UNIVERSIDAD NACIONAL DE COLOMBIA

SINGULAR POINT TRACKING: A METHOD FOR THE ANALYSIS OF SLIDING BIFURCATIONS IN NONSMOOTH SYSTEMS

SEGUIMIENTO DE PUNTOS SINGULARS: UN METODO PARA EL ANALISIS DE BIFURCACIONES DESLIZANTES EN SISTEMAS NO SUAVES

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Notation

f($\boldsymbol{x}$): vector field (set of expressions for a system dynamics)

ff: a double vector with equal characteristics.

x_i: space state variables of a dynamic system

DB: discontinuity boundary (in the text)

Σ: symbol of the DB in the figures and formula

H_j: set of equations defining the DB

VF: vector field (in the text)

Z_j: notation for the sub-space or sub-region in the state space

f_{i_N}: normal component of $\mathbf{f}_i$

f_{i_T}: tangential component of $\mathbf{f}_i$

Θ: angular range

Φ: orbit or trajectory

Φs: sliding orbit or trajectory

Ω: point belonging to the state space

Ωc: point belonging to the DB with a crossing dynamics

Ωs: point belonging to the DB with a sliding dynamics

i: sub-index indicating the order of a system

j: sub-index indicating the number of sub-spaces

α: parameter of a system

/// duplication of the symbol /// in a sequence means that are presenting many points of the same type.

sC sequence of elements including at least one sliding segment.

β sequence of cycles that belong to a global bifurcation

segment: name for a piece of orbit or trajectory that could be flat or a curve.

Note: In this work, the words ‗special points' are used as ‗isolated points'. Thus for every special point in the DB one can find a sufficiently small neighbourhood without any other special point in it.

Summary

The main objective of this research is to develop a new method for improving the understanding of the behaviour and the analysis of sliding bifurcations in nonsmooth systems. This method is called singular point tracking (SPT) because it is based on the identification and synthesis of certain points with unique characteristics. These points have been not given much attention, thus motivating an in-depth study on the subject. As a result of this research, it has been found that the information about singular points combined with the information on regular ones can be used for finding or detecting nonsmooth bifurcations.

The results of the abovementioned method have been presented at international events, published in specialized journals, and constitute the main part of this document. In the publications listed below, only the format and notation has been changed slightly in comparison with the original published versions. In the rest of this document, I discuss how the abovementioned method was developed.

I. Integration-Free Analysis of Nonsmooth Local Dynamics in Planar Filippov Systems. International Journal of Bifurcations and Chaos. Year: 2009 Vol: 19 Issue: 3, March 2009 Pages: 947–975.

II. Continuation of Nonsmooth Bifurcations in Filippov Systems Using Singular Point Tracking. International Journal of Applied Mathematics and Informatics. Year: 2007, Vol: 1 Issue 1, August 2007, Pages: 36–49.

III. Characteristic Point Sequences in Local and Global Bifurcation Analysis of Filippov Systems. WSEAS Transactions on Systems. Year: 2008, Vol: 7 Issue 10, October 2008, Pages: 840–854.

IV. Numerical Analysis of Sliding Dynamics in Three-Dimensional Filippov Systems Using SPT Method. International Journal of Mathematical Models and Methods in Applied Sciences. Year: 2008, Vol: 2 Issue 1, August 2008, Pages: 342–354.

V. Characterizing Points on Discontinuity Boundary of Filippov Systems. Modelling, Identification, and Control (MIC 2008). February 2008, Innsbruck, Austria.

VI. Analyzing Sliding Bifurcations on Discontinuity Boundary of Filippov Systems. Recent Advances on Applied Mathematics. Proceedings of the American Conference on Applied Mathematics (Math '08). 24–26 March 2008, Cambridge, Massachusetts, USA.

VII. SPTCont 1.0: A LabView Toolbox for Bifurcation Analysis of Filippov Systems. New Aspects of Systems. Proceedings of the 12th WSEAS International Conference on Systems. 22–24 July 2008, Heraklion, Greece.

VIII. Detecting Sliding Areas in Three-Dimensional Filippov Systems Using an Integration-Free Method. New Aspects on Computing Research. Proceedings of the

2nd European Computing Conference. (ECC'08) 11–13 September 2008, Malta, Pages: 160–166.

IX. Localization of sliding bifurcations in a rotational oscillator with double cam. Journal Dyna. Nro 167, pag 160-168, 2011.

X. Detection and Continuation of Filippov Sliding Limit Cycles Bifurcation in Multiple DB Systems. Submitted to IJBC in 2010.

This document begins with the definitions of the simplest elements belonging to a system. Then, in each section, the complexity of these elements and their interactions is increased. In the first part, the classification of singular points in the discontinuity boundary (DB) is expanded. This classification has been expanded because it has been found that the DB of nonsmooth systems contains a considerable amount of useful information, which can be used for determining the dynamics of the system. Using groups of points, I build curves and surfaces that are included in the classification. In order to include each element in the classification, an exclusive condition of existence has been generated. Boolean-valued functions based on geometric criteria are used for formulating these conditions. Each element, point, curve, or surface has been illustrated with didactic symbols and colours.

In the subsequent sections, changes in the groups of elements have been associated with changes in the parameters. Therefore, an identification of the changes in the groups of elements will help to determine the occurrence of bifurcations. The identification procedure begins with a complete analysis of planar systems where the groups of singular and special points build chains that provide information related to the dynamics of these systems. Then, this method is extended to three-dimensional systems where the groups of singular points form 2-dimensional networks. For n-dimensional systems, the groups of singular points form a (n-1)-dimensional hyper-networks. As a result, the SPT method is very precise with regards to the detection of a local sliding bifurcation and helps in the location of a useful initial point for tracking a global sliding bifurcation. Then, during the detection of global bifurcations, the singular points are considered a part of the sequences that determine each class of an orbit or a cycle. Using this characteristic, I have developed an extension of the SPT method that allows continuation of sliding cycles.

In order to demonstrate the use of this method, published papers include illustrative examples. The classical oscillator that is used for demonstrating most of the new methods in nonsmooth systems has also been used in this case. Further, the Cassini model (1), (2) which is quite new and has been demonstrated to be useful for the validation of methods related to systems with multiple DBs, has been used. Moreover, new models have been developed to enrich other fields of analysis.

The complete catalogue of local bifurcations for Filippov systems (a subclass of nonsmooth systems) presented by Kuznetsov (3) has been used for validating the SPT method. The result is promising; further, new features have been added to this method in order to increase its applications. Chapter 5 describes the additions to the method that have proven to be useful for tracking sliding cycles in order to detect the bifurcations of sliding cycles.

The SPT method has some interesting features: after the method was implemented in numerical solutions, the computing time was reduced in comparison with the traditional methods. The new form to present the results, using diagrams and a symbolic representation to indicate the sequences of the elements and the dynamics opens a door to a wider graphic tool for analysing complex systems. Finally, in order to support the validation of the abovementioned method, a toolbox of numerical functions that correspond to the equations presented in this document was developed. With this toolbox, it was possible to perform comparisons, develop new examples, and confirm the utility and the power of the SPT method.

Keyworks: Nonsmooth systems, sliding bifurcations, continuation, limit cycle, Filippov systems, singular point.

Resumen

El principal tópico de esta investigación está relacionado con un nuevo método para mejorar el entendimiento del comportamiento y el análisis de bifurcaciones deslizantes en sistemas no suaves. Este método ha sido llamado seguimiento de puntos singulares (Singular Point Tracking) debido a que está basado en la identificación y seguimiento de algunos puntos con características únicas. Estos puntos han sido poco estudiados motivando el deseo de realizar un más profundo estudio acerca del tema. Como resultado de esta investigación, se ha encontrado que la información de los puntos singulares unida a la información de los puntos regulares puede ser usada para encontrar o detectar bifurcaciones no suaves.

Los resultados del método han sido presentados en eventos internaciones, publicados en revistas especializadas y son el principal bloque de este documento. En las publicaciones listadas abajo sólo se ha realizado un pequeño cambio en el formato, en comparación con las versiones publicadas originalmente. En el resto del documento, utilizando un formato de catálogo, se presenta cómo fue desarrollado el método.

1. Integration-Free Analysis of Nonsmooth Local Dynamics in Planar Filippov Systems. International Journal of Bifurcations and Chaos. Year: 2009 Vol: 19 Issue: 3, March 2009 Pages: 947–975.
2. Continuation of Nonsmooth Bifurcations in Filippov Systems Using Singular Point Tracking. International Journal of Applied Mathematics and Informatics. Year: 2007, Vol: 1 Issue 1, August 2007, Pages: 36–49.

3. Characteristic Point Sequences in Local and Global Bifurcation Analysis of Filippov Systems. WSEAS Transactions on Systems. Year: 2008, Vol: 7 Issue 10, October 2008, Pages: 840–854.
4. Numerical Analysis of Sliding Dynamics in Three-Dimensional Filippov Systems Using SPT Method. International Journal of Mathematical Models and Methods in Applied Sciences. Year: 2008, Vol: 2 Issue 1, August 2008, Pages: 342–354.
5. Characterizing Points on Discontinuity Boundary of Filippov Systems. Modelling, Identification, and Control (MIC 2008). February 2008, Innsbruck, Austria.
6. Analyzing Sliding Bifurcations on Discontinuity Boundary of Filippov Systems. Recent Advances on Applied Mathematics. Proceedings of the American Conference on Applied Mathematics (MATH'08). 24–26 March 2008, Cambridge, Massachusetts, USA.
7. SPTCont 1.0: A LabView Toolbox for Bifurcation Analysis of Filippov Systems. New Aspects of Systems. Proceedings of the 12th WSEAS International Conference on Systems. 22–24 July 2008, Heraklion, Greece.
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9. Localization of sliding bifurcations in a rotational oscillator with double cam. Journal Dyna. Nro 167, pag 160-168, 2011.
10. Detection and Continuation of Filippov Sliding Limit Cycles Bifurcation in Multiple DB Systems. Submitted to IJBC in 2010.

El documento comienza con la deducción de los elementos más simples pertenecientes al sistema y de ahí comienza un proceso progresivo en el cual, en cada sección, la complejidad de los elementos y sus interacciones es incrementada. En la primera parte es expandida la clasificación de los puntos singulares en el límite de discontinuidad. Esta clasificación ha sido ampliada porque se ha encontrado que el límite de discontinuidad de un sistema no suave guarda mucha información útil para determinar la dinámica del sistema. Utilizando estos grupos de puntos han sido creadas cadenas y mallas que también admiten clasificación. Para incluir cada elemento en la clasificación han sido generadas condiciones exclusivas de existencia. Para formular estas condiciones han sido usadas funciones booleanas basadas en criterios geométricos. Cada elemento, punto, curva o superficie ha sido caracterizada con símbolos y colores didácticos

En la secciones subsecuentes los cambios de los grupos de elementos han sido asociados con los cambios de parámetros y, de esta forma, ellos permiten la ocurrencia de bifurcaciones. El procedimiento comienza con un completo análisis en los sistemas planos donde los grupos de puntos singulares forman cadenas que dan información acerca de la dinámica del sistema. Luego, el método es extendido a sistemas tridimensionales donde los grupos de puntos singulares forman mallas. En sistemas n-dimensionales los grupos de puntos singulares forman hiper-superficies (n-1)-dimensionales. Como resultado, el método

SPT es muy preciso en la detección de bifurcaciones deslizantes locales y ayuda en la localización del punto inicial útil para realizar el seguimiento de bifurcaciones deslizantes globales. Ya dentro de la detección de bifurcaciones globales, los puntos singulares son parte de las secuencias que determinan cada clase de órbita o ciclo. Usando esta característica ha sido desarrollada una extensión del método SPT que permite la continuación de ciclos deslizantes.

Para demostrar la forma de usar este método, los artículos publicados incluyen ejemplos ilustrativos. El oscilador clásico que es usado para mostrar la mayoría de los nuevos métodos en sistemas no suaves también ha sido usado en este caso. También ha sido considerado el modelo de Cassini (2), (1) el cual es reciente, y está demostrando ser útil para validar métodos relacionados con sistemas de múltiples límites de discontinuidad. Nuevos modelos también han sido desarrollados para enriquecer otros campos de análisis.

El catálogo completo de bifurcaciones locales para sistemas de Filippov (una subclase de los sistemas no suaves) presentado por Kuznetsov (3) ha sido usado para validar el método SPT. Los resultados son prometedores y desde entonces el método ha recibido nuevas capacidades para ampliar sus usos. Ha sido anexada una extensión en el Capítulo 5 que es útil para seguir ciclos, con el fin de detectar bifurcaciones de ciclos deslizantes.

El método SPT tiene algunos rasgos interesantes: después de que el método fue implementado en soluciones numéricas, en comparación con los métodos tradicionales, el tiempo de computación fue reducido. La nueva forma de presentar los resultados, utilizando mapas y representación simbólica para indicar las secuencias de elementos y dinámicas, abren una puerta hacia una más amplia herramienta gráfica para analizar sistema complejos. Finalmente, como soporte para validar el método, ha sido desarrollada una caja de herramientas de funciones numéricas que corresponden a las ecuaciones presentadas a lo largo de este documento. Con esta caja de herramientas, ha sido posible hacer comparaciones, desarrollar nuevos ejemplos y confirmar la utilidad y el poder del método SPT.

Palabras claves: Nonsmooth systems, sliding bifurcations, continuation, limit cycle, Filippov systems, singular point.

Chapter 1

Introduction

The main objective of this research is to develop a new method for improving the understanding of the behaviour and the analysis of sliding bifurcations in nonsmooth systems. This method is called singular point tracking (SPT) because it is based on the identification and synthesis of certain points related to unique characteristics. These points have been not given much attention, thus motivating an in-depth study on the subject.

This Ph.D. Thesis fits into the global picture of research in nonsmooth systems showing sliding dynamics. This study deepens the results presented by Kuznetsov *et al.,* (4) using the geometrical characteristics of points in the DB. Also we study the relation among sequences (or chains) of points over the DB and the sequence of sets over the DB, with sliding bifurcations. Further, the results consistently fit considering the relation with points which are part of limit cycles.

This research topic was selected on the basis of the suggestions of certain researchers in their papers. In the following paragraph, I have discussed the complaints, recommendations for future work, and points emphasized by some of the leading researchers in this field. I have presented the context of these comments and then I have explored the possibilities to contribute to any of the needs. A more detailed explanation of these comments is presented in Section 2.3 (State of the Art of Nonsmooth Systems).

In (3), an organized classification of the local sliding bifurcations was discussed for the first time. Global bifurcations have been reported in detail in
(5), (6), (7), (8), and others. Here are some of the comments on the topic:

- There is an increasing number of interesting results on the bifurcations of periodic solutions in three-dimensional Filippov systems specifically in general n-dimensional Filippov systems. Considerably little is known about local bifurcations in n-dimensional systems
- The bifurcation analysis of piecewise-smooth systems (PSS) has received considerable attention in the last few years. However, in most cases, the study was restricted to continuous PSS or to bifurcations of Filippov systems without sliding. This greatly simplifies the analysis, since, as we will see, sliding bifurcations are

many and of a considerably subtle nature. Indeed, the appearance or disappearance of sliding at a particular parameter value is a bifurcation, even if it leaves the attractors of the system unchanged.

- Very little is known about the normal forms and the numerical analysis of sliding bifurcations.
- Apart from numerous applications, there are two natural directions in which the analysis presented in this paper can be extended: to more dimensions and to higher codimension.

SlideCont, presented by Dercole *et al.*, (9) is a software application of a method focusing on the sliding bifurcation. SlideCont is a suite of routines accompanying Auto97 (10), which allow one to perform bifurcation analysis of generic discontinuous piecewise-smooth autonomous systems of ordinary differential equations. Some comments of the authors of these papers are as follows:

- There are several directions in which this work could be extended. First of all, there are interesting global sliding bifurcations in n-dimensional Filippov systems that involve multiple sliding and which are therefore unsupported in SlideCont 2.0. In the planar case, further efforts are required for implementing a more systematic detection of codimension-2 local and global bifurcations and branch switching at such bifurcations.
- The current version has two major limitations. The first is related to the fact that an orbit of a Filippov system might cross the DB p times and involve q sliding sets. The continuation of a corresponding solution in Auto97 would require a boundary value problem with (p+1+q)n differential equations, a number that is not determined a priori. One way to allow the continuation of a generic solution is the automatic generation of the defining system at run-time, a feature that has not been implemented yet. However, as a partial remedy to this limitation, all boundary-value problems involving only one standard set have also been implemented in a modified way, with two standard sets (one in S_1 and the other in S_2) joined at the DB.

  The second limitation is that the stability of pseudo-equilibriums, and sliding and crossing cycles is not computed. However, limit and branch points are detected by Auto97 along a solution branch. Further, limit points can be continued with one extra control parameter. It has also been noted that SlideCont does not support the time-integration of Filippov systems. Finally, SlideCont runs only in command mode with Auto97 and has no graphic user interface (GUI). The development of software that supports all of the abovementioned computational tasks for n-dimensional Filippov systems in one integrated user-friendly graphic environment is another (admittedly ambitious) goal of the future work.

In (11) Merillas gives his concept about SlideCont. The application has the ability to continue equilibriums, limit cycles, and their sliding bifurcations but, to date, it lacks the capability to perform direct numerical simulations of Filippov systems and automatically switch between sliding and non-sliding motions. Moreover, this tool is only useful for Filippov systems, and therefore, mechanical systems with impact and other nonsmooth dynamical systems are not included.

In 2006, various researchers wrote a paper (7) in which is treated in ordered way the recent achievements in the field. It refers to two-parameter nonsmooth bifurcations of limit cycles, classification and open problems. Some of the points presented in the abovementioned paper are as follows:

- Judging from experience with smooth bifurcations, another vital ingredient necessary to develop the applications of the theory of nonsmooth systems is a robust numerical framework for analyzing both regular and C- bifurcations in piecewise-smooth systems. Software such as AUTO (10) and CONTENT (12), when used in the standard mode will in general fail at grazing points. Instead, one needs to build a suite of routines that are specifically designed to compute through a discontinuity set, to accurately detect points of intersections within these sets, and to detect and follow parameter values at which grazings occur.

Further, after an analysis of some methods that can lead to discontinuous systems directly, these eight researchers concluded the following: many problems related to the numerical analysis of nonsmooth systems are yet to be solved.

SICONOS (Simulation and Control of Nonsmooth Dynamical Systems) software development was part of a European project involving different research teams and it was focussed to modelling, simulation, analysis, and control of nonsmooth dynamical systems (NSDS). Basically, the SICONOS platform aims at providing a general and common tool for NSDS present in several scientific fields such as applied mathematics, electrical networks, mechanics, and robotics. In the SICONOS platform, several things need to be implemented. We need to add routines for the continuation of periodic orbits and the detection of bifurcation points. Further, it is necessary to implement more examples in order to test the platform (11).

Thus, there are many directions in which to act, such as the applications towards a specific field of knowledge, the classification of local and global bifurcations, and numerical methods to support the new ideas. Some of these directions require more qualification, so the first decision was to pursue the direction of the applications in the field of the electromechanical systems. In the course of the first studies, when the paper of Kuznetsov *et al.* (4) was studied using some simple algorithms, it was found that there exists a deeper discrimination of singular, tangential, and other points that could be further used in one of the possible research works. Nonsmooth systems, due to its unique nature involving the DB, which has characteristics of more than one vector field, has a considerable amount of information that could be used for studying these class of systems in a different way from the one used thus far.

Then, the direction of the research was slightly changed; we focused on obtaining information related to special elements on the DB of nonsmooth systems and the relation among them when the values of the parameters are changed.

1.1 Objective

Bifurcation analysis in smooth systems has been very well studied, and we have now many tools that let us deal with almost all smooth systems. Recently, smooth bifurcations generated in nonsmooth systems have also received considerable attention. Currently, the number of tools used for bifurcation analysis is increasing. However non-smooth bifurcations such as those involving sliding sets, generated in nonsmooth systems have not been fully studied and the state-of–the-art is still in the classification process. The first available tools for the analysis are being improved and at present only cover low-dimensional and relatively simple systems. Although the theory of smooth systems plays still an important role in the framework of nonsmooth systems, the vast majority of problems in nonsmooth dynamical systems call for completely new methodologies.

On the basis of the previous paragraph, a practical objective is to contribute to the process of developing better tools or new methods for analysing those bifurcations which appear only in nonsmooth systems. A modified direction from that of the analysis of smooth systems should be taken due to the difference in the mathematical approximations between both types of systems. The mathematical initiative must deal with the unusual behaviour of nonsmooth systems.

Therefore, the main objective of this work is to develop methods for the detection of changes in the flow, orbits or sub-spaces of nonsmooth systems mainly using the information of the elements that constitute them. This method should be based on the geometrical changes due to the definition of bifurcations for these classes of systems given in the literature.

The steps required for achieving this aim are as follows:

- Study the different characteristics of singular, tangential, collinear, sliding, crossing, and regular points in order to find differences that allow us to easily distinguish among these different types of points.
- To deduce mathematical and geometrical relations and develop a method to identify and detect each type of points.
- Study the sequences of points in the DB and the changes in the set when a parameter value is changed, on the basis of previous works.
- Study the conformation of the well-known sliding cycles.
- Develop a method for recognizing and comparing cycles.
- Develop a method for recognizing and comparing sequences of cycles.

1.2 Main result(s).

This research has revealed that the information related to singular points when combined with the information related to regular points can be used for finding or detecting nonsmooth bifurcations.

As results, we have a new method, which we called singular point tracking (SPT). It has been presented and tested. The tests and verifications have proved that this method shows information that will improve our understanding of the behaviour and analysis of sliding bifurcations in nonsmooth systems.

Research was expanded to include the classification of 36 singular and special points on the DB. This expansion becomes the stone to support the method. Over the DB, the sequences of singular and special points with segments composed of crossing and sliding points become closely related with the dynamics of the neighbourhood of the DB. Through Poincaré maps we observed that when the parameter values were changed, the singular points transformed into curves between areas in which all points had the same dynamics. These dynamics differed from those of the next area. On the basis of this information, we could find the parameter values that generate bifurcations. Each element, point, line, or surface has been illustrated with didactic symbols and colours with good results. Taking a look at the symbols in the figures and taking into account the labels nonlinear phenomena can be easily analyzed.

The results of the SPT method are promising and have been presented at international events and published in specialized journals. To demonstrate the way to use this method, these articles include illustrative examples. The classical oscillator used for demonstrating most of the new methods in nonsmooth systems has been used in this study. The Casini model (13), (2), has been demonstrated to be useful in validating methods related to systems with multiple DBs, has also been used. New models have also been developed to explore other fields of analysis. The complete catalogue of the local bifurcation for Filippov systems (a smaller class of nonsmooth systems) presented by Kuznetsov (4) was used for validating the SPT method. The result was hopeful, and since then, new features have been added to this method in order to increase its applications. The extension discussed in Chapter 5 to track the sliding cycles for detecting bifurcations has been proved as a new way to study global dynamics in nonsmooth systems. An advantage of the SPT method is that it is based on previously collected information, and the performance gets better with an increase in the amount of information in the database. In this document, we have presented a large group of sequences of the dynamics in the DB and the sequences of periodic solutions that enrich the database and were recollected in the format of a catalogue. It is useful to know how this method was developed in a sequential form in order to carry out in-depth consultations.

The SPT method has some interesting features: when the tracking was implemented in the numerical solutions, the computing time decreased in comparison with brute- force methods. Further, the new way to present the results, using diagrams and the symbolic representation to indicate sequences of element and dynamics, opens a door to a wider graphic tool for analysing complex systems. Finally, as a support to validate the SPT method, a toolbox of numerical functions that correspond to the equations presented in this document was developed. Therefore, it was possible to perform comparisons, develop new examples, and confirm the utility and the power of the SPT method.

Finally, we can conclude that the SPT method is very precise regarding direct detection of most of local bifurcations catalogued and helps in the location of initial points used for tracking global bifurcations. Some cases require a study of the analytical differences in which the dynamics shows the same code, especially when the special points involved are saddle-nodes. Now, in the detection of global bifurcations, the singular and special points are a part of the sequences that determine each class of orbits or cycles.

At this moment, the SPT method needs to be improved further in order to obtain the benefit of a tool which runs fast in your research. Addition of more information to the database is needed which will increase the reliability and performance of the method. This information would consist of new sequence codes for local dynamics and for sequences of periodic solutions. As the method runs with more information, the user needs to perform less previous analysis to study any system. One branch of applications where we found many possibilities is the one associated with mixed phenomena such as dry friction and impact. From the geometrical and numerical points of view, there are also improvements to be made in the software application that is used for driving the method. It is mandatory to expand the analysis to systems with a DB composed of non flat surfaces. Finally, a comparison of some known models simulated with the SPT method has revealed that some numerical problems could be improved.

The SPT method differs from other approaches in terms of the identification and manipulation of the system information. While the methods in (9), (14) consider a system as one entity to be solved by a group of equations, the SPT method is based on previously collected information of the orbit in the evolution of the system. For that reason the first step it to break the orbit into entities. Then, the method identifies the entities of the orbits evaluating several characteristics such as its position, order, number, and ownership. This information is compared with the previously collected information and determines which situation is presented and how to change parameters and initial conditions in order to obtain more results. In the case of the first group of methods that uses boundary value problems as the main tool, the main difficulty is that practical systems can have many DBs and a high dimension. This situation leads to a large group of equations that need to be manipulated and solved. The SPT method was conceptualized for systems with many discontinuity boundaries and a high dimension; the procedure for the comparison is the same for both simple and complex systems.

Another difference of the SPT method is the tool used for validating the approach developed through this work. Other approaches (9), (14) have used Auto (10), which has provided a considerable number of successful results in the treatment of problems in the field of smooth bifurcations. However, some characteristics are restricted: the graphical user interface (GUI) is not very friendly. Another approach (15) uses the environment of Matlab®, which is a tool with all the characteristics required for developing any application. This includes a group of toolboxes that allow one to solve differential equations using various methods, a group of functions to detect events, and a wide GUI with a good set of graphics that aid the comprehension of the results. This would have been also a good tool selected for solving boundary value problems. The SPT method was tested using a slow, high-level language, Labview®, graphically programmed, with similar characteristics to Matlab and with the possibility to share codes. Labview was selected because the highly developed editing and debugging functions, allowing the shortest development times.

1.3 Motivation

With the advances in mechatronic technology, the ways to design, to make, and to merchandise machines and devices have changed. The electrical drivers have been integrated with the mechanisms, sensors, and controls in order to build up highly integrated servomechanisms of a growing intelligence. These servomechanisms are being manufactured and sold to be incorporated in multiple systems. Moreover, servomechanisms are now common elements in electrical appliances, vehicles, and all types of machinery. Rapidly, the servomechanism has changed from being a expensive and exclusive subsystem only included in spaceships, airplanes, and sophisticated machinery to be a part of many common systems such as car brakes, steering, computer disks, and cameras.

Servomechanisms consist of elements that have mass. They are elastic and have friction with each other. These servomechanisms move and spin. They get separated and then move close to each other upon impact. When a mathematical model is built, it is found that its responses are not linear. Further, when the model includes friction or mechanical backlash, the equations that represent the system are nonsmooth. These behaviours generate multiple challenges with regards to the development of models and studies which are needed to improve the systems performance.

Finally, research on servomechanisms is a dynamic field where new results are presented every year. These advances are due to the demand for more robust and economical machines and equipment with better performance and greater efficiency.

1.4 Justification

Discontinuous events characterize the behaviour of a large number of dynamical systems of relevance in applied science and engineering. The field of nonsmooth systems with emphasis on mechanical, biological, electronic, economical, and social systems are branches of the knowledge and has many theoretical and practical problems to be solved. In the last ten years, many theoretical achievements and new numerical methods for nonsmooth bifurcation analysis have been reported, but it is still almost in the start point since each new solution reveals its own set of problems.

Non-smooth mechanical nonlinearities are abundant in nature and they are usually related to a combination of phenomena which are difficult to isolate. They can be friction, systems with a discontinuous restoring force due to rigid stops or set-up elastic stops, simple impact, and the discontinuous characteristics of intermittent contacts between some components.

Nonsmooth systems appear in many types of engineering systems and in everyday life. Examples include the stick-slip oscillations of a violin string and grating brakes (16). Some related phenomena such as chatter and squeal cause serious problems in many industrial applications and, generally, these sorts of vibrations are undesirable because of their detrimental effects on the operation and performance of mechanical systems (17). Brakes are one of the most important safety and performance components in vehicles. Appropriately, ever since the advent of vehicles, the development of brakes has focused on increasing the braking power and reliability. However, the refinement of vehicle acoustics and comfort through the improvement in other aspects of vehicle design has dramatically increased the relative contribution of brake noise to these aesthetic and environmental concerns. Brake noise is an irritant to consumers who believe that it is symptomatic of a defective brake and signals a warranty claim, even though the brake operates exactly as designed in all other aspects. Therefore, noise generation and suppression have become prominent considerations in the design and manufacture of brake parts. Indeed, many material producers for brake pads spend up to 50% of their engineering budgets on issues related to noise, vibration, and harshness. A wide array of brake noise and vibration phenomena is described by an even wider array of terminology. Squeal, groan, chatter, judder, moan, hum, and squeak are just a few of the terms found in the existing literature (18). Another element related to brakes and friction is the clutch, an element of the power train vehicles. In recent years, the research effort on power train systems has mostly been concentrated upon driveline vibrations. Clutch vibrations have been studied to a relatively less extent. The refinement in the design of vehicle power train systems for reducing vibration and noise is greatly assisted by dynamic modelling and analysis, and there are many complex phenomena that need to be analyzed in an entire power train. One of them is the judder and stick-slip phenomenon that occurs during clutch engagement. Judder is a friction-induced vibration between masses with sliding contact. Stick-slip is the non-linear intermittent sliding stiction (sticking) at a contact surface.

Numerous studies in recent years have highlighted the dynamics of mechanical systems with a discontinuous restoring force due to rigid stops, set-up elastic stops, and dry friction (19), (20), (21), (22) , (23). Compared with the dynamical systems having a smooth vector field, the systems with a discontinuous vector field behave in a more complicated way. As pointed out by Nordmark *et al.* (24), for instance, the motion of a harmonically-forced impact oscillator may become chaotic all of a sudden when it grazes a rigid stop with a variation of a control parameter. Further analysis based on local mappings reported by

Nordmark revealed the nature of the grazing phenomena of the impact oscillator with a rigid stop (25), (20).

Mechanical systems subject to impact effects have been recently the focus of renewed interest in the mechanical engineering and applied mathematics scientific communities. Over the past decades, significant attention has been paid to vibro-impact problems because of their significance on the performance and life of mechanical systems. However, it is extremely difficult to solve these vibro-impact problems with strong non-linearities and non-smoothness (26). The principle of operation of vibration hammers, impact dampers, shakers, pile drivers, and machinery for compacting, milling, and forming is based on the impact action for moving bodies. Impacts also occur in the case of other equipment such as mechanisms with clearances, heat exchangers, fuel elements of nuclear reactors, gears, piping systems, and wheel-rail interaction of high-speed railway coaches, but these impacts are undesirable as they lead to failures, strain, reduced service life, and increased noise levels. Research on repeated impact dynamics is important regarding optimization design of machinery with rigid obstacles or clearances, noise suppression, and reliability analyses. The physical process during impacts has strongly non-linear and discontinuous characteristics. The presence of the non-linearity and the discontinuity considerably complicates the dynamic analysis of repeated impact systems. However, these systems can be described theoretically and numerically by discontinuities that are in good agreement with the reality. Compared with a single impact, vibro-impact dynamics are more complicated and hence have received significant attention (22), (23). In the context of actual mechanical systems, this can lead to a sudden fatal loss of control and the subsequent system failure. For example, high-precision mechanics notwithstanding the inevitable introduction of the play in joints and cylinders, leads to noise production, fatigue, fracture, and loss of kinematics control. When train cars travel at speeds above a critical value, the so-called hunting motions are initiated. These motions correspond to lateral oscillations of the cars relative to the track. At large amplitudes, these oscillations result in impacts with the rails that are believed to be responsible for reports of discomfort to train passengers. Discontinuities are sometimes unavoidable, as in the presence of limits on the motion of the parts of a mechanism; or sometimes they are desirable, as in the presence of ground that allows sustained gait. Their occurrence warrants the development of new nonlinear tools for their treatment (27). In a broad sense, there are two different methods to solve the impact problem in multi-body systems, namely, the continuous and the discontinuous approaches described in specialized publications (28), (29), (30), (31), (32), (26). However, the most interesting part is that advanced dynamics have also been studied, and the level of analysis is increasing from systems with a unilateral constraint and low dimension to multi-body systems and higher dimension (33), (22), (23), (27), (34), (35), (36).

A research subject derived from vibro-impact systems is backlash. This is characterized by a low clearance in elastic transmission systems. The effect of backlash on dynamics has been investigated in the literature, which includes bi-linear or piecewise-smooth systems as well. For instance, a bi-linear model is used for studying the dynamics of compliant off-shore structures for sub-harmonic resonances and chaos (37). Periodically forced bi-linear oscillators were studied (38). The long-term response of models with bi-linear stiffness and damping is studied for the existence and stability of boundary-crossing periodic orbits (39). Phenomena that characterize the response is also investigated. The most general n-periodic solutions and their stability are also studied for tri-linear systems with harmonic forcing (40). Chaos in these systems is also analyzed experimentally, and the results are compared with the theoretical solutions (41).

In many problems related to chemical engineering, a discontinuity may appear. This drives the attention of researchers towards the study of new methods to solve complex dynamics (42), (43), (44). The discontinuity may be a result of an external activity. For example, it is obtained by the addition of a control element or can directly be a natural part of a model under consideration. This could be the case of a closed dynamical system, more precisely, an ideal gas-liquid system. The theory of Filippov systems can be applied to these systems. The dependence of the solution on a given parameter set is studied, namely, the dependence of the solution on the molar in-flow of the gas. It is shown that local sliding bifurcations can appear on the DB.

There are many trends, not only in the comprehension of the phenomena, but in models, that accurately capture the dynamics of the system over the entire range of the system operation. These trends reveal the importance of design, simulation, training, and performance optimization of nonsmooth systems.

Finally, normal forms and the numerical analysis of sliding bifurcations (4) are not completely known. Consequently, any improvement in the field of nonsmooth systems rebounds in a wide range of knowledge, and this is a justification for a further study.

1.5 Methodology

Bifurcations are associated to changes in the behaviour of physical systems. However, since physical systems are represented using mathematical expressions, the changes can be found in the mathematical expressions. But the mathematical representation of a nonsmooth system can be quite complex. Therefore, this forces us to initiate a process of dividing the system into smaller elements and then to analyze these elements. The first step is to carry out an individual analysis of the differential equations representing each vector field. Then, we should analyze the DB since the dynamics are influenced by the elements of the neighbouring regions.

The strategy for comprehension and for future dissemination uses mathematical notation and graphical symbols for representing elements. Graphical symbols allow readers to use a relatively big part of their brains to obtain a faster understanding. Graphic symbols are associated with the main characteristic of each element. The colour and the shape of the symbols will also be a tool in this methodology. Therefore, a report is presented with more than 100 figures, including the mathematical representation.

The results have been presented with a catalogue that includes most of the possibilities so that the SPT method can be used in the study of many phenomena. The new concepts are initially explained and then are reinforced by a continuous presentation in the different systems where they are applicable.

1.6 Outline of the document

The first chapter of this report contains an introduction to the topics. Motivation, justification, objectives, and methodology are also presented. In the second chapter is given a short summary of the necessary basic concepts to support the new ideas that will be presented in the subsequent chapters. The third chapter contains a summary of the problems faced by researchers in the field of nonsmooth systems. The presentation is done from the theoretical and numerical points of view.

Chapter 4 presents the contribution of this work to the field of the localization and analysis of local bifurcations in nonsmooth systems, especially those of the Filippov type.

Chapter 5 presents the contribution of this work to the field of the localization and analysis of global bifurcations in nonsmooth systems, especially those of the Filippov type.

Chapter 6 is focussed on the conclusions and then some appendixes follow for completeness. References are included in the final part of this document.

Chapter 2

General Framework

This chapter briefly outlines some issues, which are introductory knowledge provided regarding the objective of developing methods for analysing or identifying bifurcations in nonsmooth systems (NSS). We discuss dynamic systems (DS) with an emphasis on discontinuous characteristics. Three classes of nonsmooth (also called variable-structure, or hybrid) systems are mostly considered. Numerically, we can think also on a classification (event-driven systems, time-driven systems). One of the most important characteristics that can be generated by the interaction between two vector fields is sliding dynamics, which can be solved using the Filippov or the Utkin method.

For completeness purposes the state-of-the-art in the field of nonsmooth bifurcations, including sliding bifurcations, is presented at the end of this chapter.

2.1 Dynamic system

The engineering models in which the phenomena of friction, mechanical clearances, and impact are involved, can be modelled by assuming that all elements have concentrated parameters. Therefore, ordinary differential equations are used in these models.

A dynamical system is a mathematical representaction of a deterministic process (45). All possible states of a system can be represent by points in a set X called state space in a way $X = \{\boldsymbol{x} : \boldsymbol{x} \quad with \quad \boldsymbol{x} \in \mathrm{R}^{\mathrm{n}}$ is a state of the dynamcal system $\}$. The state space is a space of real vectors of dimension *n*. The evolution of a dynamical system supposes a change of state in a time $t \in T$, where T is an ordered set.

2.1.1 Classes of discontinuous systems

Physical systems can operate in different modes and the transition from one mode to other some times is idealized as instantaneous, discrete transition. In this, the time scale of the transition from one mode to another is much smaller than the scale of the dynamics of the modes. The mathematical modelling of physical systems therefore may lead to discontinuous dynamical systems, which switch between different modes, where the dynamics in each mode is associated with a different set of differential equations (46).

Since approximately 2000, discontinuous systems have been organized depending on the discontinuity degree of their orbits and vector fields. In each class, theoretical and experimental works have been generated in order to improve upon the existing knowledge.

Grade 0, systems with jumps in the values of the state-space variables. They are typical of systems in which there is an impact and are modelled by assuming a negligible deformation of the bodies and a very short contact time (47), (48), (28), (29), (26), (30).

Grade 1, systems described by differential equations with discontinuous right hand side. These systems are also called Filippov systems and can develop so-called sliding motions. The vector fields of these systems are discontinuous and typical of systems with dry friction (40), (49), (8), (24), (5), (7).

Grade 2, systems in which the orbits are continuously differentiable but with discontinuities in the derivatives of the first order. An example is the group of systems so called mass-spring-damper with an elastic movement limiter in one or both sides of the strokes (19), (50) , (22), (23).

Higher grade are assigned to system with discontinuities in the second derivate or higher.

2.1.2 Other types of systems related to nonsmooth systems

Due to the richness in the types of systems present in nature and engineering, different mathematical representations are used to study the dynamical systems. Some of these representations are useful for the study of nonsmooth systems, but because of their similarity nature we will not describe them in detail. For more information, see Thiele (51), and Schutter and Heemels (52).

2.1.5 Hybrid systems

We can have combinations of continuous and discrete states, of continuous and discrete time. The resulting systems are called hybrid systems. A hybrid system is essentially a system with a combination of a discrete and continuous states and a combination of time-driven and event-driven evolutions. Typically, the phases of a time-driven evolution are separated by discrete time instants when something happens. Usually, the mode is then characterized by a discrete state variable. The time-driven nature is evident in both discrete and continuous time (53) .

2.1.6 Variable-structure systems

Variable-structure systems (VSSs) with a sliding mode were first proposed and elaborated in the early 1950s in the Soviet Union by Emelyanov *et al.* in their pioneer works. Since then, VSS has be developed into a general design method and is being examined for a wide spectrum of system types, including nonlinear systems, discontinuous systems, multi-input/multi-output systems, discrete-time models, large-scale and infinite dimensional systems, and stochastic systems (54).

VSS is a discontinuous nonlinear system of the form $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},t)$ where $\mathbf{x}=[x_1,x_2,\ldots,x_n] \in R^n$ is the state vector, $t \in R$ is the time variable, and $F(\mathbf{x},t)=[\mathbf{f}_1(\mathbf{x},t),\mathbf{f}_2(\mathbf{x},t),\ldots,\mathbf{f}_n(\mathbf{x},t)]:R^{n+1}\rightarrow R^n$ is a piecewise continuous function. Because of the piecewise continuity of these systems, they behave like different continuous nonlinear systems in different regions of their state space. At the boundaries of these regions, their dynamics switch abruptly. Hence, their structure varies over different parts of their state space.

2.1.7 Piecewise-smooth dynamical systems

A piecewise-smooth dynamical system (PWS) is a discrete or continuous-time dynamical system whose phase space is partitioned into different regions, each associated with a different functional form of the system vector field.

2.1.8 Event-driven systems

The state of the system changes because of the occurrence of an event. An event corresponds to the start or the end of an activity. In general, event-driven systems are asynchronous, and the event occurrence times are not equidistant (52). Typical examples of event-driven mechanisms in mechanical systems are multi-body mechanisms where the event is confirmed when two of the bodies make or finish contact or in the case of bodies with elastic barriers to the movement when the action of the barrier begins or ends.

2.1.9 Discontinuous time-driven systems

The even driven systems and the discontinuous time-driven systems more than a classification are systems in which the adopted name is related with the method of calculus more than the characteristics of the system.

Time-driven systems change state in response to uniformly progressing physical time (51). Further, a change in the state of the discontinuous system introduces a change in the state space. This class of systems is typical of the switched electronic systems (39).

2.2 Solution of sliding systems

Zigzag movement is possible in VSSs. In these systems, the orbits have the tendency to consecutively cross the DB toward the both neighbourhood regions because of the effect of the dynamics of the respective vector fields or the effect of the algorithm of control.

The zigzagging segments are characterized by the following behaviour (see figure .1):

1. If the initial conditions are inside the region Z_1, close to DB (which indicates that equation $\mathbf{f}_1(x)$ is been used to calculate the vector field) the orbit evolves toward the region Z_2, crossing the DB.
2. If the initial conditions are inside the region Z_2, close to DB (which indicates that equation $\mathbf{f}_2(x)$ is been used to calculate the vector field) the orbit evolves toward the region Z_1, crossing the DB.

These two movements can be alternate by great number of times. This type of combination of continuous movement toward the opposite region is called zigzagging because the system evolves along the DB forming orbits with shape similar to a saw tooth. See top of figure 2.1.

This movement is not real in a system triggered by events. The zigzag evolution is due to the size of the step of integration or in other terms the delta of time used for each sample in the process of integration. Independently of how small the delta of time be choose, the form is evenly presented. Moreover, the selection of relatively small deltas of time leads to another problem; the lack of efficiency in the process of integration.

In VSSs commuted by time, the zigzag movement between two vector fields is real. In these cases, the solution that used extremely short delta of time in the simulation to eliminates the zigzag movement get in a problem. The results obtained are unreal due to the physical characteristic of the switches. These devices have low efficient over specified values of frequency. The solution proposed by V. I. Utkin (55), (56) is used for this class of systems.

Note: The concept of zigzagging movement do no avoid that some special system have a real dynamic that is similar to movement described.

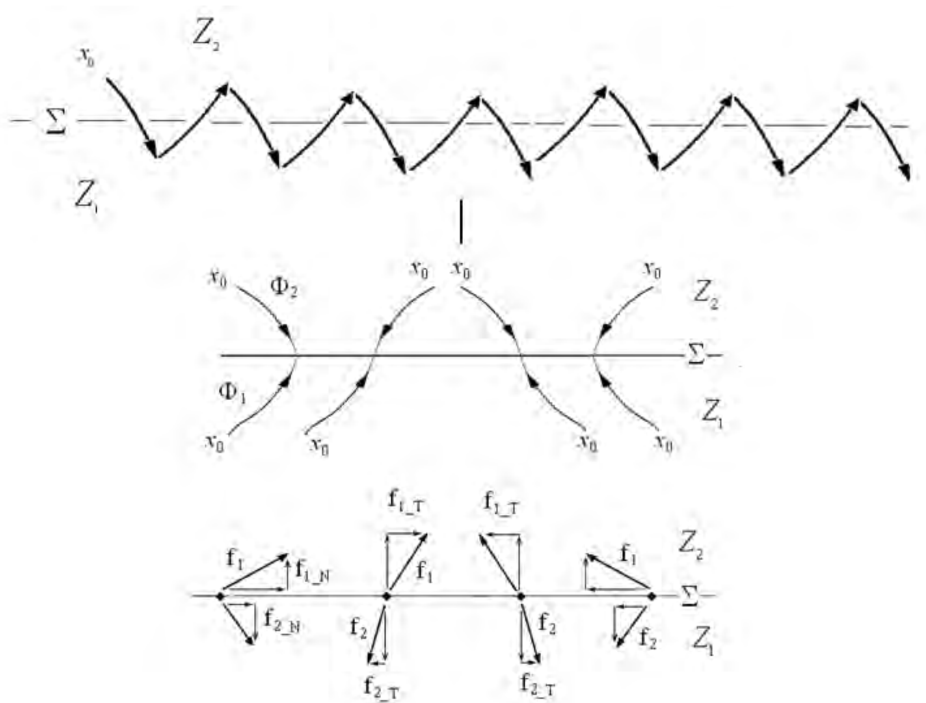


Figure 1. Zigzagging movement and attractive trajectories that generate sliding movements

Figure 2.1 (botton) shows the four possibilities that are observed when the orbits of both sides exhibit the tendency to cross the DB. Further, the vectors normal to the trajectories and the components $\mathbf{f}_{\_T}$ in the direction of the DB are shown. The tangential component to the DB gives a clue of what class of geometric solution that should be used in order to obtain a more real simulation of the movement. The solution more adjusted to the problem in the discontinuity boundary of nonsmooth systems is the proposed by A. F. Filippov (57).

2.2.1 Filippov's method for attractive movements

Filippov systems (57) are defined as n-dimensional systems with a number j of structures. However, in order to simplify the concept, in this section, a planar system (n= 2) with two structures or region where the function is defined, will be used. **X** is a set, also called state space. **x** is a vector indicating a state.

$$\dot{\mathbf{X}} = \{ \quad \mathbf{f}_1(\boldsymbol{x}, \alpha), \ \forall x \in Z_1, \quad \mathbf{f}_2(\boldsymbol{x}, \alpha) \ \forall \ \boldsymbol{x} \in Z_2 \quad \} \tag{2.1}$$

Where $\mathbf{f}_1$ and $\mathbf{f}_2$ are smooth vector functions, Z_1 and Z_2 are the corresponding regions where the vector fields are defined, $\alpha \in \mathbf{R}^1$ is a parameter. The regions are open domains of the planar state space and are defined using the scalar function $H(\boldsymbol{x}, \alpha)$.

$$Z_1 = \{\boldsymbol{x} \in \mathbf{R}^n \text{ such that: } H(\boldsymbol{x}, \alpha) > 0\} \tag{2.2}$$

$$Z_2 = \{\boldsymbol{x} \in \mathbf{R}^n \text{ such that: } H(\boldsymbol{x}, \alpha) < 0\} \tag{2.3}$$

Between Z_1 and Z_2, the state space has a DB, a curve symbolized by $\Sigma_{1,2}$, which is a smooth function from R^n to R , defined as

$$\Sigma_{1,2} = \{\boldsymbol{x} \in \mathbf{R}^n \text{ such that: } H(\boldsymbol{x}, \alpha) = 0\} \tag{2.4}$$

The Filippov solution (57) for a point $\boldsymbol{x}$ located on the DB ($\boldsymbol{x} \in \Sigma_{1,2}$) gives as a result a vector field $\mathbf{G}(x)$ (called the Filippov vector field), which is a convex combination of the vector fields $\mathbf{f}_1(x)$ and $\mathbf{f}_2(x)$.

$$\mathbf{G}(x) = \lambda\, \mathbf{f}_1(\boldsymbol{x}) + (1 - \lambda)\, \mathbf{f}_2(\boldsymbol{x}) \tag{2.5}$$

where λ is defined as a function of the vector fields $\mathbf{f}_1(\boldsymbol{x})$ and $\mathbf{f}_2(\boldsymbol{x})$. In order to obtain the value of λ, the normal vector to the DB is used. The symbol $< \;,\; >$ denotes the standard scalar product and $H_x(\boldsymbol{x})$ a gradient (a normal vector to a point in a curve or surface).

Thus

$$\lambda = \langle\, H_x(\boldsymbol{x}), \mathbf{f}_2(\boldsymbol{x}) \,\rangle \,/\, \langle\, H_x(\boldsymbol{x}), \mathbf{f}_2(\boldsymbol{x}) - \mathbf{f}_1(\boldsymbol{x}) \,\rangle \tag{2.8}$$

with a denominator different of zero

$$x^{\circ} = \mathbf{G}(\boldsymbol{x}); \quad \boldsymbol{x} \in \Sigma$$

defines a scalar differential equation on the sliding set $\sum s$ of the discontinuity boundary, which is smooth on one-dimensional sliding intervals of $\sum s$. Solutions of this equation are called sliding solutions (4).

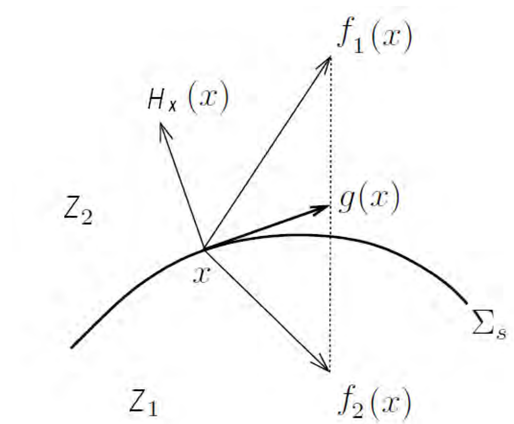


Figure 2. Solution of Filippov.

The Filippov solution is only applicable to points Ωs with attractive or repulsive dynamics. See figure 2. The sliding set ∑s is defined as:

$$\sum_S = \{(\boldsymbol{x} \in \Sigma_{1,2}) \text{ such that: } (\sigma(\boldsymbol{x}) \leq 0)\} \quad (2.9)$$

where $\sigma(x)$ is a scalar interpretation of a geometric condition

$$\sigma(x) = \{\langle H_x(\boldsymbol{x}), \mathbf{f}_1(\boldsymbol{x})\rangle \ \langle H_x(\boldsymbol{x}), \mathbf{f}_2(\boldsymbol{x})\rangle\} \quad (2.10)$$

Acording to the ubication of the regions Z_1 and Z_2 in the figure 2.2 the sliding segment is stable (or attractor) if

$$\langle H_x(\boldsymbol{x}), \mathbf{f}_1(\boldsymbol{x})\rangle > 0 \text{ and } \langle H_x(\boldsymbol{x}), \mathbf{f}_2(\boldsymbol{x})\rangle < 0 \quad (2.11)$$

The segment is unstable (or repulsive or scaping) if

$$\langle H_x(\boldsymbol{x}), \mathbf{f}_1(\boldsymbol{x})\rangle < 0 \text{ and } \langle H_x(\boldsymbol{x}), \mathbf{f}_2(\boldsymbol{x})\rangle > 0 \quad (2.12)$$

Crossing points, which are points where $\sigma(\boldsymbol{x}) > 0$, are also found on the DB. At these points, the evolution of the trajectory will not be within the DB. Instead, it crosses from the region in which has been evolving to the other. The crossing set is defined as:

$$\sum_C = \{(\boldsymbol{x} \in \Sigma_{1,2}) \text{ such that: } (\sigma(\boldsymbol{x}) > 0)\} \quad (2.13)$$

According to the definition in (2.9), it is also possible that a point $\boldsymbol{x}$ located on the DB has the associated vectors with the normal component $f\__N(\boldsymbol{x})$ without magnitude. This is because the vector is tangential to the DB or it vanishes. At such points, either both vectors $\mathbf{f}_1(\boldsymbol{x})$ and $\mathbf{f}_2(\boldsymbol{x})$ are tangent to the DB, or one of them vanishes while the other is tangent to the DB, or they both vanish. Then, the point is within the confines of the definition of a sliding point and is called a singular sliding point (Ωsss).

For this $\sigma(\boldsymbol{x}) = 0$.

$$\Omega sss = \{(\boldsymbol{x} \in \Sigma_{1,2}) \text{ such that: } \langle H_x(\boldsymbol{x}), (\mathbf{f}_2(\boldsymbol{x}) - \mathbf{f}_1(\boldsymbol{x}) \rangle = 0\} \quad (2.14)$$

From (2,9) and (2.12) the crossing set is open, while the sliding set is the union of closed sliding segments and isolated sliding points.

Other special points that define important dynamics in the sliding dynamics G(x) are:

1. Equilibria. Both vector $\mathbf{f}_1(\boldsymbol{x})$ and $\mathbf{f}_2(\boldsymbol{x})$ are transversal to the DB.
2. Quasi-equilibria. Both vector $\mathbf{f}_1(\boldsymbol{x})$ and $\mathbf{f}_2(\boldsymbol{x})$ are anticollinear.
3. Boundary equilibria. One of the vector $\mathbf{f}_1(\boldsymbol{x})$ or $\mathbf{f}_2(\boldsymbol{x})$ vanishes.
4. Tangent. One of the vector $\mathbf{f}_1(\boldsymbol{x})$ or $\mathbf{f}_2(\boldsymbol{x})$ is tangent to the DB.

Note: In this work, the term ‗special points' is used for referring to the points that are alone on the DB. This implies that the points in the neighbourhood are different. Other term also used is isolated point.

Here only introductory concepts of the Filippov concepts are outlined. In next chapter we will do a deeper work. For a more references dealing with different aspects, see (57), (5), (4), (15), (9), (46), (58), (59), (44), (60), (61), (17), (62), (1), (63).

2.2.2 Solution of Utkin

Utkin's equivalent control of Utkin (56) is a solution for controlled VSSs using fixed or variable spaces of time in which each structure is active or valid. These systems are defined in a manner similar to the Filippov systems with the following solution:

$$G(x) = (\mathbf{f}_2(\boldsymbol{x})/2 + \mathbf{f}_1(\boldsymbol{x})/2) + (\mathbf{f}_2(\boldsymbol{x})/2 - \mathbf{f}_1(\boldsymbol{x})/2)\lambda \quad (2.15)$$

In this case, the coefficient λ is defined as

$$\lambda = (\langle H_x(\boldsymbol{x}), \mathbf{f}_2(\boldsymbol{x})\rangle + \langle H_x(\boldsymbol{x}), \mathbf{f}_1(\boldsymbol{x})\rangle) / (\langle H_x(\boldsymbol{x}), \mathbf{f}_2(\boldsymbol{x})\rangle - \langle H_x(\boldsymbol{x}), \mathbf{f}_1(\boldsymbol{x})\rangle) \quad (2.16)$$

with a denominator different of zero.

2.3 State of the art of nonsmooth systems

The study of dynamical systems started with the mathematical representation of the functional relations among the different variables of the considered physical phenomenon. Usually, the result of the modelling process is a high-order differential equation or a set of first-order differential equations. First analysis was carried out using the closed-form solutions of the equations. Later, due to lack of linearity of most of the equations, an approximated solution was obtained using numerical methods and was usually called a simulation.

On the other hand, the main focus of numerical continuation and bifurcation analysis, as opposed to a simulation, is to numerically-compute the continuum families of stationary solutions for the system equations as one or several parameters are varied. Such computational results lead to a deeper understanding of the system behaviour. Moreover, stability, multiplicity of solutions, and bifurcations often provide direct links to the underlying mathematical theories (64).

In mathematics and physics, bifurcation theory has been developed over several decades. Within these disciplines, the emphasis was traditionally placed on theoretical research. Inspired by the discovery of numerous applications and stimulated by new emerging fields, researchers had to renew and reshape some of the results of bifurcation theory. The availability of more powerful computers and the simultaneously appearance of catastrophe and singularity theory were the driving forces that gave bifurcation theory a practical significance (65).

The task of bifurcation classification is to determine the class of systems in which a bifurcation is presented. Therefore, a first rough division is between bifurcations of smooth and nonsmooth systems.

Bifurcations of smooth systems generally produce a change in the number of equilibrium points, periodic orbits, and invariant sets of the system. Now, these bifurcations are called smooth bifurcations in order to differentiate them from the new types of bifurcations that have been discovered subsequently in nonsmooth systems. Smooth bifurcations are present in both classes of systems, smooth and nonsmooth. However, in nonsmooth systems and, with more emphasis in Filippov systems, new types of bifurcations called non-smooth (or discontinuity-induced) bifurcations are present. These non-smooth bifurcations are found only in nonsmooth systems.

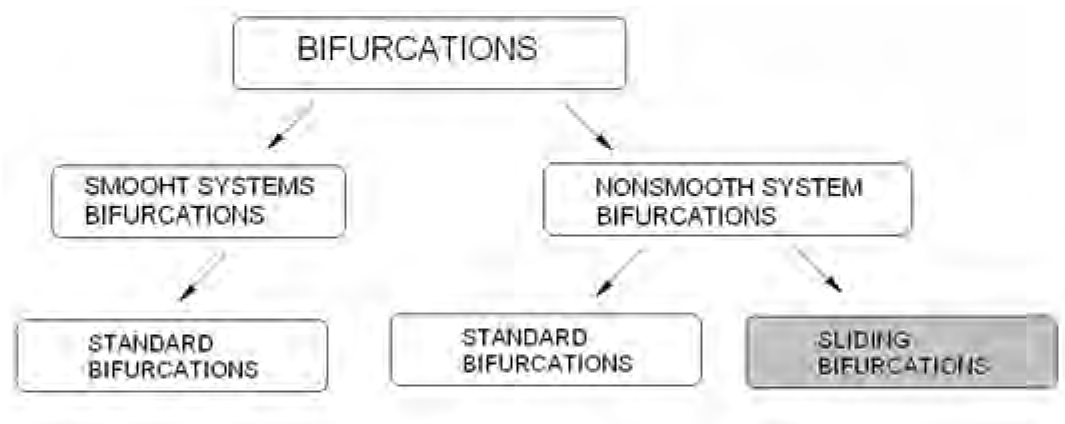


Figure 3. Presence of bifurcations in smooth and nonsmooth systems.

The study of smooth bifurcations in smooth systems is still incomplete as more work is required in relation to the classification of bifurcations of a high co-dimension, but the mathematical and practical background is abundant. Pre-graduate students can access many academic publications. See for example (66), (45).

Moreover due to their robustness, numerical continuation or path-following methods are widely used in scientific applications for smooth systems. With the recent advances in this subject regarding new adaptations, applications, and analysis of efficiency and complexity, numerical continuation is the standard method used for studying bifurcations in smooth systems.

Using the principles and methods of numerical continuation discussed in (64), (65), (67) (68), (12), (69), (70)and others, computer software for the study of bifurcations in smooth systems have been possible. Some examples follow.

MatCont is a continuation package for the interactive numerical study of a range of parameterized nonlinear dynamical systems, in particular ODEs. There is also an interactive graphical package developed in a Matlab environment with MatCont, a command-line version. Both packages allow the computation of equilibrium curves, limit points, Hopf points, limit cycles, and the flip, fold, and Torus bifurcation points of limit cycles.

Locally, in (70), a method with the capacity to locate and to identify local bifurcations of co-dimension one in continuous and discrete autonomous dynamical systems is presented. In this work, direct methods or test functions were used for the detection of the five generic bifurcations of dynamical systems: fold and Hopf for continuous-time dynamical systems; and fold, flip, and Neimark-Sacker for discrete-time dynamical systems. Indirect methods are based on simulation techniques, and in turn, the algorithms are based on continuation techniques that use predictor, corrector, and adjustment stages, using several sorts of parametrization. The software application derived from this method shows a graphical identifier for bifurcations and has been endowed with a graphical interface in a Matlab environment. It allows one to observe the behaviour of the system through variables versus time and phase-space plots. It also shows the appearance of bifurcations in bifurcation diagrams as one or two parameters are varied.

Bifurcations in nonsmooth systems can also be driven by phase-space transitions like those triggered by smooth bifurcations but, often, additional transitions not observed in smooth systems also occur. They are due to discontinuities or jumps in the vector field which represent a very different sort of nonlinearity. The main concept for bifurcation does not change, but careful definitions must be introduced and the methods and test functions used for detecting stability and bifurcations are quite different. Regarding these topics, Leine, in (46), (19) attempts the bifurcations of periodic solutions in discontinuous systems of Filippov type. Furthermore, bifurcations of fixed points in non-smooth continuous systems are addressed. Filippov‘s theory for the definition of solutions of discontinuous systems is surveyed, and the jumps in the fundamental solution matrices are discussed. The way in which the jumps in the fundamental solution matrix lead to the jumps of the Floquet multipliers of periodic solutions is discussed. Floquet multipliers can jump through the unit circle causing discontinuous bifurcations. Numerical examples that exhibit various discontinuous bifurcations are attempted. Further, the question of infinite number of unstable periodic solutions is addressed.

A large group of systems that are relevant to engineering applications is modelled using sets of ordinary differential equations where the right-hand side is discontinuous. Moreover, some systems have trajectories that show jumps in the space state due to the impact of elements of the physical system. Or, in other systems, the trajectories show discontinuities in the derivatives. This group of systems becomes a significant class of nonsmooth systems, and in addition to smooth bifurcations, it shows a class of bifurcations that are unique, such as grazing and transitions to sliding motion.

The classification of these new bifurcations was first attempted by Bautin and Leontovich (71), who obtained an incomplete classification since they did not allow for sliding. Around the same time, Feigin (72) published his work on C-bifurcations.

The next major contribution was made by Filippov (57), who classified singular points in planar discontinuous systems and identified all codim 1 local singularities. However, some unfoldings of local singularities were missing in Filippov‘s work, and bifurcations of sliding cycles were not considered at all. The next contributions were made after 1999, and during all these years, Di Bernardo et al. worked on theoretical and practical problems using the Zero-Time Discontinuity mapping method (20), (5), (7), (62). Finally, Kuznetsov *et al.* (3) presented a catalogue of local and global bifurcations in planar systems. In particular, since the 1990s, this research has been involved in not only the discovery of phenomena and theories (45) but also all important achievements related to the conversion of theoretical knowledge into applications to be used by all researchers (69), (12), (68), (9), (15). Currently, the existing contributions on the sliding bifurcations of cycles refer either to specific bifurcations (27), (73) or to particular classes of systems, like mechanical systems of the stick-slip type (49), (40), (38) (24), (34), relay systems (74), (75), and piecewise linear systems.

From the fundamental work done by Filippov (57) and based on its solution, researchers have developed algorithms, methods, and tools useful for the analysis of sliding systems. Before outlining some of them, we present an alternative to the discontinuous vector field, which is often approximated toward a smoothened vector field. See for example (76). In it for instance, sgn(x) is approximated by $[[2/\pi]\arctan(\varepsilon x)]$. A smooth approximation normally yields a good approximation for large values of ε although difficulties can be expected at repulsion sliding modes. It should be noted that the smooth approximation always has the existence and uniqueness of solutions, whereas this is not the case for a discontinuous system. However, the main disadvantage of the smoothing method is the fact that it yields stiff differential equations, which are expensive to solve (46). The smooth approximation also could avoid the capture of phenomena and that the real systems shows and do not behave in the way that nonsmooth bifurcation can be registered.

In the study of n-dimensional piecewise-smooth dynamical systems, the derivation of appropriate local mappings in a neighbourhood of a grazing event is often required. One of these maps is the so-called Discontinuity Map (DM), introduced in (24). This map can be defined as the correction to be made to the system trajectories in order to account for the presence of a switching manifold in the phase space. One type of DM takes into account the fact that the manifold has been crossed and computes the final part of the trajectory from the corrected initial point on the desired Poincaré section. It is termed Poincaré Discontinuity Mapping (PDM). In (20), another type of DM is discussed. This is the so-called Zero-time Discontinuity Mapping (ZDM). It is obtained by considering the zero-time correction needed to take into account the presence of the boundary. ZDM will be particularly useful for describing the local dynamics of non-autonomous systems (periodically forced), while PDM will be more suitable for the construction of Poincaré maps for periodic orbits that graze. In this sense, DM represents the correction brought about by the presence of the switching manifold. The construction of both PDM and ZDM requires the knowledge of both the flows on each side of the switching manifold. Using these mappings, the local dynamics of a system close to grazing can be effectively described analytically.

The new methods respond to a mathematical or numerical manipulation for obtaining results in the research or the analysis of a bifurcation. Further, for using these methods in the area of engineering, they are better to be converted into software applications. Next, we outline some ideas and methods that end up in software applications.

The simulation of routines detecting sliding modes and simulating the equivalent dynamics is the main characteristic of the method proposed in (77) and (78). The analysis and the design of the method are based on the computation of piecewise-quadratic Lyapunov functions. The computations are performed using convex optimization in terms of linear matrix inequalities. The method was converted in a software application named PWStool, which is a toolbox for computational analysis of piecewise-linear systems. The key features of the toolbox are modelling, simulation, analysis and optimal control for piecewise-linear systems. This software application was developed on a Matlab platform.

The occurrence of a sliding mode poses a difficulty for numerical integration. For solving this problem, Leine (46) proposed a numerical technique for the integration of differential inclusions with sliding modes. The idea of Leine is to construct a band or a boundary layer around the DB, namely, a sub-region Z_Σ, to allow for an efficient numerical approximation. In the sub-region Z_Σ, the vector field is such that the solution is pushed to the middle of the band, to the DB Σ. The sub-region Z_Σ ends when the vector field in Z_1 or Z_2 becomes parallel to Σ. The width of Z_Σ should be sufficiently small to yield a good approximation. A similar approach is used in (15) by Piiroinen *et al.* but adding the condition that the vectors inside the sub-region of approximation are not tangent to the DB boundary, instead they have a little inclination that move the dynamics toward the separatrix line. The result is a faster numerical solution.

SICONOS is an application with a method to solve the equations of dynamical systems using, among others, the principles of complementary formalism. It is dedicated to modelling, simulation, analysis, and control of nonsmooth dynamical systems (NSDSs). Basically, the SICONOS platform aims at providing a general and common tool for NSDSs present in various scientific fields such as applied mathematics, electrical networks, mechanics, and robotics. Currently, researchers in different areas of engineering and applied science often write their own numerical codes for dealing with systems characterized by nonsmooth nonlinearities. These codes are typically specific to the system of interest. The SICONOS platform will attempt to fill this gap, thereby creating new software for solving nonsmooth problems under a common framework. With respect to the SICONOS platform, several things need to be implemented. Routines need to be added for the continuation of periodic orbits and the detection of bifurcation points. Further, it is necessary to implement more examples in order to test the platform. Finally, a detailed study of the consequences of approximating nonsmooth systems with smooth ones is required in order to obtain a better understanding of this technique (11).

SlideCont, the most recent and complete work, has been developed by Dercole *et al.* (9) and it is based on boundary-value problems. It is made up of a suite of routines accompanying Auto97, which allow one to perform bifurcation analysis of generic discontinuous piecewise-smooth autonomous systems of ordinary differential equations (57), of Filippov type, with special attention to planar systems. The complete SlideCont can be used for performing a partial bifurcation analysis of n-dimensional Filippov systems and a considerably more complete bifurcation analysis of planar Filippov systems (n = 2). In particular, SlideCont is a ready-to-use collection of the defining systems for continuing particular solutions of Filippov systems and their bifurcations with respect to at most two control parameters. According to Dercole, this work could be extended in several directions. First, there are interesting global sliding bifurcations in n-dimensional (n > 2)

Filippov systems involving multiple sliding, which are not supported in SlideCont 2.0. In the planar case (n = 2), further efforts are required for implementing a more systematic detection of co-dimension 2 local and global bifurcations and the branch switching at such bifurcations. The current version 2.0 of SlideCont has two major limitations. The first one is related to the fact that an orbit of a Filippov system might cross the DB p times and involve q sliding segments. The continuation of a corresponding solution in Auto97 would require a boundary-value problem with $(p + 1 + q)n$ differential equations, a number that is not determined a priori. One way of allowing the continuation of a generic solution is the automatic generation of the defining system at run-time, a feature that has not yet been implemented. However, as a partial remedy to this limitation, all boundary-value problems involving only one standard segment have also been implemented in a modified form with two standard segments (one in Z_1 and the other in Z_2) connected at the DB. The second limitation, which will also be removed in the forthcoming versions of SlideCont, is that the stability of pseudo-equilibria and of the sliding and crossing cycles is not computed. However, limit and branch points are detected by Auto97 along a solution branch, and limit points can be continued with one extra control parameter. Note that in planar Filippov systems, periodic solutions with a stable sliding segment are always super-stable (i.e., have zero multiplier). It should also be noted that SlideCont does not support the time integration of Filippov systems. Such integration should be based on automatic switching from the computation of an orbit of $\mathbf{f}_i$ to the integration of the system or its modification, and back. Finally, SlideCont runs only in the command mode of Auto97 and has no GUI. The development of a software system that supports all of the above-mentioned computational tasks for n-dimensional Filippov systems in one integrated user-friendly graphic environment is another (admittedly ambitious) goal of their future work. Finally, this method is only useful for Filippov systems and consequently mechanical systems, including impact and other NSDs, are not included.

The method selected by Piiroinen *et al.* (15) for analysing Filippov systems is similar to the hybrid-system approach, where integrations of smooth ODEs are mixed with discrete maps and vector field switches. The idea is to present a numerical algorithm where the user only provides the different vector fields and information about the discontinuity surfaces, and then, the vector fields for the sliding regions are automatically computed by a routine using the descriptive equations. The introduced methods provide a simple way to automatically simulate generic orbits of Filippov systems using a hybrid-system approach. Although many research groups have developed simulation environments for specific nonsmooth problems, we are unaware of a general Filippov system solver as the one proposed here. Therefore, this software c can be used by both applied mathematicians and engineers for such simulations and hopefully make the entire community think about these problems in a more general setting. However, one problem with the hybrid-system approach is that the combinatory complexity of the code increases rapidly with an increase in the number of discontinuity surfaces. Further, for even more efficient calculations, it would be useful to have a script that initially generates a code for the user-specific problem in terms of the state-space dimension and the number of discontinuity surfaces. As mentioned earlier, the algorithm can also be used as a building block for a continuation algorithm that follows both periodic orbits in one parameter and co-dimension-one bifurcations in two parameters using Poincaré maps. The method lacks a general simulation and a continuation interactive environment that supports both discontinuous vector fields and state jumps, so that the user must specify a surface as continuous, Filippov, or impact.

Chapter 3

Problems Formulation in the Analysis of Nonsmooth Systems

The study of the dynamical systems advances has many challenges. Some of them are: the unification of the mathematical representations of the different subtypes of dynamical systems. A second challenge is to find and to classify phenomena that influence the performance of the systems. On the basis of the last challenge, bifurcations that maintain correlations of the changes in the dynamics caused by the changes in the parameters of the system are presented. The third challenge is to find efficient methods for analysis and visualising of the mathematical results and especially the most representative dynamics such as bifurcations. Therefore, a part of this work is to look for functions that can quickly detect the values of the parameters that generate substantial changes in the dynamics, in the mathematical representations.

All the advances are obtained in parallel; an advance in numerical methods allows the detection of new bifurcations that outlines new challenges in the theoretical formulation of problems that require numerical methods again. Therefore, a continuous study in each one of the abovementioned fields is necessary.

3.1 Problems with mathematical modelling

Many researchers have focused on the development of an equivalent method for analysing a general dynamical system. They have selected the special subclasses of hybrid dynamical systems. Some systems included in this class are complementary systems (LCSs), mixed logical dynamical systems (MLDs), piecewise affine systems (PWAs), variable structure systems (VSSs), and systems with inclusions. Each subclass has its own advantages over the others (53). The nonsmooth systems can be classified as a sub-class of the hybrid dynamical systems on the basis of the mathematical representation of the change in the structure when an event is presented.

3.2 Trends in search and classification of bifurcations

In particular, in the field of sliding bifurcations, there are open problems and limitations that still remain. Nonsmooth systems undergo bifurcations more dramatically than smooth systems. This is attributed to the discontinuities or jumps in the vector fields. At present, there is no complete catalogue of sliding bifurcations in *n*-dimensional systems. Moreover, there is no complete knowledge on normal forms and on numerical analysis of sliding bifurcations (9), (15), (4).

3.3 Numerical problems with analysis of nonsmooth systems

The advances in the theoretical field of nonsmooth bifurcations is bundled according to the availability of numerical tools for testing new ideas or concepts and this is the reason why almost all development presented in the numerical field boosted the theoretical field.

In the study of dynamical systems, the research effort on the classification and development of functions for the detection of nonsmooth bifurcations has been slowed mainly by the following reasons:

1. The tradition of evaluating any behaviour in the dynamics using a mathematical tool that provides an orbit or an output to the phase diagram. This tradition is derived from the good results obtained in the study of smooth systems, but it has delayed the development and standardization of new tools, more appropriate with the characteristics of nonsmooth systems.
2. The lack of a graph that represents the vector fields but shows the interaction between two or more vector fields, including sliding, and impacts dynamics.
3. The difficulty to obtain high precision results in the integration processes of nonsmooth systems makes that the results of a research group have acceptation in proportion to the methods and numeric tools they are using.

In the following paragraph, we briefly describe some problems found by researchers when they are using generic mathematical software to develop routines for the integration and detection of bifurcations of nonsmooth systems.

Curly or zigzag lines between vector fields

A problem that is not completely solved by commercial simulation tools is that of computing in systems where the state space is divided into sub-regions and the orbits cross the limits between several (more than two) different sub-regions. This is one of the reasons why many researchers develop their own tools. They attempt to capture complex phenomena that are not visible if the numerical tools use simple algorithms.

Let a system be represented as:

$$\dot{\mathbf{x}} = \mathbf{f}_j(\mathbf{x}, \alpha) \tag{3.1}$$

n: dimension of the state space
m: number of vector fields
Z_j: regions where the vector fields are defined
α: parameter of the system

For this analysis, if $i = 3$ and $j = 2$, then

$$\dot{\mathbf{x}} = (\mathbf{f}_1(\boldsymbol{x}, \alpha), \mathbf{f}_2(\boldsymbol{x}, \alpha)) \tag{3.2}$$

$\mathbf{f}_1$ is defined in the Z_1 region, and $\mathbf{f}_2$ is defined in the Z_2 region.

$H(\mathbf{x},\alpha)$ is a smooth scalar function that determines the DB, symbolized by Σ and is located between the regions Z_1 and Z_2.

$$\Sigma = \{x \in \mathbf{R}^3 : H(\boldsymbol{x}, \alpha) = 0\} \qquad 3.3)$$

and

$$Z_1 = \{x \in \mathbf{R}^3 : H(\boldsymbol{x}, \alpha) < 0\} \tag{3.4}$$

$$Z_2 = \{x \in \mathbf{R}^3 : H(\boldsymbol{x}, \alpha) > 0\} \tag{3.5}$$

Let $\boldsymbol{x}_0$ be the initial conditions for the system of interest. In order to obtain the trajectory for the condition $\boldsymbol{x}_0 \in Z_1$, then $\mathbf{f}_1(x, \alpha)$ is integrated numerically.

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \int_{to}^{t1} \mathbf{f}_1(x_0, \alpha)\, dt \tag{3.6}$$

If $\boldsymbol{x}_1 \in Z_2$

the next point on the orbit will be

$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + \int_{t1}^{t2} \mathbf{f}_2(x_1, \alpha)\, dt \tag{3.8}$$

In contrast,

if $\boldsymbol{x}_1 \in Z_1$

then

$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + \int_{t1}^{t2} \mathbf{f_1}(x_1, \alpha)\, dt \tag{3.9}$$

In some cases, the orbit is surpassed from one side to the other. Then, after a switching in the set of equations, the trajectory can return to the initial region and so forth. In these cases, the simulation shows a shape that sometimes does not correspond to the real phenomena. Zigzagging is presented by the practical impossibility to reduce time step in the integration process to a zero value. Zigzagging looks like a curly line with teeth. See Figure 4.

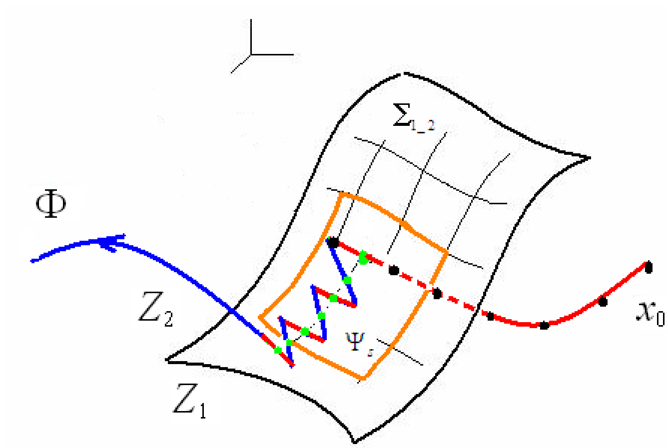


Figure 4. Zigzagging movement in 3D systems.

The zigzagging produced in the continuous and repeatedly changing regions moves according two factors: the size of the time step and the slope of the segments with relation to the DB.

In some cases, the slope of the segments is almost perpendicular to the DB. In these cases, the advance is very gradual until the orbit reaches to a point where it can leave another sector where the dynamics are different. See Figure 5.

Let $\boldsymbol{x}_0$ be the point representing the initial condition of an orbit that after evolving during a dt time arrives to point $\boldsymbol{x}_1$ (In figure 5). In the next integration, the orbit arrives to point $\boldsymbol{x}_2$. Now, the integration from point $\boldsymbol{x}_2$ results in a point very close to point $\boldsymbol{x}_1$. As a consequence, the real advance of the orbit is very small as compared to the time lapsed. The previous phenomena lead to few effective simulations.

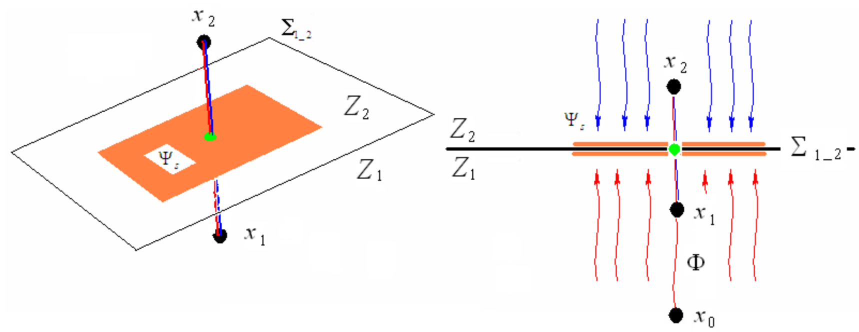


Figure 5. Perpendicular to DB zigzag movement.

The simulation process in which zigzagging is allowed generates some errors. One of them corresponds to when the orbit reaches a sector with attractive dynamics, as is illustrated in Figure 6.

Let $\boldsymbol{x}_0$ be a point representing the initial conditions of an orbit that after some time reaches point $\boldsymbol{x}_1$. After the next integration, the result is point $\boldsymbol{x}_2$. This trajectory is not the real trajectory. The real behaviour evolves from point $\boldsymbol{x}_0$ to point $\boldsymbol{x}_3$ on the DB, as is shown in Figure 6. From point $\boldsymbol{x}_3$, the flow line indicates that the following point should be $\boldsymbol{x}_4$ instead of point $\boldsymbol{x}_1$ or the subsequent point $\boldsymbol{x}_2$.

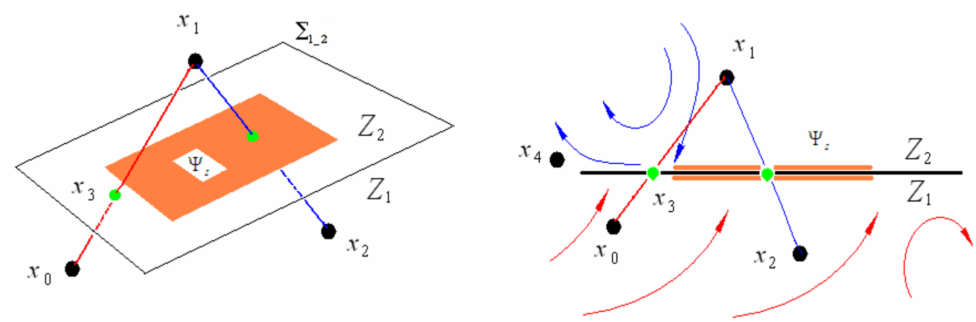


Figure 6. Error generated by an orbit near a sliding sector in the DB.

The zigzagging movement also generates errors when the orbits leave a sector where the dynamics are attractive. Let $\boldsymbol{x}_0$ be the initial condition of an orbit that after some time reaches the DB $\Sigma_{1,2}$ at point $\boldsymbol{x}_1$. In this case, point $\boldsymbol{x}_1$ shows attractive dynamics. In Figure 7, two solutions can be observed:

1. A zigzagging orbit allowing a continuous commutation of the dynamics between the vector fields.
2. A curve which fits the real phenomena

In the first option, a zigzagging segment ($\boldsymbol{x}_1, \boldsymbol{x}_2^{\bullet}$) (red-blue lines) is obtained through the continuous commutation of equations $x_1^{\bullet} = \mathbf{f}_1(\boldsymbol{x}_i,\alpha)$ and $x_2^{\bullet} = \mathbf{f}_2(\boldsymbol{x}_i,\alpha)$ and this is observed until the orbit reaches point $\boldsymbol{x}_2^{\bullet}$. At this point, crossing dynamics from Z_1 to Z_2 are allowed, and the evolution results in the orbit Φ_a.

In the second option, a curve ($\boldsymbol{x}_1, \boldsymbol{x}_2$) that evolves from the orbit Φs on the DB (orange line) until it reaches point $\boldsymbol{x}_2$ is presented. At this point, crossing dynamics from Z_1 to Z_2 are allowed, and the evolution result in the orbit Φ_b.

The error is the distance or the difference between $\boldsymbol{x}_2$ and $\boldsymbol{x}_2^{\bullet}$, which is small on the DB but generates a greater error when the orbit evolves in the Z_2 region. One alternative to minimize the error is to decrease the time step in the integration process. In the limit, when the integration step tends to 0, provided that there are no singular points within the sliding region, the zigzagging trajectory can move arbitrarily close to the sliding trajectory, but the time taken for the integration process will increase.

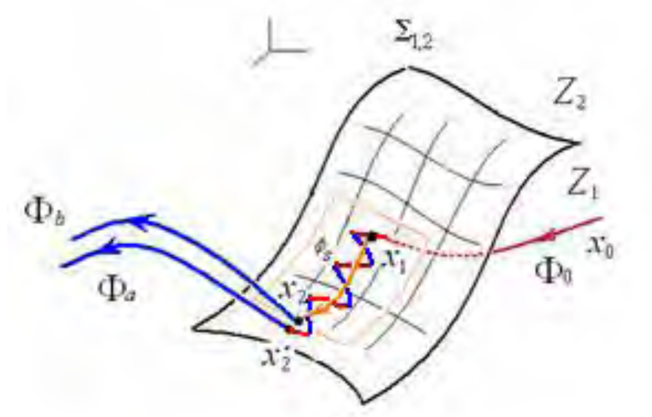


Figure 7. Error generated by an orbit leaving sliding sector in the DB.

Problems in the detection of the change of regions of a trajectory.

The numerical integration process is discrete, and the result of this process is a group of distanced points at which it is assumed that the trajectory is a straight line. As mentioned in the previous paragraph, a change in the vector field where the time step does not produce a point on the DB, generates an error in the trajectory. Then, the correct procedure implies the use of an additional function that performs the following tasks:

1. Detection of a change of region of a trajectory. See Figure 8.
2. Backward movement from point $\boldsymbol{x}_{n+1}$ until the detection of the crossing point $\boldsymbol{x}_c$.
3. Forward movement from point $\boldsymbol{x}_n$ with a time step that assures a new point in the same position as that of point $\boldsymbol{x}_c$.
4. Evaluation of whether at $\boldsymbol{x}_c$ we have crossing or sliding dynamics (or impact).

5. If the dynamics indicates a crossing point, the integration process is restarted from this point using the set of equations $\mathbf{f}_2$. Then, the correct trajectory Φ_b is generated.
6. If the dynamics indicates a sliding point, a Filippov solution is used for obtaining the correct trajectory Φc (if it is a impact point, to runs the law of impact to obtain the jump and the reset to the same region).

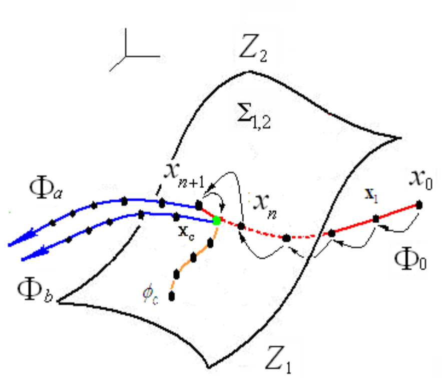


Figure 8. Backward movement close to a crossing point.

The detection of the arrival of the simulation values to the discontinuity boundary where some event can be generated is common in various types of problems related to dynamical systems. Therefore, an arrival detection function is important. Some of the events in which it is important to detect the arrival to a boundary are as follows:

1. Poincaré maps.
2. Limit between two neighbouring region of a state space.
3. Stability of one-dimensional system.
4. Systems with impacts.
5. Start and end of sliding segments in Filippov systems.
6. Zero-speed crossing in the switching representation of a system with friction (Variable structure representation).
7. Continuation of bifurcations using the predictor-corrector method in the correction phase.

Problem of a restart after an impact

This problem is different from the ones discussed thus far. It is a problem related to systems with impacts in which the non-conservative Newton restitution law is used for computing the post-impact velocity. The simplest system of this type is one that has a unique vector field and a limit where the vector field loses its validity and an algebraic function is activated. The representation of this type of system is

$$\dot{\mathbf{X}} = \mathbf{f}_j(\boldsymbol{x}, \alpha) \qquad (3.10)$$

n:	order of the system and equal to $i = 1, 2, \ldots, n$
$m \in$ R:	number of vector fields with $j = 1, 2, 3, \ldots, m$
Z_j:	subspaces where vector fields are valid
α:	parameter of the system
γ:	restitution coefficient

For this analysis, if $i = 3$ and $j = 1$, then

$$\dot{\mathbf{X}} = (\mathbf{f}_1(\boldsymbol{x}, \alpha), \quad \mathbf{f}_\Sigma(\dot{x}, \gamma)) \qquad (3.11)$$

$\mathbf{f}_1$: set of equations describing the vector field on Z_1
f_Σ: scalar equation of the DB

H(x,α) is a smooth scalar function that determines the DB (Symbolized Σ), in such a way that a point in this DB fulfils

$$\Sigma = \{\boldsymbol{x} \in \mathbf{R}^3 : H(\boldsymbol{x}, \alpha) = 0\} \qquad (3.12)$$

and

$$Z_1 = \{x \in \mathbf{R}^3 : H(\boldsymbol{x}, \alpha) < 0\} \qquad (3.13)$$

With

$$f_\Sigma: \quad \Sigma \to \Sigma$$
$$(x, \dot{x}^-) \to (x, \dot{x}^+)$$
$$\dot{x}^+ = -\gamma \dot{x}^- \qquad (3.14)$$

where $\dot{x}^-$ is the arrival speed of the trajectory to the DB, $\dot{x}^+$ is the bounce speed of the trajectory, and γ is the restitution coefficient for this type of phenomena. The value of $\boldsymbol{x}$ corresponding to the place where the impact occurs does not vary.

Let $\boldsymbol{x}_0$ be the initial conditions for the system considered. In order to obtain a trajectory with these initial conditions, an integration process with $\boldsymbol{x}_0 \in Z_1$ is initiated. This process uses $\mathbf{f}_1(x, \alpha)$ with a small time step, if we compare it with the time system response. The result is a trajectory Φ_a, as illustrated in Figure 9.

Once the orbit arrives at the DB at point $\boldsymbol{x}_n$, the vector field $\mathbf{f}_1$ is changed with the function $\mathbf{f}_\Sigma$, which generates point $\boldsymbol{x}_{n+1}$. At this point, $\mathbf{f}_\Sigma$ should be changed again with the original vector field $\mathbf{f}_1$ in order to generate the trajectory Φ_b.

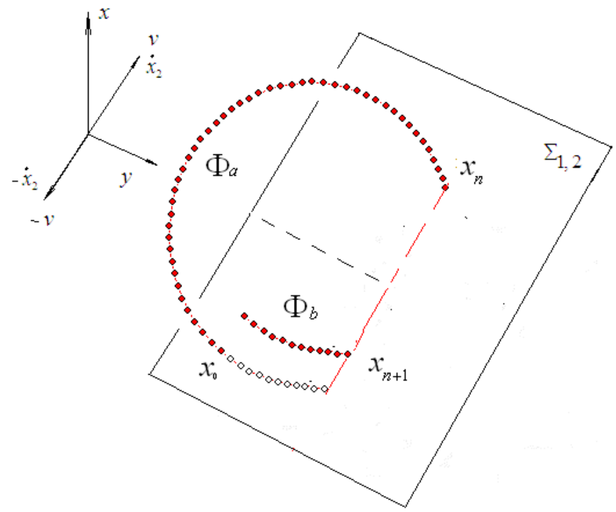


Figure 9. Trajectory of an impact and its bounce.

The problem occurs at point $\boldsymbol{x}_{n+1}$. This point belongs to the DB, and from the definition for piecewise smooth systems used for this class of systems the function $\mathbf{f}_\Sigma$ should be applied to every point belonging to the DB. The result is a point that belongs to the DB. Then, $\mathbf{f}_\Sigma$ captures the dynamics within the DB, which are different from what happens in the real phenomena. Figure 10 shows the error caused by the simulation.

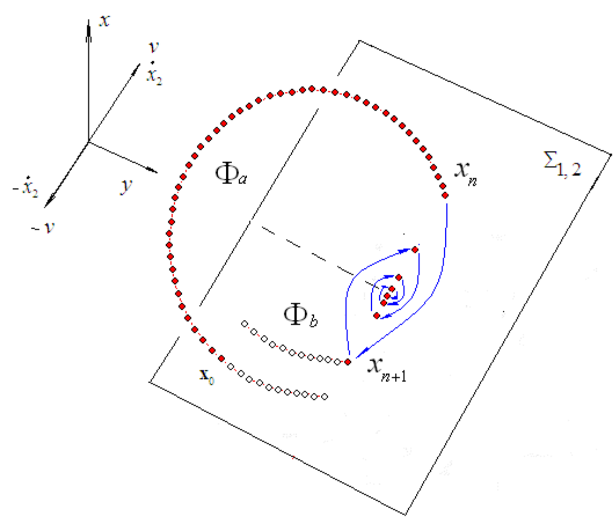


Figure 10. Error in the trajectory of a bounce.

A simple but not exactly solution extend the region of impact to $H(\boldsymbol{x}, \alpha) \geq 0$. In order to solve the difficulty, in numerical applications, a more complex representation that is not available in most of the commercial tools for simulation is required. The evaluation of this additional representation is used as a detector that identifies whether an impact is occurring and proceeds to run a routine of reset once the bounce speed has been calculated.

Problem of impact with infinite bounce

There are many systems where the impact is caused by the separation of two bodies that usually work together. These bodies separate because of a calibration failure, but after some time, they get together again in a sequence of impacts.

The cam-follower is a case in which the follower should remain on the cam, but due to a design problem, the changes in the shape of the cam accelerate the follower over its gravity-producing jumps. The landing of the follower is elastic and generates bounces. See Figure 11.

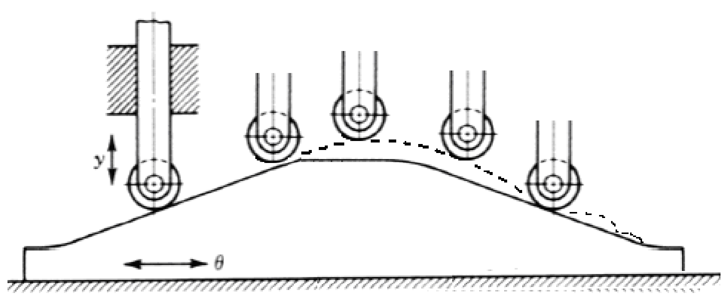


Figure 11. Scheme of a bounce in a cam-follower system.

The analysis of systems with impact and elastic bounce, which are modelled using formulas with the restitution coefficient, reveals a decreasing bounce speed. However, since the restitution coefficient is usually a percentage, in mathematical sense, the bounce speed never reaches zero. This simulation result, compared with the real behaviour, is different because in the real phenomena, after some time, the element comes to rest. This large number of smaller and smaller bounces is sometimes referred as chattering and is characterized by an infinite number of impacts occurring in finite time (79). Numerically is needed take some speed value as a limit. When the bounce speed is under this value is suppose the ball is resting and the iteration is stopped.

Note: chattering takes different meanings according to different knowledge areas (see mechanical systems with friction and vibration (17), mechanical systems with impacts (79), (11), sliding control (80), and so on).

Let w be the number of bounces, the friction of the ball with the air is negligible and the first impact with a speed $\dot{x}_1^-$ different of zero, then mathematically, the final absolute speed $\dot{x}_f$ is equal to

$$\dot{x}_f = \dot{x}_1^+ [\gamma]^w \quad (3.15)$$

From the structure of the equation (3.15) is concluded that the absolute final speed $\dot{x}_f$ is different of zero for any value of w.

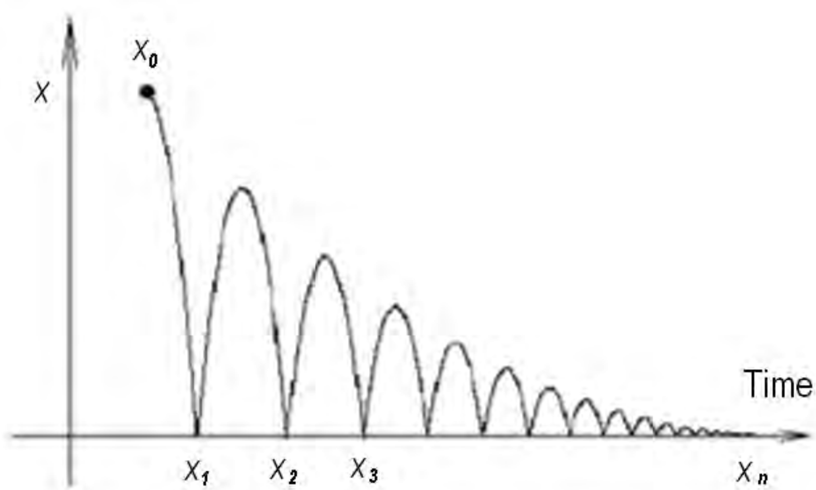


Figure 12. Scheme of an infinite bounce.

Chapter 4

Application of Singular Point Tracking (SPT) Method to Identification of Local Bifurcations

This chapter is dedicated to a progressive presentation of what we have called the singular point tracking (SPT) method. This name is coined on the basis of the fact that in all the relations found in this work, we have considered only a singular or a special point. A special point is assumed to be an isolated point of an orbit or an isolated point in a line of analysis of the Discontinuity boundary. In the first part, the geometrical concepts that are useful in the following sections in order to determine whether a point has associated vectors pointing toward an important direction are established. Then, on the basis of the quantity of information that is contained in a point that is within the DB, each point is coded with a numeric, textual, or graphic symbol that gives information the user of the characteristics of the vectors accompanying the point. Then, is developed a syntactic process in which all types of points are listed and then grouped according to the dynamics. The next step is to study the sequences of points and their respective dynamics along the DB. These sequences are listed to be used in the next step as clues of the changes in the dynamics that can be considered bifurcations due to the changes in the parameter values. At this point, the SPT method is established, and the catalogue of local bifurcations is used for the validation of the method. Finally, an introduction to a three dimension system is done to study the spread ability of the method.

4.1 Assignation of ranges to the angles of the vectors

The main characteristics of a singular or a special point are as follows:

1. It is accompanied on both sides only by groups of points of the same type.
2. The orientation of one or both of its vectors is special. For example, perpendicular or parallel to the tangent to the DB.

Then, in order to determine whether a point is a singular or a special point, we select the angle as a tool because it lets us define arcs in two-dimensional systems and caps in three-dimensional systems and so on in n-dimensional systems. These arcs are used for determining whether a trajectory is advancing toward the range of the influence of a determined region of a nonsmooth system. Another event that is determined using ranges is the evolution of a sliding trajectory on the DB.

Ranges in two-dimensional systems

Let Σ be the symbol to represent de discontinuity boundary or the curve that determines the limits between regions Z_1 and Z_2, as illustrated in Figure 13. Let **H** be the set of equations that describes the curve Σ and H_T be the tangential vector to Σ at point $\boldsymbol{x}_1$.

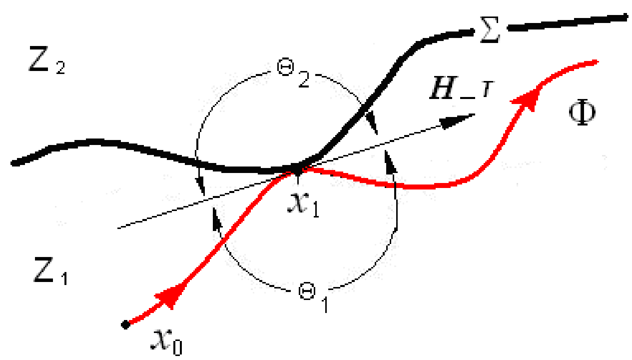


Figure 13. Assignation of ranges according to the orientation of the DB.

In order to determine whether a trajectory that was initiated at point $\boldsymbol{x}_0$ in the region Z_1, that has evolved and at the moment is on the DB in the point $\boldsymbol{x}_1$, is pointing toward the region Z_1 or the region Z_2, it is necessary to define two angular openings Θ_1 and Θ_2 (See figure 14) using the following notation $\Theta=(\angle \text{ ini}, \angle \text{ end})$.

$$\Theta_2 = (\angle H\_T, \angle H\_T + \pi) \tag{4.1}$$

and

$$\Theta_1 = (\angle H\_T + \pi, \angle H\_T + 2\pi) \tag{4.2}$$

These ranges in turn are divided into quadrants for detecting whether the orientations of the movements are toward the right or toward the left of the DB and whether there are any orientations perpendiculars or tangent relative to the DB.

$$\Theta_{2R} = (\angle H\_T, \angle H\_T + \pi/2) \tag{4.3}$$

$\Theta_{2L} = (\angle H\_{T} + \pi/2, \angle H\_{T} + \pi)$ (4.4)

$\Theta_{1L} = (\angle H\_{T} + \pi, \angle H\_{T} + 3/2\pi)$ (4.5)

$\Theta_{1R} = (\angle H\_{T} + 3/2\pi, \angle H\_{T} + 2\pi)$ (4.6)

Ranges in three-dimensional systems

Let Σ be the symbol to represent de discontinuity boundary or the surface that determines the limits between regions Z_1 and Z_2 as it is illustrated in Figure 14. Let **H** be the function that describes the surface Σ and Hzx_1 be the vector normal to the surface at point $\boldsymbol{x}_1$. This vector also is used to define a plane Hz with two axis Hxx_1 and Hyx_1 forming a transformed Cartesian system of coordinates.

The characteristics of Hxx_1 are related to those of the surface Σ. When the vector Hzx_1 is collinear with the Z axis of the normal Cartesian space, the selection of Hxx_1 and Hyx_1 is the same as that of the X and Y axes of the normal Cartesian space.

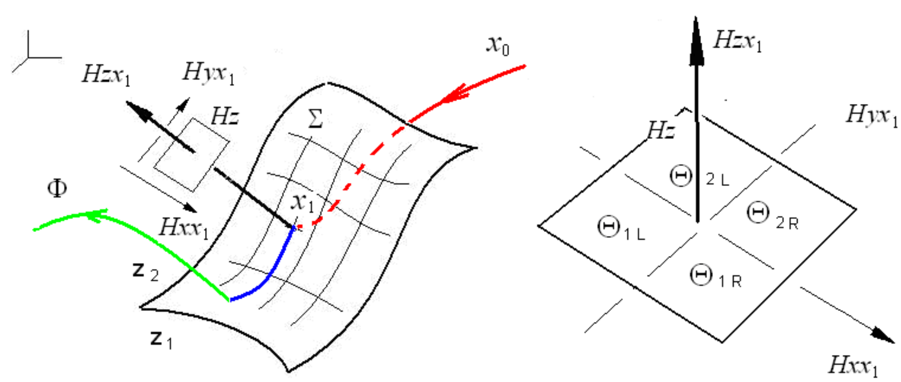


Figure 14. Assignation of directions according to the DB in 3D systems.

In three dimensions are declared four ranges over the plane Hz. With these four ranges and the combination of the positive and negative direction of Hzx_1, are declared eight three-dimensional ranges or caps. In three dimensional spaces a cap is defined using spherical coordinates that in turn uses two ranges associated with two planes of the Cartesian coordinates, the XY plane and the XZ plane. Then the upper and lower caps are:

$\Theta_2 = (\angle Hxx_1, \angle Hxx_1 + 2\pi)_{xy\text{-plane}} \wedge (\angle Hyx_1, \angle Hyx_1 + \pi)_{xz\text{-plane}}$ (4.7)

and

$$\Theta_1 = (\angle \mathrm{Hx}x_1, \angle \mathrm{Hx}x_1 + 2\pi)_{\text{xy-plane}} \wedge (\angle \mathrm{Hyx}_1 + \pi, \angle \mathrm{Hyx}_1 + 2\pi)_{\text{xz-plane}} \qquad (4.8)$$

4.2 Assignation of codes to the ranges

Vectors with an important orientation

The vectors associated to a point located in the DB can be oriented in a direction parallel or perpendicular to the curve or the plane that forms the DB. These orientations are important because the dynamics of the trajectory can be singular. Other vector fields, which are tangential but their size is zero have relation with equilibria points. The combination of vectors fields, $\mathbf{f}_1$ and $\mathbf{f}_2$ of both regions that become anti-collinear are also important.

Then, in planar systems there are four important orientations (0, 90, 180 and 270 degrees) and two possibilities related to the size of the vectors: with size and without. See Figure 15.

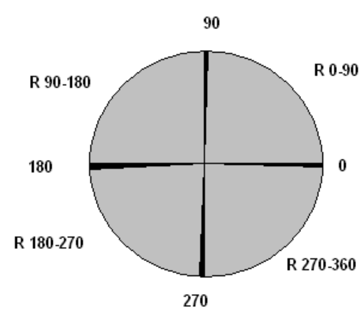


Figure 15. Ranges for analysis of vector orientations.

The numerical methods are unable to identify when an angle of a vector is equal to an exact value. In order to avoid this problem, the angle of reference is converted into a very small range, then if the angle of a vector is in this range, it is accepted that the angle is equal to the angle of reference. The range is selected by adding and subtracting a small σ to the angle of reference.

$$\angle_i \in \Theta \quad \text{if} \quad (\angle(\text{ref} - \sigma) < \angle_i < \angle(\text{ref} + \sigma)\} \qquad (4.9)$$

Therefore, the four orthogonal directions are transformed into four ranges. With the other ranges defined previously (for planar systems), there are now eight ranges. The first four ranges previously defined should give a little part of their wideness to let the new ranges be defined inside them.

If each point in the DB have associated two vectors each one from each vector field $\mathbf{f}_1$ and $\mathbf{f}_2$, and each vector can point in the direction of one of the eight ranges, then there are 64 combinations or possibilities. When the size of the vectors is considered, there are 17 additional possibilities.

Some other characteristics that make differences between both vectors are also included. For example, a size comparison of the tangential or normal components of the vectors. Consequently, the number of possibilities for analyzing the vectors associated to a point reaches more than a hundred.

For the analysis, a point $\boldsymbol{x}_1$ on the DB is taken, and the vectors associated to the point are calculated with the equation of the vector fields $\mathbf{f}_1$ and $\mathbf{f}_2$ that represents the dynamics of system in the regions Z_1 and Z_2; the point is evaluated and coded according to the geometric characteristics mentioned in the last paragraphs. We emphasize the coding process because it is useful in the construction of a numerical function for detecting bifurcations. The presence of dynamics such as crossing, attractive sliding, start or end of a sliding, and changes in the direction of a sliding segment is determined by using the information revealed by the orientation and the size of the vectors fields.

In order to recognize each point on the DB, it is possible to assign a three-digit number to each point in which the value is related with the characteristics of the vectors.

The first and the second digits (from left to right) denote the orientation of the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$.

0: range 0°
1: range Θ_{2R}
2: range 90°
3: range Θ_{2L}
4: range 180°
5: range Θ_{1L}
6: range 270°
7: range Θ_{1R}

The third digit provides information about the size and other characteristics.

0: both vectors have a size other than zero
1: both vectors have a size equal to zero.
2: vector $\mathbf{f}_1$ vanishes.
3: vector $\mathbf{f}_2$ vanishes.
4: both tangential components $\mathbf{f}_{1\_T}$ and $\mathbf{f}_{2\_T}$ have the same size.
5: both normal components $\mathbf{f}_{1\_N}$ and $\mathbf{f}_{2\_N}$ have the same size.
6: tangential component $\mathbf{f}_{1\_T}$ is bigger than $\mathbf{f}_{2\_T}$.
7: tangential component $\mathbf{f}_{2\_T}$ is bigger than $\mathbf{f}_{1\_T}$.
8: vector $\mathbf{f}_1$ and $\mathbf{f}_2$ are anti-collinear.

If two characteristics are presented at the same time, the first one is considered.

Examples:

In figure 16, we have a special point (boundary equilibrium in (3)) located where attractive sliding is initialized. This point in the region Z_1, has an associated a vector that vanishes. However, before reaching this stage, the vector was tangential with an angle in the range equal to zero. From the vector field $\mathbf{f}_2$, the vector has an angle in the range of (270°, 360°). The third digit informs that $\mathbf{f}_1$ is a null vector; therefore, the numeric classification of this point is 072.

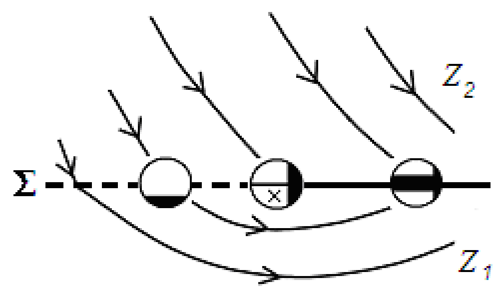


Figure 16. Point on the DB with numerical classification 072 indicating the beginning of a sliding segment.

In figure 17 is shown at the centre of the DB, a special point (tangent point in (3)) found at the end of an attractive sliding segment. This point, from the region Z_1 has an associated tangent vector with an angle equal to zero. From the region Z_2, it has an associated vector with an angle equal to 270°, and thus, the number associated with this point is 070.

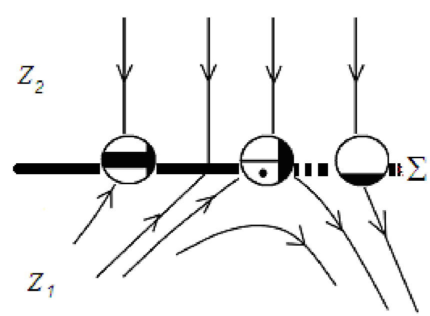


Figure 17. Point in the DB with numeric classification 070 indicating the end of a sliding segment.

4.3 Assignation of symbols to points on DB

The assignment of numbers to the points on the DB is useful from the numeric point of view since it allows the development of algorithms that help to identify and automatically classify the points. From a didactic point of view, the assignment of numbers hinders the compression, and therefore, a group of graphic symbols has been developed to be used in the figures in a way that allows us to draw a quick visual correlation of the dynamics of each point. The points will also have a textual indicator that has two parts, the Greek letter omega Ω accompanied with an index that indicates the class of the point.

Graphic symbols for the points on DB and the regions

Next, an introductory list of some of the classified points and their corresponding symbols is presented. In the next sections, these symbols are further explained, and the geometrical conditions for their existence are found. As has been mentioned before, the classification is mainly taken by the direction and the size of the vector calculated using the function of the vector fields $\mathbf{f}_1$ and $\mathbf{f}_2$ in the point and are called associated vectors.

1. A crossing point from Z_1 to Z_2.
2. A crossing point from Z_2 to Z_1.
3. An attractive point with both vectors perpendicular to the DB.
4. An attractive sliding point with movement toward the right.
5. A repulsive point pointing toward the right.
6. An attractive sliding point with movement toward the left.
7. A repulsive point pointing toward the left.
8. A double tangency point with movement toward the right.
9. A double tangency point with movement toward the left.
10. A double tangency point with double direction and.
11. An attractive-tangent point with direction to the left and.
12. An attractive-tangent point with direction to the right and.

13. An attractive anti-collinear point with inclination to the right and to the left.

14. A repulsive anti-collinear point with inclination to the right and to the left.

15. Perpendicular repulsive points , , and .

16. Atractive-nul points , , and

17. Repulsive-null points , , and

18. Tangent-null points , , and

19. Null- null points

20. Perpendicular-null points and

21. A stable node or focus.

22. A saddle node.

23. An unstable node or focus.

24. Point belonging to an orbit in Z_1 and rotating in the CW direction. (clockwise)

25. Point belonging to an orbit in Z_1 and rotating in the CCW direction. (counter clockwise)

26. Point belonging to an orbit in Z_2 and rotating in the CW direction.

27. Point belonging to an orbit in Z_2 and rotating in the CCW direction.

28. Point of arriving and bounce of an impact and

29. Impact point at a speed near zero

30. Smooth crossing points , , and .

Conditions of existence of the classified points

Classified points are characterized using Boolean-valued functions B(.) that return True or False when their arguments are evaluated. In these functions we use the logical connectives AND, OR and NOT denoted by $\wedge$, $\vee$ and $\neg$, respectively. We also evaluated variables using partial orders denoted by $\geq$, $\leq$, and =. The theory of Sets is used to group arguments or subfunctions and to evaluate membership. A simplification of the Boolean symbolism is to indicate that a point is of one type implies ($\Rightarrow$) that the arguments conforming the Boolean function are True.

Suppose two neighbouring region Z_1 and Z_2 with a DB with an analysis point $\boldsymbol{x}_1$ and vectors fields $\mathbf{f}_1$ and $\mathbf{f}_2$.

In order to simplify the explanation but without a loss of generality, it is assumed that the DB is parallel with the X axle of Cartesian plane. Other orientation can be calculated through transformation of coordinates. As consequence, the region Z_1 occupies quadrants 3 and 4 of the Cartesian plane and the region Z_2 occupies quadrants 1 and 2. The analysis of a combination of the two vectors associated to the point $\boldsymbol{x}_1$ gives the following classification (see also (81) (82)).

Crossing points

The crossing points Ωc have the following characteristics:

$$\Omega c \Rightarrow \{[(\mathbf{f}_{1\_N} > 0) \wedge (\mathbf{f}_{2\_N} > 0)] \vee [(\mathbf{f}_{1\_N} < 0) \wedge (\mathbf{f}_{2\_N} < 0)]\} \quad (4.10)$$

with those that cross from Z_1 to Z_2  ($\Omega c_{1,2}$)

$$\Omega c_{1,2} \Rightarrow \{(\mathbf{f}_{1\_N} > 0) \wedge (\mathbf{f}_{2\_N} > 0)\} \quad (4.11)$$

with those that cross from Z_2 to Z_1  ($\Omega c_{2,1}$)

$$\Omega c_{2,1} \Rightarrow \{(\mathbf{f}_{1\_N} < 0) \wedge (\mathbf{f}_{2\_N} < 0)\} \quad (4.12)$$

The points of the 11x, 12x, 13x, 21x, 22x, 23x, 31x, 32x and 33x type are crossing points from Z_1 to Z_2.

The x included in the codes indicates a coherent value. For example, the point 22x cannot be 225 since it is a contradiction because the point 22x has a vanished vector in contrast to the point x25 that represents a point with no vanished vectors.

The points of the 55x, 56x, 57x, 65x, 66x, 67x, 75x, 76x and 77x type are the crossing points from Z_2 to Z_1.

Note: when the angular orientation of $\mathbf{f}_1$ and $\mathbf{f}_2$ with relation to the vector H_T is the same, we say that the point not only is crossing but also smooth and the symbol is used. The type of crossing and the orientation defines between four possibilies.

Sliding regular points and sliding special points.

The remaining points are divided into the sub-classifications sliding points Ωs and special points that again are divided into other sub-classifications.

The sliding points are divided into sliding attractive points and repulsive points.

Each one of the previous points in turn is divided in terms of the movements toward the right or toward the left. The movement to the right or to the left is related with the evaluation of the Filippov's solution.

Let $\mathbf{G}_{ij}$ be the vector that is the result of the Filippov evaluation.

MR = 1 (movement toward the right) otherwise ML = 1 (movement toward the left)

$$MR = 1 \text{ if } \{(\angle G_{IJ} \in \Theta_{1R}) \vee (\angle G_{IJ} \in \Theta_{2R})\} \quad (4.13)$$

$$ML = 1 \text{ if } \{(\angle G_{IJ} \in \Theta_{1L}) \vee (\angle G_{IJ} \in \Theta_{2L})\} \quad (4.14)$$

The sliding points Ωs have the following characteristics:

$$\Omega s \Rightarrow \{[(\mathbf{f}_{1\_N} > 0) \wedge (\mathbf{f}_{2\_N} < 0)] \vee [(\mathbf{f}_{1\_N} < 0) \wedge (\mathbf{f}_{2\_N} > 0)]\} \quad (4.15)$$

The attractive or stable sliding points (Ωss_R) or (Ωss_L)

$$\Omega ss \Rightarrow \{(\mathbf{f}_{1\_N} > 0) \wedge (\mathbf{f}_{2\_N} < 0)\} = SS \quad (4.16)$$

The points of the 15x, 16x, 17x, 25x, 27x, 35x, 36x, and 37x type are attractive sliding points. The point of the 26x type meets the abovementioned conditions but is a singular point.

The repulsive or unstable points (Ωsu_R) or (Ωsu_L).

$$\Omega su \Rightarrow \{(\mathbf{f}_{1\_N} < 0) \wedge (\mathbf{f}_{2\_N} > 0)\} = SU \quad (4.17)$$

The points of the 51x, 52x, 53x, 61x, 63x, 71x, 72x, and 73x type are repulsive points. The point 62x meets the required conditions but is a singular point.

The attractive sliding points with movement toward the right ($\Omega ss^{\rightarrow}$) or (Ωss_R)

$$\Omega ss^{\rightarrow} \Rightarrow \{SS \wedge MR\} \tag{4.18}$$

The points of the 16x, 17x, and 27x type belong to this classification.

The attractive sliding points with movement toward the left ($\Omega ss^{\leftarrow}$) or (Ωss_L)

$$\Omega ss^{\leftarrow} \Rightarrow \{SS \wedge ML\} \tag{4.19}$$

The point of the 36x, 35x, and 25x type belong to this classification.

The repulsive points do not move over the DB but show from the Filippov solution a tendency that we call “point to”. Instead, for the attractive sliding segment we say, “they are moving to”.

The repulsive points addressing toward the right ($\Omega su^{\rightarrow}$) or (Ωsu_R)

$$\Omega su^{\rightarrow} \Rightarrow \{SU \wedge ML\} \tag{4.20}$$

The points of the 61x, 71x, and 72x type belong to this classification.

The repulsive points addressing toward the left ($\Omega su^{\leftarrow}$) or (Ωsu_L)

$$\Omega su^{\leftarrow} \Rightarrow \{SU \wedge ML\} \tag{4.21}$$

The points of the 52x, 53x, and 63x type belong to this classification.

The numerical classification does not assure in some cases the sliding direction. The points 15x, 37x, 51x, and 73x should be intercepted with the function defined in (4.13) and (4.14) equations in order to determine whether the direction is MR or ML.

The special points are subdivided into points with one tangential vector, points with both tangential vectors (singular), points with one tangential vector and one vanished vector (singular), points with both vanished vectors (singular), point with one vector vanish (boundary equilibria), and points with anti-collinear vectors having different inclinations and vectors with a combination of the above (pseudo-equilibria).

The analytical definitions of tangential, anti-collinear, perpendicular, and vanished vectors are presented next:

Note: the symbol **ff** means a duplicate characterisitic.

Points on the DB with a tangential vector $\mathbf{ff}_{\_T}$ imply that

$$\mathbf{ff}_{1\_T} \Rightarrow \{(\mathbf{f}_{1\_N} = 0) \wedge (\mathbf{f}_{1\_T} \neq 0)\} \vee \mathbf{ff}_{2\_T} \Rightarrow \{(\mathbf{f}_{2\_N} = 0) \wedge (\mathbf{f}_{2\_T} \neq 0)\} \quad (4.22)$$

Points on the DB with a vanished vector $\mathbf{ff}_{\_X}$ imply that

$$\mathbf{ff}_{1\_X} \Rightarrow \{(\mathbf{f}_{1\_N} = 0) \wedge (\mathbf{f}_{1\_T} = 0)] \vee \mathbf{ff}_{2\_X} \Rightarrow \{(\mathbf{f}_{2\_N} = 0) \wedge (\mathbf{f}_{2\_T} = 0)\} \quad (4.23)$$

Points on the DB with anti-collinear vectors $\mathbf{ff}_{12\_AC}$ (not tangents) imply that

$$\mathbf{ff}_{12\_AC} \Rightarrow \{[(\mathbf{f}_{1\_T}/\mathbf{f}_{1\_N})=(\mathbf{f}_{2\_T}/\mathbf{f}_{2\_N})] \wedge (\mathbf{f}_{1\_N} \neq 0) \wedge (\mathbf{f}_{2\_N} \neq 0)\} \quad (4.24)$$

Points on the DB with perpendicular vectors $\mathbf{ff}_{12\_NN}$ imply that

$$\mathbf{ff}_{12\_NN} \Rightarrow \{(f_{1\_T} = \mathbf{f}_{2\_T} = 0) \wedge (\mathbf{f}_{1\_N} \neq 0) \wedge (\mathbf{f}_{2\_N} \neq 0)\} \quad (4.25)$$

Sliding tangent points

A sliding tangential vector refers to the case when one of the vectors is tangential and the other determines the characteristics (stable or unstable) of the point.

If the vector $\mathbf{f}_1$ is tangential, $\mathbf{f}_2$ determines the point.

If $\mathbf{f}_{2\_N} < 0$, then according to the direction of the vectors, we have the following possibilities:

Point  ($\Omega ss^{\rightarrow}{}_T$)

$$\Omega ss^{\rightarrow}{}_T \Rightarrow \{\mathbf{ff}_{1\_T} \wedge (\mathbf{f}_{2\_N} < 0) \wedge \text{MR}\} \quad (4.26)$$

The points of the 07x type belong to that classification.

Point  ($\Omega ss^{\leftarrow}{}_T$)

$$\Omega ss^{\leftarrow}{}_T \Rightarrow \{\mathbf{ff}_{1\_T} \wedge (\mathbf{f}_{2\_N} < 0) \wedge \text{ML}\} \quad (4.27)$$

The points of the 45x type belong to that classification.

If $\mathbf{f}_{2\_N} > 0$

Point  ($\Omega su^{\rightarrow}{}_{T}$)

$$\Omega su^{\rightarrow}{}_{T} \Rightarrow \{\mathbf{ff}_{1\_T} \wedge (\mathbf{f}_{2\_N} > 0) \wedge \mathrm{MR}\} \qquad (4.28)$$

The points of the 01x type belong to that classification.

Point  ($\Omega su^{\leftarrow}{}_{T}$)

$$\Omega su^{\leftarrow}{}_{T} \Rightarrow \{\mathbf{ff}_{1\_T} \wedge (\mathbf{f}_{2\_N} > 0) \wedge \mathrm{MR}\} \qquad (4.29)$$

The points of the 43x type belong to that classification.

If the vector $\mathbf{f}_2$ is tangential, $\mathbf{f}_1$ determines the characteristics.

If $\mathbf{f}_{1\_N} > 0$, then according to the direction of the vectors, we have the following possibilities:

Point  ($\Omega ss^{\rightarrow T}$)

$$\Omega ss^{\rightarrow T} \Rightarrow \{\mathbf{ff}_{2\_T} \wedge (\mathbf{f}_{1\_N} > 0) \wedge \mathrm{MR}\} \qquad (4.30)$$

The points of the 10x type belong to that classification.

Point  ($\Omega ss^{\rightarrow T}$)

$$\Omega ss^{\rightarrow T} \Rightarrow \{\mathbf{ff}_{2\_T} \wedge (\mathbf{f}_{1\_N} > 0) \wedge \mathrm{ML}\}$$

The points of the 34x type belong to that classification.

If $\mathbf{f}_{1\_N} < 0$

Point  ($\Omega su^{\rightarrow T}$)

$$\Omega su^{\rightarrow T} \Rightarrow \{\mathbf{ff}_{2\_T} \wedge (\mathbf{f}_{1\_N} < 0) \wedge \mathrm{MR}\} \qquad (4.31)$$

The points of the 70x type belong to that classification.

Point  ($\Omega su^{\leftarrow T}$)

$$\Omega su^{\leftarrow T} \Rightarrow \{\mathbf{ff}_{2\_T} \wedge (\mathbf{f}_{1}\_n < 0) \wedge \mathrm{ML}\} \quad (4.32)$$

The following point belongs to that classification: 54x.

Null points (vanished vector fields)

If $\mathbf{f}_1$ vanishes at the analysis point, then $\mathbf{f}_1 = 0$ and $\mathbf{f}_2$ determines the characteristics (stable or unstable) of the point. The letter V is used to mean vanished and as a sub-index; the letter X indicates the vector field that generated the null vector.

If $\mathbf{f}_{1\_N} < 0$

Point  ($\Omega ssv^{\rightarrow}{}_{X}$)

$$\Omega ssv^{\rightarrow}{}_{X} \Rightarrow \{\mathbf{ff}_{1\_X} \wedge (\mathbf{f}_{2\_N} < 0) \wedge \mathrm{MR}\} \quad (4.33)$$

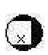
The points of the 54x, 072 and 472 types belong to that classification.

Point  ($\Omega ssv^{\leftarrow}{}_{X}$)

$$\Omega ssv^{\leftarrow}{}_{X} \Rightarrow \{\mathbf{ff}_{1\_X} \wedge (\mathbf{f}_{2\_N} < 0) \wedge \mathrm{ML}\} \quad (4.34)$$

The points of the 452 and 052 type belong to that classification.

If $\mathbf{f}_{2\_N} > 0$

Point  ($\Omega suv^{\rightarrow}{}_{X}$)

$$\Omega suv^{\rightarrow}{}_{X} \Rightarrow \{\mathbf{ff}_{1\_X} \wedge (\mathbf{f}_{2\_N} > 0) \wedge \mathrm{MR}\} \quad (4.35)$$

The points of the 012 and 412 types belong to that classification.

Point  ($\Omega suv^{\leftarrow}{}_{X}$)

$$\Omega suv^{\leftarrow}{}_{X} \Rightarrow \{\mathbf{ff}_{1\_X} \wedge (\mathbf{f}_{2\_N} > 0) \wedge \mathrm{ML}\} \quad (4.36)$$

The points of the 432 and 032 types belong to that classification.

If $\mathbf{f}_2$ becomes null at the analysis point, $\mathbf{f}_1$ determines the characteristics of the point.

If $\mathbf{f}_{1\_N} > 0$

Point c  ($\Omega ssv^{\rightarrow X}$)

$$\Omega ssv^{\rightarrow X} \Rightarrow \{\mathbf{ff}_{2\_X} \wedge (\mathbf{f}_{1\_N} > 0) \wedge \text{MR}\} \quad (4.37)$$

The points of the 103 and 143 types belong to that classification.

Point  ($\Omega ssv^{\leftarrow X}$)

$$\Omega ssv^{\leftarrow X} \Rightarrow \{\mathbf{ff}_{2\_X} \wedge (\mathbf{f}_{1\_N} > 0) \wedge \text{ML}\} \quad (4.38)$$

The points of the 343 and 303 types belong to that classification.

If $\mathbf{f}_{1\_N} < 0$

Point  ($\Omega suv^{\rightarrow X}$)

$$\Omega suv^{\rightarrow X} \Rightarrow \{\mathbf{ff}_{2\_X} \wedge (\mathbf{f}_{1\_N} < 0) \wedge \text{MR}\} \quad (4.39)$$

The points of the 703 and 743 types belong to that classification.

Point  ($\Omega suv^{\leftarrow X}$)

$$\Omega suv^{\leftarrow X} \Rightarrow \{\mathbf{ff}_{2\_X} \wedge (\mathbf{f}_{1\_N} < 0) \wedge \text{ML}\} \quad (4.40)$$

The points of the 543 and 503 types belong to that classification.

Tangential null points

If $\mathbf{f}_1$ is a tangent vector and $\mathbf{f}_2$ becomes null, it is not possible to assure that the point is attractive or repulsive. Then, this is a common point in the transitions of the flow crossing from $\mathbf{f}_1$ to $\mathbf{f}_2$ or vice versa.

Point ($\Omega_{STV}{}^{\rightarrow X}$)

$$\Omega_{STV}{}^{\rightarrow X} \Rightarrow \{\mathbf{ff}_{1\_T} \wedge \mathbf{ff}_{2\_X} \wedge \text{MR}\} \qquad (4.41)$$

The points of the 003 and 043 types belong to that classification.

Point ($\Omega_{STV}{}^{\leftarrow X}$)

$$\Omega_{STV}{}^{\leftarrow X} \Rightarrow \{\mathbf{ff}_{1\_T} \wedge \mathbf{ff}_{2\_X} \wedge \text{ML}\} \qquad (4.42)$$

The points of the 443 and 403 types belong to that classification.

Point ($\Omega_{SVT}{}^{\rightarrow}{}_{X}$)

$$\Omega_{SVT}{}^{\rightarrow}{}_{X} \Rightarrow \{\mathbf{ff}_{2\_T} \wedge \mathbf{ff}_{1\_X} \wedge \text{MR}\} \qquad (4.43)$$

The points of the 002 and 042 types belong to that classification.

Point ($\Omega_{SVT}{}^{\leftarrow}{}_{X}$)

$$\Omega_{SVT}{}^{\leftarrow}{}_{X} \Rightarrow \{\mathbf{ff}_{2\_T} \wedge \mathbf{ff}_{1\_X} \wedge \text{ML}\} \qquad (4.44)$$

The points of the 442 and 042 types belong to that classification.

Tangent-tangent points

If both vectors are tangent, we can not classify it as a point attractive or repulsive; however, the segment of tangent-tangent points moves the evolution along the DB, and hence, it can be classified as a pseudo-sliding point or a singular as in (3).

Point ($\Omega_{STT}{}^{\rightarrow}$)

$$\Omega_{STT}{}^{\rightarrow} \Rightarrow \{\mathbf{ff}_{1\_T} \wedge \mathbf{ff}_{2\_T} \wedge (\mathbf{f}_1 > 0) \wedge (\mathbf{f}_2 > 0)\} \qquad (4.45)$$

The points of the 000, 004, 006 and 007 types belong to that classification.

Point ($\Omega_{STT}{}^{\leftarrow}$)