Interactions between non-commutative algebraic geometry and skew PBW extensions

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Abstract: We study some relations and interactions between non-commutative algebraic geometry and the skew PBW extensions. For this we will introduce a new class of non-commutative rings, the semi-graded rings, and for them we will prove a generalization of the famous Serre-Artin-Zhang-Verevkin theorem. Semi-graded rings extend not only skew PBW extension but also graded rings.

Resumen: Estudiamos algunas relaciones entre geometría algebraica no conmutativa y las extensiones PBW torcidas. Para esto, introducimos una nueva clase de anillos, los anillos semi-graduados, y para ellos demostraremos una versión generalizada del famoso teorema de Serre-Artin-Zhang-Verevkin. Los anillos semi-graduados no solo generalizan las extensiones PBW torcidas sino también los anillos graduados.

Keywords: Skew PBW extensions, quantum algebras, graded and semigraded rings, filtration, non-commutative algebraic geometry, non-commutative schemes, quasi-coherent sheaves, coherent sheaves, abelian categories.

Palabras clave: Extensiones PWB torcidas, álgebras cuánticas, anillos graduados y semi-graduados, filtraciones, geometría algebraica no conmutativa, esquemas no conmutativos, haces cuasi-coherentes, haces coherentes, categorías abelianas.

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Contents

Introduction

The classical algebraic geometry have been built upon the correspondence between algebraic objects such as rings and ideals and the geometric structures of curves, surfaces, and more general varieties and schemes. Since 1950 all the advantage of categorical language have been adapted using all the developments of Grothendieck and his school [14]. The rich and beautiful of this subject is integral to modern expositions of the field, such as [7], [15] and [31], and has been essential to most recent progress in both algebraic geometry and commutative algebra.

A basic idea in algebraic geometry, relying on the duality between commutative rings R and their spectra $X = SpecR$, is to study algebraic problems via geometrical methods/results and viceversa. In special, by the affine version of the famous **Serre's global sections theorem**, the category of R-modules is equivalent to the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules for a geometric object X. A natural question is whether something similar holds for some non-commutative noetherian rings; many authors have asked if for certain quantum algebras and non-commutative rings any scheme theory in the sense of Grothendieck is available. However, try to use the ideals in this case is a useless work. A less ambitious goal, but still worthy to be considered, would be to use the *commutative intuition* to generalize some of results which might improve our understanding of certain concrete problems in non-commutative algebra. This is in a sense the basic philosophy and driving force behind a lot of work done in recent years in search of *non-commutative algebraic geometry*.

One of the most important active areas in non-commutative algebraic geometry is the classification of non-commutative projective objects. The use of geometric techniques to study non-commutative graded rings has had important applications to noncommutative algebra. A lot of ideas, techniques and doctoral research has been done in the last twenty years (see [9], [17], [25], [28]). These include Artin and Stafford's classification of *non-commutative projective curves* [27] and the use of geometric techniques to study and classify the non-commutative analogues of $\mathbb{P}^2$, and in general setting $\mathbb{P}^n$. The work of M. Artin and J.J. Zhang in which they propose a model for non-commutative projective scheme, inspired by the projective version of Serre's theorem [2], is considered fundamental for beginners. For another approach, via Ore sets and the micro-localization techniques using the natural filtrations on the considered rings, we find the propose of F. Van Oystaeyen [32].

With the work of M. Artin and J.J. Zhang [2] as corner stone, non-commutative projective algebraic geometry is done taken a non-commutative graded algebra A and associating a category $Proj(A)$ in which one can do geometry. More specifically, $Proj(A)$ is a quotient category of the module category GrA of graded modules by the dense subcategory of direct limits of finite-dimensional modules. This is not a new idea. In the 50's Serre taught us that the projective algebraic geometry of a commutative graded ring R is the study of a quotient category $ProjR$ of the graded module category GrR. More precisely, let R be a commutative connected $\mathbb{N}$-graded K-algebra, where K is an algebraically closed field, and assume R is generated by R_1 as a K-algebra. Let $(X, \mathcal{O}_X) = ProjR$ be the projective scheme defined by R. We define an equivalence relation on graded R-modules by $M \sim M'$ if there is an integer n for which $M_{\geq n} \cong M'_{\geq n}$ where $M_{\geq n} = \oplus_{d \geq n} M_d$.

On the other hand, in their seminal paper [19] O. Lezama and C. Gallego propose a new way of understand non-commutative rings through the use of the technique of filtration and graduation. This procedure has been essential to show good properties as Hilbert basis theorem, Goldie theorem, and more algebraic and homological properties [20]. They generalized the PBW (Poincaré-Birkhoff-Witt) extensions and Ore extensions and include in this classification many important classes of non-commutative rings and algebras coming from quantum mechanics. Quantum polynomials algebras and universal enveloping algebras of finite-dimensional Lie algebras are examples of skew PBW (Poincaré-Birkhoff-Witt) extensions (see [2], [4], [13], [20] and [23]). Recently, exists special interest in developing the non-commutative projective algebraic geometry for finitely graded algebras (see [12], [17] and [23]. However, for non $\mathbb{N}$-graded algebras, in particular, for many important classes of skew PBW extensions, only few works have been realized (see [9]).

In this thesis we gives the basic steps to develop the non-commutative projective algebraic geometry for skew PBW extensions and some non $\mathbb{N}$-graded algebras and rings, defining a new class of rings: the *semi-graded rings*. As we will see, the semi-graded rings generalize the finitely graded algebras and the skew PBW extensions. Based in the work of Artin and Zhang and collaborators [23], [25], [27]. we generalized all the non-commutative techniques existent for graded algebras. We will discuss the most basic problems on non-commutative algebraic geometry for semi-graded rings. The problems to be discussed are around the following topics: generalized Hilbert series and Hilbert polynomial, generalized Gelfand-Kirillov dimension, non-commutative schemes associated to semi-graded rings. Our main result is a new version of Artin-Zhang theorem for semi-graded algebras over non-commutative rings with special restrictions.

We start a new perspective for the recent work to understand the geometry inherent to non-commutative algebras, objects as Artin-Schelter regular algebras, Calabi-Yau algebras, skew Calabi-Yau algebras are of special interest in mathematical physics. With our propose we try to extend the power of Artin-Zhang methods for algebras do not studied before. In this new framework main questions are adapted to the semi-graded case: how can we describe the geometric data ?, which modules can be adapted as component of this data?, is possible to extend the point modules technique?. We hope that this ideas can be studied and applied as a new tool in noncommutative algebraic geometry for more algebras, this document is the first step in this mission.

In the remainder of this introduction, we give a overview of the context and main results of this thesis. In the first chapter we review some basic notions about graded algebras and its modules (see [23]), we introduce the classical notions of Hilbert polynomials and Hilbert series, and classical Gelgand-Kirillov dimension (see [6]). We complete this chapter with the definition of skew PBW extensions introduced in [19].

In the second chapter we introduced our main goal: we define the semi-graded rings as a generalization of skew PBW extensions and non-commutative graded algebras. We include a generalization of Hilbert series and Gelfand-Kirillov dimension for this structures. Finally, in third chapter we include the main result concerning to semi-graded rings, we will adapt a categorical equivalence for some skew PBW extensions under a special restriction. We present the theoretical tools for obtain a new general version of the Serre-Artin-Zhang-Verevkin theorem in a categorical setting, some elementary examples are included have not been studied before in the literature.

Chapter 1

Preliminaries and basic tools

In this chapter we review some basic notions about graded algebras and its modules (see [23]), we introduce the classical notions of Hilbert polynomials and Hilbert series, and classical Gelgand-Kirillov dimension (see [6]). We complete this chapter with the definition of skew PBW extensions introduced in [19]; some basic facts and theorems relative to skew PBW extensions are shown.

All the techniques and definitions are introduced for support the main results of this thesis that will be present in 2.1 and 3.1.

1.1 Finitely graded algebras: definition and basic properties

In Definition 1.4 of [23] is recalling the concept of finitely graded algebra. We start with this concept and we present some classical properties of this type of algebras needed in any approach of non-commutative algebraic geometry.

Definition 1.1.1. *Let K be a field. It is said that a K-algebra A is finitely graded if the following conditions hold:*

(i) *A is $\mathbb{N}$-graded: $A = \bigoplus_{n\geq 0} A_n$.*

(ii) *A is connected, i.e., $A_0 = K$.*

(iii) *A is finitely generated as K-algebra.*

Proposition 1.1.2. *Let A be a connected $\mathbb{N}$-graded K-algebra. A is finitely generated as K-algebra if and only if $A \cong K\langle x_1, \dots, x_m\rangle/I$, where I is a proper homogeneous two-sided ideal of $K\langle x_1, \dots, x_m\rangle$. Moreover, for every $n \in \mathbb{N}$, $\dim_K A_n < \infty$.*

Proof. $\Leftarrow$): Note that the free algebra $F := K\langle x_1, \dots, x_m\rangle$ is $\mathbb{N}$-graded with graduation given by

$$F_n :=_K \langle w | w \text{ is a word of length } n \text{ in the alphabet } \{x_1, \dots, x_m\}\rangle,\ n \in \mathbb{N}.$$

Since I is homogeneous, i.e., graded, then F/I es $\mathbb{N}$-graded with graduation given by $(F/I)_n := (F_n + I)/I$. Note that F/I is connected since $(F/I)_0 = K$. Moreover F/I is finitely generated as K-algebra by the elements $\overline{x_i} := x_i + I$, $1 \leq i \leq m$.
Observe that F_n is finitely generated as K-vector space, whence, $(F/I)_n$ is also finitely generated as K-vector space, i.e., $\dim_K((F/I)_n) < \infty$.
$\Rightarrow$): Let $a_1, \dots, a_m \in A$ be a finite collection of elements that generate A as K-algebra; by the universal property of the free algebra $K\langle x_1, \dots, x_m\rangle$, there exists a K-algebra homomorphism $f : K\langle x_1, \dots, x_m\rangle \to A$ with $f(x_i) := a_i$, $1 \leq i \leq m$; it is clear that f is surjective. Let $I := \ker(f)$, then I is a proper two-sided ideal of $K\langle x_1, \dots, x_m\rangle$ and

$$A \cong K\langle x_1, \dots, x_m\rangle/I. \tag{1.1.1}$$

Since A is $\mathbb{N}$-graded, we can assume that every a_i is homogeneous, $a_i \in A_{d_i}$ for some $d_i \geq 1$, moreover, at least one of generators is of degree 1. We define a new graduation for $F := K\langle x_1, \dots, x_m\rangle$: we put weights d_i to the variables x_i and we set

$$F'_n :=_K \langle x_{i_1} \cdots x_{i_m} | \textstyle\sum_{j=1}^m d_{i_j} = n\rangle,\ n \in \mathbb{N}.$$

This implies that f is graded, and from this we obtain that I is homogeneous. In fact, let $X_1 + \cdots + X_t \in I$, where $X_l \in F'_{n_l}$, $1 \leq l \leq t$, so $f(X_1) + \cdots + f(X_t) = 0$, and hence, $f(X_l) = 0$ for every l, i.e., $X_l \in I$.
Finally, note that under the isomorphism $\widetilde{f}$ in (1.1.1), $\widetilde{f}((F'_n + I)/I) = A_n$, so $\dim_K(A_n) < \infty$. □

Definition 1.1.3. *Let A be a finitely graded algebra; it is said that A is finitely presented if the two-sided ideal I of relations in Proposition 1.1.2 is finitely generated.*

Proposition 1.1.4. *Let A be a finitely graded algebra and M be a $\mathbb{Z}$-graded A-module which is finitely generated. Then, for every $n \in \mathbb{Z}$, $\dim_K M_n < \infty$.*

Proof. Since M is finitely generated, M is generated by a finite set of homogeneous elements, $m_1, \dots, m_r$, with $m_i \in M_{d_i}$, $1 \leq i \leq r$. Let $m \in M_n$, then there exist $a_1, \dots, a_r \in A$ such that $m = a_1 \cdot m_1 + \cdots + a_r \cdot m_r$, from this we can assume that $a_i \in A_{n-d_i}$, but as was observed in Proposition 1.1.2, every A_{n-d_i} is finitely generated as K-vector space, this implies that M_n is finitely generated over K, i.e., $\dim_K M_n < \infty$. □

1.2 Classical Hilbert series and polynomials

The proposition 1.1.4 shows that any finitely generated graded A-module M over a finitely graded algebra A has $dim_k M_n < \infty$ for all n and $dim_k M_n = 0$ for $n \gg 0$, and so the following definitions makes sense.

Definition 1.2.1. *Let A be a finitely graded algebra and M a finitely generated $\mathbb{Z}$-graded A-module. The Hilbert series of M is defined by*

$$h_M(t) := \textstyle\sum_{n\in\mathbb{Z}}(\dim_K M_n)t^n.$$

In particular,

$$h_A(t) := \textstyle\sum_{n=0}^{\infty}(\dim_K A_n)t^n.$$

Example 1.2.2. In the following examples A is considered as A-module:

1. $A = K[x]$, $\dim_K A_n = 1$ for every $n \geq 0$, so
$$h_A(t) = 1 + t + t^2 + t^3 + \cdots = \tfrac{1}{1-t}.$$
2. $A = K[x_1, \ldots, x_m]$, $\dim_K A_n = \binom{m+n-1}{n}$ for every $n \geq 0$, whence
$$h_A(t) = 1 + \tbinom{m}{1}t + \tbinom{m+1}{2}t^2 + \tbinom{m+3}{3}t^3 + \cdots = \tfrac{1}{(1-t)^m}.$$
3. $A = K\langle x_1, \ldots, x_m\rangle$, $\dim_K A_n = m^n$ for every $n \geq 0$, so
$$h_A(t) = 1 + mt + m^2t^2 + m^3t^3 + \cdots = \tfrac{1}{1-mt}.$$

1.3 Classical Gelfand-Kirillov dimension

If B is a commutative ring and M is any B-module, then the Krull dimension of M is defined to be the length of a maximal chain of prime ideals in the ring $(B/annM)$. Many non-commutative rings, including those of interest in this thesis, tend to have relatively few prime ideals, and so some different definitions of dimension are more useful in the non-commutative setting. The basic dimension functions we shall use below is the *Gelfand-Kirillov dimension*, or *GK*-dimension for short.
Recall that if A is a K-algebra, then the *Gelfand-Kirillov dimension* of A is defined by

$$\text{GKdim}(A) := \sup_V \overline{\lim_{n\to\infty}} \log_n \dim_K V^n, \qquad (1.3.1)$$

where V ranges over all frames of A and $V^n := {}_K\langle v_1 \cdots v_n | v_i \in V\rangle$ (a *frame* of A is a finite dimensional K-subspace of A such that $1 \in V$; since A is a K-algebra, then $K \hookrightarrow A$, and hence, K is a frame of A of dimension 1).

Proposition 1.3.1. *Let A be a K-algebra. If A has a generating frame V, i.e, A is generated by V as K-algebra, $A = K[V]$, then*

$$\text{GKdim}(A) = \overline{\lim_{n\to\infty}} \log_n(\dim_K V^n).$$

Moreover, this equality does not depend on the generating frame V.

Proof. It is clear that $\overline{\lim}_{n\to\infty} \log_n(\dim_K V^n) \leq \text{GKdim}(A)$. Let W be any frame of A; since $\dim_K W < \infty$, then there exists m such that $W \subseteq V^m$, so for every n we have $W^n \subseteq V^{nm}$, and hence $\dim_K W^n \leq \dim_K V^{nm}$. From this we get that $\log_n(\dim_K W^n) \leq \log_n(\dim_K V^{nm}) = (1 + \log_n m) \log_{nm}(\dim_K V^{nm})$. Taking lim we have $\overline{\lim}_{n\to\infty} \log_n(\dim W^n) \leq \overline{\lim}_{n\to\infty} \log_{nm}(\dim V^{nm})$ since $\overline{\lim}_{n\to\infty}(1 + \log_n m) = 1$. In addition, note that $\overline{\lim}_{n\to\infty} \log_{nm}(\dim V^{nm}) \leq \overline{\lim}_{n\to\infty} \log_n(\dim V^n)$, whence

$$\text{GKdim}(A) = \sup_W \overline{\lim_{n\to\infty}} \log_n(\dim W^n) \leq \overline{\lim_{n\to\infty}} \log_n(\dim V^n).$$

The proof of the second statement is completely similar. □

Proposition 1.3.2. *Let A be a finitely graded K-algebra. Then,*

$$\operatorname{GKdim}(A) = \overline{\lim_{n\to\infty}} \log_n(\sum_{i=0}^{n} \dim_K A_i).$$

Proof. See Lemma 6.1 in [18]. □

A classical result in commutative algebra about the Gelfand-Kirillov dimension is the following.

Proposition 1.3.3. *Let A be a finitely generated commutative K-algebra. Then,*

$$\operatorname{GKdim}(A) = \operatorname{Kdim}(A).$$

Proof. [18], Theorem 4.5(a). □

Example 1.3.4. Let K be a field. Then, $\operatorname{GKdim}(K[x_1, \ldots, x_m]) = m$.

Remark 1.3.5. In [22] is presented the exact computation of the Gelfand-Kirillov dimension for the most important examples of skew PBW extensions. In section 2.3. 2.3 we will give a more general version of this dimension for finitely semi-graded noetherian domains.

In [23] are considered the following finitely presented algebras, which are bijective skew PBW extensions:

1. The quantum polynomial ring (Example 1.10, [23]): given any constants $0 \neq q_{i,j} \in K, 1 \leq i < j \leq n$,

   $$A = K\langle x_1, \ldots, x_n\rangle / \langle x_j x_i - q_{ij} x_i x_j\rangle.$$

   Thus, $A = \sigma(K)\langle x_1, \ldots, x_n\rangle$, with

   $$x_j x_i = q_{ij} x_i x_j,\ 0 \neq q_{i,j} \in K,\ 1 \leq i < j \leq n.$$

   In this case $h_A(t) = \frac{1}{(1-t)^n}$ and $\operatorname{GKdim}(A) = n$.

2. The quantum plane (Example 1.10, [23]): given $0 \neq q \in K$,

   $$A = K\langle x, y\rangle / \langle yx - qxy\rangle.$$

   Thus, $A = \sigma(K)\langle x, y\rangle$, with

   $$yx = qxy,\ 0 \neq q \in K.$$

   In this case $h_A(t) = \frac{1}{(1-t)^2}$ and $\operatorname{GKdim}(A) = 2$.

3. The Jordan plane (Example 1.10, [23]):

   $$A = K\langle x, y\rangle / \langle yx - xy - x^2\rangle.$$

Thus, $A = \sigma(K[x])\langle y\rangle$, with

$$yx = xy + x^2.$$

For the Jordan plane holds $h_A(t) = \frac{1}{(1-t)^2}$ and $\mathrm{GKdim}(A) = 2$.

4. The Sklyanin algebra (Example 1.14, [23]): let $a, b, c \in K$, then

$$S = K\langle x, y, z\rangle/\langle ayx + bxy + cz^2, axz + bzx + cy^2, azy + byz + cx^2\rangle.$$

Note if $c = 0$ and $a, b \neq 0$, then $S = \sigma(K)\langle x, y, z\rangle$:

$$yx = -\tfrac{b}{a}xy, zx = -\tfrac{a}{b}xz, zy = -\tfrac{b}{a}yz.$$

In this case $h_S(t) = \frac{1}{(1-t)^3}$ and $\mathrm{GKdim}(S) = 3$.

5. The quantum polynomial ring in three variables in Example 1.16, [23]:

$$A = K\langle x, y, z\rangle/\langle f_1, f_2, f_3, \text{ with } f_1 := yx - pxy,\ f_2 := zx - qxz,\ f_3 := zy - ryz$$

In this case $h_A(t) = \frac{1}{(1-t)^3}$ and $\mathrm{GKdim}(A) = 3$.

6. The K-algebra in Example 1.18 of [23]:

$$A = K\langle x, y, z\rangle/\langle z^2 - xy - yx, zx - xz, zy - yz\rangle.$$

Then, $A = \sigma(K[z])\langle x, y\rangle$, where

$$yx = -xy + z^2, zx = xz, zy = yz.$$

In this example $h_A(t) = \frac{1}{(1-t)^3}$ and $\mathrm{GKdim}(A) = 3$.
The finitely graded K-algebra in Example 1.17 of [23] can not be viewed as a skew PBW extension:

$$A = K\langle x, y\rangle/\langle yx^2 - x^2y, y^2x - xy^2\rangle.$$

The same is true for the Sklyanin algebra with $c \neq 0$.
Closely related with the Hilbert series and the Gelfand-Kirillov dimension of an algebra A is the Hilbert polynomial of A.

Definition 1.3.6. *Let A be a finitely graded K-algebra. We say that A has Hilbert polynomial if there exists a polynomial $p_A(t) \in \mathbb{Q}[t]$ such that for all n sufficiently large, $p_A(n) = \dim_K A_n$. In this case $p_A(t)$ is called the Hilbert polynomial of A. Thus,*

$$\dim_K A_n = p_A(n), \text{ for all } n \ggg 0.$$

Proposition 1.3.7. *Let A be a finitely graded K-algebra. If $p_A(t)$ exists, then*

$$\mathrm{GKdim}(A) = \deg(p_A(t)) + 1.$$

The Hilbert series and the Hilbert polynomial have interesting applications in the study and classification of non-commutative algebras, and also in non-commutative algebraic geometry (see [11]).

1.4 Graded Hom and Ext

The next topic consider in [23] is the graded version of Hom and Ext.

Definition 1.4.1. *Let A be a $\mathbb{N}$-graded ring and let M be a $\mathbb{Z}$-graded A-module, $M = \bigoplus_{n\in\mathbb{Z}} M_n$. Let $i \in \mathbb{Z}$, the $\mathbb{Z}$-graded module $M(i)$ defined by $M(i)_n := M_{i+n}$ is called a shift of M, i.e.,*

$$M(i) = \bigoplus_{n\in\mathbb{Z}} M(i)_n = \bigoplus_{n\in\mathbb{Z}} M_{i+n}$$

Proposition 1.4.2. *Let A be a $\mathbb{N}$-graded ring and let M be a $\mathbb{Z}$-graded A-module. Then, $M \cong M(i)$ as A-modules.*

Proof. The isomorphism is given by

$$M = \bigoplus_{n\in\mathbb{Z}} M_n \to \bigoplus_{n\in\mathbb{Z}} M(i)_n = M(i)$$
$$m_{n_1} + \cdots + m_{n_t} \in M_{n_1} + \cdots + M_{n_t} \mapsto m_{n_1} + \cdots + m_{n_t} \in M(i)_{n_1-i} + \cdots + M(i)_{n_t-i}.$$

□

Proposition 1.4.3. *Let A be a finitely graded algebra. Then $A_{\geq 1} := \bigoplus_{n\geq 1} A_n$ is the unique homogeneous two-sided maximal ideal of A.*

Proof. It is clear that $A_{\geq 1}$ is a homogeneous two-sided ideal of A; the homomorphism defined by taking the homogeneous component of degree zero of any element of A is surjective and has kernel equals $A_{\geq 1}$, so $A/A_{\geq 1} \cong K$, and whence, $A_{\geq 1}$ is maximal. Let I be another two-sided homogeneous maximal ideal of A, then since I is proper, $I \cap A_0 = I \cap K = 0$; let $x \in I$, then $x = x_0 + x_1 + \cdots + x_n$, with $x_i \in A_i$, $1 \leq i \leq n$, but since I is homogeneous, $x_i \in I$ for every i, so $x_0 = 0$, and hence, $x \in A_{\geq 1}$. Thus, $I \subseteq A_{\geq 1}$, but I is maximal, so $I = A_{\geq 1}$. □

Definition 1.4.4. *Let A be a finitely graded algebra and let M, N be $\mathbb{Z}$-graded A-modules.*

(i) $Hom_{gr-A}(M, N) := \{f : M \to N | f \text{ is a graded } A\text{-homomorphism}\}$.

(ii) $\underline{Hom}_A(M, N) := \bigoplus_{d\in\mathbb{Z}} Hom_{gr-A}(M, N(d))$.

(iii) $\underline{Ext}^i_A(M, N)$ *is defined by taking a graded free resolution of M, applying the functor $\underline{Hom}_A(\ , N)$ and taking the i^{th} homology.*

Proposition 1.4.5. *Let A be a finitely graded K-algebra and let M, N be $\mathbb{Z}$-graded A-modules.*

(i) $Hom_{gr-A}(M, N)$ *is a K-vector space.*

(ii) $\underline{Hom}_A(M, N)$ *is a graded K-vector space.*

(iii) $\underline{Ext}^i_A(M, N)$ *is a graded K-vector space.*

(iv) *There is a natural inclusion $\underline{Hom}_A(M, N) \hookrightarrow Hom_A(M, N)$. If M is finitely generated, then*

$$\underline{Hom}_A(M, N) \cong Hom_A(M, N) \text{ and } \underline{Ext}^i_A(M, N) \cong Ext^i_A(M, N).$$

Proof. (i) It is clear that $Hom_{gr-A}(M,N)$ is an abelian group; moreover, if $f \in Hom_{gr-A}(M,N)$ and $a \in K$, then $a \cdot f \in Hom_{gr-A}(M,N)$, with $(a \cdot f)(m) := a \cdot f(m)$, for $m \in M$.
(ii) Let $S := \underline{Hom}_A(M,N)$, then for $d \in \mathbb{Z}$, $S_d := Hom_{gr-A}(M,N(d))$ defines a $\mathbb{Z}$-graduation of S with respect to K (note that K is a trivial finitely graded K-algebra).
(iii) This follows from (ii).
(iv) Observe that $Hom_{gr-A}(M,N(d)) \subseteq Hom_A(M,N(d)) \cong Hom_A(M,N)$, thus the inclusion $\underline{Hom}_A(M,N) \hookrightarrow Hom_A(M,N)$ is given by

$$(\dots,0,\phi_{d_1},\dots,\phi_{d_t},0,\dots) \mapsto \phi_{d_1} + \cdots + \phi_{d_t}.$$

Note that $\phi_{d_1} + \cdots + \phi_{d_t} = 0$ if and only if $\phi_{d_1} = \cdots = \phi_{d_t} = 0$. Indeed, let $m \in M$ be homogeneous of degree p, then $0 = (\phi_{d_1} + \cdots + \phi_{d_t})(m) = \phi_{d_1}(m) + \cdots + \phi_{d_t}(m) \in N_{d_1+p} \oplus \cdots \oplus N_{d_t+p}$, whence, for every j, $\phi_{d_j}(m) = 0$. This means that $\phi_{d_j} = 0$ for $1 \leq j \leq t$.
If M is finitely generated, then M is generated by a finite set of homogeneous elements $m_1,\dots,m_t$ of degrees $d_1,\dots,d_t$, respectively, and in this case,

$$\underline{Hom}_A(M,N) \cong Hom_A(M,N).$$

In fact, let $f \in Hom_A(M,N)$ and $f(m_j) = n_{j1} + \cdots + n_{jl}$, with $d_{j1} := \deg(n_{j1}) < \cdots < d_{jl} := \deg(n_{jl})$; we define the functions $h_{ji} : M \to N$ given by

$$h_{ji}(m_k) := \begin{cases} 0, & \text{if } k \neq j \\ n_{ji}, & \text{if } k = j. \end{cases}$$

Note that $h_{ji} \in Hom_{gr-A}(M,N(d_{ji}-d_j))$:
Thus, $f = \sum_{j=1,i=1}^{t,l} h_{ji} \in \bigoplus_{j=1,i=1}^{t,l} Hom_{gr-A}(M,N(d_{ji}-d_j)) \hookrightarrow \underline{Hom}_A(M,N)$, so $Hom_A(M,N) \hookrightarrow \underline{Hom}_A(M,N)$.

□

1.5 Artin-Schelter regular algebras

Now we come to the definition of Artin-Schelter regular algebras. This class of graded algebras was introduced by Artin and Schelter [3] in 1987 and classified a few years later by Artin, Tate and Van den Bergh [4], [5] and Stephenson [29], [30] . They may be considered as non-commutative analogues of polynomial rings.

Definition 1.5.1. *A connected graded K-algebra A is called an Artin-Schelter regular algebra of dimension d if it has the following properties:*

1. (i) A has finite global dimension $gld(A) = d < \infty$,

2. (ii) A has finite Gelfand-Kirillov dimension $GKdim(A) = d < \infty$,

3. (iii) A is Gorenstein, meaning there is an integer l such that

$$\underline{Ext}^i_A({}_AK_A, {}_AA) \cong \begin{cases} K(l)_A, & \text{if } i = d \\ 0, & \text{otherwise.} \end{cases}$$

where l is called the *Gorenstein parameter* of A.

Remark 1.5.2. Is well know that the third condition is equivalent to

$$\underline{Ext}^i_A(K_A, A_A) \cong \begin{cases} {}_AK(l), & \text{if } i = d \\ 0, & \text{otherwise.} \end{cases}$$

in addition, since $K \cong A/A_{\geq 1}$ is finitely generated as left A-module, then we can replace $\underline{Ext}^i_A({}_AK, {}_AA)$ by $Ext^i_A({}_AK, {}_AA$.

The polynomial ring $k[x_1, ..., x_n]$ is an AS(for short)-regular algebra of global dimension n. We think AS-regular algebras as non-commutative deformations of $k[x_1, ..., x_n]$, and the following definition is accepted as the quantum version, or non-commutative version, of the projective space $\mathbb{P}^n$ (see [24] and also [13]).

Definition 1.5.3. *A quantum projective space $\mathbb{P}^n$ is an AS algebra of global dimension n with $h_A(t) = \frac{1}{(1-t)^n}$.*

The main work developed in the last thirty years for AS-regular algebras turn around classification problem (see [27]). At this moment this is still unknown for $n \geq 4$, but completely solved for $n \leq 3$.

Examples 1.5.4. (i) If $n = 1$ then $A \cong K[x]$.

(ii) If $n = 2$ then A [28] Lema 2.2.5 is either isomorphic to

$$K\langle x, y\rangle/(ax^2 + byx + cxy + dy^2), \ \ where \ \ a, b, c, d \in K \ \ and \ \ ad - cb \neq 0$$

(in this case $\deg(x) = \deg(y) > 0$) or A is isomorphic to the skew polynomial ring $K[x][y; \sigma, \delta]$, where σ is a graded morphism and δ is a σ-derivation. If we restrict to the case where A is generated in degree one then A is either isomorphic to so-called quantum plane

$$K\langle x, y\rangle/(yx - qxy) \ \ where \ \ q \in K \setminus \{0\}$$

or to the Jordan quantum plane

$$K\langle x, y\rangle/(x^2 - yx + xy),$$

and the category $gr(A)$ is equivalent with $gr(K[x, y])$, see [36].

(iii) If $n = 3$ then there also exists a complete classification for AS-regular algebras of dimension three [3], [4], [5], [29], [30]. They are all left and right noetherian domains with Hilbert series of a weighted polynomial ring $K[x, y, z]$.

- *Homogenizations of the first Weyl algebra*

$$A_1 = K\langle x, y\rangle/(xy - yx - 1)$$

(**1**) Introduce a third variable z which commutes with x and y, and for which $yx-xy-z^2 = 0$. Thus $\deg z = 1$, and we obtain the quadratic AS-algebra

$$H = H_q = K\langle x, y, z\rangle/(yz - zy, zx - xz, xy - yx - z^2)$$

to which we refer as the *homogenized Weyl algebra*.
(**2**) Introduce a third variable z which commutes with x and y and for which $xy-yx-z=0$. Thus $\deg z = 2$ and we obtain the enveloping algebra of Heisenberg-Lie algebra, which is a cubic AS algebra

$$H_c = K\langle x, y, z\rangle/(yz - zy, xz - zx, xy - yx - z)$$

We refer to H_c as the *enveloping algebras* for short.

- The generic three dimensional AS algebras generated in degree one are so-called type A-algebras [3] , they are of the form:

  (**1**) quadratic:

  $$A = K\langle x, y, z\rangle/(f_1, f_2, f_3)$$

  where f_1, f_2, f_3 are the quadratic equations

  $$\begin{cases} f_1 = ayz + bzy + cz^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2. \end{cases}$$

  (**2**) cubic:

  $$K\langle x, y\rangle/(f_1, f_2)$$

  where f_1, f_2 are the cubic equations

  $$\begin{cases} f_1 = ay^2x + byxy + axy^2 + cx^3 \\ f_2 = ax^2y + bxyx + ayx^2 + cy^3. \end{cases}$$

  where $(a, b, c) \in \mathbb{P}^2 \backslash F$ where F is some finite set. In order to describe F, we recall from [4], Theorem 1 that the regular algebras of global dimension three generated in degree one are exactly the non-degenerated standard algebras. Quadratic algebras of generic type A are also called *three dimensional Sklyanin algebras*.

1.6 Geometry via non-commutative schemes

A famous Serre's theorem on commutative projective algebraic geometry states that the category of coherent sheaves over the projective n-space $\mathbb{P}^n$ is equivalent to a category of noetherian graded modules over a graded commutative polynomial ring. The fourth topic considered in the Rogalski's paper is the study of this equivalence for non-commutative finitely graded noetherian algebras.

Definition 1.6.1. *Let K be a field and let A be a left noetherian finitely graded K-algebra. Let* $\mathrm{gr} - A$ *be the abelian category of finitely generated $\mathbb{Z}$-graded left A-modules. We define a new abelian category* $\mathrm{qgr} - A$*. The objects are the same as the objects in* $\mathrm{gr} - A$*, and we let*

$\pi : \mathrm{gr}-A \to \mathrm{qgr}-A$ be the identity map on objects. The morphisms in $\mathrm{qgr}-A$ are defined in the following way:

$$Hom_{\mathrm{qgr-A}}(\pi(M),\pi(N)) := \varinjlim Hom_{\mathrm{gr}-A}(M_{\geq n}, N),$$

where the direct limit is taken over maps of abelian groups $Hom_{\mathrm{gr}-A}(M_{\geq n}, N) \to Hom_{\mathrm{gr}-A}(M_{\geq n+1}, N)$ induced by the inclusion homomorphism $M_{\geq n+1} \to M_{\geq n}$.

The pair $(\mathrm{qgr-A}, \pi(\mathrm{A}))$ is called the ***non-commutative projective scheme associated to*** *A, and often denoted simple by $\mathrm{qgr-A}$. The object $\pi(A)$ is called the distinguished object and plays the role of the sheaf. The map $\pi : \mathrm{gr}-A \to \mathrm{qgr}-A$, called the quotient functor, which sends the morphism $f : M \to N$ to $f|_{M\geq 0} \in Hom_{\mathrm{gr}-A}(M_{\geq 0}, N)$ in the direct limit. The category $\mathrm{qgr}-A$ is usually understood as $\mathrm{qgr}-A = \mathrm{gr}-A/\mathrm{tor}-A$.*

Now, we present some remarks about how noncommutative projective schemes generalize commutative projective schemes. Some concrete examples of non-commutative projective schemes will be presented later below.

Remark 1.6.2. (i) If X is a fixed topological space, the collection of all sheaves on X is a category: the objects of this category are the sheaves on X; if $\mathcal{F},\mathcal{G}$ are two sheaves on X, then a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is defined by the morphism of the corresponding schemes $(X,\mathcal{F}) \xrightarrow{\phi} (X,\mathcal{G})$ such that the continuous function $X \to X$ is the identical function and given an open set U of X we have a ring homomorphism $\psi_U : \mathcal{G}(U) \to \mathcal{F}(U)$. If the objects $\mathcal{F}(U)$ are not rings but groups, vector spaces or modules, then the ψ_U are morphisms of the correspondent structure.

(ii) Let A be a commutative finitely graded K-algebra generated in degree 1 by elements $x_0,\dots,x_n$. Let $\mathrm{proj}(A) := \{(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n | f(a_0,\dots,a_n) = 0 \text{ for all homogeneous } f \in J\}$. The *category of coherent sheaves* on $\mathrm{proj}(A)$, denoted by $\mathrm{coh}(\mathrm{proj}(A))$, can be described in the following way (see Lemma 4.2 in [23]). Every coherent sheaf on $\mathrm{proj}(A)$ is isomorphic to some sheaf $\widetilde{M}$ on $\mathrm{proj}(A)$, where M is some finitely graded A-module, and the sheaf $\widetilde{M}$ is defined as follows: Firstly observe that $\mathrm{proj}(A)$ has a finite open cover $\{U_0,\dots,U_n\}$; $\widetilde{M}(U_i) := M_{(x_i)}$, $0 \leq i \leq n$, with $M_{(x_i)}$ is the homogeneous component of degree 0 of the graded module M_{x_i} (the localization of M by the ring A_{x_i}, and this last one is the localization of the ring A by the powers of x_i). Note that in this case $\widetilde{M}(U_i)$ is not a ring but a K- vector space.

(iii) (***Serre's theorem***) *Let A be a commutative finitely graded K-algebra generated in degree 1. Then, there exists an equivalence of categories*

$$\mathrm{qgr}-A \simeq \mathrm{coh}(\mathrm{proj}(A)).$$

In particular,

$$\mathrm{qgr}-K[x_0,\dots,x_n] \simeq \mathrm{coh}(\mathbb{P}^n).$$

Thus, the commutative projective schemes (or more exactly, their categories of coherent sheaves) are particular cases of non-commutative projective schemes.

1.7 The χ condition

In this section we describe one property, which, while technical, are needed to extend important techniques from commutative to noncommutative geometry. Because in their absence one's tools are relatively limited, it is important to understand when this property hold.
We recall the Artin-Zhang χ conditions [2].

Definition 1.7.1. *Let R be a finitely $\mathbb{N}$-graded k-algebra, and fix $j \in \mathbb{N}$. We say that R satisfies right χ_j if, for all $i \leq j$ and for all finitely generated graded right R-modules M, we have that*

$$dim_k \underline{Ext}^i_R(k, M) < \infty$$

We say that R satisfies right χ if R satisfies right χ_j for all $j \in \mathbb{N}$. We similarly define left χ_j and left χ; we say that R satisfies χ if it satisfies left and right χ.

By [2] Corollary 8.2, any commutative noetherian ring satisfies χ. It is an easy exercise to see that R satisfies right χ_0 if and only if R is right noetherian.

The most important of the χ conditions is χ_1. Artin and Zhang discovered that its presence allows one to reconstruct R from $Proj - R$. This procedure will be formalized in the next section.

1.8 Classical Serre-Artin-Zhang-Verevkin theorem

Given a non-commutative finitely graded algebra A, it is an interesting and challenging problem to describe its associated non-commutative projective scheme qgr − A. Of course the first attempt is to investigate the non-commutative version of Serre's theorem. Some definitions are needed first.

Definition 1.8.1. *It is said that a category $\mathcal{C}$ is K-linear abelian if it satisfies the following conditions:*

(i) *$\mathcal{C}$ is abelian.*

(ii) *For every C_1, C_2 objects of $\mathcal{C}$, $Mor(C_1, C_2)$ is a K-vector space.*

We have to consider triples of the form $(\mathcal{C}, C, s)$, where $\mathcal{C}$ is a K-linear abelian category, C is a fixed special object of $\mathcal{C}$ and s is an autoequivalence of $\mathcal{C}$, i.e., $s : \mathcal{C} \to \mathcal{C}$ is a faithful, full and representative functor. For example, if A is a left noetherian finitely graded K-algebra we have the triple $(\text{qgr} - A, \pi(A), s)$, where s is the autoequivalence of q$\text{g}r - A$ defined by the shifts of degrees.

Definition 1.8.2. *Let $(\mathcal{C}, C, s)$ be a triple; the global section functor H^0 on $(\mathcal{C}, C, s)$ is defined by*

$$H^0(M) := Mor(C, M),\ M \in Ob(\mathcal{C}).$$

Observe that $H^0(C)$ is a K-algebra.

Definition 1.8.3. *Let $(\mathcal{C}, C, s)$ be a triple; it is defined*

$$\Gamma(C)_{\geq 0} := \bigoplus_{d=0}^{\infty} Mor(C, s^d(C)).$$

Observe that $\Gamma(C)_{\geq 0}$ is a graded K-algebra.

Definition 1.8.4. *Let $(\mathcal{C}, C, s)$ be a triple; s is ample if the following conditions hold:*

(i) *For every $M \in Ob(\mathcal{C})$, there exist positive integers $l_1, \dots, l_p$ and an epimorphism $\bigoplus_{i=1}^{p} C(-l_i) \to M$.*

(ii) *For every epimorphism $M \to N$ there exists an integer n_0 such that for every $n \geq n_0$ the function $H^0(M(n)) \to H^0(N(n))$ is surjective.*

The following is the non-commutative version of Serre's theorem.

Theorem 1.8.5 (Artin-Zhang; [2], Theorem 4.5)**.** *Let $(\mathcal{C}, C, s)$ be a triple that satisfies the following conditions:*

(a) *C is noetherian.*

(b) *$H^0(C)$ is left noetherian and $H^0(M)$ is a finitely generated $H^0(C)$-modulo for every $M \in Ob(\mathcal{C})$.*

(c) *s is ample.*

Then $\Gamma(C)_{\geq 0}$ is a finitely graded left noetherian K-algebra that satisfies χ_1 and there exists an equivalence of categories

$$\mathcal{C} \simeq \mathrm{qgr} - \Gamma(C)_{\geq 0}.$$

Conversely, if A is a finitely graded left noetherian K-algebra that satisfies χ_1, then the triple $(\mathrm{qgr} - A, \pi(A), s)$, where s is the autoequivalence of $\mathrm{qgr} - A$ defined by the shifts of degrees, satisfies the following conditions:

(a) *$\pi(A)$ is left noetherian.*

(b) *$H^0(\pi(A))$ is left noetherian and $H^0(\pi(M))$ is a finitely generated $H^0(\pi(A))$-modulo for every $M \in Ob(\mathrm{qgr} - A)$.*

(c) *s is ample.*

Moreover, there exists an equivalence of categories

$$\mathrm{qgr} - A \simeq \mathrm{qgr} - \Gamma(\pi(A))_{\geq 0}. \tag{1.8.1}$$

Corollary 1.8.6 ([2], Theorem 8.1)**.** *Let A be a left noetherian AS algebra. Then A satisfies χ and hence Artin-Zhang theorem holds for A.*

Remark 1.8.7. Verevkin gave the same definition in [33] of projective scheme $proj(A)$ for a K-graded algebra A but he works without the assumption of notherianity (see, [34]).

It seems to be an open problem to describe $\mathrm{qgr} - A$ for arbitrary AS algebras, or instead, to establish if any AS algebra is noetherian, and then to apply the previous corollary.

Next we present two particular examples investigated in the specialized literature.

Example 1.8.8. There are interesting examples of non-commutative graded rings whose non-commutative projective schemes are isomorphic to commutative projective schemes, i.e., Serre's theorem holds.

(i) ([23]) Let A be the quantum plane. Then,

$$\mathrm{qgr} - A \simeq \mathrm{coh}(\mathbb{P}^1).$$

The same is true for the Jordan plane. Remember that these two examples are AS algebras.
(ii) ([23]) Consider the quantum polynomial ring A in three variables with $pqr = 1$. Then,

$$\mathrm{qgr} - A \simeq \mathrm{coh}(\mathbb{P}^2).$$

If $pqr \neq 1$, it is known that $\mathrm{qgr} - A$ is not equivalent to the category of coherent sheaves on any commutative projective scheme. These two examples are AS algebras. Thus, there exist AS algebras whose non-commutative projective schemes are isomorphic to commutative projective schemes, but also there exist AS algebras whose non-commutative projective schemes are not isomorphic to commutative projective schemes.

Remark 1.8.9. A recent Ph.D. thesis (2014), and its corresponding preprint titled *Noncommutative projective Calabi-Yau schemes* [17], by Atsushi Kanazawa, give a description of $\mathrm{qgr} - A$ for finitely graded CY algebras. With this in mint, understand and extend this technique have recent interest for others non-commutative objects.

1.9 Skew PBW extensions

Now we recall the definition of skew PBW extensions defined firstly in [19]; many important algebras coming from mathematical physics are particular examples of skew PBW extensions: $\mathcal{U}(\mathcal{G})$, where $\mathcal{G}$ is a finite dimensional Lie algebra, the algebra of q-differential operators, the algebra of shifts operators, the additive analogue of the Weyl algebra, the multiplicative analogue of Weyl algebra, the quantum algebra $U'(so(3, K))$, the 3-dimensional skew polynomial algebra, the dispin algebra, the Woronowicz algebra, the q-Heisenberg algebra, are particular examples of skew PBW (see [20]).

Definition 1.9.1. *Let R and A be rings. We say that A is a skew PBW extension of R (also called a $\sigma - PBW$ extension of R) if the following conditions hold:*

(i) $R \subseteq A$.

(ii) *There exist finitely many elements $x_1, \ldots, x_n \in A$ such A is a left R-free module with basis*

$$\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}, \text{ with } \mathbb{N} := \{0, 1, 2, \ldots\}.$$

The set $Mon(A)$ is called the set of standard monomials of A.

(iii) *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that*

$$x_i r - c_{i,r} x_i \in R. \tag{1.9.1}$$

(iv) *For every* $1 \leq i, j \leq n$ *there exists* $c_{i,j} \in R - \{0\}$ *such that*

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{1.9.2}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$.

Associated to a skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$, there are n injective endomorphisms $\sigma_1, \ldots, \sigma_n$ of R and σ_i-derivations, as the following proposition shows.

Proposition 1.9.2. *Let* A *be a skew* PBW *extension of* R*. Then, for every* $1 \leq i \leq n$*, there exist an injective ring endomorphism* $\sigma_i : R \to R$ *and a* σ_i*-derivation* $\delta_i : R \to R$ *such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r),$$

for each $r \in R$.

Proof. See [19], Proposition 3. □

A particular case of skew PBW extension is when all derivations δ_i are zero. Another interesting case is when all σ_i are bijective and the constants c_{ij} are invertible. We recall the following definition (cf. [19]).

Definition 1.9.3. *Let* A *be a skew* PBW *extension.*

(a) *A is quasi-commutative if the conditions* (*iii*) *and* (*iv*) *in Definition 1.9.1 are replaced by*

(iii') *For every* $1 \leq i \leq n$ *and* $r \in R - \{0\}$ *there exists* $c_{i,r} \in R - \{0\}$ *such that*

$$x_i r = c_{i,r} x_i. \tag{1.9.3}$$

(iv') *For every* $1 \leq i, j \leq n$ *there exists* $c_{i,j} \in R - \{0\}$ *such that*

$$x_j x_i = c_{i,j} x_i x_j. \tag{1.9.4}$$

(b) *A is bijective if* σ_i *is bijective for every* $1 \leq i \leq n$ *and* $c_{i,j}$ *is invertible for any* $1 \leq i < j \leq n$.

Observe that quasi-commutative skew PBW extensions are $\mathbb{N}$-graded rings but arbitrary skew PBW extensions are semi-graded rings as we will se below. Actually, the main motivation for constructing the non-commutative algebraic geometry of semi-graded rings is due to arbitrary skew PBW extensions.

Many properties of skew PBW extensions have been studied in previous works (see [1], [20]). For example, the global, Krull and Goldie dimensions of bijective skew PBW extensions were estimated in [20]. The next theorem establishes two classical ring theoretic results for skew PBW extensions.

Theorem 1.9.4. *Let* A *be a bijective skew* PBW *extension of a ring* R.

(i) (*Hilbert Basis Theorem*) *If* R *is a left* (*right*) *Noetherian ring then* A *is also left* (*right*) *Noetherian.*

(ii) (*Ore's theorem*) *If R is a left Ore domain R. Then A is also a left Ore domain.*

We conclude this introductory section fixing some notation. If not otherwise noted, all modules are left modules; B will denote a non-commutative ring; K will be a field; $A := \sigma(R)\langle x_1, \dots, x_n\rangle$ will represent a skew PBW extension.

1.9.1 Examples

1. PBW **extensions**: any PBW extension is a bijective skew PBW extension since in this case $\sigma_i = i_R$ for each $1 \leq i \leq n$, and $c_{i,j} = 1$ for every $1 \leq i, j \leq n$. Thus, for PBW extension is hold $A = i(R)\langle x_1, ..., x_n\rangle$. Examples of PBW extensions are the following:

   (a) *Any skew polynomial ring of derivation type* $A = R[x; \sigma, \delta]$ i.e. with $\sigma = i_R$.

   (b) *The Weyl algebra* $A_n(k) := k[t_1, ..., t_n][x_1; \partial/\partial t_1, ..., \partial/\partial t_n]$.

   (c) Let k be a commutative ring and $\mathcal{G}$ a finite dimensional Lie algebra over k with basis $\{x_1, ..., x_n\}$; the *Universal enveloping algebra* of $\mathcal{G}$, $\mathcal{U}(\mathcal{G})$, is a PBW extension of k.

2. **Ore extensions of bijective type**: any *skew polynomial ring* $R[x; \sigma, \delta]$ *of bijective type*, i.e. with σ bijective, is a bijective skew PBW extension. More generally, let $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ be an *iterated skew polynomial ring of bijective type* with its standard conditions then, $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$is a bijective skew PBW extension.

   Some concrete examples of Ore algebras of bijective type are the following

   (a) *The algebra of q-differential operators* $D_{q,h}[x, y]$: let $q, h \in k, q \neq 0$; consider $k[y][x; \sigma, \delta], \sigma(y) := qy$ and $\delta(y) := h$. By definition of skew polynomial ring we have $xy = \sigma(y)x + \delta(y) = qyx + h$, and hence $xy - qyx = h$, Therefore, $D_{q,h}[x, y] \cong \sigma(k[y])\langle x\rangle$.

   (b) *The algebra of shift operators* S_h: let $h \in k$. The algebra of shift operators is defined by $S_h := k[t][x_h; \sigma_h, \delta_h]$, where $\sigma_h(p(t)) := p(t - h)$, and $\delta_h := 0$. Thus $S_h \cong \sigma(k[t])\langle x_h\rangle$.

3. **Quantum algebras**

   (a) *Additive analogue of the Weyl algebra* The k-algebra $A_n(q_1, ..., q_n)$ generated by $x_1, ..., x_n, y_1, ..., y_n$ and its inner relations. Then $A_n(q_1, ..., q_n) \cong \sigma(k[x_1, ..., x_n])\langle y_1, ..., y_n\rangle$.

   (b) *Multiplicative analogue of the Weyl algebra* The k-algebra $\mathcal{O}_n(\lambda_{ji})$ generated by $x_1, ..., x_n$ subject to the relations $x_i x_j = \lambda_{ji} x_i x_j$ for $1 \leq i < j \leq n$, We note that $\mathcal{O}_n(\lambda_{ji}) \cong \sigma(k[x_1])\langle x_2, ..., x_n\rangle$.

   (c) *Quantum algebra* $\mathcal{U}'(\mathfrak{so}(3, k))$. It is the k-algebra generated by I_1, I_2, I_3 subject to the relations

   $$I_2 I_1 - q I_1 I_2 = -q^{1/2} I_3, I_3 I_1 - q^{-1} I_1 I_3 = q^{-1/2} I_2, I_3 I_2 - q I_2 I_3 = -q^{1/2} I_1$$

   In this way, $\mathcal{U}'(\mathfrak{so}(3, k)) \cong \sigma(k)\langle I_1, I_2, I_3\rangle$.

(d) *Woronowicz algebra* $\mathcal{W}_v(\mathfrak{sl}(2,k))$. This algebra was introduced by Woronowicz in [35] and in is genrated by x, y, z subject to the relations

$$xz - v^2zx = (1+v^2)x, xy - v^2yx = vz, zy - v^4yz = (1+v^2)y$$

Then $\mathcal{W}_v(\mathfrak{sl}(2,k)) \cong \sigma(k)\langle x, y, z\rangle$

(e) *q- Heisenberg algebra* The k-algebra $H_n(q)$ is generated by the set of variables $x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_n$ subject to its inner relations. Note that $H_n(q) \cong \sigma(k[y_1, ..., y_n])\langle x_1, ..., x_n, z_1, ..., z_n\rangle$

These and more examples can be find in more detail in [20].
With the above discussion we can formulate the next problem.

Problem 1.9.5. Associate to a skew PBW extension a non-commutative scheme and adapt the equivalence (1.8.1) in theorem 1.8.5.

Chapter 2

Semi-graded rings

In this chapter we introduced our main goal: we define the semi-graded rings as a generalization of skew PBW extensions and non-commutative graded algebras. We include a generalization of Hilbert series and Gelfand-Kirillov dimension for this structures. Some examples are included for explain and support our results.

2.1 Semi-graded rings and modules

In this section we introduce the semi-graded rings and modules, we prove some elementary properties of them, and we will show that graded rings, finitely graded algebras and skew PBW extensions are particular cases of this new type of non-commutative rings.

Definition 2.1.1. *Let B be a ring. We say that B is semi-graded (SG) if there exists a collection $\{B_n\}_{n\geq 0}$ of subgroups B_n of the additive group B^+ such that the following conditions hold:*

(i) $B = \bigoplus_{n\geq 0} B_n$.

(ii) *For every $m, n \geq 0$, $B_m B_n \subseteq B_0 \oplus \cdots \oplus B_{m+n}$.*

(iii) $1 \in B_0$.

The collection $\{B_n\}_{n\geq 0}$ is called a semi-graduation of B and we say that the elements of B_n are homogeneous of degree n. Let B and C be semi-graded rings and let $f : B \to C$ be a ring homomorphism, we say that f is homogeneous if $f(B_n) \subseteq C_n$ for every $n \geq 0$.

Definition 2.1.2. *Let B be a SG ring and let M be a B-module. We say that M is a $\mathbb{Z}$-semi-graded, or simply semi-graded, if there exists a collection $\{M_n\}_{n\in\mathbb{Z}}$ of subgroups M_n of the additive group M^+ such that the following conditions hold:*

(i) $M = \bigoplus_{n\in\mathbb{Z}} M_n$.

(ii) *For every $m \geq 0$ and $n \in \mathbb{Z}$, $B_m M_n \subseteq \bigoplus_{k\leq m+n} M_k$.*

We say that M is positively semi-graded, also called $\mathbb{N}$-semi-graded, if $M_n = 0$ for every $n < 0$. Let $f : M \to N$ be an homomorphism of B-modules, where M and N are semi-graded B-modules; we say that f is homogeneous if $f(M_n) \subseteq N_n$ for every $n \in \mathbb{Z}$.

As for the case of rings, the collection $\{M_n\}_{n\in\mathbb{Z}}$ is called a *semi-graduation* of M and we say that the elements of M_n are *homogeneous* of degree n.

Let B be a semi-graded ring and let M be a semi-graded B-module, let N be a submodule of M, let $N_n := N \cap M_n$, $n \in \mathbb{Z}$; observe that the sum $\sum_n N_n$ is direct. This induces the following definition.

Definition 2.1.3. *Let B be a SG ring and M be a semi-graded module over B. Let N be a submodule of M, we say that N is a semi-graded submodule of M if $N = \bigoplus_{n\in\mathbb{Z}} N_n$.*

Note that if N is semi-graded, then $B_m N_n \subseteq \bigoplus_{k\leq m+n} N_k$, for every $n \in \mathbb{Z}$ and $m \geq 0$: In fact, let $b \in B_m$ and $z \in N_n$, then $bz \in B_m M_n \subseteq \bigoplus_{k\leq m+n} M_k$ and $bz = z_1 + \cdots + z_l$, with $z_i \in N_{n_i} \subseteq M_{n_i}$, but since the sum is direct, then $n_i \leq m + n$ for every $1 \leq i \leq l$.

Finally, we introduce an important class of semi-graded rings that includes finitely graded algebras and skew PBW extensions.

Definition 2.1.4. *Let B be a ring. We say that B is finitely semi-graded (FSG) if B satisfies the following conditions:*

(i) *B is SG.*

(ii) *There exists finitely many elements $x_1, \ldots, x_n \in B$ such that the subring generated by B_0 and $x_1, \ldots, x_n$ coincides with B.*

(iii) *For every $n \geq 0$, B_n is a free B_0-module of finite dimension.*

Moreover, if M is a B-module, we say that M is finitely semi-graded if M is semi-graded, finitely generated, and for every $n \in \mathbb{Z}$, M_n is a free B_0-module of finite dimension.

Remark 2.1.5. Observe if B is FSG, then $B_0 B_p = B_p$ for every $p \geq 0$, and if M is finitely semi-graded, then $B_0 M_n = M_n$ for all $n \in \mathbb{Z}$.

From the definitions above we get the following conclusions.

Proposition 2.1.6. *Let $B = \bigoplus_{n\geq 0} B_n$ be a SG ring and I be a proper two-sided ideal of B semi-graded as left ideal. Then,*

(i) *B_0 is a subring of B. Moreover, for any $n \geq 0$, $B_0 \oplus \cdots \oplus B_n$ is a $B_0 - B_0$-bimodule, as well as B.*

(ii) *B has a standard $\mathbb{N}$-filtration given by*

$$F_n(B) := B_0 \oplus \cdots \oplus B_n. \qquad (2.1.1)$$

(iii) *The associated graded ring satisfies*

$$Gr(B)_n \cong B_n, \textit{ for every } n \geq 0 \textit{ (isomorphism of abelian groups).}$$

(iv) *Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a semi-graded B-module and N a submodule of M. The following conditions are equivalent:*

(a) *N is semi-graded.*

(b) *For every $z \in N$, the homogeneous components of z are in N.*

(c) *M/N is semi-graded with semi-graduation given by*

$$(M/N)_n := (M_n + N)/N,\ n \in \mathbb{Z}.$$

(v) *B/I is SG.*

(vi) *If B is FSG and $I \cap B_n \subseteq IB_n$ for every n, then B/I is FSG.*

Proof. (i) and (ii) are obvious. For (iii) observe that $Gr(B)_n = F_n(B)/F_{n-1}(B) \cong B_n$ for every $n \geq 0$ (isomorphism of abelian groups); in addition, note how acts the product: let $z := \overline{b_0 + \cdots + b_m} \in Gr(B)_m$, $z' := \overline{c_0 + \cdots + c_n} \in Gr(B)_n$, then

$$zz' = \overline{b_m}\,\overline{c_n} = \overline{d_0 + \cdots + d_{m+n}} = \overline{d_{m+n}} \in Gr(B)_{m+n} \cong B_{n+m}.$$

(iv) (a)⇔(b) is obvious.
(b)⇒(c): Let $\overline{M}_n := (M/N)_n := (M_n + N)/N$, $n \in \mathbb{Z}$, then $\overline{M} := M/N = \bigoplus_{n\in\mathbb{Z}} \overline{M}_n$. In fact, let $z \in M$, then $\overline{z} \in \overline{M}$ can be written as $\overline{z} = \overline{z_1 + \cdots + z_l} = \overline{z_1} + \cdots + \overline{z_l}$, with $z_k \in M_{n_k}$, $1 \leq k \leq l$, thus, $\overline{z} \in \sum_{n\in\mathbb{Z}} \overline{M}_n$, and hence, $\overline{M} = \sum_{n\in\mathbb{Z}} \overline{M}_n$. This sum is direct since if $\overline{z_1} + \cdots + \overline{z_l} = \overline{0}$, then $z_1 + \cdots + z_l \in N$, so by (b) $z_k \in N$, i.e., $\overline{z_k} = \overline{0}$ for every $1 \leq k \leq l$. Now, let $b_m \in B_m$ and $\overline{z_n} \in \overline{M}_n$, then $b_m\overline{z_n} = \overline{b_m z_n} = \overline{d_1 + \cdots + d_p}$, with $d_i \in M_{n_i}$ and $n_i \leq m + n$, so $b_m\overline{z_n} = \overline{d_1} + \cdots + \overline{d_p} \in \bigoplus_{k\leq m+n} \overline{M}_k$. We have proved that $\overline{M}$ is semi-graded.
(c)⇒(b): Let $z = z_1 + \cdots + z_l \in N$, with $z_i \in M_{n_i}$, $1 \leq i \leq l$, then $\overline{0} = \overline{z_1} + \cdots + \overline{z_l} \in \overline{M} = \bigoplus_{n\in\mathbb{Z}} \overline{M}_n$, therefore, $\overline{z_i} = \overline{0}$, and hence $z_i \in N$ for every i.
(v) The proof is similar to (b)⇒(c) in (iv).
(vi) By (v), $\overline{B}$ is SG. Let $x_1, \ldots, x_n \in B$ such that the subring generated by B_0 and $x_1, \ldots, x_n$ coincides with B, then it is clear that the subring of $\overline{B}$ generated by $\overline{B}_0$ and $\overline{x_1}, \ldots, \overline{x_n}$ coincides with $\overline{B}$. Let $n \geq 0$ and $\{z_1, \ldots, z_l\}$ be a basis of the free left B_0-module B_n, then $\{\overline{z_1}, \ldots, \overline{z_l}\}$ is a basis of $\overline{B}_n$: in fact, let $\overline{z} \in \overline{B}_n$ with $z \in B_n$, then $z = c_1z_1 + \cdots + c_lz_l$, with $c_i \in B_0$, $1 \leq i \leq l$, and hence, $\overline{z} = \overline{c_1}\,\overline{z_1} + \cdots + \overline{c_l}\,\overline{z_l}$, i.e., $\overline{B}_n$ is generated by $\overline{z_1}, \ldots, \overline{z_l}$ over $\overline{B}_0$; now, if $\overline{c_1}\,\overline{z_1} + \cdots + \overline{c_l}\,\overline{z_l} = \overline{0}$, with $c_i \in B_0$, then $c_1z_1 + \cdots + c_lz_l \in I \cap B_n \subseteq IB_n$ and hence we can write

$$c_1z_1 + \cdots + c_lz_l = d_1z_1 + \cdots + d_lz_l, \text{ with } d_i \in I,\ 1 \leq i \leq l.$$

From this we get that $c_i = d_i$, so $\overline{c_i} = \overline{0}$ for every i. □

Note that the condition imposed to I in (vi) is of type Artin-Rees (see [McConnell]).

Proposition 2.1.7. (i) *Any $\mathbb{N}$-graded ring is SG.*

(ii) *Let K be a field. Any finitely graded K-algebra is a FSG ring.*

(iii) *Any skew PBW extension is a FSG ring.*

Proof. (i) and (ii) follow directly from the definitions.
(iii) Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, then $A = \bigoplus_{k\geq 0} A_k$, where

$$A_k :=_R \langle x^\alpha \in Mon(A) | \deg(x^\alpha) = k\rangle.$$

Thus, A_k is a free left R-module with

$$\dim_R A_k = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}. \tag{2.1.2}$$

We are assuming that R is an $\mathcal{IBN}$ ring (*Invariant basis number*), and hence, A also satisfies this condition, see [10]. □

2.2 Generalized Hilbert series and Hilbert polynomial

In this section we introduce the notion of generalized Hilbert series and generalized Hilbert polynomial for semi-graded rings. As in the classical case of finitely graded algebras over fields, these notions depends on the semi-graduation, in particular, they depend on the ring B_0. We will compute these tools for skew PBW extensions.

Definition 2.2.1. *Let $B = \sum_{n\geq 0} \oplus B_n$ be a FSG ring and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely semi-graded B-module. The generalized Hilbert series of M is defined by*

$$Gh_M(t) := \textstyle\sum_{n\in\mathbb{Z}} (\dim_{B_0} M_n) t^n.$$

In particular,

$$Gh_B(t) := \textstyle\sum_{n=0}^{\infty} (\dim_{B_0} B_n) t^n.$$

We say that B has a generalized Hilbert polynomial if there exists a polynomial $Gp_B(t) \in \mathbb{Q}[t]$ such that

$$\dim_{B_0} B_n = Gp_B(n), \textit{ for all } n \ggg 0.$$

In this case $Gp_B(t)$ is called the generalized Hilbert polynomial of B.

Remark 2.2.2. (i) Note that if K is a field and B is a finitely graded K-algebra, then the generalized Hilbert series coincides with the habitual Hilbert series, i.e., $Gh_B(t) = h_B(t)$; the same is true for the generalized Hilbert polynomial.

(ii) Observe that if a semi-graded ring B has another semi-graduation $B = \bigoplus_{n\geq 0} C_n$, then its generalized Hilbert series and its generalized Hilbert polynomial can change, i.e., the notions of generalized Hilbert series and generalized Hilbert polynomial depend on the semi-graduation, in particular on B_0. For example, consider the habitual real polynomial ring in two variables $B := \mathbb{R}[x, y]$, then $Gh_B(t) = \frac{1}{(1-t)^2}$ and $Gp_B(t) = t+1$; but if we view this ring as $B = (\mathbb{R}[x])[y]$ then $C_0 = \mathbb{R}[x]$, its generalized Hilbert polynomial series is $\frac{1}{1-t}$ and its generalized Hilbert polynomial is 1.

For skew PBW extensions the generalized Hilbert series and the generalized Hilbert polynomial can be computed explicitly.

Theorem 2.2.3. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be an arbitrary skew PBW extension. Then,*

(i)
$$Gh_A(t) = \frac{1}{(1-t)^n}. \tag{2.2.1}$$

(ii)
$$Gp_A(t) = \frac{1}{(n-1)!}[t^{n-1} - s_1t^{n-2} + \cdots + (-1)^r s_r t^{n-r-1} + \cdots + (n-1)!]. \tag{2.2.2}$$

where $s_1, \dots, s_k, \dots, s_{n-1}$ are the elementary symmetric polynomials in the variables $1-n, 2-n, \dots, (n-1)-n$.

Proof. (i) We have

$$Gh_A(t) = \sum_{k=0}^{\infty}(\dim_R A_k)t^k = \sum_{k=0}^{\infty}\binom{n+k-1}{k}t^k = \frac{1}{(1-t)^n}.$$

(ii) Note that

$$\begin{aligned}\dim_R A_k = \binom{n+k-1}{k} &= \frac{(n+k-1)!}{k!(n-1)!}\\ &= \frac{(k+n-1)(k+n-2)(k+n-3)\cdots(k+n-(n-1))(k+n-n)!}{k!(n-1)!}\\ &= \frac{(k+n-1)(k+n-2)(k+n-3)\cdots(k+n-(n-1))}{(n-1)!}\\ &= \frac{1}{(n-1)!}[k^{n-1} - s_1k^{n-2}\cdots + (-1)^r s_r k^{n-r-1} + \cdots + (-1)^{n-1}s_{n-1}k^{n-n}]\\ &= \frac{1}{(n-1)!}[k^{n-1} - s_1k^{n-2}\cdots + (-1)^r s_r k^{n-r-1} + \cdots + (n-1)!],\end{aligned}$$

where $s_1, \dots, s_k, \dots, s_{n-1}$ are the elementary symmetric polynomials in the variables $1-n, 2-n, \dots, (n-1)-n$. Thus, we found a polynomial $Gp_A(t) \in \mathbb{Q}[t]$ of degree $n-1$ such that

$$\dim_R A_k = Gp_A(k) \text{ for all } k \geq 0. \tag{2.2.3}$$

□

From Theorem 2.2.3, and considering the numeral (ii) in Remark 2.2.2, we can compute the generalized Hilbert series and the generalized Hilbert polynomial for all examples of skew *PBW* extensions described in [lezamareyes1]. In addition, for the skew quantum polynomials, we can interpreted some of them as quasi-commutative bijective skew *PBW* extensions of the r-multiparameter quantum torus. Thus, we have the following table:

2.3 Generalized Gelfand-Kirillov dimension

With respect to the Gelfand-Kirillov dimension, the classical definition over fields is not good since, in general, for a ring R a finitely generated R-module is not free. Whence, we have to replace the classical dimension of free modules with other invariant. Next we will show that for our purposes the Goldie dimension works properly, assuming that R is a left noetherian domain. A similar problem was considered in [6] for algebras over commutative noetherian domains replacing the vector space dimension with the reduced rank.
The following two remarks induce our definition.

(i) If R is a left noetherian domain, then R is a left Ore domain and hence $\operatorname{udim}({}_RR) = 1$. From this we get the following conclusion: let V be a free R-module of finite dimension, i.e., $\dim_R V = k$, then $\operatorname{udim}(V) = k$: in fact, $V \cong R^k$, and from this we obtain $\operatorname{udim}(V) = \operatorname{udim}({}_RR \oplus \cdots \oplus_R R) = \operatorname{udim}({}_RR) + \cdots + \operatorname{udim}({}_RR) = k$ (see [21]).

(ii) Let R be a left noetherian domain and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of R, then (2.2.1) takes the following form:

$$Gh_A(t) = \sum_{k=0}^{\infty}(\operatorname{udim} A_k)t^k = \sum_{k=0}^{\infty}(\dim_R A_k)t^k = \sum_{k=0}^{\infty}\binom{n+k-1}{k}t^k = \frac{1}{(1-t)^n}.$$

Definition 2.3.1. *Let B be a FSG ring such that B_0 is a left noetherian domain. The generalized Gelfand-Kirillov dimension of B is defined by*

$$\operatorname{GGKdim}(B) := \sup_V \overline{\lim_{k\to\infty}} \log_k \operatorname{udim} V^k,$$

where V ranges over all frames of B and $V^k := {}_{B_0}\langle v_1 \cdots v_k | v_i \in V\rangle$ (a frame of B is a finite dimensional B_0-free submodule of B such that $1 \in V$).

Remark 2.3.2. (i) Note that B has at least one frame: B_0 is a frame of dimension 1. We say that V is a *generating frame* of B if the subring of B generating by V and B_0 is B. For example, if R is a left noetherian domain and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, then $V :=_R \langle 1, x_1, \dots, x_n\rangle$ is a generating frame of A.

(ii) In a similar way as was observed in Remark 2.2.2, the notion of generalized Gelfand-Kirillov dimension of a finitely semi-graded ring B depends on the semi-graduation, in particular, depends on B_0. Note that this type of consideration was made in [6] for an alternative notion of the Gelfand-Kirillov dimension using the reduced rank.

(iii) If B is a finitely graded K-algebra, then the classical Gelfand-Kirillov dimension of B coincides with the just defined above notion, i.e., $\operatorname{GGKdim}(B) = \operatorname{GKdim}(B)$.

Proposition 2.3.3. *Let B be a FSG ring such that B_0 is a left noetherian domain. Let V be a generating frame of B, then*

$$\operatorname{GGKdim}(B) = \overline{\lim_{k\to\infty}} \log_k(\operatorname{udim} V^k). \tag{2.3.1}$$

Moreover, this equality does not depend on the generating frame V.

Proof. It is clear that $\overline{\lim}_{k\to\infty}\log_k(\mathrm{udim}V^k)\leq \mathrm{GGKdim}(B)$. Let W be any frame of B; since $\dim_{B_0} W<\infty$, then there exists m such that $W\subseteq V^m$, and hence for every k we have $W^k\subseteq V^{km}$, but observe that V^{km} is a finitely generated left B_0-module, and since B_0 is left noetherian, then V^{km} is a left noetherian B_0-module, so $\mathrm{udim}V^{km}<\infty$. From this, $\mathrm{udim}W^k\leq \mathrm{udim}V^{km}$. Therefore, $\log_k(\mathrm{udim}W^k)\leq\log_k(\mathrm{udim}V^{km})=(1+\log_k m)\log_{km}(\mathrm{udim}V^{km})$. Since $\overline{\lim}_{k\to\infty}(1+\log_k m)=1$, we get that $\overline{\lim}_{k\to\infty}\log_k(\mathrm{udim}W^k)\leq\overline{\lim}_{k\to\infty}\log_{km}(\mathrm{udim}V^{km})$. But observe that $\overline{\lim}_{k\to\infty}\log_{km}(\mathrm{udim}V^{km})\leq\overline{\lim}_{k\to\infty}\log_k(\mathrm{udim}V^k)$, whence

$$\mathrm{GGKdim}(B)=\sup_W\overline{\lim_{k\to\infty}}\log_k(\mathrm{udim}W^k)\leq\overline{\lim_{k\to\infty}}\log_k(\mathrm{udim}V^k).$$

The proof of the second statement is completely similar. □

Next we present the main result of the present section.

Theorem 2.3.4. *Let R be a left noetherian domain and $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension of R. Then,*

$$\mathrm{GGKdim}(A)=\overline{\lim_{k\to\infty}}\log_k(\sum_{i=0}^{k}\dim_R A_i)=1+\deg(Gp_A(t))=n.$$

Proof. According to (2.3.1), $\mathrm{GGKdim}(A)=\overline{\lim}_{k\to\infty}\log_k(\mathrm{udim}V^k)$, with

$$V:={}_R\langle 1,x_1,\dots,x_n\rangle=A_0\oplus A_1;$$

note that $V^k\subseteq A_0\oplus A_1\oplus\cdots\oplus A_k$, from this and using (2.2.3) we get

$$\begin{gathered}\mathrm{GGKdim}(A)\leq\overline{\lim}_{k\to\infty}\log_k(\mathrm{udim}(\textstyle\sum_{i=0}^k\oplus A_i))=\overline{\lim}_{k\to\infty}\log_k(\sum_{i=0}^k\mathrm{udim}A_i)=\\ \overline{\lim}_{k\to\infty}\log_k(\textstyle\sum_{i=0}^k\dim_R A_i)=\overline{\lim}_{k\to\infty}\log_k(\sum_{i=0}^k Gp_A(i))=\\ \overline{\lim}_{k\to\infty}\log_k(Gp_A(0)+Gp_A(1)+\cdots+Gp_A(k)),\end{gathered}$$

but according to (2.2.2), every coefficient in $Gp_A(t)$ is positive, so $Gp_A(i)$ is positive for every $0\leq i\leq k$, moreover, $Gp_A(i)\leq Gp_A(k)$, so $Gp_A(0)+Gp_A(1)+\cdots+Gp_A(k)\leq(k+1)Gp_A(k)$ and hence

$$\begin{gathered}\mathrm{GGKdim}(A)\leq\overline{\lim}_{k\to\infty}\log_k((k+1)Gp_A(k))=\overline{\lim}_{k\to\infty}\log_k(k+1)+\overline{\lim}_{k\to\infty}\log_k(Gp_A(k))=\\ 1+\overline{\lim}_{k\to\infty}\log_k(Gp_A(k)).\end{gathered}$$

Observe that every summand of $Gp_A(k)$ in the bracket of (2.2.2) is $\leq k^{n-1}$ for k enough large, so $Gp_A(k)\leq\frac{n}{(n-1)!}k^{n-1}$ for $k\ggg 0$ and this implies that

$$\mathrm{GGKdim}(A)\leq 1+\overline{\lim}_{k\to\infty}\log_k\tfrac{n}{(n-1)!}+\overline{\lim}_{k\to\infty}\log_k k^{n-1}=1+0+n-1=1+\deg(Gp_A(t))=n.$$

Now we have to prove that $\mathrm{GGKdim}(A)\geq n$. Note that $W:=V^{k^{n-1}}$ is a frame of A and $\mathrm{udim}W^k=\mathrm{udim}V^{k^n}\geq k^n$, therefore, $\log_k(\mathrm{udim}V^{k^n})\geq\log_k k^n=n$, and hence

$$\mathrm{GGKdim}(A)\geq\overline{\lim}_{k\to\infty}\log_k(\mathrm{udim}W^k)=\overline{\lim}_{k\to\infty}\log_k(\mathrm{udim}V^{k^n})\geq\overline{\lim}_{k\to\infty}n=n.$$

□

Ring	$Gh_A(t)$	$Gp_A(t)$
Habitual polynomial ring $R[x_1,\dots,x_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Ore extension of bijective type $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Weyl algebra $A_n(K)$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Extended Weyl algebra $B_n(K)$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Enveloping algebra of a Lie algebra $\mathcal{G}$ of dimension n, $\mathcal{U}(\mathcal{G})$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Tensor product $R\otimes_K\mathcal{U}(\mathcal{G})$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Crossed product $R*\mathcal{U}(\mathcal{G})$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Algebra of q-differential operators $D_{q,h}[x,y]$	$\frac{1}{1-t}$	1
Algebra of shift operators S_h	$\frac{1}{1-t}$	1
Mixed algebra D_h	$\frac{1}{(1-t)^2}$	$t+1$
Discrete linear systems $K[t_1,\dots,t_n][x_1,\sigma_1]\cdots[x_n;\sigma_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Linear partial shift operators $K[t_1,\dots,t_n][E_1,\dots,E_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Linear partial shift operators $K(t_1,\dots,t_n)[E_1,\dots,E_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
L. P. Differential operators $K[t_1,\dots,t_n][\partial_1,\dots,\partial_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
L. P. Differential operators $K(t_1,\dots,t_n)[\partial_1,\dots,\partial_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
L. P. Difference operators $K[t_1,\dots,t_n][\Delta_1,\dots,\Delta_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
L. P. Difference operators $K(t_1,\dots,t_n)[\Delta_1,\dots,\Delta_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
L. P. q-dilation operators $K[t_1,\dots,t_n][H_1^{(q)},\dots,H_m^{(q)}]$	$\frac{1}{(1-t)^m}$	$\frac{1}{(m-1)!}[t^{m-1}+\cdots+1]$
L. P. q-dilation operators $K(t_1,\dots,t_n)[H_1^{(q)},\dots,H_m^{(q)}]$	$\frac{1}{(1-t)^m}$	$\frac{1}{(m-1)!}[t^{m-1}+\cdots+1]$
L. P. q-differential operators $K[t_1,\dots,t_n][D_1^{(q)},\dots,D_m^{(q)}]$	$\frac{1}{(1-t)^m}$	$\frac{1}{(m-1)!}[t^{m-1}+\cdots+1]$
L. P. q-differential operators $K(t_1,\dots,t_n)[D_1^{(q)},\dots,D_m^{(q)}]$	$\frac{1}{(1-t)^m}$	$\frac{1}{(m-1)!}[t^{m-1}+\cdots+1]$
Diffusion algebras	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Additive analogue of the Weyl algebra $A_n(q_1,\dots,q_n)$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$	$\frac{1}{(1-t)^{n-1}}$	$\frac{1}{(n-2)!}[t^{n-2}+\cdots+1]$
Quantum algebra $\mathcal{U}'(\mathfrak{so}(3,K))$	$\frac{1}{(1-t)^3}$	$\frac{1}{2}[t^2+3t+1]$
3-dimensional skew polynomial algebras	$\frac{1}{(1-t)^3}$	$\frac{1}{2}[t^2+3t+1]$
Dispin algebra $\mathcal{U}(osp(1,2))$	$\frac{1}{(1-t)^3}$	$\frac{1}{2}[t^2+3t+1]$
Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2,K))$	$\frac{1}{(1-t)^3}$	$\frac{1}{2}[t^2+3t+1]$
Complex algebra $V_q(\mathfrak{sl}_3(\mathbb{C}))$	$\frac{1}{(1-t)^6}$	$\frac{1}{120}[t^5+15t^4+85t^3+217t^2+274t+120]$
Algebra $\mathbf{U}$	$\frac{1}{(1-t)^{2n}}$	$\frac{1}{(2n-1)!}[t^{2n-1}+\cdots+1]$
Manin algebra $\mathcal{O}_q(M_2(K))$	$\frac{1}{(1-t)^3}$	$\frac{1}{2}[t^2+3t+1]$
Coordinate algebra of the quantum group $SL_q(2)$	$\frac{1}{(1-t)^3}$	$\frac{1}{2}[t^2+3t+1]$
q-Heisenberg algebra $\mathbf{H}_n(q)$	$\frac{1}{(1-t)^{2n}}$	$\frac{1}{(2n-1)!}[t^{2n-1}+\cdots+1]$
Quantum enveloping algebra of $\mathfrak{sl}(2,K)$, $\mathcal{U}_q(\mathfrak{sl}(2,K))$	$\frac{1}{(1-t)^2}$	$t+1$
Hayashi algebra $W_q(J)$	$\frac{1}{(1-t)^{2n}}$	$\frac{1}{(2n-1)!}[t^{2n-1}+\cdots+1]$
Differential operators on a quantum space $S_{\mathbf{q}}$, $D_{\mathbf{q}}(S_{\mathbf{q}})$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Witten's Deformation of $\mathcal{U}(\mathfrak{sl}(2,K)$	$\frac{1}{1-t}$	1
Quantum Weyl algebra of Maltsiniotis $A_n^{\mathbf{q},\lambda}$	$\frac{1}{(1-t)^2}$	$t+1$
Quantum Weyl algebra $A_n(q,p_{i,j})$	$\frac{1}{(1-t)^2}$	$t+1$
Multiparameter Weyl algebra $A_n^{Q,\Gamma}(K)$	$\frac{1}{(1-t)^2}$	$t+1$
Quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}(K^{2n}))$	$\frac{1}{(1-t)^2}$	$t+1$
Quadratic algebras in 3 variables	$\frac{1}{1-t}$	1
n-Multiparametric skew quantum space $R_{\mathbf{q},\sigma}[x_1,\dots,x_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
n-Multiparametric quantum space $R_{\mathbf{q}}[x_1,\dots,x_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
n-Multiparametric skew quantum space $K_{\mathbf{q},\sigma}[x_1,\dots,x_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
n-Multiparametric quantum space $K_{\mathbf{q}}[x_1,\dots,x_n]$	$\frac{1}{(1-t)^n}$	$\frac{1}{(n-1)!}[t^{n-1}+\cdots+1]$
Ring of skew quantum polynomials $R_{\mathbf{q},\sigma}[x_1^{\pm1},\dots,x_r^{\pm1},x_{r+1},\dots,x_n]$	$\frac{1}{(1-t)^{n-r}}$	$\frac{1}{(n-r-1)!}[t^{n-r-1}+\cdots+1]$
Ring of quantum polynomials $R_{\mathbf{q}}[x_1^{\pm1},\dots,x_r^{\pm1},x_{r+1},\dots,x_n]$	$\frac{1}{(1-t)^{n-r}}$	$\frac{1}{(n-r-1)!}[t^{n-r-1}+\cdots+1]$
Algebra of skew quantum polynomials $K_{\mathbf{q},\sigma}[x_1^{\pm1},\dots,x_r^{\pm1},x_{r+1},\dots,x_n]$	$\frac{1}{(1-t)^{n-r}}$	$\frac{1}{(n-r-1)!}[t^{n-r-1}+\cdots+1]$
Algebra of quantum polynomials $\mathcal{O}_{\mathbf{q}}=K_{\mathbf{q}}[x_1^{\pm1},\dots,x_r^{\pm1},x_{r+1},\dots,x_n]$	$\frac{1}{(1-t)^{n-r}}$	$\frac{1}{(n-r-1)!}[t^{n-r-1}+\cdots+1]$

TABLE 2.1. Hilbert series and Hilbert polynomials of some skew quantum polynomials.

Chapter 3

Non-commutative algebraic geometry for semi-graded rings

In this chapter we include the main result concerning to semi-graded rings, we will solve 1.9.5 for some skew PBW extensions under the restriction $B_0 \subset Z(B)$. We present the theoretical tools for obtain a new general version of the Serre-Artin-Zhang-Verevkin theorem in a categorical setting, some elementary examples are included have not been studied before in the literature.

3.1 Non-commutative projective schemes

The purpose of this section is to extend the notion of non-commutative projective scheme defined in 1.6.1 to the case of semi-graded rings. We will assume that the ring B satisfies the following conditions:

(C1) B is a left noetherian SG.

(C2) B_0 is left noetherian.

(C3) For every n, B_n is a finitely generated left B_0-module.

(C4) $B_0 \subset Z(B)$.

Remark 3.1.1. (i) From (C4) we have that B_0 is commutative noetherian ring.

(ii) All important examples of skew PBW extensions satisfy (C1) and (C2). Indeed, let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension of R, assuming that R is a left noetherian, then A is also a left noetherian (see Theorem 1.9.4 ; in addition, by Proposition 2.1.7, A also satisfies (C3).

(iii) With respect to condition (C4), it is satisfied for finitely graded K-algebras since in such case $B_0 = K$. On the other hand, let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of R, then in general $R = A_0 \nsubseteq Z(A)$, unless A be a K-algebra, with $A_0 = K$ a commutative ring.

(iv) It is important to remark that some results below can be proved without assuming all of conditions (C1)-(C4). For example, for the Definition 3.1.6 we only need (C1).

Proposition 3.1.2. *Let* $\mathrm{sgr} - B$ *be the collection of all finitely generated semi-graded* B*-modules, then* $\mathrm{sgr} - B$ *is an abelian category where the morphisms are the homogeneous* B*-homomorphisms.*

Proof. It is clear that $\mathrm{sgr} - B$ is a category. $\mathrm{sgr} - B$ has kernels and co-kernels: Let M, M' be objects of $\mathrm{sgr} - B$ and let $f : M \to M'$ be an homogeneous B-homomorphism; let $K := \ker(f)$, since B is left noetherian and M is finitely generated, then K is a finitely generated semi-graded B-module; let $M'/Im(f)$ be the co-kernel of f, note that $Im(f)$ is semi-graded, so $M'/Im(f)$ is a semi-graded finitely generated B-module.

$\mathrm{sgr} - B$ is normal and co-normal: Let $f : M \to M'$ be a monomorphism in $\mathrm{sgr} - B$, then f is the kernel of the canonical homomorphism $j : M' \to M'/Im(f)$. Now, let $f : M \to M'$ be an epimorphism in $\mathrm{sgr} - B$, then f is the co-kernel of the inclusion $\iota : \ker(f) \to M'$.

$\mathrm{sgr} - B$ is additive: the trivial module 0 is an object of $\mathrm{sgr} - B$; if $\{M_i\}$ is a finite family of objects of $\mathrm{sgr} - B$, then its co-product $\bigoplus M_i$ in the category of left B-modules is an object of $\mathrm{sgr} - B$, with semi-graduation given by

$$(\bigoplus M_i)_p := \bigoplus (M_i)_p,\ p \in \mathbb{Z}.$$

Thus, $\mathrm{sgr} - B$ has finite co-products. Finally, for any objects M, M' of $\mathrm{sgr} - B$, $Mor(M, M')$ is an abelian group and the composition of morphisms is bilinear with respect the operations in these groups. □

Definition 3.1.3. *Let* M *be an object of* $\mathrm{sgr} - B$.

(i) *For* $s \geq 0$, $B_{\geq s}$ *is the least two-sided ideal of* B *that satisfies the following conditions:*
   (a) $B_{\geq s}$ *contains* $\bigoplus_{p \geq s} B_p$.
   (b) $B_{\geq s}$ *is semi-graded as left ideal of* B.
   (c) $B_{\geq s}$ *is a direct sumand of* B.

(ii) *An element* $x \in M$ *is torsion if there exist* $s, n \geq 0$ *such that* $B^n_{\geq s} x = 0$*; the set of torsion elements of* M *is denoted by* $T(M)$*;* M *is torsion if* $T(M) = M$ *and torsion-free if* $T(M) = 0$.

(iii) *For* $s, n \geq 0$, $M_{s,n}$ *will denote the least semi-graded submodule of* M *containing* $B^n_{\geq s} M$.

Remark 3.1.4. (i) Observe that if B is $\mathbb{N}$-graded, then $B_{\geq s} = \bigoplus_{p \geq s} B_p$.

(ii) Note that $T(M)$ is a submodule of M: In fact, let $x, y \in T(M)$, then there exist $r, s, n, m \geq 0$ such that $B^n_{\geq s} x = 0$ and $B^m_{\geq r} y = 0$; observe that $B_{\geq r+s} \subseteq B_{\geq s}, B_{\geq r}$, so $B^{n+m}_{\geq r+s} x \subseteq B^n_{\geq s} x = 0$ and $B^{n+m}_{\geq r+s} y \subseteq B^m_{\geq r} y = 0$, whence $B^{n+m}_{\geq r+s}(x + y) = 0$, i.e., $x + y \in T(M)$; if $b \in B$, then

$B_{\geq s}^n b \subseteq B_{\geq s}^n$, so $B_{\geq s}^n bx \subseteq B_{\geq s}^n x = 0$, i.e., $bx \in T(M)$.

(iii) Since M is noetherian, $M_{s,n}$ is finitely generated, i.e., $M_{s,n}$ is finitely generated, i.e. $M_{s,n}$ is a direct summand of M. Moreover, $M/M_{s,n}$ is torsion because $B_{\geq s}^n M \subseteq M_{s,n}$. In addition, note that $M_{s,n}$ is a direct summand of M.

(iv) If we assume that B is a domain, and hence, a left Ore domain, an alternative notion of torsion can be defined as in the classical case of commutative domains : An element $x \in M$ is torsion if there exists $b \neq 0$ in B such that $bx = 0$; the set $t(M)$ of torsion elements of M is in this case also a submodule of M. In addition, note that $T(M) \subseteq t(M)$: Since $B_{\geq s} \neq 0$, let $b \neq 0$ in $B_{\geq s}$, then $b^n x = 0$ and $b^n \neq 0$, i.e., $x \in t(M)$.

(v) It is clear that the collection $\mathcal{T}$ of modules M in $\text{sgr} - B$ such that $t(M) = M$ conforms a full subcategory of $\text{sgr} - B$. Moreover, let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence in $\text{sgr} - B$; it is obvious that $t(M) = M$ if and only if $t(M') = M'$ and $t(M'') = M''$, i.e., the collection $\mathcal{T}$ is a Serre subcategory of $\text{sgr} - B$. The next lemma shows that this property is satisfied also by the torsion modules introduced in Definition 3.1.3.

Theorem 3.1.5. *The collection* $\text{stor}-B$ *of torsion modules forms a Serre subcategory of* $\text{sgr}-B$*, and the quotient category*

$$\text{qsgr} - B := \text{sgr} - B/\text{stor} - B$$

is abelian.

Proof. It is obvious that $\text{stor} - B$ is a full subcategory of $\text{sgr} - B$. Let $0 \to M' \xrightarrow{\iota} M \xrightarrow{j} M'' \to 0$ be a short exact sequence in $\text{sgr} - B$.
Suppose that M is in $\text{stor} - B$ and let $x' \in M'$, then $\iota(x') \in M$ and there exist $s, n \geq 0$ such that $\iota(B_{\geq s}^n x') = B_{\geq s}^n \iota(x') = 0$, but since ι is injective, then $B_{\geq s}^n x' = 0$. This means that $x' \in T(M')$, so $T(M') = M'$, i.e., M' is in $\text{stor} - B$. Now let $x'' \in M''$, then there exists $x \in M$ such that $j(x) = x''$; there exist $r, m \geq 0$ such that $B_{\geq r}^m x = 0$, whence $B_{\geq s}^n x'' = 0$, this implies that $x'' \in T(M'')$. Thus, $T(M'') = M''$, i.e., M'' is in $\text{stor} - B$.
Conversely, suppose that M' and M'' are in $\text{stor} - B$; let $x \in M$, then there exist $s, n \geq 0$ such that $B_{\geq s}^n j(x) = 0$, i.e., $j(B_{\geq s}^n x) = 0$. Therefore, $B_{\geq s}^n x \subseteq \ker(j) = Im(\iota)$, but since M' is torsion, then $Im(\iota)$ is also a torsion module. Because B is left noetherian, there exist $a_1, \dots, a_l \in B_{\geq s}^n$ such that $B_{\geq s}^n = Ba_1 + \cdots + Ba_l$; there exist $r_i, m_i \geq 0$, $1 \leq i \leq l$, such that $B_{\geq r_i}^{m_i} a_i x = 0$. Without lost of generality we can assume that $r_1 \geq r_i$ for every i, so $B_{\geq r_1} \subseteq B_{\geq r_i}$ and hence $B_{\geq r_1}^{m_1} \subseteq B_{\geq r_1}^{m_1}, B_{\geq r_1}^{m_2} \subseteq B_{\geq r_2}^{m_2}, \dots, B_{\geq r_1}^{m_l} \subseteq B_{\geq r_l}^{m_l}$; from this we get that $B_{\geq r_1}^{m_1} a_1 x = 0, B_{\geq r_1}^{m_2} a_2 x = 0, \dots, B_{\geq r_1}^{m_l} a_l x = 0$, let $m := \max\{m_1, \dots, m_l\}$, then $B_{\geq r}^m a_i x = 0$ for every $1 \leq i \leq l$, with $r := r_1$. Therefore, $B_{\geq r}^m B_{\geq s}^n x = B_{\geq r}^m (Ba_1 + \cdots + Ba_l)x = B_{\geq r}^m a_1 x + \cdots + B_{\geq r}^m a_l x = 0$, i.e., $B_{\geq r}^m B_{\geq s}^n x = 0$. Observe that $B_{\geq r+s}^m \subseteq B_{\geq r}^m$ and $B_{\geq r+s}^n \subseteq B_{\geq s}^n$, so $B_{\geq r+s}^{m+n} \subseteq B_{\geq r}^m B_{\geq s}^n$ and hence $B_{\geq r}^{m+n} x = 0$, i.e., $x \in T(M)$. We have proved that $T(M) = M$, i.e., M is in $\text{stor} - B$.
The second statement of the theorem is a well known property of abelian categories. We want to recall that the objects of $\text{qsgr} - B$ are the objects of $\text{sgr} - B$; moreover, given M, N objects of $\text{qsgr} - B$ the set of morphisms from M to N in the category $\text{qsgr} - B$ is defined by

$$Hom_{\text{qsgr}-\text{B}}(M, N) := \varinjlim Hom_{\text{sgr}-B}(M', N/N'),$$

where the direct limit is taken over all $M' \subseteq M$, $N' \subseteq N$ in $\mathrm{sgr} - B$ with $M/M' \in \mathrm{stor} - B$ and $N' \in \mathrm{stor} - B$ (see [14], [8], or also [26] Proposition 2.13.4). More exactly, the limit is taken over the set $\mathcal{P}$ of all pairs (M', N') in $\mathrm{sgr} - B$ such that $M' \subseteq M$, $N' \subseteq N$, $M/M' \in \mathrm{stor} - B$ and $N' \in \mathrm{stor} - B$. The set $\mathcal{P}$ is partially ordered with order defined by

$$(M', N') \leq (M'', N'') \text{ if and only if } M'' \subseteq M' \text{ and } N' \subseteq N''.$$

$\mathcal{P}$ is directed: Indeed, given $(M', N'), (M'', N'') \in \mathcal{P}$ we apply Proposition 2.1.6 and the fact that B is left noetherian to conclude that $(M' \cap M'', N' + N'') \in \mathcal{P}$, and this couple satisfies $(M', N') \leq (M' \cap M'', N' + N'')$, $(M'', N'') \leq (M' \cap M'', N' + N'')$.

□

We have all ingredients in order to define non-commutative schemes associated to semi-graded rings.

Definition 3.1.6. *We define*

$$\mathrm{sproj}(B) := (\mathrm{qsgr} - B, \pi(B))$$

and we call it the ***non-commutative semi-projective scheme*** *associated to* B.

3.2 Serre-Artin-Zhang-Verevkin theorem for semi-graded rings

We conclude the paper investigating the non-commutative version of Serre-Atin-Zhang theorem for semi-graded rings. For this goal some preliminaries are needed.

Definition 3.2.1. *Let* M *be a semi-graded* B*-module,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$*. Let* $i \in \mathbb{Z}$*, the semi-graded module* $M(i)$ *defined by* $M(i)_n := M_{i+n}$ *is called a shift of* M*, i.e.,*

$$M(i) = \bigoplus_{n\in\mathbb{Z}} M(i)_n = \bigoplus_{n\in\mathbb{Z}} M_{i+n}.$$

Remark 3.2.2. Note that for every $i \in \mathbb{Z}$, $M \cong M(i)$ as B-modules. The isomorphism is given by

$$M = \bigoplus_{n\in\mathbb{Z}} M_n \xrightarrow{\phi_i} \bigoplus_{n\in\mathbb{Z}} M_{i+n} = M(i)$$
$$m_{n_1} + \cdots + m_{n_t} \in M_{n_1} + \cdots + M_{n_t} \mapsto m_{n_1} + \cdots + m_{n_t} \in M(i)_{n_1-i} + \cdots + M(i)_{n_t-i}.$$

ϕ_i is not homogeneous for $i \neq 0$.

The next proposition shows that the shift of degrees is an auto-equivalence.

Proposition 3.2.3. *Let* $s : \mathrm{sgr} - B \to \mathrm{sgr} - B$ *defined by*

$$M \mapsto M(1),$$
$$M \xrightarrow{f} N \mapsto M(1) \xrightarrow{f(1)} N(1),$$
$$f(1)(m) := f(m),\ m \in M(1).$$

Then,

(i) *s is an auto-equivalence.*

(ii) *For every* $d \in \mathbb{Z}$, $s^d(M) = M(d)$.

(iii) s *induces an auto-equivalence of* $\text{qsgr} - B$ *also denoted by* s.

Proof. (i) and (ii) are evident. For (iii) we only have to observe that if M is an object of $\text{stor} - B$, then $s(M)$ is also an object of $\text{stor} - B$. □

Definition 3.2.4. *Let* M, N *be objects of* $\text{sgr} - B$. *Then*

(i) $\underline{Hom}_B(M,N) := \bigoplus_{d\in\mathbb{Z}} Hom_{\text{sgr}-B}(M, N(d))$.

(ii) $\underline{Ext}^i_B(M,N) := \bigoplus_{d\in\mathbb{Z}} Ext^i_{\text{sgr}-B}(M, N(d))$.

Remark 3.2.5. Note that $\underline{Hom}_B(M,N) \hookrightarrow Hom_B(M,N)$. In fact, we have the group homomorphism $\iota : \underline{Hom}_B(M,N) \to Hom_B(M,N)$ given by $(\dots, 0, f_{d_1}, \dots, f_{d_t}, 0, \dots) \mapsto f_{d_1} + \cdots + f_{d_t}$; observe that $f_{d_1} + \cdots + f_{d_t} = 0$ if and only if $f_{d_1} = \cdots = f_{d_t} = 0$. Indeed, let $m \in M$ be homogeneous of degree p, then $0 = (f_{d_1} + \cdots + f_{d_t})(m) = f_{d_1}(m) + \cdots + f_{d_t}(m) \in N_{d_1+p} \oplus \cdots \oplus N_{d_t+p}$, whence, for every j, $f_{d_j}(m) = 0$. This means that $f_{d_j} = 0$ for $1 \leq j \leq t$, and hence, ι is injective.

Proposition 3.2.6. *Let* M *and* N *be semi-graded* B*-modules such that every of its homogeneous components are* B_0*-modules. Then,*

(i) $Hom_{\text{sgr}-B}(M,N)$ *is a* B_0*-module.*

(ii) $\underline{Hom}_B(M,N)$ *is a* B_0*-module.*

(iii) $Ext^i_{\text{sgr}-B}(M,N)$ *is a* B_0*-module for every* $i \geq 1$.

(iv) $\underline{Ext}^i_B(M,N)$ *is a* B_0*-module for every* $i \geq 1$.

Proof. (i) If $f \in Hom_{\text{sgr}-B}(M,N)$ and $b_0 \in B_0$, then product $b_0 \cdot f$ defined by $(b_0 \cdot f)(m) := b_0 \cdot f(m)$, $m \in M$, is an element of $Hom_{\text{sgr}-B}(M,N)$: In fact, $b_0 \cdot f$ is obviously additive; let $b \in B$, then $(b_0 \cdot f)(b \cdot m) = b_0 \cdot f(b \cdot m) = b_0[b \cdot f(m)] = (b_0 b) \cdot f(m) = (b b_0) \cdot f(m) = b \cdot (b_0 \cdot f(m)) = b \cdot (b_0 \cdot f)(m)$; $b_0 \cdot f$ is homogeneous: Let $m \in M_p$, then $(b_0 \cdot f)(m) = b_0 \cdot f(m) \in b_0 \cdot N_p \subseteq N_p$, for every $p \in \mathbb{Z}$. It is easy to check that $Hom_{\text{sgr}-B}(M,N)$ is a B_0-module with the defined product.
(ii) This follows from (i).
(iii) Taking a projective resolution of M in the abelian category $\text{sgr}-B$ and applying the functor $Hom_{\text{sgr}-B}(-,N)$, it is easy to verify using (i) that in the complex defining $Ext^i_{\text{sgr}-B}(M,N)$ the kernels and the images are B_0-modules, i.e., every abelian group $Ext^i_{\text{sgr}-B}(M,N)$ is a B_0-module.
(iv) This follows from (iii). □

Definition 3.2.7. *Let* $i \geq 0$*; we say that* B *satisfies the* ***s-***χ_i ***condition*** *if for every finitely generated semi-graded* B*-module* N *and for any* $j \leq i$, $\underline{Ext}^j_B(B/B_{\geq 1}, N)$ *is finitely generated as* B_0*-module. The ring* B *satisfies the* ***s-***χ ***condition*** *if it satisfies the s-*χ_i *condition for all* $i \geq 0$.

Remark 3.2.8. (i) By Proposition 3.2.6, $\underline{Ext}_B^j(B/B_{\geq 1}, N)$ is a B_0-module.
(ii) In the theory of graded rings and modules the conditions defined above are usually denoted simply by χ_i and χ. In this situation, $B/B_{\geq 1} \cong B_0$.
(iii) Observe that in the case of finitely graded K-algebras, $B_0 = K$, $B/B_{\geq 1} \cong K$ and the the condition $s - \chi_i$ means that $\dim_K \underline{Ext}_B^j(K, N) < \infty$.

Definition 3.2.9. *Let s be the auto-equivalence of* $\text{qsgr} - B$ *defined by the shifts of degrees. We define*

$$\Gamma(\pi(B))_{\geq 0} := \bigoplus_{d=0}^{\infty} Hom_{\text{qsgr}-B}(\pi(B), s^d(\pi(B))).$$

Following the ideas in the proof of Theorem 4.5 in [Artin2] and Proposition 4.11 in [Rogalski] we get the following key lemma.

Lemma 3.2.10. *Let B be a ring that satisfies* (C1)-(C4).

(i) $\Gamma(\pi(B))_{\geq 0}$ *is a* $\mathbb{N}$*-graded ring.*

(ii) *Let* $\underline{B} := \bigoplus_{d=0}^{\infty} Hom_{\text{sgr}-B}(B, s^d(B))$*. Then,* $\underline{B}$ *is a* $\mathbb{N}$*-graded ring and there exists a ring homomorphism* $\underline{B} \to \Gamma(\pi(B))_{\geq 0}$.

(iii) *For any object M of* $\text{sgr} - B$

$$\Gamma(M)_{\geq 0} := \bigoplus_{d=0}^{\infty} Hom_{\text{sgr}-B}(B, s^d(M))$$

is a graded $\underline{B}$*-module, and*

$$\Gamma(\pi(M))_{\geq 0} := \bigoplus_{d=0}^{\infty} Hom_{\text{qsgr}-B}(\pi(B), s^d(\pi(M)))$$

is a graded $\Gamma(\pi(B))_{\geq 0}$*-module.*

(iv) $\underline{B}$ *has the following properties:*

  (a) $(\underline{B})_0 \cong B_0$ *and* $\underline{B}$ *satisfies* (C2).

  (b) $\underline{B}$ *satisfies* (C3)*. More generally, let N be a finitely generated graded* $\underline{B}$*-module, then every homogeneous component of N is finitely generated over* $(\underline{B})_0$.

  (c) $\underline{B}$ *satisfies* (C1).

  (d) *If B is a domain, then* $\underline{B}$ *is also a domain.*

(v) *If B is a domain, then*

  (a) $\Gamma(\pi(B))_{\geq 0}$ *satisfies* (C2).

  (b) $\Gamma(\pi(B))_{\geq 0}$ *satisfies* (C3)*. More generally, let N be a finitely generated graded* $\Gamma(\pi(B))_{\geq 0}$*-module, then every homogeneous component of N is finitely generated over* $(\Gamma(\pi(B))_{\geq 0})_0$.

  (c) $\Gamma(\pi(B))_{\geq 0}$ *satisfies* (C1).

  (d) *If* $\underline{B}$ *satisfies* $\mathcal{X}_1$*, then* $\Gamma(\pi(B))_{\geq 0}$ *satisfies* $\mathcal{X}_1$.

  (e) $\Gamma(\pi(B))_{\geq 0}$ *is a domain.*

Proof. (i) Since $\text{qsgr}-B$ is an abelian category, $Hom_{\text{qsgr}-B}(\pi(B), s^d(\pi(B)))$ is an abelian group; the product in $\Gamma(\pi(B))_{\geq 0}$ is defined by distributive law and the following rule:

$$\text{If } f \in Hom_{\text{qsgr}-B}(\pi(B), s^n(\pi(B))) \text{ and } g \in Hom_{\text{qsgr}-B}(\pi(B), s^m(\pi(B))), \text{ then}$$
$$f \star g := s^n(g) \circ f \in Hom_{\text{qsgr}-B}(\pi(B), s^{m+n}(\pi(B))).$$

This product is associative: In fact, if $h \in Hom_{\text{qsgr}-B}(\pi(B), s^p(\pi(B)))$, then

$$(f \star g) \star h = [s^n(g) \circ f] \star h = s^{m+n}(h) \circ s^n(g) \circ f = f \star (g \star h).$$

It is clear that the product is $\mathbb{N}$-graded and the unity of $\Gamma(\pi(B))_{\geq 0}$ is i_B taken in $d = 0$ (observe that we have simplified the notation avoiding the bar notation for the morphisms in the category $\text{qsgr} - B$).

(ii) The proof of that $\underline{B}$ is a $\mathbb{N}$-graded ring is as in (i). For the second assertion we can apply the quotient functor π to define the function

$$\begin{aligned} \underline{B} &\xrightarrow{\rho} \Gamma(\pi(B))_{\geq 0} \\ (f_0, \dots, f_d, 0, \dots) &\mapsto (\pi(f_0), \dots, \pi(f_d), 0, \dots) \end{aligned} \tag{3.2.1}$$

which is a ring homomorphism since π is additive (π is exact) and $s\pi = \pi s$.

(iii) The proof of both assertions are as in (i), we only illustrate the product in the first case:

$$\text{If } f \in Hom_{\text{sgr}-B}(B, s^n(B)) \text{ and } g \in Hom_{\text{sgr}-B}(B, s^m(M)), \text{ then}$$
$$f \star g := s^n(g) \circ f \in Hom_{\text{sgr}-B}(B, s^{m+n}(M)).$$

(iv) (a) Note that $(\underline{B})_0 = Hom_{\text{sgr}-B}(B, B)$, and consider the function

$$B_0 \xrightarrow{\alpha} Hom_{\text{sgr}-B}(B, B),\ \alpha(x) = \alpha_x,\ \alpha_x(b) := bx,\ x \in B_0,\ b \in B;$$

since $B_0 \subset Z(B)$ this function is a ring homomorphism, moreover, bijective. Thus, $(\underline{B})_0$ is a commutative noetherian ring, so $(\underline{B})_0$ satisfies (C2). In addition, observe that the structure of B_0-module of $Hom_{\text{sgr}-B}(B, B)$ induced by α coincides with the structure defined in Proposition 3.2.6.

(b) Note that the function $Hom_{\text{sgr}-B}(B, B(d)) \xrightarrow{\lambda} B_d$ defined by $f \mapsto f(1)$ is an injective B_0-homomorphism. Since B_0 is noetherian and B satisfies C3, then $Hom_{\text{sgr}-B}(B, B(d))$ is finitely generated over $B_0 \cong (\underline{B})_0$.

For the second part, let N be generated by $x_1, \dots, x_r$, with $x_i \in N_{d_i}$, $1 \leq i \leq r$. Let $x \in N_d$, then there exist $f_1, \dots, f_r \in \underline{B}$ such that $x = f_1 \cdot x_1 + \cdots + f_r \cdot x_r$, from this we can assume that $f_i \in (\underline{B})_{d-d_i}$; by the just proved property (C3) for $\underline{B}$ we obtain that every $(\underline{B})_{d-d_i}$ is finitely generated as $(\underline{B})_0$-module, this implies that N_d is finitely generated over $(\underline{B})_0$.

(c) By (ii), $\underline{B}$ is not only SG but $\mathbb{N}$-graded.

$\underline{B}$ is left noetherian: We will adapt a proof given in [2]. Let I be a graded left ideal of $\underline{B}$; let $f \in \underline{B}$ be homogeneous of degree d_f, then f induces a morphism $s^{-d_f}(B) \xrightarrow{f_-} B$; thus, given a finite set F of homogeneous elements of I, let $P_F := \bigoplus_{f \in F} s^{-d_f}(B)$, $f_F := \sum_{f \in F} f_- : P_F \to B$ and let $N_F := Im(f_F)$. Since B is left noetherian we can choose a finite set F_0 such that N_{F_0} is maximal among such images. Let $N := N_{F_0}$ and $P := P_{F_0}$; we define $N'' := \Gamma(N)_{\geq 0} := \bigoplus_{d=0}^{\infty} Hom_{\text{sgr}-B}(B, s^d(N))$. According to (iii), N'' is a $\mathbb{N}$-graded $\underline{B}$-module. Given any element $f \in I$ homogeneous of degree d_f we have the morphism f_-, but since N is maximal the image of this morphism is included in N, and this implies that $f \in N''$, so $I \subseteq N''$. On the other

hand, given $f \in I$ homogeneous of degree d_f the $\mathbb{N}$-graded $\underline{B}$-homomorphism $s^{-d_f}(\underline{B}) \xrightarrow{f_-} \underline{B}$ defined by $\boldsymbol{f}_-(h) := hf$ has his image in I. Therefore, $N' \subseteq I$, where N' is the image of the induced morphism $P'' \to \underline{B}$, with $P'' := \bigoplus_{f \in F_0} s^{-d_f}(\underline{B})$. Thus, we have $N' \subseteq N''$, where both are $\mathbb{N}$-graded $\underline{B}$-modules, whence we have the $\mathbb{N}$-graded $\underline{B}$-module N''/N'. If we prove that N''/N' is noetherian, then since $I/N' \subseteq N''/N'$ we get that I/N' is also noetherian, whence, I/N' is finitely generated; but N' is a finitely generated left ideal of $\underline{B}$, so I is finitely generated.
N''/N' is noetherian: Note first that N''/N' is a module over $(\underline{B})_0$; if we prove that N''/N' is noetherian over $(\underline{B})_0$, then it is also noetherian over $\underline{B}$. According to (a), we only need to show that N''/N' is finitely generated over $(\underline{B})_0$. But this follows from (b) since N''/N' is right bounded (i.e., there exists $n \gg 0$ such that the homogeneous component of N''/N' of degree $k \geq n$ is zero, see [2]).
(d) If B is a domain, then $\underline{B}$ is also a domain: Suppose there exist $f, g \neq 0$ in $\underline{B}$ such that $f \star g = 0$, let $f_n \neq 0$ and $g_m \neq 0$ the nonzero homogeneous components of f and g of lowest degree, thus $f_n \in Hom_{\text{sgr}-B}(B, s^n(B))$, $g_m \in Hom_{\text{sgr}-B}(B, s^m(B))$ and $0 = f_n \star g_m = s^n(g_m) \circ f_n \in Hom_{\text{sgr}-B}(B, s^{m+n}(B))$; since $f_n \neq 0$ we have $f_n(1) \neq 0$, also $g_m(1) \neq 0$ and hence $s^n(g_m)(1) \neq 0$, so $0 = s^n(g_m)(f_n(1)) = f_n(1)s^n(g_m)(1)$, but this is impossible since B is a domain.
(v) We set $\Gamma := \Gamma(\pi(B))_{\geq 0}$. Then,
(a) Γ satisfies (C2): We divide the proof of this statement in two steps.
Step 1. Adapting the proof of Proposition 5.3.7 in [26] we will show that

$$\Gamma_0 = Hom_{\text{qsgr}-B}(\pi(B), \pi(B)) = \varinjlim Hom_{\text{sgr}-B}(B_{s,n}, B),$$

where the direct limit is taken over the homomorphisms of abelian groups

$$Hom_{\text{sgr}-B}(B_{s,n}, B) \to Hom_{\text{sgr}-B}(B_{r,m}, B)$$

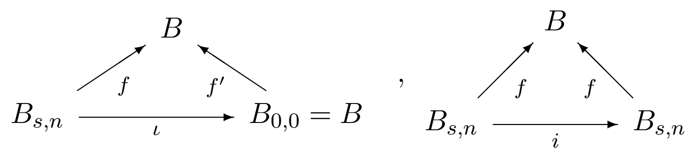
induced by the inclusion homomorphism $B_{r,m} \to B_{s,n}$, with $r \geq s$ and $m \geq n$. Observe that the collection of couples (s, n) is a partially ordered directed set.
Note first that $Hom_{\text{qsgr}-B}(\pi(B), \pi(B)) = \varinjlim Hom_{\text{sgr}-B}(M', B)$, where the direct limit is taken over all $M' \subseteq B$ with $B/M' \in \text{stor} - B$. In fact, we know that

$$Hom_{\text{qsgr}-\text{B}}(\pi(B), \pi(B)) = \varinjlim Hom_{\text{sgr}-B}(M', B/N'),$$

where the direct limit is taken over all $(M', N') \in \mathcal{P}$, but since B is a domain, $N' = 0$.
Now let $\overline{f} \in \varinjlim Hom_{\text{sgr}-\text{B}}(M', B)$, so $f \in Hom_{\text{sgr}-\text{B}}(M', B)$ for some $M' \subseteq B$ such that $T(B/M') = B/M'$; since B/M' is finitely generated, we can reasoning as in the proof of Theorem 3.1.5 and find $s, n \geq 0$ such that $B_s^n B \subseteq M'$, i.e., $B_{s,n} \subseteq M'$. From this we get that $\overline{f} = \overline{f'}$, where $\overline{f'} \in Hom_{\text{sgr}-B}(B_{s,n}, B)$, with $f' := f\iota$ and $\iota : B_{s,n} \hookrightarrow M'$ the inclusion. Since $B/B_{s,n}$ is torsion we obtain that $\varinjlim Hom_{\text{sgr}-B}(M', B) = \varinjlim Hom_{\text{sgr}-B}(B_{s,n}, B)$.
Step 2. Considering $s, n = 0$ in the limit above we obtain a ring homomorphism

$$(\underline{B})_0 = Hom_{\text{sgr}-B}(B, B) \xrightarrow{\gamma} Hom_{\text{qsgr}-B}(\pi(B), \pi(B)) = \Gamma_0;$$

since $(\underline{B})_0$ is noetherian we can prove that γ is surjective. Let $\overline{f} \in \Gamma_0$ with $f \in Hom_{\text{sgr}-B}(B_{s,n}, B)$, consider the commutative triangles



where f' is defined by $f'(x+l) := f(x)$, with $x \in B_{s,n}$, $l \in L$ and $B = B_{s,n} \oplus L$. Thus, $i^*(f) = fi = f$ and $\iota^*(f') = f'\iota = f$, so $\overline{f} = \overline{f'} = \gamma(f')$.
From this we conclude that Γ_0 is a commutative noetherian ring, and hence, Γ satisfies (C2).
(b) Γ satisfies (C3): Since Γ is graded, Γ_d is a Γ_0-module for every d, but by (a) we have a ring homomorphism $B_0 \cong (\underline{B})_0 \xrightarrow{\gamma} \Gamma_0$, so the idea is to prove that Γ_d is finitely generated over B_0. For this we will show that there exists a surjective B_0-homomorphism $(\underline{B})_d \xrightarrow{\beta} \Gamma_d$. Note that $\Gamma_d = Hom_{\text{qsgr}-B}(\pi(B), \pi(B(d))) = \varinjlim Hom_{\text{sgr}-B}(B_{s,n}, B(d))$ (the proof of this is as the step 1 in (a)); let $f \in (\underline{B})_d = Hom_{\text{sgr}-B}(B, B(d))$, we define $\beta(f) := \overline{f\iota}$, where $\iota : B_{s,n} \to B = B_{0,0}$; we can repeat the proof of the step 2 in (a) and conclude that β is a surjective B_0-homomorphism.
Additionally, let N be a finitely generated graded Γ-module, says N generated by a finite set of homogeneous elements $x_1, \ldots, x_r$, with $x_i \in N_{d_i}$, $1 \leq i \leq r$. Let $x \in N_d$, then there exist $f_1, \ldots, f_r \in \Gamma$ such that $x = f_1 \cdot x_1 + \cdots + f_r \cdot x_r$, from this we can assume that $f_i \in \Gamma_{d-d_i}$, but as was observed before, every Γ_{d-d_i} is finitely generated as Γ_0-module, so N_d is finitely generated over Γ_0 for every d.
(c) Γ satisfies (C1): By (iii), Γ is not only SG but $\mathbb{N}$-graded.
Γ is left noetherian: We will adapt the proof of (iv)-(c). Let I be a graded left ideal of Γ; let $f \in \Gamma$ be homogeneous of degree d_f, then f induces a morphism $s^{-d_f}(\pi(B)) \xrightarrow{f_-} \pi(B)$; thus, given a finite set F of homogeneous elements of I, let $P_F := \bigoplus_{f \in F} s^{-d_f}(\pi(B))$, $f_F := \sum_{f \in F} f_- : P_F \to \pi(B)$ and let $N_F := Im(f_F)$. Since $\pi(B)$ is a noetherian object of $\text{qsgr} - B$ we can choose a finite set F_0 such that N_{F_0} is maximal among such images. Let $\pi(N) := N_{F_0}$ and $\pi(P) := P_{F_0}$; we define $N'' := \Gamma(\pi(N))_{\geq 0} := \bigoplus_{d=0}^{\infty} Hom_{\text{qsgr}-B}(\pi(B), s^d(\pi(N)))$. According to (iii), N'' is a $\mathbb{N}$-graded Γ-module. Given any element $f \in I$ homogeneous of degree d_f we have the morphism f_-, but since N is maximal the image of this morphism is included in N, and this implies that $f \in N''$, so $I \subseteq N''$. On the other hand, given $f \in I$ homogeneous of degree d_f the $\mathbb{N}$-graded Γ-homomorphism $s^{-d_f}(\Gamma) \xrightarrow{\boldsymbol{f}_-} \Gamma$ defined by $\boldsymbol{f}_-(h) := hf$ has his image in I. Therefore, $N' \subseteq I$, where N' is the image of the induced morphism $P'' \to \Gamma$, with $P'' := \bigoplus_{f \in F_0} s^{-d_f}(\Gamma)$. Thus, we have $N' \subseteq N''$, where both are $\mathbb{N}$-graded Γ-modules, whence we have the $\mathbb{N}$-graded Γ-module N''/N'. If we prove that N''/N' is noetherian, then since $I/N' \subseteq N''/N'$ we get that I/N' is also noetherian, whence, I/N' is finitely generated; but N' is a finitely generated left ideal of Γ, so I is finitely generated.
N''/N' is noetherian: Note first that N''/N' is a module over Γ_0; if we prove that N''/N' is noetherian over Γ_0, then it is also noetherian over Γ. According to (a), we only need to show that N''/N' is finitely generated over Γ_0. But this follows from (b) since N''/N' is right bounded.
(d) Γ satisfies $\mathcal{X}_1$: Let N be a finitely generated graded Γ-module, we have $\underline{Ext}^j_{\Gamma}(\Gamma/\Gamma_{\geq 1}, N) = \underline{Ext}^j_{\Gamma}(\Gamma_0, N)$, so we must prove that $\underline{Ext}^j_{\Gamma}(\Gamma_0, N)$ is finitely generated as Γ_0-module for $j = 0, 1$.

By the surjective homomorphism $(\underline{B})_0 \xrightarrow{\gamma} \Gamma_0$ in the step 2 in (a), it is enough to show that $\underline{Ext}^j_\Gamma(\Gamma_0, N)$ is finitely generated over $(\underline{B})_0$. Observe that γ is also a graded homomorphism of left $(\underline{B})$-modules; moreover, N is a finitely generated graded left $(\underline{B})$-module since the homomorphism ρ in (ii) is surjective; the proof of this last statement is as in the step 2 of (a), using of course that B is a domain, we include it for completeness: It is enough to consider $\overline{f} \in \Gamma_d = \varinjlim Hom_{\text{sgr}-B}(B_{s,n}, B(d))$, with $f \in Hom_{\text{sgr}-B}(B_{s,n}, B(d))$ for some $s, n \geq 0$; we define $f' : B_{0,0} \to B(d)$, $f'(x) := f(y)$, where $B = B_{s,n} \oplus L$ and $x = y + l$ with $y \in B_{s,n}$ and $l \in L$; therefore, $\rho(f') = \pi(f') = \overline{f}$ since we have $f'\iota = f$.
Now we can apply the functor $\underline{Ext}^j_{\underline{B}}(\,, N)$ and get the injective homomorphism of left $(\underline{B})_0$-modules $\underline{Ext}^j_{\underline{B}}(\Gamma_0, N) \to \underline{Ext}^j_{\underline{B}}((\underline{B})_0, N)$, but since $\underline{B}$ satisfies $\mathcal{X}_1$, $\underline{Ext}^j_{\underline{B}}((\underline{B})_0, N)$ is finitely generated over $(\underline{B})_0$, so $\underline{Ext}^j_{\underline{B}}(\Gamma_0, N)$ is finitely generated since $(\underline{B})_0$ is left noetherian. From the injective $(\underline{B})_0$-homomorphism $\underline{Ext}^j_\Gamma(\Gamma_0, N) \to \underline{Ext}^j_{\underline{B}}(\Gamma_0, N)$ we conclude that $\underline{Ext}^j_\Gamma(\Gamma_0, N)$ is also finitely generated over $(\underline{B})_0$.
(e) Γ is a domain: Suppose there exist $f, g \neq 0$ in Γ such that $f \star g = 0$, let $f_n \neq 0$ and $g_m \neq 0$ the nonzero homogeneous components of f and g of lowest degree, thus

$$f_n \in Hom_{\text{qsgr}-B}(\pi(B), s^n(\pi(B))),\; g_m \in Hom_{\text{qsgr}-B}(\pi(B), s^m(\pi(B)))$$

and $0 = f_n \star g_m = s^n(g_m) \circ f_n \in Hom_{\text{qsgr}-B}(\pi(B), s^{m+n}(\pi(B)))$; note that the representative elements of f_n and g_m in $Hom_{\text{sgr}-B}(B_{0,0}, B(n)) \cong Hom_{\text{sgr}-B}(B, B) = (\underline{B})_0$ and $Hom_{\text{sgr}-B}(B_{0,0}, B(m)) \cong Hom_{\text{sgr}-B}(B, B) = (\underline{B})_0$, respectively, are non zero, but this is impossible since $(\underline{B})_0$ is a domain and the representative element of $f \star g$ in $Hom_{\text{sgr}-B}(B_{0,0}, B(n+m)) \cong Hom_{\text{sgr}-B}(B, B) = (\underline{B})_0$ is zero. □

Proposition 3.2.11. *Let S be a commutative noetherian ring and $\rho : C \to D$ be a homomorphism of $\mathbb{N}$-graded left noetherian S-algebras. If the kernel and cokernel of ρ are right bounded, then $D \underline{\otimes}_C -$ defines an equivalence of categories* $\text{qgr}-C \simeq \text{qgr}-D$*, where $\underline{\otimes}$ denotes the graded tensor product.*

Proof. The proof of Proposition 2.5 in [2] applies since it is independent of the notion of torsion. □

We are prepared for proving the main theorem of the present section.

Theorem 3.2.12. *If B is a domain that satisfies* (C1)-(C4) *and $\underline{B}$ satisfies the condition $\mathcal{X}_1$ then there exists an equivalence of categories*

$$\text{qgr} - \underline{B} \simeq \text{qgr} - \Gamma(\pi(B))_{\geq 0}.$$

Proof. Note that the ring homomorphism in (3.2.1) satisfies the conditions of Proposition 3.2.11, with $S = B_0$, $C = \underline{B}$ and $D = \Gamma(\pi(B))_{\geq 0}$. In fact, from Lemma 3.2.10 we know that $\underline{B}$ and $\Gamma(\pi(B))_{\geq 0}$ are $\mathbb{N}$-graded left noetherian rings and B_0-modules; moreover, they are B_0-algebras: We check this for $\underline{B}$, the proof for $\Gamma(\pi(B))_{\geq 0}$ is similar. If $f \in Hom_{\text{sgr}-B}(B, B(n)), g \in Hom_{\text{sgr}-B}(B, B(m))$, $x \in B_0$ and $b \in B$, then

$$[x \cdot (f \star g)](b) = x \cdot (s^n(g) \circ f)(b) = xg(n)(f(b));$$
$$[f \star (x \cdot g)](b) = [s^n(x \cdot g) \circ f](b) = (x \cdot g)(n)(f(b)) = xg(n)(f(b)).$$

Finally, we can apply the proof of part S10 in Theorem 4.5 in [2] to conclude that the kernel and cokernel of ρ are right bounded. □

Remark 3.2.13. Considering the above developed theory for graded rings and right modules instead of semi-graded rings and left modules it is possible to prove that $\underline{B} \cong B$. Thus, in such case we get from the previous theorem the Serre-Artin-Zhang-Verevkin equivalence $\text{qgr} - B \simeq \text{qgr} - \Gamma(\pi(B))_{\geq 0}$.

Example 3.2.14. The examples of skew PBW extensions below are semi-graded (non $\mathbb{N}$-graded) domains and satisfy the conditions (C1)-(C4); in each case we will prove that $\underline{B}$ satisfies the condition $\mathcal{X}_1$; therefore, for these algebras Theorem 3.2.12 is true. In every example $B_0 = K$ is a field, we indicate the relations defining B (see [20]) and the associated graded ring $Gr(B)$ (Proposition 2.1.6):

(i) Enveloping algebra of a Lie K-algebra $\mathcal{G}$ of dimension n, $\mathcal{U}(\mathcal{G})$:

$$x_ik - kx_i = 0,\ k \in K;$$
$$x_ix_j - x_jx_i = [x_i, x_j] \in \mathcal{G} = Kx_1 + \cdots + Kx_n,\ 1 \leq i, j \leq n;$$
$$Gr(B) = K[x_1, \ldots, x_n].$$

(ii) Quantum algebra $\mathcal{U}'(so(3, K))$:

$$x_2x_1 - qx_1x_2 = -q^{1/2}x_3, \quad x_3x_1 - q^{-1}x_1x_3 = q^{-1/2}x_2, \quad x_3x_2 - qx_2x_3 = -q^{1/2}x_1,\ q \in K - \{0\};$$

in this case $Gr(B) = K_{\mathbf{q}}[x_1, x_2, x_3]$ is the 3-multiparametric quantum space, i.e., a quantum polynomial ring in 3 variables, with

$$\mathbf{q} = \begin{bmatrix} 1 & q & q^{-1} \\ q^{-1} & 1 & q \\ q & q^{-1} & 1 \end{bmatrix}.$$

(iii) Dispin algebra $\mathcal{U}(osp(1, 2))$:

$$x_2x_3 - x_3x_2 = x_3, \quad x_3x_1 + x_1x_3 = x_2, \quad x_1x_2 - x_2x_1 = x_1;$$
$$Gr(B) = K_{\mathbf{q}}[x_1, x_2, x_3], \text{ with } \mathbf{q} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

(iv) Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2, K))$, where $\nu \in K - \{0\}$ is not a root of unity:

$$x_1x_3 - \nu^4x_3x_1 = (1 + \nu^2)x_1, \quad x_1x_2 - \nu^2x_2x_1 = \nu x_3, \quad x_3x_2 - \nu^4x_2x_3 = (1 + \nu^2)x_2;$$
$$Gr(B) = K_{\mathbf{q}}[x_1, x_2, x_3], \text{ with } \mathbf{q} = \begin{bmatrix} 1 & \nu^{-2} & \nu^{-4} \\ \nu^2 & 1 & \nu^4 \\ \nu^4 & \nu^{-4} & 1 \end{bmatrix}.$$

(v) Nine types of 3-dimensional skew polynomial algebras, $\alpha, \beta, \gamma \in K - \{0\}$:

$$x_2x_3 - \alpha x_3x_2 = 0, \ \ x_3x_1 - \beta x_1x_3 = 0, \ \ x_1x_2 - \gamma x_2x_1 = 0;$$
$$x_2x_3 - x_3x_2 = x_3, \ \ x_3x_1 - \beta x_1x_3 = x_2, \ \ x_1x_2 - x_2x_1 = x_1;$$
$$x_2x_3 - x_3x_2 = 0, \ \ x_3x_1 - \beta x_1x_3 = x_2, \ \ x_1x_2 - x_2x_1 = 0;$$
$$x_2x_3 - x_3x_2 = x_3, \ \ x_3x_1 - \beta x_1x_3 = 0, \ \ x_1x_2 - x_2x_1 = x_1;$$
$$x_2x_3 - x_3x_2 = x_3, \ \ x_3x_1 - \beta x_1x_3 = 0, \ \ x_1x_2 - x_2x_1 = 0;$$
$$x_2x_3 - x_3x_2 = x_1, \ \ x_3x_1 - x_1x_3 = x_2, \ \ x_1x_2 - x_2x_1 = x_3;$$
$$x_2x_3 - x_3x_2 = 0, \ \ x_3x_1 - x_1x_3 = 0, \ \ x_1x_2 - x_2x_1 = x_3;$$
$$x_2x_3 - x_3x_2 = -x_2, \ \ x_3x_1 - x_1x_3 = x_1 + x_2, \ \ x_1x_2 - x_2x_1 = 0;$$
$$x_2x_3 - x_3x_2 = x_3, \ \ x_3x_1 - x_1x_3 = x, \ \ x_1x_2 - x_2x_1 = 0;$$
$$Gr(B) = K_{\mathbf{q}}[x_1, x_2, x_3], \text{ where } \mathbf{q} \text{ is an appropiate matrix in every case.}$$

Observe that in every example, $Gr(B)$ is a noetherian Artin-Schelter regular algebra, and hence, $Gr(B)$ satisfies the $\mathcal{X}_1$ condition (see [23]). From this we will conclude that $\underline{B}$ also satisfies such condition.
In fact, note first that in general there is an injective $\mathbb{N}$-graded homomorphism of B_0-algebras $\eta : \underline{B} \to Gr(B)$ defined by

$$\bigoplus_{d=0}^{\infty} Hom_{\text{sgr}-B}(B, B(d)) \xrightarrow{\eta} \bigoplus_{d=0}^{\infty} Gr(B)_d = \bigoplus_{d=0}^{\infty} \frac{B_0 \oplus \cdots \oplus B_d}{B_0 \oplus \cdots \oplus B_{d-1}}$$
$$f_0 + \cdots + f_d \mapsto \overline{f_0(1)} + \cdots + \overline{f_d(1)},$$

with $f_i \in Hom_{\text{sgr}-B}(B, B(i))$, $0 \leq i \leq d$. We only check that η is multiplicative, the other conditions can be proved also easily: $\eta(f_n \star g_m) = \eta(s^n(g_m) \circ f_n) = \overline{(s^n(g_m) \circ f_n)(1)} = \overline{s^n(g_m)(f_n(1))} = \overline{g_m(f_n(1))} = \overline{f_n(1)g_m(1)} = \overline{f_n(1)}\ \overline{g_m(1)} = \eta(f_n)\eta(g_m)$.
Thus, in the examples above $K = B_0$, $(\underline{B})_0 \cong B_0 \cong Gr(B)_0$ and the kernel and cokernel of η are right bounded, so we can apply the part (5) of Lemma 8.2 in [2] and conclude that $\underline{B}$ satisfies $\mathcal{X}_1$.
We finish remarking that for the listed examples we can apply Proposition 3.2.11 and Theorem 3.2.12 and obtain that

$$\text{qgr} - K[x_1, x_2, x_3] \simeq \text{qgr} - \Gamma(\pi(B))_{\geq 0}, \text{ with } B = \mathcal{U}(\mathcal{G});$$
$$\text{qgr} - K_{\mathbf{q}}[x_1, x_2, x_3] \simeq \text{qgr} - \Gamma(\pi(B))_{\geq 0},$$

with $B = \mathcal{U}'(so(3, K)), \mathcal{U}(osp(1, 2)), \mathcal{W}_\nu(\mathfrak{sl}(2, K))$ or any of nine types of 3-dimensional skew polynomial algebras above, and $\mathbf{q}$ an appropiate matrix in every case.

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