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CENTRALIZERS OF p-SUBGROUPS IN SIMPLE LOCALLY FINITE GROUPS

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Abstract. In Ersoy et al. [*J. Algebra* **481** (2017), 1–11], we have proved that if G is a locally finite group with an elementary abelian p-subgroup A of order strictly greater than p^2 such that $C_G(A)$ is Chernikov and for every non-identity $\alpha \in A$ the centralizer $C_G(\alpha)$ does not involve an infinite simple group, then G is almost locally soluble. This result is a consequence of another result proved in Ersoy et al. [*J. Algebra* **481** (2017), 1–11], namely: if G is a simple locally finite group with an elementary abelian group A of automorphisms acting on it such that the order of A is greater than p^2, the centralizer $C_G(A)$ is Chernikov and for every non-identity $\alpha \in A$, the set of fixed points $C_G(\alpha)$ does not involve an infinite simple groups then G is finite. In this paper, we improve this result about simple locally finite groups: Indeed, suppose that G is a simple locally finite group, consider a finite non-abelian subgroup P of automorphisms of exponent p such that the centralizer $C_G(P)$ is Chernikov and for every non-identity $\alpha \in P$ the set of fixed points $C_G(\alpha)$ does not involve an infinite simple group. We prove that if $Aut(G)$ has such a subgroup, then $G \cong PSL_p(k)$ where $char\ k \neq p$ and P has a subgroup Q of order p^2 such that $C_G(P) = Q$.

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1. Introduction. In [2], we have proved the following result:

THEOREM 1.1. [2, Theorem 1.1]. *Let p be a prime and G a locally finite group containing an elementary abelian p-subgroup A of rank at least 3 such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $a \in A^\#$. Then G is almost locally soluble.*

To prove Theorem 1.1, we gave the following characterization of $PSL_p(k)$ where $char k \neq p$.

THEOREM 1.2. [2, Theorem 1.2]. *An infinite simple locally finite group G admits an elementary abelian p-group of automorphisms A such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $a \in A^\#$ if and only if G is isomorphic to $PSL_p(k)$ for some locally finite field k of characteristic different from p and A has order p^2.*

In this paper, we will improve Theorem 1.2. Indeed, we will prove a similar result without assuming A is an elementary abelian, but instead, we prove for any subgroup of exponent p.

THEOREM 1.3. *Let G be an infinite simple locally finite group, P a subgroup of automorphisms of exponent p such that*

(1) $C_G(P)$ is Chernikov,

(2) *For every $\alpha \in P\backslash\{1\}$, the set of fixed points $C_G(\alpha)$ does not involve an infinite simple group.*

Then $G \cong PSL_p(k)$ where k is an infinite locally finite field of characteristic p and P has a subgroup Q of order p^2 such that $C_G(P) = C_G(Q) = Q$.

2. Preliminaries.

Let us recall some definitions of the concepts mentioned in the theorems. First, consider $C_{p^n} = \{x \in \mathbb{C} : x^{p^n} = 1\}$. Here, $(C_{p^n}, .)$ defines a group isomorphic to a cyclic group of order p^n. Observe that if $m|n$ then $C_{p^m} \leq C_{p^n}$, and with the inclusion maps, these sets form a direct system, where the direct limit

$$\lim_{n\in\mathbb{N}} C_{p^n}$$

is denoted by C_{p^∞}, which consists of all complex p^n-th roots of unity, and forms a group under complex multiplication. This group is called the quasi-cylic p-group.

DEFINITION 2.1. A group is called a Chernikov group if it is a finite extension of a direct product of finitely many copies of some quasi-cyclic p_i-groups, for possibly distinct primes p_i.

DEFINITION 2.2. Let χ be a group-theoretical property. If a group G has a normal subgroup of finite index satisfying χ, then G is called almost χ.

DEFINITION 2.3. Let G and H be two groups. If G has a normal subgroup K such that G/K has a subgroup isomorphic to H, then G is said to involve a subgroup isomorphic to H.

DEFINITION 2.4. A group satisfies the minimal condition, namely *min*, if any non-empty set of subgroups has a minimal subgroup. A group satisfies *min-p* if any non-empty set of p-subgroups has a minimal subgroup.

Kegel–Wehrfritz and Sunkov proved independently that a locally finite group satisfying minimal condition is a Chernikov group (see [5, 9]). For detailed discussion of groups satisfying *min* and *min-p*, see [6].

3. Main results.

First, we need the following proposition:

PROPOSITION 3.1. *Let $\overline{G}$ be a simple linear algebraic group of adjoint type over the algebraic closure of $\mathbb{F}_q$, let $g \in \overline{G}$ be an element of prime order $p \neq q$ such that $C_{\overline{G}}(g)$ is a non-abelian group which does not involve any infinite simple groups. Then*

(i) *The identity component $C_{\overline{G}}(g)^0$ of the centralizer of g in $\overline{G}$ is a maximal torus of $\overline{G}$,*
(ii) *$\overline{G} \cong PGL_p(\overline{\mathbb{F}_q})$.*

Proof. Since $\overline{G}$ is a simple linear algebraic group of adjoint type over the algebraic closure of $\mathbb{F}_q$ and $g \in G$ a semisimple element, g is contained in a maximal torus T of $\overline{G}$. By [7, Propositions 14.1 and 14.2], $C_{\overline{G}}(g)^0$ is connected reductive, containing a maximal torus T, and involving no infinite simple groups. Hence, $C_{\overline{G}}(g)^0 = T$. By [7, Proposition 14.20], the exponent of $C_{\overline{G}}(g)/C_{\overline{G}}(g)^0$ divides p, hence either $C_{\overline{G}}(g)$ is connected, and hence a torus, or $C_{\overline{G}}(g)/C_{\overline{G}}(g)^0$ is a finite group of exponent p.

Since $C_{\overline{G}}(g)$ is not abelian, one has $C_{\overline{G}}(g)$ a finite extension of an abelian group T, so it has finite rank. Recall that an infinite group G is said to have finite rank r if every finitely generated subgroup is r-generated. In [1, Theorem 1.8], we have shown that when a simple linear algebraic group $\overline{G}$ over the algebraic closure of $\mathbb{F}_q$ has an element g of order p with $C_{\overline{G}}(z)$ has finite rank, then one of the following cases occur:

(1) $\overline{G}$ is of type A_l and $p > l$,
(2) $\overline{G}$ is of type B_l, C_l, and $p > 2l - 1$,
(3) $\overline{G}$ has type D_l and $p > 2l - 3$,
(4) $\overline{G}$ is isomorphic to one of E_6, E_7, E_8, F_4, or G_2 and $p > 11, 17, 29, 17$, or 5, respectively.

On the other hand, since $C_{\overline{G}}(g)/C_{\overline{G}}(g)^0$ has exponent p, by [10, Corollary 4.4] and [7, Proposition 14.20], we get p is a torsion prime. The list of torsion primes of linear algebraic groups is defined as follows: for type A_l, these are the primes that divide $l+1$. For types B_l, C_l, D_l, G_2, the prime is 2. For types E_6, E_7, F_4, the primes are 2 and 3, and for type E_8, the primes are 2, 3, 5 (see [8]).

Hence, one deduce that the only possible case that may occur is $\overline{G}$ has type A_{p-1}, indeed $\overline{G} \cong PGL_p(\overline{\mathbb{F}_q})$. □

THEOREM 3.2. *Let G be an infinite simple locally finite group with a finite non-abelian p-group of automorphisms P such that*

(1) $C_G(P)$ is Chernikov,
(2) For every $\alpha \in P\backslash\{1\}$, the set of fixed points $C_G(\alpha)$ does not involve an infinite simple group

Then, G is isomorphic to $PSL_p(k)$ where k is a locally finite field of characteristic $q \neq p$ and P is metabelian.

Proof. Since P is a finite p-group and $C_G(P)$ satisfies *min-p*, by [2, Lemma 2.1], G satisfies *min-p*. Then, by [4, Theorem B], G is a simple group of Lie type over a locally finite field k of characteristic q. Now assume that $q = p$. Clearly G contains a root subgroup, which is an infinite elementary abelian p-subgroup. Hence, G can not satisfy *min-p*. Hence, $q \neq p$, that is, G is isomorphic to a simple group of Lie type over an infinite locally finite field of characteristic $q \neq p$.

Now, by [3, Lemma 4.3], there exists a simple linear algebraic group $\overline{G}$ of adjoint type, a Frobenius map σ on $\overline{G}$ and a sequence of natural numbers $n_i | n_{i+1}$ such that

$$G = \bigcup_{i \in \mathbb{N}} O^{p'}(\overline{G}_{\sigma^{n_i}}).$$

By assumption, the centralizer of any non-identity element does not involve an infinite simple group, so [2, Lemma 2.3] implies that P consists of inner-diagonal automorphisms of G. Hence, $P \leq \bigcup_{i \in \mathbb{N}} \overline{G}_{\sigma^{n_i}}$. Therefore, $P \leq \overline{G}_{\sigma^{n_j}}$ for some $j \in \mathbb{N}$.

Choose $1 \neq z \in Z(P)$. Clearly, $P \leq C_{\overline{G}}(z)$. Now, $C_G(z) = \bigcup_{i \in \mathbb{N}} O^{p'}(C_{\overline{G}}(z)_{\sigma^{n_i}})$.

By assumption, $C_G(z)$ does not involve an infinite simple group. Now, suppose that $C_{\overline{G}}(z)$ involves a simple linear algebraic group H. Consider the union of fixed points of σ^{n_i} on H, denoted by $H_i = H_{\sigma^{n_i}}$. Clearly, $H_i \leq H_{i+1}$ and infinitely many of H involves finite simple groups such that their union form an infinite locally finite simple group. Hence, we get a contradiction and we deduce $C_{\overline{G}}(z)$ does not involve a simple linear algebraic group. By [2, Lemma 2.4], $C_{\overline{G}}(z)$ is metabelian. Hence, P is metabelian. On the other hand, since P is not abelian, $C_{\overline{G}}(z)$ is not abelian.

By Proposition 3.1, $\overline{G}$ is isomorphic to $PGL_p(\overline{\mathbb{F}_q})$. Hence, G is isomorphic to either $PSL_p(k)$ or $PSU_p(k)$. Following the argument in the proof of Theorem 1.2 in [2], since the Weyl group of $PSU_p(k)$ has no elements of order p, and PT/T embeds in the Weyl group, $PSU_p(k)$ has no such non-abelian subgroup P. Therefore, $G \cong PSL_p(k)$ where k is an infinite locally finite field of characteristic $q \neq p$. □

Then, we prove the main result of the paper:

Proof of Theorem 1.3. Assume first that P is abelian. Then by Theorem 1.2, the result follows with $|P| = p^2$.

Now, assume P is non-abelian. By Theorem 3.2, $G \cong PSL_p(k)$ where k is a locally finite field of characteristic $q \neq p$. Let $1 \neq z \in Z(P)$, observe that $P \leq C_G(z) \leq C_{\overline{G}}(z)$ where $\overline{G}$ is the corresponding simple linear algebraic group and σ is the Frobenius map such that $G = \bigcup_{i \in \mathbb{N}} O^{p'}(\overline{G}_{\sigma^{n_i}})$, which exist by [**3**, Lemma 4.3]. Denote the maximal torus of $\overline{G}$ containing z by T. By Proposition 3.1(i), $C_{\overline{G}}(z)^0 = T$. Indeed, by [**10**, Corollary 1.7], T is the unique maximal torus containing z. Since P is not abelian, $C_{\overline{G}}(z)/C_{\overline{G}}(z)^0$ can not be 1, hence by [**7**, Proposition 14.20], it has exponent p. Let y be any element of $C_{\overline{G}}(z) \backslash C_{\overline{G}}(z)^0$. Then, $Q = \langle y, z \rangle$ has order p^2. Indeed, $C_{\overline{G}}(z)^0 = T$, and $y \in N_{\overline{G}}(T)$. Hence, y induces an element w of order p in the Weyl group. Now, $z \in C_T(w)$. The computation in the proof of Theorem 1.2 in [**2**] shows that indeed $C_T(w)$ has order p, hence $C_{\overline{G}}(Q) = Q$. This Q is the required subgroup. □

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