Quantitative Risk Management under the Interplay of Insurance and Financial Risks

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Abstract

In this thesis, we tackle some problems concerning the interplay of insurance and financial risks.

First, we consider an insurance or financial company intending to allocate the risk capital withheld for its overall investment portfolio among its constituents. Shortly, we assume that the company computes the risk capital through the Haezendonck–Goovaerts risk measure, and we establish the unique capital allocation rule consistent with a RORAR (return on risk-adjusted capital) approach. Besides, we present some asymptotics and propose a consistent estimator for the capital allocation rule. Finally, we conduct some numerical studies.

Then, we solve the problem of valuing some mortality-linked derivatives by employing the utility indifference pricing approach. Succinctly, we suppose that the mortality risk emanates from a portfolio of life insurance policyholders, whose remaining lifetimes are modeled as conditionally independent random times. By adapting some results from credit risk theory, we compute an explicit expression for the utility indifference price when the derivative is a linear combination of pure endowments. By considering a more general contingent claim, we use techniques of backward stochastic differential equations (BSDE) to characterize the indifference price in terms of a solution to a non-linear BSDE with a non-Lipschitz generator.

Finally, we consider an individual aiming to optimally choose its investment, consumption, and life insurance purchase strategies in a complete financial market. By assuming that the optimality criterion is the maximization of the individual's expected state-dependent utility, we solve the optimal choice problem in a general setup, which includes several utility functions employed in the literature.

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Chapter 1

Objectives and outline of the thesis

In this thesis, we analyze quantitatively some insurance and financial risks, whose interplay impacts the decisions making of a rational economic agent. Specifically, we first study how an insurance or financial company should allocate the risk capital among constituents of its overall portfolio. Subsequently, we consider the utility indifference pricing of some mortality-linked contingent claims. Lastly, we investigate optimal consumption, investment, and life insurance acquisition strategies for an individual making rational decisions under a state-dependent utility approach.

In Chapter 2, we consider an insurance or financial company, which aims to manage aggregate risk. Precisely, we study the problem of allocating economic capital among constituents of the overall portfolio of the company. To that purpose, we assume that the company quantifies its risks through the Haezendonck–Goovaerts risk measure, which is a coherent and law invariant tail-based risk measure. In such a case, we show that the Haezendonck–Goovaerts risk measure is directionally differentiable, and we provide a quasi-explicit formula for its directional derivatives. Thus, we derive the unique capital allocation rule consistent with a RORAC (return on risk-adjusted capital) criterion, which coincides with the Euler principle. Moreover, we give a consistent plug-in estimator for the capital allocation rule. Under the regular variation structure, we study the asymptotic behavior of the capital allocation rule as the confidence level in the definition of the Haezendonck–Goovaerts risk measure approaches one, which rewords into regulators' excessive prudence. To illustrate, we consider the capital allocation rule under some specific distributional assumptions: elliptical, independent exponential marginals, and multivariate Pareto.

In Chapter 3, we consider an insurance market where the financial and mortality risks coexist independently. Specifically, we aim to price some mortality-linked securities. The financial risk arises from a complete financial market with a finite number of securities, whose prices are driven by a Brownian motion. Besides, the mortality risk emanates from a portfolio of life insurance policyholders, whose remaining lifetimes are modeled by conditionally independent doubly stochastic random times. Since the insurance market is not complete, it is not possible to determine a unique price process for mortality-linked securities only based on the exclusion of arbitrage opportunities. To value mortality-linked securities, we rely on the utility indifference pricing approach with an exponential utility function. First, we study the pricing of a claim formed by a linear combination of pure endowments, and we provide a quasi-explicit formula for the utility indifference price. Furthermore, we illustrate how the indifference price process simplifies when the mortality-rates are affine diffusion processes. Then, we follow Hu et al. (2005) in using techniques of backward stochastic differential equations (BSDE) to characterize the indifference price process of a general mortality-linked claim in terms of a solution to a non-linear BSDE with non-Lipschitz generator. Finally, we provide an example regarding a numerical approximation for the BSDE mentioned above.

In Chapter 4, we consider the problem faced by an individual, which aims to make optimal decisions about consumption, investment, and life insurance purchase in a complete Brownian financial market. The role of life insurance is to protect the individual's family of eventually an early death. In particular, we assume that the individual may continuously acquire whole life insurance with minimum and maximum constraints on the insured sum. Since the classical Merton's approach for optimal choice seems not to be consistent with empirical data, we follow Londoño (2009) in assuming that the individual ranks risky positions conforming to a state-dependent utility approach. In this context, we solve the optimal choice problem in a general set-up, which includes several utility functions employed in the literature. To that end, we adapt the martingale methodology from Karatzas et al. (1987) and Cox and Huang (1989, 1991). Lastly, we compute the solution for the Black-Scholes model.

Chapter 2

A capital allocation rule for the Haezendonck–Goovaerts risk measure with a power Young function

2.1 Notation and abbreviations

In the following table, we provide a summary of notation and abbreviations used in this chapter:

$\mathbf{0}$	the d-dimensional vector with all components being 0
$\boldsymbol{\infty}$	the d-dimensional vector with all components being ∞
$\mathbf{1}_i$	the d-dimensional vector with the ith component being 1 and the other components being 0
1_A	the indicator of an event A
$a(q) \sim b(q)$	$\lim_{q\uparrow 1} \frac{a(q)}{b(q)} = 1$, where $f(\cdot)$ and $g(\cdot)$ are positive functions
$\mathrm{B}(a,b)$	the beta function, $\int_0^1 y^{a-1}(1-y)^{b-1}dy$, where $a,b>0$
d	a positive integer
$\overset{d}{=}$	equality in distribution
E	mathematical expectation
$\overline{F}$	$1-F$ for a distribution function F
$F^{\leftarrow}$	the generalized inverse of a distribution function F
$\Gamma(a)$	the gamma function, $\int_0^\infty y^{a-1}e^{-y}dy$, where $a>0$
HG	Haezendonck–Goovaerts

L^k	the space of random variables X such that $E(\lvert X\rvert^k) < \infty$, where $k \geq 1$
$\lVert(x_1,\ldots,x_d)\rVert$	the Euclidean norm, $\left(\sum_{i=1}^d x_i^2\right)^{\frac{1}{2}}$ for $(x_1,\ldots,x_d) \in \mathbb{R}^d$
P	probability measure
$\mathbb{R}$	$(-\infty,\infty)$
$\mathbb{R}_+$	$[0,\infty)$
$\overline{\mathbb{R}}_+$	$[0,\infty]$
s_d	$z_1 + \cdots + z_d$, where $(z_1,\ldots,z_d)$ is a generic vector in $\overline{\mathbb{R}}_+^d$
tA	$\{tx : x \in A\}$ for a real number t and a set $A \subset \mathbb{R}^d$
TVaR	Tail Value-at-Risk
$x \vee y$	$\max\{x,y\}$
$x \wedge y$	$\min\{x,y\}$
x_+	$x \vee 0$
$\boldsymbol{z}$	a generic vector in $\overline{\mathbb{R}}_+^d$ with components $z_1,\ldots,z_d$

2.2 Introduction

Let X_1, ..., X_d represent $d \geq 1$ random variables in a loss-profit style, which correspond to d different sub-portfolios constituting the overall portfolio of a financial company. Accordingly, the company incurs an aggregate loss $S = \sum_{i=1}^d X_i$. By regulation, the company must set aside some riskless investments for buffering the company portfolio, which we denominate risk capital. The purpose of the backing is to protect stakeholders from potential insolvency of the company in adverse situations. While it is pivotal to determine the risk capital requirement for the aggregate loss S, it is also crucial to allocate it amongst the associated sub-portfolios X_1, ..., X_d. Indeed, risk capital allocation is a useful tool for measuring the performance of different sub-portfolios by determining the return on allocated capital for each sub-portfolio. At the same time, comparing the performance of sub-portfolios is essential to ensure that the risk capital is invested efficiently, yielding highest returns on allocated capital.

By the arbitrariness of dependence along the sub-portfolios, the risk capital of the overall portfolio is typically less than the sum of the risk capitals required to be withheld for each sub-portfolio separately. Therefore, the capital allocation problem is a challenge, on which we shall focus throughout this chapter.

In practice, a two-step procedure for determining risk capital allocations is used. The first stage is to compute $\rho(S)$, where ρ is a particular risk measure, and the second stage is to allocate the risk capital $\rho(S)$ to the sub-portfolios conforming to some mathematical capital allocation rule Λ, such that $\rho(S) = \sum_{i=1}^{d} \Lambda(X_i, S)$. Since the literature on capital allocation rules and risk measures is extensive, a great variate of possibilities for implementing the described two-step procedure is available. Thus, to conduct capital allocation quantitatively, it is crucially important to choose a capital allocation rule and a risk measure, which are well justified and possess some appealing properties.

The capital allocation rule. To tackle the capital allocation problem, there are several approaches in the literature, respectively motivated by arguments from: axiomatic considerations (see Shapley (1953), Schmeidler (1969), Tijs and Driessen (1986), Denault (2001), Kalkbrener (2005), and Csóka and Herings (2014)), performance measurement (see Tasche (2004) and Buch et al. (2011)), market valuation of assets and liabilities (see Zanjani (2010) and Bauer and Zanjani (2015)), and optimization (see Dhaene (2003, 2012)).

One of the most prominent methodologies to allocate capital amongst sub-portfolios is the Euler allocation principle, which assumes directional differentiability of the underlying risk measure. In this context, the capital allocated to a sub-portfolio is the derivative of the underlying risk measure of the overall portfolio in the direction of the sub-portfolio. Interestingly, the Euler allocation principle can be derived from several perspectives. For instance, by assuming a positively homogeneous risk measure, Tasche (2004) demonstrates that the unique capital allocation rule suitable for performance measurement is the Euler principle. Succinctly, let $\Lambda(X_i, S)$ denote the directional derivative of the risk measure ρ at S in the direction of X_i, and let RORAC(X_i) and RORAC(S) denote the returns on allocated capital for the sub-portfolio X_i and portfolio S, namely, RORAC$(X_i) = E(X_i)/\Lambda(X_i, S)$ and RORAC$(S) = E(S)/\rho(S)$. Thus, if the sub-porfolio X_i performs better than the overall portfolio S in the RORAC metric, then the RORAC of the overall portfolio is increased if one increases slightly in the direction of X_i.

Furthermore, Kalkbrener (2005) proposes a system of axioms for capital allocation (linear aggregation, diversification, and continuity) and shows that the Euler principle is the unique capital allocation rule consistent with this axiomatization when referring to subadditive and positively homogeneous risk measures; see Denault (2001) for another argument in favor of the Euler principle, based on cooperative game theory. In consequence, its appealing economic justifications advocate us to choose the Euler principle as a sensible capital allocation rule with respect to subadditive and positively homogeneous risk measures.

The risk measure. With a sound economic justification, Artzner et al. (1997, 1999) introduce the family of coherent risk measures through a set of properties (monotonicity, subadditivity, positive homogeneity, and translation invariance) that according to the authors any reasonable risk measure should satisfy. In particular, as one of the most popular coherent risk measures, the Tail Value-at-Risk (TVaR) has been considered as a practical alternative to the well-known Value-at-Risk (VaR) to remedy its lack of subadditivity. See for instance Rockafellar and Uryasev (2000, 2002), Acerbi and Tasche (2002), Tasche (2002), and Scaillet (2004). Recently, aiming to include variability at the tail when quantifying the risk, Furman et al. (2017) introduce a family of Gini-type coherent risk measures containing TVaR and develop the corresponding capital allocation rules.

As a generalization of TVaR, the Haezendonck–Goovaerts (HG) risk measure, which is defined via a convex Young function and a confidence level parameter, is introduced by Haezendonck and Goovaerts (1982) and revisited by Goovaerts et al. (2003, 2004); TVaR corresponds to the simplest Young function, the linear one. Remarkably, the HG risk measure can be derived from a multiplicative premium principle based on the economic concept of the certainty equivalent, as we shall see later. Recently, the HG risk measure has attracted increasing attention; see for instance Bellini and Rosazza Gianin (2008a, 2008b, 2012), Ahn and Shyamalkumar (2014), Tang and Yang (2014), and Peng et al. (2015). In particular, Bellini and Rosazza Gianin (2008a) point out that the HG risk measure is law invariant and coherent.

Furthermore, in the study of the HG risk measure, a power Young function has often been assumed. See for instance Tang and Yang (2012) and Mao and Hu (2012). Under the alternative name of higher-order coherent risk measure, Krokhmal (2007) and Matmoura and Penev (2013) illustrate the advantage of using this risk measure as a risk criterion in portfolio optimization when compared with the TVaR-based model and the classical mean-variance model. Moreover, as we shall see later, the HG risk measure with a power Young function is directionally differentiable. As a consequence, the Euler allocation principle based on this risk measure is supported by axiomatic and performance measurement considerations. Thus, the HG risk measure with a power Young function is a favorable option to complete the two-step procedure for allocating capital.

In this chapter, we consider a company quantifying risks through the HG risk measure with a power Young function. In this sense, we solve the question of how to allocate reasonably the risk capital of a portfolio among its sub-portfolios. To that end, because of its well-argued justification, we rely on the Euler allocation principle.

Consistently, we prove that the HG risk measure is directionally differentiable and give a formula for its directional derivatives. By following Kalkbrener (2005), we establish the unique capital allocation principle satisfying linear aggregation, diversification, and continuity. We compute the capital allocation rule under some specific distributional assumptions: elliptical, independent exponential marginals, and multivariate Pareto. For practical applications, the statistical estimation of risk measures and capital allocation rules is a key issue; see for instance Jones and Zitikis (2003), Hong and Liu (2009), Belomestny and Krätschmer (2012), Ahn and Shyamalkumar (2014), and Dentcheva et al. (2017). Accordingly, we propose a plug-in estimator and prove its consistency. Under the regular variation structure, we establish the asymptotic behavior of the capital allocation rule as the confidence level in the definition of the HG risk measure approaches one; some related works include Asimit et al. (2011), Mao and Hu (2012), and Tang and Yang (2012, 2014). Finally, we conduct some numerical studies giving an account of the goodness of our asymptotics and the rapidness of the convergence of the plug-in estimator.

2.3 Main results

2.3.1 The Haezendonck–Goovaerts risk measure

Let X be a real-valued random variable, representing a risk variable in loss-profit style, with distribution function F on $\mathbb{R}$. Let $\varphi(\cdot)$ be a normalized Young function; that is, $\varphi(\cdot)$ is a nonnegative and convex function on $\mathbb{R}_+$ with $\varphi(0) = 0$, $\varphi(1) = 1$, and $\varphi(\infty) = \infty$. Due to its convexity, the Young function $\varphi(\cdot)$ is continuous and strictly increasing on $\{s \in \mathbb{R}_+ : \varphi(s) > 0\}$.

Recall that the Orlicz heart associated with $\varphi(\cdot)$ is defined as

$$L_0^{\varphi} = \{X : E[\varphi(c|X|)] < \infty \text{ for all } c > 0\};$$

see, e.g., p.77 of Rao and Ren (1991). By the convexity of $\varphi(\cdot)$, we know that L_0^{φ} is a convex set. Thus, for $X \in L_0^{\varphi}$, the expectation

$$E\left[\varphi\left(\frac{(X-x)_+}{h}\right)\right],$$

which frequently appears in the sequel, is finite for every $x \in \mathbb{R}$ and $h > 0$.

For a Young function $\varphi(\cdot)$ and a risk variable $X \in L_0^{\varphi}$, let $h = h(x,q)$ be the unique solution to the equation

$$E\left[\varphi\left(\frac{(X-x)_+}{h}\right)\right] = 1-q, \qquad q \in (0,1),$$

if $\overline{F}(x) > 0$ and let $h(x,q) = \infty$ if $\overline{F}(x) = 0$. For $q \in (0,1)$ the Haezendonck–Goovaerts (HG) risk measure for X is defined as

$$H_q(X) = \inf_{x \in \mathbb{R}} \{x + h(x,q)\} = x^* + h^*,$$

where (x^*, h^*) denotes the minimizer. By Proposition 3(b,d) of Bellini and Rosazza Gianin (2012), the minimizer (x^*, h^*) always exists for all $q \in (0,1)$, and it is unique if $\varphi(\cdot)$ is strictly convex.

Remark 2.3.1 *For a positive random variable Y, define its certainty equivalent by $C(Y) = \varphi^{-1}(E[\varphi(Y)])$; thus, $C(Y)$ represents the sure amount for which a decision maker remains indifferent to the risk Y with respect to the loss function $\varphi(\cdot)$. The concept of certainty equivalent plays a major role in the economics literature; see for instance Wilson (1979) and McCord and De Neufville (1986). We understand the concept of certainty equivalent conforming to a revised setting $\varphi^{-1}(E[\varphi(Y)/C(Y)]) = 1$. This understanding still keeps the essence of the concept. We can also change 1 to $\varphi^{-1}(1-q)$ to reflect penalization of the tail risk. To tackle the potential loss X, the decision maker could withhold the quantity $x \in \mathbb{R}$ and face the remaining loss $(X-x)_+$; in this scenario, the certainty equivalent for X is $x + h(x,q)$. Consequently, by definition, the HG risk measure for the random variable X results from optimizing the certainty equivalent for X with respect to the initial assignation x. We refer the reader to Ben–Tal and Teboulle (2007) for another risk measure corresponding to an optimized certainty equivalent.*

For a general Young function, it is usually hard to get an explicit expression for the HG risk measure. To simplify the study, we follow Tang and Yang (2012) to consider an important special case with a power Young function $\varphi(x) = x^k$ for some $k \geq 1$. In this case, L_0^{φ} reduces to the commonly used L^k space. Let X be a non-degenerate risk variable with $X \in L^k$. Denote by $\hat{x}$ the essential supremum of X. According to Theorem 2.1 of Tang and Yang (2012), if $k > 1$, then the HG risk measure for X is given by

$$\begin{aligned} H_q(X) &= \inf_{x\in\mathbb{R}}\left\{x+\frac{1}{(1-q)^{\frac{1}{k}}}\left(E\left[(X-x)_+^k\right]\right)^{\frac{1}{k}}\right\} \\ &= x^*+\frac{1}{(1-q)^{\frac{1}{k}}}\left(E\left[(X-x^*)_+^k\right]\right)^{\frac{1}{k}}, \qquad q\in(0,1), \end{aligned} \tag{2.1}$$

where the minimizer, $x=x^*\in(-\infty,\hat{x})$, is the unique solution to the equation

$$\frac{\left(E\left[(X-x)_+^{k-1}\right]\right)^k}{\left(E\left[(X-x)_+^k\right]\right)^{k-1}}=1-q. \tag{2.2}$$

If $k=1$, then the HG risk measure for X equals to

$$H_q(X) = \inf_{x\in\mathbb{R}}\left\{x+\frac{1}{(1-q)}E\left[(X-x)_+\right]\right\} \tag{2.3}$$

$$= F^{\leftarrow}(q)+\frac{1}{(1-q)}E\left[(X-F^{\leftarrow}(q))_+\right], \qquad q\in(0,1), \tag{2.4}$$

where $F^{\leftarrow}$ denotes the generalized inverse of the distribution F or equivalently the VaR of X; see for instance Rockafellar and Uryasev (2000, 2002). Thus, the HG risk measure is reduced to $\mathrm{TVaR}_q(X)$, the well-known Tail Value-at-Risk with confidence level q of X. In light of this, we refer to the parameter q in the definition of the HG risk measure as the confidence level, even for a general power Young function.

Remark 2.3.2 *Since the case $k=1$ is considered as a particular case, we discuss the behavior of equation (2.2) when $k\downarrow 1$. Let $X\in L^{k'}$ for some $k'>1$ and $x\in(-\infty,\hat{x})$. By the dominated convergence theorem, we derive*

$$\begin{aligned} \lim_{k\downarrow 1}E\left[(X-x)_+^{k-1}\right] &= \lim_{k\downarrow 1}E\left[(X-x)_+^{k-1}1_{(X>x)}\right] \\ &= E\left[\lim_{k\downarrow 1}(X-x)_+^{k-1}1_{(X>x)}\right] \\ &= \bar{F}(x). \end{aligned}$$

The applicability of the dominated convergence theorem is justified because the random variable $(X-x)_+^{k-1}$ is bounded from above by the integrable random variable $(X-x)_+^{k'}\vee 1$, for every $1<k\leq k'$. Furthermore, the continuity of L^p spaces with respect to

p implies

$$\lim_{k\downarrow 1}\left(E\left[(X-x)_+^k\right]\right)^{k-1} = \lim_{k\downarrow 1}\left(\left(E\left[(X-x)_+^k\right]\right)^{\frac{1}{k}}\right)^{k(k-1)} = 1.$$

That said, as $k\downarrow 1$*, equation (2.2) becomes*

$$\overline{F}(x) = 1-q, \tag{2.5}$$

which is consistent with relation (2.4). Indeed, Theorem 10 of Rockafellar and Uryasev (2002) implies that the infimum in (2.3) is attained for all x *satisfying (2.5). Thus, when* X *does not put mass at its* q*-quantile, the HG risk measure is given by relations (2.1) and (2.2), even in the limit case* $k=1$*.*

2.3.2 Capital allocation under the HG risk measure

Let $X_1, \ldots, X_d$ denote $d \geq 1$ random variables from L^k for some $k \geq 1$, which represent risk variables in loss-profit style. Hereafter, to avoid repetitive notation, we reserve the index i to represent a number belonging to the set $\{1,\ldots,d\}$. For instance, when we write "$X_i \in L^1$ for every i", we stand for the long sentence "$X_i \in L^1$ for every $i \in \{1,\ldots,d\}$". Also, when we just write "$X_i \in L^1$", we stand for the long sentence "$X_i \in L^1$ for an arbitrary but fixed $i \in \{1,\ldots,d\}$".

Let W be the subspace of L^k composed of all linear combinations of $X_1, \ldots, X_d$; we refer to the elements of W as portfolios. For $q \in (0,1)$, we define a risk capital allocation $\tilde{\Lambda}_q$ as a function from $W \times W$ to $\mathbb{R}$. For X and S from W, we interpret $\tilde{\Lambda}_q(X,S)$ as the capital allocated from portfolio S to the sub-portfolio X. Accordingly, $\tilde{\Lambda}_q$ should satisfy $\tilde{\Lambda}_q(S,S) = H_q(S)$. Furthermore, we follow Kalkbrener (2005) in requiring that a coherent capital allocation rule $\tilde{\Lambda}_q$ satisfies the next properties:

Linear aggregation:

$$\tilde{\Lambda}_q(aX+bY,S) = a\tilde{\Lambda}_q(X,S) + b\tilde{\Lambda}_q(Y,S),$$

for every $a,b \in \mathbb{R}$ and $X,Y,S \in W$.

Diversification:

$$\tilde{\Lambda}_q(X,S) \leq \tilde{\Lambda}_q(S,S),$$

for every $X,S \in W$.

Continuity:

$$\lim_{t\to 0} \tilde{\Lambda}_q(X, tX + S) = \tilde{\Lambda}_q(X, S),$$

for every $X, S \in W$.

The axioms that Kalkbrener (2005) imposes are quite intuitive. The first one guarantees that the sum of the risk capital from the sub-portfolios equals the risk capital of the overall portfolio, the second one contemplates diversification, and the third one ensures that small changes to the portfolio only have a limited effect on the risk capital of its sub-portfolios.

It turns out that existence and uniqueness properties for a coherent capital allocation rule with respect to H_q are intimately related to the existence of its directional derivatives. Accordingly, for two random variables X and S from L^k for some $k \geq 1$ and $q \in (0,1)$, denote by $\Lambda_q(X,S)$ the directional derivative of the risk measure H_q at S in the direction of X, namely,

$$\Lambda_q(X,S) = \lim_{t\to 0} \frac{H_q(tX+S) - H_q(S)}{t}. \tag{2.6}$$

Our main result below shows the existence of $\Lambda_q(X,S)$ and gives an analytical formula for it. To avoid triviality, assume that S is not degenerate because otherwise, $\Lambda_q(X,S) = H_q(X)$ trivially holds.

Theorem 2.3.1 *For some $k \geq 1$, let X and S be two random variables from L^k, with S non-degenerate. If $k = 1$, further assume that S does not put mass at its q-quantile. Moreover, let H_q denote the HG risk measure defined in relation (2.1) or (2.4) as the case may be, where $q \in (0,1)$. Then, the directional derivative $\Lambda_q(X,S)$ in (2.6) is well-defined as follows.*

(a) When $k > 1$,

$$\Lambda_q(X,S) = \frac{E\left[X(S-x^*)_+^{k-1}\right]}{E\left[(S-x^*)_+^{k-1}\right]}, \tag{2.7}$$

where $x = x^*$ is the unique solution to the equation

$$\frac{\left(E\left[(S-x)_+^{k-1}\right]\right)^k}{\left(E\left[(S-x)_+^{k}\right]\right)^{k-1}} = 1 - q. \tag{2.8}$$

(b) When $k = 1$,

$$\Lambda_q(X,S) = \frac{E\left[X 1_{(S>x^*)}\right]}{P(S > x^*)}, \tag{2.9}$$

where $x = x^*$ is the unique solution to the equation

$$P(S > x^*) = 1 - q. \tag{2.10}$$

We first present an elementary result, which will be used in the proof of Theorem 2.3.1.

Lemma 2.3.1 *For two random variables from L^k for some $k \geq 1$, define the function*

$$f(t) = E\left[(tX + S)_+^k\right], \qquad t \in \mathbb{R}.$$

(a) When $k > 1$, the function $f(t)$ is continuously differentiable over $t \in \mathbb{R}$ with

$$f'(t) = kE\left[X(tX + S)_+^{k-1}\right];$$

(b) When $k = 1$, its two-sided derivatives are finite over $t \in \mathbb{R}$ with

$$\begin{aligned} f'_+(t) &= E\left[X1_{(tX+S>0)}\right] + E\left[X1_{(tX+S=0,X>0)}\right], \\ f'_-(t) &= E\left[X1_{(tX+S>0)}\right] + E\left[X1_{(tX+S=0,X\leq 0)}\right]. \end{aligned}$$

Proof. First consider the right-hand derivative:

$$f'_+(t) = \lim_{\delta \downarrow 0} \frac{E\left[((t+\delta)X + S)_+^k\right] - E\left[(tX + S)_+^k\right]}{\delta}.$$

For the ease of notation, define $Z = tX + S$ and rewrite the relation above as

$$f'_+(t) = \lim_{\delta \downarrow 0} E\left[\frac{(\delta X + Z)_+^k - Z_+^k}{\delta}\right].$$

Observe that, for any $z \in \mathbb{R}$, the function $(x+z)_+^k$ is convex in $x \in \mathbb{R}$. Thus, for $x, z \in \mathbb{R}$ and $0 < \delta < 1$, we have

$$z_+^k - (-x+z)_+^k \leq \frac{(\delta x + z)_+^k - z_+^k}{\delta} \leq (x+z)_+^k - z_+^k.$$

This ensures the application of the dominated convergence theorem to derive

$$f'_+(t) = E\left[\lim_{\delta \downarrow 0} \frac{(\delta X + Z)_+^k - Z_+^k}{\delta}\right].$$

We split the ratio inside the expectation above into four terms

$$\frac{(\delta X+Z)_+^k - Z_+^k}{\delta} = I_1(\delta) + I_2(\delta) + I_3(\delta) + I_4(\delta), \qquad \delta > 0,$$

conforming to the ranges

$$\begin{aligned}
&(-\delta X < Z \leq 0, X > 0),\\
&(Z > 0, X > 0),\\
&(0 < Z \leq -\delta X, X \leq 0), \text{ and}\\
&(Z > -\delta X, X \leq 0),
\end{aligned}$$

respectively. Taking $\delta \downarrow 0$, the last three terms have almost sure limits as

$$\begin{aligned}
I_2(\delta) &= \frac{(\delta X+Z)^k - Z^k}{\delta} 1_{(Z>0,X>0)} \to kXZ^{k-1} 1_{(Z>0,X>0)},\\
I_3(\delta) &= \left(-\frac{Z^k}{\delta}\right) 1_{(0<Z\leq -\delta X, X\leq 0)} \to 0,\\
I_4(\delta) &= \frac{(\delta X+Z)^k - Z^k}{\delta} 1_{(Z>-\delta X, X\leq 0)} \to kXZ^{k-1} 1_{(Z>0,X\leq 0)}.
\end{aligned}$$

For the first term

$$I_1(\delta) = \frac{(\delta X+Z)^k}{\delta} 1_{(-\delta X<Z\leq 0, X>0)},$$

to derive its almost sure limit as $\delta \downarrow 0$, we need to distinguish the cases $k > 1$ and $k = 1$. For $k > 1$ we have $I_1(\delta) \to 0$, while for $k = 1$ we have

$$I_1(\delta) = \frac{\delta X + Z}{\delta} 1_{(-\delta X<Z<0, X>0)} + X 1_{(Z=0,X>0)} \to X 1_{(Z=0,X>0)}.$$

Summing up, for $k > 1$ we obtain

$$\begin{aligned}
f'_+(t) &= E\left[kXZ_+^{k-1} 1_{(Z>0,X>0)} + kXZ_+^{k-1} 1_{(Z>0,X\leq 0)}\right]\\
&= kE\left[XZ_+^{k-1}\right]\\
&= kE\left[X(tX+S)_+^{k-1}\right],
\end{aligned}$$

while for $k=1$ we obtain

$$\begin{aligned} f'_+(t) &= E\left[X1_{(Z>0,X>0)} + X1_{(Z>0,X\leq 0)} + X1_{(Z=0,X>0)}\right] \\ &= E\left[X1_{(Z>0)}\right] + E\left[X1_{(Z=0,X>0)}\right] \\ &= E\left[X1_{(tX+S>0)}\right] + E\left[X1_{(tX+S=0,X>0)}\right]. \end{aligned}$$

Next consider the left-hand derivative

$$f'_-(t) = \lim_{\delta\uparrow 0} \frac{E\left[((t+\delta)X+S)_+^k\right] - E\left[(tX+S)_+^k\right]}{\delta}.$$

Analogously, with $Z = tX+S$ we split

$$\frac{(\delta X+Z)_+^k - Z_+^k}{\delta}, \qquad \delta < 0,$$

conforming to the ranges

$$\begin{aligned} &(0 < Z \leq -\delta X, X > 0), \\ &(Z > -\delta X, X > 0), \\ &(-\delta X < Z \leq 0, X \leq 0), \text{ and} \\ &(Z > 0, X \leq 0). \end{aligned}$$

Going along the same lines as in the derivation of f'_+, for $k>1$ we obtain

$$f'_-(t) = kE\left[X\,(tX+S)_+^{k-1}\right],$$

while for $k=1$ we obtain

$$f'_-(t) = E\left[X1_{(tX+S>0)}\right] + E\left[X1_{(tX+S=0,X\leq 0)}\right].$$

Thus, we have proved all conclusions of the lemma. ■

Proof of Theorem 2.3.1. For $k>1$, let h_q be the convex function defined by

$$h_q(Y) = \frac{1}{(1-q)^{\frac{1}{k}}} \left(E\left[(Y)_+^k\right]\right)^{1/k}, \qquad Y \in L^k.$$

Let $h_q(Y;-X)$ represent the directional derivative of h_q at Y in the direction of $-X$, namely,

$$h_q(Y;-X) = \lim_{t\to 0} \frac{1}{(1-q)^{\frac{1}{k}}} \frac{\left(E\left[(t(-X)+Y)_+^k\right]\right)^{1/k} - \left(E\left[(Y)_+^k\right]\right)^{1/k}}{t}.$$

By Lemma 2.3.1(a), we have

$$\begin{aligned} h_q(Y;-X) &= \frac{1}{(1-q)^{\frac{1}{k}}} \frac{d}{dt}\left(E\left[(t(-X)+Y)_+^k\right]\right)^{1/k}\Bigg|_{t=0} \\ &= \frac{E\left[-X(Y)_+^{k-1}\right]}{E\left[(Y)_+^{k-1}\right]}. \end{aligned}$$

In conclusion, h_q is differentiable in the direction of $-X$. Now we deal with the differentiability of H_q at S. If x^* solves equation (2.8), then we have

$$\begin{aligned} H_q(t(-X)+S) &= \inf_{x\in\mathbb{R}} \left\{ x + \frac{1}{(1-q)^{\frac{1}{k}}} \left(E\left[(t(-X)+S-x)_+^k\right]\right)^{\frac{1}{k}} \right\} \\ &\leq x^* + h_q(t(-X)+S-x^*), \end{aligned}$$

whence,

$$\begin{aligned} \lim_{t\downarrow 0} \frac{H_q(t(-X)+S) - H_q(S)}{t} &\leq \lim_{t\downarrow 0} \frac{h_q(t(-X)+S-x^*) - h_q(S-x^*)}{t} \\ &= h_q(S-x^*;-X) \\ &= \frac{E\left[-X(S-x^*)_+^{k-1}\right]}{E\left[(S-x^*)_+^{k-1}\right]}. \end{aligned}$$

Since H_q is a convex functional over the normed linear space L^k, by Lemma 3.6.2 of Niculescu and Persson (2006), we derive

$$\begin{aligned} \lim_{t\downarrow 0} \frac{H_q(tX+S) - H_q(S)}{t} &\geq -\lim_{t\downarrow 0} \frac{H_q(t(-X)+S) - H_q(S)}{t} \\ &\geq \frac{E\left[X(S-x^*)_+^{k-1}\right]}{E\left[(S-x^*)_+^{k-1}\right]}. \end{aligned}$$

In respect to the left-hand side directional derivative, we obtain

$$\begin{aligned}\lim_{t\uparrow 0}\frac{H_q(tX+S)-H_q(S)}{t} &= \lim_{t\downarrow 0}\frac{H_q(t(-X)+S)-H_q(S)}{(-t)}\\ &= \frac{E\left[X(S-x^*)_+^{k-1}\right]}{E\left[(S-x^*)_+^{k-1}\right]}.\end{aligned}$$

Thus, we have proved the theorem. ■

As a consequence of Theorem 3.1 and Theorem 4.3 of Kalkbrener (2005) and Theorem 2.3.1, we derive the following result.

Corollary 2.3.1 *For some $k \geq 1$ and $q \in (0,1)$, let $\Lambda_q : W \times W \to \mathbb{R}$ be the functional defined in (2.6). If $k = 1$, assume that each $S \in W$ is deterministic or does not put mass at its q-quantile. Thus Λ_q is the unique capital allocation rule concerning H_q, which satisfies linear aggregation, diversification, and continuity.*

Remark 2.3.3 *In the case $k = 1$, Corollary 2.3.1 imposes a restriction on the joint distribution of the basic risk variables $X_1, \ldots, X_d$. Notwithstanding, $\Lambda_q(\cdot,\cdot)$ defines a linear and diversifying capital allocation rule with respect to TVaR; see Section 5.2 of Kalkbrener (2005). This capital allocation rule is commonly used in practice, and it is often preferred to the capital allocation rule based on VaR contributions; see for instance Kalkbrener et al. (2004). Although (2.9) is quite elegant, its analytical tractability for generally distributed and possibly dependent $X_1, \ldots, X_d$ remains seldom feasible. For analytical expressions of (2.9), we refer the reader to Chiragiev and Landsman (2007), Dhaene et al. (2008), Ignatieva and Landsman (2019), and Kim and Kim (2019). For the general case $k \geq 1$, in Section 2.4.1, we present some dependence structures over $X_1, \ldots, X_d$ which admit quasi-explicit expressions for (2.7) and (2.9). In particular, Example 2.4.2 and Example 2.4.3 generalize the work of Chiragiev and Landsman (2007).*

Theorem 2.3.1 already gives an algorithm on how to numerically calculate $\Lambda_q(X,S)$. Let $(X_1,S_1),\ldots,(X_n,S_n)$ be a sample of (X,S). A natural candidate for approximating x^* would be the unique solution $x = x_n^* \in (-\infty, \max_{1\leq i\leq n} S_i)$ to the equation

$$\frac{\left(\frac{1}{n}\sum_{i=1}^n (S_i-x)_+^{k-1}\right)^k}{\left(\frac{1}{n}\sum_{i=1}^n (S_i-x)_+^{k}\right)^{k-1}} = 1-q. \tag{2.11}$$

Also, an intuitive candidate for estimating the capital allocation rule $\Lambda_q(X,S)$ from the sample would be

$$\Lambda_q^{(n)} = \frac{\frac{1}{n}\sum_{i=1}^n X_i (S_i - x_n^*)_+^{k-1}}{\frac{1}{n}\sum_{i=1}^n (S_i - x_n^*)_+^{k-1}}. \tag{2.12}$$

This estimator $\Lambda_q^{(n)}$ is a strongly consistent estimator of $\Lambda_q(X,S)$, i.e., with probability one

$$\lim_{n\to\infty} \Lambda_q^{(n)} = \Lambda_q(X,S).$$

In fact, by the strong law of large numbers, for every $x \in \mathbb{R}$, it holds almost surely that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^+ (S_i - x)_+^{k-1} = E\left[X^+ (S-x)_+^{k-1}\right].$$

Moreover, from Theorem 1 of Ahn and Shyamalkumar (2014), it holds almost surely that

$$\lim_{n\to\infty} x_n^* = x^*.$$

Hence, for every $\varepsilon > 0$,

$$\Omega_\varepsilon = \left\{\omega \in \Omega : \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left(|x_n^* - x^*| \le \varepsilon\right)\right\}$$

defines an event with probability one. Restricted to Ω_ε, we have, almost surely,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^+ (S_i - x_n^*)_+^{k-1} &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^+ (S_i - x^* + \varepsilon)_+^{k-1} \\
&= E\left[X^+ (S - x^* + \varepsilon)_+^{k-1}\right]
\end{aligned}$$

and

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^+ (S_i - x_n^*)_+^{k-1} &\ge \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^+ (S_i - x^* - \varepsilon)_+^{k-1} \\
&= E\left[X^+ (S - x^* - \varepsilon)_+^{k-1}\right].
\end{aligned}$$

Since the function $E\left[X^+ (S-x)_+^{k-1}\right]$ is continuous in x, it follows that, almost surely,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^+ (S_i - x_n^*)_+^{k-1} = E\left[X^+ (S-x^*)_+^{k-1}\right].$$

The same holds true with each X_i^+ changed to X_i^- on the left and X^+ changed to X^- on the right. Hence, almost surely,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i (S_i - x_n^*)_+^{k-1} = E\left[X(S-x^*)_+^{k-1}\right].$$

Similarly, we have the almost sure convergence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} (S_i - x_n^*)_+^{k-1} = E\left[(S-x^*)_+^{k-1}\right].$$

Therefore, it holds almost surely that

$$\lim_{n\to\infty} \Lambda_q^{(n)}(S) = \frac{E\left[X(S-x^*)_+^{k-1}\right]}{E\left[(S-x^*)_+^{k-1}\right]} = \Lambda_q(X,S).$$

2.3.3 Asymptotic behavior

In the current post-financial crisis era, regulators have become excessively prudent in determining risk capital requirements, which has to a certain extent weaken VaR's status. In this sense, the so-called tail-based risk measurement has emerged as an essential tool for quantifying insurance risks while emphasizing the adverse effect of low probability but high severity tail events. From Tang and Yang (2012), the HG risk measure with a power Young function is a tail measure; in this sense, the regulators' excessive prudence is reflected by $q \uparrow 1$. In this section, by following Asimit et al. (2011), we study the asymptotic behavior of the capital allocation rule Λ_q as $q \uparrow 1$.

Now, we introduce the notion of multivariate regular variation. Let $(X_1,\dots,X_d)$ be a d-dimensional non-negative random vector with marginal distributions F_1, ..., F_d. The vector $(X_1,\dots,X_d)$ is said to follow a distribution with a multivariate regularly varying tail if there exist a distribution function F and a limit measure ν not identically 0, such that, as $x \to \infty$,

$$\frac{1}{\overline{F}(x)} P\left(\left(\frac{X_1}{x},\dots,\frac{X_d}{x}\right)\in\cdot\right) \xrightarrow{v} \nu(\cdot) \text{ on } \overline{\mathbb{R}}_+^d \backslash \{\mathbf{0}\}, \tag{2.13}$$

where $\xrightarrow{v}$ denotes vague convergence. Discussions on vague convergence can be found in, e.g., Section 3.3.5 of Resnick (2007).

For every i, we define

$$c_i = \lim_{x \to \infty} \frac{\overline{F_i}(x)}{\overline{F}(x)} = \nu(\mathbf{1}_i, \boldsymbol{\infty}].$$

Due to the nondegeneracy of the limit measure ν, there exists some i such that $c_i > 0$. Moreover, from relation (2.13), we notice that the limit measure ν is homogeneous. That is, there exists some $0 < \alpha < \infty$, representing the multivariate regular variation index, such that $\nu(tA) = t^{-\alpha}\nu(A)$ holds for every Borel set $A \subset \overline{\mathbb{R}}_+^d \backslash \{\mathbf{0}\}$.

Assumption 2.3.1 *Let $(X_1, \ldots, X_d)$ be a d-dimensional non-negative random vector with marginal distributions F_1, ..., F_d, such that (2.13) holds for the auxiliary distribution function F with regularly varying tail of index $-\alpha$ and some nontrivial limit measure ν. Further, assume that the limit measure ν satisfies $c_i > 0$ for every i.*

Define $S = \sum_{i=1}^d X_i$ and let F_S denote the distribution of S.

Theorem 2.3.2 *Let $(X_1, \ldots, X_d)$ be a random vector satisfying Assumption 2.3.1. Further, assume that $\alpha > 1$ and $k \in [1, \alpha)$. By defining $\Lambda_q(X_i, S)$ conforming to relation (2.7) or (2.9) as the case may be, as $q \uparrow 1$, it holds that*

$$\Lambda_q(X_i, S) \sim C_i F^{\leftarrow}(q) =: \Lambda_q^a(X_i, S), \tag{2.14}$$

where

$$\begin{aligned} C_i &= \left(\frac{(\nu(\boldsymbol{z} : s_d > 1)\,\mathrm{B}(\alpha - k, k))^{1-\alpha}}{(\alpha - k)^{\alpha - k} k^{k-1}} \right)^{\frac{1}{\alpha}} (I_i + c_i \mathrm{B}(\alpha - k, k)), \\ I_i &= \iint_{(s>0, 0<r\leq s+1)} \nu(\boldsymbol{z} : z_i > r, s_d > s+1)\, dr ds^{k-1}, \end{aligned}$$

and $\mathrm{B}(\cdot,\cdot)$ denotes the beta function.

Before proving Theorem 2.3.2, we present the following technical result.

Lemma 2.3.2 *Let $(X_1, \ldots, X_d)$ be a random vector satisfying Assumption 2.3.1. Further, assume that $\alpha > 1$. Then, for $k \in (0, \alpha - 1)$, it holds that*

$$\lim_{x \to \infty} \frac{E\left[X_i (S-x)_+^k\right]}{x^{k+1}\overline{F}(x)} = \int_0^\infty \int_0^\infty \nu(\boldsymbol{z} : z_i > r, s_d > s+1)\, dr ds^k \tag{2.15}$$

$$\lim_{x \to \infty} \frac{E\left[(S-x)_+^k\right]}{x^k \overline{F}(x)} = k\mathrm{B}(\alpha - k, k)\,\nu(\boldsymbol{z} : s_d > 1). \tag{2.16}$$

Proof. First, we consider relation (2.15). By Fubini's theorem, we have

$$\begin{aligned} E\left[X_i(S-x)_+^k\right] &= E\left[\int_0^\infty \int_0^\infty 1_{(X_i>u)} 1_{(S-x>v)} du dv^k\right] \\ &= \int_0^\infty \int_0^\infty P\left(X_i > u, S-x > v\right) du dv^k. \end{aligned}$$

Letting $u = rx$ and $v = sx$, we rewrite

$$E\left[X_i(S-x)_+^k\right] = x^{k+1} \int_0^\infty \int_0^\infty P\left(X_i > rx, S > (s+1)x\right) dr ds^k.$$

By the dominated convergence theorem and relation (2.13), we obtain

$$\begin{aligned} \lim_{x\to\infty} \frac{E\left[X_i(S-x)_+^k\right]}{x^{k+1}\overline{F}(x)} &= \int_0^\infty \int_0^\infty \lim_{x\to\infty} \frac{P\left(X_i > rx, S > (s+1)x\right)}{\overline{F}(x)} dr ds^k \\ &= \int_0^\infty \int_0^\infty \nu\left(\boldsymbol{z} : z_i > r, s_d > s+1\right) dr ds^k, \end{aligned}$$

where in order to apply the vague convergence, we use Lemma A.1 of Shi et al. (2017) to verify that the boundary of the set $(\boldsymbol{z} : z_i > r, s_d > s+1)$ has mass zero. In the derivation above, we justify the applicability of the dominated convergence theorem with the help of Proposition 2.6(ii) of Resnick (2007). In fact, for $0 < \varepsilon < \alpha - k$ and large enough x, we have

$$\begin{aligned} & \frac{P\left(X_i > rx, S > (s+1)x\right)}{\overline{F}(x)} \\ \leq\ & 1_{(s>0, 0<r\leq s+1)} \frac{P\left(S > (s+1)x\right)}{\overline{F}(x)} + 1_{(s>0, r>s+1)} \frac{P\left(X_i > rx\right)}{\overline{F}(x)} \\ \leq\ & C 1_{(s>0, 0<r\leq s+1)} (s+1)^{-\alpha+\varepsilon} + C 1_{(s>0, r>s+1)} r^{-\alpha+\varepsilon}, \end{aligned}$$

which is clearly integrable with respect to $dr ds^k$ over $\mathbb{R}_+^2$. Now we turn to prove relation (2.16). As when verifying relation (2.15), we write

$$\begin{aligned} E\left[(S-x)_+^k\right] &= \int_0^\infty P\left(S - x > v\right) dv^k \\ &= x^k \int_0^\infty P\left(S > (s+1)x\right) ds^k. \end{aligned}$$

Due to Assumption 2.3.1, it holds that $\overline{F}_S(x) \sim \nu(\boldsymbol{z} : s_d > 1)\overline{F}(x)$. Thus, the dominated convergence theorem implies

$$\begin{aligned}\lim_{x\to\infty}\frac{E\left[(S-x)_+^k\right]}{x^k\overline{F}(x)} &= \lim_{x\to\infty}\frac{\overline{F}_S(x)}{\overline{F}(x)}\int_0^\infty \frac{P(S>(s+1)x)}{\overline{F}_S(x)}ds^k \\ &= \nu(\boldsymbol{z} : s_d > 1)\int_0^\infty \lim_{x\to\infty}\frac{\overline{F}_S((s+1)x)\,ds^k}{\overline{F}_S(x)} \\ &= \nu(\boldsymbol{z} : s_d > 1)\int_0^\infty (s+1)^{-\alpha}ds^k,\end{aligned}$$

where the last equality follows because $\overline{F}_S$ is regularly varying with parameter $-\alpha$. By changing of variable, we have

$$\begin{aligned}\int_0^\infty (s+1)^{-\alpha}ds^k &= k\int_0^1 y^{\alpha-k-1}(1-y)^{k-1}dy \\ &= k\mathrm{B}(\alpha-k,k).\end{aligned}$$

Thus

$$\lim_{x\to\infty}\frac{E\left[(S-x)_+^k\right]}{x^k\overline{F}(x)} = k\mathrm{B}(\alpha-k,k)\,\nu(\boldsymbol{z} : s_d > 1).$$

■

Proof of Theorem 2.3.2:

From Lemma 2.2 of Tang and Yang (2012), we have that $x^* \to \infty$, as $q \uparrow 1$. Thus, by Theorem 2.3.1 and Lemma 2.3.2, as $q \uparrow 1$,

$$\Lambda_q(X_i,S) = \frac{E\left[X_i(S-x^*)_+^{k-1}\right]}{E\left[(S-x^*)_+^{k-1}\right]} \sim A_i x^*,$$

where

$$A_i = \frac{\int_0^\infty\int_0^\infty \nu(\boldsymbol{z} : z_i > r, s_d > s+1)\,dr\,ds^{k-1}}{(\alpha-k)\mathrm{B}(\alpha-k,k)\,\nu(\boldsymbol{z} : s_d > 1)}.$$

In addition, by the proof of Theorem 4.1 of Tang and Yang (2012), as $q \uparrow 1$,

$$x^* \sim \left(\frac{\mathrm{B}(\alpha-k,k)}{(\alpha-k)^{-k}k^{k-1}}\right)^{\frac{1}{\alpha}} F_S^{\leftarrow}(q).$$

Therefore, it follows that

$$\Lambda_q(X_i,S) \sim B_i F_S^{\leftarrow}(q), \text{ as } q \uparrow 1,$$

where

$$B_i = \left(\frac{(\mathrm{B}(\alpha-k,k))^{1-\alpha}}{(\alpha-k)^{\alpha-k} k^{k-1}} \right)^{\frac{1}{\alpha}} \frac{\int_0^\infty \int_0^\infty \nu(\boldsymbol{z} : z_i > r, s_d > s+1)\, dr ds^{k-1}}{\nu(\boldsymbol{z} : s_d > 1)}.$$

Moreover, notice that

$$\begin{aligned} & \int_0^\infty \int_0^\infty \nu(\boldsymbol{z} : z_i > r, s_d > s+1)\, dr ds^{k-1} \\ = & \left(\iint_{(s>0, 0<r\leq s+1)} + \iint_{(s>0, r>s+1)} \right) \nu(\boldsymbol{z} : z_i > r, s_d > s+1)\, dr ds^{k-1}. \end{aligned}$$

However, by the homogeneity of ν and a change of variable, the second term can be simplified to

$$\begin{aligned} \iint_{(s>0, r>s+1)} \nu(\boldsymbol{z} : z_i > r)\, dr ds^{k-1} &= \nu(\boldsymbol{z} : z_i > 1) \iint_{(s>0, r>s+1)} r^{-\alpha} dr ds^{k-1} \\ &= c_i \mathrm{B}(\alpha-k,k). \end{aligned}$$

Hence, B_i can be expressed as

$$B_i = \left(\frac{(\mathrm{B}(\alpha-k,k))^{1-\alpha}}{(\alpha-k)^{\alpha-k} k^{k-1}} \right)^{\frac{1}{\alpha}} \left(\frac{I_i + c_i \mathrm{B}(\alpha-k,k)}{\nu(\boldsymbol{z} : s_d > 1)} \right).$$

Besides, Assumption 2.3.1 implies that

$$\overline{F}_S(x) \sim \nu(\boldsymbol{z} : s_d > 1)\, \overline{F}(x).$$

In addition, the functions $\overline{F}$ and $\overline{F}_S$ are regularly varying with parameter $-\alpha$, therefore it holds that

$$F_S^{\leftarrow}(q) \sim (\nu(\boldsymbol{z} : s_d > 1))^{\frac{1}{\alpha}} F^{\leftarrow}(q), \text{ as } q \uparrow 1;$$

see Proposition 2.6(vi) of Resnick (2007). Then, we have

$$\Lambda_q(X_i, S) \sim C_i F^{\leftarrow}(q),$$

where,

$$C_i = \left(\frac{(\nu(\boldsymbol{z} : s_d > 1)\, \mathrm{B}(\alpha-k,k))^{1-\alpha}}{(\alpha-k)^{\alpha-k} k^{k-1}} \right)^{\frac{1}{\alpha}} (I_i + c_i \mathrm{B}(\alpha-k,k)).$$

Remark 2.3.4 *From Theorem 4.1 of Tang and Yang (2012) along with Proposition 2.6(vi) of Resnick (2007), as* $q \uparrow 1$*, it holds that*

$$H_q(S) \sim \alpha \left(\frac{\nu(\boldsymbol{z} : s_d > 1) \mathrm{B}(\alpha - k, k)}{(\alpha - k)^{\alpha - k} k^{k-1}} \right)^{\frac{1}{\alpha}} F^{\leftarrow}(q) =: H_q^a(S).$$

Moreover, the proof of Theorem 2.3.2 reveals that the asymptotics of the allocated capitals add up to the asymptotic of the initial aggregate capital, i.e., $H_q^a(S) = \sum_{i=1}^d \Lambda_q^a(X_i, S)$*. Therefore, the asymptotic formulas should be used hand-in-hand for both the allocated capital and the aggregate capital.*

The computation of the constant C_i is an issue. Now we show two extreme cases which admit a significant simplification.

The first case is that $(X_1, \ldots, X_d)$ exhibits no tail dependence, of which Gaussian copula is a special case. Then ν is concentrated on the axes only. Consequently, we have

$$\nu(\boldsymbol{z} : s_d > 1) = \sum_{i=1}^d c_i$$

and

$$\begin{aligned} I_i &= \iint_{(s>0, 0<r\leq s+1)} \nu(\boldsymbol{z} : z_i > r, s_d > s+1) \, dr ds^{k-1} \\ &= \iint_{(s>0, 0<r\leq s+1)} \nu(\boldsymbol{z} : z_i > s+1, z_j = 0 \text{ for } j \neq i) \, dr ds^{k-1} \\ &= \iint_{(s>0, 0<r\leq s+1)} c_i (s+1)^{-\alpha} \, dr ds^{k-1} \\ &= c_i (\alpha - 1) \mathrm{B}(\alpha - k, k). \end{aligned}$$

It gives us a simplified expression for C_i as

$$C_i = \frac{\alpha (\mathrm{B}(\alpha - k, k))^{\frac{1}{\alpha}}}{(\alpha - k)^{1 - \frac{k}{\alpha}} k^{\frac{k-1}{\alpha}}} \left(\sum_{i=1}^d c_i \right)^{\frac{1}{\alpha}} c_i.$$

The second case is that $(X_1, \ldots, X_d)$ is fully dependent, of which the comonotonicity is a special case. Then ν is concentrated on a straight line, which we denote by

$$z_i = m_i u, \qquad u > 0, \text{ for every } i,$$

with $m_1, \ldots, m_d$ representing strictly positive constants. In this case, we define $M = \sum_{j=1}^d m_j$. Clearly, $c_i = \left(\frac{m_i}{M}\right)^{\alpha} \nu(\boldsymbol{z} : s_d > 1)$. Then, the integrand of I_i can be rewritten as follows:

$$\begin{aligned}\nu\left(\boldsymbol{z}: z_i > r, s_d > s+1\right) &= \nu\left(\boldsymbol{z}: s_d > \frac{M}{m_i} r, s_d > s+1\right) \\ &= \nu\left(\boldsymbol{z}: s_d > \left(\frac{M}{m_i} r \vee (s+1)\right)\right) \\ &= \left(\frac{M}{m_i} r \vee (s+1)\right)^{-\alpha} \nu\left(\boldsymbol{z}: s_d > 1\right).\end{aligned}$$

Therefore, we have

$$\begin{aligned}& I_i + c_i \mathrm{B}(\alpha-k,k) \\ = \; & \int_0^\infty \int_0^\infty \nu\left(\boldsymbol{z}: z_i > r, s_d > s+1\right) dr ds^{k-1} \\ = \; & \nu\left(\boldsymbol{z}: s_d > 1\right)\left(\iint_{(s>0, r>\frac{m_i}{M}(s+1))} \left(\frac{M}{m_i} r\right)^{-\alpha} dr ds^{k-1}\right. \\ & \left. + \iint_{(s>0, 0<r\le\frac{m_i}{M}(s+1))} (s+1)^{-\alpha} dr ds^{k-1}\right) \\ = \; & \nu\left(\boldsymbol{z}: s_d > 1\right) \mathrm{B}(\alpha-k,k) \frac{m_i}{M} + \nu\left(\boldsymbol{z}: s_d > 1\right) \frac{m_i}{M} (\alpha-1) \mathrm{B}(\alpha-k,k) \\ = \; & \alpha \mathrm{B}(\alpha-k,k) \nu\left(\boldsymbol{z}: s_d > 1\right) \frac{m_i}{M}.\end{aligned}$$

Hence, we obtain a simplified expression for C_i as

$$C_i = \frac{\alpha\left(\mathrm{B}(\alpha-k,k)\right)^{\frac{1}{\alpha}}}{(\alpha-k)^{1-\frac{k}{\alpha}} k^{\frac{k-1}{\alpha}}} \left(\nu\left(\boldsymbol{z}: s_d > 1\right)\right)^{\frac{1}{\alpha}} \frac{m_i}{M}.$$

Alternatively, the constant C_i may be computed numerically. For instance, when there exists a function f_ν on $(\mathbf{0}, \boldsymbol{\infty}]$ such that

$$\nu(\boldsymbol{z}, \boldsymbol{\infty}] = \int_{(\boldsymbol{z}, \boldsymbol{\infty}]} f_\nu(\boldsymbol{w}) d\boldsymbol{w}, \text{ for } \boldsymbol{z} \in (\mathbf{0}, \boldsymbol{\infty}],$$

we can use a standard approximation argument to prove that

$$\nu\left(\boldsymbol{z}: z_i > r, s_d > s+1\right) = \int_{(\boldsymbol{z}: z_i > x, s_d > s+1)} f_\nu(\boldsymbol{z}) d\boldsymbol{z}.$$

We refer the reader to Example 2.4.4 for a non-extreme case where Theorem 2.3.2 applies.

2.4 Illustrations

2.4.1 Some examples

First, we discuss how to compute (2.7) and (2.9) by assuming that the joint distribution of the variables $X_1, \ldots, X_d$ is elliptical, independent exponential marginals, and multivariate Pareto.

Example 2.4.1 *For a measurable function $w : [0,\infty) \to [0,\infty)$, Furman and Zitikis (2008a, 2008b) define the w-weighted risk capital allocated from portfolio S to sub-portfolio X to be*

$$\tilde{\Lambda}_w(X,S) = \frac{E(Xw(S))}{E(w(S))}.$$

For $y \in \mathbb{R}$, define

$$w(y) = \begin{cases} (y-x^*)_+^{k-1}, & \text{if } k > 1, \\ 1_{(y>x^*)}, & \text{else if } k = 1, \end{cases}$$

where x^* is the unique solution to equation (2.8) or (2.10) as the case may be. Then, it is apparent that $\Lambda_q(X_i,S) = \tilde{\Lambda}_w(X_i,S)$. Moreover, it is well-known that when the vector $(X_1,\ldots,X_d)$ follows an elliptical distribution, the vector (X_i,S) so does. Hence, according to Proposition 4.2 of Furman and Zitikis (2008b), $\Lambda_q(X_i,S)$ admits a simplified expression as

$$\Lambda_q(X_i,S) = E(X_i) + C(X_i,S)(H_q(S) - E(S)),$$

where $C(X_i,S)$ is a constant depending on the parameters describing the distribution of (X_i,S). Therefore, for the elliptical case, our task is reduced to computing $H_q(S)$.

For a real-valued function $h(\cdot)$, we denote by $h[\lambda_1,\ldots,\lambda_d]$ the divided difference of order d with respect to distinct numbers $\lambda_1,\ldots, \lambda_d$. Specifically, $h[\lambda_1,\ldots,\lambda_d]$ is defined recursively as follows:

$$\begin{aligned} h[\lambda_1] &= h(\lambda_1), \\ h[\lambda_1,\ldots,\lambda_d] &= \frac{h[\lambda_2,\ldots,\lambda_d] - h[\lambda_1,\ldots,\lambda_{d-1}]}{\lambda_d - \lambda_1}, \; d \geq 2. \end{aligned}$$

It is easy to prove that

$$h[\lambda_1,\ldots,\lambda_d] = \sum_{i=1}^{d} \frac{h(\lambda_i)}{\prod_{j\neq i}(\lambda_i - \lambda_j)}. \tag{2.17}$$

Even if some of the parameters λ_1, ..., λ_d are repeated, relation (2.17) makes sense. For example,

$$h[\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_2,\lambda_3]=\frac{1}{(3-1)!}\frac{1}{(2-1)!}\frac{\partial^3}{\partial^2\lambda_1\partial\lambda_2}h[\lambda_1,\lambda_2,\lambda_3].$$

Example 2.4.2 *Let X_1, ..., X_d denote $d\geq 1$ independent and exponentially distributed random variables, each with a density function*

$$f_{X_i}(y)=\lambda_i e^{-\lambda_i y},\qquad y>0.$$

For $k\geq 1$ and $\lambda\neq\lambda_i$, we define

$$\begin{aligned} h(x,\lambda) &= \frac{e^{-\lambda x}}{\lambda^k}, \qquad x>0,\\ h_i(x,\lambda) &= (k+x\lambda_i)\frac{e^{-\lambda_i x}}{\lambda_i^{k+1}(\lambda_i-\lambda)}+\frac{\lambda^k e^{-\lambda_i x}-\lambda_i^k e^{-\lambda x}}{\lambda_i^k\lambda^k(\lambda_i-\lambda)^2}, \qquad x>0. \end{aligned}$$

Then it follows that

$$\Lambda_q(X_i,S)=\frac{h_i(x^*,[\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_d])}{h(x^*,[\lambda_1,\ldots,\lambda_d])}, \tag{2.18}$$

where $x=x^*$ is the unique solution to the equation

$$(\lambda_1\cdots\lambda_d)k^{1-k}\Gamma(k)(-1)^{d-1}h(x,[\lambda_1,\ldots,\lambda_d])=1-q, \tag{2.19}$$

and $\Gamma(\cdot)$ denotes the gamma function.

By definition, to prove relations (2.18) and (2.19), it is sufficient verifying that

$$\begin{aligned} & E\left[(S-x)_+^{k-1}\right]\\ =\ & (\lambda_1\cdots\lambda_d)\Gamma(k)(-1)^{d-1}h(x,[\lambda_1,\ldots,\lambda_d]), \qquad (2.20)\\ & E\left[X_i(S-x)_+^{k-1}\right]\\ =\ & (\lambda_1\cdots\lambda_d)\Gamma(k)(-1)^{d-1}h_i(x,[\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_d]), \qquad (2.21) \end{aligned}$$

for every $x>0$. First we consider relation (2.20). In this setting, the tail distribution of the random sum S is given by

$$P(S>s)=(\lambda_1\cdots\lambda_d)\sum_{i=1}^{d}\frac{e^{-\lambda_i s}}{\lambda_i\prod_{j\neq i}(\lambda_j-\lambda_i)},\qquad s>0.$$

By Fubini's theorem, we have

$$\begin{aligned} E\left[(S-x)_+^{k-1}\right] &= \int_0^\infty P(S>s+x)\,ds^{k-1} \\ &= (\lambda_1\cdots\lambda_d)\sum_{i=1}^d \frac{\int_0^\infty e^{-\lambda_i(s+x)}ds^{k-1}}{\lambda_i\prod_{j\neq i}(\lambda_j-\lambda_i)}. \end{aligned}$$

Therefore, by integrating, it holds that

$$\begin{aligned} E\left[(S-x)_+^{k-1}\right] &= (\lambda_1\cdots\lambda_d)\,\Gamma(k)\sum_{i=1}^d \frac{e^{-\lambda_i x}}{\lambda_i^k\prod_{j\neq i}(\lambda_j-\lambda_i)} \\ &= (\lambda_1\cdots\lambda_d)\,\Gamma(k)\,(-1)^{d-1}\,h(x,[\lambda_1,\ldots,\lambda_d]). \end{aligned}$$

Now we turn to prove relation (2.21). It is easy to prove that the tail distribution of the vector (X_i,S) is given by

$$\begin{aligned} & P(X_i>t,S>s) \\ = \ & (\lambda_1\cdots\lambda_d)\sum_{l\neq i}\frac{1}{\prod_{j\neq l}(\lambda_j-\lambda_l)}\left(\frac{e^{-(\lambda_i-\lambda_l)t}e^{-\lambda_l s}}{\lambda_l}-\frac{e^{-\lambda_i s}}{\lambda_i}\right), \qquad 0<s \text{ and } 0<t\le s. \end{aligned}$$

To compute $E\left[X_i(S-x)_+^{k-1}\right]$, we split it into two terms

$$E\left[X_i(S-x)_+^{k-1}\right] = \left(\int_0^\infty\int_0^{s+x}+\int_0^\infty\int_{s+x}^\infty\right)P(X_i>t,S>s+x)\,dtds^{k-1}.$$

For the first term, we have

$$\begin{aligned} & \int_0^\infty\int_0^{s+x}P(X_i>t,S>s+x)\,dtds^{k-1} \\ = \ & (\lambda_1\cdots\lambda_d)\,\Gamma(k)\sum_{l\neq i}\frac{1}{\lambda_l(\lambda_i-\lambda_l)\prod_{j\neq l}(\lambda_j-\lambda_l)}\left(\frac{e^{-\lambda_l x}}{\lambda_l^{k-1}}-\frac{e^{-\lambda_i x}}{\lambda_i^{k-1}}\right)-\frac{e^{-\lambda_i x}(k+x\lambda_i-1)}{\lambda_i^{k+1}\prod_{j\neq l}(\lambda_j-\lambda_l)}. \end{aligned}$$

For the second term, we derive

$$\begin{aligned} & \int_0^\infty\int_{s+x}^\infty P(X_i>t,S>s+x)\,dtds^{k-1} \\ = \ & \int_0^\infty\int_{s+x}^\infty P(X_i>t,S>t)\,dtds^{k-1} \\ = \ & (\lambda_1\cdots\lambda_d)\,\Gamma(k)\sum_{l\neq i}\frac{1}{\prod_{j\neq l}(\lambda_j-\lambda_l)}\left(\frac{1}{\lambda_l}-\frac{1}{\lambda_i}\right)\frac{e^{-\lambda_i x}}{\lambda_i^k}. \end{aligned}$$

Therefore, by manipulating algebraically, we obtain

$$\begin{aligned}
& E\left[X_i (S-x)_+^{k-1}\right] \\
= \quad & (\lambda_1 \cdots \lambda_d) \Gamma(k) \sum_{l \neq i} \frac{1}{\prod_{j \neq l,i} (\lambda_j - \lambda_l)} \left(\frac{\lambda_i^k e^{-\lambda_l x} - \lambda_l^k e^{-\lambda_i x}}{\lambda_i^k \lambda_l^k (\lambda_i - \lambda_l)^2} - \frac{(k + x\lambda_i) e^{-\lambda_i x}}{\lambda_i^{k+1} (\lambda_i - \lambda_l)} \right) \\
= \quad & (\lambda_1 \cdots \lambda_d) \Gamma(k) (-1)^{d-1} h_i (x, [\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_d]).
\end{aligned}$$

Example 2.4.3 *Let $(X_1, \ldots, X_d)$ denote a d-dimensional random vector following a multivariate Pareto distribution with parameters $\sigma_1, \ldots, \sigma_d > 0$ and $\alpha > k \geq 1$. Namely,*

$$P(X_1 > y_1, \ldots, X_d > y_d) = \left(1 + \sum_{i=1}^{d} \frac{y_i}{\sigma_i}\right)^{-\alpha},$$

where $y_i > 0$ for every i. For $\sigma \neq \sigma_i$, define

$$\begin{aligned}
g(x, \sigma) &= (\alpha - k) \sigma^{d+k-2} \left(1 + \frac{x}{\sigma}\right)^{-(\alpha - k + 1)}, \qquad x > 0, \\
g_i(x, \sigma) &= -(k\sigma_i + x\alpha) \sigma_i^k \frac{\sigma^{d-2}}{(\sigma_i - \sigma)} \left(1 + \frac{x}{\sigma_i}\right)^{-(\alpha - k + 1)} \\
& \quad - \sigma_i \frac{\sigma^{k+d-1}}{(\sigma_i - \sigma)^2} \left(1 + \frac{x}{\sigma}\right)^{-(\alpha - k)} + \sigma_i^{k+1} \frac{\sigma^{d-1}}{(\sigma_i - \sigma)^2} \left(1 + \frac{x}{\sigma_i}\right)^{-(\alpha - k)}, \qquad x > 0.
\end{aligned}$$

Therefore, it follows that

$$\Lambda_q (X_i, S) = \frac{g_i (x^*, [\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_d])}{g(x^*, [\sigma_1, \ldots, \sigma_d])}, \tag{2.22}$$

where $x = x^*$ is the unique solution to the equation

$$(\alpha - k)^{k-1} k^{1-k} \mathrm{B}(\alpha - k, k) (-1)^{d-1} g(x, [\sigma_1, \ldots, \sigma_d]) = 1 - q. \tag{2.23}$$

Clearly, to verify relations (2.22) and (2.23), it suffices to prove that

$$\begin{aligned}
& E\left[(S-x)_+^{k-1}\right] \\
= \quad & \frac{\Gamma(k)}{\Gamma(\alpha)} \Gamma(\alpha - k + 1) (-1)^{d-1} g(x, [\sigma_1, \ldots, \sigma_d]), \qquad (2.24) \\
& E\left[X_i (S-x)_+^{k-1}\right] \\
= \quad & \mathrm{B}(\alpha - k, k) (-1)^{d-1} g_i (x, [\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_d]), \qquad (2.25)
\end{aligned}$$

for every $x>0$. According to Arnold (2015), the random vector $(X_1,\ldots,X_d)$ can be represented as a proportional mixture of independent and exponentially distributed random variables. Specifically,

$$(X_1,\ldots,X_d) \stackrel{d}{=} Z^{-1}(\sigma_1 Y_1,\ldots,\sigma_d Y_d),$$

where $Z \stackrel{d}{=}$Gamma$(\alpha,1)$, and $Y_1,\ \ldots,\ Y_d$ are independent standard exponential variables. Consequently, the random variables $X_1,\ \ldots,\ X_d$ given Z are independent and exponentially distributed with rates $\lambda_1,\ \ldots,\ \lambda_d$, where $\lambda_i = \frac{Z}{\sigma_i}$. Let f_Z denote the density of the gamma random variable Z. Notice that

$$\begin{aligned} E\left[(S-x)_+^{k-1}\right] &= \int_0^\infty P(S>s+x)\,ds^{k-1} \\ &= \int_0^\infty \left(\int_0^\infty P(S>s+x|\,Z=z)\,f_Z(z)\,dz\right) ds^{k-1}, \end{aligned} \tag{2.26}$$

where $P(S>s|\,Z=z)$ represents the tail distribution of a sum of independent exponential random variables with parameters $\lambda_1,\ \ldots,\ \lambda_d$. By Fubini's theorem and the substitution $\lambda_i = \frac{z}{\sigma_i}$, we derive

$$\begin{aligned} & E\left[(S-x)_+^{k-1}\right] \\ = & \int_0^\infty \left(\int_0^\infty P(S>s+x|\,Z=z)\,ds^{k-1}\right) f_Z(z)\,dz \\ = & \int_0^\infty (\lambda_1\cdots\lambda_d)\,\Gamma(k)\sum_{i=1}^d \frac{e^{-\lambda_i x}}{\lambda_i^k \prod_{j\neq i}(\lambda_j-\lambda_i)}\frac{z^{\alpha-1}e^{-z}}{\Gamma(\alpha)}dz \\ = & \frac{\Gamma(k)}{\Gamma(\alpha)}\Gamma(\alpha-k+1)(-1)^{d-1}\sum_{i=1}^d \frac{\sigma_i^{d+k-2}}{\prod_{j\neq i}(\sigma_j-\sigma_i)}\left(1+\frac{x}{\sigma_i}\right)^{-(\alpha-k+1)}. \end{aligned} \tag{2.27}$$

By plugging relations (2.26) and (2.27), we obtain relation (2.24). By using the same argument of conditioning over the values of the random variable Z, relation (2.25) follows.

Remark 2.4.1 *Considering Example 2.4.2 and Example 2.4.3, we point out that the Orlicz quantile x satisfying equation (2.8) could be negative, see Example 12 of Bellini and Rosazza Gianin (2012). However, Lemma 2.2 of Tang and Yang (2012) establishes that x is positive for large enough q.*

In the sequel, we discuss an example referring to Theorem 2.3.2.

Example 2.4.4 *Let $(X_1,\ldots,X_d)$ denote a d-dimensional random vector following a multivariate Pareto distribution with parameters $\sigma_1,\ \ldots,\ \sigma_d > 0$ and $\alpha > k$; see Example 2.4.3. It is known that $(X_1,\ldots,X_d)$ is regularly varying with parameter α. Let ν be such that relation (2.13) holds with $\bar{F}(x) = \left(1+\frac{x}{\sigma_1}\right)^{-\alpha}$. By employing Theorem 2.3.2, we have*

$$\Lambda_q^a(X_i,S) = C_i F^{\leftarrow}(q),$$

where

$$\begin{aligned}
&(-1)^{d-1} C_i \\
&= \left(\frac{\left((-1)^{d-1} g^a([\sigma_1,\ldots,\sigma_d])\right)^{1-\alpha} \mathrm{B}(\alpha-k,k)}{(\alpha-k)^{\alpha-k} k^{k-1}}\right)^{\frac{1}{\alpha}} g_i^a([\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_d]),
\end{aligned}$$

and

$$\begin{aligned}
g^a(\sigma) &= \sigma^{\alpha+d-1}\sigma_1^{-\alpha}, \\
g_i^a(\sigma) &= -\alpha\frac{\sigma^{d-2}}{(\sigma_i-\sigma)}\sigma_i^{\alpha+1}\sigma_1^{-\alpha} - \frac{\sigma^{\alpha+d-1}}{(\sigma_i-\sigma)^2}\sigma_i\sigma_1^{-\alpha} + \frac{\sigma^{d-1}}{(\sigma_i-\sigma)^2}\sigma_i^{\alpha+1}\sigma_1^{-\alpha}.
\end{aligned}$$

Now, we provide further details concerning Example 2.4.4. Conforming to Lemma 2.3.2 and its proof, it follows that

$$\begin{aligned}
& I_i + c_i \mathrm{B}(\alpha-k,k) \\
&= \left(\iint_{(s>0,0<r\le s+1)} + \iint_{(s>0,r>s+1)}\right)\nu(\boldsymbol{z}: z_i > r, s_d > s+1)\,dr ds^{k-1} \\
&= \int_0^\infty\int_0^\infty \nu(\boldsymbol{z}: z_i > r, s_d > s+1)\,dr ds^{k-1},
\end{aligned}$$

and

$$\int_0^\infty\int_0^\infty \nu(\boldsymbol{z}: z_i > r, s_d > s+1)\,dr ds^{k-1} = \lim_{x\to\infty}\frac{E\left[X_i(S-x)_+^{k-1}\right]}{x^k\overline{F}(x)}.$$

Moreover, by letting g_i be the function defined in Example 2.4.3, relation (2.25) implies that

$$\begin{aligned}
\lim_{x\to\infty}\frac{E\left[X_i(S-x)_+^{k-1}\right]}{x^k\overline{F}(x)} &= \lim_{x\to\infty}\frac{\mathrm{B}(\alpha-k,k)(-1)^{d-1} g_i(x,[\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_d])}{x^k\overline{F}(x)} \\
&= \mathrm{B}(\alpha-k,k)(-1)^{d-1} g_i^a([\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_d]).
\end{aligned}$$

Summing up, it holds that

$$I_i + c_i \mathrm{B}(\alpha - k, k) = \mathrm{B}(\alpha - k, k)(-1)^{d-1} g_i^a([\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_d]).$$

By proceeding in a similar manner, we derive

$$\nu(\boldsymbol{z} : s_d > 1) = (-1)^{d-1} g^a([\sigma_1, \ldots, \sigma_d]).$$

2.4.2 Numerical studies

To numerically illustrate our results, we refer back to Example 2.4.3 with parameters $d = 8$ and $\sigma_i = 50,000(1+i)$ for $i \in \{1, \ldots, d\}$.

First, we investigate the performance of the asymptotic formula (2.14). By Example 2.4.4, we notice that the asymptotic contribution $\Lambda_q^a(X_i, S)$ is explicit. In addition, to compute the theoretical contribution $\Lambda_q(X_i, S)$, we need to numerically solve equation (2.23). To that end, we may employ the function `uniroot` of R; we refer the reader to the monograph of Kaas et al. (2008) for applications of R to various problems in actuarial science. With this in mind, we are prepared to numerically compare $\Lambda_q(X_i, S)$ and $\Lambda_q^a(X_i, S)$. Because of the similarity of behaviors along sub-portfolios, we only comment the comparison for the first sub-portfolio ($i = 1$). For $k = 1.1$, on the left of Figure 2.1 and Figure 2.2, we compare $\Lambda_q(X_i, S)$ and $\Lambda_q^a(X_i, S)$ by varying α. Visually, it is apparent that $\Lambda_q^a(X_i, S) \to \Lambda_q(X_i, S)$ as $q \uparrow 1$, however, the rate of convergence decreases as α increases. For $\alpha = 1.6$, on the right of Figure 2.1 and Figure 2.2, we constrast $\Lambda_q(X_i, S)$ and $\Lambda_q^a(X_i, S)$ by varying k. Again, we graphically confirm the convergence of the asymptotic allocation $\Lambda_q^a(X_i, S)$ to the theoretical allocation $\Lambda_q(X_i, S)$. We also point out that the rate of convergence increases as k decreases.

Finally, for $q = 0.95$, $\alpha = 3$, and $k = 2$, Figure 2.3 shows a comparison via simulation between the capital allocation Λ_q and the consistent estimator $\Lambda_q^{(n)}$ provided by relations (2.11) and (2.12), considering different values of the sample size n.

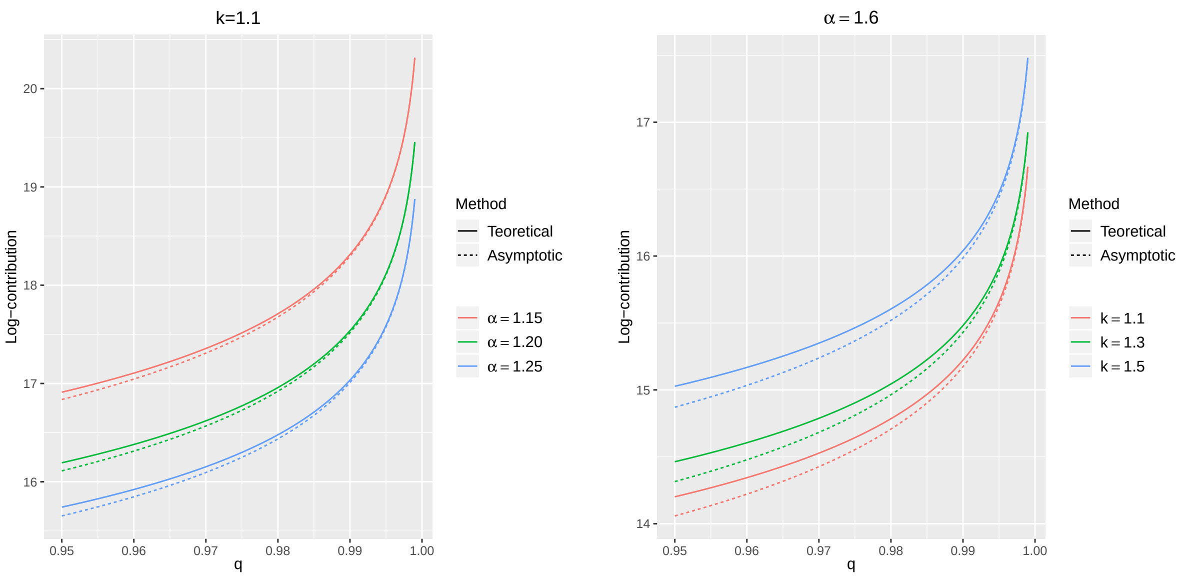

Fig. 2.1 Logarithms of the theoretical value Λ_q and the asymptotic provided by Theorem 2.3.2 Λ_q^a for different values of the parameter q

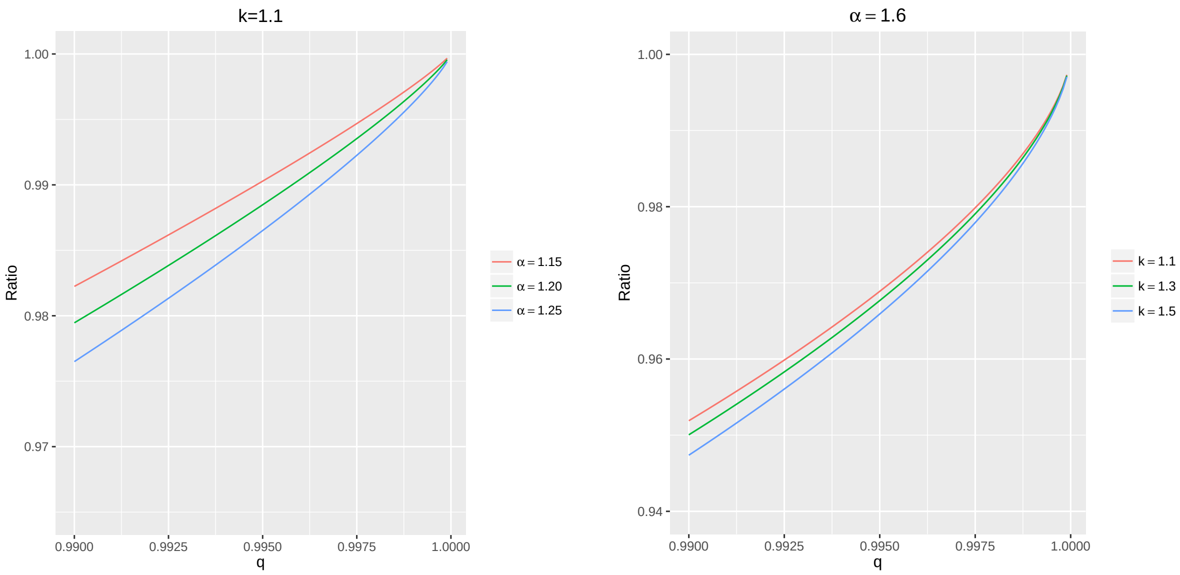

Fig. 2.2 Ratio of the approximation Λ_q^a derived from Theorem 2.3.2 to the theoretical value Λ_q for different values of the parameter q

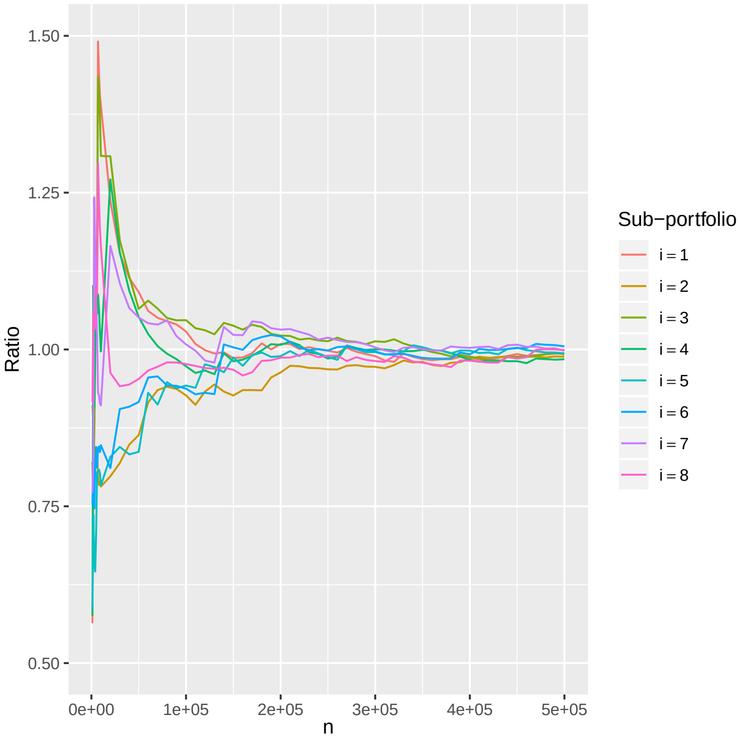

Fig. 2.3 Ratio of $\Lambda_q^{(n)}$ to Λ_q for different values of the simulation sample size n

Chapter 3

Indifference pricing of mortality-linked securities

3.1 Notation and abbreviations

In the following table, we provide a summary of notation and abbreviations used in this chapter:

$\mathbf{1}$	the d-dimensional vector with all components being 1
1_A	the indicator of an event A
BSDE	backward stochastic differential equations
CIR	Cox–Ingressol–Ross
d	a positive integer
$\mathrm{dist}_{\boldsymbol{K}}(\boldsymbol{k})$	$\inf_{\boldsymbol{l}\in\boldsymbol{K}} \lVert\boldsymbol{k}-\boldsymbol{l}\rVert$, where $\boldsymbol{K}$ is a closed subset of $\mathbb{R}^d$ and $\boldsymbol{k}\in\mathbb{R}^d$
E	mathematical expectation
$E(X\vert\mathcal{F})$	conditional expectation of X given $\mathcal{F}$
$\mathcal{F}_\infty$	$\vee_{t\geq 0}\mathcal{F}_t$, where $\{\mathcal{F}_t\}$ is a filtration
$\lvert I\rvert$	the cardinality of a finite set I, namely the number of elements in it
I_n	the set $\{1,\ldots,n\}$
$L^\infty(\mathcal{F})$	the space of $\mathcal{F}$-measurable and essentially bounded random variables
MLS	mortality-linked securities
n	a positive integer
$\lVert(x_1,\ldots,x_d)\rVert$	the Euclidean norm, $\left(\sum_{i=1}^d x_i^2\right)^{\frac{1}{2}}$ for $(x_1,\ldots,x_d)\in\mathbb{R}^d$
P	probability measure
$P(A\vert\mathcal{F})$	conditional probability of A given $\mathcal{F}$
$\Pi_{\boldsymbol{K}}(\boldsymbol{k})$	$\{\boldsymbol{l}\in\boldsymbol{K}:\ \lVert\boldsymbol{k}-\boldsymbol{l}\rVert=\mathrm{dist}_K(\boldsymbol{k})\}$, where $\boldsymbol{K}\subset\mathbb{R}^d$ is closed and $\boldsymbol{k}\in\mathbb{R}^d$

$\mathbb{R}$	$(-\infty, \infty)$
RCLL	right continuous with left limits
$\vee_{\theta\in\Theta}\mathcal{F}_\theta$	the smallest sigma-algebra containing every $\mathcal{F}_\theta$ for $\theta \in \Theta$
$x \wedge y$	$\min\{x, y\}$

3.2 Introduction

Catastrophic mortality events, such as pandemics, wreak havoc on the economy and society on a large scale despite their low likelihood of happening. For instance, the 1918 influenza pandemic took around 50 million lives. More recently, during the 2009 H1N1 influenza pandemic between 151,700 and 575,400 people perished worldwide. Concerned with such fatalities, life insurers/reinsurers are continually seeking solutions to mitigate catastrophic risk.

The uncertainty in future death rates can be divided into two components: the unsystematic mortality risk, which can be hedged by pooling, and the systematic mortality risk due to the uncertain development of future mortality rates (for instance due to a pandemic), which is inherently undiversifiable. As an alternative to traditional reinsurance, the interplay between the insurance industry and the capital market provides a vehicle for mitigating systematic mortality risk, namely, through the financial securitization. Securitization allows off-loading undiversifiable risk from the insurer and transferring it to the capital market. In particular, the mortality-linked securities (MLS) have emerged as a way to manage systematic mortality risk. Generally speaking, MLS are financial products whose payoffs depend on mortality risk. For example, mortality catastrophe bonds have been employed by some insurance and reinsurance companies to transfer extreme mortality risks to the capital market (see, e.g., Bauer and Kramer (2016)).

As the MLS market expands, pricing MLS becomes increasingly important and is attracting much research attention; see for instance Cairns et al. (2006, 2008) and Blake et al. (2008). To value an MLS, one essentially needs to specify a mortality model and to choose a pricing methodology. Because the mortality events are generally not hedgeable, the MLS market is incomplete. In consequence, it is not possible to set a unique price for an MLS by relying only on arbitrage considerations. Therefore, the election of a pricing methodology is imperative.

Various approaches have been developed in the literature to address the incompleteness of the MLS market when pricing. The exploited approaches include: arbitrage pricing theory (see Milevsky and Promislow (2001), Blanchet-Scalliet et al. (2005), and Bayraktar et al. (2009)), probability transform (see Lin and Cox (2005, 2008), and Chen and Cox (2009)), risk-minimizing strategies (see Dahl and Møller (2006), Dahl et al. (2008), and Biagini et al. (2017)), and indifference pricing (see Dahl and Møller (2006), Hainaut and Devolder (2008), and Ludkovski and Young (2008)). We refer the reader to Bauer et al. (2010) for a comparative study.

In particular, the indifference pricing approach, an extension of the notion of certainty equivalent to more general and possibly dynamic settings, allows incorporating risk aversion. Consider an agent facing a contingent claim with payoff C at future time $T > 0$. Regardless of the completeness of the market, it is possible to define the utility indifference price p as the amount at which the agent is indifferent, in the sense that his expected utility under optimal trading is unchanged, between receiving the price p now but settling the payoff C at T and neither receiving now nor paying at T something. Since its first introduction by Hodges and Neuberger (1989), the indifference pricing approach has been widely studied in the literature of pricing in incomplete markets; see for instance Davis et al. (1993), Rouge and El Karoui (2000), Hu et al. (2005), and Carmona (2009).

When employing the approach of utility indifference pricing, the resulting price has many desirable properties. For example, the price lies in the interval of prices consistent with no-arbitrage, and the price is non-linear in the number of units of the contingent claim. Moreover, the construction of an optimal hedge is implicit in the calculation of the utility indifference price. Interestingly, the value function underlying the utility indifference price has been linked to backward stochastic differential equations (BSDE). For instance, working on a Brownian filtration, Hu et al. (2005) characterize the indifference price process as the unique solution of a BSDE and describe the corresponding optimal wealth and strategy processes. Other related works include Rouge and El Karoui (2000), Frittelli (2000a, 2000b), Delbaen et al. (2002), Mania and Schweizer (2005), Becherer (2006), Jiao et al. (2013), and Kharroubi and Lim (2014). Consequently, its appealing economic justification and the power of BSDE techniques advocate us to choose the utility indifference pricing as a sensible methodology to price MLS.

In practice, the mortality risk emanates from a portfolio of life insurance policyholders. Therefore, it is pivotal to state a model for the policyholders' remaining lifetimes. Nowadays, there is a consensus that the mortality rates, which define the remaining lifetimes, are stochastic. The recent literature on this topic is abundant and essentially inspired by credit risk theory; see for instance Milevsky and Promislow (2001), Dahl (2004), Biffis (2005), Schrager (2006), Cairns et al. (2006), Wills and Sherris (2010), Blackburn and Sherris (2013), and Biagini et al. (2017). In parallel with credit risk modeling, one often assumes that the remaining lifetimes are independent, given the information on the mortality rates. Moreover, the great majority of the aforementioned works model mortality rates as affine diffusion processes because of its analytical and computational tractability. For example, Biagini et al. (2017) model mortality rates as affine diffusion processes based on Gaussian random fields, which allow capturing the cross-generational dependency structure of the portfolio.

In this chapter, we consider an insurance market in which financial and mortality risks coexist. The financial risk arises from a financial market with a finite number of securities, and the mortality risk emanates from a portfolio of life insurance policy-holders. Aiming to contemplate systematic and unsystematic mortality risks and to keep tractability, we assume that the remaining lifetimes are conditionally independent doubly stochastic random times. It is worthy of mentioning that we do not require the mortality rates to be affine. By employing the utility indifference pricing approach with an exponential utility function, we tackle the problem of valuing an MLS, which offers a payoff C at future time $T > 0$. In contrast to the related works of Moore and Young (2003), Jaimungal and Young (2005), Dahl and Møller (2006), Hainaut and Devolder (2008), and Ludkovski and Young (2008), we consider heterogeneous mortality rates allowing a portfolio composed of different age cohorts.

First, we consider the pricing of a claim formed by a linear combination of pure endowments. In this particular case, by using the independence between the financial and actuarial worlds, we provide a quasi-explicit formula for the utility indifference price. Here, our mathematical contribution consists of extending some results from credit risk theory usually referred to in the case of one random time to multiple random times. Moreover, we present some examples which admit a significant simplification for the utility indifference price by modeling the marginal mortality-rates as affine diffusion processes. Affine mortality rate models have been successfully used; see for instance Biffis (2005), Schrager (2006), Wills and Sherris (2010), and Blackburn and Sherris (2013).

Then, following Hu et al. (2005), we employ techniques of BSDE to tackle the utility indifference pricing problem by assuming a general bounded payoff C. Specifically, we characterize the optimal investment strategy and the optimal value function for the optimization problem with the solution to a non-linear BSDE with non-Lipschitz generator. Mathematically speaking, the novelty herein consists of establishing existence and uniqueness properties for the BSDE characterizing the optimization problem, which is essentially different from that one in Hu et al. (2005) and Becherer (2006) because the generator contains exponential and quadratic terms for the portfolio process.

3.3 Model

Hereafter, we work on a probability space $(\Omega, \mathcal{G}, P)$, which is rich enough to accommodate two independent Brownian motions $\boldsymbol{W^1}$ and $\boldsymbol{W^2}$ with dimensions d and m, respectively. All economic activities are assumed to take place on a time horizon $[0,T]$, where $T > 0$. For $i \in \{1,2\}$, let $\left\{\mathcal{F}_t^i\right\}$ be the P-augmentation of the filtration generated by $\boldsymbol{W}^i$. From now on, the sigma-algebras $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$ represent, respectively, the financial information and the systematic mortality risk information available up to time t. By construction, the filtrations $\left\{\mathcal{F}_t^1\right\}$ and $\left\{\mathcal{F}_t^2\right\}$ satisfy the usual conditions (namely, each of them is right continuous and contains all P-null subsets of $\mathcal{G}$) and so does $\{\mathcal{F}_t\} = \{\mathcal{F}_t^1 \vee \mathcal{F}_t^2\}$ (see Lemma 3.4.1).

3.3.1 Financial market

We consider a financial market consisting of one risk-free asset and d risky assets. As usual, we assume that the *price of the money market process*, $\tilde{S}_0(\cdot)$, evolves according to the equation

$$d\tilde{S}_0(t) = \tilde{S}_0(t)\, r(t)\, dt, \quad t \in [0,T],$$

where $r(\cdot)$ is a *risk-free rate process,* assumed to be $\{\mathcal{F}_t^1\}$-progressively measurable and bounded from below. Moreover, we introduce d *stocks* with prices per share at time t given by $\tilde{S}_1(t), \ldots, \tilde{S}_d(t)$ with $\tilde{S}_1(0), \ldots, \tilde{S}_d(0) > 0$. The processes $\tilde{S}_1(\cdot), \ldots, \tilde{S}_d(\cdot)$ satisfy the system of stochastic differential equations

$$d\tilde{S}_i(t) = \tilde{S}_i(t)\left[\tilde{\mu}_i(t)dt + \sum_{j=1}^{d} \sigma_{i,j}(t) dW_j^1(t)\right], \quad t \in [0,T],\ i \in \{1, \ldots, d\},$$

where $\tilde{\boldsymbol{\mu}}(\cdot) = (\tilde{\mu}_1(\cdot), \ldots, \tilde{\mu}_d(\cdot))^{\intercal}$ and $\boldsymbol{\sigma}(\cdot) = \{\sigma_{i,j}(\cdot)\}_{1\le i,j\le d}$ are $\left\{\mathcal{F}_t^1\right\}$-progressively measurable processes. Assume that $\boldsymbol{\mu}(\cdot) = \tilde{\boldsymbol{\mu}}(\cdot) - r(\cdot)\mathbf{1}$ and $\boldsymbol{\sigma}(\cdot)$ are uniformly bounded, and there exists $\varepsilon > 0$ such that

$$\boldsymbol{\xi}^{\intercal}\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^{\intercal}(t)\boldsymbol{\xi} \ge \varepsilon \|\boldsymbol{\xi}\|^2, \quad \boldsymbol{\xi} \in \mathbb{R}^d,\ t \in [0,T]. \tag{3.1}$$

Due to the condition (3.1), $\boldsymbol{\sigma}(t)$ has an inverse almost everywhere on $[0,T] \times \Omega$, and the process $\boldsymbol{\theta}(\cdot) = (\boldsymbol{\sigma}(\cdot))^{-1}\boldsymbol{\mu}(\cdot)$ is uniformly bounded. According to Theorem 1.4.2 of

Karatzas and Shreve (1998), the financial market is arbitrage-free. In this set-up, the process $\boldsymbol{\theta}(\cdot)$ receives the name of the *market price of risk*. Furthermore, the invertibility of the matrix-valued process $\boldsymbol{\sigma}(\cdot)$ implies that the financial market is complete; see Section 1.6 of Karatzas and Shreve (1998) for further details.

3.3.2 Mortality rates

We consider a portfolio of n life insurance policyholders whose remaining lifetimes are modeled by random variables $\tau_1, \ldots, \tau_n$.

Let $\gamma_1(\cdot), \ldots, \gamma_n(\cdot)$ be non-negative and $\{\mathcal{F}_t^2\}$-progressively measurable processes, such that $\int_0^{\cdot} \gamma_i(s)ds$ is strictly increasing and finite for each $i \in I_n$. Thus, the processes $\gamma_1(\cdot), \ldots, \gamma_n(\cdot)$ are driven by the Brownian motion $\boldsymbol{W^2}$. In addition, let $U_1, \ldots, U_n$ be independent and uniformly distributed random variables, which are independent of $\mathcal{F}_\infty$. In this chapter, we assume that the remaining lifetimes $\tau_1, \ldots, \tau_n$ are *conditionally independent doubly stochastic* random times with marginal *mortality rate processes* $\gamma_1(\cdot), \ldots, \gamma_n(\cdot)$, *i.e.*,

$$\tau_i = \inf\left\{t > 0 : e^{-\int_0^t \gamma_i(s)ds} \leq U_i\right\}, \quad i \in I_n = \{1, \ldots, n\}. \tag{3.2}$$

Under this construction, it is apparent that

$$\begin{aligned} P(\tau_i = 0) &= 0, \quad i \in I_n, \\ P(\tau_i > t) &> 0, \quad t \in [0, T], \ i \in I_n, \\ P(\tau_i = \tau_j) &= 0, \quad i, j \in I_n \text{ and } i \neq j. \end{aligned}$$

According to relation (3.2), it holds that

$$P\left(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n | \mathcal{F}_\infty^2\right) = \prod_{i \in I_n} P\left(\tau_i \leq t_i | \mathcal{F}_\infty^2\right).$$

Namely, we assume that $\tau_1, \ldots, \tau_n$ are conditionally independent, given the information $\mathcal{F}_\infty^2$. Intuitively, if the evolution of all mortality risk factors corresponding to the filtration $\left\{\mathcal{F}_t^2\right\}$ is known, then the random times $\tau_1, \ldots, \tau_n$ are independent. By the mildness of the assumptions concerning the processes $\gamma_1(\cdot), \ldots, \gamma_n(\cdot)$, our model allows the portfolio to be composed of individuals from different age cohorts, whose mortality rates may be affected by mortality risk factors differently. The idea of constructing random times through mortality rate processes has been extensively used in credit risk modeling; see for instance Lando (2009). Besides, authors such as Dalh (2004) and Biffis (2005) have exploited the tractability of this construction to price some MLS.

For $i \in I_n$, define $H_i(t) = 1_{(\tau_i \leq t)}$ for all $t \in [0,T]$, and let $\{\mathcal{H}_t^i\}$ denote the filtration generated by $H_i(\cdot)$.

3.3.3 Combined model

We situate in an extended market where derivatives, depending on both financial and mortality risks, are traded. Concretely, we assume that the information available to investors at time $t \in [0,T]$ is given by the P-augmentation of $\left\{\mathcal{F}_t^1 \vee \mathcal{F}_t^2 \vee \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^n\right\}$, which is denoted by $\{\mathcal{G}_t\}$. In our model, it is worthy to point out that any $\left\{\mathcal{F}_t^1\right\}$-martingale is a $\{\mathcal{G}_t\}$-martingale. This is a necessary condition to guarantee that there are no arbitrage possibilities in the financial market by using $\{\mathcal{G}_t\}$-adapted strategies; see for instance Elliott et al. (2000).

Remark 3.3.1 *For $i \in I_n$, let $\left\{\mathcal{G}_t^i\right\}$ denote the P-agumentation of $\{\mathcal{F}_t \vee \mathcal{H}_t^i\}$. The process $\gamma_i(\cdot)$ is called a $\left\{\mathcal{G}_t^i\right\}$-martingale intensity process for the random time τ_i because*

$$\tilde{H}_i(\cdot) = H_i(\cdot) - \int_0^{\cdot \wedge \tau_i} \gamma_i(s) ds$$

is a $\left\{\mathcal{G}_t^i\right\}$-martingale; see for instance Section 2.5 of Jeanblanc and Rutkowski (2002). Moreover, due to our construction of the random times $\tau_1, \ldots, \tau_n$, it is easy to prove that $\tilde{H}_i(\cdot)$ is also a $\{\mathcal{G}_t\}$-martingale.

3.4 Utility indifference pricing

Suppose an individual facing some liability $\tilde{C}$, whose discounted value is a $\mathcal{G}_T$-measurable and bounded random variable denoted by C. For example, the discounted value of the liability may have the form

$$C = \sum_{i=1}^{n} C_i 1_{(\tau_i > T)}, \tag{3.3}$$

where $C_1, \ldots, C_n$ denote $\mathcal{F}_T^1$-measurable bounded random variables. In this case, the insurer pays to the i-th policyholder surviving up to time T an amount $\tilde{C}_i$, equivalent to C_i shares of the cash account.

In this setting, we aim to study the pricing of a contingent claim with discounted payoff C by considering the financial market in discounted terms. To carry out the pricing task, since the expanded market is incomplete because mortality events are generally not hedgeable, we employ the utility indifference approach.

Let $t \in [0,T]$ be fixed. Consider an agent, who, starting with a capital x at t, invests an amount $\tilde{\pi}_i(u)$ at any time $u \in [t,T]$ in the i-th risky asset, $i \in \{1,\ldots,d\}$. In particular, we assume that the *strategy process* $\tilde{\boldsymbol{\pi}}(\cdot) = (\tilde{\pi}_1(\cdot),\ldots,\tilde{\pi}_d(\cdot))^\intercal$ is a $\{\mathcal{G}_t\}$-progressively measurable and $\mathbb{R}^d$-valued process, such that $\int_t^T \|\tilde{\boldsymbol{\pi}}^\intercal(s)\boldsymbol{\sigma}(s)\|^2 ds < \infty$. The *wealth process* $\tilde{X}_t^{x,\tilde{\boldsymbol{\pi}}}(\cdot)$ accumulating from t, associated with x and $\tilde{\boldsymbol{\pi}}(\cdot)$, satisfies the following stochastic differential equation

$$\begin{aligned} & d\tilde{X}_t^{x,\tilde{\boldsymbol{\pi}}}(u) \\ = \; & \tilde{X}_t^{x,\tilde{\boldsymbol{\pi}}}(u)r(u)du + \tilde{\boldsymbol{\pi}}^\intercal(u)\Big((\boldsymbol{\mu}(u) - r(u)\mathbf{1})\,du + \boldsymbol{\sigma}(u)\,d\boldsymbol{W}^{\mathbf{1}}(u)\Big), \quad t \le u \le T, \end{aligned}$$

whose solution is given by

$$\begin{aligned} & D_0(t,u)\tilde{X}_t^{x,\tilde{\boldsymbol{\pi}}}(u) \\ = \; & x + \int_t^u D_0(t,s)\tilde{\boldsymbol{\pi}}^\intercal(s)\Big((\tilde{\boldsymbol{\mu}}(s) - r(s)\mathbf{1})\,ds + \boldsymbol{\sigma}(s)\,d\boldsymbol{W}^{\mathbf{1}}(s)\Big), \quad t \le u \le T, \end{aligned}$$

where

$$D_0(t,u) = \left(\frac{\tilde{S}_0(u)}{\tilde{S}_0(t)}\right)^{-1}, \quad t \le u \le T.$$

We now introduce our financial market in discounted terms. It is easy to see that the discounted asset prices $S_1(\cdot) = \frac{\tilde{S}_1(\cdot)}{\tilde{S}_0(\cdot)}$, $\ldots$, $S_d(\cdot) = \frac{\tilde{S}_d(\cdot)}{\tilde{S}_0(\cdot)}$ satisfy

$$dS_i(t) = S_i(t)\left[\mu_i(t)dt + \sum_{j=1}^{d}\sigma_{i,j}(t)dW_j^1(t)\right], \quad t \in [0,T],\ i \in \{1,\ldots,d\},$$

where $\mu_i(\cdot)$ denotes the i-th component of the process $\boldsymbol{\mu}(\cdot)$. In terms of the discounted asset prices $S_1(\cdot)$, $\ldots$, $S_d(\cdot)$, the wealth process can be rewritten as

$$\begin{aligned} X_t^{x,\boldsymbol{\pi}}(u) &= x + \int_t^u \boldsymbol{\pi}^\intercal(s)\Big(\boldsymbol{\mu}(s)\,ds + \boldsymbol{\sigma}(s)\,d\boldsymbol{W}^{\mathbf{1}}(s)\Big) \\ &= D_0(t,u)\tilde{X}_t^{x,\tilde{\boldsymbol{\pi}}}(u), \quad t \le u \le T, \end{aligned} \tag{3.4}$$

where $\boldsymbol{\pi}(\cdot) = D_0(t,\cdot)\tilde{\boldsymbol{\pi}}(\cdot)$. To lighten the notation, we suppress from $X_t^{x,\boldsymbol{\pi}}$ the indexes x and t when they equal 0.

Hereafter, we shall work with the financial market in discounted terms.

From now on, we assume that the agent has an exponential utility function. Namely, there exists $\alpha > 0$ such that

$$U(y) = -e^{-\alpha y}, \quad y \in \mathbb{R}.$$

We now specify the class of admissible investment strategies, which we shall denote by $\mathcal{A}$.

Definition 3.4.1 *The set of admissible strategies $\mathcal{A}$ consists of all $\left\{\mathcal{F}_t^1\right\}$-progressively measurable processes $\boldsymbol{\pi}(\cdot)$ with values in a closed subset of $\mathbb{R}^d$, which satisfy*

$$E\left(\int_0^T \left\|\boldsymbol{\pi}^\intercal(s)\boldsymbol{\sigma}(s)\right\|^2 ds\right) < \infty, \tag{3.5}$$

as well as

$$\left\{-e^{-\alpha X^{\boldsymbol{\pi}}(\rho)} : \ \rho \text{ is an } \left\{\mathcal{F}_t^1\right\}\text{-stopping time on } [0,T]\right\} \tag{3.6}$$

is a uniformly integrable family.

By following Ludkovski and Young (2008), we suppose that the strategies in Definition 3.4.1 are adapted to $\left\{\mathcal{F}_t^1\right\}$. However, in Section 3.4.3, we extend the definition by requiring the admissible strategies to be adapted to $\{\mathcal{G}_t\}$.

Remark 3.4.1 *When the closed set in Definition 3.4.1 is compact, in our model, the integrability conditions (3.5) and (3.6) are redundant. See for instance Lemma 1 of Morlais (2009).*

The utility indifference price $p(t)$ is the price at which the agent is indifferent (in the sense that his expected utility under optimal trading is unchanged) between receiving the price $p(t)$ at t but paying the liability C at T and neither receiving at t nor paying at T something. Define

$$V(t,x,C) = \sup_{\boldsymbol{\pi}\in\mathcal{A}(t)} E\left(U\left(X_t^{x,\boldsymbol{\pi}}(T) - C\right)\Big|\mathcal{G}_t\right), \tag{3.7}$$

where $\mathcal{A}(t)$ denotes the set of all restrictions to $[t,T]$ of the strategies in $\mathcal{A}$. Thus, the utility indifference price $p(t)$ is the solution to

$$V(t,x,0) = V(t,x+p(t),C), \quad t \in [0,T]. \tag{3.8}$$

In our set-up, we notice that

$$V(t,x,C) = e^{-\alpha x} V(t,0,C), \quad t \in [0,T].$$

Hence, relation (3.8) reduces to

$$e^{-\alpha x} V(t,0,0) = e^{-\alpha(x+p(t))} V(t,0,C), \quad t \in [0,T],$$

whence the indifference price process $p(\cdot)$ can be written as

$$p(t) = \frac{1}{\alpha} \log \frac{V(t,0,C)}{V(t,0,0)}, \quad t \in [0,T]. \tag{3.9}$$

It is apparent that the indifference price process $p(\cdot)$ does not depend on the initial wealth x. Thus, when computing the value function (3.7), we may set $x = 0$ without loss of generality.

3.4.1 Linear combination of pure endowments

In this section, we consider the pricing of a claim whose payout is a linear combination of pure endowments. That is to say, we assume that

$$C = \sum_{i=1}^{n} c_i 1_{(\tau_i > T)}, \tag{3.10}$$

where $c_1, \ldots, c_n$ denote real numbers.

Now, we present an elementary result, which allows rewriting the value function (3.7) conveniently; see Proposition 1.12 of Aksamit and Jeanblanc (2017).

Lemma 3.4.1 *Let $\{\mathcal{F}_t\}$ and $\left\{\tilde{\mathcal{F}}_t\right\}$ be two right-continuous filtrations such that $\mathcal{F}_\infty$ and $\tilde{\mathcal{F}}_\infty$ are independent.*

(i) *For two random variables $X \in L^\infty(\mathcal{F}_\infty)$ and $\tilde{X} \in L^\infty(\tilde{\mathcal{F}}_\infty)$,*

$$E\left(X\tilde{X} \middle| \mathcal{F}_t \vee \tilde{\mathcal{F}}_t\right) = E(X|\mathcal{F}_t)\, E\left(\tilde{X} \middle| \tilde{\mathcal{F}}_t\right), \quad t \geq 0;$$

(ii) *The filtration $\left\{\mathcal{F}_t \vee \tilde{\mathcal{F}}_t\right\}$ is right-continuous.*

Let $\boldsymbol{\pi} \in \mathcal{A}$ be any admissible strategy. From Lemma 3.4.1, it holds that

$$\begin{aligned} E\left(U\left(X_t^{\boldsymbol{\pi}}(T)-C\right)\middle|\,\mathcal{G}_t\right) &= E\left(-e^{-\alpha\left(X_t^{\boldsymbol{\pi}}(T)-C\right)}\middle|\,\mathcal{G}_t\right) \\ &= E\left(-e^{-\alpha X_t^{\boldsymbol{\pi}}(T)}\middle|\,\mathcal{F}_t^1\right)E\left(e^{\alpha C}\middle|\,\mathcal{F}_t^2\vee\mathcal{H}_t\right), \quad t\geq 0. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} V(t,0,C) &= \sup_{\boldsymbol{\pi}\in\mathcal{A}(t)} E\left(U\left(X_t^{\boldsymbol{\pi}}(T)-C\right)\middle|\,\mathcal{G}_t\right) \\ &= E\left(e^{\alpha C}\middle|\,\mathcal{F}_t^2\vee\mathcal{H}_t\right)\sup_{\boldsymbol{\pi}\in\mathcal{A}(t)} E\left(-e^{-\alpha X_t^{\boldsymbol{\pi}}(T)}\middle|\,\mathcal{F}_t^1\right). \end{aligned} \tag{3.11}$$

Moreover, from Theorem 7 of Hu et al. (2005), the optimization problem

$$\sup_{\boldsymbol{\pi}\in\mathcal{A}(t)} E\left(-e^{-\alpha X_t^{\boldsymbol{\pi}}(T)}\middle|\,\mathcal{F}_t^1\right)$$

has a solution. In consequence, by using (3.9) and (3.11), the utility indifference price process $p(\cdot)$ is given by

$$p(t) = \frac{1}{\alpha}\log\left(E\left(e^{\alpha\sum_{i=1}^n c_i 1_{(\tau_i>T)}}\middle|\,\mathcal{F}_t^2\vee\mathcal{H}_t\right)\right). \tag{3.12}$$

That said, we present the main result of this section.

Proposition 3.4.1 *Let $p(\cdot)$ denote the utility indifference price process for the contingent claim whose payoff C is given by (3.10). Then*

$$p(t) = q\left(\sum_{j=1}^{n}\sum_{I\subset I_n,\ |I|=j}\left(\prod_{i\in I}(e^{\alpha c_i}-1)1_{(\tau_i>t)}E\left(e^{-\sum_{i\in I}\int_t^T\gamma_i(s)ds}\middle|\,\mathcal{F}_t^2\right)\right)\right), \tag{3.13}$$

where

$$q(x) = \frac{1}{\alpha}\log(1+x).$$

To prove Proposition 3.4.1, we first present two results frequently cited in the literature of credit risk in the case $n=1$; see for instance Jeanblanc and Rutkowski (2002).

For $\emptyset \neq I \subset I_n$, let $\left\{\mathcal{H}_t^I\right\} = \left\{\vee_{i\in I}\mathcal{H}_t^i\right\}$ and let $\mathcal{G}_t^I$ denote the P-agumentation of $\mathcal{F}_t \vee \mathcal{H}_t^I$.

Lemma 3.4.2 *For $\emptyset \neq I \subset I_n$, let $\{\mathcal{I}_t\}$ be a filtration such that $\mathcal{I}_t \subset \mathcal{G}_t^I$ for every $t \geq 0$. Then $\{\mathcal{I}_t\} \subset \left\{\mathcal{J}_t^I\right\}$, where*

$$\mathcal{J}_t^I = \{A \in \mathcal{G} : \ \exists B \in \mathcal{F}_t, \quad A \cap (\cap_{i \in I}(\tau_i > t)) = B \cap (\cap_{i \in I}(\tau_i > t))\}, \ t \geq 0.$$

Proof. It is apparent that $\mathcal{J}_t^I$ is a sub-sigma-algebra of $\mathcal{G}$. Then, to prove that $\{\mathcal{I}_t\} \subset \left\{\mathcal{J}_t^I\right\}$, it is sufficient to verify that $\{\mathcal{F}_t\} \subset \left\{\mathcal{J}_t^I\right\}$ and $\left\{\mathcal{H}_t^I\right\} \subset \left\{\mathcal{J}_t^I\right\}$. Namely, we need to show that: if either $A \in \mathcal{F}_t$ or $A = (\cap_{i \in J}(\tau_i \leq u_i)) \cap \left(\cap_{i \in I \setminus J}(\tau_i > t)\right)$ for some constants $u_i \leq t$ and $J \subset I$, then there exists $B \in \mathcal{F}_t$ such that $A \cap (\cap_{i \in I}(\tau_i > t)) = B \cap (\cap_{i \in I}(\tau_i > t))$. For the first case, we may take $B = A$, and for the later $B = \emptyset$ if $J \neq \emptyset$ or $B = \Omega$ otherwise. ∎

Lemma 3.4.3 *Let Y represent any random variable on $(\Omega, \mathcal{G}, P)$. Assume that $\mathcal{F}$ is any sub-sigma-algebra of $\mathcal{G}$, $\emptyset \neq I \subset I_n$, and $t_i \geq t$ for $i \in I$. Then*

$$E\left(1_{(\cap_{i \in I}(\tau_i > t_i))} Y \middle| \mathcal{F} \vee \mathcal{H}_t^I\right) = 1_{(\cap_{i \in I}\{\tau_i > t\})} \frac{E\left(1_{(\cap_{i \in I}(\tau_i > t_i))} Y \middle| \mathcal{F}\right)}{P\left(\cap_{i \in I}(\tau_i > t) \middle| \mathcal{F}\right)}. \tag{3.14}$$

Proof. To prove relation (3.14), we need to show that

$$E\left(1_{C_t^I} Y P\left(C_t^I \middle| \mathcal{F}\right) \middle| \mathcal{F} \vee \mathcal{H}_t^I\right) = E\left(1_{C_t^I} E\left(1_{C_t^I} Y \middle| \mathcal{F}\right) \middle| \mathcal{F} \vee \mathcal{H}_t^I\right),$$

where $C_t^I = \cap_{i \in I}(\tau_i > t_i)$. Namely, we have to verify that for every $A \in \mathcal{F} \vee \mathcal{H}_t^I$

$$\int_A 1_{C_t^I} Y P\left(C_t^I \middle| \mathcal{F}\right) dP = \int_A 1_{C_t^I} E\left(1_{C_t^I} Y \middle| \mathcal{F}\right) dP.$$

From Lemma 3.4.2, we can choose $B \in \mathcal{F}$ such that $A \cap (\cap_{i \in I}(\tau_i > t)) = B \cap (\cap_{i \in I}(\tau_i > t))$. Thus, we obtain

$$\begin{aligned}
\int_A 1_{C_t^I} Y P\left(C_t^I \middle| \mathcal{F}\right) dP &= \int_{A \cap C_t^I} Y P\left(C_t^I \middle| \mathcal{F}\right) dP \\
&= \int_B 1_{C_t^I} Y P\left(C_t^I \middle| \mathcal{F}\right) dP \\
&= \int_B E(1_{C_t^I} Y \big| \mathcal{F}) P\left(C_t^I \middle| \mathcal{F}\right) dP \\
&= \int_B E\left(1_{C_t^I} E(1_{C_t^I} Y \big| \mathcal{F}) \middle| \mathcal{F}\right) dP \\
&= \int_{B \cap C_t^I} E(1_{C_t^I} Y \big| \mathcal{F}) dP \\
&= \int_A 1_{C_t^I} E\left(1_{C_t^I} Y \middle| \mathcal{F}\right) dP.
\end{aligned}$$

So, we have proved the claim. ■

Proof of Proposition 3.4.1. Let $I \subset I_n$ and $t_i \geq t$ for every $i \in I$. From Lemma 3.4.3, it follows that

$$P\left(\cap_{i\in I}(\tau_i > t_i)|\,\mathcal{F}_t^2 \vee \mathcal{H}_t\right) = 1_{(\cap_{i\in I}(\tau_i > t))}\frac{P\left(\cap_{i\in I}(\tau_i > t_i)|\,\mathcal{F}_t^2 \vee \mathcal{H}_t^{I_n\setminus I}\right)}{P\left(\cap_{i\in I}(\tau_i > t)|\,\mathcal{F}_t^2 \vee \mathcal{H}_t^{I_n\setminus I}\right)}.$$

By recalling that the random times $\tau_1, \ldots, \tau_n$ are $\mathcal{F}_\infty^2$-conditionally independent and exponentially distributed with mortality rate processes $\gamma_1(\cdot), \ldots, \gamma_n(\cdot)$, it holds that

$$\begin{aligned} P\left(\cap_{i\in I}(\tau_i > t_i)|\,\mathcal{F}_t^2 \vee \mathcal{H}_t\right) &= 1_{(\cap_{i\in I}(\tau_i > t))}\frac{P\left(\cap_{i\in I}(\tau_i > t_i)|\,\mathcal{F}_t^2\right)}{P\left(\cap_{i\in I}(\tau_i > t)|\,\mathcal{F}_t^2\right)} \\ &= 1_{(\cap_{i\in I}(\tau_i > t))}E\left(e^{-\sum_{i\in I}\int_t^{t_i}\gamma_i(s)ds}\Big|\,\mathcal{F}_t^2\right). \end{aligned} \tag{3.15}$$

Now, we notice that

$$\begin{aligned} e^{\alpha\sum_{i=1}^n c_i 1_{(\tau_i > T)}} &= \prod_{i=1}^n \left(1 + (e^{\alpha c_i} - 1)1_{(\tau_i > T)}\right) \\ &= 1 + \sum_{j=1}^n \sum_{I\subset I_n,\ |I|=j}\left(\prod_{i\in I}(e^{\alpha c_i} - 1)1_{(\tau_i > T)}\right). \end{aligned}$$

Thus, by taking conditional expectation with respect to the filtration $\left\{\mathcal{F}_t^2 \vee \mathcal{H}_t\right\}$, we derive

$$\begin{aligned} & E\left(e^{\alpha\sum_{i=1}^n c_i 1_{(\tau_i > T)}}\Big|\,\mathcal{F}_t^2 \vee \mathcal{H}_t\right) \\ = \ & 1 + \sum_{j=1}^n \sum_{I\subset I_n,\ |I|=j}\left(\prod_{i\in I}(e^{\alpha c_i} - 1)E\left(\prod_{i\in I}1_{(\tau_i > T)}\Big|\,\mathcal{F}_t^2 \vee \mathcal{H}_t\right)\right) \\ = \ & 1 + \sum_{j=1}^n \sum_{I\subset I_n,\ |I|=j}\left(\prod_{i\in I}(e^{\alpha c_i} - 1)1_{(\tau_i > t)}E\left(e^{-\sum_{i\in I}\int_t^T\gamma_i(s)ds}\Big|\,\mathcal{F}_t^2\right)\right), \end{aligned}$$

where, the last step is due to relation (3.15). Finally, the intended result follows in light of relation (3.12). ■

In Proposition 3.4.1, we assume that the agent chooses investment strategies only based on the information arising from the financial market, and we suppose that financial and actuarial worlds are independent. Accordingly, the indifference price process $p(\cdot)$, corresponding to the contingent claim with payout C, only depends on the actuarial information.

To gain some insights into how the dependence among the mortality rate processes $\gamma_1(\cdot)$, $\ldots$, $\gamma_n(\cdot)$ affect the process $p(\cdot)$, we shall study some examples where the conditional expectations appearing in (3.13) admit explicit representations. In particular, by following Biffis (2005), we model the mortality rate processes $\gamma_1(\cdot)$, $\ldots$, $\gamma_n(\cdot)$ as affine diffusion processes.

Example 3.4.1 (Affine homogeneous intensities) *First, we consider a case in which the policyholders share the same affine mortality rate process. Succinctly, for $i \in I_n$, suppose that the mortality rate process $\gamma_i(\cdot)$ satisfies*

$$\gamma_i(t) = \lambda(t), \quad t \in [0,T],$$

where $\lambda(\cdot)$ is an $\left\{\mathcal{F}_t^2\right\}$-adapted affine diffusion process as described in Section A.1. For $I \subset I_n$ and $|I| = j$, it follows that

$$\begin{aligned} E\left(e^{-\sum_{i\in I}\int_t^T \gamma_i(s)ds}\middle|\mathcal{F}_t^2\right) &= E\left(e^{-j\int_t^T \lambda(s)ds}\middle|\mathcal{F}_t^2\right) \\ &= e^{\psi^j(T-t)+\beta^j(T-t)\lambda(t)}, \end{aligned}$$

where $\psi^j(\cdot)$ and $\beta^j(\cdot)$ are the deterministic functions in the representation (A.2) with $a = j$. Then $p(\cdot)$ simplifies to

$$p(t) = q\left(\sum_{j=1}^{n} e^{\psi^j(T-t)+\beta^j(T-t)\lambda(t)} \sum_{I\subset I_n,\ |I|=j}\left(\prod_{i\in I}(e^{\alpha c_i}-1)1_{(\tau_i>t)}\right)\right).$$

Example 3.4.2 (Affine heterogeneous intensities) *Now, we describe a case in which the policyholders have heterogeneous but correlated affine mortality rate processes. Specifically, for $i \in I_n$, assume that the mortality rate process $\gamma_i(\cdot)$ satisfies*

$$\gamma_i(t) = \lambda_0(t) + \lambda_i(t), \quad t \in [0,T],$$

where $\lambda_0(\cdot)$, $\lambda_1(\cdot)$, $\ldots$, $\lambda_n(\cdot)$ are independent $\left\{\mathcal{F}_t^2\right\}$-adapted CIR processes with respective parameters (κ, φ_0, ν), $\ldots$, (κ, φ_n, ν). Expressly, suppose that the Brownian motion $\boldsymbol{W}^{\mathbf{2}}(\cdot)$ generating the information $\left\{\mathcal{F}_t^2\right\}$ has dimension $m = n+1$ with each component $W_i^2(\cdot)$ underlying the CIR process $\lambda_i(\cdot)$. One may view $\lambda_0(\cdot)$ as a state variable containing common aspects of mortality rates, and $\lambda_i(\cdot)$ as a state variable *governing the idiosyncratic mortality specific to the policyholder i. It is not difficult to see that the process $\gamma_i(\cdot)$ is itself a CIR process with parameters $(\kappa, \varphi_0+\varphi_i, \nu)$. In this setting, for $t \in [0,T]$, it holds that*

$$\begin{aligned} & p(t) \\ = \; & q\left(\sum_{j=1}^{n} e^{\psi_0^j(T-t)+\beta_0^j(T-t)\lambda_0(t)} \sum_{I\subset I_n,\ |I|=j} \left(\prod_{i\in I}(e^{\alpha c_i}-1)1_{(\tau_i>t)}e^{\psi_i(T-t)+\beta_i(T-t)\lambda_i(t)}\right)\right), \end{aligned}$$

where $\psi_0^1(\cdot)$, $\beta_0^1(\cdot)$, $\dots$, $\psi_0^n(\cdot)$, $\beta_0^n(\cdot)$ and $\psi_1(\cdot)$, $\beta_1(\cdot)$, $\dots$, $\psi_n(\cdot)$, $\beta_n(\cdot)$ are deterministic functions. See Section A.2 for further details.

As follows, we discuss a case in which mortality rates do not have an affine structure.

Example 3.4.3 (quadratic-Gaussian heterogeneous intensities) *To substantiate the importance of allowing heterogeneous mortality rate processes, we consider a portfolio consisting of two policyholders, whose mortality rate processes correspond to the squares of two correlated Ornstein–Uhlenbeck processes with the same parameters. Specifically, assume that the Brownian motion $\boldsymbol{W^2}(\cdot)$ generating the information $\left\{\mathcal{F}_t^2\right\}$ has dimension $m=2$, and that $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ satisfy*

$$\begin{aligned} \gamma_1(t) &= \chi_1^2(t), \\ \gamma_2(t) &= \chi_2^2(t), \end{aligned}$$

with

$$\begin{aligned} d\chi_1(t) &= \kappa(\varphi-\chi_1(t))dt+\nu dW_1^2(t), \\ d\chi_2(t) &= \kappa(\varphi-\chi_2(t))dt+\nu\left(\rho dW_1^2(t)+\sqrt{1-\rho^2}dW_2^2(t)\right), \end{aligned}$$

where $\kappa>0$, $\nu>0$, $\varphi\in\mathbb{R}$, and $\rho\in[0,1]$ are parameters and t varies on $[0,T]$. By construction

$$\begin{aligned} E\left(e^{-\int_0^T \gamma_1(s)ds}\right) &= E\left(e^{-\int_0^T \gamma_2(s)ds}\right), \\ E\left(e^{-2\int_0^T \gamma_1(s)ds}\right) &> E^2\left(e^{-\int_0^T \gamma_1(s)ds}\right). \end{aligned}$$

Referring to Proposition 3.4.1 with $c_1, c_2 \geq 0$, let $p_\rho(0)$ represent the indifference price process at $t=0$, where $\rho\in[0,1]$ determines the dependence between $\chi_1(\cdot)$ and $\chi_2(\cdot)$. Therefore, it is clear that $p_0(0)\neq p_1(0)$. Moreover, according to Corollary 20 from Albanese and Lawi (2007), $p_0(0)$ and $p_1(0)$ admit closed-form expressions.

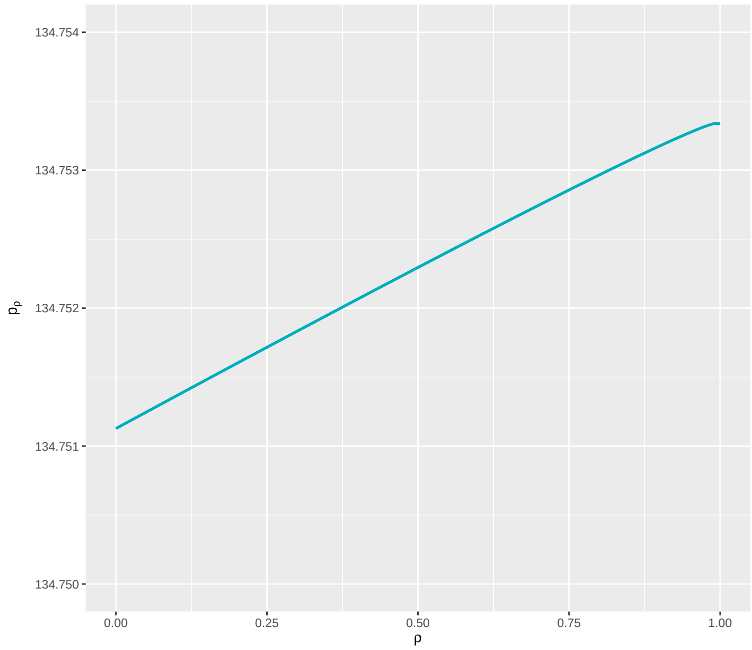

Fig. 3.1 Indifference price $p_\rho(0)$ by varying the correlation parameter $\rho \in [0,1]$

For illustrative purpose, we numerically compute $p_\rho(0)$ by varying $\rho \in [0,1]$. For the mortality rate processes, we assume the parameters $\kappa = 0.60$, $\nu = 0.06$, $\varphi = 0.68$, and $T = 10$. In addition, by following Buccola (1982), we suppose that the money is expressed in \$1000 units, and we set the absolute risk aversion parameter to be $\alpha = 0.0012$. To simulate the processes $\chi_1(\cdot)$ and $\chi_2(\cdot)$, we employ the function `sde.sim` *of R; we refer the reader to Iacus (2009) for further details. The size of the mesh in the simulation of the processes $\chi_1(\cdot)$ and $\chi_2(\cdot)$ is 5000, and the sample size to compute the expectations appearing in $p_\rho(0)$ is 5000. By choosing $c_1 = c_2 = 100$, we display the function $p_\rho(0)$ on Figure 3.1.*

3.4.2 An alternative definition of admissibility

By restricting investment strategies conforming to Definition 3.4.1, we rule out arbitrage opportunities. However, in the mathematical finance literature, there are alternative ways to define an arbitrage-free financial market. In particular, one often defines a strategy $\boldsymbol{\pi}(\cdot)$ to be admissible whenever its associated wealth $X^{\boldsymbol{\pi}}(\cdot)$ is bounded from below and satisfies an integrability condition; see for instance Section 1.4 of Karatzas and Shreve (1998). In this case, by considering a more general version of the payoff given in relation (3.10), we compute the static indifference price $p(0)$ in (3.9).

Specifically, by employing a duality approach, we price a contingent claim whose payoff C is given by relation (3.3).

Our hypotheses imply that

$$E\left(\int_0^T \|\boldsymbol{\theta}(s)\|^2 \, ds\right) < \infty.$$

Thus, we may define the *state-price-density* process

$$H_0(t) = \frac{Z_0(t)}{S_0(t)}, \quad t \in [0,T],$$

where

$$Z_0(t) = \exp\left(-\int_0^t \boldsymbol{\theta}^\intercal(s) d\boldsymbol{W}^{\mathbf{1}}(s) - \frac{1}{2}\int_0^t \|\boldsymbol{\theta}(s)\|^2 \, ds\right), \quad t \in [0,T].$$

Since $\boldsymbol{\theta}(\cdot)$ is bounded, Novikov's condition implies that the positive local martingale $Z_0(\cdot)$ is a true martingale. In consequence, we define the *standard martingale measure* P_0 on $\mathcal{F}_T^1$ by

$$P_0(A) = E(Z_0(T)1_A), \quad A \in \mathcal{F}_T^1.$$

In a consistent manner, the expectation operator with respect to P_0 is denoted by $E_0(\cdot)$.

Besides, using the following definition, we stipulate an alternative set of strategies available to the investor.

Definition 3.4.2 *The set of admissible strategies $\mathcal{A}$ consists of all $\left\{\mathcal{F}_t^1\right\}$-progressively measurable and $\mathbb{R}^d$-valued processes $\boldsymbol{\pi}(\cdot)$ satisfying*

$$E\left(\int_0^T \|\boldsymbol{\pi}^\intercal(s)\boldsymbol{\sigma}(s)\|^2 \, ds\right) < \infty,$$

as well as

$$X^{\boldsymbol{\pi}}(t) \geq b, \quad t \in [0,T],$$

for some constant b.

For $\boldsymbol{\pi} \in \mathcal{A}$, by relation (3.4), we see that $X^{\boldsymbol{\pi}}(\cdot)$ may be rewritten as

$$X^{\boldsymbol{\pi}}(t) = \int_0^t \boldsymbol{\pi}^\intercal(s)\boldsymbol{\sigma}(s)\, d\boldsymbol{W}_{\mathbf{0}}(s),$$

where

$$\boldsymbol{W}_{\mathbf{0}}(t) = \boldsymbol{W}^{\mathbf{1}}(t) + \int_0^t \boldsymbol{\theta}(s) ds, \quad t \in [0,T].$$

By Girsanov's theorem, the process $\boldsymbol{W_0}(\cdot)$ is a Brownian motion under the martingale measure P_0, relative to the filtration $\left\{\mathcal{F}_t^1\right\}$. This implies that $X^{\boldsymbol{\pi}}(\cdot)$ is a local martingale bounded from below under P_0. Then $X^{\boldsymbol{\pi}}(\cdot)$ is a supermartingale by Fatou's lemma. Hence, it follows that the strategy $\boldsymbol{\pi}(\cdot)$ must satisfy the budget constraint

$$E_0\left(X^{\boldsymbol{\pi}}(T)\right) = E\left(H_0(T)\, X^{\boldsymbol{\pi}}(T)\right) \leq 0. \tag{3.16}$$

For mathematical convenience, we shall rewrite the analog of the static value function in (3.7) at $t=0$. To that end, let $\boldsymbol{\pi} \in \mathcal{A}$. By partitioning, notice that

$$E\left(-e^{-\alpha\left(X^{\boldsymbol{\pi}}(T)-C\right)}\right) = \sum_{I\subset I_n} E\left(-e^{-\alpha X^{\boldsymbol{\pi}}(T)} e^{\alpha\left(\sum_{i\in I} C_i\right)} 1_{A^I} 1_{B^I}\right), \tag{3.17}$$

where

$$\begin{aligned} A_T^I &= \cap_{i\in I}(\tau_i > T), \\ B_T^I &= \cap_{i\in I_n\setminus I}(\tau_i \leq T). \end{aligned}$$

Because of the independence between vector the $(\tau_1,\ldots,\tau_n)$ and the filtration $\left\{\mathcal{F}_t^1\right\}$, it holds that

$$\begin{aligned} & \sum_{I\subset I_n} E\left(-e^{-\alpha X^{\boldsymbol{\pi}}(T)} e^{\alpha\left(\sum_{i\in I} C_i\right)} 1_{A_T^I} 1_{B_T^I}\right) \\ = & \sum_{I\subset I_n} E\left(-e^{-\alpha X^{\boldsymbol{\pi}}(T)} e^{\alpha\left(\sum_{i\in I} C_i\right)}\right) P\left(A_T^I \cap B_T^I\right) \\ = & \; E\left(-e^{-\alpha X^{\boldsymbol{\pi}}(T)} \sum_{I\subset I_n} e^{\alpha\left(\sum_{i\in I} C_i\right)} P\left(A_T^I \cap B_T^I\right)\right). \end{aligned} \tag{3.18}$$

Therefore, by using relations (3.17) and (3.18), we may rewrite

$$\begin{aligned} V(0,C) &= \sup_{\boldsymbol{\pi}\in\mathcal{A}} E\left(-e^{-\alpha\left(X^{\boldsymbol{\pi}}(T)-C\right)}\right) \\ &= \sup_{\boldsymbol{\pi}\in\mathcal{A}} E\left(J\left(X^{\boldsymbol{\pi}}(T),\omega\right)\right), \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} J(y,\omega) &= -e^{-\alpha y} e^{\alpha L(\omega)}, \quad (y,\omega) \in \mathbb{R}\times\Omega, \\ L(\omega) &= \frac{1}{\alpha} \log \sum_{I\subset I_n} e^{\alpha\left(\sum_{i\in I} C_i(\omega)\right)} P\left(A_T^I \cap B_T^I\right), \quad \omega \in \Omega. \end{aligned} \tag{3.20}$$

Thus, $V(0,C)$ is the value function of the optimization problem corresponding to the state-dependent utility function $J(\cdot,\cdot)$ and the family of attainable wealths $\{X^{\boldsymbol{\pi}}(T)\}_{\boldsymbol{\pi}\in\mathcal{A}}$.

To solve the optimization problem in (3.19), we employ a martingale methodology; see for instance Chapter 3 of Karatzas and Shreve (1998). To that purpose, we first present some definitions. Let $J'(\cdot,\cdot)$ denote the derivative of $J(\cdot,\cdot)$ with respect to the first variable. Moreover, let $I(\cdot,\cdot)$ denote the inverse of $J'(\cdot,\cdot)$ with respect to the first variable. Thus, it follows that

$$I(z,\omega) = -\frac{1}{\alpha}\log\left(\frac{z}{\alpha}\right) + L(\omega), \quad z > 0,\ \omega \in \Omega.$$

In these terms, the convex dual of $J(\cdot,\cdot)$, which is denoted by $\tilde{J}(\cdot,\cdot)$, can be represented as

$$\begin{aligned}\tilde{J}(z,\omega) &= \sup_{y} J(y,\omega) - yz \\ &= J(I(z,\omega),\omega) - zI(z,\omega), \quad z > 0,\ \omega \in \Omega. \end{aligned} \tag{3.21}$$

As usual, assume that the random variable $H_0(T)$ has finite entropy, namely,

$$E\left(H_0(T)\log\left(H_0(T)\right)\right) < \infty.$$

Therefore, we may define the function

$$\mathcal{X}(z) = E(H_0(T)I(zH_0(T),\omega)), \quad z > 0.$$

Under our assumptions, $\mathcal{X}(z) < \infty$ for all $z > 0$. Accordingly, one can easily prove that the function $\mathcal{X}(\cdot)$, which maps $(0,\infty)$ onto $(-\infty,\infty)$, is continuous and strictly decreasing. Moreover, it holds that $\mathcal{X}\left(0^+\right) = \lim_{y\to 0^+}\mathcal{X}(y) = \infty$ and $\mathcal{X}(\infty) = \lim_{y\to\infty}\mathcal{X}(y) = -\infty$. Let $\mathcal{Y}(\cdot)$ denote the inverse of $\mathcal{X}(\cdot)$, i.e., $\mathcal{Y}(\cdot) = \mathcal{X}^{-1}(\cdot)$.

Now, we are ready to present the main result of this section.

Theorem 3.4.1 *Define*

$$\xi^* = I\left(\mathcal{Y}(0)\,H_0(T),\omega\right).$$

Let $\boldsymbol{\pi}^* \in \mathcal{A}$ be such that $X^{\boldsymbol{\pi}^*}(T) = \xi^*$. Then $\boldsymbol{\pi}^*(\cdot)$ is optimal for the optimization problem given in (3.19):

$$V(0,C) = E\left(-e^{-\alpha\left(X^{\boldsymbol{\pi}^*}(T)-C\right)}\right).$$

Moreover, for all $t \in [0,T]$, the corresponding wealth and strategy processes are respectively given by

$$\begin{aligned} X^{\boldsymbol{\pi}^*}(t) &= (H_0(t))^{-1} E\left(H_0(T)\xi^* | \mathcal{F}_t^1\right), &(3.22)\\ \boldsymbol{\sigma}^\intercal(t)\boldsymbol{\pi}^*(t) &= (H_0(t))^{-1}\boldsymbol{\varphi}(t) + X^{\boldsymbol{\pi}^*}(t)\boldsymbol{\theta}(t), &(3.23)\end{aligned}$$

where $\boldsymbol{\varphi}(\cdot)$ is the integrand in the stochastic integral representation

$$M(t) = x + \int_0^t \boldsymbol{\varphi}^\intercal(s)\, d\boldsymbol{W}^{\mathbf{1}}(s)$$

of the martingale

$$M(t) = E\left(H_0(T)\xi^* | \mathcal{F}_t^1\right).$$

To prove Theorem 3.4.1, we first recall a standard result; see for instance Theorem 3.3.5 of Karatzas and Shreve (1998).

Lemma 3.4.4 *Let ξ be an $\mathcal{F}_T^1$-measurable and bounded from below random variable such that*

$$E(H_0(T)\xi) = 0.$$

Then there exists a strategy process $\boldsymbol{\pi} \in \mathcal{A}$ such that $X^{\boldsymbol{\pi}}(T) = \xi$. Moreover, for all $t \in [0,T]$, the corresponding wealth and strategy processes are respectively given by

$$\begin{aligned} X^{\boldsymbol{\pi}}(t) &= (H_0(t))^{-1} E\left(H_0(T)\xi | \mathcal{F}_t^1\right),\\ \boldsymbol{\sigma}^\intercal(t)\boldsymbol{\pi}(t) &= (H_0(t))^{-1}\boldsymbol{\varphi}(t) + X^{\boldsymbol{\pi}}(t)\boldsymbol{\theta}(t),\end{aligned}$$

where $\boldsymbol{\varphi}(\cdot)$ is the integrand in the stochastic integral representation

$$M(t) = x + \int_0^t \boldsymbol{\varphi}^\intercal(s)\, d\boldsymbol{W}^{\mathbf{1}}(s)$$

of the martingale

$$M(t) = E\left(H_0(T)\xi | \mathcal{F}_t^1\right).$$

Proof of Theorem 3.4.1. Clearly, the random variable ξ^* satisfies the budget constraint

$$E(H_0(T)\xi^*) = \mathcal{X}(\mathcal{Y}(0)) = 0. \tag{3.24}$$

From Lemma 3.4.4, there exists a strategy process $\boldsymbol{\pi}^* \in \mathcal{A}$ such that $X^{\boldsymbol{\pi}^*}(T) = \xi^*$. To prove that $\boldsymbol{\pi}^*(\cdot)$ is optimal, let $\boldsymbol{\pi}(\cdot)$ be any strategy in $\mathcal{A}$. By using relation (3.21), we obtain

$$\begin{aligned} J(\xi^*,\omega) - \mathcal{Y}(0)\,H_0(T)\xi^* &= \tilde{J}(\mathcal{Y}(0)\,H_0(T),\omega) \\ &\geq J(X^{\boldsymbol{\pi}}(T),\omega) - \mathcal{Y}(0)\,H_0(T)X^{\boldsymbol{\pi}}(T). \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} E(J(\xi^*,\omega)) &\geq E(J(X^{\boldsymbol{\pi}}(T),\omega)) + \mathcal{Y}(0)\left(E(H_0(T)\xi^*) - E(H_0(T)X^{\boldsymbol{\pi}}(T))\right) \\ &\geq E(J(X^{\boldsymbol{\pi}}(T),\omega)). \end{aligned}$$

The last inequality follows because of relation (3.24) and the budget constraint (3.16) satisfied by $\boldsymbol{\pi}(\cdot)$. Finally, the representations in (3.22) and (3.23) also follow by Lemma 3.4.4. ■

According to relation (3.9), the indifference price $p(0)$ corresponding to the liability C can be represented as

$$p(0) = \frac{1}{\alpha}\log\frac{V(0,C)}{V(0,0)}.$$

Thus, by computing $V(0,0)$ and $V(0,C)$ with the help of Theorem 3.4.1, we derive the next result.

Corollary 3.4.1 *The indifference price $p(0)$ for the liability C is given by*

$$p(0) = E_0(L(\omega)). \tag{3.25}$$

Notice that the quasi-explicit pricing formula (3.25) holds even when $\tau_1, \ldots, \tau_n$ are not conditionally independent. Indeed, it applies provided that the random times $\tau_1, \ldots, \tau_n$ are independent of the filtration $\left\{\mathcal{F}_t^1\right\}$. In particular, we may choose an arbitrary dependence structure to the auxiliary random variables $U_1,\ldots,U_n$ in the definition of the random times $\tau_1, \ldots, \tau_n$ given by (3.2).

Example 3.4.4 *Assume a financial market with one risky asset, that is, $d=1$. Here, the asset price process $S_1(\cdot)$ shall represent the market value of an equity fund in which the policyholders may invest. Let $\delta > 0$ denote the annualized rate at which fees are deducted from the investment account. Therefore, the account value at time t for the individual i, denoted by $F_i(t)$, is given by*

$$F_i(t) = F_i(0)\frac{S_1(t)}{S_1(0)}e^{-\delta t}, \quad t \in [0,T].$$

For an integer $m \geq 1$, define $\Delta = \frac{T}{m}$. Furthermore, let $t_0 = 0 < t_1 < \cdots < t_m = T$ be the m-regular partition of the interval $[0,T]$, i.e., $t_j = j\Delta$. Suppose that the insurer pays to the ith policyholder the discounted amount

$$\begin{aligned} &\left(g_i^0(F_i(T)) - \sum_{k=1}^{m} g_i^1(F_i(t_{k-1}))\Delta\right) 1_{(\tau_i > T)} \\ - &\sum_{j=1}^{m}\left(\sum_{k=1}^{j} g_i^1(F_i(t_{k-1}))\right) 1_{(t_{j-1} < \tau_i \leq t_j)}\Delta, \end{aligned} \tag{3.26}$$

where g_i^0 and g_i^1 are measurable functions. In such a case, $g_i^0(F_i(T))$ denotes the amount received by the policyholder when surviving to time T, and $\sum_{k=1}^{j} g_i^1(F_i(t_{k-1}))$ denotes the incurred fees by the policyholder surviving to time t_{j-1}. For instance, let $a_i \in \mathbb{R}$ and $\delta_e > 0$. When $g_i^0(x) = (a_i - x)_+$ and $g_i^1(x) = \delta_e x$ for $x \in \mathbb{R}$, the expression in (3.26) may represent the payment of a plain-vanilla guaranteed minimum maturity benefits including fees; see Feng (2014). By slightly modifying the expression (3.20), the quasi-explicit representation for the static indifference price $p(0)$ from Corollary 3.4.1 holds, when referring to a claim with a payoff given by (3.26). Precisely, we need to change the subsets indexing the sum in relation (3.20) to incorporate, for every i, the information about the interval $(t_{j_i-1}, t_{j_i}]$ at which τ_i belongs whenever $\tau_i \leq T$.

3.4.3 General mortality-linked securities

In this section, we consider the pricing of a general claim with a bounded payout C. By referring to Definition 3.4.1, for the sake of generality, we slighty modify the set of admissible strategies by asking them only to be $\{\mathcal{G}_t\}$-adapted.

Definition 3.4.3 *The set of admissible strategies $\mathcal{A}$ consists of all $\{\mathcal{G}_t\}$-progressively measurable processes $\boldsymbol{\pi}(\cdot)$ with values in a closed subset of $\mathbb{R}^d$, which satisfy*

$$E\left(\int_0^T \left\|\boldsymbol{\pi}^\intercal(s)\,\boldsymbol{\sigma}(s)\right\|^2 ds\right) < \infty,$$

as well as

$$\left\{-e^{-\alpha X^{\boldsymbol{\pi}}(\rho)} :\ \rho \text{ is a } \{\mathcal{G}_t\}\text{-stopping time on } [0,T]\right\}$$

is a uniformly integrable family.

Instead of providing an explicit expression for the indifference price process, in this general setting, we deal with the solution of the dynamic optimization problem in (3.7) by employing techniques of BSDE. For simplicity, we consider the problem at $t = 0$ only. Notwithstanding, we point out that the procedure presented below also gives us the solution to the dynamic optimization problem in (3.7) by making obvious adjustments.

First, we define some stochastic process spaces:

- $\mathcal{S}^2$ denotes the set of $\{\mathcal{G}_t\}$-adapted RCLL processes Y with $E\left(\sup_{t\in[0,T]} Y(t)\right) < \infty$,
- $\mathcal{L}_+^\infty$ denotes the set of positive $\{\mathcal{G}_t\}$-adapted and essentially bounded RCLL processes,
- $\mathcal{L}^{2,d}(\boldsymbol{W^1})$ denotes the set of $\{\mathcal{G}_t\}$-adapted RCLL processes $\boldsymbol{Z}$ taking values in $\mathbb{R}^d$ with $E\left(\int_0^T \|\boldsymbol{Z}(s)\|^2 ds\right) < \infty$,
- $\mathcal{L}^{2,m}(\boldsymbol{W^2})$ denotes the set of $\{\mathcal{G}_t\}$-adapted RCLL processes $\boldsymbol{Z}$ taking values in $\mathbb{R}^m$ with $E\left(\int_0^T \|\boldsymbol{Z}(s)\|^2 ds\right) < \infty$,
- $\mathcal{L}_\gamma^{2,n}(\boldsymbol{H})$ denotes the set of $\{\mathcal{G}_t\}$-adapted RCLL processes $\boldsymbol{U}$ taking values in $\mathbb{R}^n$ with $E\left(\sum_{i=1}^n \int_0^T U_i^2(s)\gamma_i(s)ds\right) < \infty$.

To find the value function and an optimal strategy for the optimization problem in (3.7) at $t = 0$, by following Hu et al. (2005), we seek a family of stochastic processes $\{R^{\boldsymbol{\pi}}\}_{\boldsymbol{\pi}\in\mathcal{A}}$ with the following properties:

(i) $R^{\boldsymbol{\pi}}(T) = -e^{-\alpha(X^{\boldsymbol{\pi}}(T)-C)}$ for all $\boldsymbol{\pi} \in \mathcal{A}$,

(ii) $R^{\boldsymbol{\pi}}(0) = R_0$ is constant for all $\boldsymbol{\pi} \in \mathcal{A}$,

(iii) $R^{\boldsymbol{\pi}}(\cdot)$ is a supermartingale for all $\boldsymbol{\pi} \in \mathcal{A}$, and there exists $\boldsymbol{\pi}^* \in \mathcal{A}$ such that $R^{\boldsymbol{\pi}^*}(\cdot)$ is a martingale.

If $\{R^{\boldsymbol{\pi}}\}_{\boldsymbol{\pi}\in\mathcal{A}}$ is a family satisfying the aforementioned properties, it follows that

$$E\left(-e^{-\alpha(X^{\boldsymbol{\pi}}(T)-C)}\right) \le R_0 = E\left(-e^{-\alpha(X^{\boldsymbol{\pi}^*}(T)-C)}\right) = V(0,0,C),$$

whence $\boldsymbol{\pi}^*$ is an optimal strategy. To construct such a family, for all $t \in [0,T]$ and $\boldsymbol{\pi} \in \mathcal{A}$, set

$$R^{\boldsymbol{\pi}}(t) = -e^{-\alpha(X^{\boldsymbol{\pi}}(t)-Y(t))}, \tag{3.27}$$

where

$$(Y, \boldsymbol{Z_1}, \boldsymbol{Z_2}, \boldsymbol{U}) = (Y, \boldsymbol{Z_1}, \boldsymbol{Z_2}, U_1, \ldots, U_n) \in \mathcal{S}^2 \times \mathcal{L}^{2,d}(\boldsymbol{W^1}) \times \mathcal{L}^{2,m}(\boldsymbol{W^2}) \times \mathcal{L}_\gamma^{2,n}(\boldsymbol{H})$$

is a solution to the BSDE

$$\begin{aligned} Y(t) &= C + \int_t^T f(s, \boldsymbol{Z_1}(s), \boldsymbol{Z_2}(s), \boldsymbol{U}(s))ds - \int_t^T \boldsymbol{Z_1}^\intercal(s) d\boldsymbol{W^1}(s) \\ &- \int_t^T \boldsymbol{Z_2}^\intercal(s) d\boldsymbol{W^2}(s) - \sum_{i=1}^n \int_t^T U_i(s) d\tilde{H}_i(s), \quad t \in [0,T]. \end{aligned} \tag{3.28}$$

In these terms, our task consists of finding a function f^* for which $R^{\boldsymbol{\pi}}(\cdot)$ is a supermartingale for all $\boldsymbol{\pi} \in \mathcal{A}$ and a strategy $\boldsymbol{\pi}^* \in \mathcal{A}$ such that $R^{\boldsymbol{\pi}^*}(\cdot)$ is a martingale. To that end, for all $\boldsymbol{\pi} \in \mathcal{A}$, we write $R^{\boldsymbol{\pi}}(\cdot)$ as a function involving a local martingale $M^{\boldsymbol{\pi}}(\cdot)$ and a bounded variation process $A^{\boldsymbol{\pi}}(\cdot)$. Let $\boldsymbol{\pi} \in \mathcal{A}$. From (3.4) and (3.28), notice that

$$\begin{aligned} & -\alpha(X^{\boldsymbol{\pi}}(t) - Y(t)) \\ = \; & \alpha Y(0) - \alpha \int_0^t \left(f(s) + \boldsymbol{\pi}^\intercal(s) \boldsymbol{\mu}(s)\right) ds - \alpha \int_0^t \left(\boldsymbol{\pi}^\intercal(s) \boldsymbol{\sigma}(s) - \boldsymbol{Z_1}^\intercal(s)\right) d\boldsymbol{W^1}(s) \\ + \; & \alpha \int_0^t \boldsymbol{Z_2}^\intercal(s) d\boldsymbol{W^2}(s) + \alpha \sum_{i=1}^n \int_0^t U_i(s) d\tilde{H}_i(s), \quad t \in [0,T]. \end{aligned}$$

Now, for $t \in [0,T]$, define

$$\begin{aligned} A^{\boldsymbol{\pi}}(t) &= \int_0^t \Bigg(-\alpha f(s) - \alpha \boldsymbol{\pi}^\intercal(s) \boldsymbol{\mu}(s) + \frac{1}{2}\alpha^2 \left\| \left(\boldsymbol{\pi}^\intercal(s) \boldsymbol{\sigma}(s) - \boldsymbol{Z_1}^\intercal(s)\right) \right\|^2 \\ &\quad + \frac{1}{2}\alpha^2 \left\| \boldsymbol{Z_2}(s) \right\|^2 - \sum_{i=1}^n \left(\alpha U_i(s) - e^{\alpha U_i(s)} + 1\right) \gamma_i(s) \Bigg) ds, \end{aligned} \tag{3.29}$$

$$\begin{aligned} & M^{\boldsymbol{\pi}}(t) \\ = \; & \exp\Bigg(-\int_0^t \alpha \left(\boldsymbol{\pi}^\intercal(s) \boldsymbol{\sigma}(s) - \boldsymbol{Z_1}^\intercal(s)\right) d\boldsymbol{W^1}(s) - \int_0^t \frac{1}{2}\alpha^2 \left\| \left(\boldsymbol{\pi}^\intercal(s) \boldsymbol{\sigma}(s) - \boldsymbol{Z_1}^\intercal(s)\right) \right\|^2 ds \\ + \; & \int_0^t \alpha \boldsymbol{Z_2}^\intercal(s) d\boldsymbol{W^2}(s) - \int_0^t \frac{1}{2}\alpha^2 \left\| \boldsymbol{Z_2}(s) \right\|^2 ds + \sum_{i=1}^n \int_0^t \alpha U_i(s) d\tilde{H}_i(s) \\ + \; & \sum_{i=1}^n \int_0^t \left(\alpha U_i(s) - e^{\alpha U_i(s)} + 1\right) \gamma_i(s) ds \Bigg). \end{aligned} \tag{3.30}$$

Thus, $R^{\boldsymbol{\pi}}(\cdot)$ can be written as follows:

$$R^{\boldsymbol{\pi}}(t) = R^{\boldsymbol{\pi}}(0) M^{\boldsymbol{\pi}}(t) \exp\left(A^{\boldsymbol{\pi}}(t)\right), \quad t \in [0,T].$$

If for all $\boldsymbol{\pi} \in \mathcal{A}$ the process $A^{\boldsymbol{\pi}}(\cdot)$ is increasing, then the supermartingale condition in (iii) holds as we shall see later. We can accomplish this requirement by setting

$$\begin{aligned}\alpha f^*(t) &= \min_{\boldsymbol{\pi}\in\mathcal{A}}\left\{-\alpha\boldsymbol{\pi}^\intercal\boldsymbol{\mu}(t)+\frac{1}{2}\alpha^2\left\|\boldsymbol{\pi}^\intercal\boldsymbol{\sigma}(t)-\boldsymbol{Z_1}^\intercal(t)\right\|^2\right\}\\ &+ \frac{1}{2}\alpha^2\left\|\boldsymbol{Z_2}(t)\right\|^2-\sum_{i=1}^{n}\left(\alpha U_i(t)-e^{\alpha U_i(t)}+1\right)\gamma_i(t), \quad t\in[0,T].\end{aligned}$$

Moreover, by defining

$$\boldsymbol{\pi}^*(t)=\arg\min_{\boldsymbol{\pi}\in\mathcal{A}}\left\{-\alpha\boldsymbol{\pi}^\intercal\boldsymbol{\mu}(t)+\frac{1}{2}\alpha^2\left\|\boldsymbol{\pi}^\intercal\boldsymbol{\sigma}(t)-\boldsymbol{Z_1}^\intercal(t)\right\|^2\right\}, \quad t\in[0,T],$$

we obtain that $A^{\boldsymbol{\pi}^*}(t)=1$ for all $t\in[0,T]$. That said, we are left to prove that the BSDE (3.28) with generator f^* has a solution, $M^{\boldsymbol{\pi}^*}(\cdot)$ is a true martingale, and $R^{\boldsymbol{\pi}}(\cdot)$ is a supermartingale for all $\boldsymbol{\pi}\in\mathcal{A}$. In light of this, we present the main result of this section.

Theorem 3.4.2 *Let $\boldsymbol{K}$ denote the closed set of Definition 3.4.3. Assume that $\boldsymbol{K}$ is bounded and that $\gamma_i(\cdot)$ is uniformly bounded for every i. Then for every $C\in L^\infty(\mathcal{G}_T)$:*

(i) *There exists a unique solution $(Y,\boldsymbol{Z_1},\boldsymbol{Z_2},\boldsymbol{U})\in\mathcal{S}^2\times\mathcal{L}^{2,d}(\boldsymbol{W^1})\times\mathcal{L}^{2,m}(\boldsymbol{W^2})\times\mathcal{L}^{2,n}_\gamma(\boldsymbol{H})$ to the BSDE*

$$\begin{aligned}&Y(t)\\ =\;& C+\int_t^T\left(\min_{\boldsymbol{\pi}\in\mathcal{A}}\left\{-\boldsymbol{\pi}^\intercal\boldsymbol{\mu}(s)+\frac{1}{2}\alpha\left\|\boldsymbol{\pi}^\intercal\boldsymbol{\sigma}(s)-\boldsymbol{Z_1}^\intercal(s)\right\|^2\right\}\right.\\ +\;& \left.\frac{1}{2}\alpha\left\|\boldsymbol{Z_2}(s)\right\|^2-\sum_{i=1}^{n}\frac{1}{\alpha}\left(\alpha U_i(s)-e^{\alpha U_i(s)}+1\right)\gamma_i(s)\right)ds\\ -\;& \int_t^T\boldsymbol{Z_1}^\intercal(s)d\boldsymbol{W^1}(s)-\int_t^T\boldsymbol{Z_2}^\intercal(s)d\boldsymbol{W^2}(s)-\sum_{i=1}^{n}\int_t^T U_i(s)d\tilde{H}_i(s), \quad (3.31)\end{aligned}$$

where $t\in[0,T]$.

(ii) *The optimal admissible investment strategy $\boldsymbol{\pi}^*\in\mathcal{A}$ for the utility maximization problem in (3.7) at $t=0$ satisfies*

$$\boldsymbol{\pi}^*(t)\in\Pi_{\boldsymbol{\sigma}^\intercal(t)\boldsymbol{K}}\left(\frac{1}{\alpha}\boldsymbol{\theta}(t)+\boldsymbol{Z_1}(t)\right), \quad t\in[0,T].$$

The optimal value function of the optimization problem in (3.7) at $t=0$ *equals* $-e^{\alpha Y(0)}$.

To prove Theorem 3.4.2, we need the following result concerning the stochastic integral representation of square integrable martingales; see for instance Proposition 5.4 of Jeanblanc and Rutkowski (2002).

Lemma 3.4.5 *Let* $M(\cdot)$ *be a square integrable* $\{\mathcal{G}_t\}$*-martingale. Then there are unique* $\{\mathcal{G}_t\}$*-predictable process* $(\boldsymbol{Z_1},\boldsymbol{Z_2},\boldsymbol{U})\in\mathcal{L}^{2,d}(\boldsymbol{W^1})\times\mathcal{L}^{2,m}(\boldsymbol{W^2})\times\mathcal{L}^{2,n}_{\gamma}(\boldsymbol{H})$ *such that*

$$M(t)=M(0)+\int_0^t \boldsymbol{Z_1}^\intercal(s)d\boldsymbol{W^1}(s)+\int_0^t \boldsymbol{Z_2}^\intercal(s)d\boldsymbol{W^2}(s)+\sum_{i=1}^n\int_0^t U_i(s)d\tilde{H}_i(s).$$

Proof of Theorem 3.4.2. First, we prove Theorem 3.4.2(i). Consider the following BSDE

$$\begin{aligned}
&\hat{Y}(t)\\
=\;& e^{\alpha C}+\int_t^T \min_{\boldsymbol{\pi}\in\mathcal{A}}\left\{-\alpha\boldsymbol{\pi}^\intercal\left(\boldsymbol{\mu}(s)\hat{Y}(s)+\boldsymbol{\sigma}(s)\boldsymbol{\hat{Z}_1}(s)\right)+\frac{1}{2}\alpha^2\left\|\boldsymbol{\pi}^\intercal\boldsymbol{\sigma}(s)\right\|^2\hat{Y}(s)\right\}ds\\
-\;& \int_t^T \boldsymbol{\hat{Z}_1}^\intercal(s)d\boldsymbol{W^1}(s)-\int_t^T \boldsymbol{\hat{Z}_2}^\intercal(s)d\boldsymbol{W^2}(s)-\sum_{i=1}^n\int_t^T \hat{U}_i(s)d\tilde{H}_i(s),
\end{aligned}\tag{3.32}$$

where $t\in[0,T]$. For all $(t,y,\boldsymbol{z_1},\boldsymbol{z_2})\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^m$, the generator $g(\cdot)$ of the BSDE (3.32) is

$$g(t,y,\boldsymbol{z_1},\boldsymbol{z_2})=\min_{\boldsymbol{\pi}\in\mathcal{A}}\left\{g^{\boldsymbol{\pi}}(t,y,\boldsymbol{z_1},\boldsymbol{z_2})\right\},$$

where

$$g^{\boldsymbol{\pi}}(t,y,\boldsymbol{z_1},\boldsymbol{z_2})=-\alpha\boldsymbol{\pi}^\intercal\left(\boldsymbol{\mu}(t)y+\boldsymbol{\sigma}(t)\boldsymbol{z_1}^\intercal\right)+\frac{1}{2}\alpha^2\left\|\boldsymbol{\pi}^\intercal\boldsymbol{\sigma}(t)\right\|^2 y.$$

Referring to the variables y, $\boldsymbol{z_1}$, $\boldsymbol{z_2}$, the function $g(\cdot)$ can be represented as the infimum of the Lipschitz continuous functions $\{g^{\boldsymbol{\pi}}(\cdot)\}_{\boldsymbol{\pi}\in\mathcal{A}}$, which have a global Lipschitz constant. Thus, $g(\cdot)$ is Lipschitz continuous. According to Proposition 3.2 of Becherer (2006) and Lemma 3.4.5, there exists a unique solution $(\hat{Y},\boldsymbol{\hat{Z}_1},\boldsymbol{\hat{Z}_2},\hat{U})\in\mathcal{S}^2\times\mathcal{L}^{2,d}(\boldsymbol{W^1})\times\mathcal{L}^{2,m}(\boldsymbol{W^2})\times\mathcal{L}^{2,n}_{\gamma}(\boldsymbol{H})$ to the BSDE (3.32). Besides, by employing a measurable selection argument, there exists a strategy $\boldsymbol{\hat{\pi}}\in\mathcal{A}$ satisfying

$$g(t,\hat{Y}(t),\boldsymbol{\hat{Z}_1}(t),\boldsymbol{\hat{Z}_2}(t))=g^{\boldsymbol{\hat{\pi}}}(t,\hat{Y}(t),\boldsymbol{\hat{Z}_1}(t),\boldsymbol{\hat{Z}_2}(t)),\quad t\in[0,T],$$

see for instance Lemma 11 of Hu et al. (2005). Therefore, for all $t \in [0,T]$, $\hat{Y}(\cdot)$ has the following representation

$$\hat{Y}(t) = E_Q\left(\exp\left(C + \int_t^T \left(-\alpha\hat{\boldsymbol{\pi}}^\intercal(s)\boldsymbol{\mu}(s) + \frac{1}{2}\alpha^2 \left\|\hat{\boldsymbol{\pi}}^\intercal(s)\boldsymbol{\sigma}(s)\right\|^2\right) ds\right)\middle|\mathcal{G}_t\right),$$

where

$$\frac{dQ}{dP} = \exp\left(\int_0^T -\alpha\hat{\boldsymbol{\pi}}^\intercal(s)\boldsymbol{\sigma}(s)d\boldsymbol{W}^{\mathbf{1}}(s) - \int_0^T \frac{1}{2}\alpha^2 \left\|\hat{\boldsymbol{\pi}}^\intercal(s)\boldsymbol{\sigma}(s)\right\|^2 ds\right).$$

This representation allows us to conclude that $\hat{Y}(\cdot)$ is strictly positive, bounded from above, and bounded away from zero. Consequently, $\hat{U}_1(\cdot)$, ..., $\hat{U}_n(\cdot)$ are bounded because they constitute the jump part of the bounded process $\hat{Y}(\cdot)$. For $t \in [0,T]$, we change the random variables $\hat{Y}(t)$, $\hat{\boldsymbol{Z}}_{\mathbf{1}}(t)$, $\hat{\boldsymbol{Z}}_{\mathbf{2}}(t)$, $\hat{\boldsymbol{U}}(t)$ into

$$\begin{aligned}
Y(t) &= \frac{1}{\alpha}\log\left(\hat{Y}(t)\right), \\
\boldsymbol{Z}_{\mathbf{1}}(t) &= \frac{1}{\alpha}\frac{\hat{\boldsymbol{Z}}_{\mathbf{1}}(t)}{Y(t)}, \\
\boldsymbol{Z}_{\mathbf{2}}(t) &= \frac{1}{\alpha}\frac{\hat{\boldsymbol{Z}}_{\mathbf{2}}(t)}{Y(t)}, \\
U_i(t) &= \frac{1}{\alpha}\log\left(\frac{\hat{U}_i(t)}{Y(t-)} + 1\right), \qquad \text{for all } i \in I_n.
\end{aligned}$$

By using the Ito formula, we notice that the processes $Y(\cdot)$, $\boldsymbol{Z}_{\mathbf{1}}(\cdot)$, $\boldsymbol{Z}_{\mathbf{2}}(\cdot)$, $\boldsymbol{U}(\cdot)$ satisfy the BSDE (3.31). Thus, we have proved Theorem 3.4.2(i).

Now, we turn to prove Theorem 3.4.2(ii). Let $\boldsymbol{\pi} \in \mathcal{A}$. Since $M^{\boldsymbol{\pi}}(\cdot)$ is a positive local martingale, there exists a sequence of stopping times $\{\rho_k\}_{k\geq 1}$ with $\lim_{k\to\infty}\rho_k = \infty$, such that $M^{\boldsymbol{\pi}}(\cdot \wedge \rho_k)$ is a positive martingale. Also, by the choice of $(f^*, \boldsymbol{\pi}^*)$, the process $A^{\boldsymbol{\pi}}(\cdot)$ is increasing. Then $R^{\boldsymbol{\pi}}(\cdot \wedge \rho_k)$ is a supermartingale, namely, for every $s \leq t$ and $A \in \mathcal{G}_s$, we have

$$E\left(R^{\boldsymbol{\pi}}(t \wedge \rho_k) 1_A\right) \leq E\left(R^{\boldsymbol{\pi}}(s \wedge \rho_k) 1_A\right). \tag{3.33}$$

Besides, from the uniformly integrability of the familly

$$\left\{-e^{-\alpha X^{\boldsymbol{\pi}}(\rho)} : \ \rho \text{ is a } \{\mathcal{G}_t\}\text{-stopping time}\right\}$$

and the boundeness of Y, we obtain that the family

$$\{R^{\boldsymbol{\pi}}(\rho): \ \rho \text{ is a } \{\mathcal{G}_t\}\text{-stopping time}\}$$

is uniformly integrable. Therefore, we can take the limit as $k \to \infty$ in (3.33) to obtain the supermartingale property of $R^{\boldsymbol{\pi}}(\cdot)$. Finally, we proceed to prove that $R^{\boldsymbol{\pi}^*}(\cdot)$ is a martingale. Since $A^{\boldsymbol{\pi}^*}(\cdot) = 0$, we have to show that $M^{\boldsymbol{\pi}^*}(\cdot)$ is a true martingale. To that purpose, by writing $\mathcal{E}(X)$ for the stochastic exponential of a process X, we notice that

$$\begin{aligned} M^{\boldsymbol{\pi}^*}(t) &= \mathcal{E}\Bigg(-\int_0^t \alpha\left((\boldsymbol{\pi}^*)^{\mathsf{T}}(s)\boldsymbol{\sigma}(s) - \boldsymbol{Z_1}^{\mathsf{T}}(s)\right) d\boldsymbol{W^1}(s) \\ &+ \int_0^t \alpha \boldsymbol{Z_2}^{\mathsf{T}}(s) d\boldsymbol{W^2}(s) \sum_{i=1}^n \int_0^t \left(e^{\alpha U_i(s)} - 1\right) d\tilde{H}_i(s)\Bigg). \end{aligned} \tag{3.34}$$

By Lemma 12 of Hu et al. (2005) and the boundness of $U_1, \ldots, U_n$, the argument of the stochastic exponential appearing in (3.34) is a martingale of bounded mean oscillation (BMO). Then by Kazamaki's criterion, $M^{\boldsymbol{\pi}^*}(\cdot)$ is a martingale; see for instance Kazamaki (2006). Consequently, we get

$$\begin{aligned} V(t,0,C) &= \sup_{\boldsymbol{\pi}\in\mathcal{A}(t)} E\left(-e^{-\alpha(X_t^{\boldsymbol{\pi}}(T)-C)}\Big|\mathcal{G}_t\right) \\ &= \sup_{\boldsymbol{\pi}\in\mathcal{A}(t)} E\left(-e^{-\alpha(X_t^{\boldsymbol{\pi}}(T)-Y(T))}\Big|\mathcal{G}_t\right) \\ &\leq E\left(-e^{\alpha Y(t)}\Big|\mathcal{G}_t\right) \\ &= E\left(-e^{-\alpha\left(X_t^{\boldsymbol{\pi}^*}(T)-C\right)}\Bigg|\mathcal{G}_t\right). \end{aligned}$$

It proves Theorem 3.4.2(ii). ■

Remark 3.4.2 *By referring to Theorem 3.4.2, we point out that the compactness assumption on* $\boldsymbol{K}$ *can be removed at the cost of some mathematical complexity. Indeed, by following Morlais (2009), we may construct a solution* $(Y, \boldsymbol{Z_1}, \boldsymbol{Z_2}, \boldsymbol{U})$ *to the BSDE (3.31) as the limit of a sequence of approximating solutions. Specifically, for* $k \geq 1$*, the k-th approximating solution* $(Y^k, \boldsymbol{Z_1}^k, \boldsymbol{Z_2}^k, \boldsymbol{U}^k)$ *solves the BSDE (3.31) when the closed set of Definition 3.4.3 equals* $\boldsymbol{K} \cap [-k,k]^d$*. Then we may show that*

$$E\left(\sup_{t\in[0,T]}\left|Y - Y^k\right|\right) + \left|\boldsymbol{Z_1} - \boldsymbol{Z_1}^k\right|_{\mathcal{L}^{2,d}(\boldsymbol{W^1})} + \left|\boldsymbol{Z_2} - \boldsymbol{Z_2}^k\right|_{\mathcal{L}^{2,m}(\boldsymbol{W^2})} + \left|\boldsymbol{U} - \boldsymbol{U}^k\right|_{\mathcal{L}^{2,n}_{\gamma}(\boldsymbol{H})} \to 0,$$

where $(Y, \boldsymbol{Z_1}, \boldsymbol{Z_2}, \boldsymbol{U}) \in \mathcal{L}^\infty \times \mathcal{L}^{2,d}(\boldsymbol{W^1}) \times \mathcal{L}^{2,m}(\boldsymbol{W^2}) \times \mathcal{L}^{2,n}_\gamma(\boldsymbol{H})$ *solves the BSDE (3.31).*

Example 3.4.5 *We aim to describe how to numerically apply Theorem 3.4.2. To that end, we assume a financial market consisting of one risky asset and an individual life insurance policy, i.e.,* $d = n = 1$. *In this setting, we make the following change of variable in the BSDE (3.31)*

$$\begin{aligned}
\hat{Y}(t) &= e^{\alpha Y(t)}, \\
\hat{Z}_1(t) &= \alpha Y(t) Z_1(t), \\
\hat{Z}_2(t) &= \alpha Y(t) Z_2(t), \\
\hat{U}(t) &= Y(t-)\left(e^{\alpha U(t)} - 1\right).
\end{aligned}$$

It is not difficult to see that $\hat{Y}(\cdot)$, $\hat{Z}_1(\cdot)$, $\hat{Z}_2(\cdot)$, $\hat{U}(\cdot)$ *satisfy*

$$\begin{aligned}
\hat{Y}(t) &= e^{\alpha C} + \int_t^T \hat{f}(s, \hat{Y}(s), \hat{Z}_1(s), \hat{U}(s)) ds - \int_t^T \hat{Z}_1(s) dW^1(s) \\
&- \int_t^T \hat{Z}_2(s) dW^2(s) - \int_t^T \hat{U}(s) dH(s), \quad t \in [0, T],
\end{aligned} \tag{3.35}$$

where

$$\alpha \hat{f}(t, y, z_1, u) = \min_{\boldsymbol{\pi} \in \mathcal{A}} \left\{ -\alpha\pi\left(\mu(t) y + \sigma(t) z_1\right) + \frac{1}{2}\alpha^2 \left(\pi\sigma(t)\right)^2 y \right\} + u\gamma(t), \quad t \in [0, T].$$

Moreover, we suppose that the terminal condition $\hat{C} = e^{\alpha C}$ *has the form*

$$\hat{C} = \hat{c}(\hat{X}(T)),$$

where $\hat{c}$ *is a measurable function, and* $\hat{X}(\cdot)$ *solves the following forward stochastic differential equation (FSDE)*

$$\begin{aligned}
\hat{X}(t) &= x + \int_0^t g(s, \hat{X}(s)) ds + \int_0^t h^1(s, \hat{X}(s)) dW^1(s) \\
&+ \int_0^t h^2(s, \hat{X}(s)) dW^2(s) + \int_0^t v(s, \hat{X}(s-)) dH(s), \quad t \in [0, T].
\end{aligned} \tag{3.36}$$

As usual, we assume regularity conditions on g, h^1, h^2, *and* v, *ensuring that equation (3.36) admits a unique solution. By numerically solving the BSDE (3.35) and then pulling back to the original variables* $Y(\cdot)$, $Z_1(\cdot)$, $Z_2(\cdot)$, $U(\cdot)$, *we may approximate* $Y(0)$. *From Theorem 1 of Kharroubi and Lim (2015), the solution to the FSDE (3.36)*

$\hat{X}(\cdot)$ can be written as

$$\hat{X}(t) = \hat{X}_0(t) 1_{(t<\tau)} + \hat{X}_1(t,\tau) 1_{(\tau \le t)}, \quad t \in [0,T],$$

where $\hat{X}_0(\cdot)$ and $\hat{X}_1(\cdot)$ solve the following FSDE

$$\begin{aligned}\hat{X}_1(t,\varphi) &= x + \int_0^t g(s,\hat{X}_1(s,\varphi))ds + \int_0^t h^1(s,\hat{X}_1(s,\varphi))dW^1(s) \\ &+ \int_0^t h^2(s,\hat{X}_1(s,\varphi))dW^2(s) + v(\varphi,\hat{X}_1(\varphi-,\varphi))1_{(\varphi\le t),}\end{aligned} \tag{3.37}$$

$$\begin{aligned}\hat{X}_0(t) &= x + \int_0^t g(s,\hat{X}_0(s))ds + \int_0^t h^1(s,\hat{X}_0(s))dW^1(s) \\ &+ \int_0^t h^2(s,\hat{X}_0(s))dW^2(s),\end{aligned} \tag{3.38}$$

where $\varphi, t \in [0,T]$. Also, the process $\hat{Y}(\cdot)$ can be decomposed as

$$\hat{Y}(t) = \hat{Y}_0(t) 1_{(t<\tau)} + \hat{Y}_1(t,\tau) 1_{(\tau \le t)}, \quad t \in [0,T],$$

where $\hat{Y}_0(\cdot)$ and $\hat{Y}_1(\cdot)$ solve the following BSDE

$$\begin{aligned}\hat{Y}_1(t,\varphi) &= \hat{c}(\hat{X}^1(T,\varphi)) + \int_t^T \hat{f}(s,\hat{Y}_1(s,\varphi),\hat{Z}_{1,1}(s,\varphi),0)ds \\ &- \int_t^T \hat{Z}_{1,1}(s,\varphi)dW^1(s) - \int_t^T \hat{Z}_{1,2}(s,\varphi)dW^2(s),\end{aligned} \tag{3.39}$$

$$\begin{aligned}\hat{Y}_0(t) &= \hat{c}(\hat{X}^0(T)) + \int_t^T \hat{f}(s,\hat{Y}_0(s),\hat{Z}_{0,1}(s),\hat{Y}_1(s,s)-\hat{Y}_0(s))ds \\ &- \int_t^T \hat{Z}_{0,1}(s)dW^1(s) - \int_t^T \hat{Z}_{0,2}(s)dW^2(s),\end{aligned} \tag{3.40}$$

where $0 \le \varphi \le t \le T$. For an integer $m \ge 1$, define $\Delta = \frac{T}{m}$. Besides, let $t_0 = 0 < t_1 < \cdots < t_m = T$ be the m-regular partition of the interval $[0,T]$, i.e., $t_i = i\Delta$. For $\varphi \in [0,T]$, define $t^m(\varphi) = \max\{t_i : t_i \le \varphi\}$. Now, we consider the typical backward Euler scheme as a discrete-time approximation to the infinite-dimensional system in relations (3.37), (3.38), (3.39) and (3.40). On the one hand, the FSDE in (3.37) and

(3.38) is approximated by means of

$$\begin{aligned}
\hat{X}_0^m(0) &= x, \\
\hat{X}_0^m(t_{i+1}) &= \hat{X}_0^m(t_i) + g(t_i, \hat{X}_0(t_i))\Delta + h^1(t_i, \hat{X}_0(t_i))\Delta W_{i+1}^1 \\
&+ h^2(t_i, \hat{X}_0(t_i, \varphi))\Delta W_{i+1}^2;
\end{aligned}$$

$$\begin{aligned}
\hat{X}_1^m(0, t^m(\varphi)) &= x + v(0,x)1_{(t^m(\varphi)=0)}, \\
\hat{X}_1^m(t_{i+1}, t^m(\varphi)) &= \hat{X}_1^m(t_i, t^m(\varphi)) + g(t_i, \hat{X}_1(t_i, \varphi))\Delta + h^1(t_i, \hat{X}_1(t_i, \varphi))\Delta W_{i+1}^1 \\
&+ h^2(t_i, \hat{X}_1(t_i, \varphi))\Delta W_{i+1}^2 + v(t_i, \hat{X}_1(t_i, \varphi))1_{(t^m(\varphi)=t_{i+1})},
\end{aligned}$$

where $\Delta W_{i+1}^1 = W_{i+1}^1 - W_i^1$ *and* $\Delta W_{i+1}^2 = W_{i+1}^2 - W_i^2$*. On the other hand, the BSDE in (3.39) and (3.40) is approximated as follows:*

$$\begin{aligned}
\hat{Y}_1^m(T, t^m(\varphi)) &= \hat{c}(\hat{X}^1(T, \varphi)), \\
\hat{Y}_1^m(t_i, t^m(\varphi)) &= E\left(\hat{Y}_1^m(t_{i+1}, t^m(\varphi))\middle|\mathcal{F}_{t_i}\right) + \hat{f}(t_i, \hat{Y}_1^m(t_i, t^m(\varphi)), \hat{Z}_{1,1}^m(t_i, t^m(\varphi)), 0)\Delta, \\
\hat{Z}_{1,1}^m(t_i, t^m(\varphi)) &= \frac{1}{\Delta}E\left(\hat{Y}_1^m(t_{i+1}, t^m(\varphi))\Delta W_{i+1}^1\middle|\mathcal{F}_{t_i}\right), \\
\hat{Z}_{1,2}^m(t_i, t^m(\varphi)) &= \frac{1}{\Delta}E\left(\hat{Y}_1^m(t_{i+1}, t^m(\varphi))\Delta W_{i+1}^2\middle|\mathcal{F}_{t_i}\right), \\
t^m(\varphi) &\leq t_i,\ i = 1, \ldots, m;
\end{aligned}$$

$$\begin{aligned}
\hat{Y}_0^m(T) &= \hat{c}(\hat{X}^0(T)), \\
\hat{Y}_0^m(t_i) &= E\left(\hat{Y}_0^m(t_{i+1})\middle|\mathcal{F}_{t_i}\right) + \hat{f}(t_i, \hat{Y}_0^m(t_i), Z_{0,1}^m(t_i), Y_1^m(t_i, t_i) - Y_0^m(t_i))\Delta, \\
\hat{Z}_{0,1}^m(t_i) &= \frac{1}{\Delta}E\left(\hat{Y}_0^m(t_{i+1})\Delta W_{i+1}^1\middle|\mathcal{F}_{t_i}\right), \\
\hat{Z}_{0,2}^m(t_i) &= \frac{1}{\Delta}E\left(\hat{Y}_0^m(t_{i+1})\Delta W_{i+1}^2\middle|\mathcal{F}_{t_i}\right).
\end{aligned}$$

Under some regularity conditions, including the Lipschitzity of generator $\hat{f}$, Kharroubi and Lim (2015) establish the convergence of the described approximation scheme. However, this scheme is not implementable because the involved conditional expectations have to be estimated. Under the Lipschitz condition, it is known that the process $\hat{Y}_0(\cdot)$ and $\hat{Y}_1(\cdot)$ can be expressed as functions of $\hat{X}_0(\cdot)$ and $\hat{X}_1(\cdot)$, respectively. In consequence, for the estimation purpose, we may employ the method from Gobet et al. (2005), based on iterative regressions on function bases, whose coefficients are evaluated using Monte Carlo simulations.

Chapter 4

Optimal consumption, investment, and life insurance purchase: a state-dependent utility approach

4.1 Notation and abbreviations

In the following table, we provide a summary of notation and abbreviations used in this chapter:

$\mathbf{1}$	the n-dimensional vector with all components being 1
1_A	the indicator of an event A
d	a positive integer
E	mathematical expectation
$E(X\|\mathcal{F})$	conditional expectation of X given $\mathcal{F}$
$f(T^-)$	$\lim_{t\uparrow T} f(t)$, where $f(\cdot)$ is a function with left limit at T
$\ker(\sigma)$	the null space of a matrix σ
$M^\perp$	the orthogonal complement of a subspace M
n	a positive integer
$\\|(x_1,\ldots,x_d)\\|$	the Euclidean norm, $\left(\sum_{i=1}^d x_i^2\right)^{\frac{1}{2}}$ for $(x_1,\ldots,x_d)\in\mathbb{R}^d$
P	probability measure
proj_M	the orthogonal projection onto a subspace M
$\mathbb{R}$	$(-\infty,\infty)$
RCLL	right continuous with left limits
$\vee_{\theta\in\Theta}\mathcal{F}_\theta$	the smallest sigma-algebra containing every $\mathcal{F}_\theta$ for $\theta\in\Theta$
$x\wedge y$	$\min\{x,y\}$

$x \vee y$	$\max\{x, y\}$
x^-	$-(x \wedge 0)$

4.2 Introduction

The problem of optimal consumption and investment for a "small investor," whose actions do not influence the market prices, is at the core of portfolio management. The modern treatment of this problem begins with the seminal work of Merton (1969, 1971). Expressly, let $c(\cdot)$ and $X(\cdot)$ denote the investor's consumption and wealth processes, respectively. In Merton's model, the investor seeks to maximize

$$E\left(\int_0^T U_1(s, c(s))\,dt + U_3(X(T))\right),$$

where $T > 0$, and $U_1(\cdot,\cdot)$ and $U_3(\cdot)$ represent the individual's preferences for consumption and terminal wealth, respectively. Assuming HARA (Hyperbolic Absolute Risk Aversion) utility functions, Merton explicitly solves the optimal choice problem above, under uncertainty. Subsequently, some authors extend Merton's work by weakening the assumptions in terms of the utility functions and the model of asset prices. For instance, modeling asset prices as semi-martingales, Pliska (1986) decomposes the optimal choice problem into two sub-problems and uses a martingale technique to solve them. Moreover, Karatzas et al. (1987) and Cox and Huang (1989) independently employ martingale techniques for showing how to decompose the Hamilton–Jacobi–Bellman equation arising from the optimization problem into linear partial differential equations.

Departing from Merton's model, some authors have constructed quantitative models to understand the determinants of life insurance demand for an individual, which decides under uncertainty. For instance, aiming to include mortality risk, Richard (1975) generalizes Merton's work by assuming that the investor's remaining lifetime follows an arbitrary but known distribution, which is supported on a bounded interval. In Richard's model, the individual may acquire whole life insurance by paying premiums at the rate $p(\cdot)$ continuously. Specifically, the insurer pays to the policyholder $p(t)/\lambda(t)$ in case of death at time t, where $\lambda(\cdot)$ is a function stipulated in the insurance contract. In respect to the optimal choice problem, the individual's objective is to maximize

$$E\left(\int_0^\tau U_1(s, c(s))\,ds + U_2(\tau, Z(\tau))\right),$$

where $\tau \in [0,T]$ represents the individual's uncertain lifetime, $Z(\cdot)$ represents the individual's legacy process, and $U_2(\cdot,\cdot)$ represents the individual's time-dependent preferences for legacy. By using dynamic programming techniques, Richard (1975) solves the optimization problem with CRRA (Constant Relative Risk Aversion) utility functions. Alternatively, Pliska and Ye (2007) develop a model in which the individual's remaining lifetime is a random variable supported on the interval $(0,\infty)$, and the constant T represents the retirement time. In this context, the individual's objective is to maximize

$$E\left(\int_0^{T\wedge\tau} U_1(s,c(s))\,ds + U_2(Z(\tau),\tau)\,1_{(\tau\leq T)} + U_3(X(T))\,1_{(\tau>T)}\right).$$

By relying on the dynamic programming approach, Pliska and Ye find analytical solutions to the optimization problem with CRRA utility functions.

Since then, several variants of the optimal choice problem have been considered in the actuarial science literature. See for instance Huang et al. (2008), Nielsen and Steffensen (2008), Duarte et al. (2011, 2014), Pirvu and Zhang (2012), Shen and Wei (2014), Guambe and Kufakunesu (2015), and Kronborg and Steffensen (2015). In particular, Nielsen and Steffensen (2008) and Kronborg and Steffensen (2015) assume that the individual's objective is to maximize

$$E\left(\int_0^{T\wedge\tau} U_1(s,c(s))\,ds + U_2(D(\tau),\tau)\,1_{(\tau\leq T)} + U_3(X(T))\,1_{(\tau>T)}\right),$$

where $D(\cdot)$ represents the insured sum to be paid if the policyholder dies prematurely, without taking into account the current wealth at death. Precisely, Nielsen and Steffensen (2008) derive optimal strategies under minimum and maximum constraints on the insured sum, and Kronborg and Steffensen (2015) compute optimal strategies with a surrender option guarantee.

Despite the lots of literature arising from the pioneering Merton's work, authors such as Londoño (2009) have pointed out some disadvantages of adopting this model. To obtain numerical solutions, it is often necessary to solve partial differential equations numerically. Notwithstanding, numerical methods for partial differential equations are challenging to implement in high dimensions. Also, the classical Merton's approach is not consistent with empirical data. Some authors document this lack of consistency under the name of several puzzles: the equity premium puzzle (see Mehra and Prescott (1985)), the risk-free rate puzzle (see Weil (1989)), and the risk aversion puzzle (see Jackwerth (2000)).

As an alternative, Londoño (2009) proposes to face the optimal choice problem by considering state-dependent utilities. In Londoño's framework, the individual's utility functions reflect his preferences for future flows of money, which are valued conforming to the state of the market (state-dependent). In other words, people tend to value money conforming to their social and economic context, instead of just looking at quantitative values. For instance, people appreciate more having enough money to pay off their debts in recession times than buying luxuries in boom times. In this context, the individual's objective is to maximize

$$E\left(\int_0^T U_1\left(s, H\left(s\right) c\left(s\right)\right) dt + U_3\left(H\left(T\right) X\left(T\right)\right)\right),$$

where $H(\cdot)$ is the state price density process. By considering the valuation and arbitrage theory presented in Londoño (2008), Londoño (2009) develops a martingale methodology to obtain complete solutions to the problem in a quite general setting.

In this work, we situate in a state-complete financial market; see Londoño (2008) for further details. In addition, we follow Londoño (2009) to consider an agent which ranks risky positions according to his expected state-dependent utilities. Thus, we generalize the work of Londoño (2009) by solving the problem of optimal consumption, investment, and life insurance acquisition. Specifically, we tackle the optimization problem considered by Nielsen and Steffensen (2008) and Kronborg and Steffensen (2015). To that purpose, we adopt the martingale methodology of Karatzas et al. (1987) and Cox and Huang (1989, 1991). We obtain full solutions for the optimal choice problem under minimum and maximum constrains on the life insurance purchase in a general set-up, which includes several utility functions employed in the literature.

4.3 Financial market

For the sake of completeness, we state some usual assumptions for financial markets in which asset prices evolve conforming to a Brownian filtration. Hereafter, we try to follow as closely as possible the notation in Karatzas and Shreve (1998).

From now on, we work on a probability space $(\Omega, \mathcal{F}, P)$, which accommodates a d-dimensional Brownian motion $\boldsymbol{W}(\cdot) = (W_1(\cdot), \ldots, W_d(\cdot))^\intercal$. In addition, we suppose that all economic activity takes place on a finite horizon $[0, T]$, where $T > 0$. Let $\left\{\mathcal{F}_t^{\boldsymbol{W}}\right\}$ denote the filtration generated by $\boldsymbol{W}(\cdot)$, and let $\mathcal{N}$ represent the P-null subsets of $\mathcal{F}_T^{\boldsymbol{W}}$. Subsequently, we shall use the augmented filtration $\{\mathcal{F}_t\} = \left\{\mathcal{F}_t^{\boldsymbol{W}} \vee \mathcal{N}\right\}$.

We introduce a bounded from below *risk-free rate process* $r(\cdot)$, an n-dimensional *mean rate of return process* $\boldsymbol{b}(\cdot)$, an n-dimensional *dividend rate process* $\boldsymbol{\delta}(\cdot)$, and an $(n \times d)$-matrix-valued *volatility process* $\boldsymbol{\sigma}(\cdot)$. In particular, we assume that $r(\cdot)$, $\boldsymbol{b}(\cdot)$, $\boldsymbol{\delta}(\cdot)$, and $\boldsymbol{\sigma}(\cdot)$ are progressively measurable processes satisfying the integrability condition

$$\int_0^T \left(|r(s)| + \|\boldsymbol{b}(s)\| + \|\boldsymbol{\delta}(s)\| + \sum_{i=1}^d \sum_{j=1}^d \sigma_{i,j}^2(s) \right) ds < \infty.$$

As usual, the *price of the money market process* $B(\cdot)$ evolves conforming to the equation

$$dB(t) = B(t) r(t) dt, \quad t \in [0,T].$$

As well, we consider n *stocks* with prices per share $S_1(t), \ldots, S_n(t)$ at time $t \in [0,T]$, such that $S_1(0), \ldots, S_n(0) > 0$. The processes $S_1(\cdot), \ldots, S_n(\cdot)$ are continuous and strictly positive solutions to the system of stochastic differential equations

$$dS_i(t) = S_i(t) \left[b_i(t) dt + \sum_{j=1}^d \sigma_{i,j}(t) dW_j(t) \right], \quad t \in [0,T], i = 1, \ldots, n.$$

We refer to the described financial market as $\mathcal{M} = (r(\cdot), \boldsymbol{b}(\cdot), \boldsymbol{\delta}(\cdot), \boldsymbol{\sigma}(\cdot), \boldsymbol{S}(0))$, where $\boldsymbol{S}(0) = (S_1(0), \ldots, S_n(0))$.

Given $\mathcal{M}$, define a *portfolio process* $(\pi_0(\cdot), \boldsymbol{\pi}(\cdot)) = (\pi_0(\cdot), \pi_1(\cdot), \ldots, \pi_n(\cdot))^\intercal$ to be an $\mathbb{R} \times \mathbb{R}^n$-valued and $\{\mathcal{F}_t\}$-progressively measurable process such that

$$\begin{aligned} \int_0^T \left(|\pi_0(s) + \boldsymbol{\pi}^\intercal(s) \mathbf{1}| \, |r(s)| \right) ds &< \infty, \\ \int_0^T |\boldsymbol{\pi}^\intercal(s) (\boldsymbol{b}(s) + \boldsymbol{\delta}(s) - r(s) \mathbf{1})| \, ds &< \infty, \qquad (4.1) \\ \int_0^T \|\boldsymbol{\pi}^\intercal(s) \boldsymbol{\sigma}(s)\|^2 ds &< \infty. \qquad (4.2) \end{aligned}$$

Accordingly, define the *gains process* $G(\cdot)$ associated to $(\pi_0(\cdot), \boldsymbol{\pi}(\cdot))$ by

$$\begin{aligned} G(t) = & \int_0^t (\pi_0(s) + \boldsymbol{\pi}^\intercal(s) \mathbf{1}) r(s) ds + \int_0^t \boldsymbol{\pi}^\intercal(s) (\boldsymbol{b}(s) + \boldsymbol{\delta}(s) - r(s) \mathbf{1}) ds \\ & + \int_0^t \boldsymbol{\pi}^\intercal(s) \boldsymbol{\sigma}(s) d\boldsymbol{W}(s), \quad t \in [0,T]. \end{aligned} \qquad (4.3)$$

In addition, define a *cumulative income process* $\{\Gamma(t); 0 \leq t \leq T\}$ to be a finite-variation RCLL process, which represents the cumulative wealth received by the investor throughout the time interval $[0,T]$. Let $\Gamma(\cdot) = \Gamma_+(\cdot) - \Gamma_-(\cdot)$ be the representation of $\Gamma(\cdot)$ as the difference between its positive and negative variation processes $\Gamma_+(\cdot)$ and $\Gamma_-(\cdot)$, respectively.

Following the standard literature, the *wealth process* $X(\cdot)$ associated with $(\Gamma(\cdot), \pi_0(\cdot), \boldsymbol{\pi}(\cdot))$ is

$$X(t) = \Gamma(t) + G(t), \quad t \in [0,T],$$

where $G(\cdot)$ is the gains process of (4.3). Also, the portfolio $(\pi_0(\cdot), \boldsymbol{\pi}(\cdot))$ is said to be $\Gamma(\cdot)$-financed provided that

$$X(t) = \pi_0(t) + \boldsymbol{\pi}^{\intercal}(t)\mathbf{1}, \quad t \in [0,T].$$

In particular, the portfolio $(\pi_0(\cdot), \boldsymbol{\pi}(\cdot))$ is said to be self-financed whenever $\Gamma(\cdot) = 0$.

Apparently, for a $\Gamma(\cdot)$-financed portfolio $(\pi_0(\cdot), \boldsymbol{\pi}(\cdot))$, the *wealth process* $X(\cdot)$ verifies

$$\begin{aligned} \gamma(t) X(t) &= \Gamma(0) + \int_{(0,t]} \gamma(s)\, d\Gamma(s) + \int_0^t \gamma(s) \boldsymbol{\pi}^{\intercal}(s) \boldsymbol{\sigma}(s)\, d\boldsymbol{W}(s) \\ &\quad + \int_0^t \gamma(s) \boldsymbol{\pi}^{\intercal}(s) (\boldsymbol{b}(s) + \boldsymbol{\delta}(s) - r(s)\mathbf{1})\, ds, \quad t \in [0,T], \end{aligned}$$

where $\gamma(\cdot)$ is defined by

$$\gamma(t) = \frac{1}{B(t)}, \quad t \in [0,T].$$

To emphasize that $X(\cdot)$ depends on $(\Gamma(\cdot), \boldsymbol{\pi}(\cdot))$, also refer to the wealth process as $X^{x,\boldsymbol{\pi},\Gamma}(\cdot)$ provided that $\Gamma(0) = x$.

Referring to the financial market $\mathcal{M}$, the *market price of risk process* $\boldsymbol{\theta}(\cdot)$ is defined as the unique progressively measurable and $\mathbb{R}^d$-valued process $\boldsymbol{\theta}(\cdot) \in \ker^{\perp}(\boldsymbol{\sigma}(\cdot))$ satisfying

$$(\boldsymbol{b}(t) + \boldsymbol{\delta}(t) - r(t)\mathbf{1}) - \mathrm{proj}_{\ker(\boldsymbol{\sigma}^{\intercal}(t))}(\boldsymbol{b}(t) + \boldsymbol{\delta}(t) - r(t)\mathbf{1}) = \boldsymbol{\sigma}(t)\boldsymbol{\theta}(t), \quad t \in [0,T].$$

To make sure that the process $\boldsymbol{\theta}(\cdot)$ is progressively measurable, we refer the reader to Section 1.4 of Karatzas and Shreve (1998). In the sequel, suppose that

$$\int_0^T \|\boldsymbol{\theta}(s)\|^2\, ds < \infty.$$

Therefore, it is possible to define the *state price density process* as follows:

$$H_0(t) = \gamma(t) Z_0(t), \quad t \in [0,T], \tag{4.4}$$

where

$$Z_0(t) = \exp\left(-\int_0^t \boldsymbol{\theta}^\intercal(s)\, d\boldsymbol{W}(s) - \frac{1}{2}\int_0^t \|\boldsymbol{\theta}(s)\|^2 ds\right), \quad t \in [0,T].$$

The name "state price density process" is usually given to the process defined by relation (4.4) when the financial market is standard, see for instance Karatzas and Shreve (1998). In that case, the process $Z_0(\cdot)$ is a martingale, and $Z_0(T)$ is indeed a density. However, in this setting, we allow that $E(Z_0(T)) < 1$.

4.3.1 State tameness and state arbitrage

In this section, we stipulate our notions of tameness and arbitrage by following Londoño (2008).

Let $\Gamma(\cdot)$ be a cumulative income process. Considering a $\Gamma(\cdot)$-financed portfolio process $(\pi_0(\cdot), \boldsymbol{\pi}(\cdot))$, it is said that $(\pi_0(\cdot), \boldsymbol{\pi}(\cdot))$ is $\Gamma(\cdot)$-*state-tame* provided that the associated process

$$H_0(\cdot) X(\cdot) - \int_{(0,\cdot]} H_0(s)\, d\Gamma(s)$$

is uniformly bounded from below. In particular, denominate *state-tame* to any self-financed portfolio process $(\pi_0(\cdot), \boldsymbol{\pi}(\cdot))$, whose discounted gains process $H_0(\cdot) G(\cdot)$ is uniformly bounded from bellow. In a financial market $\mathcal{M}$, a state-tame portfolio process $(\pi_0(\cdot), \boldsymbol{\pi}(\cdot))$ is said to be an *state-arbitrage opportunity* whenever the associated gains process $G(\cdot)$ satisfies $H_0(T) G(T) \geq 0$ almost surely, and $H_0(T) G(T) > 0$ with positive probability. A financial market $\mathcal{M}$ in which no such arbitrage opportunities exist is said to be *state-arbitrage-free.*

The next theorem is a characterization of the nonexistence of state-arbitrage opportunities. For a proof, we refer the reader to Londoño (2008).

Theorem 4.3.1 *A market $\mathcal{M}$ is state-arbitrage-free if and only if the market price of risk $\boldsymbol{\theta}(\cdot)$ satisfies*

$$\boldsymbol{b}(t) + \boldsymbol{\delta}(t) - r(t)\mathbf{1} = \boldsymbol{\sigma}(t)\boldsymbol{\theta}(t), \quad t \in [0,T]. \tag{4.5}$$

Remark 4.3.1 *Provided that $\boldsymbol{\theta}(\cdot)$ satisfies relation (4.5), the wealth process $X(\cdot)$, associated with $(\Gamma(\cdot), \pi_0(\cdot), \boldsymbol{\pi}(\cdot))$, can be rewritten as*

$$\begin{aligned} H_0(t) X(t) &= x + \int_{(0,t]} H_0(s)\, d\Gamma(s) \\ &+ \int_0^t H_0(s) \left(\boldsymbol{\sigma}(s)^\intercal \boldsymbol{\pi}(s) - X(s) \boldsymbol{\theta}(s) \right)^\intercal d\boldsymbol{W}(s), \quad t \in [0,T]. \end{aligned} \tag{4.6}$$

4.3.2 Completeness of financial markets

We proceed to present our notion of completeness. Let $\mathcal{M}$ be a state-arbitrage-free financial market.

An *European state-contingent claim (ESCC),* with expiration date T, is any RCLL process of finite variation $\Gamma(\cdot)$, such that $-\int_{(0,T]} H_0(s)\, d\Gamma_-(s)$ is bounded from below and

$$E\left(\int_{(0,T]} H_0(s) \left(d\Gamma_-(s) + d\Gamma_+(s) \right) \right) < \infty.$$

In addition, an European state-contingent claim $\Gamma(\cdot)$, with expiration date T, is called *attainable* if there exists a $(-\Gamma)$-state-tame portfolio process $\boldsymbol{\pi}(\cdot)$ with

$$X^{x,\boldsymbol{\pi},-\Gamma}\left(T^-\right) = \Gamma(T) - \Gamma\left(T^-\right),$$

where

$$x = E\left(\int_{(0,T]} H_0(s)\, d\Gamma(s) \right).$$

We say that a financial market $\mathcal{M}$ is *state-complete* if every European state-contingent claim is attainable.

The next theorem provides a characterization for state-complete financial markets.

Theorem 4.3.2 *A financial market $\mathcal{M}$ is state-complete if and only if the volatility matrix $\boldsymbol{\sigma}(t)$ has maximal rank for Lebesgue a.e. $t \in [0,T]$ almost surely.*

Proof of necessity. Let $\Gamma(\cdot)$ be an European state-contingent claim with expiration date $T > 0$. Define

$$\tilde{X}(t) = H_0^{-1}(t)\, E\left(\int_{(t,T]} H_0(s)\, d\Gamma(s) \middle| \mathcal{F}_t \right), \quad t \in [0,T].$$

Note that $\tilde{X}\left(T^-\right) = \Gamma(T) - \Gamma\left(T^-\right)$. From the standard representation theorem for martingales as stochastic integrals (see, e.g., Karatzas and Shreve (2012)), there exists an

$\{\mathcal{F}_t\}$-progressively measurable and $\mathbb{R}^d$-valued process $\boldsymbol{\varphi}(\cdot)$ satisfying $\int_0^T \|\boldsymbol{\varphi}(s)\|^2 ds < \infty$, such that

$$\tilde{X}(t) H_0(t) + \int_{(0,t]} H_0(s) d\Gamma(s) = x + \int_0^t \boldsymbol{\varphi}^\intercal(s) d\boldsymbol{W}(s), \quad t \in [0,T].$$

Denote by $\boldsymbol{\pi}(\cdot)$ the unique $\{\mathcal{F}_t\}$-progressively measurable and $\mathbb{R}^n$-valued process satisfying

$$\boldsymbol{\sigma}^\intercal(t) \boldsymbol{\pi}(t) = H_0^{-1}(t) \boldsymbol{\varphi}(t) + \tilde{X}(t) \boldsymbol{\theta}(t), \quad t \in [0,T].$$

The existence and uniqueness of such a process follow from the hypotheses. Also, it is easy to prove that $\boldsymbol{\pi}(\cdot)$ verifies (4.1) and (4.2). By defining $\pi_0(\cdot) = \tilde{X}(\cdot) - \boldsymbol{\pi}^\intercal(\cdot)\mathbf{1}$, we have

$$\begin{aligned} dH_0(t)\tilde{X}(t) &= \boldsymbol{\varphi}^\intercal(t) d\boldsymbol{W}(t) - H_0(t) d\Gamma(t), \quad t \in [0,T], \\ dZ_0^{-1}(t) &= Z_0^{-1}(t)\left(\boldsymbol{\theta}^\intercal(t) d\boldsymbol{W}(t) + \|\boldsymbol{\theta}(t)\|^2 dt\right), \quad t \in [0,T]. \end{aligned}$$

Thus, by applying the Itô's rule to the product $Z_0^{-1}(\cdot) H_0(\cdot) \tilde{X}(\cdot) = \gamma(\cdot)\tilde{X}(\cdot)$, we obtain that $\tilde{X}(\cdot)$ defines a wealth process with associated triple $(x, \boldsymbol{\pi}, -\Gamma)$. Because Γ is an European state-contingent claim,

$$H_0(\cdot)\tilde{X}(\cdot) + \int_{(0,\cdot]} H_0(s) d\Gamma^{fv}(s)$$

is uniformly bounded from below. In consequence, every European state-contingent claim is attainable. ■

Proof of sufficiency. Let $f : \mathbb{R}^{n\times d} \to \mathbb{R}^d$ be a bounded and Borel-measurable mapping, such that for every $\boldsymbol{\sigma} \in \mathbb{R}^{n\times d}$, $f(\boldsymbol{\sigma}) \in \ker(\boldsymbol{\sigma})$ and $f(\boldsymbol{\sigma}) \neq 0$ whenever $\ker(\boldsymbol{\sigma}) \neq \{0\}$. To substantiate the existence of such a function, see Karatzas and Shreve (1998). Let $\boldsymbol{\Psi}(\cdot)$ be the bounded and progressively measurable process defined by $\boldsymbol{\Psi}(t) = f(\boldsymbol{\sigma}(t))$, for all $t \in [0,T]$. For an arbitrary but fixed $k \in \{1,2,\ldots\}$, define the stopping time

$$\tau_k = \inf\left\{t : \int_0^t \boldsymbol{\Psi}^\intercal(s) d\boldsymbol{W}(s) \leq -k\right\} \wedge T,$$

and the finite variation process

$$\Gamma_k(t) = 1_{(t=T)} \frac{1}{H_0(T)} \int_0^{\tau_k} \boldsymbol{\Psi}^\intercal(s) d\boldsymbol{W}(s), \quad t \in [0,T].$$

By definition, $\Gamma_k(\cdot)$ is an European state-contingent claim. According to the completeness of the financial market, there exists a $(-\Gamma_k)$-state-tame portfolio process $\boldsymbol{\pi}_k(\cdot)$ with associated wealth $X_k(\cdot)$ such that

$$X_k^{0,\boldsymbol{\pi}_k,-\Gamma_k}\left(T^-\right) = \Gamma_k(T) - \Gamma_k\left(T^-\right).$$

From (4.6), for all $0 \leq t \leq T$, it holds that

$$H_0(t)X_k(t) + \int_{(0,t]} H_0(s)\,d\Gamma_k(s) = \int_0^t H_0(s)\left(\boldsymbol{\sigma}(s)^\intercal \boldsymbol{\pi}_k(s) - X_k(s)\boldsymbol{\theta}(s)\right)^\intercal d\boldsymbol{W}(s).$$

Also, by using that $\boldsymbol{\pi}_k(\cdot)$ finances $\Gamma_k(T)$ (in a bounded manner), one proves that the local martingale

$$\int_0^t H_0(s)\left(\boldsymbol{\sigma}(s)^\intercal \boldsymbol{\pi}_k(s) - X_k(s)\boldsymbol{\theta}(s)\right)^\intercal d\boldsymbol{W}(s)$$

is a true martingale taking the value $H_0(T)\Gamma_k(T)$ at $t = T$. Hence,

$$\begin{aligned}\int_0^t H_0(s)\left(\boldsymbol{\sigma}(s)^\intercal \boldsymbol{\pi}_k(s) - X_k(s)\boldsymbol{\theta}(s)\right)^\intercal d\boldsymbol{W}(s) &= \int_0^{t\wedge\tau_k} \boldsymbol{\Psi}^\intercal(s)\,d\boldsymbol{W}(s)\\ &= \int_0^t 1_{(0,\tau_k]}(s)\,\boldsymbol{\Psi}^\intercal(s)\,d\boldsymbol{W}(s).\end{aligned}$$

From the uniqueness of the representation theorem for martingales as stochastic integrals, for Lebesgue a.e. $t \in [0,T]$ almost surely on the set $\{\omega : \tau_k(\omega) = T\}$, we have

$$\begin{aligned}\boldsymbol{\Psi}(t) &= H_0(t)\left(\boldsymbol{\sigma}(t)^\intercal \boldsymbol{\pi}_k(t) - X_k(t)\boldsymbol{\theta}(t)\right) \in \ker^\perp(\boldsymbol{\sigma}(t)) \cap \ker(\boldsymbol{\sigma}(t))\\ &= \{0\}.\end{aligned}$$

Finally, by noting that $\Omega = \cup_{k\geq 0}\{\omega : \tau_k(\omega) = T\}$, we conclude that $\boldsymbol{\sigma}(t)$ has maximal rank for Lebesgue a.e. $t \in [0,T]$ almost surely. ■

4.4 Utility maximization

Let $\mathcal{M}$ be a state-arbitrage-free and state-complete financial market.

4.4.1 Model

In this work, we consider an individual whose remaining lifetime is modeled as a non-negative random variable τ, defined on the probability space $(\Omega, \mathcal{F}, P)$. Furthermore, we suppose that the individual owns a life insurance policy during the period $[0,T]$. Here, T represents the time of retirement decided beforehand by the policyholder.

As concerns the remaining lifetime, assume that τ is independent of the filtration $\{\mathcal{F}_t\}$ and continuously distributed. Moreover, suppose that the *hazard rate* of τ, which is denoted by $\lambda(\cdot)$, is a positive function on $[0,\infty)$ satisfying for some $t>0$

$$\int_0^t \lambda(s)\,ds < \infty$$

and

$$\int_0^\infty \lambda(s)\,ds = \infty.$$

In these terms, the *survival function* $\overline{F}(\cdot)$ and the *density function* $f(\cdot)$ associated with τ can be respectively expressed as

$$\begin{aligned}
\overline{F}(t) &= P(\tau > t) = \exp\left(-\int_0^t \lambda(s)\,ds\right), \quad t>0, \\
f(t) &= -\frac{d}{dt}\overline{F}(t) = \lambda(t)\exp\left(-\int_0^t \lambda(s)\,ds\right), \quad t>0.
\end{aligned}$$

For a technical purpose, assume that $\lambda(t)>0$ for all $t\in[0,T]$.

In respect to the life insurance product, we suppose that the policyholder has a pension account through which he performs all his financial interactions. Let $X(\cdot)$ denote the policyholder's wealth process or reserve process, and let $D(\cdot)$ represent the insured sum to be paid when the policyholder dies prematurely. The pension company maintains and invests the reserve, however, the policyholder can continuously choose his consumption, investment, and sum insured to be paid out upon death before time T. By selecting $D(\cdot)$, in the case of premature death at time $t\in[0,T]$, the policyholder agrees to hand over the amount of money $X(t)-D(t)$ to the pension company. Namely, the pension company keeps the reserve $X(t)$ for themselves and pays out $D(t)$ as life insurance. Thus, the actuarial risk premium rate to pay for the life insurance $D(\cdot)$ at time $t\in[0,T]$ is $\lambda(t)(D(t)-X(t))$.

To discuss in detail the dynamics of the reserve process $X(\cdot)$, consider the following definition. From now on, a *consumption process* $c(\cdot)$, a *life insurance process* $D(\cdot)$, and an *endowment process* $\epsilon(\cdot)$ denote non-negative and $\{\mathcal{F}_t\}$-progressively measurable processes satisfying

$$\int_0^T (c(s)+\lambda(s)D(s)+\epsilon(s))\,ds < \infty, \quad a.s.$$

Assume that the policyholder consumes conforming to a consumption process $c(\cdot)$, invests conforming to a portfolio process $\boldsymbol{\pi}(\cdot)$, purchases life insurance conforming to a life insurance process $D(\cdot)$, and receives continuously an endowment process $\epsilon(\cdot)$. Consequently, the dynamics of the wealth process $X(\cdot)$ at the state alive may be represented as

$$\begin{aligned} dX(t) = &(r(t)X(t) + \epsilon(t) - c(t) - \lambda(t)(D(t) - X(t)))\,dt \\ &+ \boldsymbol{\pi}^{\mathsf{T}}(t)\boldsymbol{\sigma}(t)(\boldsymbol{\theta}(t)\,dt + d\boldsymbol{W}(t)), \quad t \in [0,T]. \end{aligned} \tag{4.7}$$

The unique solution to the stochastic differential equation (4.7) is given by

$$\begin{aligned} \gamma^{\lambda}(t)X(t) &= x + \int_{(0,t]} \gamma^{\lambda}(s)(\epsilon(s) - c(s) - \lambda(s)D(s))\,ds \\ &\quad + \int_0^t \gamma^{\lambda}(s)\boldsymbol{\pi}^{\mathsf{T}}(s)\boldsymbol{\sigma}(s)(\boldsymbol{\theta}(s)\,ds + d\boldsymbol{W}(s)), \quad t \in [0,T], \end{aligned}$$

where $\gamma^{\lambda}(\cdot)$ denotes the *actuarial-adjusted discounting process* defined by

$$\gamma^{\lambda}(t) = \overline{F}(t)\gamma(t), \quad t \in [0,T].$$

By using the Itô's formula, notice that $X(\cdot)$ also satisfies

$$\begin{aligned} H_0^{\lambda}(t)X(t) &= x + \int_{(0,t]} H_0^{\lambda}(s)(\epsilon(s) - c(s) - \lambda(s)D(s))\,ds \\ &\quad + \int_0^t H_0^{\lambda}(s)\left(\boldsymbol{\sigma}(s)^{\mathsf{T}}\boldsymbol{\pi}(s) - X(s)\boldsymbol{\theta}(s)\right)^{\mathsf{T}} d\boldsymbol{W}(s), \quad t \in [0,T], \end{aligned}$$

where $H_0^{\lambda}(\cdot)$ denotes the *actuarial-adjusted state price density process defined* by

$$H_0^{\lambda}(t) = \overline{F}(t)H_0(t), \quad t \in [0,T].$$

Abusing of notation, let $X^{x,\Gamma,\boldsymbol{\pi}}(\cdot)$ represent hereafter the process in (4.6) by replacing the short rate process $r(\cdot)$ with the *actuarial-adjusted short rate process* $r(\cdot) + \lambda(\cdot)$, where $\Gamma(\cdot) = \int_0^{\cdot}(\epsilon(s) - c(s) - \lambda(s)D(s))\,ds$. Apparently, $X^{x,\Gamma,\boldsymbol{\pi}}(\cdot)$ equals the process $X(\cdot)$ described by relation (4.7).

4.4.2 Admissibility

Let $\epsilon(\cdot)$ be an endowment process such that $\int_0^T H_0(s)\epsilon(s)\,ds$ is bounded. Thus, define

$$x^{\epsilon} = -E\left(\int_0^{\tau\wedge T} H_0(s)\epsilon(s)\,ds\right).$$

By employing the Fubini–Tonelli theorem, it is easy to see that

$$x^{\epsilon} = -E\left(\int_0^{T} H_0^{\lambda}(s)\epsilon(s)\,ds\right).$$

In the following definition, we stipulate the set of strategies available for the policyholder. Our definition of admissibility is the same as in Nielsen and Steffensen (2008), Kwak and Lim (2014), and Shen and Wei (2016).

Definition 4.4.1 A consumption/life insurance/portfolio process triple $(c, D, \boldsymbol{\pi})$ is called *admissible at* $x \in \mathbb{R}$ and written $(c, D, \boldsymbol{\pi}) \in \mathcal{A}(x)$, if $X^{x,\Gamma,\boldsymbol{\pi}}(\cdot)$ satisfies

$$X^{x,\Gamma,\boldsymbol{\pi}}(t) + E\left(\int_t^T \frac{H_0^{\lambda}(s)}{H_0^{\lambda}(t)}\epsilon(s)\,ds\middle|\mathcal{F}_t\right) \geq 0, \quad t\in[0,T], \tag{4.8}$$

where $\Gamma(\cdot) = \int_0^{\cdot}(\epsilon(s) - c(s) - \lambda(s)D(s))\,ds$.

According to Definition 4.4.1, the agent is restricted to consumption/life insurance/portfolio strategies such that the balance of the pension account does not drop below the actuarial value of his future income flows. As Remark 4.4.1 substantiates, the policyholder's future income flows equals the conditional expectation appearing in relation (4.8).

Remark 4.4.1 *At first glance, the actuarial value of the agent's future income flows at time* $t \in [0,T]$ *is*

$$E\left(\int_t^{\tau\wedge T} \frac{H_0(s)}{H_0(t)}\epsilon(s)\,ds\middle|\mathcal{F}_t, \tau > t\right).$$

Notwithstanding, it holds that

$$E\left(\int_t^{\tau\wedge T} \frac{H_0(s)}{H_0(t)}\epsilon(s)\,ds\middle|\mathcal{F}_t, \tau > t\right) = E\left(\int_t^{T} \frac{H_0^{\lambda}(s)}{H_0^{\lambda}(t)}\epsilon(s)\,ds\middle|\mathcal{F}_t\right). \tag{4.9}$$

In fact, let $A \in \mathcal{F}_t$. Due to the independence between the random time τ and the sigma-algebra $\mathcal{F}_T$, the repetitive use of the Fubini–Tonelli theorem implies that

$$\begin{aligned} E\left(1_{(\tau>t)}1_A\int_t^{\tau\wedge T}\frac{H_0(s)}{H_0(t)}\epsilon(s)\,ds\right) &= E\left(\int_t^T 1_{(\tau>s)}1_A\frac{H_0(s)}{H_0(t)}\epsilon(s)\,ds\right)\\ &= \int_t^T E\left(1_{(\tau>s)}1_A\frac{H_0(s)}{H_0(t)}\epsilon(s)\right)ds\\ &= \int_t^T \overline{F}(s)E\left(1_A\frac{H_0(s)}{H_0(t)}\epsilon(s)\right)ds\\ &= E\left(1_A\int_t^T \overline{F}(s)\frac{H_0(s)}{H_0(t)}\epsilon(s)\,ds\right). \end{aligned}$$

Thus, relation (4.9) follows.

Remark 4.4.2 *Let us fix an admissible consumption/life insurance/portfolio process triple $(c, D, \boldsymbol{\pi}) \in \mathcal{A}(x)$. From (4.8), we obtain that the local martingale*

$$H_0^\lambda(\cdot)\,X^{x,\Gamma,\boldsymbol{\pi}}(\cdot) - \int_{(0,\cdot]} H_0^\lambda(s)\,d\Gamma(s)$$

is bounded from below, so it is a supermartingale. Indeed, for all $t \in [0,T]$, we have

$$\begin{aligned} & H_0^\lambda(t)\,X^{x,\Gamma,\boldsymbol{\pi}}(t) - \int_{(0,t]} H_0^\lambda(s)\,d\Gamma(s)\\ \geq\ & H_0^\lambda(t)\,X^{x,\Gamma,\boldsymbol{\pi}}(t) - \int_{(0,t]} H_0^\lambda(s)\,\epsilon(s)\,ds\\ =\ & H_0^\lambda(t)\,X^{x,\Gamma,\boldsymbol{\pi}}(t) + E\left(\int_t^T H_0^\lambda(s)\,\epsilon(s)\,ds\middle|\mathcal{F}_t\right) - E\left(\int_0^T H_0^\lambda(s)\,\epsilon(s)\,ds\middle|\mathcal{F}_t\right)\\ \geq\ & -E\left(\int_0^T H_0^\lambda(s)\,\epsilon(s)\,ds\middle|\mathcal{F}_t\right). \end{aligned}$$

Hence, it follows that the triple $(c, D, \boldsymbol{\pi})$ must satisfy the budget constraint

$$x \geq E\left(\int_0^T H_0^\lambda(s)\,(c(s)+\lambda(s)\,D(s)-\epsilon(s))\,ds + H_0^\lambda(T)\,X^{x,\Gamma,\boldsymbol{\pi}}\left(T^-\right)\right), \tag{4.10}$$

where $\Gamma(\cdot) = \int_0^\cdot (\epsilon(s) - c(s) - \lambda(s)\,D(s))\,ds$.

In reference to characterizing admissible strategies, we introduce the following result, which corresponds to hedging in complete financial markets. The proof of this result goes along the lines of the proof of Theorem 3.5 in Karatzas and Shreve (1998).

Proposition 4.4.1 *Let ξ be a non-negative and $\mathcal{F}_T$-measurable random variable such that*

$$E\left(H_0^{\lambda}(T)\xi\right) < \infty.$$

Let (c,D) be a consumption/life insurance process pair satisfying

$$E\left(\int_0^T H_0^{\lambda}(s)\,d\Gamma(s)\right) = x > 0,$$

where

$$\Gamma(s) = \int_0^{\cdot} (c(s) + \lambda(s)D(s) - \epsilon(s))\,ds + \xi 1_{(\cdot = T)}.$$

Then, there exists a portfolio process $\boldsymbol{\pi}(\cdot)$ such that $(c,D,\boldsymbol{\pi}) \in \mathcal{A}(x)$ and $X^{x,\boldsymbol{\pi},-\Gamma}\left(T^-\right) = \xi$. In particular, for $t \in [0,T]$, the corresponding wealth and portfolio processes are respectively given by

$$\begin{aligned} X^{x,\boldsymbol{\pi},-\Gamma}(t) &= \left(H_0^{\lambda}\right)^{-1}(t)\,E\left(\int_t^T H_0^{\lambda}(s)\,d\Gamma(s)\middle|\mathcal{F}_t\right), \\ \boldsymbol{\sigma}^{\mathsf{T}}(t)\,\boldsymbol{\pi}(t) &= \left(H_0^{\lambda}\right)^{-1}(t)\,\boldsymbol{\varphi}(t) + X^{x,\boldsymbol{\pi},-\Gamma}(t)\,\boldsymbol{\theta}(t), \end{aligned}$$

where $\boldsymbol{\varphi}(\cdot)$ is the integrand in the stochastic integral representation

$$M(t) = x + \int_0^t \boldsymbol{\varphi}^{\mathsf{T}}(s)\,d\boldsymbol{W}(s)$$

of the martingale

$$M(t) = E\left(\int_0^T H_0^{\lambda}(s)\,d\Gamma(s)\middle|\mathcal{F}_t\right).$$

Let $\underline{d} : [0,T] \to [0,\infty)$ and $\overline{d} : [0,T] \to [0,\infty]$ be continuous functions such that $\underline{d}(t) \leq \overline{d}(t), \quad t \in [0,T]$. The role of the functions $\underline{d}(\cdot)$ and $\overline{d}(\cdot)$ is to impose minimum-maximum constraints on the life insurance process $D(\cdot)$. Accordingly, the set of admissible strategies available to the agent modifies to

$$\mathcal{B}(x) = \left\{(c,D,\boldsymbol{\pi}) \in \mathcal{A}(x) : \quad t \in [0,T],\, D(t) \in \left[\underline{d}(t), \overline{d}(t)\right]\right\}.$$

4.4.3 The problem

Now, we introduce a key ingredient in the formulation of the optimization problem considered in this paper: the concept of utility function.

Definition 4.4.2 Consider a continuously differentiable function $U : (0,\infty) \to \mathbb{R}$, which is strictly increasing and strictly concave with $U'(\infty) = \lim_{x\to\infty} U'(x) = 0$ and $U'\left(0^+\right) = \lim_{x\to 0^+} U'(x) = \infty$. A function having these properties will be called *utility function.*

For every utility function $U(\cdot)$, denote by $I(\cdot)$ the inverse of the derivative $U'(\cdot)$. This function is strictly decreasing and map $(0,\infty)$ onto itself with $I(\infty) = U'(\infty) = 0$ and $I\left(0^+\right) = U'\left(0^+\right) = \infty$. The *convex dual* of $U(\cdot)$ is the function

$$\tilde{U}(y) = \sup_{x\in(0,\infty)} \{U(x) - xy\}, \quad y \in \mathbb{R}. \tag{4.11}$$

Evidently, $\tilde{U}(\cdot)$ verifies

$$\tilde{U}(y) = U(I(y)) - yI(y), \quad y \in (0,\infty). \tag{4.12}$$

Since the agent's preferences may vary through time, we introduce the notion of time-dependent utility functions.

Definition 4.4.3 Consider a function $U : [0,T] \times (0,\infty) \to \mathbb{R}$ of class $C^{0,1}$ on its domain, such that for every $t \in [0,T]$, $U(t,\cdot)$ is a utility function in the sense of Definition 4.4.2. A function fulfilling these properties is called a *time-dependent utility function.* For every $t \in [0,T]$, denote by $I(t,\cdot)$ the inverse of $U'(t,\cdot)$ and by $\tilde{U}(t,\cdot)$ the convex dual of $U(t,\cdot)$.

Definition 4.4.4 *Let $U_1(\cdot,\cdot)$, $U_2(\cdot,\cdot)$ be two time-dependent utility functions as in Definition 4.4.3, and let $U_3(\cdot)$ be a utility function as in Definition 4.4.2. If $U_2(\cdot,\cdot)$ is non-negative or non-positive, refer to the triple (U_1, U_2, U_3) as a preference structure.*

Let (U_1, U_2, U_3) be a preference structure representing the state-dependent attitude towards the risk of the economic agent. At this point, we are ready to introduce the optimization problem which we aim to solve.

Problem 4.4.1 *Find an optimal triple $(c, D, \boldsymbol{\pi}) \in \mathcal{C}(x)$ for the problem*

$$\begin{aligned} V(x) = \sup_{(c,D,\boldsymbol{\pi})\in\mathcal{C}(x)} E\Bigg(&\int_0^{\tau\wedge T} U_1(s, H_0(s)c(s))\,ds + U_2(\tau, H_0(\tau)D(\tau))\,1_{(\tau\le T)} \\ &+ U_3\left(H_0(T)X^{x,\boldsymbol{\pi},\Gamma}\left(T^-\right)\right)1_{(\tau>T)}\Bigg), \end{aligned} \tag{4.13}$$

where

$$\mathcal{C}(x) = \left\{ (c, D, \boldsymbol{\pi}) \in \mathcal{B}(x) : E\left(\int_0^{\tau \wedge T} U_1^-(s, H_0(s)\, c(s))\, ds + U_2^-(\tau, H_0(\tau)\, D(\tau))\, 1_{(\tau \leq T)} \right.\right.$$
$$\left.\left. + U_3^-\left(H_0(T)\, X^{x, \boldsymbol{\pi}, \Gamma}\left(T^-\right)\right)\right) 1_{(\tau > T)} < \infty \right\}. \tag{4.14}$$

By means of Remark 4.4.3 below, we point out that the value function (4.13) and the set of admissible solutions (4.14) admit a convenient representation, which allows to employ a martingale methodology in solving Problem 4.4.1.

Remark 4.4.3 *The value function and the set of admissible solutions for Problem 4.4.1 can be equivalently rewritten as follows*

$$V(x) = \sup_{(c, D, \boldsymbol{\pi}) \in \mathcal{C}(x)} E\left(\int_0^T \left(\overline{F}(s)\, U_1(s, H_0(s)\, c(s)) + f(s)\, U_2(s, H_0(s)\, D(s)) \right) ds \right.$$
$$\left. + \overline{F}(T)\, U_3\left(H_0(T)\, X^{x, \boldsymbol{\pi}, \Gamma}\left(T^-\right)\right)\right), \tag{4.15}$$

$$\mathcal{C}(x) = \left\{ (c, D, \boldsymbol{\pi}) \in \mathcal{B}(x) : E\left(\int_0^T \left(\overline{F}(s)\, U_1^-(s, H_0(s)\, c(s)) + f(s)\, U_2^-(s, H_0(s)\, D(s)) \right) ds \right.\right.$$
$$\left.\left. + \overline{F}(T)\, U_3^-\left(H_0(T)\, X^{x, \boldsymbol{\pi}, \Gamma}\left(T^-\right)\right)\right) < \infty \right\}. \tag{4.16}$$

For mathematical convenience, henceforward, when referring to the value function and the set of admissible strategies for Problem 4.4.1, we consider its representations given by relations (4.15) and (4.16), respectively.

To check the validity of the statements in Remark 4.4.3, let $(c, D, \boldsymbol{\pi}) \in \mathcal{C}(x)$. To prove relation (4.15), we show the following equalities

$$\begin{aligned} & E\left(\int_0^{\tau \wedge T} U_1(s, H_0(s)\, c(s))\, ds \right) \\ = \ & E\left(\int_0^T \overline{F}(s) U_1(s, H_0(s)\, c(s))\, ds \right), \end{aligned} \tag{4.17}$$

$$\begin{aligned}
& E\left(U_2\left(\tau, H_0\left(\tau\right) D\left(\tau\right)\right) 1_{(\tau\leq T)}\right) \\
= \;& E\left(\int_0^T f(s) U_2\left(s, H_0\left(s\right) D\left(s\right)\right) ds\right),
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
& E\left(U_3\left(H_0\left(T\right) X^{x,\boldsymbol{\pi},\Gamma}\left(T^-\right)\right) 1_{(\tau>T)}\right) \\
= \;& E\left(\overline{F}(T) U_3\left(H_0\left(T\right) X^{x,\boldsymbol{\pi},\Gamma}\left(T^-\right)\right)\right).
\end{aligned} \tag{4.19}$$

To that end, we mainly use the independence between τ and $\mathcal{F}_T$ along with the Fubini–Tonelli theorem. In respect to equality (4.17), it is easy to see that

$$E\left(\int_0^{\tau\wedge T} U_1\left(s, H_0\left(s\right) c\left(s\right)\right) ds\right) = E\left(\int_0^T 1_{(\tau>s)} U_1\left(s, H_0\left(s\right) c\left(s\right)\right) ds\right).$$

Moreover, it holds that

$$\begin{aligned}
E\left(\int_0^T 1_{(\tau>s)} U_1\left(s, H_0\left(s\right) c\left(s\right)\right) ds\right) &= \int_0^T E\left(1_{(\tau>s)} U_1\left(s, H_0\left(s\right) c\left(s\right)\right)\right) ds \\
&= \int_0^T \overline{F}(s) E\left(U_1\left(s, H_0\left(s\right) c\left(s\right)\right)\right) ds \\
&= E\left(\int_0^T \overline{F}(s) U_1\left(s, H_0\left(s\right) c\left(s\right)\right) ds\right).
\end{aligned}$$

Thus, equality (4.17) follows. Besides, by recalling that τ admits the density $f(\cdot)$, equality (4.18) can be derived as follows

$$\begin{aligned}
E\left(U_2\left(\tau, H_0\left(\tau\right) D\left(\tau\right)\right) 1_{(\tau\leq T)}\right) &= \int_0^\infty f(s) E\left(U_2\left(\tau, H_0\left(\tau\right) D\left(\tau\right)\right) 1_{(\tau\leq T)} \middle| \tau = s\right) ds \\
&= \int_0^\infty f(s) E\left(U_2\left(s, H_0\left(s\right) D\left(s\right)\right) 1_{(s\leq T)}\right) ds \\
&= E\left(\int_0^T f(s) U_2\left(s, H_0\left(s\right) D\left(s\right)\right) ds\right).
\end{aligned}$$

Lastly, equality (4.19) is immediate. Also, the (4.16) follows by using similar reasoning.

Define the function

$$\begin{aligned}
\mathcal{X}\left(y\right) = \;& E\left(\int_0^T f\left(s\right)\left(H_0\left(s\right) \underline{d}\left(s\right) \vee \left(I_2\left(s,y\right) \wedge H_0\left(s\right) \overline{d}\left(s\right)\right)\right) ds\right) \\
& + \int_0^T \overline{F}\left(s\right) I_1\left(s,y\right) ds + \overline{F}\left(T\right) I_3\left(y\right), \; y \in (0,\infty).
\end{aligned}$$

In order to solve Problem 4.4.1, we will need the following technical assumption.

Assumption 4.4.1

(i)

$$\mathcal{X}(y) < \infty, y \in (0,\infty),$$

(ii)

$$E\left(\int_0^T f(s) U_2^-\left(s, H_0(s)\overline{d}(s)\right) ds\right) < \infty.$$

If the Assumption 4.4.1 (i) holds, one can easily prove that the function $\mathcal{X}(\cdot)$, which maps $(0,\infty)$ onto itself, is continuous and strictly decreasing. Moreover, $\mathcal{X}\left(0^+\right) = \lim_{y\to 0^+} \mathcal{X}(y) = \infty$ and $\mathcal{X}(\infty) = \lim_{y\to\infty} \mathcal{X}(y) = 0$. The inverse of the function $\mathcal{X}(\cdot)$ is denoted by $\mathcal{Y}(\cdot)$.

By means of Remark 4.4.4, we exhibit sufficient and reasonable conditions under which Assumption 4.4.1 holds.

Remark 4.4.4 *If we assume that the interest rate process is bounded from below by $-\eta$ for some $\eta > 0$, then we have*

$$\begin{aligned}
& E\left(\int_0^T f(s)\left(H_0(s)\underline{d}(s) \vee \left(I_2(s,y) \wedge H_0(s)\overline{d}(s)\right)\right) ds\right) \\
\leq\ & E\left(\int_0^T f(s)\left(H_0(s)\underline{d}(s) \vee I_2(s,y)\right) ds\right) \\
\leq\ & \int_0^T f(s)\left(E(H_0(s))\underline{d}(s) + I_2(s,y)\right) ds \\
\leq\ & \int_0^T f(s)\left(e^{\eta s} E(Z_0(s))\underline{d}(s) + I_2(s,y)\right) ds \\
\leq\ & \int_0^T f(s)\left(e^{\eta s}\underline{d}(s) + I_2(s,y)\right) ds.
\end{aligned}$$

Consequently, it holds that

$$\mathcal{X}(y) \leq \int_0^T (\overline{F}(s) I_1(s,y) + f(s)(e^{\eta s}\underline{d}(s) + I_2(s,y)))ds + \overline{F}(T) I_3(y).$$

Therefore, by the continuity of f and $\underline{d}$, Assumption 4.4.1 (i) holds provided that

$$\int_0^T (\overline{F}(s) I_1(s,y) + f(s) I_2(s,y))ds < \infty.$$

Obviously, when U_2 is positive or $\overline{d}(\cdot)=\infty$, Assumption 4.4.1 (ii) holds.

Our main result below shows the existence of an admissible strategy solving Problem 4.4.1 and gives a quasi-explicit formula for it.

Theorem 4.4.1 *Suppose that Assumption 4.4.1 holds, and let $x > x^{\epsilon}$ be given. Define*

$$\begin{aligned}
\hat{y} &= \mathcal{Y}(x-x^{\epsilon}), \\
\hat{c}(t) &= H_0^{-1}(t)\, I_1(t,\hat{y}), \\
\hat{D}(t) &= \underline{d}(t) \vee \left(H_0^{-1}(t)\, I_2(t,\hat{y}) \wedge \overline{d}(t)\right), \\
\hat{\xi} &= H_0^{-1}(T)\, I_3(\hat{y}).
\end{aligned}$$

Then, there exists a portfolio process $\hat{\boldsymbol{\pi}}(\cdot)$ such that the triple $\left(\hat{c},\hat{D},\hat{\boldsymbol{\pi}}\right)$ belongs to the class $\mathcal{C}(x)$ in (4.14) and attains the supremum $V(x)$ in (4.13), i.e., $\left(\hat{c},\hat{D},\hat{\boldsymbol{\pi}}\right)$ is optimal for Problem 4.4.1. Moreover, for every $t\in[0,T]$, the corresponding wealth and portfolio processes are respectively given by

$$X^{x,\hat{\boldsymbol{\pi}},-\hat{\Gamma}}(t) = \left(H_0^{\lambda}\right)^{-1}(t)\, E\left(\int_t^T H_0^{\lambda}(s)\, d\hat{\Gamma}(s) \middle| \mathcal{F}_t\right), \tag{4.20}$$

$$\boldsymbol{\sigma}^{\intercal}(t)\,\hat{\boldsymbol{\pi}}(t) = \left(H_0^{\lambda}\right)^{-1}(t)\,\boldsymbol{\varphi}(t) + X^{x,\hat{\boldsymbol{\pi}},-\hat{\Gamma}}(t)\,\boldsymbol{\theta}(t), \tag{4.21}$$

where

$$\hat{\Gamma}(t) = \int_0^t \left(\hat{c}(s)+\lambda(s)\,\hat{D}(s)-\epsilon(s)\right)ds + \hat{\xi}1_{(\cdot=T)},$$

and $\boldsymbol{\varphi}(\cdot)$ is the integrand in the stochastic integral representation $M(t)=x+\int_0^t \boldsymbol{\varphi}^{\intercal}(s)\, d\boldsymbol{W}(s)$ of the martingale

$$M(t) = E\left(\int_0^T H_0^{\lambda}(s)\, d\hat{\Gamma}(s) \middle| \mathcal{F}_t\right).$$

Proof. We divide the proof into two parts. In Part A, we show that $\left(\hat{c},\hat{D},\hat{\boldsymbol{\pi}}\right)$ belongs to the class $\mathcal{C}(x)$, and we verify the validity of relations (4.20) and (4.21). Then, in Part B, we prove that $\left(\hat{c},\hat{D},\hat{\boldsymbol{\pi}}\right)$ attains the supremum in (4.15).

A. Clearly, the triple $\left(\hat{c},\hat{\xi},\hat{D}\right)$ satisfies

$$E\left(\int_0^T H_0^{\lambda}(s)\left(\hat{c}(s)+\lambda(s)\,\hat{D}(s)-\epsilon(s)\right)ds + H_0^{\lambda}(T)\,\hat{\xi}\right) = \mathcal{X}(\hat{y}) + x^{\epsilon} = x. \tag{4.22}$$

From Proposition 4.4.1, there exists a portfolio process $\hat{\boldsymbol{\pi}}(\cdot)$ such that the triple $\left(\hat{c},\hat{D},\hat{\boldsymbol{\pi}}\right)$ belongs to $\mathcal{A}(x)$ and $X^{x,\hat{\boldsymbol{\pi}},-\hat{\Gamma}}\left(T^{-}\right)=\hat{\xi}$. By definition, we have that

$\left(\hat{c},\hat{D},\hat{\boldsymbol{\pi}}\right)$ belongs to $\mathcal{B}(x)$, accordingly, it only remains to prove that $\left(\hat{c},\hat{D},\hat{\boldsymbol{\pi}}\right)$ satisfies the integrability condition in (4.16). Let $c>0$ and $t\in[0,T]$, from relations (4.11) and (4.12), we obtain

$$\begin{aligned}\overline{F}(t)\left(U_1(t,H_0(t)\hat{c}(t))-\hat{y}H_0(t)\hat{c}(t)\right) &= \overline{F}(t)\tilde{U}_1(t,\hat{y})\\ &\geq \overline{F}(t)\left(U_1(t,c)-\hat{y}c\right),\end{aligned}$$

$$\begin{aligned}\overline{F}(T)\left(U_3\left(H_0(T)\hat{\xi}\right)-\hat{y}H_0(T)\hat{\xi}\right) &= \overline{F}(T)\tilde{U}_3(\hat{y})\\ &\geq \overline{F}(T)\left(U_3(c)-\hat{y}c\right).\end{aligned}$$

Hence, by using the continuity of the utility functions, we derive

$$\begin{aligned}& E\left(\int_0^T\left(0\wedge\overline{F}(s)U_1(s,H_0(s)\hat{c}(s))\right)ds+\left(0\wedge\overline{F}(T)U_3\left(H_0(T)\hat{\xi}\right)\right)\right)\\ \geq\ & \int_0^T\left(0\wedge\overline{F}(s)U_1(s,c)\right)ds+\left(0\wedge\overline{F}(T)U_3(c)\right)-\hat{y}\left(\overline{F}(T)+\int_0^T\overline{F}(s)\,ds\right)c\\ >\ & -\infty.\end{aligned}$$

In consequence, it follows that

$$E\left(\int_0^T\overline{F}(s)U_1^-(s,H_0(s)\hat{c}(s))\,ds+\overline{F}(T)U_3^-\left(H_0(T)\hat{\xi}\right)\right)>-\infty. \tag{4.23}$$

Therefore, relation (4.23) along with Assumption 4.4.1 (ii) enable us to conclude that $\left(\hat{c},\hat{D},\hat{\boldsymbol{\pi}}\right)$ is a member of the class $\mathcal{C}(x)$. Moreover, relations (4.20) and (4.21) are immediate consequences of Proposition 4.4.1. This completes the proof of Part A.

B. Let $(c,D,\boldsymbol{\pi})$ be any triple in $\mathcal{C}(x)$ with corresponding final wealth ξ. By using (4.12), we obtain

$$\begin{aligned}& \overline{F}(t)\left(U_1(t,H_0(t)\hat{c}(t))-\hat{y}H_0(t)\hat{c}(t)\right)\\ \geq\ & \overline{F}(t)\left(U_1(t,H_0(t)c(t))-\hat{y}H_0(t)c(t)\right),\end{aligned} \tag{4.24}$$

$$\begin{aligned}& \overline{F}(T)\left(U_3\left(H_0(T)\hat{\xi}\right)-\hat{y}H_0(T)\hat{\xi}\right)\\ \geq\ & \overline{F}(T)\left(U_3(H_0(T)\xi)-\hat{y}H_0(T)\xi\right).\end{aligned} \tag{4.25}$$

Additionally, we notice that

$$
\begin{aligned}
&H_0(t)\hat{D}(t) = I_2(t,\hat{y}) \text{ on } \left\{H_0(t)\underline{d}(t) \le I_2(t,\hat{y}) \le H_0(t)\overline{d}(t)\right\},\\
&I_2(t,\hat{y}) < H_0(t)\hat{D}(t) \le H_0(t)D(t) \text{ on } \left\{I_2(t,\hat{y}) < H_0(t)\underline{d}(t)\right\},\\
&I_2(t,\hat{y}) > H_0(t)\hat{D}(t) \ge H_0(t)D(t) \text{ on } \left\{I_2(t,\hat{y}) > H_0(t)\overline{d}(t)\right\}.
\end{aligned}
$$

Hence, as the function $U_2(t,x)-yx$ is concave down on $x\in\mathbb{R}$ and attains the maximum on $I_2(t,y)$, we get

$$
\begin{aligned}
& f(t)\left(U_2\left(t,H_0(t)\hat{D}(t)\right)-\hat{y}H_0(t)\hat{D}(t)\right)\\
\ge\ & f(t)\left(U_2(t,H_0(t)D(t))-\hat{y}H_0(t)D(t)\right). \qquad (4.26)
\end{aligned}
$$

By employing relations (4.24), (4.25), and (4.26), we retrieve

$$
\begin{aligned}
&E\left(\int_0^T\left(\overline{F}^\lambda(s)U_1(s,H_0(s)\hat{c}(s))+f^\lambda(s)U_2\left(s,H_0(s)\hat{D}(s)\right)\right)ds\right.\\
&\left.+\overline{F}^\lambda(T)U_3\left(H_0(T)\hat{\xi}\right)\right)\\
&\ge E\left(\int_0^T\left(\overline{F}^\lambda(s)U_1(s,H_0(s)c(s))+f^\lambda(s)U_2(s,H_0(s)D(s))\right)ds\right.\\
&\quad\left.+\overline{F}^\lambda(T)U_3(H_0(T)\xi)\right)\\
&\quad+\hat{y}E\left(\int_0^T H_0^\lambda(s)\left(\hat{c}(s)+\lambda(s)\hat{D}(s)-\epsilon(s)\right)ds+H_0^\lambda(T)\hat{\xi}\right)\\
&\quad-\hat{y}E\left(\int_0^T H_0^\lambda(s)\left(c(s)+\lambda(s)D(s)-\epsilon(s)\right)ds+H_0^\lambda(T)\xi\right)\\
&\ge E\left(\int_0^T\left(\overline{F}^\lambda(s)U_1(s,H_0(s)c(s))+f^\lambda(s)U_2(s,H_0(s)D(s))\right)ds\right.\\
&\quad\left.+\overline{F}^\lambda(T)U_3(H_0(T)\xi)\right).
\end{aligned}
$$

Last inequality follows because of (4.22) and the budget constraint (4.10) satisfied by $(c,D,\boldsymbol{\pi})$. This finishes the proof of Part B.

■

Example 4.4.1 *Consider a case in which the agent does not have constraints on the life insurance purchase, i.e.,* $\underline{d}\equiv 0$ *and* $\bar{d}\equiv\infty$*. Moreover, assume that the parameters*

$r(\cdot)$ and $\epsilon(\cdot)$ are deterministic functions. For every $(t,x) \in [0,T] \times (0,\infty)$, let

$$\begin{aligned} U_1(t,x) &= e^{-\alpha t}\frac{x^p}{p}, \\ U_2(t,x) &= e^{-\alpha t}\frac{x^p}{p}, \\ U_3(x) &= e^{-\alpha T}\frac{x^p}{p}, \end{aligned}$$

where $p \in (-\infty,1)\backslash\{0\}$ and $\alpha > 0$. Apparently, for all $(t,y) \in [0,T] \times (0,\infty)$, it holds that

$$\begin{aligned} I_1(t,y) &= e^{\frac{\alpha}{p-1}t} y^{\frac{1}{p-1}}, \\ I_2(t,y) &= e^{\frac{\alpha}{p-1}t} y^{\frac{1}{p-1}}, \\ I_3(y) &= e^{\frac{\alpha}{p-1}T} y^{\frac{1}{p-1}}. \end{aligned}$$

Furthermore, the auxiliary functions $\mathcal{X}(\cdot)$ and $\mathcal{Y}(\cdot)$ can be written as

$$\begin{aligned} \mathcal{X}(y) &= \left(\int_0^T (\overline{F}(s) + f(s)) e^{\frac{\alpha}{p-1}s} ds + \overline{F}(T) e^{\frac{\alpha}{p-1}T}\right) y^{\frac{1}{p-1}}, \ y \in (0,\infty), \\ \mathcal{Y}(x) &= \left(\frac{x}{\mathcal{X}(1)}\right)^{p-1}, \ x \in (0,\infty). \end{aligned}$$

It is easy to see that

$$\begin{aligned} \hat{c}(t) &= H_0^{-1}(t) e^{\frac{\alpha}{p-1}t} \left(\frac{x - x^\epsilon}{\mathcal{X}(1)}\right), \\ \hat{D}(t) &= H_0^{-1}(t) e^{\frac{\alpha}{p-1}t} \left(\frac{x - x^\epsilon}{\mathcal{X}(1)}\right), \\ \hat{\xi} &= H_0^{-1}(T) e^{\frac{\alpha}{p-1}T} \left(\frac{x - x^\epsilon}{\mathcal{X}(1)}\right). \end{aligned}$$

As well, the value function $V(\cdot)$ is given by

$$V(x) = \frac{1}{p}(\mathcal{X}(1))^{1-p}(x - x^\epsilon)^p, \ x > x^\epsilon.$$

Since $Z_0(\cdot)$ is a martingale, we derive

$$\begin{aligned} M(t) &= E\left(\int_0^T H_0^\lambda(s)\, d\hat{\Gamma}(s) \middle| \mathcal{F}_t\right) \\ &= (x - x^\epsilon) - E\left(\int_0^T \overline{F}(s)\gamma(s)\, Z_0(s)\, \epsilon(s)\, ds \middle| \mathcal{F}_t\right) \\ &= (x - x^\epsilon) - Z_0(t) \int_t^T \overline{F}(s)\gamma(s)\, \epsilon(s)\, ds - \int_0^t \overline{F}(s)\gamma(s)\, Z_0(s)\, \epsilon(s)\, ds, \end{aligned}$$

which implies

$$\begin{aligned} dM(t) &= -\int_t^T \overline{F}(s)\gamma(s)\, \epsilon(s)\, dZ_0(t) \\ &= \left(\int_t^T \overline{F}(s)\gamma(s)\, \epsilon(s)\, ds\right) Z_0(t)\, \boldsymbol{\theta}^\intercal(t)\, d\boldsymbol{W}(t)\,. \end{aligned}$$

Therefore, the optimal investment process $\hat{\boldsymbol{\pi}}(\cdot)$ admits the following explicit representation

$$\boldsymbol{\sigma}^\intercal(t)\, \hat{\boldsymbol{\pi}}(t) = \left(\left(H_0^\lambda\right)^{-1}(t) \left(\int_t^T \overline{F}(s)\gamma(s)\, \epsilon(s)\, ds\right) Z_0(t) + X^{x,\hat{\boldsymbol{\pi}},-\hat{\Gamma}}(t)\right) \boldsymbol{\theta}(t)\,.$$

Appendix A

A.1 Affine diffusions

Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting a standard Brownian motion $W(\cdot)$ whose corresponding natural filtration is denoted by $\{\mathcal{F}_t\}$. For a subset $D \subset \mathbb{R}$, let $\zeta : D \to \mathbb{R}$ and $\eta : D \to \mathbb{R}$. Consider the following stochastic differential equation

$$d\lambda(t) = \zeta(\lambda(t))\, dt + \eta(\lambda(t))\, dW(t), \quad t \in [0, T]. \tag{A.1}$$

By assuming some regularity conditions on $\zeta(\cdot)$ and $\eta(\cdot)$, equation (A.1) admits a unique strong solution $\lambda(\cdot)$. In particular, if the functions $\zeta(\cdot)$ and $\eta^2(\cdot)$ are affine, namely,

$$\begin{aligned}
\zeta(x) &= \zeta_0 + \zeta_1 x, \quad x \in D, \\
\eta^2(x) &= \eta_0 + \eta_1 x, \quad x \in D,
\end{aligned}$$

where ζ_0, ζ_1, η_0, η_1 are real numbers, then the process $\lambda(\cdot)$ is called an affine diffusion. By employing the Itô formula, for any real number a, we have

$$E\left(e^{-a \int_t^T \lambda(s) ds} \middle| \mathcal{F}_t \right) = e^{\psi(T-t) + \beta(T-t)\lambda(t)}, \tag{A.2}$$

where $\psi(\cdot)$ and $\beta(\cdot)$ are deterministic functions satisfying the following system of Riccati ordinary differential equations

$$\frac{d\psi}{dx}(t) = -\zeta_0 \beta(t) - \frac{1}{2}\eta_0 \beta^2(t), \tag{A.3}$$

$$\frac{d\beta}{dx}(t) = -\zeta_1 \beta(t) - \frac{1}{2}\eta_1 \beta^2(t) + a, \tag{A.4}$$

$$\psi(T) = \beta(T) = 0. \tag{A.5}$$

In some applications, for instance in Example 3.4.2, we find explicit solutions to the system. In other cases, we must find solutions numerically. See Duffie et al. (2000) for a generalization of this model.

A.2 Details of Example 3.4.2

First, we define the Cox–Ingressol–Ross (CIR) process. Let $W(\cdot)$ be a standard Brownian motion with respect to its natural filtration $\left\{\mathcal{F}_t^W\right\}$. For strictly positive parameters κ, φ, ν, the CIR process, $\lambda(\cdot)$, is the unique strong solution to the equation

$$d\lambda(t) = \kappa(\varphi - \lambda(t))dt + \nu\sqrt{\lambda(t)}dW(t), \quad t \in [0,T].$$

To preclude $\lambda(\cdot)$ taking nonpositive values we take $2\kappa\varphi \geq \nu^2$. For any positive number a, as a member of the class of affine processes, the CIR process satisfies

$$E\left(e^{-a\int_t^T \lambda(s)ds}\middle|\mathcal{F}_t^W\right) = e^{\psi(T-t)+\beta(T-t)\lambda(t)}, \tag{A.6}$$

where $\psi(\cdot)$ and $\beta(\cdot)$ are deterministic functions satisfying the system of differential equations in (A.3)-(A.5). For $s \in [0,T]$, by defining $b^2 = \kappa^2 + 2a\nu^2$, it is not difficult to prove that

$$\begin{aligned}
\psi(s) &= -\frac{2\kappa\varphi}{\nu^2}\log\left(\frac{2be^{\frac{1}{2}(b+\kappa)s}}{(b+\kappa)\left(e^{bs}-1\right)+2b}\right),\\
\beta(s) &= \frac{2a\left(e^{bs}-1\right)}{(b+\kappa)\left(e^{bs}-1\right)+2b}.
\end{aligned}$$

Now, we give additional details concerning the formula for the process $p(\cdot)$. Let $I \subset I_n$ and $j = |I|$. By the independence of the processes $\lambda_0(\cdot), \ldots, \lambda_n(\cdot)$, we obtain

$$\begin{aligned}
E\left(e^{-\sum_{i\in I}\int_t^T \gamma_i(s)ds}\middle|\mathcal{F}_t^2\right) &= E\left(e^{-j\int_t^T \lambda_0(s)ds}\prod_{i\in I} e^{-\int_t^T \lambda_i(s)ds}\middle|\mathcal{F}_t^2\right)\\
&= E\left(e^{-j\int_t^T \lambda_0(s)ds}\middle|\mathcal{F}_t^2\right)\prod_{i\in I} E\left(e^{-\int_t^T \lambda_i(s)ds}\middle|\mathcal{F}_t^2\right).
\end{aligned}$$

Let $\psi_0^j(\cdot)$, $\beta_0^j(\cdot)$ correspond to the functions $\psi(\cdot)$, $\beta(\cdot)$ appearing in equation (A.6) with $a = j$ and $\lambda(\cdot) = \lambda_0(\cdot)$. Similarly, for $i \in I$, associate the functions $\psi_i(\cdot)$, $\beta_i(\cdot)$ with the functions $\psi(\cdot)$, $\beta(\cdot)$ appearing in equation (A.6) with $a = 1$ and $\lambda(\cdot) = \lambda_i(\cdot)$. Then it holds that

$$E\left(e^{-\sum_{i\in I}\int_t^T \gamma_i(s)ds} \middle| \mathcal{F}_t^2 \right) = e^{\psi_0^j(T-t)+\beta_0^j(T-t)\lambda_0(t)} \prod_{i\in I} e^{\psi_i(T-t)+\beta_i(T-t)\lambda_i(t)}. \tag{A.7}$$

Finally, by plugging equation (A.7) in (3.13), we obtain the desired explicit formula for $p(\cdot)$.

References

[1] Acerbi, C. and Tasche, D., 2002. On the coherence of expected shortfall. Journal of Banking & Finance, 26(7), pp.1487-1503.

[2] Ahn, J.Y. and Shyamalkumar, N.D., 2014. Asymptotic theory for the empirical Haezendonck–Goovaerts risk measure. Insurance: Mathematics and Economics, 55, pp.78-90.

[3] Aksamit, A. and Jeanblanc, M., 2017. Enlargement of Filtration with Finance in View. Switzerland: Springer.

[4] Albanese, C. and Lawi, S., 2007. Laplace transforms for integrals of Markov processes. arXiv preprint arXiv:0710.1599.

[5] Arnold, B.C., 2015. Pareto Distributions. Chapman and Hall/CRC.

[6] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D., 1997. Thinking coherently. Risk, 10(11), pp.68-71.

[7] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D., 1999. Coherent measures of risk. Mathematical Finance, 9(3), pp.203-228.

[8] Asimit, A.V., Furman, E., Tang, Q. and Vernic, R., 2011. Asymptotics for risk capital allocations based on conditional tail expectation. Insurance: Mathematics and Economics, 49(3), pp.310-324.

[9] Bauer, D., Börger, M. and Ruß, J., 2010. On the pricing of longevity-linked securities. Insurance: Mathematics and Economics, 46(1), pp.139-149.

[10] Bauer, D. and Kramer, F., 2016. The risk of a mortality catastrophe. Journal of Business & Economic Statistics, 34(3), pp.391-405.

[11] Bauer, D. and Zanjani, G., 2015. The marginal cost of risk, risk measures, and capital allocation. Management Science, 62(5), pp.1431-1457.

[12] Bayraktar, E., Milevsky, M.A., Promislow, S.D. and Young, V.R., 2009. Valuation of mortality risk via the instantaneous Sharpe ratio: applications to life annuities. Journal of Economic Dynamics and Control, 33(3), pp.676-691.

[13] Becherer, D., 2006. Bounded solutions to backward SDEs with jumps for utility optimization and indifference hedging. The Annals of Applied Probability, 16(4), pp.2027-2054.

[14] Bellini, F. and Rosazza Gianin, E., 2008a. On Haezendonck risk measures. Journal of Banking & Finance, 32(6), pp.986-994.

[15] Bellini, F. and Rosazza Gianin, E., 2008b. Optimal portfolios with Haezendonck risk measures. Statistics & Decisions, 26(2), pp.89-108.

[16] Bellini, F. and Rosazza Gianin, E., 2012. Haezendonck–Goovaerts risk measures and Orlicz quantiles. Insurance: Mathematics and Economics, 51(1), pp.107-114.

[17] Belomestny, D. and Krätschmer, V., 2012. Central limit theorems for law-invariant coherent risk measures. Journal of Applied Probability, 49(1), pp.1-21.

[18] Ben–Tal, A. and Teboulle, M., 2007. An old-new concept of convex risk measures: the optimized certainty equivalent. Mathematical Finance, 17(3), pp.449-476.

[19] Biagini, F., Botero, C. and Schreiber, I., 2017. Risk-minimization for life insurance liabilities with dependent mortality risk. Mathematical Finance, 27(2), pp.505-533.

[20] Biagini, F., Fouque, J.P., Frittelli, M. and Meyer-Brandis, T., 2019. A unified approach to systemic risk measures via acceptance sets. Mathematical Finance, 29(1), pp.329-367.

[21] Biffis, E., 2005. Affine processes for dynamic mortality and actuarial valuations. Insurance: Mathematics and Economics, 37(3), pp.443-468.

[22] Blackburn, C. and Sherris, M., 2013. Consistent dynamic affine mortality models for longevity risk applications. Insurance: Mathematics and Economics, 53(1), pp.64-73.

[23] Blake, D., Cairns, A. and Dowd, K., 2008. The birth of the life market. Asia-Pacific Journal of Risk and Insurance, 3(1).

[24] Blanchet-Scalliet, C., El Karoui, N. and Martellini, L., 2005. Dynamic asset pricing theory with uncertain time-horizon. Journal of Economic Dynamics and Control, 29(10), pp.1737-1764.

[25] Buccola, S.T., 1982. Portfolio selection under exponential and quadratic utility. Western Journal of Agricultural Economics, pp.43-51.

[26] Buch, A. Dorfleitner, G. and Wimmer, M., 2011. Risk capital allocation for RORAC optimization. Journal of Banking & Finance, 35(11), pp.3001-3009.

[27] Cairns, A., Blake, D. and Dowd, K., 2006. Pricing death: frameworks for the valuation and securitization of mortality risk. ASTIN Bulletin, 36(1), pp.79-120.

[28] Cairns, A., Blake, D. and Dowd, K., 2008. Modelling and management of mortality risk: a review. Scandinavian Actuarial Journal, 2008(2-3), pp.79-113.

[29] Carmona, R. ed., 2009. Indifference Pricing: Theory and Applications. Princeton University Press.

[30] Chen, H. and Cox, S.H., 2009. Modeling mortality with jumps: applications to mortality securitization. Journal of Risk and Insurance, 76(3), pp.727-751.

[31] Chiragiev, A. and Landsman, Z., 2007. Multivariate Pareto portfolios: TCE-based capital allocation and divided differences. Scandinavian Actuarial Journal, 2007(4), pp.261-280.

[32] Cox, J.C. and Huang, C.F., 1989. Optimal consumption and portfolio policies when asset prices follow a diffusion process. Journal of Economic Theory, 49(1), pp.33-83.

[33] Cox, J.C. and Huang, C.F., 1991. A variational problem arising in financial economics. Journal of mathematical Economics, 20(5), pp.465-487.

[34] Csóka, P. and Herings, P.J.J., 2014. Risk allocation under liquidity constraints. Journal of Banking & Finance, 49, pp.1-9.

[35] Dahl, M., 2004. Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts. Insurance: Mathematics and Economics, 35(1), pp.113-136.

[36] Dahl, M., Melchior, M. and Møller, T., 2008. On systematic mortality risk and risk-minimization with survivor swaps. Scandinavian Actuarial Journal, 2008(2-3), pp.114-146.

[37] Dahl, M. and Møller, T., 2006. Valuation and hedging of life insurance liabilities with systematic mortality risk. Insurance: Mathematics and Economics, 39(2), pp.193-217.

[38] Davis, M.H., Panas, V.G. and Zariphopoulou, T., 1993. European option pricing with transaction costs. SIAM Journal on Control and Optimization, 31(2), pp.470-493.

[39] Delbaen, F., Grandits, P., Rheinländer, T., Samperi, D., Schweizer, M. and Stricker, C., 2002. Exponential hedging and entropic penalties. Mathematical Finance, 12(2), pp.99-123.

[40] Denault, M., 2001. Coherent allocation of risk capital. Journal of Risk, 4, pp.1-34.

[41] Dentcheva, D., Penev, S. and Ruszczyński, A., 2017. Statistical estimation of composite risk functionals and risk optimization problems. Annals of the Institute of Statistical Mathematics, 69(4), pp.737-760.

[42] Dhaene, J., Goovaerts, M.J. and Kaas, R., 2003. Economic capital allocation derived from risk measures. North American Actuarial Journal, 7(2), pp.44-56.

[43] Dhaene, J., Henrard, L., Landsman, Z., Vandendorpe, A. and Vanduffel, S., 2008. Some results on the CTE-based capital allocation rule. Insurance: Mathematics and Economics, 42(2), pp.855-863.

[44] Dhaene, J., Tsanakas, A., Valdez, E.A. and Vanduffel, S., 2012. Optimal capital allocation principles. Journal of Risk and Insurance, 79(1), pp.1-28.

[45] Duarte, I., Pinheiro, D., Pinto, A.A. and Pliska, S.R., 2011. An overview of optimal life insurance purchase, consumption and investment problems. In Dynamics, Games and Science I (pp. 271-286). Springer, Berlin, Heidelberg.

[46] Duarte, I., Pinheiro, D., Pinto, A.A. and Pliska, S.R., 2014. Optimal life insurance purchase, consumption and investment on a financial market with multi-dimensional diffusive terms. Optimization, 63(11), pp.1737-1760.

[47] Duffie, D., Pan, J. and Singleton, K., 2000. Transform analysis and asset pricing for affine jump-diffusions. Econometrica, 68(6), pp.1343-1376.

[48] Elliott, R.J., Jeanblanc, M. and Yor, M., 2000. On models of default risk. Mathematical Finance, 10(2), pp.179-195.

[49] Feng, R., 2014. A comparative study of risk measures for guaranteed minimum maturity benefits by a PDE method. North American Actuarial Journal, 18(4), pp.445-461.

[50] Frittelli, M., 2000a. Introduction to a theory of value coherent with the no-arbitrage principle. Finance and Stochastics, 4(3), pp.275-297.

[51] Frittelli, M., 2000b. The minimal entropy martingale measure and the valuation problem in incomplete markets. Mathematical Finance, 10(1), pp.39-52.

[52] Furman, E. and Zitikis, R., 2008a. Weighted premium calculation principles. Insurance: Mathematics and Economics, 42(1), pp.459-465.

[53] Furman, E. and Zitikis, R., 2008b. Weighted risk capital allocations. Insurance: Mathematics and Economics, 43(2), pp.263-269.

[54] Furman, E., Wang, R. and Zitikis, R., 2017. Gini-type measures of risk and variability: Gini shortfall, capital allocations, and heavy-tailed risks. Journal of Banking & Finance, 83, pp.70-84.

[55] Gobet, E., Lemor, J.P. and Warin, X., 2005. A regression-based Monte Carlo method to solve backward stochastic differential equations. The Annals of Applied Probability, 15(3), pp.2172-2202.

[56] Goovaerts, M.J., Kaas, R., Dhaene, J. and Tang, Q., 2003. A unified approach to generate risk measures. ASTIN Bulletin, 33(2), pp.173-191.

[57] Goovaerts, M., Kaas, R., Dhaene, J. and Tang, Q., 2004. Some new classes of consistent risk measures. Insurance: Mathematics and Economics, 34(3), pp.505-516.

[58] Guambe, C. and Kufakunesu, R., 2015. A note on optimal investment-consumption-insurance in a Lévy market. Insurance: Mathematics and Economics, 65, pp.30-36.

[59] Haezendonck, J. and Goovaerts, M., 1982. A new premium calculation principle based on Orlicz norms. Insurance: Mathematics and Economics, 1(1), pp.41-53.

[60] Hainaut, D. and Devolder, P., 2008. Mortality modelling with Lévy processes. Insurance: Mathematics and Economics, 42(1), pp.409-418.

[61] Hodges, S., and Neuberger , A., 1989. Optimal replication of contingent claims under transaction costs. Review Futures Market, 8(2), pp.222-239.

[62] Hong, L.J. and Liu, G., 2009. Simulating sensitivities of conditional value at risk. Management Science, 55(2), pp.281-293.

[63] Hu, Y., Imkeller, P. and Müller, M., 2005. Utility maximization in incomplete markets. The Annals of Applied Probability, 15(3), pp.1691-1712.

[64] Huang, H., Milevsky, M.A. and Wang, J., 2008. Portfolio choice and life insurance: the CRRA case. Journal of Risk and Insurance, 75(4), pp.847-872.

[65] Iacus, S.M., 2009. Simulation and Inference for Stochastic Differential Equations: with R Examples. Springer Science & Business Media.

[66] Ignatieva, K. and Landsman, Z., 2019. Conditional tail risk measures for the skewed generalised hyperbolic family. Insurance: Mathematics and Economics, 86, pp.98-114.

[67] Jackwerth, J.C., 2000. Recovering risk aversion from option prices and realized returns. The Review of Financial Studies, 13(2), pp.433-451.

[68] Jaimungal, S. and Young, V.R., 2005. Pricing equity-linked pure endowments with risky assets that follow Lévy processes. Insurance: Mathematics and Economics, 36(3), pp.329-346.

[69] Jeanblanc, M. and Rutkowski, M., 2002. Default risk and hazard process. In Mathematical Finance-Bachelier Congress 2000 (pp. 281-312). Springer, Berlin, Heidelberg.

[70] Jiao, Y., Kharroubi, I. and Pham, H., 2013. Optimal investment under multiple defaults risk: a BSDE-decomposition approach. The Annals of Applied Probability, 23(2), pp.455-491.

[71] Jones, B.L. and Zitikis, R., 2003. Empirical estimation of risk measures and related quantities. North American Actuarial Journal, 7(4), pp.44-54.

[72] Kaas, R., Goovaerts, M., Dhaene, J. and Denuit, M., 2008. Modern Actuarial Risk Theory: Using R (Vol. 128). Springer Science & Business Media.

[73] Kalkbrener, M., Lotter, H. and Overbeck, L., 2004. Sensible and efficient capital allocation for credit portfolios. Risk, 17(1), pp.S19-S24.

[74] Kalkbrener, M., 2005. An axiomatic approach to capital allocation. Mathematical Finance, 15(3), pp.425-437.

[75] Karatzas, I., Lehoczky, J.P. and Shreve, S.E., 1987. Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. SIAM journal on control and optimization, 25(6), pp.1557-1586.

[76] Karatzas, I. and Shreve, S.E., 1998. Methods of Mathematical Finance (Vol. 39). New York: Springer.

[77] Karatzas, I. and Shreve, S., 2012. Brownian Motion and Stochastic Calculus (Vol. 113). Springer Science & Business Media.

[78] Kazamaki, N., 2006. Continuous Exponential Martingales and BMO. Springer.

[79] Kharroubi, I. and Lim, T., 2014. Progressive enlargement of filtrations and backward stochastic differential equations with jumps. Journal of Theoretical Probability, 27(3), pp.683-724.

[80] Kharroubi, I. and Lim, T., 2015. A decomposition approach for the discrete-time approximation of FBSDEs with a jump. Random Operators and Stochastic Equations, 23(2), pp.81-109.

[81] Kim, J.H. and Kim, S.Y., 2019. Tail risk measures and risk allocation for the class of multivariate normal mean-variance mixture distributions. Insurance: Mathematics and Economics, 86, pp.145-157.

[82] Krokhmal, P.A., 2007. Higher moment coherent risk measures. Quantitative Finance, 7(4), pp.373–387.

[83] Kronborg, M.T. and Steffensen, M., 2015. Optimal consumption, investment and life insurance with surrender option guarantee. Scandinavian Actuarial Journal, 2015(1), pp.59-87.

[84] Kwak, M. and Lim, B.H., 2014. Optimal portfolio selection with life insurance under inflation risk. Journal of Banking & Finance, 46, pp.59-71.

[85] Lando, D., 2009. Credit Risk Modeling: Theory and Applications. Princeton University Press.

[86] Lin, Y. and Cox, S.H., 2005. Securitization of mortality risks in life annuities. Journal of risk and Insurance, 72(2), pp.227-252.

[87] Lin, Y. and Cox, S.H., 2008. Securitization of catastrophe mortality risks. Insurance: Mathematics and Economics, 42(2), pp.628-637.

[88] Londoño, J.A., 2008. A more general valuation and arbitrage theory for Itô processes. Stochastic Analysis and Applications, 26(4), pp.809-831.

[89] Londoño, J.A., 2009. State-dependent utility. Journal of Applied Probability, 46(1), pp.55-70.

[90] Ludkovski, M. and Young, V.R., 2008. Indifference pricing of pure endowments and life annuities under stochastic hazard and interest rates. Insurance: Mathematics and Economics, 42(1), pp.14-30.

[91] Mania, M. and Schweizer, M., 2005. Dynamic exponential utility indifference valuation. The Annals of Applied Probability, 15(3), pp.2113-2143.

[92] Mao, T. and Hu, T., 2012. Second-order properties of the Haezendonck–Goovaerts risk measure for extreme risks. Insurance: Mathematics and Economics, 51(2), pp.333-343.

[93] Matmoura, Y. and Penev, S., 2013. Multistage optimization of option portfolio using higher order coherent risk measures. European Journal of Operational Research, 227(1), pp.190-198.

[94] McCord, M. and De Neufville, R., 1986. "Lottery equivalents": reduction of the certainty effect problem in utility assessment. Management Science, 32(1), pp.56-60.

[95] Mehra, R. and Prescott, E.C., 1985. The equity premium: a puzzle. Journal of Monetary Economics, 15(2), pp.145-161.

[96] Merton, R.C., 1969. Lifetime portfolio selection under uncertainty: the continuous-time case. The Review of Economics and Statistics, pp.247-257.

[97] Merton, R.C., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory, 3(4), pp.373-413.

[98] Milevsky, M.A. and Promislow, S.D., 2001. Mortality derivatives and the option to annuitise. Insurance: Mathematics and Economics, 29(3), pp.299-318.

[99] Moore, K.S. and Young, V.R., 2003. Pricing equity-linked pure endowments via the principle of equivalent utility. Insurance: Mathematics and Economics, 33(3), pp.497-516.

[100] Morlais, M.A., 2009. Utility maximization in a jump market model. Stochastics, 81(1), pp.1-27.

[101] Niculescu, C. and Persson, L.E., 2006. Convex Functions and Their Applications. New York: Springer.

[102] Nielsen, P.H. and Steffensen, M., 2008. Optimal investment and life insurance strategies under minimum and maximum constraints. Insurance: Mathematics and Economics, 43(1), pp.15-28.

[103] Peng, L., Wang, X. and Zheng, Y., 2015. Empirical likelihood inference for Haezendonck–Goovaerts risk measure. European Actuarial Journal, 5(2), pp.427-445.

[104] Pirvu, T.A. and Zhang, H., 2012. Optimal investment, consumption and life insurance under mean-reverting returns: the complete market solution. Insurance: Mathematics and Economics, 51(2), pp.303-309.

[105] Pliska, S.R., 1986. A stochastic calculus model of continuous trading: optimal portfolios. Mathematics of Operations Research, 11(2), pp.371-382.

[106] Pliska, S.R. and Ye, J., 2007. Optimal life insurance purchase and consumption/investment under uncertain lifetime. Journal of Banking & Finance, 31(5), pp.1307-1319.

[107] Rao, M.M. and Ren, Z.D., 1991. Theory of Orlicz Spaces. New York: M. Dekker.

[108] Resnick, S.I., 2007. Heavy-Tail Phenomena: Probabilistic and Statistical Modeling. Springer Science & Business Media.

[109] Richard, S.F., 1975. Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. Journal of Financial Economics, 2(2), pp.187-203.

[110] Rockafellar, R.T. and Uryasev, S., 2000. Optimization of conditional value-at-risk. Journal of Risk, 2, pp.21-42.

[111] Rockafellar, R.T. and Uryasev, S., 2002. Conditional value-at-risk for general loss distributions. Journal of Banking & Finance, 26(7), pp.1443-1471.

[112] Rouge, R. and El Karoui, N., 2000. Pricing via utility maximization and entropy. Mathematical Finance, 10(2), pp.259-276.

[113] Scaillet, O., 2004. Nonparametric estimation and sensitivity analysis of expected shortfall. Mathematical Finance, 14(1), pp.115-129.

[114] Schmeidler, D., 1969. The nucleolus of a characteristic function game. SIAM Journal on Applied Mathematics, 17(6), pp.1163-1170.

[115] Schrager, D.F., 2006. Affine stochastic mortality. Insurance: Mathematics and Economics, 38(1), pp.81-97.

[116] Shapley, L.S., 1953. A value for n-person games. In Contributions to the Theory of Games, 2(28), pp.307-317. Princeton University Press, Princeton.

[117] Shen, Y. and Wei, J., 2016. Optimal investment-consumption-insurance with random parameters. Scandinavian Actuarial Journal, 2016(1), pp.37-62.

[118] Shi, X., Tang, Q. and Yuan, Z., 2017. A limit distribution of credit portfolio losses with low default probabilities. Insurance: Mathematics and Economics, 73, pp.156-167.

[119] Tang, Q. and Yang, F., 2012. On the Haezendonck–Goovaerts risk measure for extreme risks. Insurance: Mathematics and Economics, 50(1), pp.217-227.

[120] Tang, Q. and Yang, F., 2014. Extreme value analysis of the Haezendonck–Goovaerts risk measure with a general Young function. Insurance: Mathematics and Economics, 59, pp.311-320.

[121] Tasche, D., 2002. Expected shortfall and beyond. Journal of Banking & Finance, 26(7), pp.1519-1533.

[122] Tasche, D., 2004. Allocating portfolio economic capital to sub-portfolios. In Economic Capital: A Practitioner Guide (pp.275-302). Risk Books.

[123] Tijs, S.H. and Driessen, T.S., 1986. Game theory and cost allocation problems. Management Science, 32(8), pp.1015-1028.

[124] Weil, P., 1989. The equity premium puzzle and the risk-free rate puzzle. Journal of Monetary Economics, 24(3), pp.401-421.

[125] Wills, S. and Sherris, M., 2010. Securitization, structuring and pricing of longevity risk. Insurance: Mathematics and Economics, 46(1), pp.173-185.

[126] Wilson, R., 1979. Auctions of shares. The Quarterly Journal of Economics, pp.675-689.

[127] Zanjani, G., 2010. An economic approach to capital allocation. Journal of Risk and Insurance, 77(3), pp.523-549.