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LENGTHS IN SEMI-FREE GROUPS

by

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§1. Introduction. Lyndon [3] introduced the idea of a length function on a group in order to give a free product structure for the group in terms of the length function. Work has been done on length functions by Harrison [1], Hoare [2] and others with the objective of studying the structure of groups equipped with length functions. In this paper we introduce the concept of a semi-free group and prove that a group G, with a length function satisfying certain conditions, can ba embedded in a semi-free group.

§2. Length functions. A length function $|\ |:G \to \mathbb{R}$ assigns to each element x of a group G a real number $|x|$ such that the following axioms are satisfied for all $x,y,z \in G$:

A_1. $|x| = 0$ if and only if $x = 1 \in G$.

A_2. $|x^{-1}| = |x|$.

A_4. $d(x,y) < d(x,z)$ implies $d(y,z) = d(x,y)$ where $d(x,y) = \frac{1}{2}(|x|+|y|-|xy^{-1}|)$.

An equivalent formulation of A_4 is that $d(x,y) \geq m$ and $d(y,z) \geq m$ imply $d(x,z) \geq m$. In either form, the two samllest numbers in the triple of real numbers are equal.

The numbering of the axioms is that of Lyndon [3], Hoare [2] requires a length function on a group to satisfy only the axioms A_2 and A_4 given by Lyndon in [3]. Lyndon's axiom A_3, which states that $d(x,y) \geqslant 0$ is a consequence of A_4 since $d(x,1) = d(1,y) = 0$ and, it follows from this that $d(x,x) = |x| \geqslant 0$. It is an immediate consquence of A_2 that $d(x,y) = d(y,x)$.

Among other consequences of the axioms, Lyndon proved in [3] the following proposition:

PROPOSITION 1. *$d(x,y) + d(x^{-1},y^{-1}) \geqslant |x| = |y|$ implies $|(xy^{-1})^2| \leqslant |xy^{-1}|$.*

Thus a group G with a length function might contain non-trivial elements x with $|x^2| \leqslant |x|$. Let $N = \{x \in G: |x^2| \leqslant |x|\}$.

PROPOSITION 2. *Let x be an element of G. Then $x \in N$ implies $|x^n| \leqslant |x|$ for all integers $n \geqslant 0$.*

Proof. The result holds trivially for $n = 0,1$ and by definition for $n = 2$. Assume the result holds for all non-negative integers $\leqslant n$. That is, $|x^{n-1}|$, $|x^n| \leqslant |x|$. Now

$$2d(x^n,x) = |x^n| + |x| - |x^{n-1}| \geqslant |x^n|,$$

$$2d(x,x^{-1}) = |x| + |x| - |x^2| \geqslant |x| \geqslant |x^n|.$$

Hence by A_4, $2d(x^n,x^{-1}) \geqslant |x^n|$; giving $|x^{n+1}| \leqslant |x|$, and therefore, the result holds for all integers $n \geqslant 0$. ▲

The Proposition implies that for $x \in N$, the lengths $|x^n|$ are bounded by $|x|$. For the case x not in N, it is proved in [1] and [3] that the lengths $|x^n|$ are unbounded.

§3. Semi-free groups. Let X be a set of symbols x_i, x_i^{-1}; where i is in some index set I, with a one to tone correspondence $\mu: X \to X$ such that $\mu x_i = x_i^{-1}$ and $\mu x_i^{-1} = x_i$. Then μ

is an involution on X; and we will say that X is a *set with involution*.

We form words on X as finite products of elements from X, and denote the empty word by e. For every $i \in I$ and every $x_i \in X$ we set $x_i x_i^{-1} = e$, $x_i^{-1} x_i = e$ and call x_i, x_i^{-1} an inverse pair. A group G having a presentation:

$$G = (X : xx^{-1} = e,\ x, x^{-1} \in X)$$

shall be called a *semi-free group on X*.

We note that if $x \neq x^{-1}$ for every $x \in X$, then G is the free group on X. Here we are taking $X = X^{-1}$ and we do not require that x, x^{-1} be distinct for every $x \in X$. That is, $\mu x = x$ for some $x \in X$ is possible, and therefore the relation $x = x^{-1}$ might hold in G. Thus the class of free groups is a subclass of the class of semi-free groups in the above sense.

The semi-free group on a given set X with involution is constructed in the same way we construct the free group on a given set. To make this more precise we give the following

DEFINITION 1. Let $X = X^{-1}$ be a subset of a group G. Then G is *semi-free* on X if and only if the following two conditions hold:

(1) X generates G,

(2) If $x \in G$, $x = x_1 x_2 \ldots x_n$, $n \geqslant 1$
where $x_i^{\pm 1} \in X$, $x_i x_{i+1} \neq 1$ for $1 \leqslant i \leqslant n-1$, then $x \neq 1$ in G.

DEFINITION 2. Let G be a semi-free group on $X = X^{-1}$. The word $x_1 x_2 \ldots x_n$, $n \geqslant 1$, $x_i^{\pm 1} \in X$ is said to be a *reduced* word on X if $x_i x_{i+1} \neq 1$ (the identity of G) for $1 \leqslant i \leqslant n-1$.

The empty word is thus a reduced word by definition; and we point out here the known fact in combinatorial group theory that a reduced word on a generating set X is unique.

Therefore, we may consider a reduced word on X as representing an element of the group G, with the empty word e representing the identity 1 of G.

DEFINITION 3. If $x \in G$, $x = x_1 \dots x_n$ and $x_1 \dots x_n$ is a reduced word on X, then the length $|x|$ of x is defined to be n.

PROPOSITION 3. *$|\ |$ of definition 3 is a length function on G satisfying the axioms A_1, A_2, A_4 and the following conditions:*

Co. *$d(x,y)$ is an integer for all $x,y \in G$,*

C1. *$x \neq 1$, $|x^2| \leqslant |x|$ implies $|x|$ is odd,*

CN. *$|x^2| \leqslant |x|$ implies $x = x^{-1}$.*

Proof. The verification of A_1 and of A_2 for all x in G is immediate upon writing x as a reduced word on X. The condition Co. is clearly satisfied since $|x|$ is an integer for all x in G and $d(x,y)$ is the number of cancellations (deleting inverse pairs) in the product xy^{-1}.

To verify A_4 for any $x,y,z \in G$, let x,y,z be expressed as reduced words on X as follows:

$$x = x_1 \dots x_n; \qquad y = y_1 \dots y_m; \qquad z = z_1 \dots z_k.$$

Let $d(x,y) = r$, $d(x,z) = t$, $d(y,z) = s$. When $r < t$, then from $yz^{-1} = yx^{-1}xz^{-1}$, we have

$$y_1 \dots y_{m-s} z_{k-s}^{-1} \dots z_1^{-1} = y_1 \dots y_{m-r} x_{n-r}^{-1} \dots x_1^{-1} x_1 \dots x_{n-t} z_{k-t}^{-1} \dots z_1^{-1}.$$

Since $t > r$, we have $n-r > n-t$ and the number of factors left from the product $x_{n-r}^{-1} \dots x_{n-t}$, after cancellations, is at most $t-r$. Taking into consideration the possibility of further cancellations, we have $m-s+k-s \leqslant (m-r)+(t-r)+(k-t)$, giving $s \geqslant r$.

Now, since x,y,z are any elements in G, then when $d(y,z) < d(x,z)$; that is, when $s < t$, we have from $yx^{-1} = yz^{-1}zx^{-1}$

$$y_1 \dots y_{m-r} x_{n-r}^{-1} \dots x_1^{-1} = y_1 \dots y_{m-s} z_{k-s}^{-1} \dots z_1^{-1} z_1 \dots z_{k-t} x_{n-t}^{-1} \dots x_1^{-1}.$$

Since $t > s$, we have $k-s > k-t$ and a similar argument as above yields $m-r+n-r \leqslant (m-s)+(t-s)+(n-t)$, giving $r \geqslant s$. Thus $s = r$; that is, $d(y,z) = d(x,y)$.

To verify C1, let $x \neq 1$, $|x^2| \leqslant |x|$ and suppose $|x|$ is even. Let $x = x_1 \dots x_{2m}$, a reduced word on X. Then

$$|x_1 x_2 \dots x_{2m} x_1 x_2 \dots x_{2m}| \leqslant 2m, \qquad m \geqslant 1.$$

Therefore, the number of cancellations in $x_1 \dots x_{2m} x_1 \dots x_{2m}$ is greater or equal than m. Thus $x_1 x_{2m} = 1$, $x_1 x_{2m-1} = 1, \dots$ $\dots, x_m\ x_{m+1} = 1$, giving $x = 1$ in G and contradicting that $x \neq 1$. Hence $|x|$ is odd.

For the condition CN, assume that $|x^2| \leqslant |x|$. If $x = 1$ in G, then $x = x^{-1}$ holds trivially. Therefore, assume that $x \neq 1$, $|x^2| \leqslant |x|$, and suppose $x \neq x^{-1}$. Then $x^2 \neq 1$ and by A_1, $|x^2| > 0$. By definition, $x \in N$ and by the remark following Proposition 2, since the lengths $|x^n|$ are bounded by $|x|$ for all integers $n \geqslant 0$, we can conclude that $x^2 \in N$. $|x^2|$ is odd by C1. On the other hand, by C0, $d(x,x^{-1}) = |x| - \frac{1}{2}|x^2|$ is an integer; that is, $|x^2|$ is even, acontradiction, and so $x = x^{-1}$. Moreover, this shows that C0. and C1. imply CN. ▲

The length function of Proposition 3 is called *the natural length function* on the semi-free group G with respect to X.

§4. Embedding theorem. This section considers an embedding problem for a group G equipped with a length function satisfying A_1, A_2, A_4, C0, C1, and a fortiori CN.

For $1 \neq x \in G$, $m \in Z$, let $S = \{(x,m) : 1 \leqslant m \leqslant |x|\}$, and define $\mu : S \to S$ by

$$\mu : (x,m) \to (x^{-1}, |x| - m+1).$$

Then μ is an involution on S, and we write

$$(x,m)^{-1} = (x^{-1}, |x| - m+1).$$

DEFINITION 4. For $x,y \in G$, we define

$$(x,m) \overset{*}{\sim} (y,n) \quad \text{if} \quad d(x,y) \geqslant m = n,$$

$$(x,m) \overset{**}{\sim} (y,n) \text{ if } d(x^{-1},y^{-1}) \geqslant |x| - m+1 = |y| - n+1.$$

$\overset{*}{\sim}$ and $\overset{**}{\sim}$ are equivalence relations on S, A_4 gives transitivity. Let $\sim$ be the transitive closure of the union of the two relations.

LEMMA 1. *For $x,y,z \in G$ and integers $r,s > 0$,*

$$(x,r) \overset{*}{\sim} (y,r) \overset{**}{\sim} (z,s) \rightleftarrows (x,r) \overset{**}{\sim} (xy^{-1}z,s) \overset{*}{\sim} (z,s).$$

Proof. By symmetry, it is sufficient to prove only "→".

$$d(x,y) \geqslant r \text{ and } d(y^{-1},z^{-1}) \geqslant |y| - r+1 = |z| - s+1$$

Since

$$d(z^{-1}y,y) = |y| - d(y^{-1},z^{-1}) \leqslant r-1 < d(x,y),$$

we have by A_4, $d(x,z^{-1}y) = d(z^{-1}y,y)$; hence $|xy^{-1}z| = |z| - |y| + |x| = s-r+|x|$, and

$$\begin{aligned} d(xy^{-1}z,z) &= \tfrac{1}{2}(|xy^{-1}z| + |z| - |xy^{-1}|) \\ &= |x| - |y| + \tfrac{1}{2}(|x| + |y| - |xy^{-1}|) \\ &= s-r + d(x,y) \geqslant s. \end{aligned}$$

Therefore $(xy^{-1}z,s) \overset{*}{\sim} (z,s)$.

$$\begin{aligned} d(z^{-1}yx^{-1},x^{-1}) &= \tfrac{1}{2}(|xy^{-1}z| + |x| - |y^{-1}z|) \\ &= |x| - |y| + \tfrac{1}{2}(|y| + |z| - |y^{-1}z|) \\ &= |x| - |y| + d(y^{-1},z^{-1}) \\ &\geqslant |x| - r+1 = |xy^{-1}z| - s+1. \end{aligned}$$

Therefore $(x,r) \overset{**}{\sim} (xy^{-1}z,s)$.

Lemma 1 allows us to collect the relations $\overset{*}{\sim}$ and $\overset{**}{\sim}$, and so the following two consequences are immediate.

COROLLARY 1. $(x,r) \sim (y,s)$ *if and only if there is a* (z,r) *such that* $(x,r) \overset{*}{\sim} (z,r) \overset{**}{\sim} (y,s)$.

COROLLARY 2. $(x,r) \sim (y,s) \rightleftarrows (x,r)^{-1} \sim (y,s)^{-1}$.

Let $[x,r]$ be the equivalence class of (x,r) under the relation $\sim$, and define $[x,r]^{-1} = [(x,r)^{-1}]$. Let

$$K = \{[x,r] : x \in G,\ 1 \leqslant r \leqslant |x|,\ r \in Z\}.$$

Then K is a set with involution given by $\mu:[x,r] \mapsto [x,r]^{-1}$.

LEMMA 2. $[x,r] \neq [x,r]^{-1}$ *unless* $[x,r] = [y,r]$ *for some* $y = y^{-1} \in G$ *and* $r = \frac{1}{2}(|y|+1)$.

Proof. Suppose $[x,r] = [x,r]^{-1}$. This implies that $(x,r) \sim (x^{-1},|x|-r+1)$. By Corollary 1, there is (y,r) such that $(x,r) \overset{*}{\sim} (y,r) \overset{**}{\sim} (x^{-1},|x|-r+1)$. That is, $d(x,y) \geqslant r$ and

$$d(y^{-1},x) \geqslant |y|-r+1 = |x^{-1}| - |x| + r - 1 + 1 = r .$$

Whence, $r = \frac{1}{2}(|y|+1)$.

Now $d(x,y) \geqslant r$ and $d(y^{-1},x) \geqslant r$ imply by A_4, $d(y,y^{-1}) \geqslant r$. That is, $\frac{1}{2}(|y|+|y|-|y^2|) \geqslant r$, giving $|y| \geqslant |y^2| + 1$, which yields $|y| > |y^2|$. Thus, either $y = 1 \in G$ and hence $r = 0$, which gives a contradiction, or $|y|$ is odd by C1 and $y = y^{-1}$ by CN. Hence the conclusions of the lemma. ▲

Let F_k be the set of all reduced words on K. Then the empty word e is in F_k and $K \subseteq F_k$. Since K is a set with involution μ, which may have fixed elements in K, we have that F_k is, in fact, a semi-free group on K.

DEFINITION 5. Define $\psi:G \to F_k$ by $\psi(1) = e$ and $\psi(x) = [x,|x|]\ldots[x,1]$, $x \neq 1$.

LEMMA 3. $\psi(x)$ *as defined is a reduced word in* F_k.

Proof. $\psi(1) = e$ is a reduced word in F_k. For $x \neq 1 \in G$, suppose $\psi(x)$ is not a reduced word. Then $[x,r] = [x,r+1]^{-1}$

for $1 \leqslant r \leqslant |x| - 1$; and this means that there is (y,r) such that $(x,r) \overset{*}{\sim} (y,r) \overset{**}{\sim} (x^{-1},|x|-r)$. That is, $d(x,y) \geqslant r$ and $d(y^{-1},x) \geqslant |y| - r+1 = |x^{-1}| - |x| + r+1 = r+1$; hence $r = \frac{1}{2}|y|$, $|y|$ is even. But by A_4 we have that, $d(y,y^{-1}) \geqslant r$, giving $|y|-\frac{1}{2}|y^2| \geqslant r = \frac{1}{2}|y|$, which gives $|y^2| \leqslant |y|$. Therefore, either $y = 1$ and hence $r = 0$, which is a contradiction; or $y \neq 1$, $y = y^{-1}$ by CN and $|y|$ is odd by C1, which contradicts that $|y|$ is even. Hance there is no such (y,r), and $\psi(x)$ is reduced.

LEMMA 4. *ψ as defined is a monomorphism.*

Proof. We first show that ψ is a homomorphism. Let $d(x,y) = r$. Then $|x|, |y| \geqslant r$ and $|xy^{-1}| = |x|+|y|-2r$.

$$\psi(xy^{-1}) = |xy^{-1}, |x|+|y|-2r| \ldots |xy^{-1},1|.$$

Now for $t \leqslant r$, $d(x,y) \geqslant t$; therefore,

$$[x,t] = [y,t] = [y^{-1},|y| - t + 1]^{-1}.$$

Also,

$$d(x^{-1},yx^{-1}) = |x| - d(x,y) = |x| - r,$$
$$d(xy^{-1},y^{-1}) = |y| - d(x,y) = |y| - r,$$

Therefore,

$$[x,|x| - t+1] = [xy^{-1},|xy^{-1}| - t+1], t \leqslant |x| - r ;$$
$$[y^{-1},t] = [xy^{-1},t], \; t \leqslant |y| - r.$$

Hence

$$\begin{aligned}\psi(x)\psi(y^{-1}) &= [x,|x|]\ldots[x,1][y^{-1},|y|]\ldots[y^{-1},1]\\ &= [x,|x|]\ldots[x,r+1][y^{-1},|y|-r]\ldots[y^{-1},1]\\ &= [xy^{-1},|xy^{-1}|]\ldots[xy^{-1},|xy^{-1}|-|x|+r+1]\\ &= [xy^{-1},|y|-r]\ldots[xy^{-1},1]\\ &= \psi(xy^{-1}).\end{aligned}$$

Now by Lemma 3, $\psi(x)$ is a reduced word in F_k for all $x \in G$, and so Ker$\psi = 1$. Hence ψ is a monomorphism of G into F_k. ▲

Finally, by Lemma 3, $|\psi(x)| = |x|$ for every $x \in G$, and because of the involution μ on K, $|\psi(x)|$ is the corresponding length function on F_k. By Lemma 4, G is embedded in F_k. Identifying G with its image $\psi(G)$ in F_k, we have that the restriction of the length function on F_k to G is the given length function on G. We have therefore proved the following theorem (recall that C0, C1 imply CN).

THEOREM 1. *Let G be a group with a length function satisfying* A_1, A_2, A_4, C0 *and* C1. *Then G can be embedded in a semi-free group* F_k *on a set K constructed above. Moreover, the given length function on G is the restriction to G of the length function on* F_k *with respecto to K.*

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