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EIGENVALUE CONTINUITY AND GERŠGORIN'S THEOREM*

CHI-KWONG LI[†] AND FUZHEN ZHANG[‡]

Abstract. Two types of eigenvalue continuity are commonly used in the literature. However, their meanings and the conditions under which continuities are used are not always stated clearly. This can lead to some confusion and needs to be addressed. In this note, the Geršgorin disk theorem is revisited and the issue concerning the proofs of the theorem by continuity is clarified.

Key words. Eigenvalue continuity, Geršgorin disk theorem, Root continuity.

AMS subject classifications. 15A18, 65F15.

1. Introduction. In his seminal paper in 1931 [9], Geršgorin presented an important result about the localization of the eigenvalues of matrices. He showed that (1) all eigenvalues of a square matrix lie in the union of the later so-called *Geršgorin disks* and (2) if some, say m , of the disks are disjoint from the remaining disks, then the union of these m disks contains exactly m eigenvalues (counted with algebraic multiplicities). The result was named after Geršgorin as the *Geršgorin disk theorem* due to its importance and applications for estimating and localizing eigenvalues.

Let $A = (a_{ij})$ be an $n \times n$ complex matrix and let $r_i = \sum_{j \neq i} |a_{ij}|$, $i = 1, \dots, n$. The set $D_i = \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i\}$ is referred to as a Geršgorin disk of A . Let A_0 be the diagonal matrix that has the same main diagonal as A . Geršgorin proved the second part of his theorem by considering the matrix $A(t) = A_0 + t(A - A_0)$, $t \in [0, 1]$, and letting t increase continuously from 0 to 1. Intuitively, the concentric Geršgorin disks of $A(t)$ centered at a_{ii} ($i = 1, \dots, n$) get larger and larger as t increases from 0 to 1. He stated that “Since the eigenvalues of the matrix depend continuously on its elements, it follows that m eigenvalues must always lie in the disks ...”. Geršgorin used as a fact without justification that *eigenvalues are continuous functions of the entries of matrices*.

Such a statement is often seen in the literature when it comes to the proof of the second part of the Geršgorin disk theorem. For instance, here are a few widely-cited and comprehensive references. In the first edition of Horn and Johnson's book *Matrix Analysis* [12], page 345, it asserts that “the eigenvalues are continuous functions of the entries of A (see Appendix D)...”, in Rahman and Schmeisser's *Analytic Theory of Polynomials* [22], page 55, it states that “The eigenvalues of $A(t)$ are continuous functions of t ...”, in Varga's *Geršgorin and His Circles* [27], page 8, it is written that “the eigenvalues $\lambda_i(t)$ of $A(t)$ also vary continuously with t ...”, and in Wilkinson's *The Algebraic Eigenvalue Problem* [28], page 72, it says that “the eigenvalues all traverse continuous paths”.

What does it really mean to say that eigenvalues are continuous functions? Geršgorin's proof by continuity may lead one to imagine continuous curves of the eigenvalues evolving on the complex plane, or one may trace the curves continuously. But that is not as easy as it sounds. First, ordering eigenvalues with a

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parameter can be tricky and difficult; second, the eigenvalue curves may merge (or split) and the algebraic multiplicities of eigenvalues can change as the parameter varies.

EXAMPLE 1. Let $A(t) = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$, $t \in [-1, 1]$. Each t produces a set of two eigenvalues. How does one order the eigenvalues as functions of t ? It is natural to order the eigenvalues of $A(t)$ as $\lambda_1(t) = t$, $\lambda_2(t) = -t$. Notice that $A(t)$ is real symmetric. For real symmetric (or complex Hermitian) matrices, we usually want the eigenvalues to be in a non-increasing (or non-decreasing) order. So we would order the eigenvalues as $\mu_1(t) = |t| \geq \mu_2(t) = -|t|$. (Unlike $\lambda_1(t)$ and $\lambda_2(t)$, $\mu_1(t)$ and $\mu_2(t)$ are not differentiable. Of course, there are infinitely many ways to parameterize the eigenvalues as non-continuous functions.)

The eigenvalues in Example 1 are parameterized as continuous functions of t . Is this always possible? The answer is *yes* for t on a real interval (but why? see Theorem 3) and *no* for t on a complex domain containing the origin (see Example 2).

Geršgorin's original proof by continuity is more like "hand-waving" than a rigorous proof and it has led to some confusion or ambiguity [8]. A rigorous proof of the theorem using eigenvalues as continuous functions requires creating or referencing some heavy machinery that was absent from all the classical sources. This issue deserves attention and clarification for both teaching and research.

Additionally, matrices depending on a parameter play important roles in scientific areas. In some studies such as stability problems and adiabatic quantum computing, one may consider a real parameter t joining matrix A and matrix B by $(1 - t)A + tB$ and analyze the change of the eigenvalues as t varies.

In Section 2, we briefly recap the eigenvalue continuity in the topological sense. In Section 3, we summarize a celebrated result of Kato on the continuity of eigenvalues as functions. In Section 4, we discuss the existing proofs of the Geršgorin disk theorem and present a proof with topological continuity and a proof with functional continuity. We end the paper by including a short and neat proof of the second part of the Geršgorin disk theorem by using the argument principle.

2. Topological continuity of eigenvalues. Are eigenvalues of a matrix continuous functions of the matrix? Since eigenvalue problems of matrices are essentially root problems of (characteristic) polynomials, one immediately realizes that the question is a bit subtle and needs careful formulation. It is known that the roots of a polynomial vary continuously as a function of the coefficients. In [10], the authors gave a nice proof for the result concerning the continuity of zeros of complex polynomials. In fact, the map sending a monic polynomial $f(z) = z^n + a_1 z^{n-1} + \dots + a_n$ to the multi-set of its zeros $\pi(f) = \{\lambda_1, \dots, \lambda_n\}$ is continuous in the following sense. For monic polynomials $f(z) = z^n + a_1 z^{n-1} + \dots + a_n$ and $\tilde{f}(z) = z^n + \tilde{a}_1 z^{n-1} + \dots + \tilde{a}_n$ with multi-sets of zeros $\pi(f) = \{\lambda_1, \dots, \lambda_n\}$ and $\pi(\tilde{f}) = \{\tilde{\lambda}_1, \dots, \tilde{\lambda}_n\}$, one can use the metrics

$$\|f - \tilde{f}\| = \max\{|a_j - \tilde{a}_j| : 1 \leq j \leq n\}$$

and

$$d(\pi(f), \pi(\tilde{f})) = \min_J \left\{ \max_{1 \leq j \leq n} |\lambda_j - \tilde{\lambda}_{j_\epsilon}| : J = (j_1, \dots, j_n) \text{ is a permutation of } (1, \dots, n) \right\}.$$

Then π is (pointwise) continuous; that is, for fixed f and for any given $\varepsilon > 0$, there exists $\delta > 0$ (depending on f) such that $\|f - \tilde{f}\| < \delta$ implies $d(\pi(f), \pi(\tilde{f})) < \varepsilon$. Moreover, if ξ is a zero of $f(z)$ with algebraic multiplicity m , then \tilde{f} has exactly m zeros in the disk centered at ξ with radius ε .

If we identify f with the associated n -tuple $(a_1, \dots, a_n) \in \mathbb{C}^n$, then π is a homeomorphism between \mathbb{C}^n (with the usual topology) and the quotient space \mathbb{C}^n_{\sim} (with the induced quotient topology), the unordered n -tuples (see [3, p. 153]).

Applying this result to matrices, one gets the eigenvalue continuity as the eigenvalues of an $n \times n$ matrix A are the zeros of the characteristic polynomial

$$p_A(z) = \det(zI - A) = z^n + a_1 z^{n-1} + \cdots + a_n,$$

where a_j is $(-1)^j$ times the sum of the $j \times j$ principal minors of A .

To be more specific, with M_n for the space of $n \times n$ complex matrices, we consider the eigenvalue function $\sigma : M_n \rightarrow \mathbb{C}_\infty^n$ that maps a matrix $A \in M_n$ to its spectrum $\sigma(A) \in \mathbb{C}_\infty^n$. For the continuity of σ , we can use any (fixed) norm $\|\cdot\|$ on M_n .

The function σ is continuous, i.e., for fixed $A \in M_n$ and for any given $\varepsilon > 0$, there is $\delta > 0$ (depending on A) such that $d(\sigma(A), \sigma(\tilde{A})) < \varepsilon$ whenever $\|A - \tilde{A}\| < \delta$. Such an eigenvalue continuity may be referred to as *eigenvalue topological continuity* or *eigenvalue matching continuity*. Thus, eigenvalues are always continuous in the topological sense.

A nice proof regarding eigenvalue topological continuity for the discrete case (i.e., matrix sequences) is available in [1, pp. 138–140]. The same continuity of eigenvalues is also studied in [13, p. 121]) by using Schur triangularization and compactness of the unitary group. Closely related to eigenvalue continuity are eigenvalue perturbation (variation) results with norm bounds involving the entries of matrices (see [18, 21] and [13, p. 563, Appendix D]).

There is another possible way of thinking of the eigenvalue continuity problem. Let $A(t)$ be a family of $n \times n$ matrices depending continuously on a parameter t over a domain in the complex plane or on a real interval. Then do there exist n continuous complex functions of t that represent eigenvalues of $A(t)$? We discuss the question in the next section.

3. Parametrization of eigenvalues as continuous functions. In some applications, one needs to consider a continuous function $A : D \rightarrow M_n$, where $A(t) \in M_n$ and D is a certain subset of \mathbb{C} (say, a domain); and one wants to parametrize the eigenvalues of $A(t)$ as n continuous functions $\lambda_1(t), \dots, \lambda_n(t)$ with $t \in D$. We refer to such continuity as *eigenvalue functional continuity* provided there exist n continuous functions of t that represent eigenvalues of $A(t)$.

Eigenvalue functional continuity is widely used in the proof of the second part of the Geršgorin disk theorem; similar ideas are needed in the perturbation theory of Hermitian matrices, stable matrices, etc. However, such a parametrization is not always possible over a complex domain ([3, p. 154], [15, p. 64; p. 108]).

EXAMPLE 2. Let $A(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$, $t \in D = \{z \in \mathbb{C} : |z| < 1\}$. It is impossible to have two continuous functions $\lambda_1(t), \lambda_2(t)$ on D representing the eigenvalues of $A(t)$. This is because each eigenvalue λ of $A(t)$ satisfies $\lambda^2 = t$; thus, the desired continuous functions $\lambda_1(t)$ and $\lambda_2(t)$ have to satisfy $(\lambda_1(t))^2 = (\lambda_2(t))^2 = t$ for all t on the open unit disk, which is impossible (as is known, there is no continuous function f on a disk D containing the origin such that $(f(z))^2 = z$ for all $z \in D$).

However, as $t \rightarrow 0$, $A(t)$ approaches $A(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (entrywise), which has repeated eigenvalue 0. Any small disk that contains the origin will contain two eigenvalues of $A(t)$ when t is close enough to 0. (This is what topological continuity means.)

The difference between topological continuity and functional continuity is that the eigenvalues (as a whole) are always topologically continuous but need not be continuous as individual functions. The two continuities for $A(t)$ are equivalent when the parameter t belongs to a real interval (see [3, 15]).

In [15, p. 109, Theorem 5.2], the following remarkable result is shown.

THEOREM 3. (Kato, 1966) *Suppose that $D \subset \mathbb{C}$ is a connected domain and that $A : D \rightarrow M_n$ is a continuous function. If (1) D is a real interval, or (2) $A(t)$ has only real eigenvalues, then there exist n eigenvalues (counted with algebraic multiplicities) of $A(t)$ that can be parameterized as continuous functions $\lambda_1(t), \dots, \lambda_n(t)$ from D to \mathbb{C} . In the second case, one can set $\lambda_1(t) \geq \dots \geq \lambda_n(t)$.*

The study of eigenvalue functional continuity can be traced back at least as early as 1954 [23]. Rellich [24, p. 39] showed that individual eigenvalues are continuous functions when the matrices are Hermitian (in such case all eigenvalues are necessarily real). In his well-received book, Kato [15, p. 109] showed that topological continuity implies functional continuity when the parameter is restricted to a real interval or if all the eigenvalues of the matrices are real, i.e., Theorem 3.

It is tempting to extend Kato's result on a real interval for the parameter to a domain (with interior points) on the complex plane. However, this is impossible. Let $z_0 \neq 0$ and let D_{z_0} be an open disk centered at z_0 that does not contain the origin. Considering $A(z) = \begin{pmatrix} 0 & 1 \\ z-z_0 & 0 \end{pmatrix}$, $z \in D_{z_0}$, we see that there does not exist a continuous eigenvalue function of $A(z)$ on D_{z_0} . Suppose, otherwise, there is a continuous eigenvalue function $\lambda(z)$ on D_{z_0} , then $(\lambda(z))^2 = z - z_0$ for all $z \in D_{z_0}$. This leads to a continuous function $f(z) = \lambda(z + z_0)$ defined on the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ such that $(f(z))^2 = z$ for all $z \in D$, a contradiction.

So, in a sense, the result of Kato is the best possible with respect to eigenvalue functional continuity.

4. Proofs of the Geršgorin disk theorem. Geršgorin's disk theorem is a useful result for estimating and localizing the eigenvalues of a matrix. Usually and traditionally, the second part of the theorem is proved by considering the matrix $A(t) = A_0 + t(A - A_0)$ (where A_0 is the diagonal matrix that has the same main diagonal as A) and by using eigenvalue continuity (see [6], [11, p. 23], [12, p. 345], [14, p. 74], [17, p. 372], [20, p. 499], [22, p. 55], [26, p. 169], [27, p. 8], [28, p. 72]), and [29, p. 70]). However, in these references, it is not always clear which types of eigenvalue continuity conditions were used. If it is topological continuity, then one needs to add some details in the proofs to justify why the total number of eigenvalues in an isolated region remains the same when t increases from 0 to 1 (note that the algebraic multiplicity of an eigenvalue may change); if it is functional continuity (which is the case in most texts), then it would be nice to state Kato's result (or other references) as evidence of the existence of continuous functions that represent the eigenvalues. In the following, we state the Geršgorin disk theorem and give two different proofs. One (Proposition 5) uses eigenvalue functional continuity (Kato's theorem) and exploits the fact that a continuous function takes a connected set into a connected set; the other (Proposition 6) uses eigenvalue topological continuity and exploits the fact that a continuous function on a compact set is *uniformly continuous*. In the latter, we completely avoid the continuity of each eigenvalue as a function.

THEOREM 4. (Geršgorin [9], 1931) *Let $A = (a_{ij}) \in M_n$ and define the disks*

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}, \quad i = 1, \dots, n.$$

Then: (1) All eigenvalues of A are contained in the union $\cup_{i=1}^n D_i$. (2) If $\cup_{i=1}^n D_i$ is the union of k disjoint connected regions R_1, \dots, R_k , and R_r is the union of m_r of the disks D_1, \dots, D_n , then R_r contains exactly m_r eigenvalues of A , $r = 1, \dots, k$.

Part (1) says that every eigenvalue of A is contained in a Geršgorin disk. Its proof is easy, standard, and omitted here. Part (2) is immediate from Propositions 5 and 6. We call a union of some Geršgorin disks a *Geršgorin region* (which in general need not be connected). In particular, the singletons of diagonal entries are degenerate Geršgorin regions. By a *curve* we mean the image (range) of a continuous map from a real closed interval to the complex plane $\gamma : [a, b] \mapsto \mathbb{C}$.

PROPOSITION 5. *Let $A = (a_{ij}) \in M_n$ and let $A(t) = A_0 + t(A - A_0)$, where $t \in [0, 1]$ and $A_0 = \text{diag}(a_{11}, \dots, a_{nn})$. Then each continuous eigenvalue curve of $A(t)$ lies entirely in a connected Geršgorin region of A .*

Proof. By Kato's result (Theorem 3), there exists a selection of n eigenvalues $\lambda_1(t), \dots, \lambda_n(t)$ of $A(t)$ that are continuous functions in t on the real interval $[0, 1]$. Moreover, part (1) of the Geršgorin disk theorem ensures that $\lambda_1(t), \dots, \lambda_n(t)$ are contained in $\cup_{i=1}^k R_i$ for every $t \in [0, 1]$, and each set $\lambda_j([0, 1])$ is connected.

Let $r \in \{1, \dots, k\}$. Since R_r comprises m_r disks (not necessarily different) whose centers are m_r elements of the diagonal matrix A_0 , m_r of the continuous eigenvalue curves $\lambda_1(t), \dots, \lambda_n(t)$ are in R_r at $t = 0$. If $\lambda_j(0) \in R_r$, then the connected set

$$\lambda_j([0, 1]) = \lambda_j([0, 1]) \cap \cup_{i=1}^k R_i = (\lambda_j([0, 1]) \cap R_r) \cup (\lambda_j([0, 1]) \cap \cup_{i \neq r}^k R_i)$$

is the union of two disjoint closed sets, the first of which is nonempty. Therefore, the second set is empty, and hence, $\lambda_j([0, 1]) \subset R_r$. \square

The following proposition considers the eigenvalues as a whole in a Geršgorin region rather than focusing on an individual eigenvalue as a function. That is, we use eigenvalue topological continuity and avoid entirely (the difficult issue of) eigenvalue functional continuity (which is not needed) to prove the assertion.

PROPOSITION 6. *Let $A = (a_{ij}) \in M_n$ and let $A(t) = A_0 + t(A - A_0)$, where $t \in [0, 1]$ and $A_0 = \text{diag}(a_{11}, \dots, a_{nn})$. Then a connected Geršgorin region of A contains the same number of eigenvalues of $A(t)$ for all $t \in [0, 1]$.*

Proof. Every entry of $A(t)$ is a continuous function of $t \in [0, 1]$ and each Geršgorin disk of $A(t)$ ($0 \leq t \leq 1$) is contained in a Geršgorin disk of $A = A(1)$ with the same corresponding center. Let R_r , $r = 1, \dots, k$, be the connected Geršgorin regions of A . (The number of connected Geršgorin regions for $A(t)$ may vary depending on t .) Suppose that R_r contains m_r diagonal entries of A , i.e., m_r eigenvalues of $A_0 = A(0)$ (counted with algebraic multiplicities). We claim that R_r contains m_r eigenvalues of $A(t)$ for all $t \in [0, 1]$ (that is, the sum of the algebraic multiplicities of the eigenvalues of $A(t)$ remains constant on each connected Geršgorin region of A as t varies from 0 to 1).

Since the eigenvalues are topologically continuous over the compact set $[0, 1]$, the continuity is uniform. To be precise, the map $\varphi : [0, 1] \mapsto \mathbb{C}^n$ defined by $\varphi(t) = \sigma(A(t))$ is uniformly continuous.

Let $\varepsilon > 0$ be such that $|x - y| > 2\varepsilon$ for all x, y lying in any two disjoint Geršgorin regions of A . There is $\delta > 0$ (depending only on ε) such that for any t_1 and t_2 satisfying $0 \leq t_1 < t_2 \leq 1$ and $t_2 - t_1 < \delta$, the eigenvalues of $A(t_1)$ and $A(t_2)$ can be labeled as $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_n such that $|\lambda_j - \mu_j| < \varepsilon$ for $j = 1, \dots, n$.

We divide the interval $[0, 1]$ into N subintervals: $0 = t_0 < t_1 < \dots < t_N = 1$ such that $t_i - t_{i-1} < \delta$ for $i = 1, \dots, N$. We show that on each of the intervals $A(t)$ has m_r eigenvalues in R_r for $t \in [t_i, t_{i+1}]$, $i = 0, 1, \dots, N - 1$.

By assumption, $A(t_0) = A_0 = A(0)$ has exactly m_r eigenvalues in R_r . For any $t \in [t_0, t_1]$, since $t - t_0 < \delta$, $A(t)$ has exactly m_r eigenvalues each of which is located in some disk centered at an eigenvalue of $A(t_0)$ with radius ε . By our choice of ε , all the m_r eigenvalues of $A(t)$ are contained in R_r (i.e., not in other regions) for all $t \in [t_0, t_1]$.

Because $t_2 - t_1 < \delta$, for any $t \in [t_1, t_2]$, $A(t)$ has exactly m_r eigenvalues that are close (with respect to ε) to the m_r eigenvalues of $A(t_1)$ in R_r . Again, by our choice of ε , all these m_r eigenvalues of $A(t)$ are also contained in R_r for all $t \in [t_1, t_2]$.

Repeating the arguments for $[t_2, t_3], \dots, [t_{N-1}, t_N]$, we see that $A(t)$ has exactly m_r eigenvalues in R_r for each $t \in [0, 1]$. Thus, $A(t_N) = A(1) = A$ has exactly m_r eigenvalues in the region R_r . \square

5. A proof of Geršgorin theorem using the argument principle. The Geršgorin disk theorem is a statement about counting eigenvalues according to their algebraic multiplicities; it is essentially about counting zeros of a polynomial that depends on a parameter. Thus, Rouché's theorem would be a much more natural and effective tool since it focuses squarely on what the theorem says about numbers of eigenvalues. This approach does not require the parameter t to be real and it does not need the concept of any eigenvalue continuity, functional or topological.

There is a short and neat proof of the second part of the Geršgorin disk theorem that uses the argument principle. This approach was adopted in the second edition of Horn and Johnson's book *Matrix Analysis* [13, p. 389] (see also [25, p. 103]), while the proof by continuity used in the first edition [12, p. 345] was abandoned.

Let Γ be a simple contour in the complex plane that surrounds the Geršgorin region to be considered. Let $p_t(z)$ be the characteristic polynomial of $A(t)$ for each given $t \in [0, 1]$. By the argument principle [7, p. 123], the number of zeros (counted with algebraic multiplicities) of $p_t(z)$ inside Γ is

$$m(t) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{p'_t(z)}{p_t(z)} dz.$$

On the other hand, $f(t, z) := \frac{p'_t(z)}{p_t(z)}$ is a continuous function from $[0, 1] \times \Gamma$ to \mathbb{C} . By Leibniz's rule [7, p. 68], $m(t)$ is a continuous function on $[0, 1]$. As $m(t)$ is an integer, it has to be a constant. Thus, $m(0) = m(1)$, which is the number of eigenvalues of A in the Geršgorin region.

Similar ideas using Rouché's theorem or winding numbers have been employed in the study of localization for nonlinear eigenvalue problems [4, 5, 11]. Eigenvalues as functions deserve study and it is an interesting (and classical) problem. There is a well-developed theory on the smoothness of roots of polynomials (see [2], [15, Chap. II, §4], [16], and [19]).

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