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THE CLOSURE OF A MODEL CATEGORY

por

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§ 0 Introduction.

The concept of model category is due to Quillen [1]. It represents an axiomatic aproach to homotopy in which not only homotopy itself but also several of the concepts of Algebraic Topology are developed, such as fibrations, loop and suspension functors, homology and homotopy sequences, among others. Thus in order to precise the aims of this paper we first give the definition of a model category.

0.1. Definition: A model category consists of a category A together with three clases of maps: fibrations (F), cofibrations (C), and weak equivalences (WE) such that:

M.0. A is closed under finite projective and inductive limits.

M.1. Given a solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where $i \in C$ and $p \in F$, and where i or p belong to WE then the dotted arrow exists.

M.2. Any map f can be factored as $f = pi$, where i is a cofibration and p is a fibration and weak equivalence. Also $f = pi$, whith i a cofibration and weak equivalence and p a fibration.

M.3. Fibrations are closed under composition, base change, and any isomorphism is a fibration. Cofibrations are closed under composition, co-base change and any isomorphism is a cofibration.

M.4. The base change of a map which is both a fibration and a weak equivalence is a weak equivalence. The co-base change of a map which is a cofibration and a weak equivalence, is a weak equivalence.

M.5. Any isomorphism is a weak equivalence, and if in a conmutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & Z \arrow[ur] & \end{tikzcd}$$

two of the maps are weak equivalences, so is the third.

As it was mentioned before, the basic objective of the definition of model categories was the axio̲matic developement of homotopy. In fact the word model stands for model for homotopy . However, it is not homotopy that we are concerned with here, but rather with conditions on the model category under which the classes of maps involved in the axioms admit precise characterizations which are, in general, missing. This may be the reason why much, if not all, of the later developements and aplications of model categories is being done using a special kind of model categories where F, C, WE, F∩WE, and C∩WE admit characterization by means of liftings. They are called **closed model categories**, they are defined by Quillen [1] and [2]) as follows:

0.2. <u>Definition</u>: **A closed model category**, consists of a category A, and three classes of maps F, C, WE, such that:

C.M.1. A is closed under finite projective and inductive limits.

C.M.2. Whenever in a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & Z \arrow[ur] & \end{tikzcd}$$

two of the maps belog to WE, then so does the third.

C.M.3. F, C, and WE are closed under retracts.

C.M.4. In a commutative solid arrow diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ Z \arrow[r] \arrow[ur, dashed] & K \end{tikzcd}$$

the dotted arrow exists in either of the following situations:

i) i belongs to C and p to WE∩F.

ii) i belongs to C∩WE and p to F.

C.M.5. Any map f can be factored in two ways:
$f = p \circ i$ with i in C and p in F∩WE, and
$f = p \circ i$ with i in C∩WE and p in F.

The class F∩WE will be called the class of trivial fibrations and will be denoted by TF. Similarly C∩WE will be called the class of trivial cofibrations and will be denoted by TC. The classes F, C, WE, TF, and TC will be refered to as the classes of basic morphisms of the model or closed model category. The advantage of closed model categories is the characterization of the classes of basic morphisms, except for WE, by means of liftings:

0.3. Proposition: For a model category (A, F, C, WE) the following statements are equivalent:

i) (A, C, F, WE) is a closed model category.

ii) The classes of basic morphisms admit the following characterizations:

f is a fibration if an only if it has the R.L.P. for TC.

f is a trivial fibration if and only if it has R.L.P. for C.

f is a cofibration if and only if it has L.L.P. for TF.

f is a trivial cofibration if and only if it has L.L.P. for F.

f is a weak equivalence if and only if $f = pi$ where i is a trivial cofibration and p is a trivial fibration.

In proposition 0.3. we have used the following nomenclature:

R.L.P. stands for right lifting property and L.L.P. stands for left lifting property. The proof of this proposition can be found in Quillen [2].

The aims of this paper are basically the following:

i) A sugestion is given for the axiomatization of the theory of liftings or categories with theories of liftings. This is done by introducing the concept of premodel category, which is basically a category A together with five classes of maps F, C, WE, TF, and TC for which the conditions of definition 0.3. part ii), hold.

We consider this to be the ideal situation, as far as liftings is concerned, first because it is highly workable, and second it represents not only the setting of liftings of the most used model categories, the closed model ones, but also because there happens to exist a unique structure of this kind associated to a model category. In fact:

ii) It will be shown that given a category A with model structure (F, C, WE), there exists on A a premodel structure ($\overline{F}$, $\overline{C}$, $\overline{WE}$, $\overline{TF}$, $\overline{TC}$), and only one, for which the following property holds: if $\mathcal{Q}$ stands for any of the classes of basic maps of the model structure, and $\overline{\mathcal{Q}}$ for the corresponding of the premodel category, then $\mathcal{Q} \subseteq \overline{\mathcal{Q}}$.

This premodel structure will be called the closure of A and will be denoted by $\tilde{A}$. It will be very useful for the third purpose of this paper. In order to explain it, let us recall that, associated to a model category (A, F, C, WE), there exists a homotopy category, denoted by $H_o A$ and obtained by localizing the class WE. There exists, therefore, a functor $r: A \to H_o A$, which will be refered to as the homotopic functor, and such that (r, $H_o A$) has the following universal property: If f belongs to WE then r(f) is an isomorphism, and if $t : A \to B$ is a functor such that for each f in WE, t(f) is an isomorphism, then there exists a unique functor $\theta : H_o A \to B$ such that $\theta r = t$. Now, if f belongs to WE then $r(f)$ is an isomorphism, but this does not characterize the weak equivalences of A . In a closed model category, however, f belong to WE if and only if, $r(f)$ is an isomorphism. Yet this behavior of WE apparently does not characterize closed model categories.

iii) $\tilde{A}$ does provide a characterization of model categories in which weak equivalences are the only morphisms sent by r into isomorphisms. In fact, it will be shown that for a model category A the following statements are equivalent:

a) $\tilde{A}$ (the closure of A) is a closed model category.

b) f belongs to WE if and only if $r(f)$ is an isomorphism.

Categories with these (equivalent) conditions will be called **semiclosed model categories** and some other characterizations of them are provided at the end of the paper.

§ 1. Theory of Liftings.

Recall that a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\bar{\beta}} & Z \\ {\scriptstyle \bar{\alpha}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ K & \xrightarrow[\beta]{} & L \end{array}$$

in a category A is called a pull-back square if whenever a square of the kind

$$\begin{tikzcd} T \arrow[r, "\rho"] \arrow[d, "\eta"'] & Z \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"'] & L \end{tikzcd}$$

commutes, then there exists a unique morphism $i : T \to X$ such that $\bar{\beta} i = \rho$ and $\bar{\alpha} i = \eta$. Dually, a commutative square is called a push-out square if the corresponding one in A^o (the opposite category of A) is a pull-back square.

In a pull-back (resp. push-out) square

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "c"'] & Z \arrow[d, "d"] \\ K \arrow[r, "b"'] & L \end{tikzcd}$$

a and c are called the base extensions of b and d, respectively (resp. b and d are called the co-base extensions of a and c, respectively).

A morphism $f: X \to Y$ is called a retract of $g: K \to L$ if there exists a commutative diagram of the kind.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1_X"] & K \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] \arrow[rr, bend right, "1_Y"'] & L \arrow[r] & Y \end{tikzcd}$$

1.1. <u>Definition:</u> Let $\mathcal{Q}$ be a class of morphisms of a category A we say that it is a **fibration type class** if:

F.T.1. $\mathcal{Q}$ contains all the isomorphisms of A.

F.T.2. $\mathcal{Q}$ is closed under composition.

F.T.3. $\mathcal{Q}$ is closed under base extensions, i.e.

the base extension of a map in $\mathcal{Q}$ belongs to $\mathcal{Q}$.

F.T.4. $\mathcal{Q}$ in closed under retracts, i.e. any re tract of an element of $\mathcal{Q}$ belongs to $\mathcal{Q}$.

As examples of fibration type classes we have Kan fibrations in $\Delta^{o}S$ (the category of simplicial sets), Serre fibrations in Top (the category of topological spaces), and Hurewicz fibrations, among others. The fact that they are fibration ty pe classes follows (as we will see) from

1.2. <u>Proposition</u>: Let $\mathcal{Q}$ denote a non empty class of morphisms of a category A, and RLP($\mathcal{Q}$) the class of morphisms of A with right lifting property with respect to $\mathcal{Q}$. Then RLP($\mathcal{Q}$) is a fibration type class.

We omit the proof which is very simple, but we recall the definition of RLP: a morphism $f: X \to Y$ is said to have the right lifting property with respect to $g: K \to L$ if given any commutative solid arrow diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ L \arrow[r] \arrow[ur, dashed, "q"] & Y \end{tikzcd}$$

the lifting q exists, i.e., $q:L \to X$ makes the triangles commutative. Now, f has the right lifting property for a class of morphisms if f has that property for each member of the class. If f has the right lifting property for g we say also

that g has the left lifting property for f .
Again, g has the left lifting property for a class if g has that property for each member of the class.

Let us return to the examples given above; we first consider the standard simplicial simplexes $\Delta[n]$ (resp., topological simplexes $\Delta(n)$) $n = 0,1,2,\ldots,$ and we denote by $\Delta[n,k]$ (resp. $\Delta(n,k)$) the simplicial set $\bigcup_i d^i(\Delta[n-1])$ (resp. $\coprod_i \Delta(n-1)$), $i = 0,\ldots \hat{k} \ldots n$.

Thus the class of Kan fibrations is the class of simplicial functions with right lifting property for the inclusions

$$\Delta[n,k] \to \Delta[n] \ ,$$

Where $n>0$ and $o \leqslant k \leqslant n$. Similarly, in Top the class of Serre fibrations is the class of continuos functions with right lifting property for the class of inclusions

$$\Delta(n,k) \to \Delta(n) \ ,$$

where $n>0$ and $0 \leqslant k \leqslant n$.

As for Hurewicz fibrations they are precisely the class

$$\mathrm{RLP}\ \{ A \xrightarrow{J_o} A\times I \mid A \in \mathrm{Top} \}$$

Where $J_o(a) = (a,0)$.

1.3. Definition: A class $\mathcal{Q}$ of maps of a category A is said to be a cofibration type class if :

C.T.1. $\mathcal{Q}$ contains the classof isomorphims of A.

C.T.2. $\mathcal{Q}$ is closed under composition.

C.T.3. $\mathcal{Q}$ is closed under co-base extensions.

C.T.4. $\mathcal{Q}$ is closed under retracts.

As an example of cofibration type classes we have following.

1.4. <u>Proposition</u>: Let $\mathcal{Q}$ be a class of morphisms of a category A. Then the class LLP($\mathcal{Q}$) of morphisms with left lifting property for $\mathcal{Q}$, is a cofibration type class.

It can be proved (Quillen [1]) that the class of injective simplicial functions, better known as the (standard) cofibrations of $\Delta^o S$, is a cofibration type class. It is precisely

LLP (Kan fibrations $\cap$ HE) ,

where HE denotes the class of weak homotopy equivalences of $\Delta^o S$. Similarly, the class

LLP (Kan fibrations)

is, of course, a cofibration type class known as the class of trivial cofibrations of $\Delta^o S$.

It follows from 0.3 that, in general, in a closed model category F and TF are fibration type classes and C and TC are cofibration type classes.

An interesting example are the isomorphisms and the class of all the morphisms of any category. In fact, one has that, denoting by Mor A and Iso A these classes of morphisms, then

$$\text{Mor } A = \text{LLP } (\text{Iso } A) = \text{RLP } (\text{Iso } A),$$

$$\text{Iso } A = \text{LLP } (\text{Mor } A) = \text{RLP } (\text{Mor } A).$$

Note that RLP and LLP can be considered as operators from the class of parts of Mor A. Furthermore, if we complete the class of parts of Mor A into a category with the morphisms being the inclusions, then RLP and LLP are contravariant functors. That is to say (among other things), if $\mathcal{Q} \subseteq \mathcal{B}$ then $\text{RLP}(\mathcal{B}) \subseteq \text{RLP}(\mathcal{Q})$ and $\text{LLP}(\mathcal{B}) \subseteq \text{LLP}(\mathcal{Q})$.

§ 2. Premodel categories.

2.1. Definition: By a premodel category we mean a category A together with four classes of maps : F(fibrations), TF (trivial fibrations), C (cofibrations) and TC (trivial cofibrations). The class of compositions of the kind $X \xrightarrow{i} Y \xrightarrow{p} Z$, where $i \in TC$ and $p \in TF$, will be denoted by WE and its members will be called weak equivalences. The classes F, TF, C, TC, WE will be called the classes of structural maps and are subjected to the following properties:

P.M.1. $TF \subseteq F$ i.e. any trivial fibration is a fibration.

P.M.2. F, TF, C, TC, admit the following characterization by liftings:

$F = \text{RLP}(TC)$, $TF = \text{RLP}(C)$, $\text{LLP}(TF) \subseteq C$, $\text{LLP}(F) \subseteq TC$.

P.M.3. Any morphism f of A admits two factorizations: $f = kh$, where $h \in TC$ and $k \in F$, and $f = kh$, where $h \in C$ and $k \in TF$.

One has the following consequences of P.M.1. to P.M.3.

2.2. Proposition: **In a premodel category the following hold:**

i) $C = LLP(TF)$ **and** $TC = LLP(F)$

ii) F **and** TF **are fibration type classes and** C **and** TC **are cofibration type classes.**

iii) **Iso** $A \subseteq WE$.

iv) $TC \subseteq C$ **and moreover** $TC = C \cap WE$. **Also** $TF = F \cap WE$.

v) $F \cap C \cap WE =$ Iso **A**.

Proof: i) is an inmediate consequence of the relations

$$F = RLP\ (TC) \quad \text{and} \quad TF = RLP(\ C\ ).$$

As far as ii) is concerned, the characterization of F and TF by the right lifting property implies that they are fibration type classes. Similary, for C and TC, since they are characterized by the left lifting property they are cofibration type classes. For iii), since any isomorphism belong to any fibration (res. cofibration) type class, then any isomorphism belongs to TF and TC. Therefore, any isomorphism $f;X \to Y$ can be written as $f = 1_Y \circ f$, which in turn implies that $f \in WE$.

iv) Since $TF \subseteq F$, then $LLP(F) \subseteq LLP(TF)$. Hence by P.M.2., $TC \subseteq C$. Note that if $f \in TF$ (resp. $f \in TC$), then f can be factored as $f = f \circ 1_Y$ and $f = 1_Y \circ f$; therefore, TF, $TC \subseteq WE$, and since $TF \subseteq F$ and $TC \subseteq C$, then $TF \subseteq F \cap WE$ and $TC \subseteq C \cap WE$. We next prove the opposite inclusions: suppose that $f \in F \cap WE$. Since $f \in WE$, it admits a factorization $f = k \circ h$ where $h \in TC$ and $k \in TF$. We then have a solid arrow diagram

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ K \arrow[r, "k"'] \arrow[ur, dashed, "q"] & Y \end{tikzcd}$$

in which the dotted arrow exists since $h \in TC$ and $f \in F$. Then f is a retract of $k \in TF$, which is closed under retracts. That implies that $f \in TF$. The proof of $TC = C \cap WE$ is similar.

v) follows from the commutativity of the following diagram, for $f \in F \cap C \cap WE$:

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "1_Y"'] \arrow[ur, dashed, "q"] & Y \end{tikzcd}$$

That ends the proof of proposition 2.2.

<u>Remark</u>. We will say that a map $X \xrightarrow{j} K$ is a codomain restriction of $X \xrightarrow{f} Y$ if there exists an injection $K \xrightarrow{i} X$ such that $f = i \circ j$. Similarly, we will say that a map $L \xrightarrow{j} Y$ is a domain restriction of $X \xrightarrow{f} Y$ if there exists a surjection $X \xrightarrow{s} L$ such that $f = j \circ s$. In particular, if one has a composition

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[dr, "i_L"'] & X \arrow[d, "s"] \\ & L \end{tikzcd}$$

so that (i is an injection and s is a surjection), where s defines a domain restriction $L \xrightarrow{j} Y$ of $f\colon X \longrightarrow Y$, then the codomain restriction j is simply given by $L \xrightarrow{i} X \xrightarrow{f} Y$. This is the form generally used to present domain restrictions but, unfortunatelly, it is not enough for our purposes. In many useful categories the two definitions coincide.

It is very easy to verify that if $f \varepsilon RLP(\mathcal{Q})$ and g is a codomain restriction of f, then $g \varepsilon RLP(\mathcal{Q})$. Also, if $f \varepsilon LLP(\mathcal{Q})$ and g is domain restriction of f, then $g \varepsilon LLP(\mathcal{Q})$. Therefore.

2.3. <u>Proposition</u>: In a premodel category F and TF are closed under codomain restrictions and C and TC are closed under domain restrictions.

2.4. <u>Remarks</u>: i) The basic properties of a premodel category can be given diagramatically as follows:

$$\begin{tikzcd}
C \arrow[rr, shift left, "RLP"] \arrow[dd, "\cap WE"'] & & TF \arrow[ll, shift left, "LLP"] \arrow[dl] \\
& WE & \\
TC \arrow[ur] \arrow[rr, shift left, "RLP"] & & F \arrow[ll, shift left, "LLP"] \arrow[uu, "\cap WE"']
\end{tikzcd}$$

For example, the arrow $C \xrightarrow{RLP} TF$ stands for the equality $RLP(C)=TF$. The arrow $C \xrightarrow{\cap WE} TC$ for

$C \cap WE = TC$ and the diagonal for the equality $WE = TF \circ TC$ (in the sense that $f \in WE$ if and only if $f = p \circ i$, where $i \in TF$ and $p \in TC$).

ii) It is clear that any closed model category is a premodel category, but the opposite does not seem to be true. As in the case suggested by Quillen [1] (and never formalized) to build up closed model categories from model categories by omiting unnecesary arrows, there is also the open question on whether or not there is a formal procedure to associate with a premodel category (which is not closed) a closed model category. But, is possible, in the light of the results given later on in this paper, this can not be done by simple elimination. In fact, as we will see A is closed if and only if $A = \tilde{A}$, and $\tilde{A}$ is the unique premodel category associate to A such that for each one of the classes of structural maps (say $\mathcal{Q}$) one has $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$. Hence if $\tilde{A}$ is a premodel category and A is a closed model category obtained by elimination of maps, then $\tilde{A}$ becomes the closure of A, and since A is closed, then $A = \tilde{A}$, which contradicts the assumption of factual elimination of maps or the hypothesis that A is not closed.

iii) From the previous remark one is tempted to predict that premodel categories are in fact closed. But from the point of view of general model category theory one would be lead to a less enthusiastic position. In fact, it involves the axiom of model categories, namely M.5., less likely to be redundant. Yet, if accepted the equivalence closed ≡ premodel, then, at least in the closed model categories, M.5. would be redundant and by implication (from some of the results of this paper) a first choice for redundance in the general case.

The same expectation as in Quillen's work, in which no examples of non-closed model categories are given, remains alive here, except for the fact that several propositions suggest serious reasons to suspect a difference between (not only the two, but) the three concepts and thus an eventual equivalence being a surprise.

§ 3. <u>The closure of a model category</u>.

In this paragraph we will prove the existence of a premodel category (over the same underlaying category) associated to a model category. Since the part corresponding to uniqueness of the closure leaves only one possible closure, we dealt firts with this part and subsequently we prove that the only possible choice is in fact a premodel category.

3.1 <u>Proposition</u>: Let (A, F, C, WE) be a model category. Suppose further that (A, $\overline{F}$, $\overline{C}$, $\overline{TF}$, $\overline{TC}$) is a premodel category such that $F \subseteq \overline{F}$, $C \subseteq \overline{C}$, $TF \subseteq \overline{TF}$, $TC \subseteq \overline{TC}$. Then the following equalities hold:

$$\overline{F} = [F] = RLP(TC), \quad \overline{C} = [C] = LLP(TF),$$
$$\overline{TF} = [TF] = RLP(C), \quad \overline{TC} = [TC] = LLP(F).$$

where if $\mathcal{Q}$ is a class of morphisms of A then $[\mathcal{Q}]$ denotes the class of all retracts of members of $\mathcal{Q}$.

Proof: Recall that in a premodel category the classes of fibrations and trivial cofibrations are fibration type classes and therefore closed under retracts. Similarly, cofibrations and trivial cofibrations are cofibration type classes and hence also closed under retracts. Since, by hypotesis, one has inclusions $\mathcal{Q} = \bar{\mathcal{Q}}$ ($\mathcal{Q}$ = F, TF, C, TC), it follows that $[\mathcal{Q}] \subseteq \bar{\mathcal{Q}}$ ($\mathcal{Q}$ = F, TF, C, TC). We prove now that $\bar{F} \subseteq [F]$. The procedure to prove that $\overline{TF} \subseteq [TF]$ is the same and will be ommited.

Let $f: X \longrightarrow Y \in \bar{F}$. Since (A, F, C, WE) is a model category, then f can be factored as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & K \arrow[ur, "h"'] & \end{tikzcd}$$

Where $h \in F$ and $g \in TC = C \cap WE$. By hypotesis $F \subseteq \bar{F}$ and $TC \subseteq \overline{TC}$. One can then consider the following diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ K \arrow[r, "h"'] \arrow[ur, dashed, "q"] & Y \end{tikzcd}$$

where the lifting q exists, since $\bar{F} = RLP(\overline{TC})$. Thus, f is a retract of $h \in F$. Now, if we assume that $f : X \longrightarrow Y \in C$, then, from a decomposition of f in (A, F, C, WE), say
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "k"'] & & Y \\ & L \arrow[ur, "l"'] & \end{tikzcd}$$

$k \in C$ and $l \in TF$, one gets the following diagram, from which $\bar{C} \subseteq [C]$ follows:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "f"'] & L \arrow[d, "l"] \\ Y \arrow[r, "1_Y"'] \arrow[ur, dashed, "q"] & Y \end{tikzcd}$$

Using similar procedures one can prove the remaining equalities. That ends the proof of proposition 3.1.

The following corolary is obvious:

3.2. <u>Corolary</u>: Given a model category there exists at most a premodel category (over the same undelying category) such that if $\mathcal{Q}$ denotes any of the classes of structural maps of the model category and $\bar{\mathcal{Q}}$ the corresponding one of the premodel category, then $\mathcal{Q} \subseteq \bar{\mathcal{Q}}$.

We face now the task of proving that (A, $[F]$, $[C]$, $[TF]$, $[TC]$) is a premodel category. In order to simplify it we give first a lemma whose result corresponds to the general theory of liftins.

3.3. <u>Lemma</u>: Let A be a category closed under retracts. Let $\mathcal{Q}$ and $\mathcal{B}$ be two (not necessarily diffe rent) classes of morphisms of A. One has

i) If $\mathcal{Q} \subseteq RLP(\mathcal{B})$ then $[\mathcal{Q}] \subseteq RLP(\mathcal{B})$ and $\mathcal{Q} \subseteq RLP([\mathcal{B}])$.

ii) If $\mathcal{Q} \subseteq LLP(\mathcal{B})$ then $[\mathcal{Q}] \subseteq LLP(\mathcal{B})$ and $\mathcal{Q} \subseteq LLP([\mathcal{B}])$.

In words: if a morphism f has the right lifting property with respect to a class $\mathcal{Q}$, then any retract of f has the right lifting property with respect to $\mathcal{Q}$, and f has the right lifting property with respect to any retract of any morphism of $\mathcal{Q}$. Similarly for left lifting property.

Proof: Suppose that $f : X \to Y$ has the right lifting property with respect to a class β of morphisms of a category A . Let $g: K \to L$ be a retract of f given by the following commutative diagram

$$\begin{tikzcd} K \arrow[r, "\alpha_1"] \arrow[d, "g"'] & X \arrow[r, "\beta_1"] \arrow[d, "f"] & K \arrow[d, "g"] \\ L \arrow[r, "\alpha_2"'] & Y \arrow[r, "\beta_2"'] & L \end{tikzcd}$$

with $\beta_i \circ \alpha_i = 1,2$. Suppose given a commutative square

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "k"'] & K \arrow[d, "g"] \\ N \arrow[r, "\eta"'] & L \end{tikzcd}$$

with $k \varepsilon \beta$. Then the following diagram provides a lifting $q : N \to K$:

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "k"'] & K \arrow[r, "\alpha_1"] \arrow[d] & X \arrow[r, "\beta_1"] \arrow[d, "f"] & K \arrow[d, "g"] \\ N \arrow[r] \arrow[urr, dashed, "q'"] & L \arrow[r, "\alpha_2"'] & Y \arrow[r, "\beta_2"'] & L \end{tikzcd}$$

$q = \beta_1 \circ q'$, where q' exists since $k \varepsilon \beta$ and f has the right lifting property with respect to β. That proves $[\mathcal{Q}] \subseteq RLP(\beta)$. Suppose now that f has the right lifting property for β . Let $g \varepsilon \beta$ and supose that h is a retract of g given by the following diagram

$$\begin{tikzcd} K \arrow[r, "\alpha_1"] \arrow[d, "h"'] & M \arrow[r, "\beta_1"] \arrow[d, "g"] & K \arrow[d, "h"] \\ L \arrow[r, "\alpha_2"'] & N \arrow[r, "\alpha_2"'] & L \end{tikzcd}$$

with $\beta_i \circ \alpha_i = 1$. We want to prove that then f has the right lifting property for h. For this purpose consider a commutative diagram

$$\begin{tikzcd} K \arrow[r, "\rho"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ L \arrow[r, "\eta"'] & Y \end{tikzcd}$$

The lifting $q : L \to X$ is given by $q = q' \circ \beta_1$ in

$$\begin{tikzcd} K \arrow[r, "\alpha_1"] \arrow[d, "h"'] & M \arrow[r, "\beta_1"] \arrow[d, "g"] & K \arrow[r, "\rho"] \arrow[d, "h"] & X \arrow[d, "f"] \\ L \arrow[r, "\beta_1"'] & N \arrow[r, "\beta_2"'] \arrow[urr, dashed, "q'"] & L \arrow[r, "\eta"'] & Y \end{tikzcd}$$

where q' exists since f has the right lifting property with respect to g. That proves $\mathcal{Q} \subseteq RLP([\mathcal{B}])$.

Part ii) can be proved in a similar way.

3.4. <u>Proposition:</u> If (A, F, C, WE) is a model category, then (A, [F], [C], [TF], [TC]) is a premodel category.

<u>Proof:</u> We first notice that if $\mathcal{Q}$ and $\mathcal{B}$ are classes of morphisms in a category then $\mathcal{Q} \subseteq [\mathcal{Q}]$ and if $\mathcal{Q} \subseteq \mathcal{B}$, then $[\mathcal{Q}] \subseteq [\mathcal{B}]$. Now, since $TF \subseteq F$ then $[TF] \subseteq [F]$, which proves axiom P.M.1. In order to

prove P.M.2., we notice that since (A, F, C, WE) is a model category then the following inclusions hold

$F \subseteq \mathrm{RLP}\ (TC)$, $TF \subseteq \mathrm{RLP}\ (C)$.

Then by lemma 3.3 one also has that $[F] \subseteq \mathrm{RLP}\ [TC]$ and $[TF] \subseteq \mathrm{RLP}[C]$. On the other hand, since for any class $\mathcal{Q}$ one has that $\mathrm{LLP}(\mathrm{RLP}(\mathcal{Q})) \supseteq \mathcal{Q}$ and $\mathrm{RLP}(\mathrm{LLP}\ (\mathcal{Q})) \supseteq \mathcal{Q}$, then, from the inclusion $F \subseteq \mathrm{RLP}\ [TC]$, one gets that :

$\mathrm{LLP}[F] \supseteq \mathrm{LLP}(F) \supseteq \mathrm{LLP}[TC] \supseteq TC$.

Similarly, $\mathrm{LLP}[TF] \supseteq [C]$. Therefore, in order to finish the proof of P.M.2., it remains to prove that $\mathrm{RLP}[TC] \subseteq [F]$ and $\mathrm{RLP}\ [C] \subseteq [TF]$. Since the proofs are identical we only do the first one. Suppose $f: X \to Y$ has the right lifting property with respect to $[TC]$ and thus to TC (see next remark). One considers a decomposition of f, say

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & k \arrow[ur, "h"'] & \end{tikzcd}$$

with $g \in TC$ and $h \in F$. One then has the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "1_x"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ K \arrow[r, "h"'] \arrow[ur, dashed, "q"] & Y \end{tikzcd}$$

where the lifting q exists by the assumption on f. Thus f belongs to $[F]$.

Axiom P.M.3. is obvious.

The following theorem is clear now:

3.5. Theorem: Given a model category (A,F,C,WE) there exists one and only one premodel category $(A, \bar{F}, \bar{C}, \overline{TC})$ such that $F \subseteq \bar{F}$, $C \subseteq \bar{C}$, $TF \subseteq \overline{TF}$, $TC \subseteq \overline{TC}$, $WE \subseteq \overline{WE}$.

We call the premodel category associated to a model category the closure of the model category. This name is justified by the following proposition:

3.6. Proposition: A model category is closed if and only if it coincides with is closure.

§ 4. Semiclosed model categories.

In this paragraph we want to give characterizations of model categories whose closure is also a model category. It turns out that, as we will see, they are closely related to model categories whose weak equivalences are the only morphisms mapped by the homotopy functor into isomorphisms. In fact, the two characterizations are equivalent and model categories with these two equivalent properties will be called semiclosed model categories.

4.1. Definition: We will say that a model category is a semiclosed model category if in any commutative diagrams of the kind

A) $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "k"] \\ K \arrow[r, "g"'] & Z \end{tikzcd}$$ with $f \in LLP(F)$, $k \in WE$, $g \in LLP(F)$;

B)
$$\begin{tikzcd} L \arrow[r, "f'"] \arrow[d, "k'"'] & X \arrow[d, "h"] \\ M \arrow[r, "g'"'] & K \end{tikzcd}$$
with f', $g' \varepsilon \mathrm{RLP}(C)$ and $k' \varepsilon WE$,

the morphism h can factored as $h = i \circ p$, where $p \varepsilon \mathrm{LLP}(F)$ and $i \varepsilon \mathrm{RLP}(C)$.

4.2. <u>Proposition</u>: The closure of a semiclosed model category is a closed model category.

In order to prove this proposition we need some le mmata. We will use the following notarion: is a map $X \to Y$ belongs to a class $\mathcal{Q}$ of morphisms of A we write $X \xrightarrow{\mathcal{Q}} Y$.

4.3. <u>Lemma</u>: In commutative diagrams of the kind below the morphism h belongs to $\overline{WE}$.

$$\begin{tikzcd} X \arrow[rr, "{[TC]}"] \arrow[dr, "TC"'] & & Y \\ & K \arrow[ur, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "{[TF]}"] \arrow[dr, "h"'] & & Y \\ & K \arrow[ur, "TF"'] & \end{tikzcd}$$

<u>Proof</u>: Since $WE \subsetneq \overline{WE}$, the result follows from the following push-out and pull-back diagrams for the first and second situations respectively:

$$\begin{tikzcd} X \arrow[r, "{[TC]}"] \arrow[d, "TC"'] & Y \arrow[d, "WE"] \arrow[ddr, bend left, "1_X"] & \\ K \arrow[r, "{[TC]}"] \arrow[drr, bend right, "h"'] & K \amalg_X Y \arrow[dr, "j"] & \\ & & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, "i"] \arrow[drr, bend left, "1_X"] \arrow[ddr, bend right, "h"'] & & \\ & K \times_Y X \arrow[r, "WE"] \arrow[d, "{[TF]}"] & X \arrow[d, "{[TF]}"] \\ & K \arrow[r, "TF"'] & Y \end{tikzcd}$$

Note that i, $j \varepsilon WE$ by axiom M.5. Further, the cobase extension of a member of TC belongs to WE as well as the base extension of a member of TF, by axiom M.4. Finally, TC is closed under cobase extension since it is a cofibration type class, and TF is closed under base extensions since it is a fibration type class. Hence $h\varepsilon\overline{WE}$ and the result follows.

4.4. <u>Lemma</u>: In commutative diagrams of the kind below the morphism h belong to $\overline{WE}$.

$$\begin{tikzcd} X \arrow[r, "TC"] \arrow[dr, "{[TC]}"'] & Y \arrow[r, "h"] & Z \\ & K \arrow[ur, "{[TF]}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "{[TC]}"'] & Y \arrow[r, "TF"] & Z \\ & K \arrow[ur, "{[TF]}"'] & \end{tikzcd}$$

<u>Proof:</u> In the first diagram there exists a lifting $q:Y \to K$ and by the previous lemma it belongs to $\overline{WE}$. Therefore, h factors as

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "\overline{WE}"'] & & Z \\ & K \arrow[ur, "{[TF]}"'] & \end{tikzcd}$$

which implies that $h\varepsilon\overline{WE}$. For the second diagram, there exists a lifting $q': K \to Y$ which again belongs to $\overline{WE}$. Hence h factors as

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "{[TC]}"'] & & Y \\ & K \arrow[ur, "\overline{WE}"'] & \end{tikzcd}$$

which implies that $h\varepsilon\ \overline{WE}$.

<u>Proof of proposition 4.2</u>: We first prove that, with no conditions on (A, F, C, $\overline{WE}$), WE is closed under composition. Let $f:X \to Y$ and $g:Y \to Z$ be members

of $\overline{WE}$. One can pick factorings of f and g as follows

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "{[TC]}"'] & & Y \\ & K \arrow[ur, "TF"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "TC"'] & & Z \\ & K \arrow[ur, "{[TC]}"'] & \end{tikzcd}$$

according to lemma 4.4. Hence one gets an extended diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "{[TC]}"'] & & Y \arrow[rr, "g"] \arrow[dr, "TC"] & & Z \\ & K \arrow[ur, "TF"] \arrow[dr, "TC"'] & & K \arrow[ur, "{[TF]}"'] & \\ & & L \arrow[ur, "TF"'] & & \end{tikzcd}$$

by axiom M.5. Thus $g\circ f\in\overline{WE}$. Now we take in account that (A, F, C, WE) is a semiclosed model category. Suppose a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\overline{WE}"] \arrow[dr, "h"'] & & Y \\ & K \arrow[ur, "\overline{WE}"'] & \end{tikzcd}$$

is given. We want to prove that $h\in\overline{WE}$. By lemma 4.4 one can extend this diagram to

$$\begin{tikzcd} X \arrow[r, "{[TC]}"] \arrow[d, "h"'] & Y' \arrow[r, "TF"] & Y \\ K \arrow[rr, "{[TC]}"'] & & K' \arrow[u, "TF"'] \end{tikzcd}$$

thus, there exists a lifting $q:Y'\to K'$. From the diagram

$$\begin{tikzcd} Y' \arrow[rr, "TF"] \arrow[dr, "q"'] & & Y \\ & K \arrow[ur, "TF"'] & \end{tikzcd}$$

it follows (M.5) that $q \varepsilon WE$. Hence one gets a commutative diagram

$$\begin{tikzcd} X \arrow[r, "{[TC]}"] \arrow[d, "h"'] & Y \arrow[d, "q\varepsilon WE"] \\ K \arrow[r, "{[TC]}"'] & K \end{tikzcd}$$

and since $[TC]$ = LLP(F) we are in the situation A) of the hypotesis. Hence $h = i \circ p$. with $i \varepsilon RLP(C) = [TF]$ and $p \varepsilon LLP(F) = [TC]$. That implies that $h \varepsilon WE$. We now prove that in a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\overline{WE}"] \arrow[dr, "\overline{WE}"'] & & Y \\ & K \arrow[ur, "h"'] & \end{tikzcd}$$

$h \varepsilon WE$. This follows from condition B) of the hypotesis and the following extended diagram, guaranted by lemma 4.4. ,

$$\begin{tikzcd} X \arrow[r, "TC"] \arrow[dr, "TC"'] & Y \arrow[r, "{[TF]}"] & Y \\ & K \arrow[u, dashed, "WE"'] \arrow[r, "{[TF]}"'] \arrow[ur, "h"'] & K \end{tikzcd}$$

Since it is clear that a premodel category is a closed model category if and only if axiom M.5 holds for its weak equivalences, then (A, $[F]$, $[C]$, $[TF]$, $[TC]$) is a closed model category. That ends the proof of 4.2.

The converse of 4.2 is also true and obvious:

4.5 <u>Proposition</u>: If the closure of a model category is closed then the model category is semiclosed.

We now relate semiclosed model categories with the homotopic functor $r: A \to H_0 A$. We first identify a larger than the known class WE of morphisms of A whose images $r(f)$ are isomorphisms:

4.6. Proposition: In a model category (A, F, C, WE) any morphism which factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "LLP(F)"'] & & Y \\ & K \arrow[ur, "RLP(C)"'] & \end{tikzcd}$$

is sent by $r: A_0 \to H_0 A$ into an isomorphism.

Proof: Since $LLP(F) = [TC]$ and $RLP(C) = [TF]$ and any retract of an isomorphism is an isomorphism, it follows that, if $h \in [TC]$ (resp. $h \in [TF]$) then $r(h)$ is an isomorphism and if f is a retract of h, then r(f) is an isomorphism.

We have therefore that any weak equivalence of the closure of a model category is also sent by $r : A \to H_0 A$ into an isomorphism. Conversely we have:

4.7. Theorem: For a model category the following two statements are equivalent:

i) The model category is semiclosed.

ii) If r(f) is an isomorphism then f is a weak equivalence of its closure.

Proof: i) $\to$ ii). If A is semiclosed, then its clo sure is closed. We denote the homotopy category of the closure $\tilde{A}$ of A by $\tilde{H}_oA$ and the homotopy functor by $\tilde{r}$. Since $WE \subseteq \overline{WE}$, if $f \in WE$, then $\tilde{r}(f)$ is an isomorphism. Hence there exists a functor $\varphi: H_oA \to \tilde{H}_oA$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[dr, "\tilde{r}"'] & H_oA \arrow[d, "\varphi"] \\ & \tilde{H}_oA \end{tikzcd}$$

So, if $r(f)$ is an isomorphism so is $\tilde{r}(f) = \varphi r(f)$, and since the closure of A is closed then $f \in \overline{WE}$.

ii) $\Rightarrow$ i). By the note above, ii) becomes: $r(f)$ is an isomorphism if and only if $f \in \overline{WE}$. Thus M.5. holds for $\overline{WE}$. Hence $(A, [F], [C], [TF], [TC])$ is a closed model category if ii) holds, and in such a case A is semiclosed.

It is clear, from 4.7, that the property of being a semiclosed model category, for a model category, lies primarily on the good behavior of the class of its weak equivalences. We next emphasize more on this aspect:

4.8 <u>Definition</u>: A model category (A, F, C, WE) is said to be strongly semiclosed if in any diagram of the kind below, $h \in WE$.

$$\begin{tikzcd} X \arrow[r, "RLP(C)"] \arrow[d, "WE"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "RLP(C)"'] & K \end{tikzcd}$$

It is not difficult to prove that A is strongly semiclosed if and only if $WE = \overline{WE}$.

We next give some workable sufficient conditions under which a model category is strongly semiclosed.

4.9 <u>Proposition</u>: Each one of the following is a sufficient condition in order for a model category to be strongly semiclosed:

i) TF is closed under retracts.
ii) TC is closed under retracts.
iii) WE is closed under retracts.

<u>Proof:</u> i) If TF is closed under retracts then $TF = \overline{TF} = RLP(C)$.

Hence the diagram of 4.5 becomes

$$\begin{tikzcd} X \arrow[r, "TF"] \arrow[d, "WE"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "TF"'] & K \end{tikzcd}$$

Since $TF \subseteq WE$ then by axiom M.5 $h \varepsilon WE$.

ii) We will prove that $WE = \overline{WE}$. It remains to prove that $\overline{WE} \subseteq WE$. But if $f \varepsilon \overline{WE}$ one can pick a factorization of f of the kind

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "TF"'] & & Y \\ & K \arrow[ur, "TC"'] & \end{tikzcd}$$

and since TC is closed under retracts, then

$\overline{TC} = [TC] = TC$. Thus $f \in WE$. (Note that this procedure could very well be used in part i) as well).

iii) Suppose that WE is closed under retracts. We will prove that $RLP(C) = \overline{TF} \subseteq WE$, and therefore the condition of 4.5 holds. The desired inclusion follows from $\overline{TF} = [TF] \subseteq [WE] = WE$. That ends the proof of 4.9.

Notice the following equivalences of conditions i) and ii) of proposition 4.9:

TF is closed under retracts if and only if $TF = RLP(C)$.

TC is closed under retracts if and only if $TC = LLP(F)$.

So, if one of the onclusions $TF \subseteq RLP(C)$ or $TC \subseteq LLP(F)$ becomes equality in a model category, then $WE = \overline{WE}$ and the model category becomes a (strongly) semiclosed model category.

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