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ON MIKUSINSKI'S OPERATORS OF FRACTIONAL INTEGRATION

by

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ABSTRACT

In the field F of convolution quotients, h^{α} is the operator of integration of fractional order α and $h^{\alpha}f$ is the Riemann-Liouville integral of order α of f. In this paper we give a generalization of this operator, which is denoted as $h_a^{\alpha,\nu}$. Some particular cases are mentioned and the inverse operator is obtained.

1. Introduction. In a certain approach to the solution of mixed boundary value problems, an important part is played by certain operators of fractional integration. A brief account of operators of fractional integration is given in Sneddon's book [13]. Operators of fractional integration involving generalized hypergeometric functions have been defined and discussed by Kalla [6, 7, 8]. Riemann-Liouville and Weyl fractional integrals and their connections with certain integral transforms are given in [3, 4, 5]. Kalla and Saxena [9] have discussed integral operators involving Gauss hypergeometric function $_2F_1$, and they have established their connections with the Hankel operator [10] .

On the other hand, when we look into the field F of convolution quotient [1, 12] . h, the constant function $\{1\}$, which is the restriction of Heaviside's unit function to the half line $t \geq 0$, plays an important role as an operator. h^{α} is regarded as the operator of integration of fractional order α, and $h^{\alpha} f$ the Riemann-Liouville integral of order α of f.

The object of the present paper is to generalize the operator $h^{\alpha} f$. We denote the generalized operator as $h_a^{\alpha,\nu} f$. Several special cases of this operator are mentioned and the inverse operator is discussed. Integral operators involving Bessel functions can also be derived from our generalized operator.

2. *Definition and Special Cases.* We have [1, p. 14]

$$h(t) = 1, \tag{1}$$

$$h(t) * h(t) = h^2(t) = \int_0^t h(x)\, h(t-x)\, dx = t, \tag{2}$$

and

$$h^{\alpha} = \left\{ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right\}, \quad Re(\alpha) > 0 . \tag{3}$$

Thus, the Riemann-Liouville fractional integration of order α may be considered as

$$h^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \qquad Re(\alpha) > 0 \tag{4}$$

If we set

$$f(t) = t^{\nu-1} e^{at}, \qquad Re(\nu) > 0, \quad Re(a) > 0, \tag{5}$$

then [3, p.187]

$$b^{\alpha}(s-a)^{-\nu} \leftrightarrow \frac{x^{\alpha+\nu-1}}{\Gamma(\alpha+\nu)}\, {}_1F_1(\nu ; \alpha+\nu ; ax) . \tag{6}$$

where

$$b^{-1} = s \quad \text{and} \quad b^{-\alpha} = s^{\alpha} . \tag{7}$$

The relation (6) can be rewritten as

$$\frac{(s-a)^{-\nu}}{s^{\alpha}} \leftrightarrow \frac{x^{\alpha+\nu-1}}{\Gamma(\alpha+\nu)} \cdot {}_1F_1(\nu ; \alpha+\nu ; ax) \tag{8}$$

We shall denote the operator $\frac{(s-a)^{-\nu}}{s^{\alpha}}$ by $b_a^{\alpha,\nu}$; thus

$$b_a^{\alpha,\nu} \equiv \frac{(s-a)}{s^{\alpha}} \tag{9}$$

For any elements $g \in F$, we have

$$b_a^{\alpha,\nu} g = \frac{1}{\Gamma(\alpha+\nu)} \int_0^x (x-t)^{\alpha+\nu-1} \, {}_1F_1(\nu ; \alpha+\nu ; a(x-t)) . g(t)\, dt , \tag{10}$$

$Re(\nu) > 0$, $Re(\alpha) > 0$. When $\nu \to 0$

$$b_a^{\alpha,0} g = b^{\alpha} g = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t)\, dt , \quad Re(\alpha) > 0 , \tag{11}$$

we obtain the Riemann-Liouville fractional integral of order α(4) .

We mention some special cases of our generalized operator $b_a^{\alpha,\nu}$.

(i) By virtue of the relation [11, p. 271]

$${}_1F_1(\nu ; \nu ; ax) = e^{ax} \tag{12}$$

we have
$$b_a^{o,\nu} g = \frac{1}{\Gamma(\nu)} \int_o^x (x-t)^{\nu-1} e^{a(x-t)} g(t)\, dt . \qquad (13)$$

Similarly on using the special cases of the confluent hypergeometric function $_1F_1$ [11, p. 271], we obtain the following particular cases of our operator :

(ii) $$b_{c^2}^{1,\frac{1}{2}} g = \frac{2}{c\sqrt{\pi}} \int_o^x Erf. \; [\, c(x-t)^{\frac{1}{2}} \,]\, g(t)\, dt , \qquad (14)$$

where *Erf.* stands for error function [11] .

(iii) If we replace α and ν by $\nu+1$ and $\nu+\frac{1}{2}$, respectively, then we obtain

$$b_a^{\nu+1;\nu+\frac{1}{2}} g = \frac{2^{2\nu}\,\Gamma(\nu+1)}{a^{\nu}\,\Gamma(2\nu+1)} \int_o^x exp\left(\frac{a(x-t)}{2}\right) I_\nu \left(\frac{a(x-t)}{2}\right) g(t)\, dt . \qquad (15)$$

iv) If $\nu = \mu+\frac{1}{2}-k$ and $\alpha = \mu+\frac{1}{2}+k$ then we get

$$b_a^{\alpha+\frac{1}{2}+k;\mu+\frac{1}{2}-k} g = \frac{1}{a^{\mu+\frac{1}{2}}\,\Gamma(2\mu+1)} \int_o^x (x-t)^{-k-1} exp\left(\frac{a(x-t)}{2}\right) M_{k,\mu} [a(x-t)]\, g(t)dt \qquad (16)$$

3. *Inverse Operator . THEOREM : If* $Re(\alpha) > 0$, $g \in F$, $\varphi(0) = \varphi'(0) = \ldots = \varphi^{(n-1)}(0) = 0$, $n > Re(\alpha+\nu) > 0$ *and*

$$b_a^{\alpha,\nu} g = \frac{1}{\Gamma(\alpha+\nu)} \int_o^x (x-t)^{\alpha+\nu-1} {}_1F_1(\nu\,;\,\alpha+\nu\,;a(x-t)\,)\, g(t)\, dt = \varphi ,(say) . \qquad (17)$$

then

$$g(x) = \frac{1}{\Gamma(n-\alpha+\nu)} \int_o^x (x-t)^{n-\alpha+\nu-1} {}_1F_1(-\nu\,;n-\alpha+\nu\,;a(x-t))\, \varphi^{(n)}(t)\, dt , \qquad (18)$$

that is to say, if

$$h_a^{\alpha,\nu} g = \varphi \tag{19}$$

then

$$g = h_a^{n-\alpha,-\nu} \varphi^{(n)} \tag{20}$$

Proof : **We have**

$$h_a^{\alpha,\nu} \leftrightarrow \frac{(s-a)^{-\nu}}{s^{\alpha}} \tag{21}$$

hence

$$g = s^{\alpha}(s-a)^{\nu} \varphi$$

$$= s^{-\nu-(n-\alpha-\nu)} (s-a)^{\nu} (s^{n} \varphi) = h_a^{n-\alpha,-\nu} (\varphi^{(n)}), \tag{22}$$

by virtue of the result [1, p. 28] ,

$$s^{n} f = f^{n} + f^{n-1}(0) + f^{n-2}(0) . s + \cdots + f(0) . s^{n-1} \tag{23}$$

The results (17) and (18) are in agreement with those given by Wimp [14] .

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