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Special functions, Lie theory and partial differential equations

Ernest G. Kalnins
University of Waikato, New Zealand

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Abstract. In the lectures an outline is given of the close relationship between special functions and Lie group theory. In doing so the exposition is deliberately quite direct and hopefully clear. An attempt is made to give an outline of the state of the subject if only from the author's point of view. Principal references containing the important features of the subject are also given.

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0. Introduction

Before getting into the subject proper let us have a look at the basic ingredients of the subject of Lie groups. A Lie group is a group that is specified by a set of coordinates $g = (g_1, g_2, \dots, g_n)$ for which there is a prescribed rule of composition viz $gh = (\varphi_1(g,h), \dots, \varphi_n(g,h))$ which satisfies the group properties

$$g(hk) = (gh)k,$$
$$ge = eg = g,$$
$$gg^{-1} = g^{-1}g = e,$$

for suitably smooth functions $\varphi(g,h)$. The symbol e denotes the identity element. As an illustrative example which will make these ideas clear we consider the group specified by the pair of numbers $g=(a,b)$. The composition law is taken to be

$$gh=(a,b)\cdot(c,d)=(ac,ad+b)$$

where a,b,c,d are real numbers. It is in fact generally true that groups such as this are realisable as matrices. Indeed there is a mapping $g\mapsto A(g)$ which is such that $A(g)A(h)=A(gh)$ and the group laws are faithfully preserved. For the example that we are using this mapping is in fact

$$(a,b)=g\mapsto A(g)=\begin{bmatrix} a & b\\ 0 & 1\end{bmatrix}.$$

It is easy to check that correct group composition laws are obeyed for the matrices $A(g)$. From now on all the Lie groups we consider are realised in the form of a matrix groups. The relationship between Lie groups and Lie algebras is best understood through the idea of one parameter gropus in the group G. A one parameter group in G is specified by $A(g(t))$, $0<t<1$ where $g(t)$ is in G, $g(t)g(s)=g(s+t)$ and $g(0)=e$. In the case of our example we could use

$$A(g(t))=\begin{bmatrix} e^t & 0\\ 0 & 1\end{bmatrix} \quad \text{or} \quad A(h(s))=\begin{bmatrix} 1 & s\\ 0 & 1\end{bmatrix}.$$

If we consider the space of tangent vectors $\mathbf{A}=\frac{d}{dt}A(g(t))\Big|_{t=0}$ at the identity then the set of all such tangent vectors forms the Lie algebra $L(G)$ structure associated with the group G. This means that if $\mathbf{A},\mathbf{B}$ and $\mathbf{C}$ are in $L(G)$ then so is $[\mathbf{A},\mathbf{B}]=\mathbf{AB}-\mathbf{BA}$ and furthermore the conditions that $L(G)$ be a Lie algebra are satisfied, *i.e.*

$$[\mathbf{A},\mathbf{B}]=-[\mathbf{B},\mathbf{A}],$$
$$[c\mathbf{A}+d\mathbf{B},\mathbf{C}]=c[\mathbf{A},\mathbf{C}]+d[\mathbf{B},\mathbf{C}], \qquad c \text{ and } d \text{ real numbers},$$
$$[[\mathbf{A},\mathbf{B}],\mathbf{C}]+[[\mathbf{B},\mathbf{C}],\mathbf{A}]+[[\mathbf{C},\mathbf{A}],\mathbf{B}]=0 \qquad \text{(Jacobi identity)}.$$

The crucial relationship between the Lie algebra and the group itself is that for sufficiently small $\mathbf{A}$ a group element A sufficiently close to the identity can be uniquely written in the form

$$A=\exp(\mathbf{A})=\sum_{j=0}^{\infty}\frac{\mathbf{A}^j}{j!}.$$

For the group we are studying a suitable basis for $L(G)$ is

$$L_1=\frac{d}{dt}A(g(t))\Big|_{t=0}=\begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}$$
$$L_2=\frac{d}{ds}A(h(s))\Big|_{s=0}=\begin{bmatrix} 0 & 1\\ 0 & 0\end{bmatrix}$$

which follows readily from the relation

$$\exp(a_1 L_1 + a_2 L_2) = \begin{bmatrix} e^{a_1} & \frac{a_2}{a_1}(e^{a_1} - 1) \\ 0 & 1 \end{bmatrix}$$

which recovers a general group element from an element of the Lie algebra. It is then easy to see that any one parameter group $g(t)$ could be written in the form $\exp(g_1(t)L_1 + g_2(t)L_2)$ for suitable g_1 and g_2. The basis for the Lie algebra satisfies the commutation relation $[L_1, L_2] = L_2$.

The main way that Lie groups appear for the theory of special functions is via action as a transformation group. In this case the group acts on some vector space V. The action on elements x of this vector space is specified functions $f(x, g) = gx$ which faithfully represent the group laws, viz

$$g(hx) = (gh)x$$
$$ex = x.$$

In the case of our example, $gx = ax + b$, or on matrix form

$$A(g)\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} ax + b \\ 1 \end{bmatrix}.$$

The main way this is of interest to us is to consider the space of functions $f(x)$ and the effect of the transformation group is given via $T_g f(x) = f(g^{-1}x)$.

Of primary concern in the study of Lie groups is the representation theory. A representation of a Lie group on a vector space is a mapping $g \mapsto T(g)$ which homomorphically reflects the properties of the group, *i.e.*

$$T(g)T(k) = T(gk),$$
$$T(e) = \text{identity element}.$$

For the cases we have considered the vector space could be the vectors $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ or when acting as a transformation group the functions $f(x)$. In any given representation the Lie algebra elements also have their representatives which can be calculated as follows

$$\mathbf{A}v = \frac{d}{dt} T_{g(t)} v \Big|_{t=0}$$

for any one parameter group $g(t)$. In this definition v can be a finite dimensional vector (in which case $\mathbf{A}$ is a matrix) or a function of x (in which case $\mathbf{A}$ is a differential operator). For the example we have been using the differential operators that represent the Lie algebra acting on functions of x are

$$\widetilde{L}_1 = -x\frac{d}{dx},$$
$$\widetilde{L}_2 = -\frac{d}{dx}.$$

It can be easily checked that these elements satisfy the correct commutation relations.

Excellent references on the subject of Lie groups are [10], [14], [15] and [16].

1. An illustrative example: the Helmholtz equation in two dimensions

These lectures will attempt to explain how the classical special functions arise and how they can be related to Lie's theory of symmetry groups of differential equations. The essential ideas are most easily explained via crucial examples. As our first example let us consider the Helmholtz equation

$$(\Delta_2+\omega^2)\Psi=\Psi_{xx}+\Psi_{yy}+\omega^2\Psi=0,\qquad \omega \text{ real and non zero.} \tag{1}$$

If we consider solutions of a certain class of this equation *e.g.* the solutions which are analytic or perhaps square integrable, then this space is a vector space, *i.e.* if Ψ_1,Ψ_2 are solutions of (1) then so is $a\Psi_1+b\Psi_2$ if a and b are complex. It is intuitively clear that if $\Psi(x,y)$ is a solution of (1) then so are $\Psi_1=\Psi(x+a,y+b)$ and $\Psi_2=\Psi(x\cos\theta+y\sin\theta,-x\sin\theta+y\cos\theta)$. This observation means that if we translate or rotate the coordinates in any solution of (1) then we obtain another solution. The group of all translations and rotations is called the Euclidean group $E(2)$. These are just the motions that preserve the Euclidean distance between two points in the plane. The effect of a symmetry group on the solutions of (1) can be deduced from that of its Lie algebra. This is the fundamental result of the theory of continuous Lie groups. For a space of functions such as we are considering the elements of the Lie algebra are realised as partial differential operators. Indeed if we consider the operators

$$P_1=\partial_x,\quad P_2=\partial_y\quad\text{and}\quad M=y\partial_x-x\partial_y$$

then we can observe from the Taylor series expansion that

$$\exp(aP_1)\Psi(x,y)=\sum_{i=0}^{\infty}\frac{a^i}{i!}P_1^i\Psi(x,y)=\Psi(x+a,y),$$

$$\exp(bP_2)\Psi(x,y)=\sum_{i=0}^{\infty}\frac{b^i}{i!}P_2^i\Psi(x,y)=\Psi(x,y+b),$$

$$\exp(\theta M)\Psi(x,y)=\sum_{i=0}^{\infty}\frac{\theta^i}{i!}M^i\Psi(x,y)=\Psi(x\cos\theta+y\sin\theta,-x\sin\theta+y\cos\theta).$$

These partial differential operators close to form a Lie algebra under commutation. The commutation relations are

$$[P_1,P_2]=0,$$
$$[M,P_1]=P_2,$$
$$[M,P_2]=-P_1$$

The actual group $E(2)$ can be realised as the group of matrices of the form

$$g(\theta,a,b)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ a & b & 1\end{bmatrix}$$

where a and b are real and $0\le\theta<2\pi$. The rule for combining these elements *i.e.* the group composition law is

$$g(\theta,a,b)g(\alpha,c,d)=g(\theta+\alpha, a\cos\alpha+b\sin\alpha+c, -a\sin\alpha+b\cos\theta+d).$$

Acting on the space of vectors in two dimensional Euclidean space the group element transforms $x=(x,y)$ to the point

$$xg=(x\cos\theta+y\sin\theta+a, -x\sin\theta+y\cos\theta+b).$$

Indeed the matrices

$$M=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad P_1=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix},\quad P_2=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},$$

satisfy the same commutation relations as the partial differential operators we introduced previously. Indeed we can easily compute that

$$\exp(\theta M)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\exp(P_1a)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a & 0 & 1\end{bmatrix}$$

$$\exp(P_2b)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix}.$$

These are then two ways of realising the same algebra $LE(2)$, one in terms of matrices acting on the vectors $(x,y,1)$, and the other acting on functions $\Psi(x,y)$. Before proceeding we observe that the general group element can be constructed via the formula $g(\theta,a,b)=\exp(\theta M)\exp(aP_1+bP_2)$, *i.e.* a general $E(2)$ element can be written as the product of two translations $\exp(aP_1+bP_2)$ and a rotation $\exp(\theta M)$. Lie groups such as $E(2)$ are studied from the point of view of representation theory. This involves the representation of the group element $g(\theta,a,b)$ by a linear operator $T(g(\theta,a,b))$ acting on some vector space in such a way that $T(k)T(h)=T(kh)$ where $k=g(\theta,a,b)$, $h=g(\alpha,c,d)$ and if k

is the identity of the group then $T(k)$ is the identity operator. These operators mimic the group laws in a homomorphic way. On our space of functions of $\Psi(x)$ the corresponding representation is

$$T(g(\theta, a, b))\Psi(x) = \exp(\theta M)\exp(aP_1 + bP_2)\Psi(x) = \Psi(xg).$$

where $xg = (x\cos\theta + y\sin\theta + a, -x\sin\theta + y\cos\theta + b)$.

What has all this got to do with special functions? It is known from classical reference books that solutions of (1) which can be solved by separation of variables give rise to some of the classical special functions of physics. A good reference for the classical special functions and how they arise via separation of variables is [12].

In particular it can be shown that there are only four coordinate systems in which this separation occurs.

1. Cartesian coordinates, x, y. The solutions then have the form of a product of exponential functions
$$\Psi_1(x, y) = e^{\lambda x + \mu y}$$
where $\lambda^2 + \mu^2 = -\omega^2$.
2. Polar coordinates, $x = r\cos\theta, y = r\sin\theta$, $0 < r < \infty$, $0 \le \theta < 2\pi$. The solutions are then of the form
$$\Psi_2(r, \theta) = C_m(\omega r)e^{im\,\theta}$$
where $C_m(\omega r)$ is a solution of Bessel's equation.
3. Parabolic coordinates, $x = \frac{1}{2}(\xi^2 - \eta^2), y = \xi\eta$. The solutions are a product of parabolic cylinder functions.
4. Elliptic coordinates, $x = d\cosh a\cos\alpha, y = d\sinh a\sin\alpha$. The solutions in this case are then products of Mathieu functions.

We now see that we have special functions arising from separation of variables and also an associated group of symmetry transformations $E(2)$.

How are the notions of separable solutions and the existence of a group of symmetries connected? To answer this we need to look at the operators that define the Lie algebra $LE(2)$. These operators are such that for the solutions $\Psi_1(x, y)$ and $\Psi_2(r, \theta)$ the following eigenfunction equations hold

$$P_1\Psi_1(x, y) = \lambda\Psi_1(x, y),$$
$$M\Psi_2(r, \theta) = im\,\Psi_2(r, \theta).$$

Each of the solutions is the eigenfunction of an element of $LE(2)$. If we consider an element of $LE(2)$ such as $L = pP_1 + qP_2 + rM$ then the occurrence of these two coordinate systems can be explained as follows. Let's say that two elements of the form L, L_1 and L_2 say are related by a transformation in $E(2)$ acting on the x and y coordinates we then say they are equivalent. If we classify all equivalence classes of such operators we obtain only two possibilities. Suitable choices of representative from these two classes are M and P_1. We can then say that if we are looking for solutions of (1) which are also eigenfunctions of an element of $LE(2)$ then to within translations and rotations in the plane the solutions are either those obtained for Cartesian coordinates or polar coordinates, *i.e.* we could position our Cartesian coordinates or polar coordinates arbitrarily in the plane.

Is there an analogue of this result for the two other coordinates systems? The answer is yes. To see this consider the coordinate systems 3 and 4 in more detail.

3. If we rewrite (1) in terms of parabolic coordinates then the equation has the form
$$(\partial_{\xi\xi} + \partial_{\eta\eta})\Psi + \omega^2(\xi^2 + \eta^2)\Psi = 0.$$
The separable solutions $\Psi_3(\xi, \eta) = M(\xi)N(\eta)$ satisfy the separation equations
$$(\partial_{\xi\xi} + (\omega^2\xi^2 - \lambda))M(\xi) = 0,$$
$$(\partial_{\eta\eta} + (\omega^2\eta^2 + \lambda))N(\eta) = 0.$$
We can see by direct computation that these solutions satisfy the eigenvalue equation
$$(\xi^2 + \eta^2)^{-1}(\eta^2\partial_{\xi\xi} - \xi^2\partial_{\eta\eta})\Psi = \lambda\Psi.$$
This is the analogous equation to that given for the polar and Cartesian coordinate systems. The difference is that λ is the eigenvalue of a second order partial differential operator. What is the key observation is that this equation can also be written in the form
$$\{M, P_2\}\Psi = (MP_2 + P_2M)\Psi = \lambda\Psi.$$

4. Similar observations hold for elliptic coordinates. The equation (1) in these coordinates has the form
$$(\partial_{aa} + \partial_{\alpha\alpha})\Psi + d^2\omega^2(\cosh^2 a + \cos^2 \alpha)\Psi = 0.$$
Setting $\Psi = A(a)B(\alpha)$ the separation equations are
$$(\partial_{aa} + (d^2\omega^2 \cosh^2 a + \lambda))A(\alpha) = 0$$
$$(\partial_{\alpha\alpha} - (d^2\omega^2 \cos^2 \alpha + \lambda))B(\alpha) = 0.$$
The separation constant is an eigenvalue of the operator
$$(\cosh^2 a - \cos^2 \alpha)^{-1}(\cos^2 \alpha\partial_{aa} + \cosh^2 a\partial_{\alpha\alpha})\Psi = \lambda\Psi,$$
$$(M^2 + d^2P_1^2)\Psi = \lambda\Psi.$$

From this we see that all the separation equations for the four coordinates systems can be characterised as eigensolutions of a second order operator which can be expressed as a sum of symmetric products in terms of M, P_1 and P_2.

If instead of considering first order operators in the elements M, P_1 and P_2 we consider second order elements of the form $Q = r\{P_1, M\} + s\{P_2, M\} + tP_1P_2 + uP_1^2 + vP_2^2 + wM^2$ and classify all classes of such operators into equivalence classes under the Euclidean group $E(2)$ we obtain four classes. Suitable representatives of these classes are

1. P_1^2,
2. M^2,
3. $\{M, P_2\}$,
4. $M^2 + d^2P_1^2$,

to within multiples of $\Delta_2 + \omega^2 = P_1^2 + P_2^2 + \omega^2$, *i.e.* the original equation (1). The operator Δ_2 is an example of a Casimir operator in that it commutes with all the elements of $LE(2)$:

$$[\Delta_2, P_i] = 0, \quad i = 1, 2, \quad [\Delta_2, M] = 0.$$

We therefore have a very nice relationship between equivalence classes of operators of this type and separable systems of (1).

How can we use group theory to derive properties of the special functions that occur as solutions of (1)? This can be done by introducing the Fourier transform of the solutions $\Psi(x, y)$ of (1). Suppose that $\Psi(x, y)$ is represented by

$$\Psi(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(i(\omega_1 x + \omega_2 y))h(\omega_1, \omega_2)d\omega_1 d\omega_2$$

with Fourier transform $h(\omega_1, \omega_2)$. Proceeding formally this function is a solution of (1) provided that

$$(\Delta + \omega^2)\Psi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\omega^2 - \omega_1^2 - \omega_2^2)\exp(i(\omega_1 x + \omega_2 y))h(\omega_1, \omega_2)d\omega_1 d\omega_2 = 0$$

i.e., $h(\omega_1, \omega_2) = (1/\omega)\delta(\omega - s)h(\varphi)$ where $\delta(z)$ is the Dirac delta function and s and ω are polar coordinates in the ω_1, ω_2 plane. Noting that $d\omega_1 d\omega_2 = s ds d\varphi$, the s integration can be carried out for the representation of Ψ to obtain

$$\Psi(x, y) = \int_{-\pi}^{\pi} \exp(i\omega(x\cos\varphi + y\sin\varphi))h(\varphi)d\varphi = I(h).$$

The group acts on functions of x as already described. How does it act on the functions $h(\varphi)$? This is easily deduced by requiring that

$$T(g)\Psi(x, y) = \int_{-\pi}^{\pi} \exp(i\omega(x\cos\varphi + y\sin\varphi))T(g)h(\varphi)d\varphi.$$

The induced action is then

$$T(g)h(\varphi) = \exp(i\omega(a\cos(\varphi-\theta) + b\sin(\varphi-\theta))h(\varphi-\theta)$$

where we are implicitly assuming that $h(\varphi+2\pi) = h(\varphi)$. We can consider the space of functions $h(\varphi)$ with inner product

$$\langle h_1, h_2\rangle = \int_{-\pi}^{\pi} h_1^*(\varphi)h_2(\varphi)d\varphi$$

and finite norm. This will be a Hilbert space H and the operators $T(g)$ act on this space irreducibly and satisfy the group property $T(g_1g_2) = T(g_1)T(g_2)$ (irreducibility means that there are no proper subspaces on which $E(2)$ acts as a representation). The operators representing the Lie algebra on H are

$$P_1 = i\omega\cos\varphi, \quad P_2 = i\omega\sin\varphi, \quad M = -\frac{d}{d\varphi}.$$

These operators satisfy the same commutation relations as we have seen earlier and the effect of the group acting on the functions $h(\varphi)$ can be obtained via

$$T(g) = \exp(\theta M)\exp(aP_1 + bP_2).$$

In particular we should note that the transformation $\Psi \mapsto h$ is unitary *i.e.* inner product preserving, $(\Psi_1, \Psi_2) = \langle h_1, h_2\rangle$ where

$$(\Psi_1, \Psi_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_1^*(x)\Psi_2(x)dx.$$

What this observation achieves is the following. If we want to describe the space of solutions Ψ of (1) with the inner product in terms of a set of basic eigenfunctions of say M, we can just as well do it in the Hilbert space of functions $h(\varphi)$ which has one variable less and we lose nothing. Because the mapping $\Psi \mapsto h$ is unitary no spectral properties are changed. In short, if we look for a complete set of vectors f_m such that $Mf_m = im\, f_m$, then we need only solve this equation in the Hilbert space H, *i.e.*, solve

$$-\left(\frac{d}{d\varphi}\right) f_m = im\, f_m.$$

The corresponding orthonormal solutions are

$$f_m = (2\pi)^{-\frac{1}{2}}\exp(im\,\varphi), \quad m = 0, \pm1, \pm2, \ldots$$

The corresponding orthonormal functions $\Psi_m(x,y)$ are

$$\begin{aligned}\Psi_m(x,y) &= I(f_m)\\ &= (2\pi)^{-\frac{1}{2}}\int_{-\pi}^{\pi}\exp(i\omega(\cos(\varphi-\theta)))\exp(im\varphi)d\varphi\\ &= i^m(2\pi)^{\frac{1}{2}}J_m(\omega r)\exp(im\varphi)\end{aligned}$$

where $J_m(z)$ is the Bessel function of order m. From what we have just discussed we can then say that $(\Psi_m,\Psi_n)=\delta_{mn}$.

It is basically the mapping $I: h\to\Psi$ which enables one to derive relations amongst these special function from very general principles. Let's look at bases corresponding to the other three separable coordinate systems in Euclidean space.

1. Cartesian coordinates. It is sufficient to look for eigenvectors of $P_2 = i\omega\sin\varphi$. A suitable basis must satisfy $P_2 f_\alpha^{(1)} = i\omega\sin\alpha f_\alpha^{(1)}$, the choice to be made is $f_\alpha^{(1)}=\delta(\varphi-\alpha)$ where α is real. The basis vectors are then labeled by the continuous index α. The corresponding Ψ function is

$$\Psi^{(1)}(x,y)=\exp(i\omega(x\cos\alpha+y\sin\alpha)),\qquad \alpha \text{ real}.$$

2. Polar coordinates. In this case we have seen that the basis vectors $f_m^{(2)}=f_m$ are labeled by the discrete index $m=0,\pm1,\pm2,\ldots$

$$\Psi_m^{(2)}(x,y)=i^m(2\pi)^{(1/2)}J_m(\omega r)\exp(im\varphi).$$

3. In the case of parabolic coordinates the basis vectors have the form

$$f_{\mu+}^{(3)}=\begin{cases}(2\pi)^{(-1/2)}(1+\cos\varphi)^{-i\mu/2-(1/4)}(1-\cos\varphi)^{i\mu/2-1/4}, & 0<\varphi\le\pi\\ 0, & -\pi\le\varphi<0\end{cases}$$

and $f_{\mu-}^{(3)}(\varphi)=f_{\mu+}^{(3)}(-\varphi)$. There is in this case a continuous spectrum labeled by the real number μ and the indices $\pm$. The eigenfunctions satisfy

$$\{M,P_2\}f_{\mu\pm}^{(3)}=2\omega\mu f_{\mu\pm}^{(3)},\qquad \mu \text{ real}$$

and

$$\langle f_{\mu\epsilon}^{(3)},f_{\mu'\epsilon'}^{(3)}\rangle=\delta_{\epsilon\epsilon'}\delta(\mu-\mu'),\qquad \text{where } \epsilon,\epsilon'=\pm.$$

The corresponding Ψ functions are

$$\begin{aligned}\Psi_{\mu+}^{(3)}(\xi,\eta)=\left(\sqrt{2}\cos(i\mu\pi)\right)^{-1}&\left(D_{i\mu-\frac{1}{2}}(\sigma\xi)D_{-i\mu-\frac{1}{2}}(\sigma\eta)\right.\\ &\left.+D_{i\mu-\frac{1}{2}}(-\sigma\xi)D_{-i\mu-\frac{1}{2}}(-\sigma\eta)\right)\end{aligned}$$

where $\sigma = \exp(i\pi/4)(2\omega)^{1/2}$ and $D_\nu(z)$ is a parabolic cylinder function.

4. In the case of elliptical coordinates the corresponding eigenfunctions are

$$f_{nc}^{(4)} = \pi^{-\frac{1}{2}} \mathrm{ce}_n(\varphi, q), \quad n = 0, 1, 2, 3 \ldots$$

$$f_{ns}^{(4)} = \pi^{-\frac{1}{2}} \mathrm{se}_n(\varphi, q), \quad n = 1, 2, 3 \ldots$$

where the ce and se functions are even and odd periodic Mathieu functions and $q = d^2\omega^2/4$. The eigenvalues are transcendental and are labelled by the integer n. If we label this basis by $f_{nt}^{(4)}$ where $t = s, c$ then these functions satisfy the orthonormality relations

$$\left\langle f_{nt}^{(4)}, f_{n't'}^{(4)} \right\rangle = \delta_{tt'}\delta_{nn'} \quad \text{and} \quad (M^2 + d^2P_1^2) f_{nt}^{(4)} = \lambda_{nt} f_{nt}^{(4)}.$$

The corresponding Ψ functions are

$$\Psi_{nc}(x, y) = \mathrm{ce}_n(a, q)\mathrm{ce}_n(\alpha, q),$$

$$\Psi_{nc}(x, y) = \mathrm{se}_n(a, q)\mathrm{se}_n(\alpha, q)$$

to within normalisation constants.

Let us establish the formula for the generating function of a Bessel function. In the one dimensional model we can write $f_\alpha^{(1)} = \sum_{m=-\infty}^{\infty} A_{\alpha m} f_m^{(2)}$ where $A_{\alpha m} = (1/\sqrt{2\pi})e^{-im\,\alpha}$. As the transformation between one and two variable models is unitary we may write $\Psi_\alpha^{(1)} = \sum_{m=-\infty}^{\infty} A_{\alpha m} \Psi_m^{(2)}$ or explicitly

$$e^{i\omega(x \cos\alpha + y \sin\alpha)} = \sum_{m=-\infty}^{\infty} i^m J_m(\omega r) e^{im\,(\varphi - \alpha)}$$

which is the desired formula. Many other identities between classical special functions can be worked out using many of the intrinsic group properties such as group multiplication laws

$$T_{mn}^{(j)}(gh) = \sum_k T_{mk}^{(j)}(g) T_{kn}^{(j)}(h)$$

and the reduction of Tensor products.

In addition to the Helmholtz equation for real variables x and y there is also the possibility of separation of variables when the variables are complex. In this case there are six coordinate systems. They are

(1) Cartesian coordinates x, y, characterised by P_1.
(2) Polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, characterised by M.
(3) Elliptical coordinates $x = a\cosh\alpha\cos\beta$, $y = a\sinh\alpha\sin\beta$, characterised by $M^2 - a^2P_2^2$.

(4) Parabolic coordinates $x = \frac{1}{2}(\xi^2 - \eta^2)$, $y = \xi\eta$, characterised by $\{M, P_2\}$.
(5) Degenerate elliptical coordinates of type one $x = (u^2 + u^2v^2 - v^2)/2uv$, $y = i(u^2 - u^2v^2 + v^2)/2uv$, characterised by $M^2 + (P_1 + iP_2)^2$ with separable solutions the product ot two Bessel functions.
(6) Degenerate elliptical coordinates of type two

$$x = -\frac{1}{4}(w - z)^2 + \frac{1}{2}(w + z),$$
$$y = -i\left(-\frac{1}{4}(w - z)^2 - \frac{1}{2}(w + z)\right),$$

characterised by $\{M, P_1 + iP_2\} + (P_1 - iP_2)^2$ with separable solutions the product of Airy functions.

What is clear from here is that there are more coordinate systems available than in the real case. In fact if we were to consider the Klein-Gordon equation $\Psi_{xx} - \Psi_{yy} + \omega^2\Psi = 0$, there would in fact be ten real forms and correspondingly different separable systems possible.

What have we developed here? The recurring features of these lectures will be as follows.

1. We are given an equation of which three types are typical. The Helmholtz equation, the heat/Schrödinger equation, or Laplace's equation.
2. If we write this equation as

$$Q\Psi(x) = 0 \tag{$*$}$$

we the look for the vector space of operators $L = \sum_{k=1}^{n} a_k \partial_k + a_0$ which are such that if Ψ is a solution of $(*)$ then so is $L\Psi$. The set of such operators forms a Lie algebra *i.e.* it is closed under commutation.
3. The action of any Lie algebra element on the functions Ψ is given by

$$\exp(L\alpha)\Psi = \sum_{k=0}^{\infty} \frac{(L\alpha)^n}{n!}\Psi.$$

4. The candidate special functions are R separable solutions of $(*)$, viz solutions such that $\Psi = R(u_1, \dots, u_n) \prod_{k=1}^{n} \Psi_i(u_i)$. These solutions are characterised as eigenfunctions of a suitable set of operators which are often times expressible in terms of elements which are symmetric products of second order in the enveloping algebra of an associated symmetry group.
5. What is then required is a vector space of solutions of $(*)$. This could be a Hilbert space with an appropriate inner product or perhaps a space of analytic solutions of $(*)$ about a point.

6. What is then needed is a second model of this space of solutions. This generally involves fewer variables. Basis functions arising from separable solutions are usually easier to calculate in this model.
7. The group property is what is used to derive identities amongst different bases. At the Lie algebra level the equivalent commutation relations can result in identities for the solutions. These identities translate to identities for the solutions of (1).

These are the basic steps in studying special functions arising from equations of physics. The important texts on this subject are [6] and [11].

2. The Schrödinger equation, separation of variables and special functions

In quantum mechanics the state of a system is described by the Schrödinger equation in the coordinate representation. In one space dimension the time dependant Schrödinger equation has the form

$$i\hat{h}\Psi_t = (-\hat{h}^2/2m)\Psi_{xx} + V(x)\Psi$$

where $\hat{h} = h/2\pi$ and h is Planck's constant. Among the most important Schrödinger equations are those for which $V(x)$ is one of the potentials

1. $V = 0$.
2. $V = a/x^2 + kx^2$, k positive or negative.
3. $V = ax, a \neq 0$, linear potential.

The problem of relating solutions of the Schrödinger equation to special functions is again via separation of variables and the associated symmetry group. The key difference is that we now consider the concept of R separation. Previously we considered solutions that were a product of functions on individual functions of the separating variables, *i.e.* $\Psi = \Psi(x_1)\Psi(x_2)$. By R separation we mean that there is the possibility that there is a well defined function in front of this product $\Psi = R(x_1, x_2)\Psi(x_1)\Psi(x_2)$. To find out all the possible coordinate systems associated with the free particle Schrödinger equation

$$i\Psi_t + \Psi_{xx} = 0 \tag{2}$$

one has to calculate the symmetry algebra. In the case of the Helmholtz equation it was clear what group mapped solutions into solutions. In the case of the free particle Schrödinger equation this is not so. Consequently we look for all the operators $L = a(t,x)\partial_x + b(t,x)\partial_y + c(t,x)$ such that if Ψ is a solution of (2) then so is $L\Psi$. If we make this requirement then we obtain a six dimensional algebra $L(S_1)$ with basis

$$K_2 = -t^2\partial_t - tx\partial_x - t/2 + ix^2/4, \quad K_1 = -t\partial_x + ix/2$$

$$K_0 = i, \quad K_{-1} = \partial_x, \quad K_{-2} = \partial_t, \quad K^0 = x\partial_x + 2t\partial_t + 1/2.$$

If we choose the basis

$$C_1 = K_{-1}, \quad C_2 = K_1, \quad L_3 = K_{-2} - K_2,$$
$$L_1 = K^0, \quad L_2 = K_{-2} + K_2, \quad E = K_0,$$

the commutation relations are

$$[L_1, L_2] = -2L_3, \quad [L_3, L_1] = 2L_3, \quad [L_2, L_3] = 2L_1,$$
$$[C_1, C_2] = E/2,$$
$$[L_3, C_1] = C_2, \quad [L_3, C_2] = -C_1, \quad [L_2, C_1] = [C_2, L_1] = -C_2,$$
$$[L_1, C_1] = [L_2, C_2] = -C_1$$

This form of the commutation relations gives the essential structure of the algebra.

1. The $L_i, i = 1, 2, 3$ generate the Lie algebra of $SL(2, R)$ also realised as the space of traceless 2×2 matrices *i.e.* matrices $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with

$$L_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad L_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The corresponding group consists of the matrices $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, $\alpha\delta - \beta\gamma = 1$, $\alpha, \beta, \gamma, \delta$ real numbers.

2. The elements C_1, C_2 and E form a basis of the Weyl algebra W_1 also realised as the matrices

$$C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This algebra can be exponentiated via

$$B(u, v, \rho) = \begin{bmatrix} 1 & v & 2\rho + uv/2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix} = \exp\bigl((\rho + uv/4)E\bigr)\exp(uC_2)\exp(vC_2).$$

The law of group multiplication is

$$B(u, v, \rho)B(u', v', \rho,) = B(u + u', v + v', \rho + \rho' + (vu' - uv')/4).$$

The action of these subgroups on functions of x and t is

$$T(u, v, \rho)\Phi(t, x) = \exp(i\rho + i(uv + 2ux - u^2 t)/4)\Phi(t, x + v - ut),$$
$$T(A)\Phi(t, x) = \exp\left[i\frac{x^2\beta}{4(\delta + i\beta)}\right](\delta + t\beta)^{-\frac{1}{2}}\Phi\left(\frac{\gamma + t\alpha}{\delta + t\beta}, \frac{x}{\delta + t\beta}\right).$$

This symmetry group is the semi-direct product of W_1 and $SL(2,R)$, *i.e.* any element is specified by the coordinates $g = (A, w)$ where $w = (u, v, \rho)$ and a group element in a given representation by $T(g) = T(A)T(w)$. Now that we have found our symmetry group we know that it maps solutions of the Schrödinger equation into solutions. In order to find separable solutions of the Schrödinger equation we proceed in analogy with the Helmholtz equation. We can act on the vector space generated by the Lie algebra via the action of the group S_1. This is done by what is called the adjoint action. This is defined by

$$\exp(Ad(L))M = \sum_{k=0}^{\infty} \frac{(Ad(L))^k}{k!} M, \qquad L,\ M \quad \text{in} \quad L(S_1)$$

where $Ad(L)M = [L, M]$. (Exercise: show that this is what we did for the case of the Helmholtz equation). We say that two elements M_1, M_2 are equivalent if there is an L in $L(S_1)$ such that $M_1 = \exp(Ad(L))M_2$. Under this equivalence relation we find there are four equivalence classes of elements of $L(S_1)$. Typical elements of these five orbits have representatives K_{-1}, $K_{-2} - K_1$, K^0 and $K_2 - K_{-2}$. For each of these representatives a separable coordinate system is possible. This can be seen as follows.

1. Choose the Cartesian coordinates $x = u$, $t = v$. The Schrödinger equation is then $i\Psi_v + \Psi_{uu} = 0$. The solutions are then $\Psi = \exp(\lambda u - i\lambda^2 v)$ which satisfy $K_{-1}\Psi = \lambda\Psi$.

2. For the coordinates associated with $K_{-2} - K_1$ we choose coordinates $x = u + v^2/2$, $t = v$. The Schrödinger equation is then

$$\Phi_{uu} + i\Phi_v + (-u/2 + v^2/4)\Phi = 0$$

where $\Psi = \exp(iuv/2)\Phi$. The solutions are then

$$\Psi = \exp(uv/2)\exp(-(iv^3/12) - i\lambda v)\mathrm{Ai}((u/2) - \lambda))$$

where $\mathrm{Ai}(z)$ is an Airy function. In particular, $(K_{-2} - K_1)\Psi = \lambda\Psi$. We should note the if we wrote $\Phi = \exp(-iv^3/12)\Lambda$ and $v = V, u = U$ then Λ satisfies the equation $\Phi_{UU} + i\Phi_V + (-U/2)\Phi = 0$ which is the Schrödinger equation with linear potential $-U/2$.

3. For the coordinates associated with K^0 we choose $x = u\sqrt{v}$, $t = v$. The Schrödinger equation is

$$\Phi_{uu} + iv\Phi_v + (u^2/16 + i/4)\Phi = 0$$

where $\Psi = \exp(iu^2/8)\Phi$. The solutions are then

$$\Psi = \exp(iu^2/8)v^{(-i\lambda - 1/4)}D_{i\lambda - \frac{1}{2}}(-u\sqrt{i/2})$$

where $D_\nu(z)$ is a parabolic cylinder function. The solutions satisfy $K^0\Psi = i\lambda\Psi$, and we note that putting $u = U$, $v = \exp(V)$ that the equation for Φ becomes

$$\Phi_{UU} + i\Phi_V + (U^2/16)\Phi = 0$$

which is the Schrödinger equation with potential $U^2/16$.

4. For the coordinates associated with $K_2 - K_{-2}$ we choose $x = u(1+v^2)^{\frac{1}{2}}$, $t = v$. The Schrödinger equation is

$$\Phi_{uu} + i(v^2+1)\Phi_v + (-u^2/4 + iv/2)\Phi = 0$$

where $\Psi = \exp(iu^2v/4)\Phi$. The solutions are then

$$\Psi = \exp(iu^2v/4 - i(n+(1/2))\tan^{-1}v)(v^2+1)^{-\frac{1}{4}}H_n(u)$$

where $H_p(x)$ is a Hermite polynomial. The solutions satisfy

$$(K_2 - K_{-2})\Psi = i(n+(1/2))\Psi,$$

and we note that putting $u = U$, $v = \tan V$ and $\Phi = (\cos V)^{-\frac{1}{2}}\Lambda$, the equation for Λ becomes $\Lambda_{UU} + i\Lambda_V + (-U^2/4)\Lambda = 0$ which is the Schrödinger equation with potential $-U^2/4$.

From these observations we see that the Schrödinger equations

$$i\Psi_t + \Psi_{xx} + V(x)\Psi = 0$$

with potentials $V(x) = 0,\ ax,\ \pm bx^2$ all have the same symmetry group.

These results extend to the case of Schrödinger equation in n dimensions in the sense that for potentials of the form $V(x) = 0,\ ax_1,\ \pm bx\cdot x$ the corresponding symmetry groups are abstractly identical.

The classical references for the extension to n dimensions are [3], [13].

The next question is: How do we set up a model of the solutions and workout the bases corresponding to these separable solutions? Just as in the case of the Helmholtz equation there is indeed a one variable model available. It is

$$K_2 = ix^2/4,\ K_1 = ix/2,\ K_{-1} = \partial_x,\ K_{-2} = i\partial_{xx},\ K_0 = i,\ K^0 = x\partial_x + \frac{1}{2}.$$

It can be shown that given $f(x)$ then $\Psi(x,t) = \exp(tK_{-2})f(x)$ is a solution of the Schrödinger equation (2). Moreover this relationship is inner product preserving *i.e.* unitary:

$$\int_{-\infty}^{\infty} \Psi_1(t,x)\Psi_2^*(t,x)dx = \langle \exp(tK_{-2})f_1, \exp(tK_{-2})f_2\rangle$$

$$= \langle f_1, f_2\rangle = \int_{-\infty}^{\infty} f_1(x)f_2^*(x)dx.$$

3. Separation of variables and the Helmholtz equation in three dimensions

Just as we have done in the previous lectures we can look for solutions of the Helmholtz equation

$$(\Delta_3 + \omega^2)\Psi = (\Psi_{x_1x_1} + \Psi_{x_2x_2} + \Psi_{x_3x_3} + \omega^2)\Psi = 0 \qquad (3)$$

which are solvable via the separation of variables ansatz $\Psi = \prod_{k=1}^{3} \Psi_k(u_k)$. If we look for the set of partial differential operators $L = \sum_{k=1}^{3} a_k(x_1, x_2, x_3)\partial_{x_k}$ such that if Ψ is a solution of (3) then $L\Psi$ is also, we find that the space of all such operators is six dimensional with a basis

$$P_{x_i} = \partial_{x_i}, \qquad i = 1, 2, 3$$

$$J_3 = x_2\partial_{x_1} - x_1\partial_{x_2}, \quad J_2 = x_1\partial_{x_3} - x_3\partial_{x_1}, \quad J_1 = x_3\partial_{x_2} - x_2\partial_{x_3}.$$

We can easily verify that they form a Lie algebra with commutation relations

$$[J_i, J_j] = \epsilon_{ijk} J_k,$$
$$[J_l, P_m] = \epsilon_{lmn} P_n,$$
$$[P_l, P_m] = 0.$$

This algebra is the three dimensional version $LE(3)$ of the two dimensional Euclidean group which we discussed in the first lecture. It is clear that if we rotate or translate the coordinates of any solution $\Psi(x_1, x_2, x_3)$ of (3) then we again obtain another solution. Indeed we have typically the relations

$$\exp(aP_1)\Psi(x_1, x_2, x_3) = \Psi(x_1 + a, x_2, x_3),$$
$$\exp(\theta J_1)\Psi(x_1, x_2, x_3) = \Psi(x_1, x_2 \cos\theta - x_3 \sin\theta, x_3 \cos\theta + x_2 \sin\theta).$$

In matrix form the elements of $E(3)$ are represented by the 4×4 matrices

$$g(A, a) = \begin{bmatrix} A & 0 \\ a & 1 \end{bmatrix}$$

where A is a rotation matrix and $a = (a_1, a_2, a_3)$ a real three dimensional vector. $E(3)$ acts as a transformation group acting on function $\Psi(x)$ via the formula

$$T(A, a)\Psi(x) = \Psi(xA + a).$$

In order to obtain separable solutions for this equation we need to find two additional operators which describe the separable solutions. This can be done just as we did for the two dimensional case. If we classify pairs of operators which commute and are made up of symmetric products of elements of $LE(3)$ we obtain various equivalence classes with typical representatives. For each of these there corresponds a separable coordinate system and corresponding special function solutions. If the solutions are of the form $\Psi(x) = \Phi(x_1, x_2)\exp(i\omega_1 x_3)$ then the problem reduces to the two dimensional case already studied in the first lecture *i.e.* $\Phi(x_1, x_2)$ is a solution of $(\Delta_2 + \omega^2 - \omega_1^2)\Phi(x_1, x_2) = 0$. We can then choose one of the four coordinate systems associated with the first lecture if we identify $x = x_1,\ y = x_2$. The remaining coordinate systems are given below together with the separable solutions and their Lie algebra characterization.

1. Spherical coordinates $x_1 = r\sin\theta\cos\varphi,\ x_2 = r\sin\theta\sin\varphi,\ x_3 = r\cos\varphi$. If we look for solutions $\Psi = R(r)T(\theta)\Phi(\varphi)$ then the separation equations are

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + (\omega^2 - (\ell(\ell+1)/r^2)R = 0,$$

$$\frac{d^2T}{d\theta^2} + \cot\theta\frac{dT}{d\theta} + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right)T = 0,$$

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi.$$

The solutions are of the form $\Psi = r^{-1/2}C_{\ell+\frac{1}{2}}(\omega r)Y_{\ell m}(\theta,\varphi)$ where $Y_{\ell m}(\theta,\varphi)$ is a spherical harmonic and $C_\nu(z)$ is a Bessel function. These functions satisfy the eigenvalue equations

$$(J\cdot J)\Psi = (J_1^2 + J_2^2 + J_3^2)\Psi = -\ell(\ell+1)\Psi,$$

$$J_3^2\Psi = -m^2\Psi.$$

2. Prolate spheroidal coordinates,

$$x_1 = c\sinh a\sin\alpha\cos\varphi,\ x_2 = c\sinh a\sin\alpha\sin\varphi,\ x_3 = c\cosh a\cos\alpha.$$

If we look for solutions of the form $\Psi = A(a)B(\alpha)C(\varphi)$ then the separation equations are

$$\frac{d^2A}{da^2} + \coth a\frac{dA}{da} + (-\lambda + c^2\omega^2\sinh^2 a - (m^2/\sinh^2 a))A = 0,$$

$$\frac{d^2B}{d\alpha^2} + \cot\alpha\frac{dB}{d\alpha} + (\lambda + c^2\omega^2\sin^2\alpha - (m^2/\sin^2\alpha))B = 0,$$

$$\frac{d^2C}{d\varphi^2} = -m^2C.$$

The typical solutions have the form

$$\Psi = Ps_n^{|m|}(\cosh a, c^2\omega^2)Ps_n^{|m|}(\cos\alpha, c^2\omega^2)e^{im\varphi}$$

where $Ps_n^{|m|}(\cos\beta, c^2\omega^2)$ is a spheroidal wave function. These solutions are eigenfunctions of the operators

$$\begin{aligned}(J \cdot J - a^2(P_1^2 + P_2^2))\Psi &= -\lambda\Psi\\ J_3^2\Psi &= -m^2\Psi.\end{aligned}$$

3. Oblate spheroidal coordinates,

$$x_1 = c\cosh a\cos\alpha\cos\varphi,\ \ x_2 = c\cosh a\cos\alpha\sin\varphi,\ \ x_3 = c\sinh a\sin\alpha.$$

If we look for solutions of the form $\Psi = A(a)B(\alpha)C(\varphi)$ then the separation equations are

$$\begin{aligned}\frac{d^2A}{da^2} + \tanh a\frac{dA}{da} + (-\lambda + c^2\omega^2\cosh^2 a - (m^2/\cosh^2 a))A &= 0,\\ \frac{d^2B}{d\alpha^2} - \tan\alpha\frac{dB}{d\alpha} + (\lambda + c^2\omega^2\cos^2\alpha - (m^2/\cos^2\alpha))B &= 0,\\ \frac{d^2C}{d\varphi^2} &= -m^2C.\end{aligned}$$

The typical solutions have the form

$$\Psi = Ps_n^{|m|}(-i\sinh a, c^2\omega^2)Ps_n^{|m|}(\sin\alpha, -c^2\omega^2)e^{im\varphi}.$$

These solutions are eigenfunctions of the operators

$$\begin{aligned}(J \cdot J + a^2(P_1^2 + P_2^2))\Psi &= -\lambda\Psi,\\ J_3^2\Psi &= -m^2\Psi.\end{aligned}$$

4. Parabolic coordinates $x_1 = \xi\eta\cos\varphi$, $x_2 = \xi\eta\sin\varphi$, $x_3 = (\xi^2 - \eta^2)/2$. If we look for solutions of the form $\Psi = \Omega(\xi)\Lambda(\eta)C(\varphi)$ then the separation equations have the form

$$\begin{aligned}\frac{d^2\Omega}{d\xi^2} + \frac{1}{\xi}\frac{d\Omega}{d\xi} + (\omega^2\xi^2 - m^2/\xi^2 - \lambda)\Omega &= 0,\\ \frac{d^2\Lambda}{d\eta^2} + \frac{1}{\eta}\frac{d\Lambda}{d\eta} + (\omega^2\eta^2 - m^2/\eta^2 + \lambda)\Lambda &= 0,\\ \frac{d^2C}{d\varphi^2} &= -m^2C.\end{aligned}$$

These equations have the solutions

$$\Psi = (\xi\eta)^m\exp(i\omega(\xi^2+\eta^2)/2 + im\varphi)\,{}_1F_1\left(\begin{matrix}i\lambda/4\omega+(m+1)/2\\ m+1\end{matrix}\,\middle|\,i\omega\xi^2\right)\times$$
$${}_1F_1\left(\begin{matrix}i\lambda/4\omega+(m+1)/2\\ m+1\end{matrix}\,\middle|\,i\omega\eta^2\right)$$

These solutions are eigenfunctions of the operators

$$(\{J_1, P_2\} - \{J_2, P_1\})\Psi = \lambda\Psi,$$
$$J_3^2\Psi = -m^2\Psi.$$

5. Paraboloidal coordinates,

$$x_1 = 2c\cosh a\cos\alpha\sinh\gamma,$$
$$x_2 = 2c\sinh\alpha\sin\beta\cosh\gamma,$$
$$x_3 = c(\cosh 2\alpha + \cos 2\beta - \cosh 2\gamma)/2.$$

If we look for solutions of the form $\Psi = A(\alpha)B(\beta)C(\gamma)$ then the separation equations are

$$\frac{d^2A}{d\alpha^2}(-q - \lambda c\cosh 2\alpha + (\omega^2c^2/2)\cosh 4\alpha)A = 0,$$
$$\frac{d^2B}{d\beta^2}(q + \lambda c\cos 2\beta - (\omega^2c^2/2)\cos 4\beta)B = 0,$$
$$\frac{d^2C}{d\gamma^2}(-q + \lambda c\cosh 2\gamma + (\omega^2c^2/2)\cosh 4\gamma)C = 0$$

where $q = \mu - c^2\omega^2/2$. These equations have the solutions

$$\Psi = gc_n(i\alpha; 2c\omega, \lambda/2\omega)gc_n(\beta; 2c\omega, \lambda/2\omega)gc_n(i\gamma + \pi/2; 2c\omega, \lambda/2\omega)$$

for $n = 0, 1, 2, \ldots$ and $\mu = \mu_n$. The functions $gc_n(z, a, b)$ are odd periodic solutions of the Whittaker-Hill equation. The c can be replaced by s to give odd solutions as well. The eigenvalue equations satisfied by these solutions are

$$(J_3^2 - c^2P_3^2 + c\{J_2, P_1\} + c\{J_1, P_2\})\Psi = -\mu\Psi,$$
$$(cP_2^2 - cP_1^2 + \{J_2, P_1\} - \{J_1, P_2\})\Psi = \lambda\Psi.$$

6. Elliptical coordinates,

$$x_1^2 = (\mu - a)(\nu - a)(\rho - a)/a(a-1),$$
$$x_2^2 = (\mu - 1)(\nu - 1)(\rho - 1)/(1 - a),$$
$$x_3^2 = \mu\nu\rho/a.$$

If we look for solutions of the form $\Psi = A(\mu)B(\nu)C(\rho)$ then the separation equations are of the form

$$4\kappa(\kappa - 1)(\kappa - a)\frac{d^2L}{d\kappa^2} + \left[\frac{2}{\kappa} + \frac{2}{\kappa - 1} + \frac{2}{\kappa - a}\right]\frac{dL}{d\kappa} + (\lambda_1\kappa + \lambda_2 + \omega^2k^2)L = 0$$

where $L = A, B, C$ and $\kappa = \mu, \nu, \rho$ respectively. The solutions satisfy the eigenvalue equations

$$(J \cdot J + P_1^2 + aP_2^2 + (a+1)P_3^2)\Psi = \lambda_1 \Psi,$$
$$(J_2^2 + aJ_1^2 + aP_3^2)\Psi = \lambda_2 \Psi.$$

The solutions to these equations are called ellipsoidal wave functions and have been discussed to some extent in the literature.

7. Conical coordinates,

$$x_1^2 = r^2[(b\mu - 1)(b\nu - 1)/(1-b)],$$
$$x_2^2 = r^2[b(\mu - 1)(\nu - 1)/(b-1)],$$
$$x_3^2 = r^2[b\mu\nu].$$

It is convenient to rewrite these coordinates in elliptic function form, viz

$$x_1 = r(1/k')\mathrm{dn}(\alpha, k)\mathrm{dn}(\beta, k),$$
$$x_2 = ir(k/k')\mathrm{cn}(\alpha, k)\mathrm{cn}(\beta, k),$$
$$x_3 = rk\,\mathrm{sn}(\alpha, k)\mathrm{sn}(\beta, k)$$

where we have introduced Jacobian elliptic functions sn, dn, cn and the variables are in the ranges $0 \le r$, $-2K < \alpha < 2K$, $K < \beta < K + 2iK'$, $k = b^{\frac{1}{2}}$. If we now look for solutions in the form $\Psi = R(r)A(\alpha)B(\beta)$ then the separation equations satisfied by this ansatz are

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + (\omega^2 - (\ell(\ell+1)/r^2)R(r) = 0,$$
$$\frac{d^2L}{d\kappa^2} + (\lambda - \ell(\ell+1)\sin^2\kappa)L = 0$$

where $L = A, B$ and $\kappa = \alpha, \beta$ respectively. These solutions satisfy the eigenvalue equations

$$(J \cdot J)\Psi = -\ell(\ell+1)\Psi,$$
$$(J_1^2 + bJ_2^2)\Psi = \lambda\Psi.$$

The equation for R is identical with that for spherical coordinates and the solutions for L are Lame polynomials.

Just as in the case of two dimensions we have that the classification of pairs of commuting operators is exhaustive, Also there is an inner product that we can put on the space of solutions of (3), viz

$$(\Psi_1, \Psi_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_1^*(x)\Psi_2(x)dx.$$

In direct analogy we can move to a simpler model by taking the Fourier transform

$$\Psi = \int\int_{S_2} \exp(i\omega x \cdot k)h(k)dk = I(h)$$

where $k \cdot k = 1$. This relationship between h and Ψ is inner product preserving, *i.e.* $(\Psi_1, \Psi_2) = \langle h_1, h_2 \rangle = \int\int_{S_2} h_1^* h_2 dk$.

It is then possible to derive the correct spectrum from this simpler model on the sphere just as we have done in the case of two dimensions and this is often easier to do as there is one variable less. In fact acting on the functions $h(k)$ the basis elements of the Lie algebra have the form

$$P_1 = i\omega \sin\theta\cos\varphi, \quad P_2 = i\omega\sin\theta\sin\varphi, \quad P_3 = i\omega\cos\theta,$$
$$J_1 = k_3\partial_{k_2} - k_2\partial_{k_3} = \sin\varphi\partial_\theta + \cos\varphi\cot\theta\partial_\varphi,$$
$$J_2 = k_1\partial_{k_3} - k_3\partial_{k_1} = -\cos\varphi\partial_\theta + \sin\varphi\cot\theta\partial_\varphi,$$
$$J_2 = k_2\partial_{k_1} - k_1\partial_{k_2} = -\partial_\varphi$$

where $k = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$.

A classical and important paper on separation of variables for the Helmholtz equation in three dimensions is [5].

4. Separation of variables and the Laplace equation in three dimensions

Just as we have done in the previous lectures we look for solutions of Laplace's equation

$$\Delta_3\Psi = \Psi_{x_1x_1} + \Psi_{x_2x_2} + \Psi_{x_3x_3} = 0 \qquad (4)$$

which are solvable via the separation of variables ansatz

$$\Psi = R(u_1, u_2, u_3)\prod_{k=1}^{3}\Psi_k(u_k).$$

Classically this has been done by Maxime Bocher in his classical work on Potential theory [2]. The way to understand his work via group theory is by the methods we have already discussed for the Helmholtz and Schrödinger equations. If we look for the set of partial differential operators

$$L = \sum_{k=1}^{3} a_k(x_1, x_2, x_3)\partial_{x_k} + a_0(x_1, x_2, x_3)$$

such that if Ψ is a solution of (3) then $L\Psi$ is also we find that the space of all such operators is ten dimensional with a basis

$$P_{x_i} = \partial_{x_i}, \qquad i = 1,2,3,$$
$$J_3 = x_2\partial_{x_1} - x_1\partial_{x_2}, \quad J_2 = x_1\partial_{x_3} - x_3\partial_{x_1}, \quad J_1 = x_3\partial_{x_2} - x_2\partial_{x_3},$$
$$D = -\frac{1}{2} - (x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}),$$
$$K_i = x_i + (x_i^2 - x_j^2 - x_k^2)\partial_{x_i} + 2x_ix_j\partial_{x_j} + 2x_ix_k\partial_{x_k}, \quad i \neq j,\ j \neq k,\ i \neq k.$$

The elements P_{x_i}, J_k generate the algebra $LE(3)$. The remaining elements are genuinely new. The action of these elements in the large on functions $\Psi(x)$. are

$$\exp(\lambda D)\Psi(x) = \exp(-\lambda/2)\Psi(\exp(-\lambda)x),$$
$$\exp(a_1K_1 + a_2K_2 + a_3K_3)\Psi(x) = (1 - 2x\cdot a + (a\cdot a)(x\cdot x))^{-\frac{1}{2}},$$
$$\Psi((x - a(x\cdot x))/(1 - 2x\cdot a + (a\cdot a)(x\cdot x)).$$

The K_1 are referred to as special conformal transformations and the D is a dilation operator which corresponds to the obvious symmetry of the laplace equation obtained by replacing $x \to [\exp(-\lambda)]x$. There are in fact other important discrete transformations associated with the Laplace equation. These are the transformation of inversion in the sphere specified by

$$I\Psi(x) = (x\cdot x)^{-\frac{1}{2}}\Psi(x/(x\cdot x))$$

and of course reflections $R\Psi(x_1,x_2,x_3) = \Psi(-x_1,x_2,x_3)$. In fact the relations between the K_i and P_i are specifically $IP_i = -K_iI$.

In fact the geometry of this fifteen dimensional algebra of (4) can be easily understood in terms of the so called pentaspherical coordinates. These can be obtained using the projective coordinates (x,t)

$$y_1 = i(z\cdot z + t^2), \quad y_5 = z\cdot z - t^2, \quad y_i = 2z_{i-1}t, \quad i = 2,3,4$$

where the Cartesian coordinates are given by $x_i = z_i/t$. In the five dimensional space of the coordinates y_i we can get a clear interpretation of the elements of our Lie algebra. In fact the following formulas can be verified

$$J_3 = \Gamma_{32}, \quad J_2 = \Gamma_{24}, \quad J_1 = \Gamma_{43}, \quad D = \Gamma_{15},$$
$$P_1 = \Gamma_{12} + \Gamma_{25}, \quad P_2 = \Gamma_{13} + \Gamma_{35}, \quad P_3 = \Gamma_{14} + \Gamma_{45},$$
$$K_1 = \Gamma_{12} - \Gamma_{25}, \quad K_2 = \Gamma_{13} - \Gamma_{35}, \quad K_3 = \Gamma_{14} - \Gamma_{45},$$

where $\Gamma_{ij} = y_i\partial_{y_j} - y_j\partial_{y_i}$. From these relations we can see that the Lie algebra is the same as that we would get by considering all the linear transformations which preserve the quadratic form $\sum_{k=1}^{5} y_k^2 = 0$, which for the choice of coordinates given above is in fact zero. This algebra is called the conformal group and can be identified via the notation $LSO(1,4)$, the Lie algebra of pseudoorthogonal 5×5 matrices M that preserve the diagonal matrix $D = \mathrm{diag}(1,-1,-1,-1,-1)$, *i.e.* $D = MDM^t$. The geometry of the Laplace equation is closely related to this observation. To obtain separable coordinates for this equation and the corresponding special functions, quadratic surfaces need to be considered on this five dimensional space. These are curves specified by

$$\sum_{k=1}^{5} \frac{y_k^2}{\lambda - e_k} = 0, \qquad \sum_{k=1}^{5} y_k^2 = 0,$$

$\lambda = \lambda_1, \lambda_2, \lambda_3$. For these coordinates the pentaspherical coordinates are given by

$$y_k^2 = \frac{\prod_{i=1}^{3}(\lambda_i - e_k)}{\prod_{\ell\neq k}(e_\ell - e_k)}, \qquad k = 1, \ldots 5.$$

The confocal curves introduced in this way are called confocal cyclides. They form the basis of all coordinate systems for the Laplace equation (4). We can again discuss the notion of equivalence under the adjoint action of the conformal group algebra and find that pairs of operators S_1, S_2 which are symmetric and quadratic in the elements of $LSO(1,4)$ and which commute can be divided into equivalence classes that correspond exactly to separable coordinates. Let's look in some detail at the most general coordinates of cyclidic type. We can without loss of generality subject the coordinates λ_i and numbers e_ℓ to arbitrary transformations

$$\lambda \mapsto \frac{a\lambda + b}{c\lambda + d}, \quad e_k \mapsto \frac{ae_k + b}{ce_k + d}$$

and can therefore choose $e_4 = \infty$ and $e_0 = 0, e_1 = 1, e_2 = a, e_3 = b$. A suitable choice of coordinates is then

$$x_1^2 = R^{-2}[(\lambda_1 - a)(\lambda_2 - a)(\lambda_3 - a)/(b-a)(a-1)a],$$
$$x_2^2 = R^{-2}[(\lambda_1 - b)(\lambda_2 - b)(\lambda_3 - b)/(a-b)(b-1)b],$$
$$x_3^2 = R^{-2}[(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1)/(a-1)(b-1)],$$

where $R = 1 + [\lambda_1\lambda_2\lambda_3/ab]^{\frac{1}{2}}$. Separable solutions to (4) can be obtained by looking for solutions of the form $\Psi = R^{\frac{1}{2}}\prod_{j=1}^{3}\Lambda_j(\lambda_j)$. The functions $\Lambda_k(\lambda_k)$ satisfy the ordinary differential equation

$$\sqrt{P(\lambda)}\frac{d}{d\lambda}\Big(\sqrt{P(\lambda)}\frac{d}{d\lambda}\Lambda_k\Big) - \Big(\frac{3}{16}\lambda^2 + c_1\lambda + c_2\Big)\Lambda_k = 0$$

where $P(\lambda) = \lambda(\lambda-1)(\lambda-a)(\lambda-b)$, $\lambda = \lambda_1, \lambda_2, \lambda_3$. This is an equation similar to the one we have found for ellipsoidal coordinates in three dimensional Euclidean space. It is a two parameter spectral problem with parameters c_1, c_2 needing to be determined. In fact these solutions satisfy the eigenvalue equations

$$\left[\frac{a+1}{4}(P_2+K_2)^2 + \frac{b+1}{4}(P_1+K_1)^2 + \frac{a+b}{4}(P_3+K_3)^2 + J_3^2 + bJ_2^2 + aJ_1^2\right]\Psi = c_1\Psi,$$

$$\left[\frac{a}{4}(P_2+K_2)^2 + \frac{b}{4}(P_1+K_1)^2 + \frac{ab}{4}(P_3+K_3)^2\right]\Psi = c_2\Psi.$$

There is no nice Hilbert space structure on the space of solutions of (4). However we can derive identities relating separable solutions of this equation. This is done using what is known as Weisner's method. For this let us consider the expression

$$\Psi(x_1,x_2,x_3) = \int_{C_1} d\beta \int_{C_2} (dt/t) h(\beta,t) \exp[(ix\beta/2)(t+t^{-1}) + (y\beta/2)(t-t^{-1}) - \beta z] = I(h)$$

where h is analytic on the domain of integration. For any h, $\Psi = I(h)$ is a solution of Laplace's equation. By integrating by parts we can find the elements of our symmetry algebra that act on the functions $h(t,\beta)$. They are

$$P^+ = -\beta t, \quad P^- = -\beta/t, \quad P^0 = -i\beta, \quad D = \beta\partial_\beta + \frac{1}{2},$$
$$J^+ = it\partial_\beta - it^2\partial_t, \quad J^- = -i(\beta/t)\partial_t - i\partial_t, \quad J^0 = t\partial_t,$$
$$K^+ = (t/\beta)(\beta\partial_\beta - t\partial_t)(\beta\partial_\beta - t\partial_t - 1),$$
$$K^- = (1/t\beta)(\beta\partial_\beta + t\partial_t)(\beta\partial_\beta + t\partial_t - 1),$$
$$K^0 = (i/\beta)((t\partial_t)^2 - (\beta\partial_\beta)^2)$$

where $-J^\pm = \pm J_2 + iJ_1$, $J^0 = iJ_3$ with similar expressions for $P^\pm$, and $K^\pm$. This representation is obtained from the requirement that $L\Psi = I(Lh)$ for L in $LSO(4,1)$. If we introduce the functions $g_m^{(\ell)} = i^\ell \beta^\ell t^m$ it follows that on this basis

$$J^\pm g_m^{(\ell)} = (-\ell \pm m) g_{m\pm1}^{(\ell)},$$
$$J^0 g_m^{(\ell)} = m g_m^{(\ell)},$$
$$P^0 g_m^{(\ell)} = -g_m^{(\ell+1)},$$
$$-P^\pm g_m^{(\ell)} = \pm g_{m\pm1}^{(\ell+1)},$$

$$Dg_m^{(\ell)} = (\ell + (1/2))g_m^{(\ell)},$$
$$K^0 g_m^{(\ell)} = (\ell^2 - m^2)g_m^{(\ell-1)},$$
$$K^+ g_m^{(\ell)} = -(\ell - m)(\ell - m - 1)g_{m+1}^{(\ell-1)},$$
$$K^- g_m^{(\ell)} = (\ell + m)(\ell + m - 1)g_{m-1}^{(\ell-1)}.$$

If ℓ_0 is a complex number such that $\ell_0 + \frac{1}{2}$ is not an integer then if $\ell = \ell_0,\ \ell_0 \pm 1,\ \ell_0 \pm 2, \dots$ and $m = \ell,\ \ell - 1,\ \ell - 2, \dots$ then all these vectors are invariant under the action of the conformal group. This basis can be used to construct identities for Gegenbauer polynomials. If we introduce coordinates r, t, w which are spherical coordinates *i.e.*,

$$w = \frac{x_3}{r}, \quad t = \frac{x_1 + ix_2}{r}, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

and consider functions $\Psi_m^{(\ell)}(w, t, r)$ which satisfy the recurrence relations. We determine that $P \cdot P\, g_m^{(\ell)} = 0$. In this model one can establish the relations that

$$J^0 \Psi_\ell^{(\ell)} = \ell \Psi_\ell^{(\ell)}, \quad D\Psi_\ell^{(\ell)} = (\ell + \frac{1}{2})\Psi_\ell^{(\ell)}, \quad K^0 \Psi_\ell^{(\ell)} = 0$$

which implies that $\Psi_\ell^{(\ell)} = \Gamma(\ell + (1/2))(2t)^\ell (r/i)^{-\ell-1}$ to within a constant multiple. We can also deduce from the recurrence formulas that

$$\exp(-i\alpha P^0)\Psi_m^{(\ell)} = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \Psi_m^{(\ell+n)}$$

and we also know that

$$\exp(-i\alpha P^0)\Psi_m^{(\ell)}(w, t, r) =$$
$$\Psi_m^{(\ell)}\left((w + \tfrac{\alpha}{r})(1 + \tfrac{\alpha^2}{r^2} + \tfrac{2\alpha w}{r})^{-\frac{1}{2}}, t(1 + \tfrac{\alpha^2}{r^2} + \tfrac{2\alpha w}{r})^{-\frac{1}{2}}, r(1 + \tfrac{\alpha^2}{r^2} + \tfrac{2\alpha w}{r})^{-\frac{1}{2}}\right).$$

If these two expressions are to be the same then

$$\Psi_m^{(\ell)}(w, t, r) = (\ell - m)!\Gamma\left(m + \frac{1}{2}\right) C_{\ell-m}^{m+(1/2)}(w)(2t)^m \left(\frac{r}{i}\right)^{-\ell-1}$$

where $C_m^\nu(z)$ is a Gegenbauer polynomial. This can be deduced by putting $m = \ell$ in the above identity and recalling the formula for the generating function of Gegenbauer polynomials, viz

$$(1 - 2\alpha x + \alpha^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^\nu(x)\alpha^n.$$

The general identity for Gegenbauer polynomials deduced from these ideas is

$$\left(1 - 2w + \alpha^2\right)^{-\nu-k/2} C_k^\nu\left[(w - \alpha)(1 - 2w\alpha + \alpha^2)^{-\frac{1}{2}}\right] = \sum_{n=0}^{\infty} \alpha^n \binom{k+n}{n} C_{n+k}^\nu(w),$$

if $|\alpha^2 - 2\alpha w| < 1$.

5. Special functions and partial differential equations in n dimensions

Thus far we have looked at examples of the Helmholtz equation, heat equation and Laplace equation in lower dimensions and seen the many of the well known classical special functions can be characterised in Lie algebraic terms. This has notable included special functions such as for example spheroidal wave functions which are not of hypergeometric type. In this lecture we will see that these statements can be extended to n dimensional problems. In fact in the case of the n dimensional sphere all the separable equations for the corresponding Helmholtz equation are known. This equation is

$$\Delta\Psi + \sigma(\sigma+n-1)\Psi = 0 \tag{5}$$

where $\Delta = \sum_{i<j}(s_i\partial_{s_j} - s_j\partial_{s_i})^2 = \sum_{i<j} I_{ij}^2$ and $\sum_{k=1}^{n+1} s_k^2 = 1$. The basic building blocks of separable coordinates are the elliptical coordinates on the p dimensional sphere. These are given by

$$_p s_j^2 = \frac{\prod_{i=1}^{p}(u^i - e_j)}{\prod_{i\neq j}(e_i - e_j)},$$

$$j = 1,\ldots p+1,\ e_k \neq e_\ell \quad \text{if} \quad k \neq \ell;\quad e_1 < u^1 < e_2 < \ldots. < u^p < e_{p+1}$$

For these coordinates we adopt the notation $[e_1|....|e_{p+1}]$. If $p = n$ then (5) assumes the form

$$\sum_{k=1}^{n} \frac{1}{\prod_{j\neq k}(u^j - u^k)}(\sqrt{P_k(u^k)}\partial_{u^k}(\sqrt{P_k(u^k)}\partial_{u^k}))\Psi + \sigma(\sigma+n-1)\Psi = 0$$

where $P_k(u_k) = \prod_{j=1}^{n}(u^k - e_j)$. If we look for solutions of the form $\Psi = \prod_{j=1}^{n}\Psi_j(u^j)$ then the separation equations are then

$$(\sqrt{P_k(u^k)}\partial_{u^k}(\sqrt{P_k(u^k)}\partial_{u^k}))\Psi_k + \left[\sigma(\sigma+n-1)(u^k)^n + \sum_{j=2}^{n-1}\mu_j(u^j)^{n-j}\right]\Psi_k = 0.$$

The operators that describe the separable solutions are

$$I_k^n = \sum_{i<j} S_{k-1}^{ij} I_{ij}^2, \quad k = 1, ..., n.$$

where $S_\ell^{ij} = \frac{1}{\ell!}\sum_{i_1,\ldots,i_\ell\neq} e_{i_1}\ldots e_{i_\ell}$. To see how this all works let's go to the example of the two dimensional sphere $n = 2$. One of the critical questions is how can one solve the separation equations. We have been used to equations for which two term (or perhaps three term) recurrence relations are used in the series solution. In the case of ellipsoidal coordinates the best way to solve for the appropriate harmonic solutions is as follows. If we look for solutions of (4) of the form

$$\Psi = \left(\prod_{j=1}^{k} s_{\alpha_k}\right)\prod_{i=1}^{M}\sum_{p=1}^{3}\frac{s_p^2}{\theta_i - e_p} = \left(\prod_{j=1}^{k} s_{\alpha_k}\right)\frac{\prod_{i=1}^{M}\prod_{p=1}^{3}(u^p - \theta_i)}{\prod_{p=1}^{3}(\theta_i - e_p)}$$

where the α_k are all different and $k \leq 3$. The zeros of the elliptic harmonics then satisfy

$$\frac{1}{4}\sum_{j=1}^{3}\frac{1}{\theta_i - e_j} + \frac{1}{2}\sum_{j=1}^{3}\frac{1}{\theta_i - e_{\alpha_j}} + \frac{1}{4}\sum_{j\neq i}\frac{1}{\theta_i - \theta_j} = 0.$$

This achieves the solution of elliptic harmonics in three dimensions. It is relatively easy to see how the generalisation works. In the notation we have been using this corresponds to the diagram

$$[\ e_1 \ |\ e_2 \ |\ e_3\ ].$$

There is one other separable coordinate system separable on the two dimensional sphere, viz spherical coordinates. These coordinates can be constructed from two copies of elliptic coordinates on the one dimensional sphere. These coordinates can be given by

$$_2s_1^2 = \frac{u^1 - e_1}{e_2 - e_1},$$
$$_2s_2^2 = \frac{u^1 - e_2}{e_1 - e_2}$$

where $e_1 < u^1 < e_2$. Clearly if we choose $e_2 = 0$, $e_1 = 1$ and $u^1 = \sin^2\beta$ then $_2s_1 = \cos\beta$, $_2s_2 = \sin\beta$. We of course know that spherical coordinates can be written as

$$_2s_1 = \sin\alpha\cos\beta, \quad _2s_2 = \sin\alpha\sin\beta, \quad _2s_3 = \cos\alpha,$$

this coordinate system is denoted by the graph

$$\begin{array}{c}[\ 0 \ |\ 1\ ]\\ \downarrow \\ [\ 0\ |\ 1\ ].\end{array}$$

This graphical calculus sums the situation up. On the three dimensional sphere the corresponding diagrams of the six different possible coordinate systems are

1. Elliptical coordinates, $[\ e_1 \mid e_2 \mid e_3 \mid e_4\ ]$,

$$s_j^2 = \frac{\prod_{i=1}^3 u^i - e_j}{\prod_{i \neq j} e_i - e_j}, \qquad j = 1, \ldots, 4, \quad e_k \neq e_\ell \ \text{ if } \ k \neq \ell,$$

$$e_1 < u^1 < e_2 < u^2 < e_3 < u^3 < e_4.$$

2. (a) Lame rotational coordinates of type one,

$$s_1^2 =_3 s_1^2, \quad s_2^2 =_2 s_3^2 \cos^2 \varphi, \quad s_3^2 =_2 s_3^2 \sin^2 \varphi, \quad s_4^2 =_3 s_3^2.$$

$$\begin{array}{c} [\ e_1 \mid e_2 \mid e_3\ ] \\ \downarrow \\ [\ 0 \mid 1\ ]. \end{array}$$

(b) Lame rotational coordinates of type two,

$$s_1^2 =_3 s_1^2, \quad s_2^2 =_3 s_2^2, \quad s_3^2 =_3 s_3^2 \cos^2 \varphi, \quad s_4^2 =_3 s_3^2 \sin^2 \varphi.$$

$$\begin{array}{c} [\ e_1 \mid e_2 \mid e_3\ ] \\ \downarrow \\ [\ 0 \mid 1\ ] \end{array}$$

3. Lame subgroup reduction,

$$s_1^2 = \cos^2 \varphi, \quad s_2^2 = \sin^2 \varphi_3 s_1^2, \quad s_3^2 = \sin^2 \varphi_3 s_2^2, \quad s_4^2 = \sin^2 \varphi_3 s_3^2.$$

$$\begin{array}{c} [\ 0 \mid 1\ ] \\ \downarrow \\ [\ 0 \mid 1 \mid a\ ] \end{array}$$

4. Spherical coordinates,

$$s_1 = \sin \alpha \sin \theta \cos \varphi, \ s_2 = \sin \alpha \sin \theta \sin \varphi, \ s_3 = \sin \alpha \cos \theta, \ s_4 = \cos \alpha.$$

$$\begin{array}{c} [\ 0 \mid 1\ ] \\ \downarrow \\ [\ 0 \mid 1\ ] \\ \downarrow \\ [\ 0 \mid 1\ ] \end{array}$$

5. Cylindrical coordinates,

$$s_1 = \sin \alpha \cos \varphi, \quad s_2 = \sin \alpha \sin \varphi, \quad s_3 = \cos \alpha \cos \psi, \quad s_4 = \cos \alpha \sin \psi.$$

$$\begin{array}{cc} \multicolumn{2}{c}{[\ 0 \mid 1\ ]} \\ \downarrow & \downarrow \\ [\ 0 \mid 1\ ] & [\ 0 \mid 1\ ] \end{array}$$

This graphical calculus can be extended to the n sphere. It may also be extended to n dimensional Euclidean space. The basic building blocks are elliptic and parabolic coordinates. In three dimensions these correspond to

1. Elliptic coordinates

$$x_j^2 = c^2 \frac{\prod_{k=1}^{3} u^k - e_j}{\prod_{k\neq i} e_i - e_j}, \quad j = 1, 2, 3, \quad u^1 < e_1 < u^2 < e_2 < u^3 < e_3.$$

These coordinates are denoted by the symbol $\langle\ e_1 \mid e_2 \mid e_3\ \rangle$.

2. Parabolic coordinates.

$$x_1 = \frac{c}{2}(u^1 + u^2 + u^3 + e_1 + e_2),$$

$$x_j^2 = -c^2 \frac{\prod_{k=1}^{3}(u^k - e_j)}{\prod_{k\neq i}(e_i - e_j)}, \quad j = 2, 3, \quad u^1 < e_1 < u^2 < e_2 < u^3.$$

These coordinates are denoted by the symbol $(\ e_1 \mid e_2 \mid e_3\ )$.

For three dimensional Euclidean space the graphs which represent the various coordinate systems are given below.

1. Cartesian coordinates,

$$\langle\ 1\ \rangle, \langle\ 1\ \rangle, \langle\ 1\ \rangle.$$

2. Cylindrical coordinates,

$$\begin{array}{cc} \langle\ 1\ \rangle, & \langle\ 1\ \rangle \\ \downarrow & \\ [\ 0 \mid 1\ ]. & \end{array}$$

3. Elliptical cylindrical coordinates,

$$\langle\ e_1 \mid e_2\ \rangle, \langle\ 1\ \rangle.$$

4. Parabolic cylindrical,

$$(\ 0\ ), \langle\ 1\ \rangle.$$

5. Spherical coordinates,

$$\begin{array}{cc} \langle\ 1\ \rangle & \\ \downarrow & \\ [\ 0 \mid 1\ ] & \\ & \downarrow \\ & [\ 0 \mid 1\ ]. \end{array}$$

6. Prolate spheroidal coordinates,

$$\begin{array}{c} \langle\ 0 \mid 1\ \rangle \\ \downarrow \\ [\ 0 \mid 1\ ]. \end{array}$$

7. Oblate spheroidal coordinates,

$$\begin{array}{c} \langle\ 0 \mid 1\ \rangle \\ \qquad\downarrow \\ \qquad[\ 0 \mid 1\ ]. \end{array}$$

8. Parabolic coordinates,

$$\begin{array}{c} (\ 1\ ) \\ \downarrow \\ [\ 0 \mid 1\ ] \end{array}$$

9. Paraboloidal coordinates,

$$(\ e_1 \mid e_2\ ).$$

10. Ellipsoidal coordinates,

$$\langle\ e_1 \mid e_2 \mid e_3\ \rangle.$$

11. Conical coordinates,

$$\begin{array}{c} \langle\ 1\ \rangle \\ \downarrow \\ [\ 0 \mid 1 \mid a\ ] \end{array}$$

The other feature to note about these coordinates is that the graphs of the systems do not need to be connected. This is however a requirement for coordinates on the sphere. It is possible to extend these ideas for the case of separable coordinates on the hyperboloid and for Laplace's equation and the heat equation. The results obtained are complete for real manifolds which are positive definite spaces. There is however the problem of the complexification of these equations. In this case special functions can occur in a number of ways. I will discuss these in the last lecture.

6. Open problems and prospects for this subject

What are the problems that remain in any such study of the connection between special functions and partial differential equations?

1. There is the question of what are all the separable coordinates for which separation of variables can occur and give rise to "special functions". These are indeed known for the Helmholtz equation on the sphere and in Euclidean space, Laplace's equation and also the heat equation. If we can solve this problem then all real cases can be established by considering appropriate real forms. There are various possible courses of solution of this problem.

First, it is known that on a Riemmannian space the corresponding Helmholtz equation is

$$\Delta\Psi = \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \partial_{x_i}(g^{ij}\sqrt{g}\partial_{x_j})\Psi = \lambda\Psi.$$

In order that separable solutions be possible the matrix G with elements g^{ij} must have the form

$$(g^{ij})_{n\times n} = \begin{bmatrix} \delta^{ab}H_a^{-2} & 0 & 0 \\ 0 & 0 & g^{r\alpha} \\ 0 & g^{r\alpha} & g^{\alpha\beta} \end{bmatrix}$$

where

$$H_a^{-2} = S^{a1}/S,$$
$$g^{r\alpha} = \sum_{\beta} A^{r\beta}(x_r)(S^{r1}/S),$$
$$g^{\alpha\beta} = \sum_{b} B^{\alpha\beta}(x_b)(S^{b1}/S),$$
$$1 \le a \le n_1, \quad n_1+1 \le r \le n_1+n_2, \quad 1 \le b \le n_1+n_2,$$
$$\alpha,\beta = n_1+n_2+1, .., n_1+n_2+n_3.$$

There exists a so called Stackel matrix $\hat{S} = (S_{ij}(x_i))_{(n_1+n_2)\times(n_1+n_2)}$ such that S^{b1} is the b1 cofactor of $\hat{S}$ and S is the determinant. The reference for this crucial form is [1].

There is another condition for the separation of the Helmholtz equation also. In keeping with our theme of partial differential equations, Lie groups and special functions we do not pursue this much further. From this condition we could make the requirement that the space be of constant curvature, *i.e.* $R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$ where if $K = 0$ we have Euclidean space and when $K \ne 0$ then an n dimensional sphere. While this works for some of the possible coordinates it is clearly unfeasible in general (*e.g.* it is quite straightforward in the case of orthogonal coordinates). An outstanding problem is to solve in some tangible form the general problem of finding all the coordinate systems which are separable for complex spaces of constant curvature.

The second approach to this problem may be the following. It is known that each of the separable coordinates give rise to operators of which the separable solutions are eigenfunctions. These operators are mutually commuting. Could these families of commuting operators be characterised in some way when they give rise to separable coordinate systems? There has not been a great deal of progress on this problem. We also note that for the general form of the metric g^{ij} given above the solutions Ψ satisfy

$$L_\alpha \Psi = \partial_{x_\alpha} \Psi = m_\alpha \Psi$$

for all coordinates of type x_α. This means that the L_α form an abelian algebra *i.e.* $[L_\alpha, L_\beta] = 0$ for all α, β. Can one systematically classify the abelian algebras of this type? They must be subalgebras of $E(n, \mathbf{C})$ and $SO(n, \mathbf{C})$. Some progress in this direction has been made but a crucial resolution of this problem has still to be made. In addition to the abelian algebra there is also the requirement that there be a suitable number of second order operators that commute with the abelian subalgebra. As an example of a non orthogonal coordinate system we can choose coordinates in four dimensional flat space via

$$z_1 + iz_2 = x_1, \quad z_1 - iz_2 = 2x_2$$
$$z_3 + iz_4 = 2x_3x_1, \quad z_3 - iz_4 = x_4$$

The corresponding infinitesimal distance is then

$$ds^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 = 2dx_1dx_2 + 2dx_4(x_3dx_1 + x_1dx_3).$$

The corresponding Helmholtz equation is

$$\Delta\Psi = 2\big(\partial_{x_1}\partial_{x_2} + (1/x_1)(-x_3\partial_{x_3}\partial_{x_2} + \partial_{x_3}\partial_{x_4})\big)\Psi = E\Psi.$$

The separation equations are

$$\partial_{x_2}\Psi = \nu_1\Psi, \quad \partial_{x_4}\Psi = \nu_2\Psi,$$
$$2(-\nu_1 x_3 + \nu_2)\partial_{x_3}\Psi = \nu_3\Psi,$$
$$2(\nu_1\partial_{x_1} + \nu_3/x_1)\Psi = E\Psi.$$

The operators which describe this separation of variables are

$$L_1 = (P_1 + iP_2)/2,$$
$$L_2 = (P_3 + iP_4)/2,$$
$$L_3 = (1/4)\{P_3 - iP_4, I_{13} + iI_{23} + iI_{14} - I_{24}\},$$

where $I_{jk} = x_j\partial_{x_i} - x_i\partial_{x_j}$.

2. What about other sources of special functions from a group theoretic point of view? More recently we have looked at the notion of superintegrable mechanical systems. These are Schrödinger equations with potential which admits a separation of variables in more than one coordinate system. In the case of Euclidean space of dimension 2 there are four potentials of this type.

(1) $V_1(x,y) = \frac{1}{2}\frac{\omega^2(x^2+y^2)+k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}$

which separates the Schrödinger equation $-\frac{1}{2}\Delta\Psi + V_1(x,y)\Psi = E\Psi$ in

a. Cartesian coordinates x, y.

b. Polar coordinates $x = r\cos\theta, y = r\sin\theta$.

c. Elliptic coordinates

$$x^2 = c^2(u_1 - e_1)(u_2 - e_1)/(e_1 - e_2),$$

$$y^2 = c^2(u_1 - e_2)(u_2 - e_2)/(e_2 - e_1).$$

(2)
$$V_2(x,y) = \frac{1}{2}\left(\omega^2(4x^2+y^2) + \frac{k_2^2-\frac{1}{4}}{y^2}\right).$$

The corresponding Schrödinger equation then admits separable solutions in

a. Cartesian coordinates x, y.

b. Parabolic coordinates $x = \frac{1}{2}(\xi^2 - \eta^2), y = \xi\eta$.

(3) $V_3(x,y) =$

$$\frac{-\alpha}{\sqrt{x^2+y^2}} + \frac{1}{4\sqrt{x^2+y^2}}\frac{k_1^2-\frac{1}{4}}{\sqrt{x^2+y^2}+x}\frac{k_2^2-\frac{1}{4}}{\sqrt{x^2+y^2}-x}.$$

This separates in

a. Spherical coordinates $x = r\cos\theta, y = r\sin\theta$.

b. Parabolic coordinates $x = \frac{1}{2}(\xi^2 - \eta^2), y = \xi\eta$.

c. Elliptical coordinates

$$x = \frac{1}{2}\sqrt{\frac{(U_1-E_1)(U_2-E_1)}{E_2-E_1}},$$

$$y = \frac{1}{2}\sqrt{\frac{(U_1-E_2)(U_2-E_2)}{E_1-E_2}} - 2\sqrt{E_1-E_2}$$

(4) $V_4(x,y) =$

$$\frac{-\alpha}{\sqrt{x^2+y^2}} + \frac{B_1}{4}\frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}} + \frac{B_2}{4}\frac{\sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{x^2+y^2}}.$$

The separation of variables occurs it two types of parabolic coordinates.

a. Parabolic coordinates one, $x = \frac{1}{2}(\xi^2 - \eta^2)$, $y = \xi\eta$.

b. Parabolic coordinates two, $x = \mu\nu$, $y = \frac{1}{2}(\mu^2 - \nu^2)$.

The essential feature of these coordinates systems is that the boundstate energy eigenvalues and basis functions in each of the separable bases and the relations between them can be determined from algebraic criteria alone.

For details of this approach to special function properties see for instance [8].

3. I have not mentioned special functions arising from equations which are component valued such as Maxwell's equations or the Dirac equation. Also included in this are the gravitational perturbations of a Kerr black hole. The classical reference for this is [4]. For a recent update on these matters see [9].

4. There are interesting relations between modern theories of integrability and separation of variables. The last word has not been said on this subject. How is separation of variables connected with the notion of integrability. A good reference for this topic is [7].

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WAIKATO
HAMILTON, NEW ZEALAND
e-mail: MATH0236@waikato.ac.nz