using graded duals:

$$Cob^{\boldsymbol{\cdot}}(B, M, R^*) = Bar_{\boldsymbol{\cdot}}(R, B, M)^*. \text{ So, } \underline{Ext}^{i,j}_B(M, R^*) = Tor^B_{i,j}(R, M)^*, \tag{3.1.21}$$

where R^* is graded by $(R^*)_s = (R_{-s})^*$.

A particular case of duality (3.1.21) is

$$\underline{Ext}^{i,j}_B(M, \mathbb{K}) = Tor^B_{i,j}(\mathbb{K}, M)^*. \tag{3.1.22}$$

We denote the relevant bar and cobar-complex by

$$Bar_{\boldsymbol{\cdot}}(B, M) = Bar_{\boldsymbol{\cdot}}(\mathbb{K}, B, M); \quad Cob^{\boldsymbol{\cdot}}(B, M) = Cob^{\boldsymbol{\cdot}}(B, M, \mathbb{K}) = Bar_{\boldsymbol{\cdot}}(B, M)^*.$$

For $M = \mathbb{K}$ we have: $Bar_{\boldsymbol{\cdot}}(B) = Bar_{\boldsymbol{\cdot}}(B, \mathbb{K})$ and $Cob^{\boldsymbol{\cdot}}(B) = Cob^{\boldsymbol{\cdot}}(B, \mathbb{K}) = Bar_{\boldsymbol{\cdot}}(B)^*$, where $Bar_i(B) = B_+^{\otimes i}$.

Let B be a finitely graded algebra generated in degree 1; consider the *Yoneda algebra* of B defined by

$$E(B) := \bigoplus_{i\geq 0} Ext^i_B(\mathbb{K}, \mathbb{K});$$

the Ext groups here are computed in the category of graded B-modules with graded Hom spaces; the product in $E(B)$ is defined in the following way: Let $\{P_i \xrightarrow{d_i} P_{i-1}\}_{i\geq 0}$ be a graded projective resolution of $\mathbb{K}$ that defines the groups $Ext^i_B(\mathbb{K}, \mathbb{K})$, with $P_{-1} := \mathbb{K}$; moreover, let $\overline{f} \in Ext^i_B(\mathbb{K}, \mathbb{K}) = \ker(d^*_{i+1})/Imd^*_i$ with $f \in \ker(d^*_{i+1}) \subseteq Hom_B(P_i, \mathbb{K})$ and $\overline{g} \in Ext^j_B(\mathbb{K}, \mathbb{K}) = \ker(d^*_{j+1})/Imd^*_j$ with $g \in \ker(d^*_{j+1}) \subseteq Hom_B(P_j, \mathbb{K})$, then we define

$$\begin{aligned} Ext^i_B(\mathbb{K}, \mathbb{K}) \times Ext^j_B(\mathbb{K}, \mathbb{K}) &\to Ext^{i+j}_B(\mathbb{K}, \mathbb{K}) \\ (\overline{f}, \overline{g}) &\mapsto \overline{f}\ \overline{g} := \overline{fg'}, \end{aligned}$$

where $g' : P_{i+j} \to P_i$ is defined inductively by the following commutative diagrams:

$$\begin{tikzcd} & P_j \arrow[dl, dotted, "g_0"'] \arrow[d, "g"] \\ P_0 \arrow[r, "d_0"'] & \mathbb{K} \end{tikzcd} \Rightarrow \begin{tikzcd} & P_{j+1} \arrow[dl, dotted, "g_1"'] \arrow[d, "g_0 d_{j+1}"] \\ P_1 \arrow[r, "d_1"'] & Im(d_1) \end{tikzcd} \Rightarrow \cdots \Rightarrow \begin{tikzcd} & P_{j+i} \arrow[dl, dotted, "g':=g_i"'] \arrow[d, "g_{i-1} d_{j+i}"] \\ P_i \arrow[r, "d_i"'] & Im(d_i) \end{tikzcd}$$

Can be proved that this product is well defined, i.e., it does not depend of the projective resolution of $\mathbb{K}$ and the choosing of $g_0, g_1, \ldots, g_{i-1}, g_i$; moreover, $fg' \in \ker(d^*_{i+j+1})$: In fact, from the step $i+1$ in the previous inductive procedure we have that $d_{i+1}g_{i+1} = g_i d_{i+j+1}$, so $fd_{i+1}g_{i+1} = fg_i d_{i+j+1}$, i.e., $0 = d^*_{i+1}(f)g_{i+1} = d^*_{i+j+1}(fg_i)$.

Thus, $E(B)$ is a graded algebra; note that the vector space $Ext^i_B(\mathbb{K}, \mathbb{K})$ is graded

$$Ext^i_B(\mathbb{K},\mathbb{K}) = \bigoplus_{j\geq 0} Ext^{i,j}_B(\mathbb{K},\mathbb{K}),$$

with

$$Ext^{i,j}_B(\mathbb{K},\mathbb{K}) := (Ext^i_B(\mathbb{K},\mathbb{K}))_{-j} := Ext^i_B(\mathbb{K},\mathbb{K}(-j)),$$

so setting $E^{i,j}(B) := Ext^{i,j}_B(\mathbb{K},\mathbb{K})$ we get that

$$E(B) = \bigoplus_{i,j\geq 0} E^{i,j}(B)$$

is a bigraded algebra. For $i \geq 0$, we write

$$E^i(B) := \bigoplus_{j\geq 0} E^{i,j}(B);$$

in particular,

$$E^0(B) = \bigoplus_{j\geq 0} Hom^j_B(\mathbb{K},\mathbb{K}) = \bigoplus_{j\geq 0}(Hom_B(\mathbb{K},\mathbb{K}))_{-j} = \bigoplus_{j\geq 0} Hom_B(\mathbb{K},\mathbb{K}(-j)),$$

with $Hom_B(\mathbb{K},\mathbb{K}(-j)) := \{f \in Hom_B(\mathbb{K},\mathbb{K}) | f(\mathbb{K}_l) \subseteq \mathbb{K}_{l-j}, l \in \mathbb{Z}\}$.

In [62], Chapter 1, Lemma 4.1 is presented the following property of noncommutative graded algebras.

Lemma 3.1.6. *Let B be a locally finite algebra and M a locally finite B-module. Then a graded vector subspace $X \subseteq M$ generates M as an B-module if and only if the composition $X \to M \to \mathbb{K} \otimes_B M$ is surjective.*

Proof. X generates M if and only if the natural morphism $f : B \otimes X \to M$ is surjective. It is clear that this implies surjectivity of the map $\bar{f} : X \to \mathbb{K} \otimes_B M$. Conversely, assume that $\bar{f}$ is surjective and let us show that f is surjective. We can argue by induction in n that the degree n component f_n is surjective. This is true for $n \ll 0$. Assume that f_n is surjective for all $i < n$. Given an element $m \in M_n$ there exists $x \in X_n$ such that $m - x \in B_+$. Hence, by the assumption $m - x$ belongs to the image of f, so m is also in the image of f. □

It follows immediately from the above lemma that inside any generating subspace $X \subseteq M$ one can find a smaller generating subspace $X' \subseteq X$ such that the map $X' \to \mathbb{K} \otimes_B M$ is an isomorphism.

Let B be a locally finite algebra and M a locally finite B-module. A bounded above complex of free graded B-modules

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to 0 \tag{3.1.23}$$

is called *minimal* if all the induced maps $\mathbb{K} \otimes_B P_{i+1} \to \mathbb{K} \otimes_B P_i$ vanish (see [62], page 8). A resolution

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0 \tag{3.1.24}$$

of a graded B-module M by free graded B-modules is called a *linear free resolution* if each P_i is generated in degree i (see [62], page 9). A complex

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots$$

of modules over an augmented algebra B with augmentation ideal B_+ is minimal if, for each n, $\ker(d_n) \subseteq B_+P_n$ (See [80]).

Remark 3.1.7 (See [62])**.** Let M be a locally finite B-module of a locally finite algebra B.

(i) For M, one can construct a minimal free graded B-module resolution of M in the following way. Using Lemma 3.1.6 choose a generating subspace $X_0 \subseteq M$ such that $X_0 \cong \mathbb{K} \otimes_B M$. Consider the corresponding morphism $B \otimes X_0 \to M$ and let $M_1 \subseteq B \otimes X_0$ be its kernel. Choose a generating subspace $X_1 \subseteq M_1$ such that $X_1 \cong \mathbb{K} \otimes_B M_1$. It is easy to see that continuing in this manner we will find a minimal resolution of M consisting of the modules $P_i = B \otimes X_i$. Since any two free resolutions of the same module M are connected by a chain map inducing the identity on M, it follows that a minimal resolution is unique up to a nonunique isomorphism.

(ii) A linear free resolution can be written in the form

$$\cdots \to V_2 \otimes B(-2) \to V_1 \otimes B(-1) \to V_0 \otimes B \to M \to 0,$$

where V_i are vector spaces (of degree zero).

(iii) Linear free resolutions are minimal.

(iv) M admits a linear free resolution if and only if $Tor^B_{i,j}(\mathbb{K}, M) = 0$, for $i \neq j$, if and only if $Ext^{i,j}_B(M, \mathbb{K}) = 0$, for $i \neq j$.

(v) If a module M admits a linear free resolution then it is unique up to unique isomorphism. Indeed, since P_i is generated in degree i, it follows that an endomorphism of $P_.$ is determined by its action on $\mathbb{K} \otimes_B P_.$, i.e., on the spaces $Tor^B_i(\mathbb{K}, M)$. But this action is trivial for any endomorphism of $P_.$ inducing the identity on M.

(vi) B is one-generated if and only if $Ext^{1,j}_B(\mathbb{K}, \mathbb{K}) = 0$ for $j > 1$. B is quadratic if and only if $Ext^{i,j}_B(\mathbb{K}, \mathbb{K}) = 0$ for $j > i$ and $i = 1, 2$ (see [62], Chapter 1 - Corollary 5.3).

Definition 3.1.8. A locally finite pre-Koszul algebra B is called *Koszul* if the following equivalent conditions hold:

(i) $Ext^{i,j}_B(\mathbb{K}, \mathbb{K}) = 0$ for $i \neq j$;

(ii) B is one-generated and $Ext^*_B(\mathbb{K}, \mathbb{K})$ is generated by $Ext^1_B(\mathbb{K}, \mathbb{K})$;

(iii) The module $\mathbb{K}$ admits a *linear free resolution*.

Definition 3.1.9 ([63], page 43)**.** We say that a nonhomogeneous pre-Koszul algebra B is a *nonhomogeneous Koszul algebra* if $B^{(0)}$ is a Koszul algebra.

Remark 3.1.10. Notice that if B is Koszul algebra then B is nonhomogeneous Koszul. Indeed, as B is homogeneous then $B^{(0)} \cong B$ as algebras and so $B^{(0)}$ is Koszul, therefore B is nonhomogeneous Koszul.

Let $B = \mathbb{K}\langle x_1, \dots, x_n\rangle/\langle R\rangle$ be a quadratic algebra with a fixed set of generators $\{x_1, \dots, x_n\}$. For a multi-index $\alpha := (i_1, \dots, i_m)$, where $1 \leq i_k \leq n$, we denote the monomials $x^\alpha := x_{i_1}x_{i_2}\cdots x_{i_m} \in \mathbb{K}\langle x_1, \dots, x_n\rangle$. For $\alpha = \emptyset$ we set $x^\emptyset := 1$. Let us consider the *lexicographical order* on the set of multiindices of length m: $(i_1, \dots, i_m) < (j_1, \dots, j_m)$ if and only if there exists k such that $i_1 = j_1, \dots, i_{k-1} = j_{k-1}$ and $i_k < j_k$. Now let us equip the subspace L_2 with the basis consisting of the monomials $x_{i_1}x_{i_2}$. Let $S^{(1)} := \{1, 2, \dots, n\}$, $S^{(1)} \times S^{(1)}$ the cartesian product, then for $R \subseteq L_2$ we obtain the set $S \subseteq S^{(1)} \times S^{(1)}$ of pairs of indices (l, m) for which the class of $x_l x_m$ in L_2/R is not in the span of the classes of $x_r x_s$ with $(r, s) < (l, m)$, where $<$ denotes the lexicographical order ([62], 4.1-Lemma 1.1). Hence, the relations in B can be written in the following form:

$$x_i x_j = \sum_{\substack{(r,s)<(i,j)\\(r,s)\in S}} c_{ij}^{rs} x_r x_s, \qquad (i,j) \in S^{(1)} \times S^{(1)} \setminus S.$$

Define further $S^{(0)} := \{\emptyset\}$, and for $m \geq 2$,

$$S^{(m)} := \big\{(i_1, \dots, i_m) \mid (i_k, i_{k+1}) \in S,\ k = 1, \dots, m-1\big\}$$

and consider the monomials $\{x_{i_1}\cdots x_{i_m} \in B_m \mid (i_1, \dots, i_m) \in S^{(m)}\}$. Note that these monomials always span B_m as a vector space and the monomials

$$(B, S) := \big\{x_{i_1}\cdots x_{i_m} \mid (i_1, \dots, i_m) \in \bigcup_{m\geq 0} S^{(m)}\big\} \tag{3.1.25}$$

linearly span the entire B.

Definition 3.1.11. With the above notation, we call (B, S) in (3.1.25) a *PBW-basis* of B if they are linearly independent and hence form a $\mathbb{K}$-linear basis. The elements $x_1, \dots, x_n$ are called PBW-*generators* of B. A *PBW algebra* is a quadratic algebra admitting a PBW-basis, i.e., there exists a permutation of $x_1, \dots, x_n$ such that the standard monomials in $x_1, \dots, x_n$ conforms a $\mathbb{K}$-basis of B.

Proposition 3.1.12. *Let A be a semi-commutative skew PBW extension. Then A is a PBW algebra.*

Proof. If $A = \sigma(\mathbb{K})\langle x_1, \dots, x_n\rangle$ is a semi-commutative skew PBW extension, then $A = \mathbb{K}\langle x_1, \dots, x_n\rangle/\langle x_j x_i - c_{i,j}x_i x_j\rangle$ (as in (3.1.6)) is a quadratic algebra with generators $x_1, \dots, x_n$ and relations $x_j x_i - c_{i,j}x_i x_j$, $c_{i,j} \in \mathbb{K} \setminus \{0\}$, $1 \leq i, j \leq n$. Using the above notation we have that for $1 \leq i \leq j \leq n$, the class of $x_i x_j$ is not in the span of the classes of $x_r x_s$ with $(r, s) < (i, j)$, but, the class of $x_j x_i$ is in the span of the class of $x_i x_j$ with $(i, j) < (j, i)$. Therefore $S = \{(i, j) \mid 1 \leq i \leq j \leq n\} = S^{(2)}$ and $S^{(m)} = \{(i_1, \dots, i_m) \mid i_1 \leq i_2 \leq \cdots \leq i_m,\ 1 \leq i_k \leq n\}$ for $m \geq 3$. Then

$$(A, S) = \big\{x_1^{m_1}\cdots x_n^{m_n} \mid m_1, \dots, m_n \geq 0\big\} = \mathrm{Mon}(A) := \big\{x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\big\}.$$

By Definition 1.1.1-(ii), Mon(A) is a $\mathbb{K}$-basis for A and therefore A is a PBW algebra. $\square$

Theorem 3.1.13 ([62], Chapter 4 - Theorem 3.1)**.** *If B is a PBW algebra then B is a Koszul algebra.*

Proof. Let $(B, S) := \left\{x_{i_1} \cdots x_{i_m} \mid (i_1, \ldots, i_m) \in \bigcup_{m\geq 0} S^{(m)}\right\}$ be a PBW-basis with PBW-generator $x_1, \ldots, x_n$. Define a *multiindex-valued filtration* on the algebra B by the rule $F_\alpha(B_m) = \langle X^\beta \mid \beta \leq \alpha\rangle$, where length $\alpha = m$. Since the elements $(x_1, \ldots, x_n)$ generate a PBW-basis of B, the associated graded algebra $Gr(B) = \bigoplus_\alpha F_\alpha/F_{\alpha'}$ (where α' precedes α in the multiindex order, $F_\alpha = F_\alpha(B_m)$) is quadratic. Thus, $Gr(B)$ is a quadratic monomial algebra. As a monomial quadratic algebra is Koszul (see [62], Chapter 2 - Corollary 4.3) then $Gr(B)$ is Koszul. Then, from Remark 3.1.7-(iv), $Tor_{i,j}^{Gr(B)}(\mathbb{K}, \mathbb{K}) = 0$ for $i \neq j$. Thus, the same is true for $Tor_{i,j}^{B}(\mathbb{K}, \mathbb{K})$ (see [62], Chapter 4, proof of Theorem 7.1). By Remark 3.1.7-(iv), we have that $Ext_{i,j}^{B}(\mathbb{K}, \mathbb{K}) = 0$ for $i \neq j$. Then B is Koszul. $\square$

Corollary 3.1.14. *Every semi-commutative skew PBW extension is a Koszul algebra.*

Proof. Let $A = \sigma(\mathbb{K})\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension. From Definition 1.1.1-(ii), A is a left free $\mathbb{K}$-module with basis the set of standard monomials Mon$(A) := \{x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$. Therefore, if A is semi-commutative then A is locally finite. Now, by Proposition 3.1.12 we have that A is a PBW algebra. Theorem 3.1.13 says that A is a Koszul algebra. $\square$

Theorem 3.1.15. *Every pre-commutative skew PBW extension is a nonhomogeneous Koszul algebra.*

Proof. If A is a pre-commutative skew PBW extension then $A = \mathbb{K}\langle x_1, \ldots, x_n\rangle/I$, where I is the two-sided ideal generated by relations of the form

$$x_j x_i - c_{i,j} x_i x_j + \sum_{t=1}^{n} k_t x_t,$$

$c_{i,j} \in \mathbb{K} \setminus \{0\}$, $k_t \in \mathbb{K}$, $1 \leq i, j, t \leq n$ (Remark 3.1.2). By Proposition 3.1.3, A is nonhomogeneous pre-Koszul, therefore from Proposition 3.1.5, $A^{(0)} = A^\sigma = \mathbb{K}\langle x_1, \ldots, x_n\rangle/\langle x_j x_i - c_{i,j} x_i x_j\rangle$ is the associated pre-Koszul algebra of A. Note that A^σ is semi-commutative, so by Corollary 3.1.14, $A^{(0)}$ is a Koszul algebra, i.e., A is a nonhomogeneous Koszul algebra. $\square$

Corollary 3.1.16. *If A is a pre-commutative skew PBW extension then $Gr(A)$ is Koszul.*

Examples 3.1.17. Next we present some examples of Koszul skew PBW extensions. For this purpose we use Corollary 3.1.14 and the classification given in the Table 1.1, Table 1.2 and Table 1.4.

1. Classical polynomial ring over $\mathbb{K}$ (Example 1.2.1).
2. Particular Sklyanin algebra (Example 1.2.5).
3. Multiplicative analogue of the Weyl algebras (Example 1.2.20).

4. Multi-parameter quantum affine n-space (Example 1.2.31).

5. The 3-dimensional skew polynomial algebra with $|\{\alpha, \beta, \gamma\}| = 3$.

6. The Sridharan enveloping algebra of 3-dimensional Lie algebra with $[x, y] = [y, z] = [z, x] = 0$.

Examples 3.1.18. Recall that every Koszul algebra is nonhomogeneous Koszul (Remark 3.1.10), so Examples 3.1.17 are nonhomogeneous Koszul skew PBW extensions. According to Theorem 3.1.15 and the classification given in Table 1.1, Table 1.2 and Table 1.4, the next skew PBW extensions are other examples of nonhomogeneous Koszul algebras.

1. Universal enveloping algebra of a Lie algebras (Example 1.2.6).

2. Additive analogue of the Weyl algebras (Example 1.2.19).

3. Quantum algebras $\mathcal{U}'(so(3, K))$ (Example 1.2.21).

4. Dispin algebra (Example 1.2.22).

5. Woronowicz algebra (Example 1.2.23).

6. q-Heisenberg algebra (Example 1.2.28).

7. Nine types 3-dimensional skew polynomial algebras: (a); (b) items (i), (iii), (v), (vi); (e) items (i), (ii), (iv), (v) (Subsection 1.2.3).

8. Six types of Sridharan enveloping algebra of 3-dimensional Lie algebras (Subsection 1.2.4):

Type	$[x, y]$	$[y, z]$	$[z, x]$
1	0	0	0
2	0	x	0
3	x	0	0
4	0	αy	$-x$
5	0	y	$-(x + y)$
6	z	$-2y$	$-2x$

Note that some particular classes of skew PBW extensions in Example 3.1.17 and Example 3.1.18 represent the same algebra. For example Sridharan enveloping algebra of 3-dimensional Lie algebra of type 1 and the classical polynomial ring $\mathbb{K}[x, y, z]$ are the same algebra.

3.1.3 PBW deformations

Let $L = \mathbb{K}\langle x_1, \ldots, x_n\rangle$ be the free associative algebra (tensor algebra) in n generators $x_1, \ldots, x_n$. Fix a subspace $P \subseteq F_2(L) = \mathbb{K} \bigoplus L_1 \bigoplus L_2$, and let us consider the two-sided ideal $\langle P\rangle$ in L generated by P. Let $B = L/\langle P\rangle$ be a nonhomogeneous quadratic algebra. It inherits a filtration $B_0 \subseteq B_1 \subseteq \cdots B_n \subseteq \cdots$ from L, let $Gr(B)$ the associated graded algebra. Consider the natural projection $\pi : F_2(L) \to L_2$ on the homogeneous component, set $R = \pi(P)$ and consider the quadratic algebra $L/\langle R\rangle$. $L/\langle R\rangle$ is called the *homogeneous version* (or the *induced quadratic*) algebra of B determined by P. We have the natural epimorphism $p : L/\langle R\rangle \to Gr(B)$ (induced by the projection $L \to B$).

Definition 3.1.19 ([17], page 316)**.** With the above notation, a nonhomogeneous quadratic algebra $B := L/\langle P\rangle$ is a *Poincaré-Birkhoff-Witt* (PBW) *deformation* (or satisfies the PBW *property* with respect to the subspace P of $F_2(L)$) of $C := L/\langle R\rangle$ if the natural projection $p : C \to Gr(B)$ is an isomorphism.

Proposition 3.1.20 ([52], Theorem 3.6.4)**.** *Let $B = L/\langle P\rangle$ with $P \subseteq L_1 \bigoplus L_2$, $P \cap L_1 = \{0\}$ and $\{P \otimes L_1 + L_1 \otimes P\} \cap L_2 \subseteq P \cap L_2$. If B is nonhomogeneous Koszul, then the epimorphism p: $L/\langle R\rangle \to Gr(B)$ is an isomorphism of graded algebras, i.e., B is a PBW deformation of $C = L/\langle R\rangle$.*

Corollary 3.1.21 ([52], Corollary 3.6.5)**.** *Let $B = L/\langle P\rangle$, with $P \subseteq L_1 \bigoplus L_2$. If $C = L/\langle R\rangle$ is Koszul, then:*

1. $P \cap L_1 = \{0\} \Leftrightarrow \langle P\rangle \cap L_1 = \{0\}$,
2. $\{P \otimes L_1 + L_1 \otimes P\} \cap L_2 \subseteq P \cap L_2 \Leftrightarrow \langle P\rangle \cap \{L_1 \bigoplus L_2\} = P$.

Let $f \in L$ and let $hh(f)$ the highest homogeneous part of f in the graded algebra L, i.e., if $f = \sum_{i=1}^{m} f_i$ with each f_i in L_i and f_m nonzero, $hh(f) = f_m$. Now if $W \subseteq L$, $hh(W) = \{hh(f) : f \in W\}$. Therefore the homogeneous version of $B = L/\langle P\rangle$ is $L/\langle hh(P)\rangle$. Note that the associated graded algebra may be realized concretely by projecting each element in the ideal $I = \langle P\rangle$ onto its highest homogeneous part, i.e. $Gr(B) = Gr\big(L/\langle P\rangle\big) \cong L/\langle hh(I)\rangle$ (see [48], Theorem 3.2). Also, $Gr(B)$ does not depend on the choice of generators P of the ideal I of relations, so we can associate to every nonhomogeneous quadratic algebra $L/\langle P\rangle$ two different graded versions: $Gr(B) \cong L/\langle hh(I)\rangle$ and $L/\langle hh(P)\rangle = L/\langle \pi(P)\rangle = L/\langle R\rangle$. So, a nonhomogeneous quadratic algebra $L/\langle P\rangle$ is a PBW deformation of $L/\langle R\rangle$ when the above two associate graded versions coincide, and thus both give the associated graded algebra $Gr(B)$. Some authors defined a PBW-deformation of a graded algebra $C = C_0 \bigoplus C_1 \bigoplus C_2 \bigoplus \cdots$ as a filtered algebra B with an ascending filtration $0 \subseteq F_0(B) \subseteq F_1(B) \subseteq F_2(B) \subseteq \cdots$ such that the associated graded algebra $Gr(B)$ is isomorphic to C (see for example [33]).

Example 3.1.22. Let $B = \mathbb{K}\langle x, y\rangle/\langle xy - x, yx - y\rangle$ be a nonhomogeneous quadratic algebra. Let $\bar{x}$ and $\bar{y}$ a coset of x and y modulo $\langle xy - x, yx - y\rangle$ respectively, then $\overline{x}^2 = \overline{xy}^2 = \overline{xyxy} = \overline{xyy} = \overline{xy} = \overline{x}$, similarly $\overline{y}^2 = \overline{y}$ in B, i.e. $x^2 - x$, $y^2 - y \in \langle xy - x, yx - y\rangle := \langle P\rangle$, so $B = \mathbb{K}\langle x, y\rangle/\langle xy - x, yx - y\rangle = \mathbb{K}\langle x, y\rangle/\langle xy - x, yx - y, x^2 - x, y^2 - y\rangle$. The associated graded algebra $Gr(B)$ is trivial in degree two while the homogeneous version of $\mathbb{K}\langle x, y\rangle/\langle xy, yx\rangle$, $\mathbb{K}\langle x, y\rangle/\langle xy, yx\rangle$ is not (as x^2 and y^2 represent nonzero classes). Therefore the algebra B does not satisfy the PBW property with respect to the generating relations $xy - x$ and $yx - y$. B has the PBW property with respect to the generating set $\{xy - x, yx - y, x^2 - x, y^2 - y\}$ since $Gr(B) \cong \mathbb{K}\langle x, y\rangle/\langle xy, yx, x^2, y^2\rangle$ (see [79], Example 4.1).

The definition of a PBW deformation depends on P, if $L/\langle P\rangle$ is a PBW deformation of their homogeneous version then Shepler and Witherspoon in [78] proved that generating relations is always unique up to additive closure over the degree zero component of L, i.e. if B satisfies the PBW property with respect to some generating set P of relations, then the P generator subspace is unique.

Remark 3.1.23. If the algebra $L/\langle P\rangle$ is a PBW deformation of $L/\langle R\rangle$ then it satisfies the following conditions (see [17], Lemma 0.4):

(I) $P \cap F_1(L) = 0$;

(J) $(F_1(L) \cdot P \cdot F_1(L)) \cap F_2(L) = P$.

If a nonhomogeneous quadratic algebras satisfy (I) then the subspace $P \subset F_2(L)$ can be described in terms of two maps $\alpha : R \to L_1$ and $\beta : R \to \mathbb{K}$ as $P = \{x - \alpha(x) - \beta(x) \mid x \in R\}$. If $B = L/\langle P\rangle$ is a PBW deformations of its homogeneous version then P can not have relations in $F_1(L)$, so:

(i) If B is a PBW deformation of some skew PBW extension A, then B is constant.

(ii) The homogeneous version of a skew PBW extension A is the skew PBW extension B such that the conditions (i) and (ii) of Definition 1.1.1 for A are satisfied for B, and the conditions (iii) and (iv) are replaced by $x_jx_i - c_{i,j}x_ix_j = 0$, where $c_{i,j}$ are the same that for A.

(iii) The homogeneous version of a skew PBW extension is Koszul.

For example the homogeneous version for the universal enveloping algebra of a Lie algebra $\mathcal{G}$, $\mathcal{U}(\mathcal{G})$ is the symmetric algebra $\mathbb{S}(\mathcal{G})$.

Proposition 3.1.24. *Let A be a constant skew PBW extension of $\mathbb{K}$. Then A is a PBW deformation of its homogeneous version B.*

Proof. Let A be a constant skew PBW extension of $\mathbb{K}$ then, $x_jx_i - c_{i,j}x_ix_j + r_0 + r_1x_1 + \cdots + r_nx_n$ (as in Definition 1.1.1) are the generated relations of the subspace P, that is, $A = \mathbb{K}\langle x_1, \dots, x_n\rangle/\langle P\rangle$. Then the subspace $\pi(P) = R$ is generated by the relations $x_jx_i - c_{i,j}x_ix_j$, i.e, $\mathbb{K}\langle x_1, \dots, x_n\rangle/\langle R\rangle = B$ is the homogeneous version of A. Now for Theorem 2.1.2, $Gr(A) \cong A^\sigma$ where A^σ is a skew PBW extension of $\mathbb{K}$ in n variables $z_1, \dots, z_n$ defined by the relations $z_jz_i = c_{i,j}z_iz_j$, for $1 \leq i \leq n$. So by Remark 3.1.23, $A^\sigma \cong B$ and therefore $Gr(A) \cong B$, i.e., A is a PBW deformation of B. $\square$

Note that if a skew PBW extension A is not constant then Proposition 3.1.24 fails. Indeed, the homogeneous version of A is the skew PBW extension B with relations $x_jx_i - c_{i,j}x_ix_j = 0$, where $c_{i,j}$ are the same that for A, but $Gr(A)$ is defined by the relations $z_ir = c_{i,r}z_i$, $z_jz_i = c_{i,j}z_iz_j$ (see Theorem 2.1.2 and Proposition 2.1.1), so $Gr(A) \ncong B$.

Let $L/\langle P\rangle$ be a nonhomogeneous quadratic algebra. Take $R = p(P) \subseteq L_2$ and consider the corresponding quadratic algebra $B = L/\langle R\rangle$. The main theorem of [17] establishes that if B is a Koszul algebra then conditions (I) and (J) in Remark 3.1.23 imply that the algebra $L/\langle P\rangle$ is a PBW deformation of B.

Example 3.1.25. In the following examples we consider skew PBW extensions of $\mathbb{K}$, some of which are quadratic algebras and the other ones are nonhomogeneous quadratic algebras.

1. Quadratic algebras: Classical polynomial rings, particular Sklyanin algebras, multiplicative analogue of the Weyl algebras and multi-parameter quantum affine n-spaces.

2. Nonhomogeneous quadratic algebras: Universal enveloping algebras of a Lie algebras, additive analogue of the Weyl algebras, quantum algebras, Dispin algebras, Woronowicz algebras, q-Heisenberg algebras, 3-dimensional skew polynomial algebras and Sridharan enveloping algebras of 3-dimensional Lie algebras.

For nonhomogeneous quadratic algebras we have:

(a) The homogeneous version of a universal enveloping algebra of a Lie algebra is a skew PBW extension of $\mathbb{K}$ with relations $x_ix_j - x_jx_i = 0$ for $1 \leq i, j \leq n$, i.e., the classical polynomial ring. Therefore, $\mathcal{U}(\mathcal{G})$ is a PBW deformation of the classical polynomial ring (symmetric algebra $\mathbb{S}(\mathcal{G})$) as it was well-known.

(b) The homogeneous version of an additive analogue of a Weyl algebra is the algebra B generated by the variables $x_1, \dots, x_n$, $y_1, \dots, y_n$ subject to the relations $x_jx_i = x_ix_j$, $1 \leq i, j \leq n$; $y_jy_i = y_iy_j$, $1 \leq i, j \leq n$; $y_ix_j = x_jy_i$, $i \neq j$; $y_ix_i = q_ix_iy_i$, $1 \leq i \leq n$, where $q_i \in \mathbb{K} \setminus \{0\}$. $A_n(q_1, \dots, q_n)$ is a PBW deformation of B.

(c) The homogeneous version of a quantum algebra is the algebra generated by I_1, I_2, I_3, subject to relations $I_2I_1 - qI_1I_2 = 0$; $I_3I_1 - q^{-1}I_1I_3 = 0$, $I_3I_2 - qI_2I_3 = 0$, where $q \in \mathbb{K} \setminus \{0\}$. In this way $\mathcal{U}'(\mathfrak{so}(3, \mathbb{K}))$ is a PBW deformation of B.

(d) The homogeneous version of a Dispin algebra is the algebra B generated by x, y, z over $\mathbb{K}$ satisfying the relations $yz - zy = 0$, $zx + xz = 0$, $xy - yx = 0$. Thus, $\mathcal{U}(osp(1, 2))$ is a PBW deformation of B.

(e) The homogeneous version of a Woronowicz algebra is the algebra B generated by x, y, z, subject to the relations $xz - \nu^4zx = 0$; $xy - \nu^2yx = 0$; $zy - \nu^4yz = 0$, where $\nu \in \mathbb{K} \setminus \{0\}$ is not a root of unity. We have $\mathcal{W}_\nu(\mathfrak{sl}(2, \mathbb{K}))$ is a PBW deformation of B.

(f) The homogeneous version of a 3-dimensional skew polynomial algebra is the algebra $\mathcal{B}$ generated by the variables x, y, z, restricted to relations $yz - \alpha zy = 0$, $zx - \beta xz = 0$, $xy - \gamma yx = 0$, such that $\alpha, \beta, \gamma \in \mathbb{K}^*$. A 3-dimensional skew polynomial algebra is a PBW deformation of $\mathcal{B}$.

(g) The homogeneous version of a Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ of a 3-dimensional Lie algebra $\mathcal{G}$ is the algebra $B = T(\mathcal{G})/J$, where $T(\mathcal{G})$ is the tensor algebra of $\mathcal{G}$ and I is the two-side ideal of $T(\mathcal{G})$ generated by the elements $(x \otimes y) - (y \otimes x)$ for all $x, y \in \mathcal{G}$, i.e., the symmetric algebra $\mathbb{S}(\mathcal{G})$. Then a Sridharan enveloping algebra is a PBW deformation of the symmetric algebra.

Proposition 3.1.26. *If A is a PBW deformation of a skew PBW extension B, then B is Koszul.*

Proof. Let $A = \sigma(\mathbb{K})\langle x_1, \dots, x_n\rangle$ be a PBW deformation of B. Then

$$A = \mathbb{K}\langle x_1, \dots, x_n\rangle / \langle x_jx_i - c_{i,j}x_ix_j + k_0 + k_1x_1 + \cdots + k_nx_n\rangle,$$

with $c_{i,j} \in \mathbb{K} \setminus \{0\}$, $k_l \in \mathbb{K}$, $1 \leq i, j \leq n$, $0 \leq l \leq n$ and $B \cong \mathbb{K}\langle x_1, \dots, x_n\rangle / \langle x_j x_i - c_{i,j} x_i x_j\rangle$. Therefore B is a semi-commutative skew PBW extension. Then by Corollary 3.1.14 we have that B is a Koszul algebra. □

3.2 Koszul property for graded skew PBW extensions

In the literature there exist examples of Koszul algebras which are skew PBW extensions of an algebra $R \neq \mathbb{K}$. For example, the Jordan plane is an Artin-Schelter regular algebra of dimension two and therefore is a Koszul algebra, but the Jordan plane is a skew PBW extension of $\mathbb{K}[x]$. Therefore, the results given in the previous section (see also [88]) do not apply in this case. In this section we use graded skew PBW extensions (Definition 2.2.2) and study the Koszul property for these algebras.

Remark 3.2.1. Note that pre-Koszul algebras are graded algebras. So we can rewrite the Definition 3.1.8 as follows: A locally finite algebra $B = \mathbb{K} \bigoplus B_1 \bigoplus B_2 \bigoplus \cdots$ is called *Koszul* if the following equivalent conditions hold (see [62], Chapter 2, Definition 1):

(i) $Ext_B^{i,j}(\mathbb{K}, \mathbb{K}) = 0$ for $i \neq j$;

(ii) B is one-generated and $E(B)$ is generated by $E^{1,1}(B)$;

(iii) The module $\mathbb{K}$ admits a *linear free resolution*, i.e., a resolution by free B-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{K} \to 0$$

such that P_i is generated in degree i.

Proposition 3.2.2. *Let B be a graded Ore extension of R. Then B is Koszul if and only if R is Koszul.*

Proof. See [61], Corollary 1.3. □

Proposition 3.2.3. *The graded iterated Ore extension $A := R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is Koszul if and only if R is Koszul.*

Proof. From Proposition 3.2.2 the result is clear. □

Proposition 3.2.4. *Let A be a graded quasi-commutative skew PBW extension of R. Then R is a Koszul algebra if and only if A is Koszul.*

Proof. If A is a graded quasi-commutative bijective skew PBW extension of R, then by Proposition 2.1.3 A is isomorphic to a graded iterated Ore extension wherein each endomorphism is bijective. Then by Proposition 3.2.3, R is Koszul if and only if A is Koszul. □

3.2.1 PBW algebras

In Subsection 3.1.2 we use PBW algebras to study the Koszul property for skew PBW extensions over fields. The homogenized enveloping algebra $\mathcal{A}(\mathcal{G})$ (Example 1.2.40) is a PBW algebra where the elements $(zx_1, x_2, \dots, x_n)$ are PBW-generators and the corresponding PBW-basis consists of the monomials $(z^k x_1^{\alpha_1} \cdots x_n^{\alpha_n})$ with $k, \alpha_1, \dots, \alpha_n \in \mathbb{N}$. Note that $\mathcal{A}(\mathcal{G})$ is not a skew PBW extension of $\mathbb{K}$, but is a graded skew PBW extension of $\mathbb{K}[z]$. Therefore, in this subsection we use PBW algebras for study the Koszul property of graded skew PBW extensions, not necessarily over fields.

Theorem 3.2.5. *Let A be a graded skew PBW extension of a finitely presented algebra R. If R is a PBW algebra then A is a PBW algebra.*

Proof. Let R be a finitely presented PBW algebra with PBW generators $t_1, \dots, t_m$. Then by Proposition 2.3.1, $R = L^t/I$, where $L^t = \mathbb{K}\langle t_1, \dots, t_m\rangle$ and

$$I = \langle r_1, \dots, r_s\rangle \tag{3.2.1}$$

is a two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m\rangle$, generated by a finite set $r_1, \dots, r_s$ of homogeneous polynomials in $\mathbb{K}\langle t_1, \dots, t_m\rangle$ of degree two. Let

$$(R, S_t) := \{t_{i_1} \cdots t_{i_v} \mid (i_1, \dots, i_\nu) \in \bigcup_{p\geq 0} S_t^{(p)}\} \tag{3.2.2}$$

be a PBW basis of R, with $S_t^{(p)} = \{(i_1, i_2, \dots, i_p) \mid (i_k, i_{k+1}) \in S_t,\ k = 1, \dots, p-1\}$, $S_t^{(1)} := \{1, 2, \dots, m\}$ and $S_t \subseteq S_t^{(1)} \times S_t^{(1)}$ is the set of pairs of indices (i_μ, i_ν) for which the class of $t_{i_\mu} t_{i_\nu}$ in L_2^t/P (where P is the space of relations $r_1, \dots, r_s$) is not in the span of the classes of $t_r t_s$ with $(r, s) < (i_\mu, i_\nu)$. For $1 \leq d \leq s$,

$$r_d = t_{i_d} t_{j_d} = \sum_{\substack{(r_d,q_d)<(i_d,j_d)\\ (r_d,q_d)\in S_t}} c_{i_d j_d}^{r_d q_d} t_{r_d} t_{q_d}, \qquad (i_d, j_d) \in S_t^{(1)} \times S_t^{(1)} \setminus S_t. \tag{3.2.3}$$

Let $A = \sigma(R)\langle x_{m+1}, \dots, x_{m+n}\rangle$ be a graded skew PBW extension of R. As $R \subseteq A$, we have that $A = \mathbb{K}\langle t_1, \dots, t_m, x_{m+1}, \dots, x_{m+n}\rangle/J$ where

$$J = \langle r_1, \dots, r_s,\ f_{hk},\ g_{ji} \mid m+1 \leq i, j, h \leq m+n,\ 1 \leq k \leq m\rangle \tag{3.2.4}$$

is the two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m, x_{m+1}, \dots, x_{m+n}\rangle$ generated by a set $r_1, \dots, r_s,\ f_{hk},\ g_{ji}$ where $r_1, \dots, r_s$ are as in (3.2.1); let

$$f_{hk} := x_{m+h} t_k - \sigma_{m+h}(t_k) x_{m+h} - \delta_{m+h}(t_k) \tag{3.2.5}$$

with σ_{m+h} and δ_{m+h} as in Proposition 1.1.4;

$$g_{ji} := x_{m+j} x_{m+i} - c_{i,j} x_{m+i} x_{m+j} - (r_{0_{j,i}} + r_{1_{j,i}} x_{m+1} + \cdots + r_{n_{j,i}} x_{m+n}) \tag{3.2.6}$$

is as in (1.1.2) of Definition 1.1.1. As A is graded skew PBW extension then it is a quadratic algebra, since $r_1, \dots, r_s,\ f_{hk},\ g_{ji}$ are homogeneous polynomials of degree two in $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle$. Now, let $S_{tx}^{(1)} := \{1, \dots, m, m+1, \dots, m+n\}$. From the relations

(3.2.5) we obtain the set $S_{tx} := \{(k,l) \mid 1 \leq k \leq m, m+1 \leq l \leq m+n\}$. From the relations (3.2.6) we obtain the set $S_x := \{(m+i, m+j) \mid 1 \leq i \leq j \leq n)\}$. From Definition 1.1.1, we have that $R \subseteq A$ and A is a left free R-module. Then, for the algebra A, we have that

$$S^{(p)} = \{(i_1, \dots, i_k, i_{k+1}, \dots, i_p) \mid (i_1, \dots, i_k) \in S_t^{(k)} \text{ and } i_{k+1} \leq \cdots \leq i_p\}.$$

So,

$$(A, S) := \{t_{i_1} \cdots t_{i_k} x_{i_{k+1}} \cdots x_{i_p} \mid (i_1, \dots, i_k, i_{k+1}, \dots, i_p) \in \bigcup_{p \geq 0} S^{(p)}\} \tag{3.2.7}$$

span A as a vector space. As $(R, S_t) := \{t_{i_1} \cdots t_{i_v} \mid (i_1, \dots, i_\nu) \in \bigcup_{p\geq 0} S_t^{(p)}\}$ is a $\mathbb{K}$-basis for R and A is a left free R-module, with basis the basic elements

$$\begin{aligned} \{x^\alpha = x_{m+1}^{\alpha_{m+1}} \cdots x_{m+n}^{\alpha_{m+n}} \mid \alpha = (\alpha_{m+1}, \dots, \alpha_{m+n}) \in \mathbb{N}^n\} \\ = \{x_{i_{k+1}} \cdots x_{i_p} \mid m+1 \leq i_{k+1} \leq \cdots \leq i_p \leq m+n\} \cup \{1\}, \end{aligned}$$

then (A, S) is a PBW basis of A. Therefore A is a PBW algebra.

□

Remark 3.2.6. If in the free algebra $\mathbb{K}\langle x_1, \dots, x_n\rangle$ we fix the set $\{1, 2, \dots, n\}$, we implicitly understand that $x_1 < x_2 < \cdots < x_n$. For example, for $A = \mathbb{K}\langle x, y, z\rangle/\langle z^2 - xy - yx, zx - xz, zy - yz\rangle$ with $x < y < z$, i.e., $x = x_1$, $y = x_2$, $z = x_3$, we have that $S^{(1)} = \{1, 2, 3\}$, $S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\} = S^{(2)}$. Note that (A, S) is not a $\mathbb{K}$-basis for A. Indeed, $(2,1,1), (1,1,2) \in S^{(3)}$ and therefore the classes (nonzero) of $yx^2, x^2y \in (A, S)$, but $yx^2 - x^2y = yx^2 + xyx - x^2y - xyx = (xy+yx)x - x(xy+yx) = z^2x - xz^2 = 0$, since $xz = zx$ in A. But observe that $A = \mathbb{K}\langle x, y, z\rangle/\langle z^2 - xy - yx, zx - xz, zy - yz\rangle \cong \sigma(\mathbb{K}[z]\langle x, y\rangle$ is a graded skew PBW extension of the PBW algebra $\mathbb{K}[z]$ and in this case Theorem 3.2.5 fails. So it is important the order of the generators of the free algebra L as in the proof of Theorem 3.2.5; for the graded skew PBW extension $A = \sigma(\mathbb{K}[z]\langle x, y\rangle$ we have that $A = \mathbb{K}\langle z, x, y\rangle/\langle z^2 - xy - yx, zx - xz, zy - yz\rangle$, i.e., $z = x_1 < x = x_2 < y = x_3$. In this case we write the relations as $yx = -xy + z^2; xz = zx; yz = zy$, whereby $(3,2)$, $(2,1)$, $(3,1) \notin S$. So, $S = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$, $S^{(p)} = \{(i_1, i_2, \dots, i_p) \mid i_1 \leq i_2 \leq \cdots \leq i_p\}$ and $(A, S) = \{z^{\alpha_1} x^{\alpha_2} y^{\alpha_3} \mid \alpha_1, \alpha_2, \alpha_3 \geq 0\}$ is a PBW base for A.

Corollary 3.2.7. *Let A be a graded skew PBW extension of a finitely presented algebra R. If R is a PBW algebra then A is a Koszul algebra.*

Proof. This result follows immediately from Theorem 3.2.5 and Theorem 3.1.13. □

Example 3.2.8. Let $R = \mathbb{K}[t_1, \dots, t_m]$ be the classical polynomial ring. Then, from Corollary 3.2.7 every graded skew PBW extension of R is Koszul. Therefore, Examples 2.2.7 are Koszul algebras. Also, by Remark 3.2.6 and Corollary 3.2.7, we have that $A = \mathbb{K}\langle z, x, y\rangle/\langle z^2 - xy - yx, zx - xz, zy - yz\rangle$ is a Koszul algebra. Note that $A = \mathbb{K}\langle z, x, y\rangle/\langle z^2 - xy - yx, zx - xz, zy - yz\rangle = \sigma(\mathbb{K}[z])\langle x, y\rangle = \mathbb{K}[z][x; \sigma_1, \delta_1][y; \sigma_2, \delta_2]$ is a graded iterated Ore extension, where $\sigma_1(z) = z$, $\sigma_2(x) = -x$, $\delta_1(z) = 0$ and $\delta_2(x) = z^2$. So, we also can use the Proposition 3.2.3 to guarantee that A is Koszul.

Remark 3.2.9. (i) Some of the graded skew PBW extensions had already been presented by other authors as Koszul algebras using other characterizations. For example:

1. The polynomial algebra $B = \mathbb{K}[x, y]$ is a Koszul algebra (see [12], Proposition 5.2).
2. Let $B = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial algebra in n variables. Then B is a Koszul algebra (see [54], Example 1.6).
3. Let $B = \mathbb{K}\langle x, y, z\rangle/\langle yz - zy, zx - xz, xy - yx + z^2\rangle$ which is of type S_1' in the classification of three-dimensional Artin-Schelter regular algebras given in [3]. By Proposition 5.2 of [12] B is Koszul.
4. For any $n \geq 2$, let B be a non-degenerate non-commutative quadric graded algebra in n variables $x_1, \ldots, x_n$ of degree 1. Let z be an extra variable of degree 1. Let B be an algebra defined by a non-zero cubic potential w in the variables $x_1, \ldots, x_n$, z. Assume that the graded algebra B is isomorphic to a skew polynomial algebra $B[z; \sigma; \delta]$ over B in the variable z, defined by a 0-degree homogeneous automorphism σ of B and a 1-degree homogeneous $\sigma-$derivation δ of B. Then B is Koszul (see [13], Proposition 4.1).
5. The algebra $B = \mathbb{K}\langle x, y, z\rangle/\langle \alpha\beta xy + a\alpha\beta yx, \alpha zx + axz, yz + a\beta zy\rangle$ is Koszul (see [12], Proposition 5.2).
6. The quantum plane $B = \mathbb{K}\langle x, y\rangle/\langle yx - cxy\rangle \quad (c \neq 0)$ is a Koszul algebra (see [12], Proposition 5.2).
7. The Jordan plane $B = \mathbb{K}\langle x, y\rangle/\langle yx - xy - x^2\rangle$ is a quadratic algebra and $\langle yx - xy - x^2\rangle$ is a principal ideal, it follows that B is Koszul (see [22]).
8. Smith in [80], Proposition 12.1, showed that the homogenized enveloping algebra $\mathcal{A}(\mathcal{G})$ is Koszul.

(ii) The converse of Corollary 3.2.7 is false. Indeed, the algebra

$$R = \mathbb{K}\langle x, y, z\rangle/\langle x^2 + yz, x^2 + azy \mid a \neq 0, 1\rangle$$

with $\mathbb{K}$ an algebraically closed field, is Koszul (see [62], Example of page 84). So, an associated graded Ore extension $A := R[u]$ is a Koszul algebra (Proposition 3.2.2). Now, for $x < y < z$, $S^{(1)} = \{1, 2, 3\}$, $S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3)\} = S^{(2)}$. Note that $(1, 1, 2)$, $(2, 1, 1) \in S^{(3)}$ and therefore the classes (nonzero) of x^2y, $yx^2 \in (R, S)$, but $ax^2y + yx^2 = ayzy + y(-a)zy = ayzy - ayzy = 0$. Thus, (R, S) is not a $\mathbb{K}$-basis for R, i.e., R is not a PBW algebra.

(iii) Let R be as in the part (ii) above. Note that $A = R[u] \cong \mathbb{K}\langle x, y, z, u\rangle/\langle x^2 + yz, x^2 + azy, ux - xu, uy - yu, uz - zu\rangle$, with $a \neq 0, 1$. So, $x < y < z < u$ and $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 4)\}$. Therefore $(1, 1, 2)$, $(2, 1, 1) \in S^{(3)}$ and x^2y, yx^2 are nonzero monomials in A, but $a^{-1}yx^2 + x^2y = yzy - yzy = 0$. Then (A, S) is not a PBW basis, i.e., A is not a PBW algebra. So, the condition that A is a graded skew PBW extension of the Koszul algebra R does not imply that A is PBW algebra.

(iv) With the above reasoning we have that not any graded skew PBW extension is a PBW algebra.

(v) We have also that not every graded skew PBW extension are Koszul. Indeed, let $R = \mathbb{K}\langle x, y\rangle/\langle y^2 - xy, y^2\rangle$ be a quadratic non-Koszul algebra ([19], page 10), then $R[u]$ is a non-Koszul associate graded Ore extension of R, which is also a graded skew PBW extension.

3.2.2 Lattices

A *lattice* is a discrete set Ω endowed with two idempotent (i.e., $a \cdot a = a$) commutative, and associative binary operations $\wedge, \vee : \Omega \times \Omega \to \Omega$ satisfying the following *absorption identities*: $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$. A lattice is called *distributive* if it satisfies the following distributivity identity: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. We write $a \leq b$, if the equivalent conditions $a \wedge b = a$ or $a \vee b = b$ hold. A sublattice of a lattice Ω is a subset closed under both operations $\wedge$ and $\vee$. The *sublattice generated* by a subset $X \subseteq \Omega$, $[X]$, consists of all elements of Ω that can be obtained from the elements of X using these operations. Note that a finitely generated distributive lattice is finite. A lattice is called *modular*, if the absorption identities hold for any triple of its elements a, b, c such that $a \geq c$. Equivalently, one should have $a \geq c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$. Let $[1, z]$ be the set $\{1, \ldots, z\}$ ordered in increasing form, $I \subseteq [1, z]$ means that $I \subset \{1, \ldots, z\}$ is also ordered in increasing form.

Proposition 3.2.10. *Let $u, x_1, \ldots, x_z$ be elements of a modular lattice Ω. Then the set $u, x_1, \ldots, x_z$ is distributive if and only if both sets $x_i, \ldots, x_z$ and $u \vee x_1, \ldots, u \vee x_z$ are distributive and the equation $u \vee \left(\bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I} \left(u \vee x_i\right)$ holds for any subset $I \subseteq [1, z]$.*

Proof. See [62], Chapter 1, Corollary 6.5. □

Let W be a vector space. The set Ω_W of all its linear subspaces is a lattice with respect to the operations of sum and intersection. This lattice is modular but not distributive: the equation $(X+Y) \cap Z = X \cap Z + Y \cap Z$ does not hold in general. Given $X_1, \ldots, X_z$ subspaces of a vector space W, we may consider the *sublattice* of subspaces of W *generated* by $X_1, \ldots, X_z$ by the operations of intersection and summation. We will say that a *collection of subspaces $X_1, \ldots, X_z \subseteq W$ is distributive* if it generates a distributive lattice of subspaces of W.

Proposition 3.2.11 ([62], Proposition 1-7.1)**.** *Let W be a vector space and $X_1, \ldots, X_z \subseteq W$ be a collection of its subspaces. Then the following conditions are equivalent:*

(i) *The collection $X_1, \ldots, X_z$ is distributive.*

(ii) *There exists a direct sum decomposition $W = \bigoplus_{j\in J} W_j$ of the vector space W such that each of the subspaces X_i is the sum of a set of subspaces W_j.*

(iii) *There exists a basis $\mathcal{B} = \{w_i \mid i \in I\}$ of the vector space W such that each of the subspaces X_i is the linear span of a set of vectors w_i.*

(iv) *There exists a basis $\mathcal{B}$ of the vector space W such that $\mathcal{B} \cap X_i$ is a basis of the subspace X_i, for each $1 \leq i \leq z$ ([6], Lemma 1.2).*

Proof. (ii)$\Rightarrow$ (i), (ii) $\Leftrightarrow$ (iii) and (iii) $\Leftrightarrow$ (iv) are clear. Let us prove (i) $\Rightarrow$ (ii). Let $[1,z]$ be the set $\{1,2,\dots,z\}$ ordered in increasing form. For every subset $T \subseteq [1,z]$ let us choose a subspace $W_T \subseteq \bigcap_{i\in T} X_i$ such that

$$\bigcap_{i\in T} X_i = W_T \bigoplus \Big((\bigcap_{i\in T} X_i) \cap (\sum_{j\notin T} X_j) \Big).$$

Then we claim that $X_i = \sum_{i\in T} W_T$. More generally, we can prove by descending induction in a subset $S \subseteq [1,z]$ that $\sum_{S\subseteq T} W_T = \bigcap_{i\in S} X_i$. Indeed, suppose this is true for all strictly larger sets $S' \supset S$. Then using the definition of W_S and the distributivity identity one can easily derive the above equation for S. Note that in the case $S = \emptyset$ this equation states that the subspaces W_T generate W. It remains to prove that the subspaces (W_T), $T \subseteq [1,z]$, are linearly independent. Assume that $\sum_s w_{T_s} = 0$ for a set of nonzero vectors $w_{T_s} \in W_{T_s}$, where all subsets T_s are distinct. Choose s_0 such that the subset $T_{s_0} \subseteq [1,z]$ does not contain any other subsets T_s. Then we have

$$\sum_{s\neq s_0} W_{T_s} \subseteq \sum_{s\neq s_0} \bigcap_{j\in T_s} X_j \subseteq \sum_{j\notin T_{s_0}} X_j.$$

This is a contradiction since $W_{T_{s_0}}$, does not intersect $\sum_{j\notin T_{s_0}} X_j$ by the definition. $\square$

Proposition 3.2.12 ([62], Proposition 1-7.2)**.** *Let W be a vector space and $X_1,\dots,X_z \subseteq W$ be a collection of subspaces such that any proper subset $X_1,\dots,\widehat{X}_k,\dots,X_z$ is distributive ($\widehat{X}_k$ means delete). Then the collection $X_1,\dots,X_z$ is distributive if and only if the following complex of vector spaces $\mathcal{B}_{\textbf{.}}(W;X_1,\dots,X_z)$*

$$W \to \bigoplus_t W/X_t \to \cdots \to \bigoplus_{t_1<\cdots<t_{z-t}} W/\sum_{s=1}^{z-i} X_{t_s} \to \cdots \to W/\sum_s X_s \to 0 \tag{3.2.8}$$

is exact everywhere except for the leftmost term.

Proof. There is a natural exact sequence of complexes

$$\begin{aligned} 0 \to \mathcal{B}_{\textbf{.}}(W/X_1;(X_2+X_1)/X_1,\dots,(X_z+X_1)/X_1) &\to \mathcal{B}_{\textbf{.}}(W;X_1,\dots,X_z) \\ &\to \mathcal{B}_{\textbf{.}}(W;X_2,\dots,X_z)[-1] \to 0, \end{aligned}$$

where $[-1]$ denotes the shift of homological degree. Since we assume that any proper subcollection is distributive, the third complex is exact in homological degree $\neq z$. Note that $H_z\mathcal{B}_{\textbf{.}}(W;X_1,\dots,X_z) = X_1\cap\cdots\cap X_z$. It follows easily that the second complex is exact at the desired terms if and only if the first complex is exact in degree $\neq z-1$ and the connecting map

$$X_2\cap\cdots\cap X_z \to (X_2+X_1)\cap\cdots(X_z+X_1)/X_1$$

is surjective. Using induction in z we conclude that exactness of the first complex is equivalent to distributivity of the collection $X_1+X_2,\dots,X_1+X_z$. It remains to apply Proposition 3.2.10, with u corresponding to X_1. $\square$

Let $B = \mathbb{K}\langle x_1, \dots, x_n\rangle/I = L/I$, where I is a two-sided ideal generated by homogeneous elements of L_2 and let $L_+ = \bigoplus_{p>0} L_p$. The *lattice associated* to B, $\Omega(B)$ is the lattice generated by $\{L_+^\lambda I^\mu L_+^\nu \mid \lambda, \mu, \nu \geq 0\} \subseteq \{$ Subspaces of $\mathbb{K}\langle x_1, \dots, x_n\rangle\}$, where $I^0 = \mathbb{K}\langle x_1, \dots, x_n\rangle$, $I^1 = I$; $I^2 = \{\sum xy \mid x, y \in I\}$, $\cdots$ (see [6]).

Lemma 3.2.13 ([62], Theorem 2-4.1)**.** *A quadratic algebra $B = \mathbb{K}\langle x_1, \dots, x_n\rangle/\langle P\rangle$ is Koszul if and only if for all $k \geq 0$, the collection of subspaces of $\mathbb{K}\langle x_1, \dots, x_n\rangle$,*

$$X_i := L_{i-1} P L_{k-i-1} \subset L_k, \; i = 1, \dots, k-1 \tag{3.2.9}$$

is distributive. More precisely, $Ext_B^{i,j}(\mathbb{K}, \mathbb{K}) = 0$ for all $i < j \leq n$, if and only if the collection $\{X_1, \dots, X_{k-1}\}$ in L_k is distributive.

Proof. This follows from Proposition 3.2.12 by observing that the complex

$$\mathcal{B}_{\bullet}(L_k; X_1, \dots, X_{k-1})$$

can be identified with the degree-k component of the bar-complex $Bar_{\bullet}(B)$. $\square$

Lemma 3.2.14 ([6], Lemma 2.3)**.** *Let $R = \mathbb{K}\langle t_1, \dots, t_n\rangle/I$ be a quadratic algebra, let $\Omega(R)$ the lattice associated to R and let*

$$B = \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle/\langle I\rangle,$$

where $\langle I\rangle$ is the two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle$ generated by I. Then $\Omega(R)$ is distributive if and only if $\Omega(B)$ is distributive.

Proof. We put $L_t := \mathbb{K}\langle t_1, \dots, t_n\rangle$, $L := \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle$ and $L_x := \mathbb{K}\langle x_1, \dots, x_n\rangle$. Let $(L_t)_+$ the two-sided ideal of L_t generated by $\{t_1, \dots, t_m\}$, L_+ the two-sided ideal of L generated by $\{t_1, \dots, t_m, x_1, \dots, x_n\}$ and $(L_x)_+$ the two-sided ideal of L_x generated by $\{x_1, \dots, x_n\}$. Let $\lambda, \mu, \nu \geq 0$. If $\lambda + \mu + \nu > 0$ then

$$(L_t)_+^\lambda I^\mu (L_t)_+^\nu \subseteq (L_t)_+ \subset \mathbb{K}\langle t_1, \dots, t_m\rangle = (L_t)_+^0 I^0 (L_t)_+^0.$$

Thus $\Omega(R)$ is distributive if and only if

$$\Omega^+(R) = [\{(L_t)_+^\lambda I^\mu (L_t)_+^\nu \mid \lambda, \mu, \nu \geq 0 \text{ and } \lambda + \mu + \nu > 0\}]$$

is distributive. Similarly,

$$\Omega^+(B) = [\{L_+^\lambda \langle I\rangle^\mu L_+^\nu \mid \lambda, \mu, \nu \geq 0 \text{ and} \lambda + \mu + \nu > 0\}]$$

is distributive.
We identify L_t, $(L_t)_+$, I and $(L_x)_+$ with their images in $L = \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle$. We may give $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle$ a vector space graduation (which is not compatible with the product) by letting

$$L_0 := L_t, \quad L_1 := L_t L_+ L_t, \quad L_2 := L_t (L_x)_+ (L_t)_+ (L_x)_+ L_t,$$

etcetera. Then $(L_+)^\lambda(\langle I\rangle)^\mu(L_+)^\nu$ is a graded subspace of L, for any $\lambda,\mu,\nu\geq 0$; thus so is any $W\in\Omega^+(B)$. Furthermore, for $W\in\Omega^+(B)$ and $l\geq 0$ we have $W\cap(L_t)_l\in\Omega_l$, where

$$\Omega_l := [U_1(L_x)_+U_2(L_x)_+U_3\cdots L_+U_{l+1}\in\Omega^+(R)\}] \xrightarrow{\sim} \prod_{m=1}^{l+1}\Omega^+(R),$$

isomorphism of lattices, $\prod_{m=1}^{l+1}\Omega^+(R)$ the cartesian product, with sum and intersection operating component-wise. Thus

$$\Omega^+(B)\subseteq\prod_{l\geq 0}\Omega_l \xrightarrow{\sim} \prod_{l,m}\Omega^+(R),$$

whence $\Omega^+(B)$ is distributive if $\Omega^+(R)$ is.
On the other hand there is a surjective lattice homomorphism $\pi_0:\ \Omega^+(B)\twoheadrightarrow\Omega^+(R)$ defined by $\pi_0(W)=W\cap(L_t)_0$, whence $\Omega^+(R)$ is distributive if $\Omega^+(B)$ is. □

Proposition 3.2.15. *Let $B=\mathbb{K}\langle x_1,\dots,x_n\rangle/I$ be a quadratic algebra. $\Omega(B)$ is distributive if and only if for all $i,j\geq 0$, if $i\neq j$ then $Tor^B_{i,j}(\mathbb{K},\mathbb{K})=0$.*

Proof. See [6], Theorem 3.3. □

Lemma 3.2.16. *A quadratic algebra $R=\mathbb{K}\langle t_1,\dots,t_m\rangle/I$ is Koszul if and only if*

$$A=\mathbb{K}\langle t_1,\dots,t_m,x_1,\dots,x_n\rangle/\langle I\rangle$$

is Koszul, where $\langle I\rangle$ is the two-sided ideal of $\mathbb{K}\langle t_1,\dots,t_m,x_1,\dots,x_n\rangle$ generated by I.

Proof. Note that R is quadratic if and only if A is quadratic. Also, by Lemma 3.2.14, $\Omega(R)$ is distributive if and only if $\Omega(A)$ is distributive. Therefore, by Proposition 3.2.15, if $i\neq j$, $Tor^R_{i,j}(\mathbb{K},\mathbb{K})=0$ if and only if $Tor^A_{i,j}(\mathbb{K},\mathbb{K})=0$, for all $i,j\geq 0$. Then by Remark 3.1.7-(iv), $Ext^{i,j}_R(\mathbb{K},\mathbb{K})=0$ if and only if $Ext^{i,j}_A(\mathbb{K},\mathbb{K})=0$, for $i\neq j$. Therefore, R is a Koszul algebra if and only if A is a Koszul algebra. □

Related to Proposition 3.2.3 we have the following theorem.

Theorem 3.2.17. *If $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ is a graded skew PBW extension of a finitely presented Koszul algebra R, then A is Koszul.*

Proof. Let R be a finitely presented algebra; by Proposition 2.3.1,

$$R=\mathbb{K}\langle t_1,\dots,t_m\rangle/\langle P\rangle \tag{3.2.10}$$

where P is the vector space generated by homogeneous polynomials

$$r_1,\dots,r_s\in L_t:=\mathbb{K}\langle t_1,\dots,t_m\rangle. \tag{3.2.11}$$

Let $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ be a graded skew PBW extension. Then by Remark 2.3.2, A is a finitely presented algebra. So, by Proposition 2.3.1,

$$A=\mathbb{K}\langle t_1,\dots,t_m,x_1,\dots,x_n\rangle/\langle W\rangle, \tag{3.2.12}$$

where W is the vector space generated by the polynomials

$$r_1, \dots, r_s,\ x_j t_k - \sigma_j(t_k)x_i - \delta_j(t_k),\ x_j x_i - c_{i,j}x_i x_j - (r_{0_{j,i}} + r_{1_{j,i}}x_1 + \cdots + r_{n_{j,i}}x_n)$$
$$\in L := \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle, \quad (3.2.13)$$

with $1 \leq i, j \leq n$, $1 \leq k \leq m$.

Since R is a Koszul algebra then:

(i) R is a quadratic algebra, and by Remark 2.3.2, A is a quadratic algebra.

(ii) By Lemma 3.2.16, we have that $A_P := \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle/\langle P\rangle_X$ is Koszul, where $\langle P\rangle_X$ is the two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle$ generated by the polynomials as in (3.2.11). So, by Lemma 3.2.13, we have that for all $k \geq 0$, the collection of subspaces
$$X_i^P := L_{i-1}PL_{k-i-1} \subseteq L_k,\ i = 1, \dots, k-1 \quad (3.2.14)$$
is distributive. Therefore, by Proposition 3.2.11, there exists a base $\mathcal{B}_k$ of the space L_k such that $\mathcal{B}_k \cap X_i^P$ is a basis of X_i^P for each $1 \leq i \leq k-1$. Let $X_i := L_{i-1}WL_{k-i-1} \subseteq L_k$, $i = 1, \dots, k-1$, where W is the space generated by the polynomials as in (3.2.13).

Let $Y := (X_i \setminus X_i^P) \cap \mathcal{B}_k$. Since X_i^P is a subspace of X_i we claim that $\bar{Y} := \{y + X_i^P \mid y \in Y\} = \{\bar{y} \in X_i/X_i^P \mid y \in Y\}$ is a basis of X_i/X_i^P. Indeed, if $0 \neq \bar{x} \in X_i/X_i^P$, then $\bar{x} = x + X_i^P$, with $x \in X_i \setminus X_i^P$. Note that $x = k_1b_1 + \cdots + k_\rho b_\rho$, where $b_1, \dots, b_\rho$ are different nonzero elements in $\mathcal{B}_k$ and $k_1, \dots, k_\rho \in \mathbb{K}$. Then $\bar{x} = \overline{k_1b_1 + \cdots + k_\rho b_\rho} = k_1\bar{b_1} + \cdots + k_\rho\bar{b_\rho}$. If $b_\nu \in X_i^P$ for some $1 \leq \nu \leq \rho$ then $\bar{b_\nu} = 0$. So $\bar{x} = s_1\bar{v_1} + \cdots + s_\mu\bar{v_\mu}$ with $s_1, \dots, s_\mu \in \mathbb{K}$ and $v_1, \dots, v_\mu \in Y$. Now suppose that $k_1\bar{y_1} + \cdots + k_v\bar{y_v} = \bar{0}$ with $k_1, \dots, k_v \in \mathbb{K}$ and $0 \neq \bar{y_1}, \dots, 0 \neq \bar{y_v} \in \bar{Y}$. Then $y_1, \dots, y_v \notin X_i^P$, $\overline{k_1y_1 + \cdots + k_vy_v} = \bar{0}$ and so $k_1y_1 + \cdots + k_vy_v \in X_i^P$. As $X_i^P \cap \mathcal{B}_k$ is a basis of X_i^P then there are different nonzero elements $w_{v+1}, \dots, w_{v+\mu} \in X_i^P \cap \mathcal{B}_k$ such that $k_1y_1 + \cdots + k_vy_v = k_{v+1}w_{v+1} + \cdots + k_{v+\mu}w_{v+\mu}$, with $k_{v+1}, \dots, k_{v+\mu} \in \mathbb{K}$. As $y_1, \dots, y_v \notin X_i^P$ then $y_1, \dots, y_v, w_{v+1}, \dots, w_{v+\mu}$ are nonzero different elements in $\mathcal{B}_k$ such that $k_1y_1 + \cdots + k_vy_v + (-k_{v+1})w_{v+1} + \cdots + (-k_{v+\mu})w_{v+\mu} = 0$. Then $k_1 = \cdots = k_v = k_{v+1} = \cdots = k_{v+\mu} = 0$.

Therefore, by Theorem 3.33 in [81], we have that $(\mathcal{B}_k \cap X_i^P) \cup Y = \mathcal{B}_k \cap X_i$ is a basis of X_i. So, by Proposition 3.2.11 the collection of subspaces $X_1, \dots, X_{k-1}$ is distributive for each $k \geq 0$. Thus, by Lemma 3.2.13 we have that A is Koszul. □

Remark 3.2.18. Phan in [59] and [60] defined Koszul algebras for augmented algebras and R-augmented algebras. In [87] was studied a generalized Koszul property for skew PBW extensions, according to [59] and [60].

Remark 3.2.19. Generalized Koszul algebras or N-Koszul algebras are generated in degree one and all their relations are homogeneous of the same degree N (see [12], page 1). *N-Koszul* algebras are graded algebras

$$B = \mathbb{K} \oplus B_1 \oplus B_2 \oplus \cdots$$

such that there is a graded projective resolution of $\mathbb{K}$

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to \mathbb{K} \to 0,$$

such that for any $i \geq 0$, P_i is generated in degree $\delta(i)$, where

$$\delta(i) = \begin{cases} \frac{i}{2}N, & \text{if } i \text{ is even;} \\ \frac{i-1}{2}N + 1, & \text{if } i \text{ is odd,} \end{cases}$$

for some $N \geq 2$.
Since graded skew PBW extensions are quadratic algebras (2-homogeneous), then:

(i) If $N = 2$,

$$\delta(i) = \begin{cases} \frac{i}{2}2 = i, & \text{if } i \text{ is even;} \\ \frac{i-1}{2}2 + 1 = i, & \text{if } i \text{ is odd.} \end{cases}$$

Thus for $N = 2$, Koszul algebras and N-Koszul algebras are the same. Therefore, for graded skew PBW extensions N-Koszul property and Koszul property are equivalent.

(ii) The connected algebra B is said to be $\mathcal{K}_2$ if the Yoneda algebra $E(B)$ is generated as an algebra by $E^1(B)$ and $E^2(B)$. As an algebra is Koszul if and only if it is quadratic and $\mathcal{K}_2$ (see [19], Corollary 4.6), then Koszul algebras and $\mathcal{K}_2$ algebras are the same for skew PBW extensions.

CHAPTER 4

The Calabi-Yau condition for skew PBW extensions

Regular algebras were defined by Artin and Schelter in [3] and they are now known in the literature as Artin-Schelter regular algebras; Calabi-Yau algebras were defined by Ginzburg in [26], and as a generalization of them, were defined the skew (also named twisted) Calabi-Yau algebras. Reyes, Rogalski and Zhang in [75] proved that for connected algebras, skew Calabi-Yau property is equivalent to Artin-Schelter-regular property. Moreover, it is clear that Calabi-Yau algebras are skew Calabi-Yau, but we will exhibit examples of graded skew PBW extensions which are skew Calabi-Yau, but not Calabi-Yau (one of them is the Jordan Plane). Since graded quasi-commutative skew PBW extensions are isomorphic to graded iterated Ore extensions of endomorphism type (Proposition 2.3.5-(ii)), in this chapter we prove that graded quasi-commutative skew PBW extensions with coefficients in Artin-Schelter regular algebras are Artin-Schelter regular algebras, and graded skew PBW extensions of a finitely presented Auslander-regular algebras are Artin-Schelter regular. As a consequence of this, we prove that graded quasi-commutative skew PBW extensions of finitely presented skew Calabi-Yau algebras are skew Calabi-Yau, and graded skew PBW extensions of a finitely presented Auslander-regular algebras are skew Calabi-Yau. We also give a description of Nakayama automorphism for graded quasi-commutative skew PBW extensions of Artin-Schelter regular algebras, using the Nakayama automorphism of the ring of the coefficients. The main results of this chapter are Theorem 4.1.2, Theorem 4.1.3, Theorem 4.2.8, Example 4.2.9, Example 4.2.10, Theorem 4.3.3, Example 4.3.4 and Example 4.3.5.

4.1 Artin-Schelter regular algebras

In this section we will prove that graded quasi-commutative skew PBW extensions of an Artin-Schelter regular algebra are Artin-Schelter regular and that graded skew PBW extensions of a finitely presented Auslander-regular algebras are Artin-Schelter regular. These results will be appear in [85].

Definition 4.1.1. Let $B = \mathbb{K} \oplus B_1 \oplus B_2 \oplus \cdots$ be a finitely presented algebra over $\mathbb{K}$. The algebra B will be called *Artin-Schelter regular* if it has the following properties:

(i) B has finite global dimension d.

(ii) B has finite GK-dimension.

(iii) B is *Gorenstein*, meaning that $Ext^i_B(\mathbb{K}, B) = 0$ if $i \neq d$, and $Ext^d_B(\mathbb{K}, B) \cong \mathbb{K}(l)$, for some integer l.

Theorem 4.1.2. *Every graded skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ of a finitely presented Auslander-regular algebra R is Artin-Schelter regular.*

Proof. By Proposition 2.3.1, we know that $R = \mathbb{K}\langle t_1, \ldots, t_m\rangle/I$, where I is a proper two-sided ideal of $\mathbb{K}\langle t_1, \ldots, t_m\rangle$ generated by finite homogeneous polynomials $r_1, \ldots, r_s$ in $\mathbb{K}\langle t_1, \ldots, t_m\rangle$ (it is assumed that t_j has grade 1, $1 \leq j \leq m$). Then

$$A = \mathbb{K}\langle t_1, \ldots, t_m, x_1, \ldots, x_n\rangle/J,$$

where J is a two-sided ideal of $\mathbb{K}\langle t_1, \ldots, t_m, x_1, \ldots, x_n\rangle$, generated by a finite set of homogeneous polynomials $r_1, \ldots, r_s$, f_{hk} and g_{ji}, where the polynomials f_{hk} are as in (2.3.4) and g_{ji} are as in (2.3.5). Now, by Remark 2.3.2-(ii), we have that A is connected. So, by Theorem 2.2.1 and Remark 2.3.2-(ix), we know that $A = \mathbb{K} \bigoplus A_1 \bigoplus A_2 \bigoplus \cdots$ is a finitely presented algebra. Since A is bijective, then by Proposition 2.3.12, we have that A is Auslander-regular. Now, since R is a noetherian, then by Proposition 2.3.5-(i) we have that A is noetherian. Therefore, from Proposition 2.3.13 we have that A is graded Auslander-regular. From Corollary 2.3.16 we have that A has graded finite global dimension, say d. By Proposition 2.3.18-(ii), A has finite GK-dimension. Now, we have that A is Gorenstein (see [41], Theorem 6.3). Therefore A Artin-Schelter regular. $\square$

Theorem 4.1.3. *Let R be an Artin-Schelter regular algebra and let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a graded quasi-commutative skew PBW extension. Then A is Artin-Schelter regular.*

Proof. From Remark 2.3.2-(ii), we have that A is connected. So, by Theorem 2.2.1 and Remark 2.3.2-(ix), we know that $A = \mathbb{K} \bigoplus A_1 \bigoplus A_2 \bigoplus \cdots$ is a finitely presented algebra.

(i) Since R has finite global dimension, say e, then by Proposition 2.1.8 we know that $\operatorname{gld}(A) = e + n = d$, i.e., A has finite global dimension.

(ii) Let V be a subspace of R generated by $\{t_1, \ldots, t_m\}$. Note that V is a finite dimensional generating subspace of R. As σ_i is graded for all i, then $\sigma_n(V) \subseteq V$. Now, as A is bijective and R has finite GK-dimension then by Proposition 2.1.11 we have that $\mathrm{GKdim}(A) = \mathrm{GKdim}(R) + n$, i.e., A has finite GK-dimension.

(iii) From Proposition 2.3.5-(ii) and his proof, we know that A is isomorphic to a graded iterated Ore extension of endomorphism type $R[z_1; \theta_1] \cdots [z_n; \theta_n]$, where θ_i is bijective, for each i, $\theta_1 = \sigma_1$, $\theta_j : R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}] \to R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}]$ is such that $\theta_j(z_i) = c_{i,j} z_i$, $c_{i,j} \in \mathbb{K} \setminus \{0\}$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$. Then $z_1 r = \theta_1(r) z_1 = \sigma_1(r) z_1 \in R z_1$ and $r z_1 = z_1 \theta_1^{-1}(r) = z_1 \sigma_1^{-1}(r) \in z_1 R$. Hence $z_1 \in A_1$ is a nonzero normal element of $A^{(1)} := R[z_1; \theta_1]$ and $A^{(1)}/\langle z_1\rangle = R$. From the Rees lemma (see [41], Proposition 3.4-(b)) we have that

$$Ext^j_{A^{(1)}}(\mathbb{K}, A^{(1)}) \cong Ext^{j-1}_{A^{(1)}/\langle z_1\rangle}(\mathbb{K}, A^{(1)}/\langle z_1\rangle) = Ext^{j-1}_R(\mathbb{K}, R).$$

By Proposition 2.1.8 we have that $d_1 :=\mathrm{gld}(A^{(1)}) = e+1$. Since R is Gorenstein then $Ext^i_R(\mathbb{K}, R) = 0$ if $i \neq e$, and $Ext^e_R(\mathbb{K}, R) \cong \mathbb{K}(l_1)$, for some integer l_1, i.e., $Ext^{i+1}_{A^{(1)}}(\mathbb{K}, A^{(1)}) = 0$ if $i+1 \neq e+1 = d_1$ and $Ext^{d_1}_{A^{(1)}}(\mathbb{K}, A^{(1)}) \cong \mathbb{K}(l_1)$, for some integer l_1, . Then $A^{(1)} = R[z_1;\theta_1]$ is Gorenstein. Now, $z_2 \in A_1$ is a nonzero normal element of $A^{(2)} := A^{(1)}[z_2;\theta_2] = R[z_1;\theta_1][z_2;\theta_2]$ and $A^{(2)}/\langle z_2\rangle = A^{(1)}$. Thus, with the above procedure we have that $R[z_1;\theta_1][z_2;\theta_2]$ is Gorenstein. Now, $z_n \in A_1$ is a nonzero normal element of $A^{(n)} := A^{(n-1)}[z_n;\theta_n] = R[z_1;\theta_1]\cdots[z_{n-1};\theta_{n-1}][z_n;\theta_n] = A$, $A/\langle z_n\rangle = A^{(n-1)}$, $\mathrm{gld}(A^{(n-1)}) = e+n-1 := d_{n-1}$ and $\mathrm{gld}(A) = \mathrm{gld}(A^{(n)}) = e+n := d_n = d$. Assuming that $A^{(n-1)}$ is Gorenstein, we have that $Ext^{i-1}_{A^{(n-1)}}(\mathbb{K}, A^{(n-1)}) = 0$ if $i-1 \neq e+n-1$ and $Ext^{e+n-1}_{A^{(n-1)}}(\mathbb{K}, A^{(n-1)}) \cong \mathbb{K}(l_{n-1})$, for some integer l_{n-1}. From the Rees lemma, we have that $Ext^i_{A^{(n)}}(\mathbb{K}, A^{(n)}) = Ext^i_A(\mathbb{K}, A) = 0$ if $i \neq e+n = d$ and $Ext^d_A(\mathbb{K}, A) \cong \mathbb{K}(l_{n-1})$, for some integer l_{n-1}. Thus $A^{(n)} := R[z_1;\theta_1]\cdots[z_n;\theta_n] \cong A$ is Gorenstein.

Therefore, A is an Artin-Schelter regular algebra. □

4.2 Skew Calabi-Yau algebras

In this section we will prove that graded quasi-commutative skew PBW extensions of a finitely presented skew calabi-Yau algebra are skew Calabi-Yau algebras, and that graded skew PBW extensions of a finitely presented Auslander-regular algebra are skew Calabi-Yau. These results will be appear in [85].

The *enveloping algebra* of an algebra B is the tensor product $B^e = B \otimes B^{op}$, where B^{op} is the opposite algebra of B. Bimodules over B are essentially the same as modules over the enveloping algebra of B, so in particular, B and M can be considered as B^e-modules. Suppose that M and N are both B^e-modules. Then there are two B^e-module structures on $M \otimes N$, one of them is called the outer structure defined by $(a\otimes b)\cdot(m\otimes n) =^{out} am\otimes nb$, and the another one is called the inner structure defined by $(a\otimes b)\cdot(m\otimes n) =^{int} ma\otimes bn$, for any $a, b \in B$, $m \in M$, $n \in N$. Since B^e is identified with $B\otimes B$ as a $\mathbb{K}$-module (${}_{\mathbb{K}}B^e = {}_{\mathbb{K}}(B\otimes B^{op}) = {}_{\mathbb{K}}(B\otimes B)$), $B\otimes B$ endowed with the outer structure is nothing but the left B^e-module B^e. ${}_{B^e}(B\otimes B) =^{out} {}_{B^e}B^e$: In ${}_{B^e}(B\otimes B)$, $(a\otimes b)\cdot(x\otimes y) = a\cdot(x\otimes y)\cdot b =^{out} ax\otimes yb$, whereas that in ${}_{B^e}B^e$ $(a\otimes b)\cdot(x\otimes y) = ax\otimes b\circ y = ax\otimes yb$. $B\otimes B$ endowed with the inner structure is nothing but the right B^e-module B^e. ${}_{B^e}(B\otimes B) =^{int} B^e_{B^e}$: In ${}_{B^e}(B\otimes B)$, $(a\otimes b)\cdot(x\otimes y) = a\cdot(x\otimes y)\cdot b =^{int} xa\otimes by$, whereas that in $B^e_{B^e}$, $(x\otimes y)\cdot(a\otimes b) = xa\otimes y\circ b = xa\otimes by$. Hence, we often say B^e has the outer (left) and inner (right) B^e-module structures. We characterize the enveloping algebra of a skew PBW extension in [71].

An algebra B is said to be *homologically smooth* if as an B^e-module, B has a projective resolution that has finite length and such that each term in the projective resolution is finitely generated. The length of this resolution is known as the *Hochschild dimension* of B. In the next definition, the outer structure on B^e is used when computing the homology $Ext^*_{B^e}(B, B^e)$. Thus, $Ext^*_{B^e}(B, B^e)$ admits an B^e-module structure induced by the inner one on B^e.

Let M be a B-bimodule, ν, $\mu : B \to B$ be two automorphisms, the skew B-bimodule ${}^{\nu}M^{\mu}$ is equal to M as a vector space with $a \cdot m \cdot b := \nu(a) \cdot m \cdot \mu(b)$. Thus, M is a left B^e-module with product given by

$$(a \otimes b) \cdot m = a \cdot m \cdot b = \nu(a) \cdot m \cdot \mu(b).$$

In particular, for B and B^e we have the structure of left B^e-modules given by

$$(a \otimes b) \cdot x = \nu(a)x\mu(b),$$
$$(a \otimes b) \cdot (x \otimes y) = a \cdot (x \otimes y) \cdot b = \nu(a) \cdot (x \otimes y) \cdot \mu(b) = \nu(a)x \otimes y\mu(b).$$

Proposition 4.2.1. *([28], Lemma 2.1) Let ν, σ and ϕ be automorphisms of B.Then*

(i) The map

$${}^{\nu}B^{\sigma} \to {}^{\phi\nu}B^{\phi\sigma}, \quad a \mapsto \phi(a)$$

is an isomorphism of B^e-modules. In particular,

$${}^{\nu}B^{\sigma} \cong B^{\nu^{-1}\sigma} \cong {}^{\sigma^{-1}\nu}B \text{ and } B^{\sigma} \cong {}^{\sigma^{-1}}B.$$

(ii) $B \cong B^{\sigma}$ as B^e-modules if and only if σ is an inner automorphism.

Proof. (i) Is verified by straightforward computation.
(ii) Note that the left B-linear maps $\tau : B \to B$ on a ring B are those of the form $a \mapsto ax$ for some $x \in B$ (namely $x = \tau(1)$). Such a map is a B^e-module morphism $B \to B^{\sigma}$ precisely when we have $abx = ab\tau(1) = \tau(ab) = \tau(a)\sigma(b) = ax\sigma(b)$ for all $a, b \in B$, that is, $bx = x\sigma(b)$ for all $b \in B$. Suppose that x is invertible; if $\tau(a) = \tau(b)$ then $ax = bx$ and therefore $a = b$, $\tau(yx^{-1}) = y$, so τ is bijective. Now, if τ is biyective, then is x is invertible. So $\sigma(b) = x^{-1}bx$ for all $b \in B$. □

Remark 4.2.2. If σ is an inner automorphism of B, given by conjugation $x \mapsto uxu^{-1}$ by the unit u of B, then the map on B given by left multiplication by u shows that

$${}^{\nu}B^{\beta} \cong {}^{\sigma\nu}B^{\beta} \cong {}^{\nu}B^{\beta\sigma}$$

for all automorphisms ν and β of B.

Definition 4.2.3. A graded algebra B is called *skew Calabi-Yau* of dimension d if

(i) B is homologically smooth.

(ii) There exists an algebra automorphism ν of B such that

$$Ext^{i}_{B^e}(B, B^e) \cong \begin{cases} 0, & i \neq d; \\ B^{\nu}(l), & i = d. \end{cases}$$

as B^e -modules, for some integer l. If ν is the identity, then B is said to be *Calabi-Yau*.

Ungraded Calabi-Yau algebras are defined similarly but without degree shift. Sometimes condition (*ii*) is called the skew (twisted) Calabi-Yau condition. The skew Calabi-Yau condition appears to have first been explicitly defined in [18] where the authors used the term *rigid Gorenstein*. The automorphism ν is called the *Nakayama* automorphism of B.

Proposition 4.2.4. *Let B be a skew Calabi-Yau algebra with Nakayama automorphism ν. Then ν is unique up to an inner automorphism, i.e, the Nakayama automorphism is determined up to multiplication by an inner automorphism of B.*

Proof. Let B be a skew Calabi-Yau algebra with Nakayama automorphism ν and let μ another Nakayama automorphism, i.e., $Ext^d_{B^e}(B, B^e) \cong B^\mu$, then $Ext^d_{B^e}(B, B^e) \cong B^\nu \cong B^\mu$ as B^e -modules. By Proposition 4.2.1-(i), $B \cong B^{\nu^{-1}\mu}$; by Proposition 4.2.1-(ii), $\nu^{-1}\mu$ is an inner automorphism of B. Let $\nu^{-1}\mu = \sigma$ where σ is an inner automorphism of B, so $\mu = \nu\sigma$ for some inner automorphism σ of B. □

Proposition 4.2.5. *A skew Calabi-Yau algebra B is Calabi-Yau if and only if ν is an inner automorphism of B.*

Proof. Let B be a skew Calabi-Yau algebra of dimension d.
$\Rightarrow$) If B is Calabi-Yau then $Ext^d_{B^e}(B, B^e) \cong B \cong B^\nu$. By Propositiona 4.2.1-(ii) ν is an inner automorphism of B.
$\Leftarrow$) If ν is an inner automorphism of B then by Proposition 4.2.1-(ii), $B \cong B^\nu$. Therefore $Ext^d_{B^e}(B, B^e) \cong B$, and so B is Calabi-Yau algebra. □

Proposition 4.2.6 ([75], Lemma 1.2)**.** *Let B be a connected graded algebra. Then B is skew Calabi-Yau if and only if it is Artin-Schelter regular.*

Proposition 4.2.7. *Let R be a Koszul Artin-Schelter regular algebra of global dimension d with Nakayama automorphism σ.*

(i) *([33], Theorem 3.3) The skew polynomial algebra $B = R[x; \sigma]$ is a Calabi-Yau algebra of dimension $d + 1$.*

(ii) *([95], Remark 3.13) There exists a unique skew polynomial extension B such that B is Calabi-Yau.*

(iii) *([95], Theorem 3.16) If ν is a graded algebra automorphism of R, then $B = R[x; \nu]$ is Calabi-Yau if and only if $\sigma = \nu$.*

The Calabi-Yau and skew Calabi-Yau properties for graded skew PBW extensions will be next proved using the cited results presented in the literature and from our previous results.

Theorem 4.2.8. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a graded skew PBW extension of an algebra R.*

(i) *If A is quasi-commutative and R is a finitely presented skew Calabi-Yau algebra of global dimension d, then A is skew Calabi-Yau of global dimension $d + n$. Moreover, if R is Koszul and θ_i is the Nakayama automorphism of $R[x_1; \theta_1] \cdots [x_{i-1}; \theta_{i-1}]$ for $1 \leq i \leq n$, then A is Calabi-Yau of dimension $d + n$ (θ_i as in Proposition 2.3.5-(ii), $x_0 = 1$).*

(ii) *If R is finitely presented and Auslander-regular, then A is skew Calabi-Yau.*

Proof. (i) Since R is connected and skew Calabi-Yau, then by Proposition 4.2.6 we know that R is Artin-Schelter regular. From Theorem 4.1.3 we have that A is Artin-Schelter regular and, in particular, connected. Thus, using again Proposition 4.2.6, we have that A is a skew Calabi-Yau algebra. By the proof of Theorem 4.1.3 we have that the global dimension of A is $d+n$.

For the second part, we know that graded Ore extensions of Koszul algebras are Koszul algebras and, as a particular case of Theorem 4.1.3, we have that a graded Ore extension of an Artin-Schelter regular algebra is an Artin-Schelter regular algebra. Now, by Proposition 2.3.5-(ii) we have that A is isomorphic to a graded iterated Ore extension $R[x_1;\theta_1]\cdots[x_n;\theta_n]$. It is known that if A is a Calabi-Yau algebra of dimension d, then the global dimension of A is d (see for example [14], Remark 2.8). Then, using Proposition 4.2.7-(i) and applying induction on n we obtain that A is a Calabi-Yau algebra of dimension $d+n$.

(ii) From Theorem 4.1.2 we have that A is Artin-Schelter regular. Since R is connected, then by Remark 2.3.2-(ii) we have that A is connected. Then from Proposition 4.2.6 we get that A is skew Calabi-Yau.

□

Using the previous results we have the following examples of graded skew PBW extensions which are skew Calabi-Yau algebras and Artin-Schelter regular algebras.

Example 4.2.9. From Example 2.2.5 and Theorem 4.1.3 we obtain that the algebra of linear partial q-dilation operators $\mathbb{K}[t_1,\dots,t_n][H_1^{(q)},\dots,H_m^{(q)}]$ (Example 1.2.16), the multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$ (Example 1.2.20) and the multi-parameter quantum affine n-space $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ (Example 1.2.31), are Artin-Schelter regular algebras. By Theorem 4.2.8-(i), we have that the above examples are also skew Calabi-Yau algebras.

Example 4.2.10. The following examples are graded skew PBW extensions of the classical polynomial ring R with coefficients in $\mathbb{K}$, which are not quasi-commutative and where R has the usual graduation (see Example 2.2.7). By Theorem 4.2.8-(ii), these extensions are skew Calabi-Yau algebras, since R is a finitely presented Auslander-regular algebra. By proof of Theorem 4.2.8-(ii), we have that these extensions are also Artin-Schelter regular algebras.

1. The Jordan plane. $A=\mathbb{K}\langle x,y\rangle/\langle yx-xy-x^2\rangle\cong\sigma(\mathbb{K}[x])\langle y\rangle$.
2. The homogenized enveloping algebra. $\mathcal{A}(\mathcal{G})\cong\sigma(\mathbb{K}[z])\langle x_1,\dots,x_n\rangle$.
3. The Diffusion algebra. $A\cong\sigma(\mathbb{K}[x_1,\dots,x_n])\langle D_1,\dots,D_n\rangle$.
4. The algebra $U\cong\sigma(\mathbb{K}[x_1,\dots,x_n])\langle y_1,\dots,y_n;z_1,\dots,z_n\rangle$.
5. Manin algebra. $\mathcal{O}(M_q(2))\cong\sigma(\mathbb{K}[u])\langle x,y,v\rangle$.
6. Algebra of quantum matrices. $\mathcal{O}_q(M_n(\mathbb{K}))\cong\sigma(\mathbb{K}[x_{im},x_{jk}])\langle x_{ik},x_{jm}\rangle$, for $1\le i<j,k<m\le n$.

7. Quadratic algebras. If $a_1 = a_4 = 0$ then the quadratic algebra is a graded skew PBW extension of $R = \mathbb{K}[y, z]$, and if $a_5 = a_3 = 0$ then quadratic algebras are graded skew PBW extensions of $R = \mathbb{K}[x, z]$.

For some of the above algebras other authors had already studied the skew Calabi-Yau property and the Artin-Schelter regularity, but using other techniques, the novelty here consists in interpreting these algebras as skew PBW extensions and applying some its algebraic properties studied previously. For example:

(i) The polynomial algebra $B = \mathbb{K}[x, y]$ is a connected graded noetherian algebra of global dimension 2. It follows that B is Artin-Schelter regular with $GKdim(B) = 2$ (see [82], Theorem 3.5). Moreover, B is Calabi-Yau of dimension 2 (see [56]). Let $B = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial algebra in n variables. B is Calabi-Yau of dimension n (see [11], page 18) and Artin-Schelter regular (see [14], Proposition 4.3).

(ii) Let $B = \mathbb{K}\langle x, y, z\rangle/\langle yz - zy, zx - xz, xy - yx + z^2\rangle$ which is of type S_1' in the classification of three-dimensional Artin-Schelter regular algebras given in [3]. According to [12], B is 3-Calabi-Yau (see [92], Example 3.6).

(iii) The quantum plane $B = \mathbb{K}\langle x, y\rangle/\langle yx - cxy\rangle \quad (c \neq 0)$ is an Artin-Schelter regular algebra of global dimension 2 (see [3], page 172 or [76], Example 2.10).

(iv) The Jordan plane $B = \mathbb{K}\langle x, y\rangle/\langle yx - xy - x^2\rangle$ is an Artin-Schelter regular algebra of global dimension 2 (see [3], page 172). The Jordan plane B is skew Calabi-Yau, but not Calabi-Yau (see [51]).

(v) Stephenson and Zhang proved that if B is a connected graded noetherian algebra of global dimension 2, then B is Artin-Schelter regular and $\mathrm{GKdim}(B) = 2$. Moreover, B is isomorphic to either $\mathbb{K}\langle x, y\rangle/\langle ax^2+byx+cxy+dy^2\rangle$ where $ad-bc \neq 0$, or $\mathbb{K}[x][y\sigma, \delta]$ where σ is an automorphism of $\mathbb{K}[x]$ and δ is a σ-derivation (see [82], Theorem 3.5). We note that if $a = 0$, $b = -1$, $c = 1$ and $d = 0$, then $\mathbb{K}\langle x, y\rangle/\langle ax^2+byx+cxy+dy^2\rangle = \mathbb{K}\langle x, y\rangle/\langle xy-yx\rangle \cong \mathbb{K}[x, y]$; if $a = 0$, $b \neq 0$, $c = 1$ and $d = 0$, then $\mathbb{K}\langle x, y\rangle/\langle ax^2 + byx + cxy + dy^2\rangle = \mathbb{K}\langle x, y\rangle/\langle xy + byx\rangle$ is the quantum plane and if $a = -1$, $b = 1$, $c = -1$ and $d = 0$, then $\mathbb{K}\langle x, y\rangle/\langle ax^2 + byx + cxy + dy^2\rangle = \mathbb{K}\langle x, y\rangle/\langle yx - xy - x^2\rangle$ is the Jordan plane. This is another way of seeing that $\mathbb{K}[x, y]$, the quantum plane and the Jordan plane are Artin-Schelter regular algebras. These examples had already been given by other authors, for example in [76], Example 2.10.

(vi) The multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$ is Artin-Schelter (see [76], Example 2.10).

(vii) Multi-parameter quantum affine n-spaces $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ is skew Calabi-Yau (see [51], Proposition 4.1).

(viii) He, Van Oystaeyen and Zhang showed that for the 3-dimensional Lie algebra $\mathcal{G}$ with basis $\{x, y, z\}$, $\mathcal{U}(\mathcal{G})$ is a Calabi-Yau algebra if and only if the Lie bracket is given by $[x, y] = ax + by + wz$, $[x, z] = cx + vy - bz$, $[y, z] = ux - cy + az$, where $a, b, c, u, v, w \in \mathbb{K}$; and if $\mathcal{G}$ is a finite dimensional Lie algebra, $\mathcal{U}(\mathcal{G})$ is Calabi-Yau of dimension 3 if and only if $\mathcal{G}$ is isomorphic to one of the following Lie algebras (see [31], Proposition 4.5 and Proposition 4.6):

(a) The 3-dimensional simple Lie algebra $sl(2, \mathbb{K})$;

(b) $\mathcal{G}$ has a basis $\{x, y, z\}$ such that $[x, y] = y$, $[x, z] = -z$ and $[y, z] = 0$;

(c) The Heisenberg algebra, that is; $\mathcal{G}$ has a basis $\{x, y, z\}$ such that $[x, y] = z$ and $[x, z] = [y, z] = 0$;

(d) The 3$-$dimensional abelian Lie algebra.

(ix) Let $\mathcal{U}_f(\mathcal{G})$ be a Sridharan enveloping algebra of a finite dimensional Lie algebra $\mathcal{G}$. Then $\mathcal{U}_f(\mathcal{G})$ is Calabi-Yau of dimension 3 if and only if $\mathcal{U}_f(\mathcal{G})$ is isomorphic to $\mathbb{K}\langle x, y, z\rangle/\langle R\rangle$ with the commuting relations R listed in the following table (see [31], Theorem 5.5):

Case	{x,y}	{x,z}	{y,z}
1	z	$-2x$	$2y$
2	y	$-z$	0
3	z	0	0
4	0	0	0
5	y	$-z$	1
6	z	1	0
7	1	0	0

where $\{x, y\} = xy - yx$.

Let $\mathcal{G}$ be a finite dimensional Lie algebra. Then the Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ is Calabi-Yau of dimension d if and only if the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ is Calabi-Yau of dimension d (see [31], Theorem 5.3).

Possibly for the algebra of linear partial q-dilation operators, the homogenized enveloping algebra, the Diffusion algebra, the algebra U, Manin algebra and algebra of quantum matrices, the Artin-Schelter regular and the skew Calabi-Yau properties had not yet been studied.

Remark 4.2.11. Every skew Calabi-Yau algebra may be extended to a Calabi-Yau algebra, i.e., if B is a skew Calabi-Yau algebra with Nakayama automorphism σ, then $B[z, \sigma]$ is Calabi-Yau (see [28], Theorem 1.1 and Remark 5.1).

Remark 4.2.12. Note that the Calabi-Yau property is not preserved by skew PBW extensions. The Jordan plane $A = \mathbb{K}\langle x, y\rangle/\langle yx - xy - x^2\rangle \cong \sigma(\mathbb{K}[x])\langle y\rangle = \mathbb{K}[x][y; \sigma, \delta]$, where $\sigma(x) = x$ and $\delta(x) = x^2$, is a graded skew PBW extension of a Calabi-Yau algebra $\mathbb{K}[x]$, but A is not Calabi-Yau. Indeed, the Nakayama automorphism ν of the Jordan plane is given by $\nu(x) = x$ and $\nu(y) = 2x + y$ (see for example [51], page 16) and it is not inner.

4.3 Nakayama automorphism

In final section we give a description of Nakayama automorphism of a graded quasi-commutative skew PBW extensions over finitely presented skew Calabi-Yau algebras, using the Nakayama automorphism of the ring of coefficients. We use the ideas of [51], Theorem 3.3 and Remark 3.4. The results of this section also appear in [86] and were submitted for publication.

Theorem 4.3.1 ([51], Theorem 3.3)**.** *Let R be a projective K-algebra and $B = R[x; \sigma, \delta]$ be a graded Ore extension. Suppose that R is skew Calabi-Yau of dimension d, with Nakayama automorphism ν. Then the Nakayama automorphism μ of B satisfies that $\mu|_R = \sigma^{-1}\nu$ and $\mu(x) = ux + b$ with u, $b \in R$ and u invertible.*

Recall that a *projective algebra* is an K-algebra B such that the K-module B is projective.

Remark 4.3.2 ([51], Remark 3.4)**.** If in Theorem 4.3.1, σ is the identical then $\mu(x) = x+b$. If $\delta = 0$ then $\mu(x) = ux$.

Theorem 4.3.3. *Let R be a finitely presented skew Calabi-Yau algebra with Nakayama automorphism ν. Then the Nakayama automorphism μ of a graded quasi-commutative skew PBW extension $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is given by*

$$\mu(r) = (\sigma_1 \cdots \sigma_n)^{-1}\nu(r), \text{ for } r \in R, \text{ and}$$
$$\mu(x_i) = u_i \prod_{j=i}^{n} c_{i,j}^{-1} x_i, \text{ for each } 1 \leq i \leq n,$$

where σ_i is as in Proposition 1.1.4, $u_i, c_{i,j} \in \mathbb{K} \setminus \{0\}$, and the elements $c_{i,j}$ are as in Definition 1.1.1.

Proof. Note that A is skew Calabi-Yau (Theorem 4.2.8-(i)) and therefore the Nakayama automorphism of A exists. By Proposition 2.3.5-(ii) and its proof we have that A is isomorphic to a graded iterated Ore extension $R[x_1; \theta_1] \cdots [x_n; \theta_n]$, where θ_i is bijective; $\theta_1 = \sigma_1$;

$$\theta_j : R[x_1; \theta_1] \cdots [x_{j-1}; \theta_{j-1}] \to R[x_1; \theta_1] \cdots [x_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(x_i) = c_{i,j}x_i$ ($c_{i,j} \in \mathbb{K}$ as in Definition 1.1.1), $1 \leq i < j \leq n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$. Note that

$$\theta_j^{-1}(x_i) = c_{i,j}^{-1} x_i. \tag{4.3.1}$$

Now, since R is connected then by Remark 2.3.2, A is connected. So, the multiplicative group of R and also the multiplicative group of A is $\mathbb{K}\setminus\{0\}$, therefore the identity map is the only inner automorphism of A. Let μ_i the Nakayama automorphism of $R[x_1; \theta_1] \cdots [x_i; \theta_i]$. By Theorem 4.3.1 and Remark 4.3.2 we have that the Nakayama automorphism μ_1 of $R[x_1; \theta_1]$ is given by $\mu_1(r) = \sigma_1^{-1}\nu(r)$ for $r \in R$, and $\mu_1(x_1) = u_1x_1$ with $u_1 \in \mathbb{K} \setminus \{0\}$; the Nakayama automorphism μ_2 of $R[x_1; \theta_1][x_2; \theta_2]$ is given by $\mu_2(r) = \sigma_2^{-1}\mu_1(r) = \sigma_2^{-1}\sigma_1^{-1}\nu(r)$, for $r \in R$; $\mu_2(x_1) = \theta_2^{-1}\mu_1(x_1) = \theta_2^{-1}(u_1x_1) = u_1\theta_2^{-1}(x_1) = u_1c_{1,2}^{-1}x_1$ and $\mu_2(x_2) = u_2x_2$, for $u_2 \in \mathbb{K} \setminus \{0\}$; the Nakayama automorphism μ_3 of $R[x_1; \theta_1][x_2; \theta_2][x_3; \theta_3]$ is given by $\mu_3(r) = \sigma_3^{-1}\mu_2(r) = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\nu(r)$, for $r \in R$; $\mu_3(x_1) = \theta_3^{-1}\mu_2(x_1) = \theta_3^{-1}(u_1c_{1,2}^{-1}x_1) = u_1c_{1,2}^{-1}\theta_3^{-1}(x_1) = u_1c_{1,2}^{-1}c_{1,3}^{-1}x_1$; $\mu_3(x_2) = \theta_3^{-1}\mu_2(x_2) = \theta_3^{-1}(u_2x_2) = u_2c_{2,3}^{-1}x_2$ and $\mu_3(x_3) = u_3x_3$, for $u_3 \in \mathbb{K} \setminus \{0\}$.
Continuing with the procedure we have that the Nakayama automorphism of A is given by

$$\mu(r) = \mu_n(r) = \sigma_n^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\nu(r),$$

for $r \in R$; $\mu(x_1) = u_1c_{1,2}^{-1}c_{1,3}^{-1} \cdots c_{1,n}^{-1}x_1$; $\mu(x_2) = u_2c_{2,3}^{-1} \cdots c_{2,n}^{-1}x_2$. In general, for $1 \leq i \leq n$, we have that $\mu(x_i) = u_ic_{i,i+1}^{-1}c_{i,i+2}^{-1} \cdots c_{i,n}^{-1}x_i = u_i(c_{i,n} \cdots c_{i,i+2}c_{i,i+1})^{-1}x_i$, for $u_i \in \mathbb{K} \setminus \{0\}$. Note that $c_{i,i} = 1$ (Remark 1.1.3-(ii)). ∎

Example 4.3.4. Let $A = \mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ be the quantum affine n-space from Example 1.2.31. $A = \mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ is a graded quasi-commutative skew PBW extension of $\mathbb{K}[x_1]$ (see Example 2.2.5), with $\sigma_j(k) = k$ for $k \in \mathbb{K}$ and $\sigma_j(x_1) = q_{1j}x_1$, $j \geq 2$. Therefore, according to Proposition 2.3.5 and its proof, A is isomorphic to a graded iterated Ore extension $\mathbb{K}[x_1][x_2;\theta_2]\cdots[x_n;\theta_n]$, where $\theta_j(k) = k$ for $k \in \mathbb{K}$ and $\theta_j(x_i) = q_{ij}x_i$, for $1 \leq i < j \leq n$. Note that the Nakayama automorphism ν of $\mathbb{K}[x_1]$ is the identity map. Applying Theorem 4.3.3 we have that the Nakayama automorphism μ of A is given by $\mu(k) = k$ for $k \in \mathbb{K}$, $\mu(x_1) = (\sigma_1\cdots\sigma_n)^{-1}\nu(x_1) = (q_{1n}^{-1}\cdots q_{12}^{-1})x_1 = (q_{n1}\cdots q_{21})x_1$, and $\mu(x_i) = u_i q_{i(i+1)}^{-1} q_{i(i+2)}^{-1}\cdots q_{in}^{-1}x_i = u_i q_{(i+1)i}q_{(i+2)i}\cdots q_{ni}x_i = u_i q_{(i+1)i}q_{(i+2)i}\cdots q_{ni}x_i$, for each $2 \leq i \leq n$. Since μ is unique up to an inner automorphism (see Proposition 4.2.4) and the invertible elements in $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ are those nonzero scalars in $\mathbb{K}$, the identity map is the only inner automorphism of $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$. Therefore, using the same reasoning of [51] in the proof of Proposition 4.1, we have that $u_i = q_{1i}q_{2i}\cdots q_{(i-1)i}$. Then, $\mu(x_i) = (\prod_{j=1}^{n} q_{ji})x_i$, for $2 \leq i \leq n$. This result is known and can be deduced in some other way (see for example [51], Proposition 4.1).

Example 4.3.5. Let R be a Koszul Artin-Schelter regular algebra of global dimension d, with Nakayama automorphism ν. Let $A = R[x_1,\ldots,x_n;\sigma_1,\ldots,\sigma_n]$ be an iterated skew polynomial ring, with σ_i graded. A is a skew PBW extension of R with relations $x_ir = \sigma_i(r)x_i$ and $x_jx_i = x_ix_j$, for $r \in R$ and $1 \leq i,j \leq n$. As R is graded and $c_{i,j} = 1 \in R_0$, then by Proposition 2.2.4 we have that A is a graded quasi-commutative skew PBW extension. Therefore, A is a Koszul Artin-Schelter regular algebra (Proposition 3.2.3 and Theorem 4.1.3). Note that R is a finitely presented skew Calabi-Yau algebra, then A is a skew Calabi-Yau algebra (Theorem 4.2.8). By Proposition 2.3.5, $R[x_1,\ldots,x_n;\sigma_1,\ldots,\sigma_n] \cong R[x_1;\theta_1]\cdots[x_n;\theta_n]$, where $\theta_j(r) = \sigma_j(r)$ and $\theta_j(x_i) = x_i$ for $i < j$. Applying Theorem 4.3.3 we have that the Nakayama automorphism μ of A is given by $\mu(r) = (\sigma_1\cdots\sigma_n)^{-1}\nu(r)$, if $r \in R$ and $\mu(x_i) = u_i\prod_{j=i}^{n} c_{i,j}^{-1}x_i = u_ix_i$, $u_i \in \mathbb{K}\setminus\{0\}$, $1 \leq i \leq n$. This automorphism had already been calculated in Theorem 4.6 of [95], like this: $\mu(r) = (\sigma_1\cdots\sigma_n)^{-1}\nu(r)$, if $r \in R$ and $\mu(x_i) = (\mathrm{hdet}\sigma_i)x_i$, $1 \leq i \leq n$.

Future work

Lezama and Latorre in [44] introduce the semi-graded rings, which extend graded rings, skew PBW extensions and graded skew PBW extensions. For this new type of non-commutative rings they discussed some basic problems of noncommutative algebraic geometry. In particular, they proved some elementary properties of the generalized Hilbert series, Hilbert polynomial and Gelfand-Kirillov dimension. They extended the notion of non-commutative projective scheme to the case of semi-graded rings and generalized the Serre-Artin-Zhang-Verevkin theorem. A definition of Koszul, Artin-Schelter regular and Calabi-Yau algebras could be given for semi-graded algebras, to study those properties and their equivalent definitions.

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