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ON SOME SPACES OF ANALYTIC FUNCTIONS AND THEIR DUALITY RELATIONS

by

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Abstract: For each $0 \leqslant C < +\infty$ and $0 < p < +\infty$ let $E_{C,p}$ be the space of entire functions f such that, for some constant $A \geqslant 0$, $|f(z)| \leqslant Ae^{C|z|^p}$ for all z in $\mathbb{C}$. If $\|f\|_{C,p}$ is the minimun of such constants A, $\| \ \|_{C,p}$ is a Banach space norm on $E_{C,p}$. Let $0 < B \leqslant +\infty$ and denote with $\tilde{E}_{B,p}$ the inductive limit space of the Banach spaces $E_{C,p}$, $0 \leqslant C < B$. The topological dual space of $\tilde{E}_{B,p}$ is identified as the space $\mathcal{O}_{B,p}$ of analytic functions on the open disk $\mathcal{D}(0,(Bp)^{1/p})$. If $\mathcal{O}_{B,p}$ is given the topology of uniform convergence on compact sets, its topological dual is also identified as $\tilde{E}_{B,p}$. Relations between different topologies on the spaces $E_{C,p}$ and $\tilde{E}_{B,p}$ having their origin in the duality are also examined.

§1. Introduction. If Ω is an open domain of the complex plane $\mathbb{C}$, $\mathcal{O}(\Omega)$ is the space of complex *analytic functions* in Ω. We write $\mathcal{O}$ instead of $\mathcal{O}(\mathbb{C})$ to denote the space of *entire functions*. To $\mathcal{O}(\Omega)$ we will give the topology $\tau(\mathcal{O}(\Omega))$ of *uniform*

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convergence on compact subsets. This topology is defined by the seminorms

$$(1.1) \qquad \|f\|_K = \sup_{z\in K} |f(x)|$$

when K runs over the compact subset of Ω, and it is therefore a locally convex vector space topology on $O(\Omega)$ (Horvâth [2], Chap.2,§4). If (K_n) is a sequence of compact subsets of Ω such that $\phi \neq K_n \subseteq \overset{\circ}{K}_{n+1}$, $n \geqslant 0$, and $\bigcup_{n=0}^{\infty} K_n = \Omega$, the seminorms $\| \;\|_{K_n}$, $n = 1,2,\ldots$, define $\tau(O(\Omega))$. These are in fact norms. Moreover,

$$(1.2) \qquad d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{K_n}}{1+\|f-g\|_{K_n}}$$

is a distance on $O(\Omega)$ also defining $\tau(O(\Omega))$. For this distance $O(\Omega)$ is a complete metric space, so that $O(\Omega)$ is a *Frechet space* (Horvâth [2], Chap.2,§9). If $\Omega = \mathbb{C}$, we can take $K_n = \bar{D}(0,n)$, $n \geqslant 1$.

For each $0 \leqslant C < +\infty$ and each $0 < p < +\infty$, $E_{C,p}$ will be the space of entire functions f such that for some constant $A \geqslant 0$

$$(1.3) \qquad |f(z)| \leqslant Ae^{C|z|^p}, \quad z \in \mathbb{C}.$$

If $\|f\|_{C,p}$ is the minimun of those constants A for which (1.3) holds then $\| \;\|_{C,p}$ is a norm on $E_{C,p}$. Since

$$(1.4) \qquad \|f\|_{\bar{D}(0,R)} \leqslant \|f\|_{C,p} e^{CR^p}, \qquad R > 0,$$

the topology $\tau_{C,p}$ on $E_{C,p}$ defined by this norm is stronger than that induced by $\tau(O)$. Hence, a Cauchy sequence (f_n) in $E_{C,p}$ for $\| \;\|_{C,p}$ is convergent on compact subsets to an analytic function $f \in O$. If $\|f_n\|_{C,p} \leqslant M$ for all $n \geqslant 1$, then

$$(1.5) \qquad |f(z)| \leqslant (1+M)e^{C|z|^p}$$

so that $f \in E_{C,p}$. Now, let $\varepsilon > 0$, and $n_o \geqslant n$ be such that

$\|f_n - f_m\|_{C,p} \leqslant \varepsilon$ for $m,n \geqslant n_o$. Then

$$|f_n(z) - f_m(z)| \leqslant \varepsilon e^{C|z|^p}$$

for $m,n > n_o$, and passing to the limit when $m \to \infty$ we obtain

$$|f_n(z) - f(z)| \leqslant \varepsilon e^{C|z|^p}$$

for $n \geqslant n_o$. This means that $\|f_n - f\|_{C,p} \leqslant \varepsilon$ for $n \geqslant n_o$. Hence $(E_{C,p}, \| \|_{C,p})$ is a Banach space. If $C \leqslant B$ then $E_{C,p} \subseteq E_{B,p}$ and

$$\|f\|_{B,p} \leqslant \|f\|_{C,p}, \quad f \in E_{C,p}. \tag{1.6}$$

Thus, the inclusion maps $i_{B,C,p}: E_{C,p} \to E_{B,p}$ are continuous and the topology of $E_{C,p}$ is stronger than the induced topology from $E_{B,p}$.

For $0 < B \leqslant +\infty$ and $0 < p < +\infty$ let

$$\tilde{E}_{B,p} := \bigcup_{C<B} E_{C,p} \tag{1.7}$$

As a vector space, $\tilde{E}_{B,p}$ is the *inductive limit* of the vector spaces $E_{C,p}$, $0 \leqslant C < B$. We will give $\tilde{E}_{B,p}$ the *inductive limit topology* $\tilde{\tau}_{B,p}$ of the Banach space topologies of the $E_{C,p}$'s. This is the strongest locally convex topology on $\tilde{E}_{B,p}$ such that the inclusion maps $\tilde{i}_{B,C,p}$ $E_{C,p} \to E_{B,p}$, $0 \leqslant C < B$, are continuous. A fundamental system of neighborhoods of 0 for the topology $\tilde{\tau}_{B,p}$ is given by the convex, balanced and absorbing subsets V such that, for all $0 \leqslant C < B$, $V \cap E_{C,p}$ is a neighborhood of 0 in $E_{C,p}$. If E is a locally convex, topological vector space and $T: \tilde{E}_{B,p} \to E$ is a linear map, then T is continuos if and only if $T \circ \tilde{i}_{B,C,p}$ is continuous for all $0 \leqslant C < B$. Further information about inductive limit topologies can be found in Horváth [2], Chap.2, §12.

J. Rodríguez [8] has shown that the map

$$\langle , \rangle : \tilde{E} \times O \to \mathbb{C}$$

given by

$$(1.8) \qquad \langle f, g \rangle := \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) g^{(n)}(0),$$

where $\tilde{E}$ is $\tilde{E}_{+\infty,1}$, is a duality bracket (Horváth [2], Chap.3, §2). If E^* and 0^* respectively denote the topological dual spaces of $\tilde{E}$ and 0 (Horváth [2], Chap.3, 2), the duality bracket $\langle,\rangle$ allows for the identifications $\tilde{E}^* = 0$ and $0^* = \tilde{E}$ as vector spaces. He has given interesting applications of the above duality to the theory of complex differential equations.

The main purpose of this paper is to extend those duality results to $\tilde{E}_{B,p}$. This we do by means of an appropriate bracket. The arguments for the general case turn out to be subtler than those in the case $p = 1$, $B = +\infty$. Thorough study of the topologies originating in the duality is also made. Applications to complex differential equations are currently under research and will be the subject of a forthcoming paper.

§2. Basic Results.

The following result characterizes the entire functions in $\tilde{E}_{B,p}$.

THEOREM 2.1. *Let $0 \leqslant C < +\infty$, $0 < p < +\infty$. Then, for $f \in E_{C,p}$,*

$$\left|\frac{1}{n!} f^{(n)}(0)\right| \leqslant \| f \|_{C,p} \left(\frac{Cep}{n}\right)^{n/p}$$

holds for all $n \geqslant 1$. Conversely, if for some $A \geqslant 0$ and some $n_0 \geqslant 1$

$$(2.2) \qquad |a_n| \leqslant A\left(\frac{Cep}{n}\right)^{n/p}$$

holds for $n \geqslant n_0$, then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an entire function in $E_{B,p}$ for all $B > C$.

Proof. Assume $f \in E_{C,p}$. The Cauchy estimates give

(2.4) $$|1/n!f^{(n)}(0)| \leqslant \|f\|_{C,p}\frac{e^{CR^p}}{R^n}, \quad R > 0,\ n \geqslant 0,$$

and (2.1) follows from this by taking $R = (n/pC)^{1/p}$ for each $n \geqslant 1$. Assume conversely that (2.2) holds for $n \geqslant n_o$ and let f be given by (2.3). Clearly $f \in \mathcal{O}$. If

(2.5) $$F_n(r) := e^{Cr^p}/r^n, \quad r > 0,\ n \geqslant 1,$$

then $F_n'(r) = 0$ if, and only if, $r = (n/Cp)^{1/p}$. Since

(2.6) $$\lim_{r\to\infty} F_n(r) = \lim_{r\to 0} F_n(r) = +\infty,$$

it follows that

(2.7) $$F_n\left(\left(\frac{n}{Cp}\right)^{1/p}\right) = \left(\frac{eCp}{n}\right)^{n/p} \leqslant \frac{e^{Cr^p}}{r^n} = F_n(r)$$

for all $r > 0$. Therefore

$$|a_n| \leqslant A\frac{e^{Cr^p}}{r^n}, \qquad n \geqslant n_o.$$

Let $0 \leqslant C < B$ and let $\alpha = (B/C)^{1/p}$, so that $\alpha > 1$. If $z \neq 0$ and $r = \alpha|z|$ then

$$|a_n z^n| \leqslant A\ \alpha^{-n}e^{\alpha^p C|z|^p} = A\alpha^{-n}e^{B|z|^p}, \quad n \geqslant n_o.$$

Hence

$$\left|\sum_{n=n_o}^{\infty} a_n z^n\right| \leqslant A\frac{\alpha^{1-n_o}}{\alpha-1}\ e^{B|z|^p}, \qquad z \neq 0,$$

and this inequality trivialy holds if $z = 0$. Obviously

$$\left|\sum_{n=o}^{n_o-1} a_n z^n\right| \leqslant A'e^{B|z|^p}, \quad z \in \mathbb{C},$$

for some $A' \geqslant 0$. Therefore

$$|f(z)| \leqslant A''e^{B|z|^p}, \quad z \in \mathbb{C},$$

if $A'' \geqslant \max\{A', A\frac{\alpha^{1-n_o}}{\alpha-1}\}$. This proves the theorem. ▲

REMARK 2.1. Further work shows that under conditions (2.2) there is a constant $M > 0$ such that the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies

$$|f(z)| \leq M(1+|z|^p)e^{C|z|^p}, \qquad z \in \mathbb{C}, \tag{2.8}$$

where bounds for M can be given in terms of f and C. This is the best type of estimate we have been able to obtain along this lines. It does not guaranty that $f \in E_{C,p}$.

COROLLARY 2.1. *If*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n , \tag{2.9}$$

then f *is an entire function in* $\tilde{E}_{B,p}$ *if and only if for some* $0 < C < B$, *some* $A \geq 0$, *and some* $n_0 \geq 1$,

$$|a_n| \leq A\left(\frac{eCp}{n}\right)^{n/p} \tag{2.10}$$

holds for all $n \geq n_0$.

REMARK 2.2. *If* $C > 0$ *it follows that*

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{eCp}{n}\right)^{n/p} z^n \tag{2.11}$$

is an entire function in $E_{B,p}$ for all $B > C$. Let $B_1 < B_2$ and take $B_1 < C < B_2$. Then f, as given by (2.11), is in $E_{B_2,p}$. If we had $f \in E_{B_1,p}$ then for some constant $A > 0$ and some $n_0 \geq 1$

$$\left(\frac{eCp}{n}\right)^{n/p} \leq A\left(\frac{eB_1p}{n}\right)^{n/p}, \quad n \geq n_0 .$$

This is absurd as $(C/B_1)^{n/p} \to \infty$ when $n \to \infty$. It follows that $E_{B_1,p} \subseteq E_{B_2,p}$ but $E_{B_1,p} \neq E_{B_2,p}$.

Let P be the algebra of polynomials with complex coefficients. Clearly $P \subseteq E_{B,p}$ for all $0 < B < +\infty$, $0 < p < +\infty$.

Since P is dense in O, *the topology of* $E_{B,p}$ *is strictly stronger than the topology induced from* O. The estimate

$$(2.12) \qquad \|z^n\|_{B,p} \leqslant \left|\frac{n!}{(pB)^n}\right|^{1/p}$$

holds for all $n > 0$. This follows from the Taylor development of e^x, since then

$$\frac{1}{n!}|(pB)^{1/p}z|^{pn} \leqslant e^{pB|z|^p}$$

For each $\alpha > 1$ there is also $n_\alpha \geqslant 1$ such that

$$(2.13) \qquad \|z^n\|_{B,p} \leqslant \alpha \left(\frac{n}{pBe}\right)^{n/p}, \quad n \geqslant n_\alpha.$$

This follows from $(n!/n^n)^{1/n} \leqslant \alpha^p/e$ holding for large n.

THEOREM 2.2. *Let* $f \in \tilde{E}_{B,p}$, $0 < B \leqslant +\infty$, $0 < p < +\infty$. *Then the Taylor series*

$$\sum_{n=0}^{\infty} (1/n!) f^{(n)}(0) z^n$$

converges to f *in* $\tilde{E}_{B,p}$.

Proof. Assume $f \in E_{C,p}$, $0 < C < B$, and let $\alpha > 1$ be such that $\alpha^p C < B$. The Cauchy estimates give

$$\left|\sum_{k=n+1}^{\infty} 1/k!\, f^{(k)}(0) z^k\right| \leqslant \sum_{k=n+1}^{\infty} \|f\|_{C,p} \frac{e^{\alpha^p C|z|^p}}{(\alpha|z|)^k} |z|^k$$

$$\leqslant \sum_{k=n+1}^{\infty} \left(\frac{1}{\alpha}\right)^k \|f\|_{C,p} e^{\alpha^p C|z|^p},$$

so that

$$(2.14) \qquad \left\|\sum_{k=n+1}^{\infty} \frac{1}{k!} f^{(k)}(0) z^k\right\|_{\alpha^p C,p} \leqslant \left\{\sum_{k=n+1}^{\infty} \left(\frac{1}{\alpha}\right)^k\right\} \|f\|_{C,p}.$$

It follows that

$$(2.15) \qquad \sum_{k=n+1}^{\infty} \frac{1}{n!} f^{(k)}(0) z^k \to 0, \quad n \to \infty,$$

in $E_{\alpha^p C, p}$. This proves the theorem. ▲

REMARK 2.3. The proof of the above theorem shows that if $C > 0$ and $f \in E_{C,p}$ then the Maclaurin Series of f converges to f in $E_{B,p}$ for all $B > C$.

COROLLARY 2.2. *For each* $0 < B \leqslant +\infty$, $0 < p < +\infty$, P *is dense in* $\tilde{E}_{B,p}$.

For each $0 < C$, $B < +\infty$, let

$$\rho_{B,C,p} : E_{C,p} \to E_{B,p}$$

be the linear map

$$\rho_{B,C,p}(f)(z) := f((B/C)^{1/p} z) = \tilde{f}(z) \tag{2.16}$$

Since

$$|\tilde{f}(z)| \leqslant \|f\|_{C,p} e^{B|z|^p}$$

then

$$\|\tilde{f}\|_{B,p} \leqslant \|f\|_{C,p},$$

so that

$$\|\rho_{B,C,p}\| \leqslant 1 .$$

On the other hand

$$\rho_{B,C,p} \circ \rho_{C,B,p} = 1_{B,p}, \quad \rho_{C,B,p} \circ \rho_{B,C,p} = 1_{C,p} \tag{2.17}$$

are respectively the identity maps of $E_{B,p}$ and $E_{C,p}$. Then

$$\|\rho_{B,C,p}\| = 1 \tag{2.18}$$

and $\rho_{B,C,p}$ is a toplinear isomorphism of $E_{C,p}$ onto $E_{B,p}$. Clearly

$$\rho_{B,C,p}(P) = P. \tag{2.19}$$

Now we can state the following corollary of Theorem 2.2:

COROLLARY 2.3. *If* $0 < C < B < +\infty$, *the topology on* $E_{C,p}$ *induced from* $E_{B,p}$ *is strictly weaker than the topology of* $E_{C,p}$.

Proof. Assume the two topologies coincide in $E_{C,p}$. From the proof of the theorem, the Maclaurin series of $f \in E_{C,p}$ converges in $E_{B,p}$ to f. Hence, it also converges in $E_{C,p}$ to this function. Then, P is dense in $E_{C,p}$. Since $\rho_{B,C,p}$ is a homeomorphism, also P is dense in $E_{B,p}$. Therefore $E_{C,p} = E_{B,p}$, which is absurd. ▲

§3. Duality. Let $0 \leqslant B < +\infty$, $0 < p < +\infty$, and denote with $\mathcal{D}_{B,p}$ the open disk $\mathcal{D}(0,(Bp)^{1/p})$ and with $\bar{\mathcal{D}}_{B,p}$ its closure ($\bar{\mathcal{D}}_{+\infty,p} = \mathcal{D}_{+\infty,p} = \mathbb{C}$). Also, $\mathcal{O}_{B,p}$ will be the space of analytic functions in $\mathcal{D}_{B,p}$. Let

$$< \, , \, >: \tilde{E}_{B,p} \times \mathcal{O}_{B,p} \to \mathbb{C}$$

be the bilinear map

$$<f,g> := \sum_{n=0}^{\infty} \frac{(n!)^{1/p}}{(n!)^2} f^{(n)}(0) g^{(n)}(0) \tag{3.17}$$

Since

$$<f,z^n> = (n!)^{1/p-1} f^{(n)}(0), \quad <z^n,g> = (n!)^{1/p-1} g^{(n)}(0), \tag{3.2}$$

$<,>$ is non-degenerate, i.e., if $<f,g> = 0$ for all $g \in \mathcal{O}_{B,p}$ then $f = 0$, and also $g = 0$ if $<f,g> = 0$ for all $f \in \tilde{E}_{B,p}$. It is also continuous. To prove this last assertion we will show that the restriction of $<,>$ to $E_{C,p} \times \mathcal{O}_{B,p}$, $0 < C < B$, is bounded. This follows at once from (2.12) and the Cauchy estimates, for if $f \in E_{C,p}$, $g \in \mathcal{O}_{B,p}$ and $C < R < B$ then

$$|<f,g>| \leqslant M_{C,p} \|f\|_{C,p} \|g\|_{\bar{\mathcal{D}}_{R,p}} . \tag{3.3}$$

where

$$(3.4)\qquad M_{C,R} := \sum_{n=0}^{\infty} \left(\frac{n!}{n^n}\right)^{1/p} e^{n/p} \left(\frac{C}{R}\right)^{n/p} < +\infty,$$

as follows from

$$\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/np} e^{1/p} \left(\frac{C}{R}\right)^{1/p} = \left(\frac{C}{R}\right)^{1/p} < 1 .$$

Now let $\tilde{E}^*_{B,p}$ and $O^*_{B,p}$ respectively denote the topological dual spaces of $\tilde{E}_{B,p}$ and $O_{B,p}$. For each $f \in \tilde{E}_{B,p}$ and $g \in O_{B,p}$ let $T(f): O_{B,p} \to \mathbb{C}$ and $L(g): \tilde{E}_{B,p} \to \mathbb{C}$ be the maps

$$(3.5)\qquad T(f)(\Phi) := \langle f, \Phi\rangle, L(g)(\psi) := \langle \psi, g\rangle$$

Obviously $T(f)$ and $L(g)$ are linear. For $f \in E_{C,p}$, $C < B$, $g \in O_{B,p}$ and $C < R < B$,

$$(3.6)\qquad |T(f)(g)| = |L(g)(f)| \leq M_{C,R} \|f\|_{C,p} \|g\|_{\bar{D}_{R,p}}$$

holds. Thus $T(f) \in O^*_{B,p}$, $L(g) \in \tilde{E}^*_{B,p}$.

THEOREM 3.1. *The linear maps* $T: \tilde{E}_{B,p} \to O_{B,p}$ *and* $L: O_{B,p} \to \tilde{E}_{B,p}$ *given by (3.5) are 1-1 and onto. If* $\tilde{E}_{B,p}$ *and* $O^*_{B,p}$ *are respectively given their weak topologies* $\sigma(\tilde{E}^*_{B,p}, \tilde{E}_{B,p})$ *and* $\sigma(O^*_{B,p}, O_{B,p})$*, these maps are also continuous.*

Proof. For the notion of weak topology, see Horváth [2], Chap.3, §2. That T, L are 1-1 follows from the non-degeneracy of $\langle,\rangle$ and their continuity results from (3.6). It remains to prove that they are onto. Let $f^* \in O^*_{B,p}$ and define

$$(3.7)\qquad f(z) := \sum_{n=0}^{\infty} \frac{f^*(z^n)}{(n!)^{1/p}} z^n .$$

Because of the continuity of f^* there are $M > 0$ and $0 < R < B$ such that

$$|f^*(z^n)| \leq M \|z^n\|_{\bar{D}_{R,p}}, \quad n = 0,1,2,...$$

Therefore

$$\frac{|f^*(z^n)|}{(n!)^{1/p}} \leqslant M\frac{(Rp)^{n/p}}{(n!)^{1/p}} \leqslant M\left(\frac{eRp}{n}\right)^{n/p}, \quad n \geqslant 1,$$

and thus $f \in \tilde{E}_{B,p}$. Since

$$(3.8) \qquad f^*\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} b_n f^*(z^n)$$

if the series is uniformly convergent on compact subsets of $\mathcal{D}_{B,p}$, it follows that if $g(z) = \sum_{n=0}^{\infty} b_n z^n \in O_{B,p}$ then $f^*(g) = \langle f, g\rangle$ and thus $f^* = T(f)$.

Now let $g^* \in E^*_{B,p}$. From (2.12) and the continuity of g^*, given $0 < C < B$ there is $M_C > 0$ such that

$$|g^*(z^n)| \leqslant M_C \|z^n\|_{C,p} \leqslant M_C \frac{(n!)^{1/p}}{(Cp)^{n/p}} .$$

If

$$(3.9) \qquad g(z) := \sum_{n=0}^{\infty} \frac{g^*(z^n)}{(n!)^{1/p}} z^n ,$$

from

$$\left|\frac{g^*(z^n)}{(n!)^{1/p}}\right|^{1/n} \leqslant M_C^{\ 1/n} \frac{1}{(Cp)^{1/p}}$$

it follows that the series defining g has radius of convergence $\geqslant (Cp)^{1/p}$ for all $0 < C < B$. Hence $g \in O_{B,p}$. Since the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ of $f \in E_{C,p}$ converges to f in $\tilde{E}_{B,p}$,

$$(3.10) \qquad g^*(f) = \sum_{n=0}^{\infty} a_n g^*(z^n) = \langle f, g\rangle,$$

and thus $g^* = L(g)$. This proves the theorem. ▲

COROLLARY 3.1. *For each* $0 < B \leqslant +\infty$ *and each* $0 < p \leqslant +\infty$, $\tilde{E}_{B,p}$ *is a Hausdorff space.*

Proof. Assume $f(z) = \sum_{n=0}^{\infty} a_n z^n \neq 0$ is in $E_{B,p}$. Then $a_n \neq 0$ for some $n \geqslant 0$ and therefore $L(z^n)(f) \neq 0$. Since $z^n \in O_{B,p}$ then $g^* = L(z^n) \in E^*_{B,p}$, so that the sets

$$U = \{y \in \tilde{E}_{B,p} \mid |g^*(y)| < \tfrac{1}{2}|g^*(f)|\},$$

$$V = \{y \in \tilde{E}_{B,p} \mid |g^*(y)| > \tfrac{1}{2}|g^*(f)|\}$$

are disjoint open neighborhoods of 0 and f respectively. Hcen, $\tilde{E}_{B,p}$ is a Hausdorff space. ▲

REMARK 3.1. The above argument also shows that $\sigma(\tilde{E}_{B,p},\tilde{E}^*_{B,p})$ is a Hausdorff topology. An entirely similar argument proves that $\sigma(O_{B,p},O^*_{B,p})$ is also Hausdorff.

COROLLARY 3.2. *Let $0 < C < B \leqslant +\infty$, $0 < p < +\infty$. Then, the topology of $E_{C,p}$ is strictly stronger than the induced topology from $\tilde{E}_{B,p}$.*

Proof. Since the inclusion map $\tilde{i}_{B,C,p}: E_{C,p} \to \tilde{E}_{B,p}$ is continuous, the induced topology is weaker than $\tau_{C,p}$. Assume they coincide. Since $E_{C,p}$ is a Banach-space and $\tilde{E}_{B,p}$ is Hausdorff, it follows that $E_{C,p}$ is closed in $\tilde{E}_{B,p}$. Since $P \subseteq E_{C,p}$, $E_{C,p} \neq \tilde{E}_{B,p}$ and P is dense in $\tilde{E}_{B,p}$, this is absurd. ▲

COROLLARY 3.3. *Let $0 < C < B$. Then, the topology of $\tilde{E}_{C,p}$ is strictly stronger than the induced topology from $\tilde{E}_{B,p}$.*

Proof. The induced topology is clearly weaker than the topology $\tilde{\tau}_{C,p}$ of $\tilde{E}_{C,p}$. Let g be analytic on $D_{C,p}$ with a singularity on $|z| = (Cp)^{1/p}$, and assume the two topologies coincide on $\tilde{E}_{C,p}$. Since $\tilde{E}_{C,p}$ is dense in $\tilde{E}_{B,p}$ and $L(g)$ is continuous on $\tilde{E}_{C,p}$, $L(g)$ has a continuous extension h^* to $\tilde{E}_{B,p}$. From the theorem there is $h \in O_{B,p}$ such that $h^* = L(h)$. But then $h = g$ on $D_{C,p}$, which is absurd. ▲

COROLLARY 3.4. *If $0 < C < B < +\infty$ then the topology of $\tilde{E}_{C,p}$ is strictly stronger than the induced topology from $E_{B,p}$.*

Proof. The topology of $\tilde{E}_{C,p}$ is strictly stronger than the induced topology from $\tilde{E}_{B,p}$, and the latter is in its own turn stronger than the induced topology from $E_{B,p}$. ▲

Now we will compare different topologies on $E_{B,p}$ having their origin in the duality bracket. We recall that if E, F are vector spaces, a bilinear map

$$(3.10) \qquad B : E\times F \rightarrow \mathbb{C}$$

is called a *pairing* of E and F. The duality bracket <,> is a pairing of $\tilde{E}_{B,p}$ and $O_{B,p}$. Let B be a pairing of E and F. For $x \in E$ and $y \in F$

$$(3.11) \qquad |x|_y := |B(x,y)|, \quad |y|_x := |B(x,y)|$$

are respectively semi-norms on E,F. The topology on E (resp. on F) defined by the semi-norms $|\ |_y$, $y \in F$ (resp. $|\ |_x$, $x \in E$) is called the *weak topology* on E defined by B and is denoted with $\sigma(E,F)$ (resp. the *weak topology* on F defined by B is denoted with $\sigma(F;E)$). The weak topologies are locally convex, vector space topologies. Let B_F be the set of bounded subsets of F for $\sigma(F;E)$ (resp. B_E of E for $\sigma(E,F)$). The locally convex topology on E defined by the family of seminorms

$$(3.15) \qquad |x|_A := \sup_{y\in A}|B(x,y)| \; A \in B_F ,$$

is called the *strong topology* defined on E by the pairing B and is denoted with $\beta(E,F)$. The strong topology $\beta(F,E)$ on F is defined by the semi-norms

$$(3.16) \qquad |y|_A := \sup_{x\in A}|B(x,y)|, \quad A \in B_E.$$

For $x \in E$, $y \in F$, let B^x, B_y respectively denote the linear maps on F, E given by $B^x(z) = B(x,z)$ and $B_y(z) = B(z,y)$. If E', F' are the algebraic dual spaces of E and F then $B^x \in F'$ and $B_y \in E'$. Let $T:E \rightarrow F'$ and $L:F \rightarrow E'$ be given by

$$(3.17) \qquad T(x) := B^x, \quad L(y) := B_y.$$

If both T and L are 1-1, B is called a *duality bracket for E and F*. A locally convex topology τ_E on E (resp. τ_F on F)

is said to be *compatible* with B if $L(F) = E^*$, where E^* is the topological dual of E for τ_E (resp. if $T(E) = F^*$, where F^* in the topological dual of F for τ_F). Among the locally convex topologies on E (resp. F) which are compatible with a duality bracket B there is a strongest, $\tau(E,F)$, called the *Mackey topology of* E *for* B (resp. $\tau(F,E)$ is the Mackey topology of F for B). A locally convex topology τ_E on E (resp. τ_F on F) is compatible with B if and only if $\sigma(E,F) \subseteq \tau_E \subseteq \tau(E,F)$ (resp. $\sigma(F,E) \subseteq \tau_F \subseteq \tau(E,F)$). The relevant facts about duality pairings can be found in Horváth [2], which is our main reference source.

We recall that $\tau_{C,p}$ denotes the Banach space topology of $E_{C,p}$ and $\tilde{\tau}_{B,p}$ is the inductive limit topology of $\tilde{E}_{B,p}$. With $\tau(O_{B,p})$ we will denote the topology on $O_{B,p}$ of uniform convergence on compact subsets of $\mathcal{D}(0,(Bp)^{1/p})$ and we recall that $\tau(O_{B,p})$ is a Fréchet and Montel space topology on $O_{B,p}$. For the notion of a Montel space, see Horváth [2], Chap.3 §9. Both $\tilde{\tau}_{B,p}$ and $\tau(O_{B,p})$ are compatible with the duality bracket $\langle , \rangle$.

Let A be a bounded subset of $O_{B,p}$ for $\sigma(O_{B,p},E_{B,p})$ (Horváth [2], Chap.3 §4). Since $\tau(O_{B,p})$ is compatible with the pairing $\langle , \rangle$ it follows from Mackey's theorem (Horváth [2], Chap.3 §5) that A is bounded (and hence relatively compact) for $\tau(O_{B,p})$. For $f \in E_{C,p}$, $C < B$, let

$$(3.18) \qquad |f|_A := \sup_{g \in A} |\langle f,g \rangle|$$

Then, for some constant $M' > 0$ and $C < R < B$ we have

$$(3.19) \qquad |f|_A \leq M' \| f \|_{C,p} \sup_{g \in A} \| g \|_{\bar{\mathcal{D}}(0,(pR)^{1/p})}$$

and since A is bounded for $\tau(O_{B,p})$, for some constant $M > 0$ depending only on C and A we have

$$(3.20) \qquad |f|_A \leq M \| f \|_{C,p} .$$

It follows that the inclusion maps $\tilde{i}_{B,C,p}: E_{C,p} \to \tilde{E}_{B,p}$ are continuous if $E_{C,p}$ and $\tilde{E}_{B,p}$ are respectively given the topologies $\tau_{C,p}$ and $\beta(\tilde{E}_{B,p}, O_{B,p})$. Therefore

$$(3.21) \qquad \beta(\tilde{E}_{B,p}, O_{B,p}) \subseteq \tilde{\tau}_{B,p} .$$

Hence (Horvath [2], Chap.3, §8),

THEOREM 3.2. *The topology $\tilde{\tau}_{B,p}$ of $\tilde{E}_{B,p}$ coincides with the strong topology $\beta(\tilde{E}_{B,p}, O_{B,p})$ of this space.*

COROLLARY 3.5. *The space $\tilde{E}_{B,p}$ is a complete Montel space for $\tilde{\tau}_{B,p}$.*

Proof. In fact, since $O_{B,p}$ is a Frechet space, it is bornological (Horváth [2], Chap.3, §7). Hence $\tilde{E}_{B,p}$ is complete for $\beta(\tilde{E}_{B,p}; O_{B,p})$ (Hőrvath [2], Chap.3, §7) and also a Montel space (Horváth [2], Chap.3, §9). ▲

Also

COROLLARY 3.6. *On $\tilde{E}_{B,p}$ the topological $\tilde{\tau}_{B,p}$, the strong topology $\beta(\tilde{E}_{B,p}, O_{B,p})$ and the Mackey topology $\tau(\tilde{E}_{N,p}, O_{B,p})$ are all identical.*

Now observe that, for $0 < B < +\infty$, $E_{B,p}$ is subspace of $\tilde{E}_{B,p}$.

THEOREM 3.4. *The topology $\tilde{\tau}_{B,p}$ of $\tilde{E}_{B,p}$ is strictly stronger than the induced topology from $\tau_{B,p}$ of $E_{B,p}$.*

Proof. Let $\tau'_{B,p}$ be the induced topology. Then $\tau'_{B,p} \subseteq \tilde{\tau}_{B,p}$. If $\tau'_{B,p} = \tau_{B,p}$ then $\tilde{E}_{B,p}$ would be a Banach space for $\tilde{\tau}_{B,p}$. But this is absurd, as $\tilde{E}_{B,p}$ is Montel's and infinite dimensional. ▲

REMARK 3.1. The proof of the above theorem shows that no norm on $\tilde{E}_{B,p}$ can define $\tilde{\tau}_{B,p}$.

§4. Further Results and Remarks. In section §2 we have shown that the Maclaurin series of $f \in \tilde{E}_{B,p}$, $0 \leqslant B \leqslant +\infty$, $0 < p < +\infty$, converges to f for $\tilde{\tau}_{B,p}$. So far we have not been able to determine whether the Maclaurin series of $f \in E_{B,p}$, $0 < B < +\infty$, converges to f for $\tau_{B,p}$, nor whether the set P of complex polynomials is dense in $E_{B,p}$. The following results may be however of some interest.

Let $E^*_{B,p}$ denote the topological dual of $E_{B,p}$ and let $\| \|^*_{B,p}$ denote the norm of $E^*_{B,p}$ defining its strong topology $\beta(E^*_{B,p}, E_{B,p})$. Let $i: \tilde{E}_{B,p} \to E_{B,p}$ be the inclusion map. Then i is continuous, so that its transpose $\phi = {}^t i : E^*_{B,p} \to O_{B,p}$ is continuous for the topologies $\sigma(E^*_{B,p}, E_{B,p})$ on $E^*_{B,p}$ and $\sigma(O_{B,p}, \tilde{E}_{B,p})$ on $O_{B,p}$ (Horváth [2], Chap.3, §12). Furthermore

THEOREM 4.1. *The map ϕ is continuous if $E^*_{B,p}$ is given the strong topology of $\| \|^*_{B,p}$ and $O_{B,p}$ its topology $\tau(O_{B,p})$.*

Proof. If $f^* \in E^*_{B,p}$ then also $f^* \in \tilde{E}^*_{B,p}$, so that the series $\sum_{n=0}^{\infty} (f^*(z^n)/(n!)^{1/p}) z^n$ defines an analytic function in $O_{B,p}$. From $\phi(f^*)(z^n) = \langle \phi(f^*), z^n \rangle = f^*(z^n)$ it follows that

$$(4.1) \qquad \phi(f^*) = \sum_{n=0}^{\infty} \frac{f^*(z^n)}{(n!)^{1/p}} z^n .$$

If $0 < (Rp)^{1/p} < (Bp)^{1/p}$ then, from (2.12),

$$(4.2) \qquad \|\phi(f^*)\|_{\bar{D}_{R,p}} \leqslant \Big\| \sum_{n=0}^{\infty} \frac{f^*(z^n)}{(n!)^{1/p}} z^n \Big\|_{\bar{D}_{R,p}}$$

$$\leqslant \|f^*\|^*_{B,p} \sum_{n=0}^{\infty} \frac{1}{(n!)^{1/p}} \left(\frac{n!}{(pB)^n}\right)^{1/p} (pR)^{n/p}$$

$$\leqslant \frac{B^{1/p}}{B^{1/p} - R^{1/p}} \|f^*\|^*_{B,p},$$

which proves the assertion. ▲

REMARK 4.1. Two forms $f^*, g^* \in E^*_{B,p}$ define the same series $\sum_{n=0}^{\infty} a_n z^n$ in $O_{B,p}$ if and only if they coincide on the set P of complex polinomials. This follows from $f^*(z^n) = (n!)^{1/p} a_n = g^*(z^n)$.

REMARK 4.2. Let $E_{B,p}^{**}$ be the topological bidual (Horváth [2], Cahp.1,§7) of $E_{B,p}$ and let $\psi = {}^t\phi : \tilde{E}_{B,p} \to E_{B,p}^{**}$ be the transpose map of ϕ. Then ψ is the composite map $j \circ i$ of i and the inclusion map j of $E_{B,p}$ into its bidual $E_{B,p}^{**}$. If $E_{B,p}^{**}$ is given the topology $\tau_{B,p}^{**}$ of the bidual norm $\| \|_{B,p}^{**}$, then ψ is continuous.

THEOREM 4.2. *Let* (f_n) *be the sequence of partial sums of the Maclaurin series of* $f \in E_{B,p}$. *Then the following propositions are equivalent:*

1) *The sequence* (f_n) *converges to some* f^{**} *in* $E_{B,p}^{**}$ *for* $\sigma(E_{B,p}^{**}, E_{B,p}^{*})$.
2) *For any* $f^* \in E_{B,p}^*$,

$$f^*\left(\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) z^n \right) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) f^*(z^n).$$

3) *For any* $f^* \in E_{B,p}^*$, $\sum_{n=}^{\infty} 1/n! f^{(n)}(0) f^*(z^n)$ *is convergent.*

Proof. It is clear that (1) ⇒ (2) ⇒ (3). To show that (3) ⇒ (1) consider f_n as an element of $E_{B,p}^{**}$. Then

$$(4.3) \qquad \lim_{n\to\infty} f_n(f^*) = \lim_{n\to\infty} f^*(f_n) = \lim_{n\to\infty} \sum_{k=0}^{n} 1/k!\ f^{(k)}(0) f^*(z^k)$$
$$= \sum_{k=0}^{\infty} 1/k!\, f^{(k)}(0) f^*(z^k),$$

and therefore, for some constant $C(f^*)$ depending only on f^*,

$$(4.4) \qquad |f_n(f^*)| \leqslant C(f^*).$$

In view of the Banach-Steinhaus Theorem (Horváth [2], Chap. 1,§8),

$$(4.5) \qquad |f_n(f^*)| \leqslant M \| f^* \|_{B,p}^*$$

for some constant $M > 0$, independent on n and f^*. If f^{**}

denotes the linear form on $E^*_{B,p}$ given by

$$(4.6) \qquad f^{**}(f^*) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) f^*(z^k),$$

it follows from (4.5) that

$$(4.7) \qquad |f^{**}(f^*)| \leqslant M \| f^* \|^*_{B,p} ,$$

so that $f^{**} \in E^{**}_{B,p}$. Clearly $f_n \to f^{**}$ for $\sigma(E^{**}_{B,p}, E^*_{B,p})$. ▲

REMARK 4.3. Now let $0 < B < C < +\infty$. The inclusion map $\tilde{i}_{C,B}: E_{B,p} \to \tilde{E}_{C,p}$ is continuous for $\tau_{B,p}$ and $\tilde{\tau}_{C,p}$. Hence, $j_{C,B} = {}^{tt}i_{C,B}$ is also continuous for $\sigma(E^{**}_{B,p}, E^*_{B,p})$ and $\sigma(\tilde{E}^{**}_{C,p}, O^*_{C,p})$ (Horváth [2], Chap.3, §12), so that $f_n \to j_{C,B}(f^{**})$ for $\sigma(E^{**}_{C,p}, E^*_{C,p})$. But $f_n \to f$ for $\sigma(E_{C,p}, E^*_{C,p})$. Hence $f = j_{C,B}(f^{**})$ for $C > B$. However, it can not be asserted that $f^{**} = f$.

THEOREM 4.3. *Let $f \in O$, and let $p > 0$ and q be such that $1/p + 1/q = 1$. Let (f_n) be the sequence of partial sums of the Maclaurin series of f. If the series*

$$(4.8) \qquad \sum_{n=0}^{\infty} [1/n!]^{1/q} f^{(n)}(0) z^n$$

has radius of convergence $R > 0$, then, for all $B > 0$ such that $1/(pB)^{1/p} < R$, f is in $E_{B,p}$ and $f_n \to f$ in $E_{B,p}$. If $R = \infty$, the above holds for all $B > 0$.

Proof. Let $C < C' < D < B$ and assume $(pC)^{1/p} > 1/R$. Then for some n_0 and all $n \geqslant n_0$

$$\left| \frac{f^{(n)}(0)}{n!} \right|^{1/n} (n!)^{1/np} < (pC)^{1/p},$$

so that if n_0 is large enough then

$$\left| \frac{f^{(n)}(0)}{n!} \right|^{1/p} < \left(\frac{pC}{(n!)^{1/p}} \right)^{1/p} < \left(\frac{pC'e}{n} \right)^{1/p}$$

In view of Theorem 2.1, $f \in E_{\mathcal{D},p}$, so that $f \in E_{B,p}$. From Remark 2.3, we conclude that $f_n \to f$ in $E_{B,p}$. ▲

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