ESSAYS IN FINANCIAL ECONOMETRICS

A Dissertation

by

DAE HEE JEONG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2009

Major Subject: Economics

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Approved by:

Chair of Committee,	Joon Y. Park
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ABSTRACT

Essays in Financial Econometrics. (August 2009)

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I consider continuous time asset pricing models with stochastic differential utility incorporating decision makers' concern with ambiguity on true probability measure. In order to identify and estimate key parameters in the models, I use a novel econometric methodology developed recently by Park (2008) for the statistical inference on continuous time conditional mean models. The methodology only imposes the condition that the pricing error is a continuous martingale to achieve identification, and obtain consistent and asymptotically normal estimates of the unknown parameters. Under a representative agent setting, I empirically evaluate alternative preference specifications including a multiple-prior recursive utility. My empirical findings are summarized as follows: Relative risk aversion is estimated around 1.5-5.5 with ambiguity aversion and 6-14 without ambiguity aversion. Related, the estimated ambiguity aversion is both economically and statistically significant and including the ambiguity aversion clearly lowers relative risk aversion. The elasticity of intertemporal substitution (EIS) is higher than 1, around 1.3-22 with ambiguity aversion, and quite high without ambiguity aversion. The identification of EIS appears to be fairly weak, as observed by many previous authors, though other aspects of my empirical results seem quite robust.

Next, I develop an approach to test for martingale in a continuous time framework. The approach yields various test statistics that are consistent against a wide class of nonmartingale semimartingales. A novel aspect of my approach is to use a time change defined by the inverse of the quadratic variation of a semimartingale, which is to be tested for the martingale hypothesis. With the time change, a continuous semimartingale reduces to Brownian motion if and only if it is a continuous martingale. This follows immediately from the celebrated theorem by Dambis, Dubins and Schwarz. For the test of martingale, I may therefore see if the given process becomes Brownian motion after the time change. I use several existing tests for multivariate normality to test whether the time changed process is indeed Brownian motion. I provide asymptotic theories for my test statistics, on the assumption that the sampling interval decreases, as well as the time horizon expands. The stationarity of the underlying process is not assumed, so that my results are applicable also to nonstationary processes. A Monte-Carlo study shows that our tests perform very well for a wide range of realistic alternatives and have superior power than other discrete time tests.

To Lillian and Sunyoung

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CHAPTER I

INTRODUCTION

Recently, continuous time models have drawn much attention in economics and finance due to their mathematical elegance and tractability. Meanwhile, it also has long been customary to use Euler approximation and standard discrete time methods to evaluate the continuous time models. However, it is well known that the continuous time models are well approximated only with small time interval for which the errors in the disceretized model are predominantly large compared to the conditional mean. Moreover, the volatility in the error is nonlinear and nonstationary process in general, which is not very attractive to analyze in the standard discrete time framework.

My approach is based on a time change defined as the inverse of the quadratic variation of the underlying continuous time process, which I assume to be a general semimartingale whose martingale component is a.s. continuous. Under my setting, the underlying stochastic process becomes a martingale if and only if it becomes Brownian motion after the time change by the celebrated theorem by Dambis (1965), Dubins and Schwarz (1965). Using the time change, I develop a new method to test and estimate continuous time models. My approach based on time change enables us to observe a new set of samples with homogeneous volatility, and moreover, it delivers a nice and simple restrictions on the distribution of the samples.

In Chapter II, I estimate a continuous time asset pricing model with stochastic differential utility incorporating decision makers' concern with ambiguity on true probability measure. Our new time change estimation method provides us a way to gauge the empirical performances of the asset pricing models, and moreover, it helps us to identify one of the most interesting concept in modern asset pricing theory, the ambiguity. In Chapter III , I consider a test for martingale hypothesis in continuous

This dissertation follows the style of *Econometrica*.

time based on the time change framework. I suggest four classes of test statistics based on the time change and show that our tests perform very well for a wide range of realistic specifications, and have superior power than other discrete time tests.

CHAPTER II

DOES AMBIGUITY MATTER? ESTIMATING ASSET PRICING MODELS WITH A MULTIPLE-PRIORS RECURSIVE UTILITY

A. Introduction

In this paper, we empirically investigate continuous time asset pricing models with recursive preferences incorporating decision makers' concern with ambiguity on true probability measure. Our measure of ambiguity aversion following Chen and Epstein (2002) is explicitly derived by exploiting some properties of continuous time diffusion processes and measuring instantaneous conditional volatilities of asset returns plays an important role in quantifying uncertainty premium rewarded by financial markets. Alas, the theoretical model adopted here sets the pricing kernel via the intertemporal marginal rate of substitution of a representative agent, and identifying this kernel involves using macroeconomic variables such as consumption growth, which are available only at lower frequencies. Consequently, we need to handle both financial data sampled at a high frequency and macroeconomic data available only at lower frequencies in order to estimate asset pricing models written in continuous time. This motivates us to develop a new set of econometric tools for the statistical inference on continuous time conditional mean models when available data series are of mixed frequencies. With this method in hand,we quantify the extent to which financial markets price ambiguity, risk, and intertemporal substitutability.

Since the seminal papers by Hansen and Singleton (1982) and Mehra and Prescott (1985), a large body of work has sought after more relevant forms of the preferences of economic agents to explain asset market behaviors. The main reason for this direction of the study is because time-separable expected utility functions equipped with a constant relative risk aversion (CRRA) impose a potentially restrictive relationship between the risk aversion and intertemporal substitution. Specifically, under power utility models, the elasticity of intertemporal substitution (EIS) is given by the reciprocal of the coefficient of relative risk aversion, which may result in various complications, such as equity premium, volatility and interest rate puzzles. Epstein and Zin (1989, 1991) investigated an important generalization of the standard power utility model by considering a class of recursive utility functions.[1] They provide a theoretical framework in which the agent can have distinct attitudes toward intertemporal substitution and risk. This flexibility may offer a possible solution for various asset price anomalies because a high (low) risk aversion does not necessarily imply a low (high) elasticity of intertemporal substitution.[2]

Even though the recursive utility models allow the distinction between risk aversion and willingness to substitute intertemporally, the preference toward Knightian uncertainty or ambiguity is difficult to model within the original recursive utility framework due to the assumption of single prior held by investors. However, the Ellsberg paradox suggests that decision makers prefer an unambiguous situation, other things being equal. In response to this, Gilboa and Schmeidler (1989) built a multiple-priors model to incorporate ambiguity aversion in an atemporal setting.[3]

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[1]The basic structure of recursive utility is due to Koopmans (1960) and Lucas and Stokey (1984), which decompose a utility function into current consumption and future utility in a non-linear fashion. In this context, Epstein and Zin (1989) can be regarded as a stochastic extension of the recursive utility framework.

[2]In addition, this preference has a preference ordering for temporal resolution of uncertainty. Recently, Bansal and Yaron (2004) and Hansen, Heaton, and Li (2008) exploit this aspect to explain equity premium puzzle together with a time-varying, conditional mean component. Kim, Lee, Park, and Yeo (2008) develop a stochastic volatility model with two asymptotic regimes and transition regimes and show that this type of preference can explain aversion to uncertainties in regimes.

[3]Simply put, they assume that economic agents have a class of probability distribu-

Epstein and Wang (1994) develop a dynamic version of Gilboa and Schmeidler in a discrete-time framework and Epstein and Schneider (2003) provide axiomatic foundations for recursive multiple-priors utility. Chen and Epstein (2002) focused on the formulation of utility in continuous time that allows a distinction between risk aversion and ambiguity aversion, as well as the distinction from the EIS. In order to achieve the additional dimension of flexibility, they extended the continuous time version of the recursive utility (stochastic differential utility) investigated by Duffie and Epstein (1992a, 1992b), such that the model includes a set of priors rather than a single prior. According to Chen and Epstein (2002), the economic agents will have multiple prior beliefs on the state of the nature, and they form a set of expectations based on their beliefs. Due to the fact that fundamental shock processes are generated by Brownian motion, the degree of ambiguity is described by an additional term distorting the conditional mean component of the implied asset return processes and the decision maker chooses a probability measure using the maxmin principle following Gilboa and Schmeidler (1989).[4]

Despite the appealing features of the multiple-priors recursive utility model, there has been little econometric work on estimating the model compared to other utility specifications. To the best of our knowledge, this is the first paper that empirically tackles this issue under the framework of consumption-based models. The multiple-priors recursive utility model has a multi-factor beta representation of asset returns; (i) covariance between returns and consumption growth, (ii) covariance between re-

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tions, say $\mathcal{P}$ on some events in a measurable space $(\Omega, \mathcal{F})$. Then the agents will make decisions following a max-min rule. For instance, if the agent decides consumption c to maximize utility $u(c)$, she solves $\max_{c} \min_{Q\in\mathcal{P}} E^{Q}[u(c)]$

[4]There exists a related line of work on robust decision making. Hansen and Sargent (2001) and their co-authors emphasize 'model uncertainty' and the concern on the misspecification, which is similar in spirit to ambiguity aversion à la Gilboa and Schmeidler.

turns and aggregate wealth return, and (iii) covariance between returns and ambiguity.[5] However, this structure makes identification of the model difficult because aggregate wealth and volatility of returns are unobservable latent variables and more notably, there is a lack of econometric methodology for estimating continuous time models. Below, we briefly explain how we overcome these issues.

With regard to the unobservability of aggregate wealth, several approaches have been suggested. The baseline approach would be to use a market return as a proxy for the returns on aggregate wealth (e.g., Epstein and Zin (1991), Bakshi and Naka (1997) and Normandin and St-Amour (1998)). However, the aggregate wealth portfolio should be a broader measure than the financial market portfolio, because the former includes human capital, natural resources, and housing wealth etc. as well as the financial wealth. Therefore, the market return only covers a subset of the aggregate wealth returns. Another approach is to use a specific structure for the unobservable wealth by incorporating the dynamics of consumption growth and utility continuation value (e.g., Chen et al. (2008)) Given the imposed structure, the aggregate wealth is implicitly given by consumption and utility continuation value. Therefore, this approach enables them to replace the unobservable wealth return with the specific structure imposed on the consumption and the future utilities. Chen et al. (2008) exploit the Euler equation to estimate future continuation utility in a non-parametric way.

Although this method is attractive, it is difficult to use in our continuous-time framework handling mixed frequencies of data. Instead, we consider a different approach to overcome the difficulties from the unobservable aggregate wealth. The

[5]Note that this representation, especially in closed form is available only in continuous time due to Girsanov transformation which allows different subjective probability measures to be expressed via tilting the drift component in an equilibrium asset pricing equation.

aggregate wealth return is a return on the claim which gives a stream of future consumption. In this sense, the consumption of each period is financed by the aggregate wealth return, and therefore, we can think of the aggregate wealth as the sum of financial wealth and human capital, which are the two largest sources of the income in an economy. That is, the unobservability of aggregate wealth falls mostly on the human wealth. Following Campbell (1993), we assume that the proportion of the financial wealth to the human wealth is stationary, and moreover, the labor income is homogeneous of degree one with respect to the human wealth. In this case, the unobservable wealth can be substituted by a linear combination of market return and labor income growth. This simple structure makes the asset pricing formula tractable so that we can directly compare the results of alternative models.

For an empirical analysis of our model, we use the martingale regression method recently developed by Park (2008) for inference on continuous time conditional mean models. It identifies the true parameter value simply by imposing the martingale condition for pricing error, utilizing the fact that the conditional expectation of pricing error is zero for the true parameter value, whereas it is generally non-zero for other values of parameter in the pricing equation. The spirit of the methodology is therefore somewhat similar to the GMM estimation for the nonlinear Euler equation models (e.g. Hansen and Singleton (1982)). The martingale condition for pricing error can easily be handled, since any continuous martingale can be converted into Brownian motion due to the celebrated theorem by Dambis, Dubins and Schwarz. The theorem states that any continuous martingale becomes Brownian motion if it is read after time change defined by the generalized inverse of its quadratic variation. The actual martingale estimator is defined as a minimum distance estimator based on the discrepancy between the empirical distribution of normalized pricing errors after time change and the standard normal distribution.

There are several attractive features of our approach using the martingale estimation. First, the martingale estimation does not require any parametric specification of volatilities in pricing errors. However, it allows for, and is robust with respect to, the presence of a wide variety of both deterministic and stochastic volatilities in pricing errors. This is an important advantage, since many empirical researches on the financial data find strong evidences that stock returns possess time-varying and potentially stochastic volatilities, while the exact natures of these volatilities are difficult to specify more precisely. Second, the martingale estimation does not use the orthogonality condition to identify the true parameters. Instead, it only imposes the martingale condition for pricing errors, and subsequently uses the time change theorem by Dambis, Dubins and Schwarz to make the condition implementable in defining the martingale estimator. Unlike the GMM estimator, we do not need instruments to compute the martingale estimator. Yet, the martingale estimator naturally accommodates endogeneity, and it is free of any kind of endogeneity problem.

Last but not least, this method allows applied econometricians to directly tackle asset pricing models formulated in continuous time. Many asset pricing models are developed in continuous time partly because of its mathematical elegance and tractability. However, it is also true that continuous time models give better descriptions for many financial markets, which clear at very high frequencies. Choosing an empirical model to fit at a most relevant frequency will certainly reduce the possibility of data missaggregation bias and decision bias. As metioned earlier, however, macro variables are only observed at lower frequencies. Therefore, we have to deal with mixed frequencies of data in models involving both macro and financial variables. The martingale method provides an effective solution to this problem. The observations on financial variables at high frequencies are used to identify the model and also to nonparametrically correct for volatilities in pricing errors and after identification and volatility correction, the model is estimated by observations sampled at random intervals of lower frequencies.

Using daily data on asset returns and monthly and quarterly macroeconomic data from 1960 to 2006, we estimate several specifications of recursive utility framework. According to our results, the estimates of ambiguity aversion is both economically and statistically significant. This is a highly robust feature of the data and the estimates are almost invariant to specifications. In addition, relative risk aversion is estimated around 1.5-5.5 with ambiguity aversion and 6-14 without ambiguity aversion. That is, the ambiguity aversion lowers the estimates of the relative risk aversion in all cases we have considered. Suppose investors receive information from stock prices which may include some noisy signals. If those signals are hard to interpret and hence difficult to extract fundamentals, they would prefer an asset market with less ambiguous information flows and request premiums for bearing such uncertainty different from risk, in a Knightian sense. Given that, our empirical results suggest that risk aversion parameter can have an upward bias sans an adjustment for ambiguity aversion to account for high average market returns.

Another important preference parameter is the elasticity of intertemporal substitution (EIS). Recently, estimating the EIS has drawn much attention and existing studies report a wide range of values including even negative numbers. According to our estimations, the EIS is higher than 1; specifically 1.3-22 with ambiguity aversion, and quite high without ambiguity aversion. We find that values of the objective function of our minimum distance estimator measured by the Cramer-von Mises statistic is very flat around the values of the reciprocal of the EIS between 0 and 3, meaning that the EIS may be observed in a wide range between a number close to 0 and a very large positive number. Based on extensive robustness checks, we argue that the weak identification issue of the EIS parameter results from the combination of smooth variations of consumption growth and parametric restrictions imposed in preferences. One notable finding is that EIS estimations become tighter when the ambiguity aversion is incorporated and this result is robust to alternative specifications.

The remainder of the paper begins with describing our theoretical model in Section 2. For comparison, we also consider other baseline models, which can be considered as special cases of our model. Section 3 accounts for the theoretical underpinnings of our econometric methodologies Section 4 describes our empirical procedure and resulting measurements necessary for our empirical evaluation. Section 5 shows and discusses our main results. Then we conclude in Section 6.

B. A Recursive Utility Model with Ambiguity Aversion

Consider a probability space $(\Omega, \mathcal{F}, P)$ which describes the uncertain nature of the economy. Define a standard one dimensional Brownian motion (W_t) on $(\Omega, \mathcal{F}, P)$, and the Brownian filtration $(\mathcal{F}_t)_{0\leq t\leq T}$, where $\mathcal{F}_t$ is the σ-field generated by $(W_s)_{s\leq t}$. The time horizon $[0,T]$, where T is finite. Suppose that the representative decision maker does not know the true probability measure and has to choose a subjective probability measure from the set of all priors $\mathcal{P}$, which are uniformly absolutely continuous with respect to the true P in $\mathcal{P}$.[6]. Duffie and Epstein (1992a) show that for a fixed consumption process C and a probability measure $Q \in \mathcal{P}$, there exists a utility process V_t^Q uniquely solving

$$V_t^Q = \mathbb{E}^Q\left[\int_t^T f(C_s, V_s^Q)ds \middle| \mathcal{F}_t\right], \quad 0 \leq t \leq T, \tag{2.1}$$

where $\mathbb{E}^Q(\cdot|\mathcal{F}_t)$ is the conditional expectation operator and $f(C,V)$ is called a normalized aggregator function linking current consumption and the future value. From

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[6] $\mathcal{P}$ is uniformly absolutely continuous with respect to P if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $E \in \mathcal{F}$ and $P(E) < \delta$ imply $Q(E) < \varepsilon$, for all $Q \in P$.

the martingale representation theorem, we can express (2.1) in a differential form of

$$dV_t^Q = -f(C_t, V_t^Q)dt + \sigma_t^v dW_t^Q, \tag{2.2}$$

where $V_T^Q = 0$, (W_t^Q) is the standard Brownian motion under Q-measure, and σ_t^v is endogenously determined.

From now on, we use the functional form

$$f(C, V) = \frac{C^{1-\beta} - \phi(\alpha V)^{\frac{1-\beta}{\alpha}}}{(1-\beta)(\alpha V)^{\frac{1-\beta}{\alpha}}} \tag{2.3}$$

for some $\phi \geq 0$, $\beta \neq 1$, $\alpha \leq 1$. This can be regarded as the continuous-time version of Kreps and Porteus (1978) utility function in which α and β measure the degree of relative risk aversion (RRA) and the elasticity of intertemporal substitution (EIS) respectively. Specifically, the RRA is measured by $(1-\alpha)$, and the EIS is $1/\beta$. In addition, following Epstein and Zin (1989), relative sizes of these two measures are related to the investor's attitude toward the speed of resolving uncertainty: If the RRA $(1-\alpha)$ is larger (smaller) than the reciprocal of the EIS (β), the investor prefers an early (a late) resolution of uncertainty. The additional feature of this model compared to the conventional recursive utility model is that the consumer chooses a probability measure from available priors, which justifies the name, 'multiple-priors utility'. Under this extra layer of uncertainty which leads to the Ellsberg paradox, Gilboa and Schmeidler (1989) suggested the following minimax type of value function

$$V_t = \min_{Q \in \mathcal{P}} V_t^Q, \quad 0 \leq t \leq T. \tag{2.4}$$

The multiple-priors recursive utility is given by the lower envelope of the utility process (V_t^Q) which is determined by the conditional expectation of future consumption and utility values. Chen and Epstein (2002) showed that there exists a unique solution to (2.4) satisfying the dynamic consistency under certain conditions.[7] As clearly seen from (2.2), the Girsanov transformation lies at the heart of constructing a set of priors $\mathcal{P}$ on $(\Omega, \mathcal{F}_T)$. Specifically, they define a density generator (λ_t) for which the process (z_t^λ) is a P-martingale, where

$$dz_t^\lambda = -z_t^\lambda \lambda_t dW_t, \quad z_0^\lambda = 1,$$

or equivalently,

$$z_t^\lambda \equiv \exp\left(-\frac{1}{2}\int_0^t \lambda_s^2 ds - \int_0^t \lambda_s dW_s\right), \quad 0 \leq t \leq T.$$

Then, (z_t^λ) is set as the Radon-Nikodym derivative dQ/dP on $(\mathcal{F}_t)$ and $\mathcal{P}$ is defined as the set of Q-measures produced by the density generator. We assume that the Novikov condition holds to suffice the existence of such (λ_t). Since all the prior beliefs are absolutely continuous with P, we can expect from the Girsanov's theorem that any subjective utility (V_t^Q) given an equivalent measure $Q \in \mathcal{P}$ will modify the drift function of the utility continuation process by $(\lambda_t \sigma_t^v)$. This is because (W_t) is the Brownian motion under P measure, but not under Q. That is, by shifting (λ_t), we can generate a continuum of subjective utility functions differing in terms of probability distribution within the class of absolutely continuous multiple-priors. Chen and Epstein (2002) showed that the differential form of (2.4) is

$$dV_t = -\left\{f(C_t, V_t) + \max_{\lambda_t \in \mathcal{L}} \lambda_t \sigma_t^v\right\} dt + \sigma_t^v dW_t, \tag{2.5}$$

where $\mathcal{L}$ is to be defined in short.

To further analyze the additional term in (2.5), assume that (λ_t) is bounded by

[7]Dynamic consistency in this paper is defined in the following sense. If two consumption plans c and c' are the same up to a stopping time τ, and the value V_τ of c is weakly preferred to that of c' at τ almost surely, then $V_0(c) \geq V_0(c')$ almost surely with a strict inequality in the case that $P(V_\tau(c) > V_\tau(c')) > 0$ holds.

some constant $\kappa > 0$. This makes the domain of (λ_t) defined as

$$\mathcal{L} = \{(\lambda_t) : \sup\{|\lambda_t| : 0 \leq t \leq T\} \leq \kappa\}.$$

That is, the subjective beliefs have some boundary defined by a constant κ. We can interpret the multiple priors as the subjective beliefs for which the worst case scenario of the economic agents is confined by the case defined by κ. Hereafter, we examine the multiple-priors model with a boundary restriction for (λ_t) with $\kappa > 0$.[8]

Under the standard environment of the economy, first order conditions for optimal consumption choice can be expressed in terms of the supergradient of utility at the optimal consumption C.[9] Especially, C is optimal if

$$\Lambda_t = \exp\left\{\int_0^t f_v(C_s, V_s)ds\right\} f_c(C_t, V_t) z_t^{\lambda^*}, \qquad \text{for all } t, \tag{2.6}$$

where (Λ_t) is the state-price process (or intertemporal marginal rate of substitution process, IMRS) and (λ_t^*) is the maximizer of the ambiguity compensation $(\lambda_t \sigma_t^v)$ for any given (λ_t) such that $|\lambda_t| \leq \kappa$ for all $0 \leq t \leq T$. Then the IMRS in our case is given as

$$\Lambda_t = \exp\left\{\int_0^t \left(-\phi + \frac{\phi}{1-\beta} \frac{(\phi - 1 - \beta)(C_s^{(1-\beta)} - (\alpha V_s)^{(1-\beta)/\alpha}}{(\alpha V_s)^{(1-\beta)/\alpha}}\right) ds\right\} \phi C_t^{-\beta} (\alpha V_t)^{\beta/\alpha} z_t^{\kappa}.$$

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[8] Chen and Epstein (2002) call this specification "κ-ignorance" case. This is closely related to the condition guaranteeing the dynamic consistency, called rectangularity. For a nice interpretation on the rectangularity condition and dynamic consistency, see Epstein and Schneider (2003).

[9] A supergradient for V at C is a process (Λ_t) with $\mathbb{E}\left\{\int_0^T \Lambda_t \cdot (C_t' - C_t)dt\right\} \geq V(C') - V(C)$ for all admissible C'. For more details, see Duffie and Skiadas (1994) and Chen and Epstein (2002).

Using Ito's lemma and no arbitrage principle, we can show that, for an asset i,

$$\begin{aligned}\frac{dp_t^i}{p_t^i} - r_t^f dt &= \mathbb{E}\left(\frac{dp_t^i}{p_t^i}\frac{d\Lambda_t}{\Lambda_t} \middle| \mathcal{F}_t \right) + \sigma_t^i dW_t \\ &= \left\{ \frac{\beta\alpha}{1-\beta}\rho_c^i \sigma_t^c \sigma_t^i + \left(1 - \frac{\alpha}{1-\beta}\right) \rho_g^i \sigma_t^g \sigma_t^i + \kappa \sigma_t^i \right\} dt + \sigma_t^i dW_t,\end{aligned} \tag{2.7}$$

where (σ_t^g) is the volatility of aggregate wealth (G_t) for which the return is given by

$$dr_t^g = \mu_t^g dt + \sigma_t^g dW_t, \quad dG_t = G_t(dr_t^g) - C_t dt,$$

(σ_t^c) is the volatility of consumption growth, ρ_c^i and ρ_g^i are the correlation coefficients of consumption growth and aggregate wealth return with the return of an asset i, respectively.

Equation (2.7) is a three-factor CAPM of the cross-sectional asset pricing model; the risk premium of any tradable asset i with return (dp_t^i/p_t^i) is determined by the covariance between returns and consumption growth, covariance between returns and aggregate wealth, and covariance between returns and density generator. Notice that the standard CRRA utility specification, such as power utility, only has the first factor, while the single-prior recursive utility models (e.g. Epstein and Zin (1989, 1991) and Duffie and Epstein (1992a, 1992b)) have the first two factors.

In order to include unobservable wealth, we assume the wealth process (G_t) has two components - financial wealth (M_t) and human wealth (H_t),

$$G_t = M_t + H_t. \tag{2.8}$$

From Ito's lemma we have

$$\sigma_t^g = \sqrt{\pi_t^2 \left(\sigma_t^m\right)^2 + (1-\pi_t)^2 \left(\sigma_t^h\right)^2 + 2\rho_h \sigma_t^m \sigma_t^h}, \tag{2.9}$$

where $\pi_t = M_t/G_t$ is the proportion of financial wealth to the total wealth at time t,

(σ_t^g) and (σ_t^h) are the respective diffusion coefficient of (dM_t/M_t) and (dH_t/H_t), and ρ_h is the correlation coefficient between market return and labor income growth. In particular, we specify the human capital process by

$$dH_t = H_t(dr_t^h) - Y_t dt, \qquad dr_t^h = \mu_t^h dt + \sigma_t^h dW_t, \tag{2.10}$$

where Y_t is real labor income at time t. Note that the labor income Y_t is financed from the return on the human capital H_t at time t. From (2.9) and (2.10), the covariance between the aggregate wealth and individual return for an asset i is written as

$$\rho_g^i \sigma_t^g \sigma_t^i = (\pi_t \sigma_t^m + (1-\pi_t)\rho_h^i \sigma_t^h)\sigma_t^i$$

For simplicity, we assume that $\pi_t = \pi$ for all t. This is true, for instance, under the steady state of the economy, in which the proportion of aggregate wealth to the financial wealth is constant over time. Moreover, we assume that the labor income is homogeneous of degree one with respect to human capital, especially, $Y_t = \psi H_t$ for all t with some constant ψ. Given that we are interested in the market return's behavior, we set $i = m$, and therefore the fundamental asset pricing equation of the multiple-priors recursive utility model is expressed as

$$\begin{aligned} \frac{dp_t}{p_t} - r_t^f dt = {} & \frac{\beta\alpha}{1-\beta}\rho_c \sigma_t^c \sigma_t^m dt + \left(1 - \frac{\alpha}{1-\beta}\right)(\sigma_t^m)^2 \pi dt \\ & + \left(1 - \frac{\alpha}{1-\beta}\right)\rho_y \sigma_t^m \sigma_t^y (1-\pi) dt + \kappa \sigma_t^m dt + \sigma_t^m dW_t, \end{aligned} \tag{2.11}$$

where (p_t) is the price of the market index, (σ_t^y) is the instantaneous conditional volatility of labor income growth, and ρ_c, ρ_y are the correlation coefficients of consumption growth and labor income growth with the market return, respectively. From now on, we turn our attention to estimating the key preference parameters (α, β, κ) in (2.11).

Note that (2.11) nests many popular asset pricing models as special cases. Thus by imposing a priori restrictions to (2.11), we can estimate different models to compare the common set of parameters. First three models estimated are the power (CRRA) utility case (Model I), recursive utility with financial wealth only (Model II), and a multiple-priors recursive utility with financial wealth only (Model III). Specifically, we can express Model I, II, III as

$$\frac{dp_t}{p_t} - r_t^f dt = \beta\rho_c\sigma_t^c\sigma_t^m dt + \sigma_t^m dW_t, \qquad \text{(Model I)}$$

$$\frac{dp_t}{p_t} - r_t^f dt = \frac{\beta\alpha}{1-\beta}\rho_c\sigma_t^c\sigma_t^m dt + \left(1 - \frac{\alpha}{1-\beta}\right)(\sigma_t^m)^2 dt + \sigma_t^m dW_t, \qquad \text{(Model II)}$$

$$\frac{dp_t}{p_t} - r_t^f dt = \frac{\beta\alpha}{1-\beta}\rho_c\sigma_t^c\sigma_t^m dt + \left(1 - \frac{\alpha}{1-\beta}\right)(\sigma_t^m)^2 dt + \kappa\sigma_t^m dt + \sigma_t^m dW_t. \qquad \text{(Model III)}$$

As emphasized by many authors such as Campbell (1993), Bansal and Yaron (2004), and Lettau and Ludvigson (2001), financial wealth is insufficient to proxy the aggregate wealth of the representative investor. To address this issue, we include another source of risk premium resulting from labor income risk and this is the setup of (2.11). This is not a completely innocuous assumption because the fraction of human wealth to total wealth is assumed to be constant. However, it turns out that this restriction has a smaller order of effect affecting the empirical results according to our robustness checks.

One important observation from our empirical setting is that time-varying volatilities of macroeconomic variables and asset returns play key roles in both the conditional mean (drift) part and the error (diffusion) terms. Given the ample evidence that those volatilities are highly persistent, this makes identification of the models statistically challenging because of heteroskedasticity, endogeneity, and measurement problems. In addition, the equilibrium relationship (2.11) that continuously holds need to be properly treated for correct empirical evaluations with discretely sampled data points. In the below, we tackle those issues.

C. Econometric Methodology

1. Martingale Estimation of Asset Pricing Models in Continuous Time

Here we explain how to specify and estimate our model (2.11). Tentatively, we assume that the volatility processes $(\sigma_t^m), (\sigma_t^c)$ and (σ_t^y) are observed. In the next subsection, we will explain in detail how we may extract these processes. Moreover, we will set the correlation coefficients ρ_c and ρ_y of consumption and labor income growths with market returns, as well as the fraction π of financial wealth, to be known and constants.[10] These parameters will be calibrated using the values obtained or often assumed in the empirical literature. In what follows, we assume that $(\sigma_t^m), (\sigma_t^c)$ and (σ_t^y) are non-constant and time-varying, and that ρ_c and ρ_y are non-zero. These assumptions are necessary for the identification of our model.

Now we let $\theta = (\alpha, \beta, \kappa)$ be the vector parameters in our model with the true value $\theta_0 = (\alpha_0, \beta_0, \kappa_0)$, and define $(\Lambda_t(\theta))$ to be the state-price deflator (or IMRS) that is given by

$$\Lambda_t(\theta) = \frac{\beta\alpha}{1-\beta}\rho_c\sigma_t^c\sigma_t^m + \left(1 - \frac{\alpha}{1-\beta}\right)\left\{\pi\sigma_t^m + (1-\pi)\rho_y\sigma_t^y + \kappa\right\}\sigma_t^m. \qquad (2.12)$$

Subsequently, we define the pricing error process $(Z_t(\theta))$ from our model as

$$dZ_t(\theta) = \frac{dp_t}{p_t} - r_t^f dt - \Lambda_t(\theta)dt,$$

[10]This does not imply that the correlation between the aggregate wealth and market returns are constant. As shown above, it still varies over time.

and write

$$Z_t(\theta) = A_t(\theta) + U_t, \tag{2.13}$$

where $dA_t = -\{\Lambda_t(\theta) - \Lambda_t(\theta_0)\}dt$ and $dU_t = \sigma_t^m dW_t$.

It is clear that the pricing error process $(Z_t(\theta))$ is a semimartingale with the bounded variation component $(A_t(\theta))$ and the martingale component (U_t). Note in particular that (U_t) is a continuous martingale with respect to the filtration $(\mathcal{F}_t)$, to which the Brownian motion (W_t) is adapted. Furthermore, the bounded variation component $(A_t(\theta))$ vanishes if and only if $\theta = \theta_0$ under the trivial identification conditions introduced above.[11] Therefore, we may conclude that the pricing error process $(Z_t(\theta))$ becomes a continuous martingale if and only if $\theta = \theta_0$.[12]

Recently, Park (2008) developed a general methodology to estimate and test the continuous-time conditional mean model that is identified by this type of martingale condition for the error process. Below we explain how we can implement his methodology to estimate the unknown parameter θ in our model. The methodology relies on the celebrated theorem by Dambis, Dubins and Schwarz, which will be referred to the DDS theorem throughout the paper. To introduce the DDS theorem, we denote by $([U]_t)$ the quadratic variation of (U_t), which is given by

$$[U]_t = \operatorname*{plim}_{|t_k|_t \to 0} \sum_k (U_{t_k} - U_{t_{k-1}})^2,$$

where $|t_k|$ is the mesh of partition (t_k) of the interval $[0, t]$. We assume that $[U]_t \to \infty$

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[11]As can be clearly seen, we may identify up to four unknown parameters in our model. Therefore, for instance, we may regard π as unknown and estimate it as an additional unknown parameter. However, the estimate for π is unstable and unreliable.

[12]We temporarily assume that there is no jump in the pricing error process to focus on the main idea of the methodology. Indeed, it can be applied to the processes with jumps with some simple modifications, which we will explain later in this subsection.

a.s. as $t \to \infty$. Moreover, we introduce the time change (T_t), which is defined as

$$T_t = \inf\{s \geq 0 | [U]_s > t\}. \tag{2.14}$$

The DDS theorem says that if (U_t) is a continuous martingale, then there exists a standard Brownian motion B such that $U_t = B_{[U]_t}$, or equivalently,

$$U_{T_t} = B_t.$$

The Brownian motion B is called the DDS Brownian motion of U. See, e.g., Revuz and Yor (2005) for the proof and more discussions about the DDS theorem. In most applications, $([U]_t)$ is strictly increasing, in which case T is just the time inverse of $([U]_t)$. Roughly, the DDS theorem implies that if we read a continuous martingale using a clock that is running at a speed inversely proportional to its quadratic variation, it reduces to a Brownian motion.

If we apply the time change to the original pricing error process $(Z_t(\theta))$, then we may deduce from (2.13) that

$$Z_{T_t}(\theta) = A_{T_t}(\theta) + U_{T_t} = A_{T_t}(\theta) + B_t.$$

Therefore, we may now claim that $(Z_{T_t}(\theta))$ becomes the standard Brownian motion if and only if $\theta = \theta_0$, due to the DDS theorem. Obviously, the bounded variation component $(A_{T_t}(\theta))$, even after time change, vanishes when and only when $\theta = \theta_0$. The martingale method by Park (2008) uses this fact and defines the value of θ, which makes the time-changed pricing error process best approximate the standard Brownian motion, to be the martingale estimator of the unknown parameter θ_0. It is important to note that we may obtain the time change (T_t) without any knowledge on the true parameter value θ_0, since the bounded variation component contributes nothing to the quadratic variation of a semimartingale. Therefore, for instance, the quadratic variation ($[U]_t$) of the martingale component, which is required to get the time change (T_t), is identical to the quadratic variation of (P_t), say, $dP_t = dp_t/p_t - r_t^f dt$, i.e., $d[U]_t = d[P]_t = d[p]_t/p_t^2$.

To implement the methodology, we set $\Delta > 0$ to be fixed,[13] and consider the normalized increments of the pricing error process that are given by

$$z_i(\theta) = \frac{1}{\sqrt{\Delta}} \left\{ Z_{T_{i\Delta}}(\theta) - Z_{T_{(i-1)\Delta}}(\theta) \right\}$$

for $i = 1, \ldots, N$. The discrete samples ($z_i(\theta)$) of size N obtained for each $\theta \in \Theta$ are then used to estimate the unknown parameter θ_0. Recall that the samples are obtained from the pricing error processes as their increments over the random intervals $[T_{(i-1)\Delta}, T_{i\Delta}]$ for $i = 1, \ldots N$. It is quite clear that ($z_i(\theta)$) are i.i.d. normals for $\theta = \theta_0$, regardless of the choice of Δ. For all other values of $\theta \in \Theta$, this is not true at least for some value of Δ.

We let $z_i^d(\theta) = (z_i(\theta), \ldots, z_{i-d+1}(\theta))$ be the d-dimensional random vector consisting of d-adjacent samples starting from $i = 1, \ldots, N - d + 1$, so that ($z_i^d(\theta)$) is the d-dimensional standard multivariate random vector, i.e., the multivariate normal random vector with mean zero and identity covariance matrix, under $\theta = \theta_0$. Moreover, we denote by $\Phi_N(\cdot, \theta)$ the empirical distribution of ($z_i^d(\theta)$) for each $\theta \in \Theta$, and define the criterion function Q_N by

$$Q_N(\theta) = \int_{-\infty}^{\infty} \left\{ \Phi_N(x, \theta) - \Phi(x) \right\}^2 d\Phi(x),$$

where Φ is the distribution function of the d-dimensional multivariate standard normal random vector. The martingale estimator $\hat{\theta}_N$ of θ_0 is then defined as the minimizer

[13] The choice of Δ is more of an empirical matter, which we will discuss in detail later in our empirical section.

of the criterion function Q_N, i.e.,

$$\hat{\theta}_N = \operatorname*{argmin}_{\theta \in \Theta} Q_N(\theta).$$

The martingale estimator is therefore a minimum-distance estimator with the Cramer-von Mises (CvM) distance between the empirical distribution of the sample under the unknown parameter values and the distribution under the true parameter values. Park (2008) shows that this type of minimum distance estimator is consistent, and asymptotically normal, under mild regularity conditions. The asymptotic variance of the estimator can be obtained by the usual subsampling method.

To introduce the main idea of the methodology more effectively, we assume thus far that the pricing error process $(Z_t(\theta))$ is observed continuously in time for all $\theta \in \Theta$. This, of course, is not true in our analysis, as is the case for virtually all other potential applications. The methodology can be easily implemented and all the theoretical results continue to hold for discretely sampled observations, as long as the sampling intervals are sufficiently small relative to the time horizon of the samples. This was shown in Park (2008). For our empirical analysis, we use daily observations over approximately fifty years. The necessary modifications required to deal with discretely observed samples are largely trivial and obvious. To obtain the time change, for instance, we use the realized variance of (P_t), $dP_t = dp_t/p_t - r_t^f dt$, given by

$$[P]_t^\delta = \sum_{i\delta \leq t} (P_{i\delta} - P_{(i-1)\delta})^2,$$

instead of its quadratic variation $([P]_t)$, if (P_t) is observed at intervals of length $\delta > 0$ over time horizon $[0, T]$ with $T = n\delta$, where n is the size of discrete samples.

Finally, we may readily allow for the existence of jump components in our model (2.11). Indeed, we may easily deal with the presence of discrete jumps in our methodology, simply by discarding the observations of (P_t), $dP_t = dp_t/p_t - r_t^f dt$, over the random time interval $[T_{(i-1)\Delta}, T_{i\Delta}]$ that is believed to have jumps. All other procedures in our methodology are valid for the remaining observations. In our empirical studies, we use the Hausman-type test of Barndorff-Nielsen and Shephard (2006) for the detection of jumps for each of the random intervals $[T_{(i-1)\Delta}, T_{i\Delta}]$, $i = 1, \ldots, N$. Although it is well-known that the jumps are frequently observed for many intra-day samples, it appears that jumps are rare for the samples of daily or lower frequency observations. We detected some evidence of jumps in our daily observations, but their number is relatively small.

2. Measuring Volatilities of Macroeconomic Variables

Now we explain how to extract the volatility processes (σ_t^c) and (σ_t^y). It is much more challenging than to extract the volatility process (σ_t^m), since the observations on their underlying processes are available at relative low frequencies like many other macroeconomic variables. As we explained in the previous subsection, (σ_t^m) can be readily measured and estimated by the realized variance of market returns at high frequencies.[14] However, the identification and estimation of volatility for the processes that are not observed at high frequencies are not straightforward. In the paper, we directly tackle this issue in the following way. First we let the underlying process (X_t) follow an Ito-diffusion

$$\frac{dX_t}{X_t} = \mu_t dt + \sigma_t dB_t,$$

where (B_t) is the standard Brownian motion, and consider the problem of estimating (σ_t), $\sigma_t = \sigma_t^c$ or σ_t^y, under some realistic assumptions, using discrete samples (X_{t_j}) of

[14]See, e.g., Barndorff-Nielsen and Shephard (2002) for more discussions on the estimation of volatility processes using high-frequency data.

(X_t). It is assumed in our setup here that the sampling intervals $t_j - t_{j-1}$, $j = 1, \ldots, m$, are not sufficiently small.

Over the interval $[t_{j-1}, t_j]$, we have

$$\int_{t_{j-1}}^{t_j} \frac{dX_t}{X_t} = \int_{t_{j-1}}^{t_j} \mu_t dt + \int_{t_{j-1}}^{t_j} \sigma_t dB_t. \tag{2.15}$$

For many macroeconomic variables, the values of the level X_t are relatively much larger than its increment $X_{t_j} - X_{t_{j-1}}$ in any of the intervals $[t_{j-1}, t_j]$ of frequency such as monthly and quarterly. Therefore, it seems reasonable to approximate $\int_{t_{j-1}}^{t_j} dX_t/X_t$ by $(X_{t_j} - X_{t_{j-1}})/X_{t_{j-1}}$, i.e., the growth rate of (X_t) over the interval $[t_{j-1}, t_j]$, for $j = 1, \ldots, m$.[15] Moreover, if we assume the drift term (μ_t) is continuous, then there exists $s_j \in [t_{j-1}, t_j]$ such that $\mu_{s_j}(t_j - t_{j-1}) = \int_{t_{j-1}}^{t_j} \mu_t dt$ for all $j = 1, \ldots, m$, by the mean value theorem. If, furthermore, (μ_t) varies smoothly over time, then we may approximate (μ_{s_j}) by (μ_{t_j}). This appears to be realistic in our case, so we assume that (μ_t) is an exogenous function of time for which these approximations are valid. Given the assumption, the drift term (μ_t) can be consistently estimated by the standard nonparametric method applied to (2.15). We adopted the local linear estimation, using the least squares cross-validation method to obtain the optimal bandwidth parameter. The reader is referred to Li and Racine (2007, p.83) for more details.

We exploit two different approaches to extract the volatility process (σ_t^2). First, we consider

$$\left(\int_{t_{j-1}}^{t_j} \frac{dX_t}{X_t} - \int_{t_{j-1}}^{t_j} \mu_t dt \right)^2 = \int_{t_{j-1}}^{t_j} \sigma_t^2 dt + \left\{ \left(\int_{t_{j-1}}^{t_j} \sigma_t dB_t \right)^2 - \int_{t_{j-1}}^{t_j} \sigma_t^2 dt \right\}, \tag{2.16}$$

the left-hand side of which we may approximate well using discrete observations (X_{t_j})

[15] Note that the approximation error is given by $\int_{t_{j-1}}^{t_j} (X_t - X_{t_{j-1}})/(X_t X_{t_{j-1}}) dX_t$ and $(X_t - X_{t_{j-1}})/(X_t X_{t_{j-1}}) \approx 0$ for many macroeconomic variables including those we consider here.

of (X_t) as explained above. Note that

$$\mathbb{E}\left\{\left(\int_{t_{j-1}}^{t_j}\sigma_t dB_t\right)^2 - \int_{t_{j-1}}^{t_j}\sigma_t^2 dt\,\middle|\,\mathcal{F}_{t_{j-1}}\right\} = 0$$

for $j = 1, \ldots, m$.

As with the drift term (μ_t), we may regard the diffusion term (σ_t) as an exogenous function of time varying smoothly over intervals $[t_{j-1}, t_j]$ for all $j = 1, \ldots, m$. In this case, we may approximate in (2.16)

$$\int_{t_{j-1}}^{t_j}\sigma_t^2 dt = \sigma_{s_j}^2(t_j - t_{j-1}) \approx \sigma_{t_j}^2(t_j - t_{j-1}),$$

where $s_j \in [t_{j-1}, t_j]$, $j = 1, \ldots, m$, and the volatility process (σ_t) can be estimated by the standard nonparametric method such as the local linear estimation. We use this approach to extract the volatility processes (σ_t^c) and (σ_t^y), again with the optimal choice of bandwidth based on the least squares cross-validation. A potential caveat of this nonparametric method would be that this method may produce overly smooth volatility factors. We discuss more on this point in detail in our empirical section.

Second, we suppose that the volatility process is stochastic with an additional source of randomness. For this approach, we let the volatility process (σ_t) be random, but remain to be constant over each of the intervals $[t_{j-1}, t_j]$, $j = 1, \ldots, m$. More specifically, we set

$$\int_{t_{j-1}}^{t_j}\sigma_t dB_t = \sigma_j(B_{t_j} - B_{t_{j-1}}) \tag{2.17}$$

and (σ_j^2) to be driven by the logistic transformation of a latent autoregressive factor (w_j), i.e.,

$$\sigma_j^2 = a + \frac{b}{1 + \exp\{-c(w_j - d)\}}$$

with $w_j = \rho w_{j-1} + \varepsilon_j$, where (ε_j) is assumed to be an i.i.d. sequence of standard normals. Note that (2.17) is the standard Gaussian volatility model in discrete time.

We let (ε_j) be correlated with the Brownian motion (B_t) to allow for the leverage effect. The model parameters $a > 0, b > 0, c > 0$ and d determine the actual volatility function. In particular, a and $a + b$ represent the two asymptotic values of volatility, and c and d respectively the speed and location of transition.

The volatility model introduced above was developed and investigated recently by Kim, Lee and Park (2008). The model can be regarded as an extension of the usual discrete-time stochastic volatility model, which relies on the autoregressive modeling for the logarithmic transformation of volatility. The former is indeed much more flexible than the latter, and has implications that are much more realistic. The latent factor (w_j) and unknown parameters a, b, c and d can be estimated by the density-based Kalman filter, or by the Bayesian method using Gibbs sampling method. The reader is referred to Kim, Lee and Park (2008) for more details about the computation procedure and comparison with other existing discrete-time stochastic volatility models.

3. Calculating Covariances in Mixed Frequencies

Now that we have the extracted volatilities of the macroeconomic variables in their observation frequency, the calculation of the time changed covariances between the market and the macroeconomic variables are straightforward if we have the volatilities in the same frequency. However, the consumption volatility is estimated in monthly frequency and the labor income volatility in quarterly. Moreover, the market volatility σ_t^m is not estimated yet in any frequency. In this section, we describe a nonparametric interpolation method for the lower frequency macroeconomic volatilities into daily frequency, as well as, a simple way to calculate the market volatility in daily level.

In order to interpolate the daily level of volatilities from lower frequency data, we assume that the volatilities are characterized by a nonparametric function of time.

Furthermore, we assume that the extracted volatilities are realizations of the time-varying volatility functions at the mid-point of the observation interval. For instance, if the observation interval is monthly, we assume that the extracted volatilities are realizations of the volatility function at the fifteenth day of each month[16]. Based on the assumptions, the daily level of volatilities can be calculated by plugging in the time levels which correspond to the daily frequency. For instance, the corresponding daily time level for monthly interval is obtained by finding the grid infilling the monthly time interval at daily frequency. When applying the nonparametric method, we also use the local linear kernel with the smoothing parameter obtained from the least squares cross validation.

For the market volatility, we use the same idea. The difference from the macroeconomic volatilities is that the market volatility is observed in random time interval $[T_{(i-1)\Delta}, T_{i\Delta}]$. In this case, we apply the nonparametric interpolation by assuming that the estimated market volatility $\sqrt{([P]^{\delta}_{T_{i\Delta}} - [P]^{\delta}_{T_{(i-1)\Delta}})/(T_{i\Delta} - T_{(i-1)\Delta})}$ is a realization at the time $t = (T_{(i-1)\Delta} + T_{i\Delta})/2$. Finally, if the daily volatilities are obtained, we use the Riemann sum to calculate the time changed covariances.

D. Empirical Procedure and Measurement

1. Data

We use S&P500 index to calculate the market returns. The index is daily close price adjusted for the dividends and splits, and the returns are obtained by calculating the arithmetic returns of the daily close price. Once we have the daily series of the market returns, we calculate the daily excess returns of the market over the risk free rate of

[16]This mid-point assumption is not necessary and alternative point in the observation interval can be used, however it is almost impossible to identify the exact point where the mean value is reached from the lower frequency data.

return. For the risk free rate of return, three months treasury bill rates are used. The three months rates are adjusted to the daily level by dividing by 360. Since the daily series on the three months treasury bill rates can be considered as a risk free return from today to tomorrow, the daily excess return on the market portfolio is calculated by subtracting yesterday's treasury bill rate from today's return on the market.

We exclude the returns over weekends from our data set because the returns from Friday to Monday seem to have different distribution than the returns on the other day of the week. Especially, the returns on Mondays are significantly negative. This, so called the "Monday effect" or "weekends effect", has been widely investigated in the literature; see, for instance, French (1980), Lakonishok and Levi (1982) and Wang et. al. (1997) among others. Settlement effect and clearing delays, or expiration of stock options can be considered one of the possible explanations for the Monday effect, however it seems hard to include the daily seasonal effect in the asset pricing models that we considered in section 2. One might use the dummy variable for Mondays and proceed the analysis (e.g. Fortune (1999)), but in this paper we simply discard the Monday returns.

For the volatilities of macroeconomic variables, such as consumption and labor income, we use monthly real per capita consumption of nondurables plus services and quarterly real per capita labor income. The labor income is defined as wages and salaries plus transfer payments plus other labor income minus personal contributions for social insurance minus taxes, which is used in Lettau and Ludvigson (2001).

The S&P 500 index is obtained from Yahoo finance web site, and the consumption data is obtained from the Bureau of Economic Analysis. The three month treasury bill rate is from the Federal Reserve Bank of St. Louis, and the labor income from Martin Lettau's web site. The data set covers from January 1, 1960 to December 29, 2006. The summary statistics for the market, risk-free, consumption, and labor income are presented in Table I.

Table I. Summary Statistics

	Market	Risk Free	Consumption	Labor Income
Mean*	0.12839	0.05551	0.02168	0.02309
Std. Dev.*	0.13973	0.00146	0.01243	0.01713
Skewness	0.25178	1.12173	-0.04598	0.43410
Kurtosis	7.96404	4.93917	3.94341	6.65616
Auto. Coef.	0.04900	0.99910	-0.23070	-0.06390

Note: Summary statistics for market, risk-free, consumption, and labor income. Market is the daily S&P 500 index returns, risk-free is the daily three month treasury bill rate divided by 360, consumption is the monthly growth rate of real per capita non-durable plus service consumption, and labor is the quarterly growth rate of labor income defined in Lettau and Ludvigson (2001). *For the purpose of presentation, we report the annualized mean and standard deviation of the series.

2. Implementation of Time Change

Our martingale estimation framework enables us to observe the market returns in the volatility time, not in the usual calendar time, by incorporating the time change. In order to calculate the time change ($T_{i\Delta}$) with $i = 1, \ldots, N$, one needs to preset a constant volatility length Δ which determines the degree on how often the data should be observed in terms of the volatility time. Since the total quadratic variation is finite for most of asset returns observed in finite time horizon, it is easy to deduce that higher volatility length would imply lower number of samples and vice versa. Common sense will choose the smallest Δ to obtain the largest number of samples. This is because usual estimators are more efficient as the number of samples gets larger. Adopting this idea, we find the volatility length Δ which is the smallest among all the admissible values of Δ. Note that the admissible range of Δ is determined by a number of factors that are difficult to evaluate in practice. In general, extremely small values of Δ can harm the effectiveness of the time change. For instance, if Δ

is too small, then from the definition of the time change, $[T_{(i-1)\Delta}, T_{i\Delta}]$ often becomes the same as the observation interval of the data, and therefore the time changed data will have similar property as the original data.

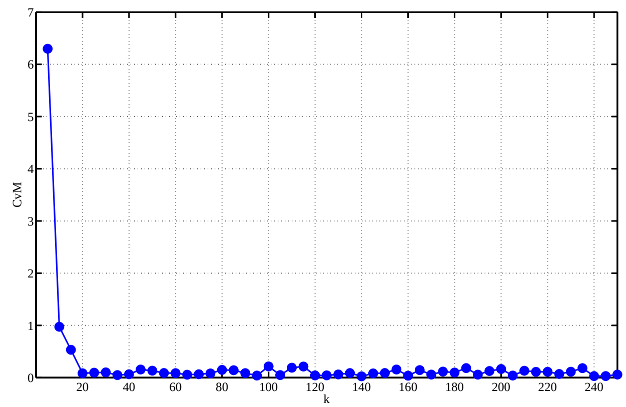

Fig. 1. Signature Plot of CvM Distance for S&P 500 Index Excess Returns

Note: The x-axis represents the number of days k included to calculate the volatility length Δ_k, i.e., $\Delta_k = [P]^{\delta}_T k/K$, where $[P]^{\delta}_t$ is the realized variance of (P_t), $dP_t = dp_t/p_t - r^f_t dt$, computed using daily observations over the time horizon $[0, T]$, and K is the total number of days. The y-axis represents the CvM distance for the standardized excess returns after the time change.

In order to find the admissible range of Δ, we use the Cramer-von Mises (or CvM) distance for the time changed market returns $\left\{\int_{T_{(i-1)\Delta}}^{T_{i\Delta}} \left(dp_t/p_t - r^f_t dt\right)\right\} / \sqrt{\Delta}$. Especially, we find the time changes based on different volatility length Δ_k, with $k = 1, \ldots, K$, and calculate the CvM distance for each Δ_k. Since the CvM distance can be interpreted as the degree on how far the empirical distribution of time changed market return departs from $N(0, 1)$, it is useful to check whether time change based on some Δ_k is effective or not. In other words, if it is close to 0, then the time change based on Δ works effectively, while if it is far from 0, then the Δ is supposedly too small or too large to produce the effective time change. Figure 1 shows the CvM distance for $k = 5, 10, \ldots, K$ with $\Delta_k = [P]_T k/K$. Note that k is the number of days to be considered to calculate the average quadratic variation and K is the total number of days in the dataset. We can see that the CvM distance drastically decreases as k increases from 5 (or 5-day average quadratic variation) to 20 and then it stabilizes around the level of 0.1 - 0.3. This implies that the time change will work effectively if k is greater or equal to 20. From this point, we set the volatility length as Δ_{20} or 20-day average quadratic variation.

It is important to note that for the actual calculation of the time change ($T_{i\Delta}$) for $i = 1, \ldots, N$ with the discrete observations, we use the minimum distance criteria for the incremental quadratic variation process. Especially, our time change for the discrete observation is defined as

$$T_{i\Delta} = \operatorname*{argmin}_{s \geq T_{(i-1)\Delta}} \left| [P]_s^\delta - [P]_{T_{(i-1)\Delta}}^\delta - \Delta \right| .$$

Our modified time change is different from the original time change in (2.14) in the sense that it finds the time when the quadratic variation from the previous time change $T_{(i-1)\Delta}$ is closest to the volatility length Δ, while the original time change finds the smallest time when the quadratic variation from initial time is greater than $i\Delta$. Note that this modified time change is not stopping time because the time with minimum distance criteria is unknown based on current information. Nevertheless, for practical purposes, it is advantageous to use the modified time change with discrete observations because it ensures that the quadratic variation on the interval $[T_{(i-1)\Delta}, T_{i\Delta}]$ is closest to Δ in the given sampling frequency. We use the modified time change in calculating the signature plot mentioned above, as well as in the following empirical analysis.

Once we find the time change ($T_{i\Delta}$) for $i = 1, \ldots, N$, we need to calculate the time changed market return, time changed risk free return, time changed covariance between market and consumption, time changed market variance, time changed covariance between market and labor income, and time changed market volatility. Basically, all the terms in (2.11) will be calculated and plugged in before estimating the preference parameters (α, β, κ). Firstly, the time changed market return $(\int dp_t/p_t$[17]$)$ and risk free return $(\int r_t^f dt)$ is easily obtained by calculating the cumulative return for the intervals $[T_{(i-1)\Delta}, T_{i\Delta}]$ based on the daily returns. Secondly, the time changed variance of the market $(\int (\sigma_t^m)^2 dt)$is calculated from the realized volatility for the interval, i.e., $[P]^{\delta}_{T_{i\Delta}} - [P]^{\delta}_{T_{(i-1)\Delta}}$. Thirdly, the macroeconomic volatilities $(\sigma_t^c,\ \sigma_t^y)$ are calculated from two different approaches discussed in section 3.2. The basic idea is to extract the macroeconomic volatilities in their observation frequency and interpolate them in daily frequency by the standard nonparametric method. Meanwhile, we also find the market volatility (σ_t^m) in the daily frequency by the same method. Once the daily measures are ready, the time changed covariances $(\int \sigma_t^m \sigma_t^c dt$ and $\int \sigma_t^m \sigma_t^y dt)$are calculated by the Rieman sum. Lastly, the time changed volatility of the market $(\int \sigma_t^m dt)$ is calculated in the same way.

3. Macroeconomic Volatilities

Following the econometric methodology developed in the previous section, we extract the volatilities of macroeconomic variables by two different approaches; (i) time-varying volatility, and (ii) nonlinear stochastic volatility.

Table II presents the estimation results of the consumption and labor income based on a Gibbs sampling. We first obtain the demeaned rate of returns for each process from the standard local linear method and exclude the outliers which is greater than 3 in absolute value after normalization. The prior distributions for the Gibbs sampling are based on Kim, Lee, and Park (2008) while the location and scale param-

[17]For simplicity, we put $\int$ as the integral on the interval $[T_{(i-1)\Delta}, T_{i\Delta}]$

Table II. Gibbs Sampling Results for Consumption and Labor Income

	Consumption			Labor Income		
	Priors	Pos.Mean	Pos.Std.Dev.	Priors	Pos.Mean	Pos.Std.Dev.
a	$G(1, 0.001)$	0.001	(0.001)	$G(1, 0.001)$	0.001	(0.001)
b	$G(1, 0.2)$	0.332	(0.036)	$G(1, 1.2)$	1.075	(0.182)
c	$G(1, 0.2)$	0.089	(0.020)	$G(1, 2)$	1.105	(0.527)
w_0	$N(40, 40^2)$	2.617	(3.913)	$N(-4, 4^2)$	2.225	(4.532)
ρ				$U[-1, 1]$	0.727	(0.187)
γ	$U[-1, 1]$	0.544	(0.219)	$U[-1, 1]$	0.052	(0.250)

Note: The table presents the estimation results of nonlinear stochastic volatility model based on a Gibbs sampling. We sample 30000 iterations and discards 15000 iterations. The sample period for consumption is from October 1959 to December 2006 and for labor income first quarter of 1959 to fourth quarter of 2006. $U[\theta_1, \theta_2]$ denotes uniform distribution with a support (θ_1, θ_2) and $G(\theta_1, \theta_2)$ denotes gamma distribution with mean $\theta_1\theta_2$ and variance $\theta_1\theta_2^2$, and $N(\theta_1, \theta_2)$ denotes normal distribution with mean θ_1 and variance θ_2. Pos. Mean and Pos. Std. Dev. denote the posterior mean and posterior standard deviation, respectively. All the parameters are estimated for the scaled data with 100 (or in percentage level).

eters for a, b, and c are chosen to be similar to the point estimates and their standard deviations of the maximum likelihood estimation[18]. For the consumption volatility, the lower and upper bound is estimated to be 0.001 and 0.333, which implies 0.1% of lower bound and 2% of upper bound in annual level. The speed parameter (c) is estimated to be 0.089, which implies that the transition from low to high volatility is relatively slow. The leverage effect is estimated to be positive and significant. On the other hand, the labor income volatility has relatively smaller level of lower bound 0.001 (or 0.06% in annual level), and similar level of upper bound 1.076 (or 2.03% in annual level). The speed parameter is estimated to be 1.105, which is much greater than the consumption volatility. This high speed implies that the shift between low and high volatility is fast, or equivalently, the volatility process is closer to a process switching two regimes. The leveage effect is small with 0.05, however it is not signifi-

[18]The results of the maximum likelihood estimation is not presented in this paper. However, the results can be provided upon request.

cant. The persistency parameter for the latent factor is estimated to be 0.727, which implies that the latent process is highly stationary[19].

Figure 2 (a) plots the extracted consumption volatilities from the two approaches and the realized volatility. For the realized volatility, we use the squared monthly consumption growth rate. The growth rate is demeaned by the similar local linear method. We can see that both volatilities from the local linear and Gibbs sampling are decreasing over the sampling period, which coincides with the overall pattern of the realized volatility. Figure 2 (b) presents the Gibbs sampling result compared with the local linear kernel result. In gerneral, the long-run trend of both volatility processes are very similar, however the short-run fluctuations in the Gibbs sampling does not exist in the local linear kernel result. This is well expected from the two different approaches because the stochastic volatility model generalizes the time-varying volatility model by adding additional shock to the volatility process.

Figure 3 (a) plots the extracted and realized volatilities for the labor income. Compared to the consumption volatility, the labor income volatility seems to move between two different states depending upon the business cycle. This feature coincides with the parameter estimates in Table II, which reports the smaller lower bound and much higher speed parameter compared to the consumption volatility. If we compare the extracted volatilities from the two different approaches (in Figure 3 (b)), we can see that they almost share the long-run trend, while the short-run movement of the Gibbs sampling result suggests a possibly an additional risk factor from the stochastic volatility.

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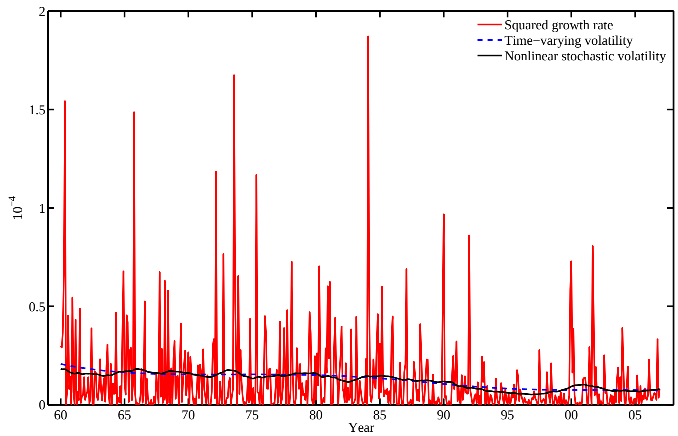
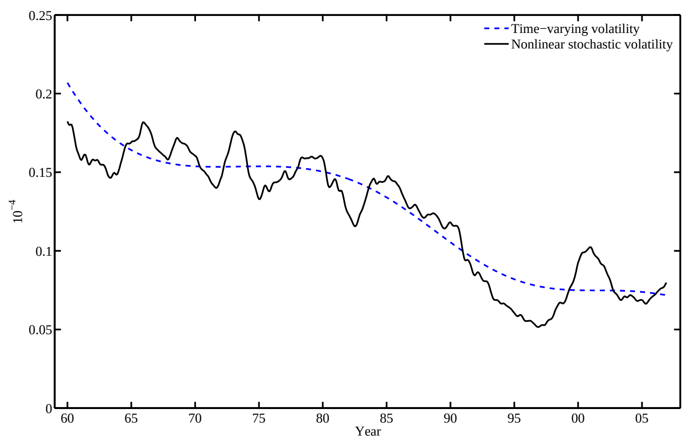
[19]For the consumption volatility, we restrict $I(1)$ latent factor. If we estimate the persistency parameter, then it is estimated to be 0.998, which is very close to 1.



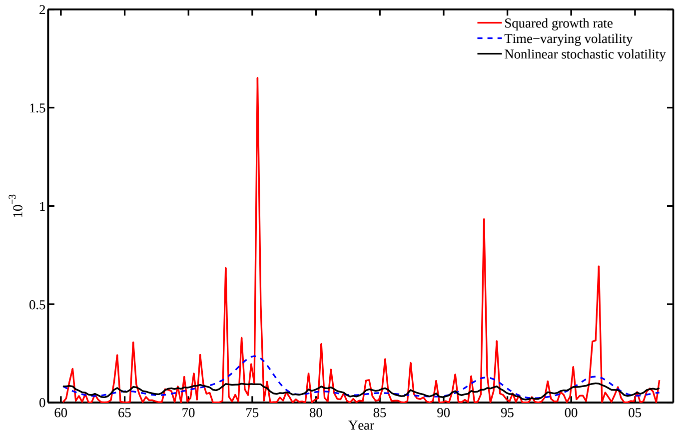
(a) Comparison between squared growth rate and estimated volatilities



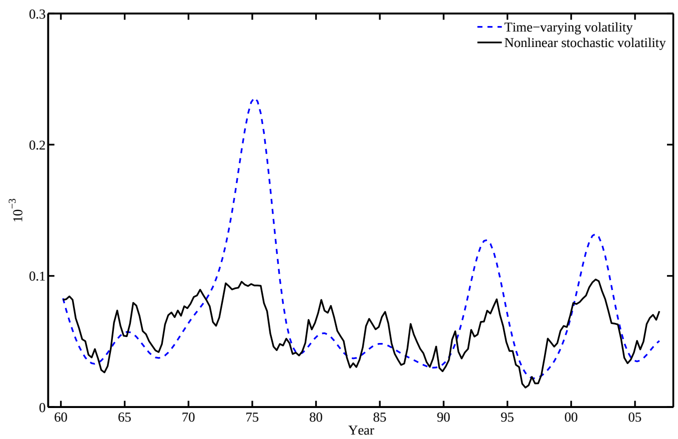
(b) Comparison between time-varying volatility and nonlinear stochastic volatility

Fig. 2. Consumption Volatilities

Note: In Figure (a), the volatilities of consumption growth rate from 1960 to 2006 are presented against the squared growth rates. The growth rates are demeaned by the local linear method. The bandwidths for the local linear kernel estimation for time-varying mean and volatilities are selected separately, and are based on the least squares cross-validation (see Li and Racine (2007, p. 83)). In Figure (b), the extracted volatility from the nonlinear stochastic volatility model is presented with the result from the time-varying volatility model. The volatility is estimated by a Gibbs sampling method and the extracted volatility is obtained by plugging in the posterior sample means for the parameters and the latent factor.



(a) Comparison between squared growth rate and estimated volatilities



(b) Comparison between time-varying volatility and nonlinear stochastic volatility

Fig. 3. Labor Income Volatilities

Note: In Figure (a), the volatilities of labor income growth rate from 1960 to 2006 are presented against the squared growth rates. The growth rates are demeaned by the local linear method. The bandwidths for the local linear kernel estimation for time-varying mean and volatilities are selected separately, and are based on the least squares cross-validation (see Li and Racine (2007, p. 83)). In Figure (b), the extracted volatility from the nonlinear stochastic volatility model is presented with the result from the time-varying volatility model. The volatility is estimated by a Gibbs sampling method and the extracted volatility is obtained by plugging in the posterior sample means for the parameters and the latent factor.

E. Ambiguity, Risk, and Intertemporal Substitution

Finally, we are led to report and discuss our empirical results on the importance of ambiguity aversion, risk aversion, and intertemporal substitutability. We first analyze the baseline case in which financial market is assumed to represent the aggregate wealth. Then, we show our main results and investigate the model (2.11) in detail.

1. Baseline Case: Financial Wealth Only

Table III presents estimation results of three configurations for the recursive utility models; power utility (Model I), stochastic differential utility (Model II), and the stochastic differential utility with ambiguity aversion (Model III). In all three settings, it is assumed that financial wealth proxies the total wealth, i.e., $\pi = 1$, is imposed. As mentioned earlier, financial wealth is only a subset of the aggregate wealth and therefore we may miss important interactions between human wealth and asset returns in this case. However, to better understand the effect of human wealth, we believe that it is necessary to compare the results from the specifications with and without human wealth. In this light, we set this as our baseline case. Specifically, Model I is used to verify if the equity premium puzzle arises in our setting and data set. The results from Model II are directly comparable to Epstein and Zin (1991), Baskshi and Naka (1997), and Normandin and St-Amour (1998) in that they also used financial wealth as the proxy for the aggregate wealth. To the best of our knowledge, Model III which is our main model has not been empirically studied.

In all three settings, we need to estimate volatilities of the consumption growth (σ_t^c), which we constructed using the method developed in a previous section. To compute conditional covariances, we assume that the correlation ρ_c between consumption growth and the market return is constant at 0.2. We obtained this value

Table III. Estimation Results for Baseline Models

	Model I		Model II		Model III	
			Panel A: Time-Varying Volatility			
β	257.991	(3.589)	0.000	-	1.290	(0.856)
α	-	-	-3.572	(0.063)	0.263	(0.719)
κ	-	-	-	-	0.360	(0.022)
RA	257.991	(3.589)	4.572	(0.063)	0.737	(0.719)
EIS	0.004	(0.000)	∞	-	0.775	(0.514)
CvM	0.035		0.034		0.031	
			Panel B: Nonlinear Stochastic Volatility			
β	258.881	(3.508)	0.000	-	1.461	(0.812)
α	-	-	-3.572	(0.063)	0.419	(0.692)
κ	-	-	-	-	0.361	(0.021)
RA	258.881	(3.508)	4.572	(0.063)	0.581	(0.692)
EIS	0.004	(0.000)	∞	-	0.684	(0.380)
CvM	0.035		0.034		0.031	

Note: The table reports the estimation results for the asset pricing models in which the aggregate wealth consists of only financial wealth. All results are for the sample 1/2/1960-12/29/2006. The first column is Model I with standard additive CRRA utility, the second column is Model II with recursive utility, and the third column is Model III with multiple-priors recursive utility. The correlation between the market return and the consumption growth (ρ_c) is set to be 0.2. The standard errors in parenthesis are obtained by the subsampling method.

by computing the sample correlation between the two variables and it is consistent with the existing studies.

Result from Model I states that the famous 'equity premium puzzle *à* la Mehra and Prescott does prevail, showing roughly 258 as the estimate of the relative risk aversion (RRA). Increasing the correlation coefficient to a counterfactual value of 1 still generates an estimate of RRA around 52, confirming that the main reason for the puzzle is the smooth consumption growth. In this case, the elasticity of intertemporal substitution (EIS) is given by the reciprocal of relative risk aversion, and it is estimated to be close to 0. This, in fact, is consistent with the existing studies estimating the EIS such as Hall (1988). In those studies, they use the consumption growth as the regressed and asset returns especially, Treasury bills as a regressor to estimate a linearized Euler equation with homoskedasticity.[20] There are also numerous studies including Hansen and Singleton (1983), estimating a non-linear Euler equation. Similar to the case of the linearized setup, some studies have reported the EIS close to zero, while others reported significantly positive numbers often greater than one. The empirical literature ascribed this mainly to weak instruments and this basically reveals the difficulty of identifying the key preference parameter. It may also result from the tight restriction imposed on the power utility function and the counterfactual assumption or treatment on the nature of the volatilities of asset returns.[21] With

[20]However, even in the linearized setup, results are mixed. For instance, Attanasio and Weber (1989) estimated the EIS around 2. Vissing-Jorgensen (2002) reported that the EIS is close to 3 for Treasury returns with bond holders' consumption and higher than 1 for stock holders. Moreover, if one estimates the linearized equation consisting of consumption growth and asset returns, switching the regressor and the regressed, a high EIS is obtained with asset returns as the regressed, while a low EIS is estimated when consumption growth is the regressed.

[21]Although some studies such as Yogo (2004), adopt a more flexible preferences relaxing this restriction, due to their linearization and homoscedasticity assumption, they virtually estimate only one parameter.

our newly developed econometric tools in hand, we believe that we can handle most of these issues aforementioned. We now examine how alternative models affect the estimation results.

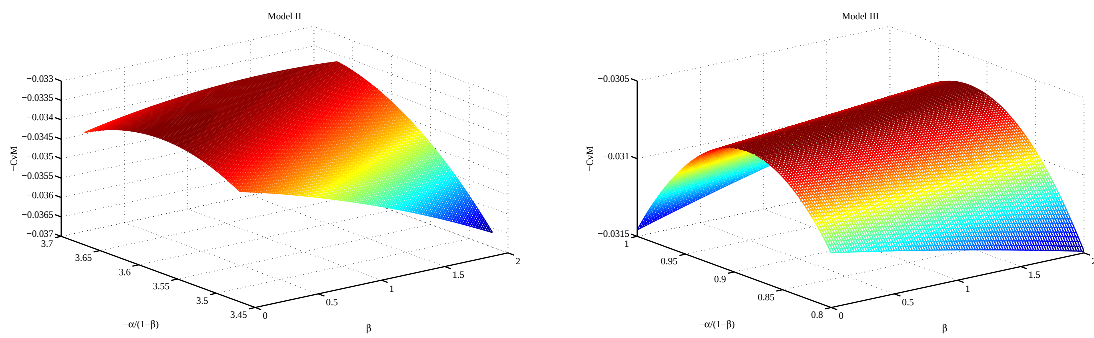
With the stochastic differential utility (Model II), the estimates of the two parameters α and β in Model II are -3.6 and 0 respectively. This implies that the estimated risk aversion $(1-\alpha)$ is dramatically decreasing to 4.6, while the estimated EIS $(1/\beta)$ is reaching a very high number. In comparison with the existing studies in a similar setting, Epstein and Zin (1991) reported the RRA around one and the EIS close to zero via GMM estimations, and Normandin and St-Amour (1998) stated the RRA around 1.4 and the EIS around 1.2 using a maximum likelihood estimation method. Our result suggests that the representative agent is more risk averse and her consumption is highly substitutable across periods compared to the previous results. Although high values of EIS are not incompatible with explaining the behaviors of stock returns or risk-free rates, high standard errors of β estimates hint that it requires further investigation. In addition, β being close to zero implies that asset returns have little link to consumption growth, which is puzzling. Thus, all of the points lead us to suspecting a weak identification problem for the EIS parameter. As briefly mentioned, a wide range of EIS estimates is not new in the related literature of estimating the preference parameter in both linearized and non-linear Euler equations. However, the literature does not offer much explanation on why it is difficult. Instead, many recent studies have been concentrating on weak instruments to overcome this issue. Although there is little doubt that this is an important task in econometrics, in this paper we focus on understanding the nature of identifying the EIS parameter given that our estimation does not make use of instrumental variables and the estimated models are of highly interpretable forms in continuous time.

First of all, although Model II is an extension of Model I by adding the conditional return variance, the two models have very different implications for linking asset returns to conditional covariances of consumption growth and returns. In case of Model I, small volatility of consumption growth without an additional explanatory variable implies a large coefficient (i.e. a small EIS) to match the average market risk premium. Meanwhile, Model II has an additional explanatory variable and the coefficients of both explanatory variables have non-linear parametric restrictions. For expositional ease, we rewrite the Model II in below:

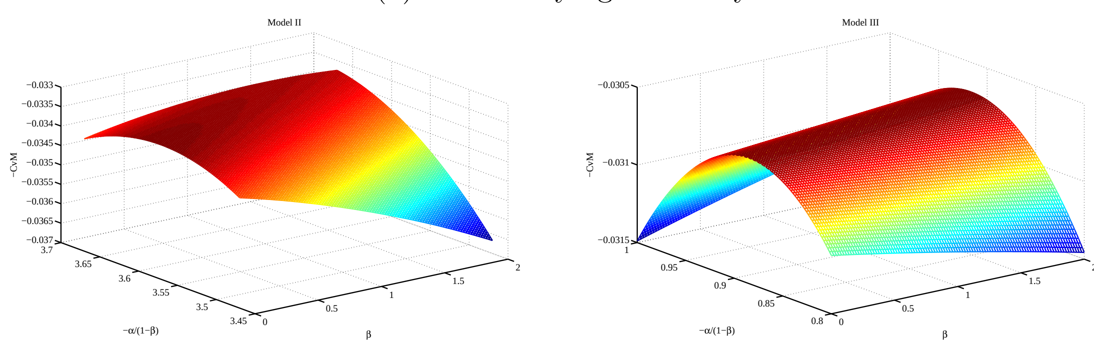
$$\frac{dp_t}{p_t} - r_t^f dt = \frac{\beta\alpha}{1-\beta}\rho_c\sigma_t^c\sigma_t^m dt + \left(1 - \frac{\alpha}{1-\beta}\right)(\sigma_t^m)^2 dt + \sigma_t^m dW_t.$$

Between two explanatory variables, it is well known that the conditional variance of market return ($(\sigma_t^m)^2$) is more volatile than the conditional covariance between consumption growth and the market return ($\rho_c\sigma_t^c\sigma_t^m$). If both variables have similar degrees of correlations with asset returns, it is possible that there exists some statistical tension for estimating two parameters, because of the relatively weak signal from the consumption growth. Especially, since the EIS is closely related to shifting consumptions across periods without uncertainty, identification of the EIS can be a more challenging task. That is, the equity premium puzzle in the context of the power utility becomes a weak identification problem when we break the tight restriction between risk aversion and intertemporal substitutability. In addition to this, note that the estimated EIS is measured by the reciprocal of β. A small perturbation of β coefficient can lead to a large swing of the EIS. For instance, $\beta = 0.1$ implies the EIS of 10, while $\beta = 2$ means 0.5. Even if it appears to be a minor issue, this can amplify the weak identification problem given the weak signal from the consumption growth. Therefore, all these factors contribute to a weak identification problem of the EIS parameter. To further analyze this issue, we draw the surfaces and contours of the

CvM measures of the model II. The left panels of Figure 4 show that estimating the risk aversion appears to be easily attained and relatively accurate, while it suggests that the elasticity of intertemporal substitution is not going to be easy to estimate due to its flat surface. The left panels in Figure 5 corroborate our conjecture. A clear pattern from the contours is that β, the reciprocal of the EIS is small and close to zero and has a very flat region ranging from 0 to 2. Thus, it is not surprising to observe such distant values of the EIS estimates in the literature.



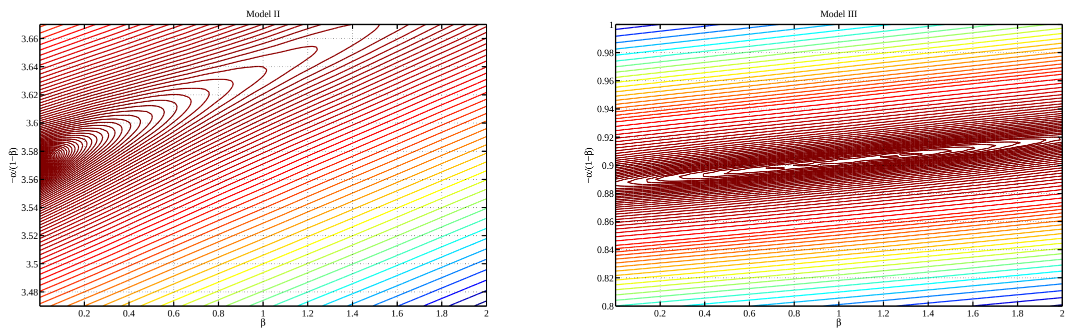
(a) Time-Varying Volatility



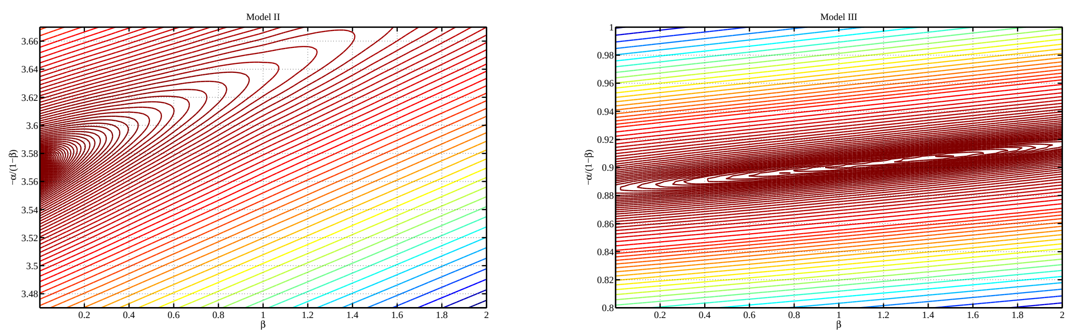
(b) Nonlinear Stochastic Volatility

Fig. 4. Surface Plots of CvM Measure for Model II and Model III

Note: The above figures plot the surfaces of the CvM distance on the parameter vector of $(\beta, -\alpha/(1-\beta))$. Figure (a) presents the result based on the macroeconomic volatilities from the time-varying volatility model, and Figure (b) presents the result from the nonlinear latent stochastic volatility model. In case of Model III, the surfaces are plotted with $\kappa = 0.360$ (in (a)), and $\kappa = 0.361$ (in (b)).



(a) Time-Varying Volatility



(b) Nonlinear Stochastic Volatility

Fig. 5. Contour Plots of CvM Measure for Model II and Model III

Note: The above figures plot the contours of the CvM distance on the parameter vector of $(\beta, -\alpha/(1-\beta))$. Figure (a) presents the result based on the macroeconomic volatilities from the time-varying volatility model, and Figure (b) presents the result from the nonlinear latent stochastic volatility model. In case of Model III, the contours are plotted with $\kappa = 0.360$ (in (a)), and $\kappa = 0.361$ (in (b)).

Next we incorporate the ambiguity aversion into the stochastic differential utility setup (Model III). The result clearly dictates that ambiguity aversion is significant both economically and statistically. Specifically, the last column in the Table II shows that the estimated RRA is around 0.58˜0.74, the EIS is estimated around 0.68˜0.78, and the ambiguity aversion parameter (κ) is estimated around 0.36. κ measures how much the representative household distorts her beliefs to prepare against a worst case scenario given the ignorance of the true conditional probability distribution. Recall that the conventional notion of the market price of risk measures the degree to which an investor will adjust her probability to be risk neutral. Thus, κ quantifies a constant adjustment of probability in order to be neutral against a Knightian sense of uncertainty. Although it is true that a more sophisticated model of ambiguity aversion such as time-varying ambiguity aversion or learning with ambiguity aversion would further clarify the nature of this new source of premium, our empirical results state that modeling uncertainty differentiated from the usual sense of risk is a first-order business to understand the behaviors of asset returns. Given that, the lower RRA estimates in Model III is understandable because ambiguity aversion captured by the conditional volatility in our setup is likely to alleviate the burden of the return variance in accounting for the average return behaviors. One caveat of the RRA estimate is that its value is somewhat too low and its measurement is noisy. We suspect that the Model III over-compensates the contributions from the market factors compared to the consumption factor. We believe that this results from not including human wealth to construct the aggregate wealth which is the right measure as shown in the model section. Related but not expected, it appears that ambiguity aversion helps identify the EIS as well. Admittedly, it is still statistically insignificant. But our numerous robustness checks in various dimensions suggest that the inclusion of ambiguity aversion provides the other two explanatory variables, the consumption growth and the rate of return from wealth, with fair chances of explaining asset returns by correctly specifying the existence of ambiguity aversion.

2. Human Wealth

Now we state our main results from our continuous-time recursive utility model with human wealth, (2.11).

$$\begin{aligned}\frac{dp_t}{p_t} - r_t^f dt =& \frac{\beta\alpha}{1-\beta}\rho_c\sigma_t^c\sigma_t^m dt + \left(1 - \frac{\alpha}{1-\beta}\right)(\sigma_t^m)^2 \pi dt \\ &+ \left(1 - \frac{\alpha}{1-\beta}\right)\rho_y\sigma_t^y\sigma_t^m(1-\pi)dt + \kappa\sigma_t^m dt + \sigma_t^m dW_t.\end{aligned}$$

This involves fixing two more parameters π and ρ_y, the fraction of financial wealth, and the correlation between labor income growth and the market return. For the former, we tried two values (1/3 and 2/3). Our robustness checks reveal that different values of π give similar results to either of the closer chosen values.[22] Regarding the value of ρ_y, there is little consensus about it. Several empirical studies report that this correlation is positive, while other studies based on structural models such as Lustig and Van Nieuwerburgh (2008), and Chen et. al. (2008) report a strong negative correlation such as -0.7. According to our computation it was 0.03. We tried different values such as 0.03 and -0.03, and the results are reported in Table IV.

The main question to be addressed is the effect of including labor income growth. Regarding ambiguity aversion, the estimates of ambiguity aversion rarely vary across settings and the estimates of κ is again, around 0.36. Note that this result measured together with the key sources of aggregate risk factors such as the consumption

[22]We also tried estimating this parameter directly and the estimated values are around 0.2-0.3 in some cases. But due to the weak identification problem, its identification is affected by alternative model settings.

Table IV. Implications of Human Wealth

	Without Ambiguity				**With Ambiguity**			
	Calibrated values							
π	0.333	0.333	0.667	0.667	0.333	0.333	0.667	0.667
ρ_y	0.030	-0.030	0.030	-0.030	0.030	-0.030	0.030	-0.030
	Panel A: Time-Varying Volatility							
β	0.000	0.000	0.000	0.000	0.107	0.046	0.186	0.645
	-	-	-	-	(0.451)	(0.449)	(0.680)	(0.660)
α	-12.628	-12.831	-5.842	-5.867	-4.205	-4.545	-1.509	-0.660
	(0.189)	(0.193)	(0.095)	(0.095)	(2.006)	(2.039)	(1.149)	(1.113)
κ	-	-	-	-	0.358	0.358	0.358	0.361
	-	-	-	-	(0.024)	(0.023)	(0.022)	(0.023)
RA	13.628	13.831	6.842	6.867	5.205	5.545	2.509	1.660
	(0.189)	(0.193)	(0.095)	(0.095)	(2.006)	(2.039)	(1.149)	(1.113)
EIS	∞	∞	∞	∞	9.331	21.792	5.382	1.550
	-	-	-	-	(39.292)	(213.227)	(19.689)	(1.586)
CvM	0.034	0.034	0.034	0.034	0.031	0.031	0.031	0.031
	Panel B: Nonlinear Stochastic Volatility							
β	0.000	0.000	0.000	0.000	0.271	0.076	0.748	0.699
	-	-	-	-	(0.469)	(0.492)	(0.649)	(0.653)
α	-12.635	-12.825	-5.843	-5.866	-3.449	-4.359	-0.469	-0.560
	(0.189)	(0.193)	(0.095)	(0.095)	(2.137)	(2.248)	(1.107)	(1.111)
κ	-	-	-	-	0.359	0.361	0.360	0.361
	-	-	-	-	(0.023)	(0.020)	(0.022)	(0.022)
RA	13.635	13.825	6.843	6.866	4.449	5.359	1.469	1.560
	(0.189)	(0.193)	(0.095)	(0.095)	(2.137)	(2.248)	(1.107)	(1.111)
EIS	∞	∞	∞	∞	3.693	13.222	1.337	1.431
	-	-	-	-	(6.392)	(85.958)	(1.159)	(1.338)
CvM	0.034	0.034	0.034	0.034	0.031	0.031	0.031	0.031

Note: The table reports the estimation results for the asset pricing models in which the aggregate wealth consists of financial wealth and human wealth. All results are for the sample 1/2/1960-12/29/2006. In each panel, each column represents the point estimates and their standard errors for the recursive utility model given the proportion of financial wealth to the aggregate wealth (π) and the correlation between the return on human wealth and financial wealth (ρ_y). The correlation between the market return and the consumption growth (ρ_c) is set to be 0.2. The standard errors in parenthesis are obtained by the subsampling method.

growth, market returns, and labor income growth. Our empirical results strongly suggest that investors will readily take uncertain bets in financial markets only if a sufficient amount of premium is given, separate from the conventional risk premium. Meanwhile, the major difference of the Table IV in comparison with Table III is that the risk aversion coefficient increases. In case of Model II counterparts (i.e., models without ambiguity), the RRA increases from 4.6 up to 14. With ambiguity aversion, the RRA increases from 1.4 up to 5.5. With the addition of the labor income growth, the total wealth becomes less volatile in comparison with the financial wealth. Thus, in order to make up for the level of variability related to the aggregate wealth return term, a higher risk aversion is needed.

Regarding the intertemporal substitutability, the EIS estimates increase like the case of risk aversion when ambiguity aversion is imposed. With $\pi = 1/3$ and $\rho_y = 0.03$, we have 9.33 with the non-paramteric volatility model, and 3.69 with the non-linear stochastic volatility model. With $\rho_y = -0.03$ instead, the estimated EIS is 21.79 with the parametric volatility and 13.22 with the non-linear volatility, but the standard erros are very big. When the fraction of financial wealth π is set to 2/3, the estimated EIS is around 1.3-5.4 depending on settings. Without ambiguity aversion, β is again close to zero and therefore, identification is fairly weak. Interestingly, when we impose a strong negative number for ρ_y as implied in the papers mentioned above, we have somewhat lower EIS around 1.2 for most cases. However, in all of the settings we have tried, the point estimates of the EIS is higher than one, meaning that economic agents will change their consumptions rather elastically when real interest rate changes. It should be also noted that all the results in Table IV show that agents prefer early resolution of uncertainty whether or not there exists ambiguity aversion. This makes economic agents unhappy about fluctuations in future utilities, often called the long-run risk channel. For more details on the mechanisms, see Bansal and Yaron (2004), Hansen, Heaton and Li (2008), and Kim et. al. (2008) on stock returns. For the term structure modeling along this line, see Piazzesi and Schneider (2006), and Gallmeyer, Gonzales, and Kim (2008).

In a summary, the recursive utility models with both financial and human wealth give most reasonable results when ambiguity aversion is included and the estimates of ambiguity aversion do not depend on alternative setting. Although the estimates of the risk aversion increase as human capital is added, those are still in an acceptable range of values.[23] The weak identification problem of the EIS is also a prevalent feature across different model specifications.

Before we conclude, we discuss the relevance of our human capital setup linking the volatility of returns from human capital and that of labor income growth. There can be some additional components that may not be fully internalized via labor income growth such as the learning-by-doing or interactions across workers. If this happens to have a strong time-varying volatilities, our setup may be misspecified in this direction. On the other hand, if it exists but has a constant conditional volatility, then this has an interesting implication for our ambiguity aversion. Suppose that the covariations between returns from human wealth and the financial market return are decomposed into a covariation term related to labor income growth and the other from the externality factor, say Z, or

$$\rho_h \sigma_t^h \sigma_t^m = \rho_y \sigma_t^y \sigma_t^m + \rho_z \sigma^z \sigma_t^m,$$

where ρ_z and σ^z are the correlation with the market return and the constant condi-

[23] According to Mehra and Prescott, the acceptable range of the relative risk aversion does not exceed 10.

tional volatility of z respectively. Then, (Model III) is extended as

$$\begin{aligned}\frac{dp_t}{p_t} - r_t^f dt =& \frac{\beta\alpha}{1-\beta}\rho_c\sigma_t^c\sigma_t^m dt + \left(1 - \frac{\alpha}{1-\beta}\right)(\sigma_t^m)^2 \pi dt \\ &+ \left(1 - \frac{\alpha}{1-\beta}\right)\rho_y\sigma_t^y\sigma_t^m(1-\pi)dt + \Upsilon\sigma_t^m dt + \sigma_t^m dW_t,\end{aligned}$$

where $\Upsilon = \left(1 - \frac{\alpha}{1-\beta}\right)\rho_z\sigma^z + \kappa$. That is, if this setup were a better approximation of reality, our estimate of ambiguity aversion may include an additional term related to the interaction between human capital and asset returns. Note, however, that the size of Υ depends on the sign of ρ_z. As mentioned in the beginning of this section, the returns on human capital is strongly negatively correlated with the market return according to the recent empirical research, while the correlation between labor income growth and market returns turns out to be a weakly positive number. If this is the case, $\rho_z < 0$ must hold. That is, given $1 > \alpha/(1-\beta)$, this implies that estimated ambiguity aversion may be downward biased. Of course a more elaborate quantitative assessment is necessary, but based on all the results, we believe that our estimates on ambiguity aversion are robust and conservative.

F. Conclusion

We began this paper with a question asking if there is an important role played by decision makers' fear on ambiguity on true probability measure. The answer to this question is positive based on our empirical analysis.

In terms of economic theory, the inclusion of ambiguity aversion is meaningful because it can overcome the Ellsberg paradox. Multiple-priors utility models were developed to incorporate such ambiguity aversion, and have a neat expression for asset prices available in continuous time. In addition, one can view that multiple-priors models as an extension of the rational expectation in that investors may be of insufficient knowledge about the true probability density. When ambiguity aversion is assumed, economic agents are basically endowed with a set of beliefs on the true probability distribution and choose the one that is the least ambiguous. Of course, whether or not the ambiguity aversion matters is an empirical and quantitative concern. Our estimation results strongly suggest that there exists premium for bearing market uncertainty separate from the conventional risk sources. Even with various specifications, the preference parameter indicating the ambiguity aversion is both economically and statistically significant. In addition, the estimates of the ambiguity aversion parameter rarely vary across alternative settings.

Another interesting finding is that the models with ambiguity aversion have lower relative risk aversion. Thus, the conjectures in Epstein and Wang (1994) or Chen and Epstein (2002) are confirmed in our empirical result. Empirically speaking, our results suggest that relative risk aversion can be estimated with an upward bias if ambiguity aversion is properly adjusted. What is even more interesting is that incorporation of ambiguity aversion does not dominate the role of risk aversion. Clearly, there exist some independent dimensions of generating premiums for bearing such risk and uncertainty. It is true that we only employed a simple case of ambiguity aversion. Therefore, it would be interesting to study links of alternative forms of ambiguity aversion to asset prices.

With regard to the elasticity of substitution, there exists a weak identification problem due to its non-linear parametric restrictions and the weak signal from consumption growth. That said, the models with ambiguity aversion still produce quite reasonable estimates of the intertemporal substitution. Therefore, ambiguity aversion not only matters in terms of explaining the behaviors of asset returns, but also helping identify key preference parameters.

In addition to the empirical findings, another contribution of our paper is that we provide a novel econometric approach estimating and testing for continuous-time asset pricing models including both financial and macroeconomic variables. In the empirical analysis of such models, it has long been a tradition that we ignore the availability of high-frequency observations on financial variables, mostly for the lack of ideas about how to use them constructively. Virtually all empirical studies of such models have been done only using lower-frequencies, at which all involved macroeconomic variables are also available. Our paper makes it clear that this is an important loss of information.

In our analysis, we use the available high-frequency observations directly to identify our model, and also nonparametrically correct for time-varying stochastic volatility in the price equation errors. It is widely known that many asset returns show strong evidence for the presence of time-varying stochastic volatility. Unless properly and carefully taken care of, the time-varying stochastic volatility may well have a fatal effect on our estimation results. We believe that our method can be used in many other interesting applications, to unravel the complicated interactions between financial markets and macroeconomy.

CHAPTER III

A TEST OF MARTINGALE IN CONTINUOUS TIME

A. Introduction

The concept of martingale plays a central role in modern economics and finance. The martingale hypothesis implies that any future value of a time series is expected to be its current value, conditional on the current information set. It has long been believed, rather vaguely, that an asset's price would follow a martingale if it has an efficient market and any new information is instantly and fully reflected in the asset's price. Modern finance theory shows that this is indeed true under the risk-neutral measure, if the market offers no arbitrage opportunity. In fact, it is now well known that the price of any financial asset should follow a martingale, if deflated by the so-called pricing kernel, to avoid any arbitrage opportunity. The martingale concept is also a crucial instrument in econometric modeling. Any conditional mean specification implies that the error in the resulting model is a martingale difference or a martingale stochastic differential respectively in discrete or continuous times. Testing for model specification in conditional mean is therefore closely related to testing the martingale hypothesis for the error in the resulting model.

Various tests for the martingale hypothesis have already been developed by many authors, which include the tests by Durlauf (1991), Hong (2000), Deo (2000), Dominguez and Lobato (2003), Kuan and Lee (2004), Hong and Lee (2003), Hong and Lee (2005), Park and Whang (2005), Escanciano and Velasco (2006) and Escanciano and Mayoral (2007), among others. All of the existing tests, however, have been developed in the discrete time framework, and examine whether or not the first differences of the given time series are martingale differences. Our approach in the paper is entirely different, in the sense that our tests are developed in the continuous time framework. Though we use the time series data observed in discrete time, we assume that they are generated by the underlying stochastic process evolving continuously in time. Many of the asset pricing models have been derived in continuous time, using continuous time models such as diffusions. Moreover, we may expect that the continuous time models are likely to become more important, since as the financial market develops transactions will be made almost continuously in time and accordingly no arbitrage condition should be imposed continuously in time.

Our approach is based on the time change defined as the inverse of the quadratic variation of the underlying stochastic process, which we assume to be a general semimartingale whose martingale component is a.s. continuous. Under our setting, the underlying stochastic process becomes a martingale if and only if it becomes Brownian motion after the time change by the celebrated theorem by Dambis, Dubins and Schwarz. To test the martingale hypothesis, we may therefore test whether or not the time changed underlying stochastic process is Brownian motion. There can be many different ways to test whether a given process is Brownian motion. In the paper, we directly test whether their increments are iid normals, using modified versions of already existing tests on multivariate normality. The idea of a time change was explored earlier by Peters and de Vilder (2006) to test whether a given sample may be viewed as being generated from a semimartingale. After removing the drift component using a smoothing method, they suggest to test whether the remaining component is a martingale using a procedure similar to ours. In contrast, we test whether a given process has any non-vanishing drift component. While they do not provide any statistical theory, we fully develop the asymptotics of our methodology.

There are several obvious advantages of our approach relying on continuous time models. First, the continuous time approach is more appropriate for observations collected at relatively high frequencies. Though the tests based on discrete time models are physically implementable also for high frequency data, their limit behaviors are not well defined as the sample interval decreases down to zero. Second, our approach allows for martingales whose increments are not necessarily weakly stationary. This is in contrast with virtually all the existing tests, which are based on stationary discrete martingale difference models. Third, we just assume that the underlying stochastic process is a continuous martingale under the null hypothesis, permitting a wide variety of stochastic volatilities. On the other hand, only a very limited class of stochastic volatilities are allowed for the existing tests. In particular, all the existing tests become invalid if, for instance, present are near-nonstationary stochastic volatilities that Jacquier, Polson and Rossi (2004) and others found in many important financial data.

The theorem by Dambis, Dubins and Schwarz, which our approach heavily relies on, does not apply if the underlying stochastic process has jumps. However, this does not imply that we cannot use our methodology for the stochastic processes with jumps. Our approach may easily accommodate the presence of jumps, as long as they are of a relatively simple type such as the one generated by a compound Poisson process. For such an underlying stochastic process, our approach provides a very simple way of testing if its continuous component is a martingale. To implement our tests for the stochastic processes with suspected jumps, we may just use the tests for jumps, e.g., the one by Barndorff-Nielsen and Shephard (2006), and simply discard the observations in the intervals which are tested positively against the presence of jumps. It would, of course, also be possible to collect the jump components and test whether they may be generated from a martingale separately.[1]

[1]We do not pursue this idea in the paper, since it appears to be extremely difficult to identify and estimate the sizes and locations of the jumps, with a precision that would make the subsequent martingale test meaningful.

The finite sample performances of our tests are evaluated through an extensive set of simulations. The finite sample performances of our tests are reasonably good in general, especially relative to other existing tests developed in the discrete time framework. The overall finite sample rejection probabilities of our tests are quite close to the nominal asymptotic sizes, even in the presence of general nonstationary stochastic volatilities. The finite sample powers of our test statistics, however, are somewhat sensitive to the specific alternatives. They vary widely against different classes of nonmartingales we consider in the paper. It is therefore recommended that multiple tests are used and compared if there is no knowledge on the alternative processes. For illustrative examples, we use our methodology to test whether there is a risk premium in the excess return of some selected set of portfolios, and to test whether the market prices of risks in two countries are identical. The presence of a risk premium is quite evident in all portfolios we investigated, and the market prices of risks appear to differ in every pair of countries we considered in the paper.

The rest of the paper is organized as follows. Section 2 introduces the main idea of our approach involving a time change, followed by a discussion on how to implement the time change in actual applications using discrete observations. It is also addressed how we may effectively deal with the presence of jumps. The test statistics are presented in Section 3. We consider three different types of the tests, i.e., goodness-of-fit tests, smooth tests and invariant tests. Section 4 develops the asymptotic null distributions of the test statistics introduced in Section 3. The asymptotic powers of the test statistics are considered subsequently in Section 5. We show in particular that the tests considered in the paper are generally consistent against nonmartingales. An extensive set of simulation results are given in Section 6. Section 7 concludes the paper. All the mathematical proofs are collected in Mathematical Appendix.

B. Preliminaries

In this section, we introduce some preliminaries that will be used in the subsequent development of our methodology and its asymptotic theory.

1. The Main Idea

We let $X = (X_t)$ be a continuous semimartingale with respect to some filtration $(\mathcal{F}_t)$, which has the representation

$$X_t = A_t + M_t, \tag{3.1}$$

where $A = (A_t)$ is a continuous process of finite variation and $M = (M_t)$ is a continuous local martingale.[2] Both (A_t) and (M_t) are assumed to be adapted to the filtration $(\mathcal{F}_t)$. Strictly, local martingales are not necessarily martingales.[3] We will, however, not distinguish them and simply refer local martingales as just to martingales throughout the paper. Temporarily in this section, we assume that X is observed continuously in time $t \in \mathbb{R}_+$, so that we can more effectively present the main ideas of our approach more effectively. In later sections, our methodology and its asymptotic theory will be derived from a discrete set of samples from X.

Using the usual notation, we denote by $[X]$ the quadratic variation of X. Throughout the paper, we assume that $[X]_t \to \infty$ a.s. as $t \to \infty$. It is well known that any continuous process of finite variation has zero quadratic variation, and therefore, we have $[A] = 0$ and

$$[X] = [M].$$

[2]It will be explained later how we may allow for jumps in X.

[3]Therefore, there are local martingales that are nonmartingales. See, e.g., Karatzas and Shreve (1991) for such examples.

Moreover, we introduce a time change $T = (T_t)$, which is defined as

$$T_t = \inf\{s \geq 0 \,|[X]_s > t\} = \inf\{s \geq 0 \,|[M]_s > t\}. \tag{3.2}$$

As is easily seen, T is a class of nonnegative and nondecreasing stopping times adapted to the filtration $(\mathcal{F}_t)$. Note that the time change T is indeed the generalized time inverse of the quadratic variation $[X]$ of X, or equivalently, of $[M]$ of M.

Now we introduce a continuous process Y, which is defined by

$$Y_t = X_{T_t} = A_{T_t} + M_{T_t}. \tag{3.3}$$

The process Y may be considered as the process X running on a different clock, given by the inverse of the quadratic variation of X. By the celebrated theorem of Dambis, Dubins and Schwarz (DDS) [see Revuz and Yor (2005, pp181)],

$$W_t = M_{T_t}$$

is the standard Brownian motion, adapted to the filtration $(\mathcal{F}_{T_t})$, and we have $M_t = W_{[M]_t}$. The Brownian motion $W = (W_t)$ will be referred to as the DDS Brownian motion of M. As a result, it follows immediately that

Lemma 2.1 *X is a continuous martingale if and only if Y is the standard Brownian motion.*

The 'only if' part if obvious, since in this case A_T term in (3.3) would vanish. To show that 'if' part, note that A_T is a process of finite variation, and that Y cannot be the standard Brownian motion unless $A_T = 0$ for all $t \in \mathbb{R}_+$.

In some special cases, we may obtain A_T more explicitly. To see this, let X be a homogeneous diffusion process that is given by $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$, where μ and σ signify respectively the drift and diffusion functions. Moreover, we assume that $[X]$ is a.s. differentiable with derivative bounded below frow zero. In this case, the time changed process Y becomes another homogeneous diffusion given by

$$dY_t = \frac{\mu(Y_t)}{\sigma(Y_t)^2} dt + dW_t, \tag{3.4}$$

where W is the DDS Brownian motion of the continuous martingale M, $dM_t = \sigma(X_t)dB_t$. This implies that the time change for a non-martingale process transforms the drift function inversely proportional to the square of the diffusion function. As expected, Y becomes the standard Brownian motion when and only when the drift term μ in X vanishes, i.e., when and only when X is a martingale.

Due to Lemma 2.1, we may now test whether Y is the standard Brownian motion, instead of directly testing the martingale hypothesis for X. There can be many different ways to test for a process being the standard Brownian motion. In the paper, we test for the independence and Gaussianity of the increments. In other words, we let

$$Z_i = \frac{1}{\sqrt{\Delta}} \left(Y_{i\Delta} - Y_{(i-1)\Delta} \right) \tag{3.5}$$

for $i = 1, \ldots, N$ and some fixed $\Delta > 0$, and test whether (Z_i) are iid standard normal. It follows immediately from Lemma 2.1 and the definition of Brownian motion that (Z_i) is a sequence of iid standard normals for all $\Delta > 0$ if and only if X is a martingale. This is the main idea of our approach.

Clearly, we may interpret the time change T as a volatility equalizing clock if X is an asset price. It runs slow (fast) when the market for X is volatile (stable), so that the time changed process Y has a constant volatility. It may be interesting to compare the clock T with other time scales considered earlier, such as business cycle time introduced by Burns and Mitchell (1946) and economic time scale by Stock (1987). See also Allais (1966), Barro (1970) and Flood and Garber (1980) for economic models that incorporate some economic time scales.

2. Implementation Using Discrete Samples

For virtually all empirical applications in economics and finance, continuous time observations are not available. Therefore, we assume in the paper that samples are collected discretely in time. More specifically, we suppose that there are n-number of observations on a continuous semimartingale X at intervals of length δ over the time interval $[0, T]$, $T = n\delta$, which are denoted by

$$X_\delta, X_{2\delta}, \ldots, X_{n\delta}.$$

Note that we have

$$[X]_T = [M]_T = N\Delta$$

and

$$T_{N\Delta} = T,$$

if we let N and Δ be respectively the number and the interval of the testing samples extracted from the time-changed process Y, as introduced in the previous subsection.

The asymptotic theory developed in the paper requires that $\delta \to 0$ and $T \to \infty$. Moreover, it is necessary to assume that $[X]_t \to \infty$ as $t \to \infty$, so that $N \to \infty$. We let Δ be fixed. To apply our theory and methodology, we must therefore have relatively high frequency data over a reasonably large time span, i.e., small δ and large T, and the quadratic variation of the underlying process should be unbounded so that the number N of testing samples is not too small. In our applications reported in the paper, we use daily observations which span the time intervals of twenty years or more. We set Δ to be approximately the same as the average monthly realized variance. For Brownian motion, we have $N\Delta = T$, and therefore, $N \to \infty$ as long as $T \to \infty$. On the other hand, our theory and methodology is not applicable for exponential Brownian motion, for which the quadratic variation is bounded and we cannot have $N \to \infty$ even when $T \to \infty$.

Let

$$[X]_t^\delta = \sum_{i\delta \le t} \left(X_{i\delta} - X_{(i-1)\delta} \right)^2$$

i.e., the sample analogue of the quadratic variation of X. Note that $[X]_t^\delta$ is nothing but the realized variance of X over the time interval $[0, t]$. Furthermore, we define

$$T_t^\delta = \inf \left\{ s \ge \middle| [X]_s^\delta > t \right\}$$

and subsequently introduce the stochastic process Y^δ by

$$Y_t^\delta = X_{T_t^\delta}$$

similarly as in (3.3). It is well expected that $[X]_t^\delta \approx [X]_t$, $T_t^\delta \approx T_t$ and $Y_t^\delta \approx Y_t$, as $\delta \approx 0$.

We assume

Assumption 2.1 For all $0 \le s \le t \le T$,

$$a_T(t - s) \le [M]_t - [M]_s \le b_T(t - s),$$

where $a_T, b_T > 0$ are some constants depending only upon T.

Assumption 2.2 For all $0 \le s \le t \le T$,

$$\sup_{0 \le t, s \le T} |A_t - A_s| \le c_T |t - s|$$

for all large T.

Assumption 2.1 is not stringent and holds for a large class of continuous martingales.

We may easily see that the condition is met for Brownian motion with $a_T = b_T = 1$. For Ito martingales given by $dM_t = \sigma_t dW_t$ with some volatility process σ and Brownian motion W, the condition holds with

$$a_T =\leq \inf_{0\leq t\leq T} \sigma_t^2 < \sup_{0\leq t\leq T} \sigma_t^2 = b_T.$$

As is well known, this condition is satisfied for many martingale diffusion processes. See Park (2008) for more discussions on the condition in Assumption 2.1. Assumption 2.2 is also very weak and satisfied by many continuous semimartingales. For instance, the condition obviously holds with $c_T = 1$ for any diffusions having bounded drift functions. It is satisfied for Ornstein-Uhlenbeck (1930) process as well, if we set $c_T = O_p((\log T)^{1/2})$.

It is shown in Park (2008) that

Lemma 2.2 *Under* Assumption 2.1, *we have*

$$\sup_{0\leq t\leq T} \left|[M]_t^\delta - [M]_t\right| = O_p\left(b_T(\delta T)^{1/2}\right)$$

for b_T *introduced in* Assumption 2.1.

Lemma 2.3 *Under* Assumptions 2.1 *and* 2.2, *we have*

$$\sup_{0\leq t\leq T} \left|[X]_t^\delta - [M]_t^\delta\right| = O_p\left((b_T^{1/2}c_T)(\delta^{1/2}T)\right) + O_p\left(c_T^2(\delta T)\right)$$

with b_T *and* c_T *introduced in* Assumptions 2.1 *and* 2.2.

As expected, the realized variance of M converges to the true quadratic variation of M if δ decreases to zero fast enough, compared to the increasing rate of T. Moreover, the realized variance of X approximates that of its martingale component M as long as δ is sufficiently small. In this case, the realized volatility of M or X obtained from the discrete samples can be used as an estimate for the quadratic variation $[M]$. It should be noted that we need to use observations at higher frequencies, as the sampling horizon expands. This is to obtain the uniform consistency of $[M]$ or $[X]$ over an expanding time horizon $[0, T]$. In general, it is required that δ decrease at a faster rate, if the maximal rates, b_T and c_T, of increase in the quadratic variation of M or the bounded variation component A.

Our tests, which will be introduced in the next section, are based on

$$Z_i^\delta = \frac{1}{\sqrt{\Delta}}(Y_{i\Delta}^\delta - Y_{(i-1)\Delta}^\delta). \tag{3.6}$$

Given the results in Lemmas 2.2 and 2.3, it is not difficult to expect that the actual testing samples (Z_i^δ) are uniformly close to the infeasible testing samples (Z_i) defined earlier in (3.5). This will be presented formally in what follows.

3. Testing for Processes with Jumps

The DDS theorem, which allows us to convert any continuous martingale to Brownian motion, no longer holds when there are discontinuous jumps in the process. However, our approach is still valid in the presence of jumps. Sometimes, the jumps in the data are pretty conspicuous and easy to detect. In this case, we may simply skip the suspected location of each jump in defining time change $(T_{i\Delta})$ and extracting test samples (Z_i), and make sure that test samples are all originated from the continuous part of the process. For instance, if τ is the location of a jump, we may define $T_{i\Delta}$, $i = 1, \ldots, m$, until it reaches τ, i.e., $T_{m\Delta} < \tau$, and then start defining $T_{(m+1)\Delta}$ and on again from $T_{(m+1)\Delta} = \tau$. If the jumps in the data are of a more complicated nature and hard to detect, we may define time change $(T_{i\Delta})$ ignoring discontinuities and test for the presence of jumps in each of the time intervals $[T_{(i-1)\Delta}, T_{i\Delta}]$. It is quite obvious that our methodology is applicable, as long as we discard the corresponding test samples Z_i from the intervals where we find positive evidence of jumps, and use only Z_i from the intervals where the underlying process is continuous. Of course, this procedure based on a pretest may result in some size distortions in finite samples. However, the deleterious effect on test size would be minimal, if we only have a small number of jumps in the process.

To detect jumps, we may use the Hausman (1978)-type jump test, proposed by Barndorff-Nielsen and Shephard (2006) that relies on the comparison between bi-power and quadratic variation of X. The bi-power variation of X is defined as

$$\{X\}_t = \operatorname*{plim}_{\delta\to 0} \sum_{i\delta\le t} |X_{i\delta} - X_{(i-1)\delta}||X_{(i-1)\delta} - X_{(i-2)\delta}|$$

for each $t \geq 0$. The concept of bi-power variation was introduced by Barndorff-Nielsen and Shephard (2004), who showed that the bi-power variation of a semimartingale with jumps is just a constant multiple of the quadratic variation of its continuous martingale component, i.e.,

$$[M]_t = \kappa\{X\}_t$$

with $\kappa = \pi/2$.

For the test of the presence of jumps in the time interval $[T_{(i-1)\Delta}, T_{i\Delta}]$, we use the test statistic given by

$$\sqrt{\frac{\delta}{\{\{X\}\}_i^\delta/(\{X\}_i^\delta)^2}}\left(\frac{\kappa\{X\}_i^\delta}{[X]_i^\delta} - 1\right),$$

where $[X]_i^\delta = \sum_j (\triangle X_{j\delta})^2$, $\{X\}_i^\delta = \sum_j |\triangle X_{j\delta}||\triangle X_{(j-1)\delta}|$ and

$$\{\{X\}\}_i^\delta = (1/\delta)\sum_j |\triangle X_{j\delta}||\triangle X_{(j-1)\delta}||\triangle X_{(j-2)\delta}||\triangle X_{(j-3)\delta}|$$

with $\triangle X_{(j-k)\delta} = X_{(j-k)\delta} - X_{(j-k-1)\delta}$ and the summation index j running over the range for which the time indices of summands are in $[T_{(i-1)\delta}, T_{i\Delta}]$. The test statistic has a limit normal distribution with mean zero and variance $\pi^2/4 + \pi - 5 \simeq 0.609$.

It is interesting to note that our approach, if applied to martingales, can be used to test for the presence of jumps. Suppose $A \equiv 0$. To test whether or not there are jumps, we may test whether our test samples (Z_i) are independent standard normals, exactly as if we tested for the martingale hypothesis. Under the maintained assumption of martingale, our test samples are or are not independent standard normals, depending upon whether or not there are jumps. To fix the idea, we consider the Poisson-based jump of Merton (1976). We let N_t be a Poisson process with intensity λ, and J_t is a Gaussian random variable with mean μ and variance σ^2, and look at the process with the Poisson jump component defined by

$$X_t = M_t + \sum_{i=1}^{N_t} J_i, \tag{3.7}$$

where M is a continuous martingale. For simplicity, we assume that M, N, and J are independent. The quadratic variation of X in (3.7) is given by

$$[X]_t = [M]_t + \sum_{i=1}^{N_t} J_i^2,$$

see, e.g., Lepingle (1976).

In the presence of jumps, the distribution of our test samples (Z_i) would not be normal. To see this, we assume that the time change (T_t) is observable. In this case, we may indeed easily deduce that the probability density ψ_i of Z_i, conditional on $\tau_i = T_{i\Delta} - T_{(i-1)\Delta}$, is given by

$$\psi_i(z) = \sum_{k=0}^{\infty} \frac{(\lambda\tau_i)^k e^{-\lambda\tau_i}}{k!} \frac{1}{\sqrt{2\pi}\sqrt{(\Delta + k\sigma^2)/\Delta}} \exp\left[-\frac{1}{2}\frac{(z - k\mu/\sqrt{\Delta})^2}{(\Delta + k\sigma^2)/\Delta}\right].$$

It is well known that the distribution given by the density $\psi_i(z)$ departs from the standard normal and exhibits excess kurtosis. See Aït-Sahalia (2004) for more discussions on the fat-tail distribution originated from the jump component.

C. Test Statistics

In our approach, we examine the martingale hypothesis by testing for independent standard normality of the test samples (Z_i). In this section, we introduce various methods to test for the independent standard normality of (Z_i). We let

$$Z_{i_m} = (Z_i, Z_{i-1}, \ldots, Z_{i-m+1})'$$

for some fixed $m \geq 1$, and propose three types of test statistics for multivariate standard normality of (Z_{i_m}). Testing based on the m-dimensional vector can jointly test the independence and the normality of the samples, and it provides power of the tests in a wider range of alternatives than the tests based marginal distributions. Note that tests based on the marginal distribution of (Z_i) may not be consistent against some Gaussian semimartingales.

1. Goodness-of-fit Tests

Define the m-dimensional empirical distribution function

$$F_N(z) = \frac{1}{N_m} \sum_{i=1}^{N_m} 1\{Z_{i_m} \leq z\}, \tag{3.8}$$

where

$$1\{Z_{i_m} \leq z\} = 1\{Z_i \leq z_1\} 1\{Z_{i-1} \leq z_2\} \cdots 1\{Z_{N-m+1} \leq z_m\},$$

$Z_{i_m} = (Z_1, \ldots, Z_m) \in R^m$, and $N_m = N - m + 1$. Put $F_0(z) = \Phi(z_1) \cdots \Phi(z_m)$, where $\Phi(\cdot)$ is the c.d.f of $N(0,1)$. Then, the m-dimensional Cramer-von Mises statistic (or CvM) is given by

$$W_m^2 = N_m \int_{R^m} \{F_N(z) - F_0(z)\}^2 \, dF_0(z)$$

and the corresponding Kolmogorov-Smirnov statistic (KS) is

$$D_m = \sqrt{N_m} \sup_{z \in R^m} |F_N(z) - F_0(z)|.$$

These are m-dimensional goodness-of-fit tests which are based on the empirical distribution function in Equation (3.8). Both the independence and normality are tested by the distributional distance between the empirical distribution function and the population distribution, which is in our case, a standard multivariate normal distribution. For all the alternatives in which the empirical distribution function differs from the population distribution function, those statistics will be consistent, and as a result, form omnibus tests.

There are several advantages of these statistics. First, they do not require any numerical integration or approximation. Therefore, we can easily calculate the statistics based on computationally equivalent forms of the statistics. For example, the univariate CvM statistic can be calculated by

$$W_1^2 = \sum_{i=1}^{N} (V_i - \frac{2i-1}{N})^2 + \frac{1}{12N},$$

where V_i are the ordered $\Phi(Z_i)$. Also, the Kolmogorov-Smirnov statistic is given by

$$D_1 = \max(\max_i (V_i - \frac{i-1}{N}), \max_i (\frac{i}{N} - V_i)).$$

Second, it is known by Stephens (1970) that the goodness-of-fit tests based on the empirical distribution function is more powerful than Pearson's chi-square test. Third, unlike the usual goodness-of-fit tests for multivariate normality, our null hypothesis is specifically given by a standard multivariate normality. Therefore, the asymptotic distributions of our statistics are much simpler and easy to simulate. In Section 4, we will discuss the asymptotic distribution of the multivariate goodness-of-fit statistics more detail.

2. Smooth Tests

Assume that under the alternative there exists a multivariate distribution function $F(\cdot)$ given by

$$F(z) = \sum_{0 \le s_1 + s_2 + \cdots + s_m \le p} \frac{(-1)^m}{\prod_{j=1}^m s_j!} E\left[\prod_{i=1}^m H_{s_i}(z_i)\right] \prod_{p=1}^m H_{s_p-1}(z_p)\phi(z_p),$$

where p is a positive integer, $H_i(\cdot)$ is the Hermite polynomial with order i, and $\phi(\cdot)$ is the density function of $N(0,1)$. This is a truncated Gram-Charlier series up to order p running from order 0. Note that the terms up to order 2 vanish in the usual Gram-Charlier series that expand the standard multivariate distribution function. In this regard, we call the above series as an augmented Gram-Charlier series. Define a q-dimensional vector given by

$$h_{m,p} = (h_{p1}, h_{p2}, \ldots, h_{pq(p)})',$$

where h_{pj} is the normalized sample analogue estimator for $\mathbb{E}[\prod_{k=1}^m H_{s_k}(z_k)]$ when $\sum_{k=1}^m s_k = p$ while $s_1 \neq 0$, and $q(p) = \binom{m+p-1}{p} - \binom{m+p-2}{p}$. For example, if $m = 2$ and $p = 2$, then we have

$$h_{2,2} = \left(\frac{1}{\sqrt{2!N_m}} \sum_{i=1}^{N_m} H_2(Z_i^\delta), \frac{1}{\sqrt{1!1!N_m}} \sum_{i=1}^{N_m} H_1(Z_i^\delta) H_1(Z_{i+1}^\delta) \right)'.$$

The smooth test for the standard multivariate normality of $(Z_{i_m}^\delta)$ is defined by

$$I_{m,P} = \sum_{p=1}^P H_{m,p}, \tag{3.9}$$

where $H_{m,p} = h'_{m,p} h_{m,p}$.

Note that each element of $h_{m,p}$ is asymptotically independent standard normal under the null hypothesis because of the orthogonality of the Hermite polynomials, and therefore, the limiting distribution of $H_{m,p}$ is given by the chi-square distribution with degrees of freedom $q(p)$. Moreover, the limiting distribution of $I_{m,P}$ is also the chi-square distribution with degrees of freedom $\sum_{p=1}^{P} q(p)$; see Theorem 4.3 for detail. Intuitively, the smooth test compares the coefficients of the augmented Gram-Charlier series with zero vector, which is the case under the null distribution. Unlike the usual smooth tests based on the Hermite polynomials (Koziol (1986) and Bogdan (1999)), the test only uses asymptotically independent coefficients in the expansion. Specifically, if the coefficients with $s_1 = 0$ is included in $h_{m,p}$, the asymptotic independence will no longer hold. The choice of truncation order P can be given by a specific prior based on the knowledge of the alternative hypothesis, or a data-driven selection rule can be adopted. Specifically, if the alternative hypothesis for the cumulative pricing error is known to be Brownian motion with drift, we can expect that $P = 1$ will be optimal in detecting non-zero mean of $F(\cdot)$. However, if such a knowledge is not available, we can use a specific type of information criteria. For example, a modification of Schwarz Bayesian Information Criterion of Bogdan (1999) can be considered as follows;

$$P = \underset{1 \le p \le d(N_m)}{\operatorname{argmin}} \{I_{m,p} - p \log(N_m) \ge I_{m,j} - j \log(N_m), j = 1, \ldots, d(N_m)\},$$

where $d(N_m)$ is the upper bound for P.

3. Invariant Tests

Define the Euclidean norm on R^m of the m-dimensional vector $Z^{\delta}_{i_m}$ by

$$R_i = (Z^{\delta}_{i_m})'(Z^{\delta}_{i_m}), \quad i = 1, \ldots, N_m. \tag{3.10}$$

Our invariant test is the Cramer-von Mises statistic given by

$$J_m = N_m \int_0^{\infty} (G_N(x) - \Psi(x))^2 \, d\Psi(x),$$

where

$$G_N(x) = \frac{1}{N_m} \sum_{i=1}^{N_m} 1\{R_i \leq x\},$$

and $\Psi(\cdot)$ is the cumulative distribution function of $\chi^2(m)$.

The main idea of this test is to compare the distribution of the squared radii of the random vector $Z^{\delta}_{i_m}$ with the chi-square distribution with degrees of freedom m. Unlike the test of Koziol (1982), the null hypothesis is completely specified as standard multivariate normality, and therefore, estimation of mean and variance-covariance matrix is not required. However, each observation R_i is serially correlated because the random vector $Z^{\delta}_{i_m}$ is correlated with $Z^{\delta}_{j_m}$ for $i < j \leq i + m - 1$. In this case, the asymptotic distribution of the Cramer-von Mises statistic will be different from the usual case because the empirical process will have different covariance kernel. We will discuss on the asymptotic distribution of the invariant test in Section 4.

4. Variance Ratio Tests

Variance ratio test is defined by

$$V_q = \frac{\bar{\sigma}^2_c(q)}{q} - 1,$$

where

$$\bar{\sigma}^2_c(q) = \frac{1}{N - q + 1} \sum_{i=q}^{N} \left(\sum_{j=1}^{q} Z_{i+j-q} \right)^2 .$$

The variance ratio test compares the variance of q partial sum of (Z_i) with q. If (Z_i) is i.i.d normal, then the variance of q partial sum of (Z_i) should be equal to q. This idea has been used to test the independence of stock returns; Cochrane (1988) and Lo and MacKinlay (1988). Under the null hypothesis, (Z_i) is i.i.d normal, hence the statistic is much simpler than the conventional variance ratio tests which considers non-zero mean and conditional heteroskedasticity. Moreover, unlike the conventional tests, this test can detect the non-zero mean as well as the non-unity variance of (Z_i). One can expect that the variance ratio test is powerful against stationary alternatives because (Z_i) under staionary alternatives are in general correlated. In our Monte Carlo simulation study, we show that the test is in fact the most powerful among our time change tests against stationary alternatives, such as Ornstein-Uhlenbeck (1930) process and Feller (1951)'s square root process.

D. Asymptotic Null Distribution

Our asymptotic theory is based on the assumption that the observation time interval δ is shrinking to 0 (or infill), while the observation time horizon T is diverging to infinity (or longspan). These two assumptions are jointly important in dealing with the asymptotic theory for test statistics based on the increment of time changed process. First of all, the "infill" assumption guarantees that we collect higher frequency data to gather more information on the volatility (or volatility time), which implies that the sample time change will converge to the true time change in the infill. Hence, we do not need to worry about the errors of time change, which might affect the distribution of the test statistics, at least in the infill setting. Secondly, the "longspan" assumption implies that $[M]_T \to \infty$ with $\Delta > 0$ fixed. Since the number of samples for the time changed process will depend on $[M]_T$, this longspan assumption enables us to rely on the usual asymptotic theory that the sample size goes to infinity.

Assumption 4.1 We assume

$$N = o_p\left(\frac{1}{\alpha_{\delta,T}\log(1/\alpha_{\delta,T})}\right)$$

with $\alpha_{\delta,T} = b_T(\delta T)^{1/2})$, for small δ and large T.

Lemma 4.1 Under Assumptions 2.1 and 4.1, we have

$$\sup_{1\leq i\leq N} \left|Z_i^\delta - Z_i\right| = o_p(N^{-1/2})$$

for large N.

Theorem 4.2 Let Assumptions 2.1 and 4.1 hold. Define D_m^δ, $W_m^{2,\delta}$, $I_{m,P}^\delta$, and J_m^δ to be the Kolmogorov-Smirnov, Cramer-von Mises statistics for goodness-of-fit, the smooth test, and the invariant test based on the test samples (Z_i^δ). Then we have

$$D_m^\delta = D_m + o_p(1), W_m^{2,\delta} = W_m^2 + o_p(1), I_{m,P}^\delta = I_{m,p} + o_p(1), J_m^\delta = J_m + o_p(1).$$

for large N.

Theorem 4.3 Let Assumptions 2.1 and 4.1 hold.

(a) Goodness-of-fit tests

$$\begin{aligned} D_m &\to_d \sup_{t\in[0,1]^m} |B(t)|, \\ W_m^2 &\to_d \int_{[0,1]^m} B(t)^2 dt, \end{aligned}$$

where $B(t) : [0,1]^m \mapsto \mathbb{R}$ is a Gaussian process with covariance kernel

$$K_B(s,t) = \lim_{N\to\infty} \mathbb{E}\left[b_N(s)b_N(t)\right],$$

where $b_N(t)$ is the empirical process given by

$$b_N(t) = \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} (1\{\Phi(Z_{i_m}) \leq t\} - t_1 \cdots t_m).$$

Here $\Phi(\cdot)$ is the distribution function of $N(0, I_m)$.

(b) Smooth tests

$$I_{m,P} \to_d \chi^2 \left(\sum_{p=1}^{P} q(p) \right),$$

where $q(p) = \binom{m+p-1}{p} - \binom{m+p-2}{p}$.

(c) Invariant tests

$$J_m \to_d \int_0^1 W(s)ds,$$

where $W(s) : [0,1] \mapsto R$ is a Gaussian process with covariance kernel

$$K_W(t,s) = \lim_{N\to\infty} \mathbb{E}\left[w_N(s)w_N(t)\right],$$

and $w_N(t)$ is the corresponding empirical process given by

$$w_N(t) = \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} \left(1\left\{\Psi(R_i) \le t\right\} - t\right).$$

Here $\Psi(\cdot)$ is the chi-square distribution function with degrees of freedom m.

(d) Variance ratio tests

$$\sqrt{N} V_q \to_d N\left(0, \frac{4q}{3} + \frac{2}{3q}\right).$$

The asymptotic distribution of D_1 and W_1^2 is well-known by Kolmogorov (1933), Smirnov (1939), Donsker (1952), Durbin and Knott (1972) and Stephens (1970). However, the limiting distribution of D_m and W_m^2 with $m > 1$ is not known in general because (i) the observations for the m-dimensional vector Z_{i_m} are correlated, or m-dependent process, and (ii) the property of the Gaussian process with dimension $m > 1$ is unknown in general. Similarly, the asymptotic distribution of J_m with $m > 1$ is unknown in general because of the same reason in (i). Even if the asymptotic distributions for the goodness-of-fit tests and the invariant tests are hard to find analytically, the critical values for the tests can be easily obtained from a simulation

Table V. Critical Values for the Test Statistics in Bivariate Case

	D_2			W_2^2			J_2		
N	10%	5%	1%	10%	5%	1%	10%	5%	1%
10	1.1443	1.3013	1.6134	0.3641	0.4873	0.8423	0.5848	0.7745	1.1985
20	1.2386	1.4064	1.7393	0.3783	0.5052	0.8429	0.6069	0.8109	1.3021
30	1.2854	1.4516	1.7876	0.3809	0.5096	0.8447	0.6158	0.8181	1.3312
40	1.3168	1.4863	1.8342	0.3829	0.5148	0.8528	0.6191	0.8301	1.3451
50	1.3393	1.5102	1.8454	0.3821	0.5131	0.8457	0.6168	0.8354	1.3534
100	1.4029	1.5735	1.9184	0.3855	0.5191	0.8436	0.6217	0.8370	1.3599
200	1.4575	1.6294	1.9778	0.3872	0.5183	0.8475	0.6216	0.8402	1.3834

Note: Monte Carlo approximation to the bivariate Goodness-of-fit tests and invariant tests. If tabulated value is λ, $\mathbb{P}\{t_N > \lambda\} = \alpha$ for the statistic t_N and significance level α. D_2 is calculated by finding the maximum distance of the distributions in the observed points, and W_2^2 is calculated by the formula presented by Zimmerman (1993). Iterations for the simulation was 100,000.

study. To find the critical values of D_m, W_m^2, and J_m, Monte-Carlo simulations based on samples from $i.i.d.N(0,1)$ with sample size $N = 10, 20, 30, 40, 50, 100, 200$ are used. Table V presents the percentiles of the statistic D_m, W_m^2, and J_m for the bivariate case.

On the other hand, the smooth tests have the chi-square distribution with $\sum_{p=1}^{P} q(p)$ degrees of freedom. This is because the statistics are constructed in such a way that all the component in $I_{m,P}$ are independent with each other. This is possible because of the orthogonality of Hermite polynomials with respect to the standard normal distribution.

E. Asymptotic Power

We assume

Assumption 5.1 We assume

$$N = o_p\left(\frac{1}{\beta_{\delta,T}\log(1/\beta_{\delta,T})} \wedge \frac{a_T^2}{c_T^2\delta^2\beta_{\delta,T}^2}\right),$$

where

$$\beta_{\delta,T} = b_T(\delta T)^{1/2} + (b_T^{1/2}c_T)(\delta^{1/2}T) + c_T^2(\delta T)$$

for small δ and large T.

Lemma 5.1 Under Assumptions 2.1, 2.2 and 5.1, we have

$$\sup_{1\leq i\leq N}\left|Z_i^\delta - Z_i\right| = o_p(N^{-1/2})$$

for large N.

Assumption 5.2 There exists a distribution function F_1 such that

$$F_N = F_1 + O_p(N^{-1/2})$$

and $F_1 \neq \Phi(z)$ on a subset of $\mathbb{R}^m$ with positive Lebesgue measure.

Theorem 5.2 Let Assumptions 2.1, 2.2 and 5.1 hold. Then the statistics D_m^δ, $W_m^{2,\delta}$, $I_{m,P}^\delta$ and J_m^δ are all consistent and diverge at the rate of $\sqrt{N}$.

F. Monte Carlo Study

In this section, we examine the finite sample performance of our test statistics based on a Monte-Carlo simulation. We investigate the size and power performances with some data generating processes (DGPs), which are calibrated to various financial data. Also, we compare our time change statistics with the martingale tests recently developed by Hong and Lee (2005) and Escanciano and Mayoral (2007). For the purpose of comparison, we use the conventional observation frequencies, such as monthly, quarterly, and yearly for the discrete time martingale tests. For the sake of reference, we describe the discrete time tests first.

Hong and Lee (2005) consider generalized spectral tests under conditional heteroskedasticity with general form. Their test statistic is given as follows:

$$\hat{M}_1(p) = \left[\sum_{j=1}^{T-1} k^2(j/p)(T-j)\int |\hat{\sigma}_j^{(1,0)}(0,v)|^2 dW(v) - \hat{C}_1(p)\right] / \sqrt{\hat{D}_1(p)},$$

where $W : \mathbb{R} \to \mathbb{R}^+$ is a nondecreasing function that weighs sets symmetric about zero equally, and

$$\begin{aligned}
\hat{C}_1(p) &= \sum_{j=1}^{T-1} k^2(j/p)\frac{1}{T-j}\sum_{t=j+1}^{T-1} X_t^2 \int |\hat{\psi}_{t-j}(v)|^2 dW(v), \\
\hat{D}_1(p) &= 2\sum_{j=1}^{T-2}\sum_{l=1}^{T-2} k^2(j/p)k^2(l/p)\int\int \left|\frac{1}{T-\max(j,l)}\right. \\
&\quad \times \left.\sum_{t=\max(j,l)+1}^{T} X_t^2 \hat{\psi}_{t-j}(v)\hat{\psi}_{t-l}(v')\right|^2 dW(v)dW(v'),
\end{aligned}$$

and $\hat{\psi}_t(v) = e^{ivX_t} - \hat{\phi}(v)$ and $\hat{\phi}(v) = T^{-1}\sum_{t=1}^{T} e^{ivX_t}$, and $W(\cdot)$ is the $N(0,1)$. Here $k : \mathbb{R} \to [-1,1]$ is a symmetric kernel function, and we choose the Bartlett kernel as suggested in their paper. The lag order p is selected from their data-driven lag order. The data-driven rule involes the choice of a preliminary lag order $\bar{p}$, and we set $\bar{p} = 5$ in our simulation experiments. We denote this test as HL.

Escanciano and Mayoral (2007)'s data-driven smooth tests are defined as follows:

$$T_{T,m} = \sum_{j=1}^{m} \hat{\varepsilon}_{j,T}^2,$$

where $\hat{\varepsilon}_{j,n}$ are the sample principle components given by

$$\begin{aligned}\hat{\varepsilon}_{j,T} &= \lambda_j^{-1/2}\int_{\mathbb{R}}\psi_j(\tau_T^2(x))R_T(x)\tau_T^2(dx)\\ &= \lambda_j^{-1/2}\frac{\sqrt{2}}{\hat{\sigma}^2T}\sum_{t=1}^{T}X_t^2R_T(X_{t-1})\sin((j-1/2)\pi\tau_T^2(X_{t-1})).\end{aligned}$$

Here $\lambda_j = 1/((j-1/2)\pi^2)$, $\psi_j(t) = \sqrt{2}\sin((j-1/2)\pi t))$, for $t \in [0,1]$, $j = 1, \ldots$, and

$$R_T(x) = \frac{1}{\hat{\sigma}\sqrt{T}}\sum_{t=1}^{T}X_t 1\{X_{t-1} \leq x\}, \qquad \text{for } x \in \mathbb{R},$$

where $\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}X_t^2$, and $\tau_T^2(x) = \hat{\sigma}^{-2}T^{-1}\sum_{t=1}^{T}X_t^2 1\{X_{t-1} \leq x\}$. We use their optimal data-driven test with $m = \tilde{m}$, where

$$\tilde{m} = \min\{m : 1 \leq m \leq d; L_m \geq L_h, h = 1, \ldots, d\},$$

with $L_m = T_{T,m} - \pi(m, T, q)$, d is an upper bound which can be arbitrary large, and

$$\pi(m,T,q) = \begin{cases} m\log T, & \text{if} \quad \max_{1\leq j\leq d}|\hat{\varepsilon}_{j,T}| \leq \sqrt{q\log T} \\ 2m, & \text{if} \quad \max_{1\leq j\leq d}|\hat{\varepsilon}_{j,T}| > \sqrt{q\log T}, \end{cases}$$

where q is some fixed positive number. We follow the suggestion from the paper and set $q = 2.4$. We denote the test as EM.

The number of Monte Carlo simulations is 5,000 for all the DGPs and we obtain the empirical size and power based on the experiments. We set the observation horizon T to be 50, and the sampling interval $\delta = 1/250$. This is equivalent to the daily observations over 50 year horizon. For our time change test statistics, we set Δ to be average monthly, quarterly, and yearly quadratic variations. For the discrete time tests, we use the conventional monthly, quarterly, and yearly observations with fixed time interval. In this case, the number of normal samples (Z_i) resulting from time change will be comparable to the number of observations in the conventional

Table VI. Data Generating Processes for Simulations

Hypothesis	Label	Data Generating Process
Null	BM	$dX_t = 0.16dB_t$
	RSSVM	$dX_t = \sigma_{s_t} dB_t$
		$s_t = 0, 1$ with $(\sigma_1, \sigma_0) = (0.236, 0.0784)$
		$1 - p = P\{s_t = 0 \| s_{t^-} = 1\} = 0.708dt$
		$1 - q = P\{s_t = 1 \| s_{t^-} = 0\} = 0.108dt$
	ASVM	$dX_t = \sigma_t dB_t$
		$d\ln\sigma_t = 1.103(-1.899 - \ln\sigma_t)dt + 1.158dV_t$
		$\mathrm{corr}(dB_t, dV_t) = -0.318dt$
Alternative	BD	$dX_t = 0.0672dt + 0.16dB_t$
	RSSV	$dX_t = 0.08dt + \sigma_{s_t} dB_t$
		$s_t = 0, 1$ with $(\sigma_1, \sigma_0) = (0.236, 0.0784)$
		$1 - p = P\{s_t = 0 \| s_{t^-} = 1\} = 0.708dt$
		$1 - q = P\{s_t = 1 \| s_{t^-} = 0\} = 0.108dt$
	ASV	$dX_t = 0.08dt + \sigma_t dB_t$
		$d\ln\sigma_t = 1.103(-1.899 - \ln\sigma_t)dt + 1.158dV_t$
		$\mathrm{corr}(dB_t, dV_t) = -0.318dt$
	OU	$dX_t = 0.25(0.07 - X_t)dt + 0.02dB_t$
	SQ	$dX_t = 0.25(0.07 - X_t)dt + 0.075\sqrt{X_t}dB_t$

Note: The table presents data generating processes for which finite sample performances of the test statistics are assessed. H_0 represents the null of continuous martingale while H_1 indicates alternative hypothesis. Each model represents a stochastic process under the hypothesis.

frequency. We call the time change intervals (in our time change statistics), or the observation intervals (in the discrete time statistics) as the test intervals in the sense that those intervals are used to test the martingale hypothesis.

Table VI presents the specifications for each model under the null and alternative hypothesis. For the null hypothesis, we consider Brownian motion (BM), regime switching stochastic volatility martingale (RSSVM), and asymmetric stochastic volatility martingale (ASVM). BM is chosen to match the standard deviation of daily log returns on Dow Jones Industrial Average (or DJIA) from 1968 to 2008. RSSVM is used in Shaller and Norden (1997) to estimate the volatility of stock returns, and ASVM is used in option pricing literatures to model the leverage effect; see Harvey and Shephard (1996). The parameter values of RSSVM is obtained by their estimation results, while those of ASVM is obtained from Yu (2005). For the alternative hypothesis, we have Brownian motion with drift (BD), regime switching stochastic volatility (RSSV), asymmetric stochastic volatility (ASV), Ornstein-Uhlenbeck (1930) process (OU), and Feller (1951)'s square root process (SQ). BD, RSSV, and ASV are non-martingale counterparts of BM, RSSVM, and ASVM in the sense that they have an added non-zero bounded variations to the martingale components. OU and SQ are widely used in term structure models and the parameter values are used in the Monte Carlo simulations of Ait-Sahalia (2002).

Table VII presents the size performances of our time change statistics and the discrete time statistics. It can be seen that all tests except for EM test have satisfactory size performances with the yearly test interval. With quarterly test interval, the empirical sizes for our tests are overall satisfactory but the invariant tests are slightly over rejecting the null hypothesis in ASVM. With monthly test interval, all of our tests except for the variance ratio tests have the tendency of over-rejecting the null. This tendency is well expected in our time change statistics because as the time change interval gets smaller, the time change ($T^{\delta}_{i\Delta}$) becomes less accurate. More precisely, with small test interval, time change error ($|T^{\delta}_{i\Delta} - T_{i\Delta}|$) becomes larger, so that the sampled increments of DDS Brownian motion $X_{T^{\delta}_{i\Delta}} - X_{T^{\delta}_{(i-1)\Delta}}$ is no longer close to the true increments $X_{T_{i\Delta}} - X_{T_{(i-1)\Delta}}$ and the deviation is larger compared to the magnitude of Δ. As a result, the time change errors result in the distortions of size performances. On the other hand, EM test has the tendency that the size performances get better as the test interval decreases. However, we can see that the size performance for ASVM is still not very satisfactory even with monthly test interval. This implies that EM test may suffer size distortion even if the number of observa-

Table VII. Size Performances of Tests

	BM						RSSVM						ASVM					
dt	M		Q		Y		M		Q		Y		M		Q		Y	
	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%
D_1	12.9	6.8	10.4	5.4	9.8	5.0	45.2	29.0	10.6	5.5	9.5	4.9	76.8	60.9	16.9	8.6	10.0	4.9
W_1^2	12.2	6.0	10.1	5.2	9.9	5.2	40.0	21.6	10.6	5.3	9.9	4.7	76.3	57.6	15.2	7.6	9.8	5.1
W_2^2	11.6	5.7	10.5	5.4	10.2	5.1	26.7	13.8	10.6	5.2	9.9	4.8	61.0	38.7	13.7	6.7	9.9	4.9
$I_{1,2}$	8.5	4.3	10.0	5.2	9.6	4.8	7.0	3.0	8.5	4.1	9.0	4.6	32.0	24.2	9.0	4.6	8.4	4.2
$I_{1,3}$	6.3	2.9	9.2	4.6	9.9	5.6	4.8	2.3	7.1	3.5	8.9	5.5	27.4	20.4	7.3	3.5	8.0	4.5
$I_{1,4}$	15.6	7.7	8.1	4.6	9.4	6.1	74.2	58.9	7.3	3.8	8.6	5.6	92.9	86.7	14.0	7.6	7.4	4.4
J_1	19.2	11.7	10.6	5.4	9.4	4.7	82.8	73.7	12.6	6.7	9.7	5.0	97.8	95.5	27.6	18.7	10.4	5.1
J_2	18.1	10.1	10.5	5.6	9.3	4.5	80.2	68.4	11.8	6.3	9.8	4.9	97.2	94.6	25.3	15.9	10.0	4.7
J_3	16.6	9.6	10.9	5.9	10.4	5.2	70.5	55.4	11.9	6.4	10.6	5.4	94.2	89.2	21.3	13.2	10.1	5.0
$V_{N/15}$	8.7	4.9	9.2	3.4	8.7	4.2	9.0	5.1	9.2	3.7	8.4	4.6	13.1	8.4	8.5	4.6	7.8	3.4
$V_{N/10}$	8.6	4.7	8.7	3.3	8.1	4.3	9.2	5.4	9.4	3.2	8.5	4.8	12.0	8.1	8.3	4.9	7.4	3.9
$V_{N/5}$	8.1	5.8	8.1	2.0	7.4	5.2	8.3	5.9	7.9	2.1	7.5	5.3	11.4	8.2	8.5	5.8	6.8	4.7
HL	11.1	7.6	10.4	7.1	9.8	6.6	10.3	7.0	9.3	5.7	7.7	4.7	8.2	5.3	7.5	4.5	6.8	4.1
EM	11.4	6.8	14.1	9.0	61.1	59.6	12.6	8.1	38.4	36.5	81.9	81.3	24.0	20.7	53.8	52.6	88.6	88.5

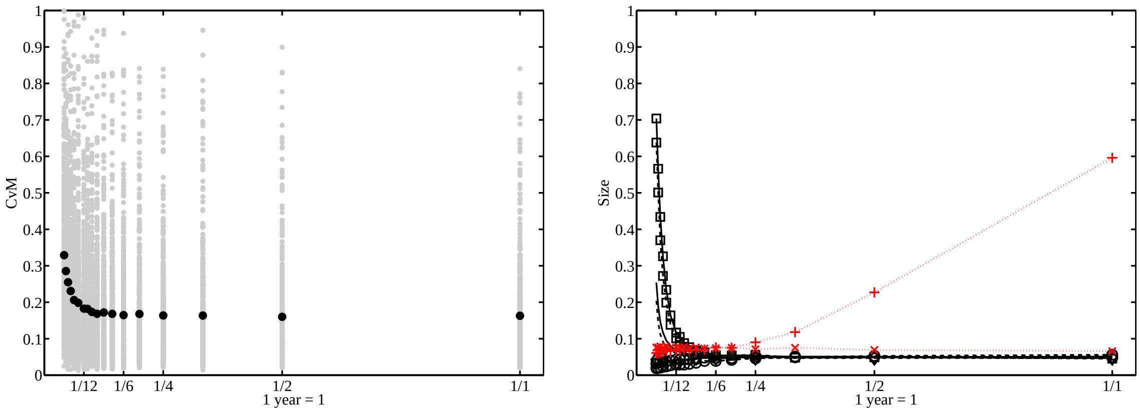
Note: Empirical sizes of the data-generating processes; BM, RSSVM, and ASVM. W_m^2 and D_m are the Cramer-von Mises and Kolmogorov-Smirnov tests of goodness-of-fit, $I_{m,P}$ is the smooth tests, J_m is the invariant test, and V_q is the variance ratio test. Here m denotes the dimension of the vector process z_{i_m} constructed from the normal samples. Δ is selected as the average monthly, quarterly, and yearly quadratic variation for each process. HL is the generalized spectral test of Hong and Lee (2005) with preliminary lag order 5, and EM is the data-driven smooth tests of Escanciano and Mayoral (2007). For HL and EM, we use the increments of the process at monthly, quarterly, and yearly frequency. Number of iterations is 5,000 and $\delta = 1/240$, $T = 50$. The numbers are proportion of rejection in percentage unit.

tions are relatively large. Meanwhile, HL test has satisfactory size performances over various test intervals.

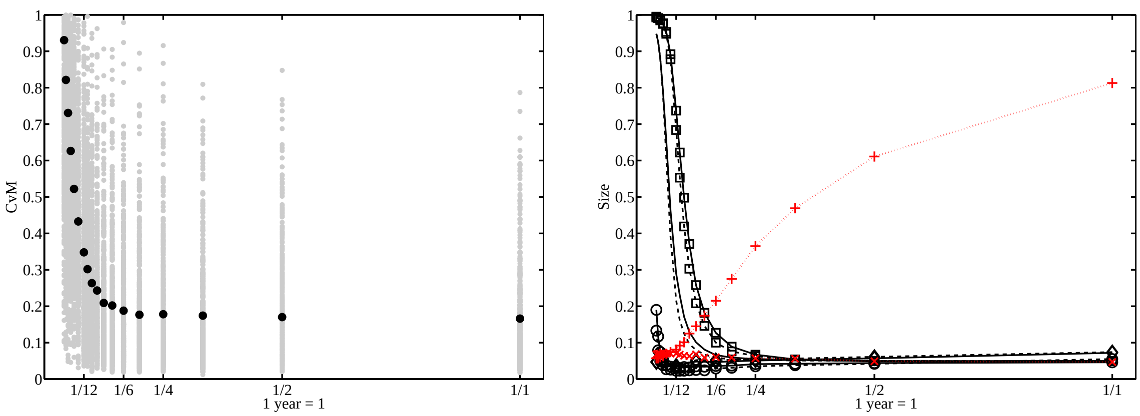
Figure 6 presents the size performances of the tests for various test intervals. It is clear that the size performances of our time change tests are not satisfactory if we choose the test interval smaller than 1/4. This is because the test interval is too small to make the time change work effectively. In order to check whether the time change work poorly for small test intervals, we calculate the Cramer-von Mises (CvM) distances of the normal samples Z_i from $N(0, 1)$ for different test intervals. For all DGPs, it can be seen that CvM is stabilizing after 1/4 (or quarterly test interval), implying that if the test interval is too small (less than 1/4), time change $T_{i\Delta}$ is erroneous and cannot be used to test the hypothesis. However, if the test interval is appropriately seletected, the time change works as expected. More specifically, we can see that BM requires smaller test interval than RSSVM and ASVM in order to reach the stabilized level of CvM. This is because BM has constant volatility while RSSVM and ASVM have stochastic volatilities. In other words, the time change error is bigger even with the same test interval if the volatility process of the DGP is stochastic. Moreover, we can see that ASVM requires greater test interval than RSSVM in order to achieve the stabilized level. This implies that as the volatility of a DGP have more fluctuations, the time change error gets larger, requiring larger test interval.

Overall, all the time change statistics have satisfactory size performances if the test interval is greater than 1/2 (or average half year quadratic variation). In fact, one can check that they have good size performances once the test interval is over some required minimum test interval. This suggests that one may use the signature of CvM for different test interval to find the minimum required level of test interval.

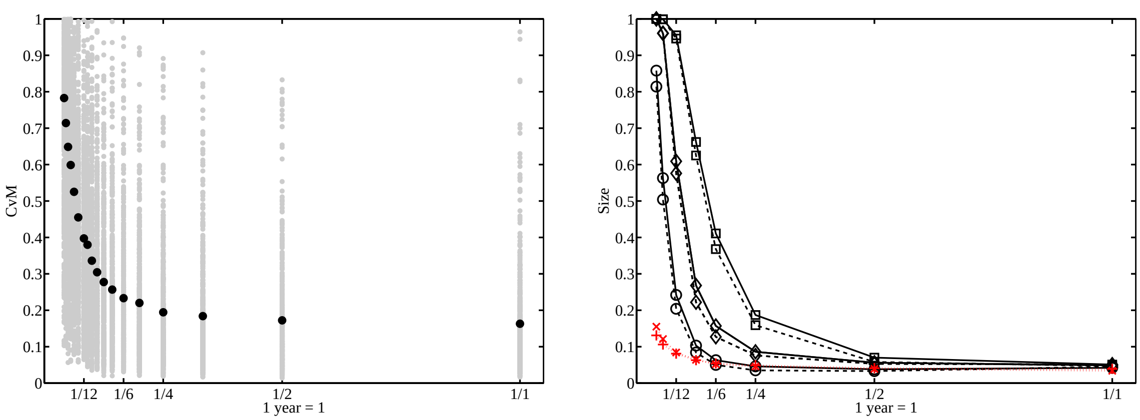
In Table VIII, we report the empirical power (size adjusted) of the tests at 5%



(a) Brownian Motion



(b) Regime Switching Stochastic Volatility Martingale



(c) Asymmetric Stochastic Volatility Martingale

Fig. 6. Signature Plots and Size Performances

Note: The above figures present the signature plots and empirical sizes for test intervals from 1/12 to 1. The black dots in the signature plots represents the mean level of 5,000 iterations. D_1-solid, W_1^2-dashed, $I_{1,2}$-solid circle, $I_{1,3}$-dashed circle, J_1-solid square, J_2-dashed square, $V_{N/15}$-solid diamond, $V_{N/10}$-dashed diamond, HL-dotted x, and EM-dotted cross.

Table VIII. Power Performances of Tests

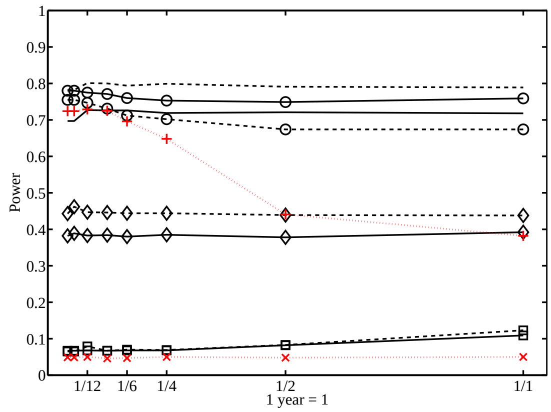
	BD			RSSV			ASV			OU			SQ		
dt	M	Q	Y	M	Q	Y	M	Q	Y	M	Q	Y	M	Q	Y
D_1	72.7	71.9	71.8	68.2	72.6	69.3	36.8	43.7	39.4	0.1	0.0	0.1	0.0	0.2	0.3
W_1^2	80.1	79.9	78.9	75.0	78.4	76.7	45.5	50.2	44.2	0.0	0.0	0.0	0.0	0.0	0.1
W_2^2	79.4	75.6	72.1	77.5	73.5	65.5	50.9	44.9	35.0	0.0	0.0	0.0	0.0	0.1	0.1
$I_{1,2}$	77.5	75.3	75.9	79.5	77.7	78.7	31.5	65.7	65.6	1.1	1.4	1.1	1.1	2.6	2.3
$I_{1,3}$	74.7	70.2	67.4	76.7	73.9	72.1	31.3	68.5	66.2	1.2	1.5	1.1	1.4	3.1	1.6
$I_{1,4}$	65.8	65.2	59.7	51.0	69.2	67.5	20.7	59.2	63.6	1.8	1.8	1.7	0.0	1.8	2.3
J_1	6.8	6.8	10.9	8.6	12.9	18.4	11.9	16.2	18.5	3.6	4.9	7.2	0.0	3.9	7.7
J_2	7.9	6.9	12.3	9.7	13.2	19.6	12.9	17.7	20.9	4.1	4.9	8.2	0.0	3.9	8.5
J_3	7.5	7.2	12.2	10.4	13.1	19.8	14.9	18.8	21.7	3.9	5.1	8.2	0.0	4.5	8.9
$V_{N/15}$	38.3	38.5	39.2	54.3	54.6	53.1	43.8	47.9	48.5	20.8	21.1	21.1	17.5	16.7	17.3
$V_{N/10}$	44.7	44.4	43.8	57.5	57.1	56.1	43.5	46.6	47.3	26.4	25.5	23.8	21.0	20.5	21.2
$V_{N/5}$	54.8	54.7	54.4	59.0	58.3	57.9	38.5	41.4	41.2	32.9	32.9	32.4	27.3	27.0	26.0
HL	5.0	5.0	5.0	4.8	5.5	5.0	5.0	5.0	5.0	4.1	5.5	6.7	3.7	5.5	9.0
EM	72.9	64.8	38.2	71.6	32.7	30.8	18.1	18.3	16.1	0.9	2.3	1.0	0.0	0.0	0.0

Note: Empirical powers (adjusted empirical sizes) of the data-generating processes; BD, RSSV, ASV, OU, and SV. W_m^2 and D_m are the Cramer-von Mises and Kolmogorov-Smirnov tests of goodness-of-fit, $I_{m,P}$ is the smooth tests, J_m is the invariant test, and V_q is the variance ratio test. Here m denotes the dimension of the vector process z_{i_m} constructed from the normal samples. Δ is selected as the average monthly, quarterly, and yearly quadratic variation for each process. HL is the generalized spectral test of Hong and Lee (2005) with preliminary lag order 5, and EM is the data-driven smooth tests of Escanciano and Mayoral (2007). For HL and EM, we use the increments of the process at monthly, quarterly, and yearly frequency. Number of iterations is 5,000 and $\delta = 1/240$, $T = 50$. The numbers are proportion of rejection in percentage unit.

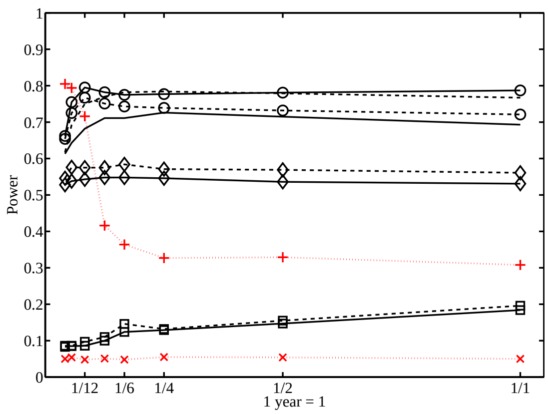
significance level. We observe that the time change tests have distinctive power for BD, RSSV, and ASV, however they have no power against stationary alternatives, such as OU and SQ, except for the variance ratio tests. The variance ratio tests are powerful against all the alternatives, and in fact, they are the most powerful for ASV, OU, and SQ. BD, RSSV, and ASV are nonstationary processes which contain time trend (or deterministic drift). Hence, the test samples (Z_i) have non-zero mean, which is easily detected by the Goodness-of-fit, smooth tests, and invariant tests. However, OU and SQ are stationary processes which generates (Z_i) with zero mean with variance close to 1. This produces considerable loss of power for those tests because the distribution of (Z_i) is very close to $N(0,1)$. On the other hand, the variance ratio tests can detect the deviations of first two moments from $N(0,1)$ as well as the deviation from independence in (Z_i). As a result, the variance ratio tests are the most powerful for OU and SQ. EM test is also powerful for BD, RSSV, and ASV while it loses power for OU and SQ. Meanwhile HL test has no power against all the DGPs that we consider in the simulations.

Figure 7 presents the power performances at different test intervals. We can see that for most test intervals, (i) the Goodness-of-fit tests are the most powerful for BD and RSSV, (ii) the time change smooth tests are the most powerful for ASV, and (iii) the variance ratio tests are the most powerful for OU and SQ. Note that with stochastic volatility models, the selection of test interval is more important because wrong choice of test interval may lead to a considerable loss of power; see the result for the time change smooth tests for ASV.

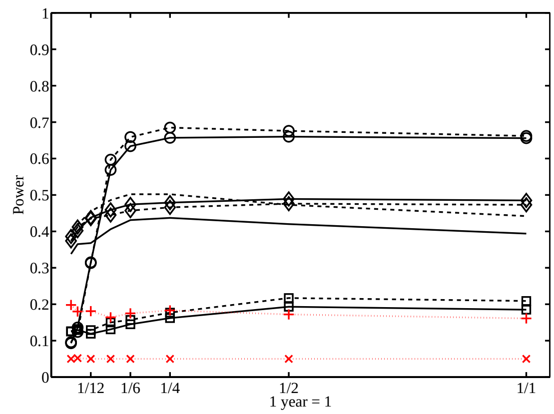
The conclusions from the simulation experiments are the following. The time change test statistics have a satisfactory size performances for most test interval and presents excellent power properties against the DGPs considered, being the most powerful test in most cases. HL test has a good size performances, however it has



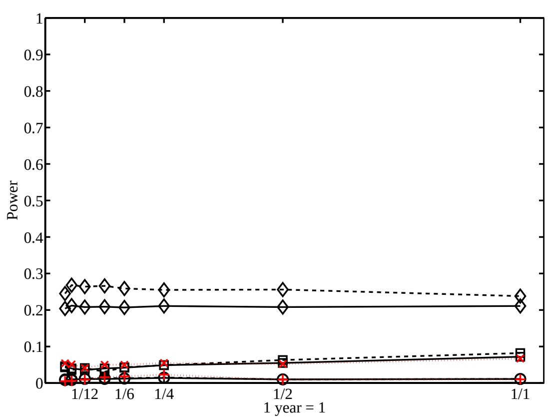
(a) Brownian Motion with Drift



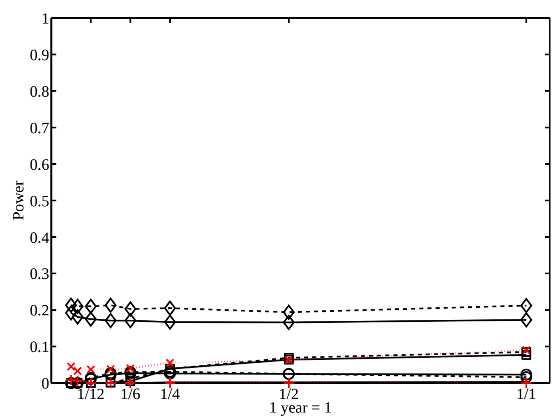
(b) Regime Switching Stochastic Volatility



(c) Asymmetric Stochastic Volatility



(d) Ornstein-Uhlenbeck (1930)



(e) Feller (1951)'s Square Root

Fig. 7. Power Performances

Note: The above figures present the empirical powers for test interval from 1/12 to 1. D_1-solid, W_1^2-dashed, $I_{1,2}$-solid circle, $I_{1,3}$-dashed circle, J_1-solid square, J_2-dashed square, $V_{N/15}$-solid diamond, $V_{N/10}$-dashed diamond, HL-dotted x, and EM-dotted cross.

no power for all the DGPs considered. EM test has a reasonable size performance with moderate sample sizes and have reasonable power for mean-shift alternatives. Additionally, the time change variance ratio tests have excellent power properties against stationary alternatives where most tests lose distinctive power. Our time change tests have satisfactory size and power performances if the test interval is greater than the level which the time change error is relatively large, and one can choose such test interval based on the signature plot. HL test seems irrelavant to the test interval, however, the performance of EM test is sensitive to the test interval (or equivalently to the sample size).

G. Conclusion

In this paper, we have investigated a methodology to test the martingale hypothesis in continuous time framework. The key feature of our methodology is that it utilize the time change theorem (Dambis (1965), Dubins and Schwarz (1965)) to obtain the samples for DDS Brownian motion, which results in a simple testing of *i.i.d.* normality. In order to test the *i.i.d.* normality, we incorporate the classical multivariate normality tests, such as goodness-of-fit tests, smooth tests, invariant tests, and variance ratio tests. Necessary modifications–weak convergence theory on the dependent process–have been introduced in order to apply the testing problem to time series applications. A Monte Carlo simulation study shows that the time change test statistics have a satisfactory size performances for most test interval and presents excellent power properties against the DGPs considered, being the most powerful test in most cases. In particular, the time change variance ratio tests have excellent power properties against stationary alternatives where most tests lose distinctive power.

CHAPTER IV

CONCLUSION

In Chapter II, I began with a title asking if there is an important role played by decision makers' concern with ambiguity on true probability measure. My answer to this question is positive from both economic and econometric perspectives. In terms of economic theory, the inclusion of ambiguity aversion can overcome the Ellsberg paradox. In addition, one can view that a multiple-priors utility as an extension of the rational expectation in that investors may be of insufficient knowledge about the true probability density. When ambiguity aversion is assumed, economic agents are basically endowed with a set of beliefs on the true probability distribution and choose the one that is the least ambiguous. My estimation results strongly suggest that this is indeed the case. Even with various specifications, the preference parameter indicating the ambiguity aversion is both economically and statistically significant. Another interesting finding is that the models with ambiguity aversion have lower relative risk aversion. With regard to the elasticity of substitution, there exists a weak identification problem due to its non-linear parametric restrictions and the weak signal from consumption growth. That said, the models with ambiguity aversion still produce quite reasonable estimates of the intertemporal substitution. Therefore, ambiguity aversion not only matters in terms of explaining the behaviors of asset returns, but also helping identify key preference parameters.

In Chapter III, I have investigated a methodology to test the martingale hypothesis in continuous time framework. It utilize the time change theorem (Dambis (1965), Dubins and Schwarz (1965)) to get the samples for DDS Brownian motion, which results in a simple testing of *i.i.d.* normality. I incorporate the classical multivariate normality tests, such as goodness-of-fit tests, smooth tests, invariant tests, and variance ratio tests to test the *i.i.d.* normality of the time changed samples. From a Monte Carlo simulation study, I show that the time change test statistics have a satisfactory size performances for most test interval and presents excellent power properties against the DGPs considered, being the most powerful test in most cases.

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APPENDIX A

MATHEMATICAL APPENDIX

The proofs of Lemmas 2.2, 2.3, 4.1 and 5.1 are essentially the same as those in Park (2008).

Lemma A Let X be a general diffusion process $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$ satisfying the Assumption 1-3. Then the time changed process $Y_s = X_{T_s}$ will have the stochastic differential equation in quadratic variation clock as

$$dY_s = \frac{\mu(Y_s)}{\sigma(Y_s)^2}ds + dW_s,$$

where W_s is the DDS Brownian motion.

Proof of Lemma A Let $Y_s = X_{T_s}$, then we can write

$$Y_s - Y_0 = X_{T_s} - X_{T_0} = \int_{T_0}^{T_s} \mu(X_t)dt + \int_{T_0}^{T_s} \sigma(X_t)dB_t.$$

Put $\nu(t) = [X]_t$. Now consider a transformation with respect to the quadratic variation $\nu(t)$, such that $\nu(T_s) = s$. It is absolutely continuous, and the time derivative $\nu'(t)$ is strictly positive from Assumption 2.1. Due to the integration by substitution with respect to $u = \nu(t)$, the drift is given by

$$\int_{T_0}^{T_s} \mu(X_t)dt = \int_0^s \mu(X_{\nu^{-1}(u)})\frac{\partial \nu^{-1}(u)}{\partial u}du. \tag{A.1}$$

Observe that

$$\frac{\partial \nu^{-1}(u)}{\partial u} = \frac{1}{\partial \nu(t)/\partial t}|_{t=\nu^{-1}(u)} = \frac{1}{\sigma(X_{\nu^{-1}(u)})^2}. \tag{A.2}$$

Recalling that $\nu^{-1}(u) = \tau(u)$, substitute (A.2) into (A.1) then we have

$$\int_{T_0}^{T_s} \mu(X_t)dt = \int_0^s \frac{\mu(Y_u)}{\sigma(Y_u)^2}du.$$

Moreover, due to Dambis (1965) and Dubins and Schwarz (1965), we have that

$$\int_{T_0}^{T_s} \sigma(X_t)dB_t = \int_0^s dW_u.$$

Lemma B Let Assumptions 2.1 and 4.1 hold. Then for $1 \leq m \leq N$ we have

$$\sup_{z \in R^m} \sqrt{N_m} |F_N^\delta(z) - F_N(z)| = o_p(1),$$

where $F_N^\delta(z) = 1/N_m \sum_{i=1}^{N_m} 1\{Z_{i_m}^\delta \leq z\}$ and $z \in R^m$.

Proof of Lemma B Let

$$e_N = \sup_{1 \leq i \leq N} |Z_{i_m}^\delta - Z_{i_m}|.$$

Then we have

$$\begin{aligned}
\sqrt{N_m}|F_N^\delta(x) - F_N(x)| &\leq \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} |1\{Z_{i_m}^\delta \leq x\} - 1\{Z_{i_m} \leq x\}| \\
&\leq \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} 1\{|Z_{i_m} - x| \leq e_N\} \\
&= \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} (1\{|Z_{i_m} - x| \leq e_N\} - \mathbb{P}\{|Z_{i_m} - x| \leq e_N\}) \\
&\quad + \sqrt{N_m} \mathbb{P}\{|Z_{i_m} - x| \leq e_N\},
\end{aligned}$$

where $N_m = N - m + 1$, $x \in R^m$, and $1\{Z_{i_m} \leq x\} = 1\{Z_i \leq x_1\} \cdots 1\{Z_{i+m-1} \leq x_m\}$. First we show that

$$\sup_x \sqrt{N} \mathbb{P}\{|Z_{i_m} - x| \leq e_N\} = o_p(1). \tag{A.3}$$

Define $p_N = \mathbb{P}\{|Z_{i_m} - x| \leq e_N\}$. Note that for all $x \in R^m$

$$\mathbb{P}\{|Z_{i_m} - x| \leq e_N\} = F_0(x + e_N) - F_0(x - e_N) = O_p(e_N) = o_p(N^{-1/2}),$$

where $F_0(\cdot)$ is the c.d.f of $N(0, I_m)$. Then, the state result in Equation (A.3) is followed. Next, we show that

$$\sup_x \left| \frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} (1\{|Z_{i_m} - x| \leq e_N\} - p_N) \right| = o_p(1), \tag{A.4}$$

Denote $w_{N,i} = 1\{|Z_{i_m} - x| \leq e_N\} - p_N$, then for any small $\epsilon > 0$ we have

$$\begin{aligned}
\mathbb{P}\{\sqrt{N}|w_{N,i}| > \epsilon\} &= (1 - p_N) 1\{\epsilon < \sqrt{N} p_N\} + p_N 1\{\epsilon < \sqrt{N}(1 - p_N)\} \\
&= O_p(p_N) \\
&= o_p(1),
\end{aligned}$$

because $\sqrt{N} p_N = o_p(1)$ and $\sqrt{N}(1 - p_N) \to \infty$. Therefore, $w_{N,i} = o_p(N^{-1/2})$, and this implies that $1/\sqrt{N} \sum_{i=1}^n w_{N,i} = o_p(1)$ for all x. Then, the stated result in Equation (A.4) will be followed.

Proof of Theorem 4.2

From Lemma B, we can expect that for all $x \in R^m$,

$$\sqrt{N_m}|F_N^\delta(x) - F_N(x)| = o_p(1).$$

Now, the multivariate Kolmogorov-Smirnov statistic D_m^δ from discrete observations will converge to D_m from continuous observations in the sense that

$$\begin{aligned}
|D_m^\delta - D_m| &= \sqrt{N_m}\left|\max_{x\in R^m}|F_N^\delta(x) - F_0(x)| - \max_{x\in R^m}|F_N(x) - F_0(x)|\right| \\
&\le \sqrt{N_m}\left|\max_{x\in R^m}|F_N^\delta(x) - F_N(x)| + \max_{x\in R^m}|F_N(x) - F_0(x)| - \max_{x\in R^m}|F_N(x) - F_0(x)|\right| \\
&= \sqrt{N_m}\max_{x\in R^m}|F_N^\delta(x) - F_N(x)| \\
&= o_p(1).
\end{aligned}$$

Similarly, the Cramer-von Mises statistic $W_m^{2,\delta}$ will converge to W_m^2 because

$$\begin{aligned}
|W_m^{2,\delta} - W_m| &= N_m\left|\int_{R^m}\left(F_N^\delta(x) - F_0(x)\right)^2 dF_0(x) - \int_{R^m}(F_N(x) - F_0(x))^2\, dF_0(x)\right| \\
&= N_m\left|\int_{R^m}\left(F_N^\delta(x) - F_N(x)\right)^2 dF_0(x)\right. \\
&\quad \left. + 2\int_{R^m}\left(F_N^\delta(x) - F_N(x)\right)(F_N(x) - F_0(x))\, dF_0(x)\right| \\
&\le N_m\int_{R^m}\left(F_N^\delta(x) - F_N(x)\right)^2 dF_0(x) \\
&\quad +2N_m\int_{R^m}\left(F_N^\delta(x) - F_N(x)\right)(F_N(x) - F_0(x))\, dF_0(x) \\
&\le N_m\int_{R^m}\left(F_N^\delta(x) - F_N(x)\right)^2 dF_0(x) \\
&\quad +2\sqrt{N_m\int_{R^m}\left(F_N^\delta(x) - F_N(x)\right)^2 dF_0(x)}\sqrt{N_m\int_{R^m}(F_N(x) - F_0(x))^2\, dF_0(x)} \\
&= o_p(1).
\end{aligned}$$

For the convergence of the smooth tests, it is sufficient to show that

$$\left|\frac{1}{\sqrt{N_m}}\sum_{i=1}^{N_m}\prod_{j=1}^{m}H_{s_j}(z_{i+j-1}^\delta) - \frac{1}{\sqrt{N_m}}\sum_{i=1}^{N_m}\prod_{j=1}^{m}H_{s_j}(z_{i+j-1})\right| = o_p(1). \tag{A.5}$$

Consider the Taylor series expansion of $H_r(z_i^\delta)$ with respect to z_i given by

$$H_r(z_i^\delta) = H_r(z_i) + \sum_{k=1}^{\infty}\frac{1}{k!}H_r^{(k)}(z_i)(z_i^\delta - z_i)^k,$$

where $H_r^{(k)}(\cdot)$ is the kth order derivative of $H_r(\cdot)$. Note that

$$\prod_{j=1}^{m} H_{s_j}(z_{i+j-1}^{\delta}) = \prod_{i=1}^{m} H_{s_j}(z_{i+j-1}) + \sum_{k=1}^{\infty}\sum_{j=1}^{m} a_{j,k}(z_{i+j-1}^{\delta} - z_{i+j-1})^k,$$

where $a_{j,k} < \infty$ is the appropriate coefficient for the equality. This implies that

$$\left|\prod_{j=1}^{m} H_{s_j}(z_{i+j-1}^{\delta}) - \prod_{j=1}^{m} H_{s_j}(z_{i+j-1})\right| = o_p(N^{-1/2}).$$

The stated result in Equation (A.5) is immediately followed by the above Equation.

In order to show the convergence of the invariant test, we will use the result of Lemma B. Especially, we will show that

$$\sup_i |r_i^{\delta} - r_i| = o_p(N^{-1/2}),$$

which implies the convergence of the invariant test. From the definition of r_i given in Equation (3.10), for any $m < \infty$, we have

$$\begin{aligned}
\left|r_i^{\delta} - r_i\right| &= \left|\sum_{j=1}^{m}\left((z_{i+j-1}^{\delta})^2 - z_{i+j-1}^2\right)\right| \\
&= \left|\sum_{j=1}^{m}\left(z_{i+j-1}^{\delta} - z_{i+j-1}\right)\left(z_{i+j-1}^{\delta} + z_{i+j-1}\right)\right| \\
&\leq \max_{j=1,\ldots,m} |z_{i+j-1}^{\delta} - z_{i+j-1}| \sum_{j=1}^{m}\left(z_{i+j-1}^{\delta} + z_{i+j-1}\right) \\
&= o_p(N^{-1/2})O_p(1) \\
&= o_p(N^{-1/2}).
\end{aligned}$$

From Lemma B, we have

$$\sup_{x\in R^+} \sqrt{N_m}\left(G_N^{\delta}(x) - G_N(x)\right) = o_p(1).$$

Therefore, we have

$$
\begin{aligned}
|J_m^\delta - J_m| &= N_m \left| \int_0^\infty \left(G_N^\delta(x) - \Psi(x)\right)^2 d\Psi(x) - \int_0^\infty (G_N(x) - \Psi(x))^2 d\Psi(x) \right| \\
&= N_m \left| \int_0^\infty \left(G_N^\delta(x) - G_N(x)\right)^2 d\Psi(x) \right. \\
&\quad \left. + 2\int_0^\infty \left(G_N^\delta(x) - G_N(x)\right)(G_N(x) - \Psi(x))\, d\Psi(x) \right| \\
&\leq N_m \int_0^\infty \left(G_N^\delta(x) - G_N(x)\right)^2 d\Psi(x) \\
&\quad +2N_m \int_{R^m} \left(G_N^\delta(x) - G_N(x)\right)(G_N(x) - \Psi(x))\, d\Psi(x) \\
&\leq N_m \int_0^\infty \left(G_N^\delta(x) - G_N(x)\right)^2 d\Psi(x) \\
&\quad +2\sqrt{N_m \int_0^\infty \left(G_N^\delta(x) - G_N(x)\right)^2 d\Psi(x)}\sqrt{N_m \int_0^\infty (G_N(x) - \Psi(x))^2 d\Psi(x)} \\
&= o_p(1).
\end{aligned}
$$

Proof of Theorem 4.3 Due to Theorem 4.2, we assume that the test statistics based on the discrete observations are close to the statistics based on continuous observations. We first prove the weak convergence of two empirical processes given by $b_N(t)$ and $w_N(t)$ in (a) and (c) respectively. Then, we prove the result in (b).

The weak convergence of $b_N(t)$ and $w_N(t)$ is proved by showing (i) convergence in distribution for given $t \in [0,1]$, (ii) finite dimensional convergence, and (iii) stochastic equicontinuity of the empirical processes (see Andrews (1993) for detail). We start by considering the empirical process $b_N(t)$. Define $c_i(t) = 1\{Z_{i_m} \leq t\} - \mathbb{E}[1\{Z_{i_m} \leq t\}]$. Note that $c_i(t)$ is m-dependent process in the sense that the correlation between $c_i(t)$ and $c_{i+j}(t)$ is zero if $j \geq m$ while it is nonzero if $0 \leq j < m$. From the usual central limit theorem, we have

$$
\frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} c_i(t) \to_d N(0, \sigma_c^2),
$$

where $\sigma_c^2 = \lim_{N\to\infty} \mathbb{V}[\frac{1}{\sqrt{N_m}} \sum_{i=1}^{N_m} c_i(t)]$. The finite dimensional convergence is followed by the multivariate central limit theorem and the covariance kernel will be given by

$$
k_B(s,t) = \lim_{N\to} \mathbb{E}[b_N(s) b_N(t)].
$$

The stochastic equicontinuity of $b_N(t)$ can be shown by checking the three sufficient conditions for the stochastic equicontinuity of m-dependent case presented by Andrews (1993, p. 199). The first condition is obvious because z_{i_m} is m-dependent. The second condition is satisfied because

$$
\max_{1\leq d\leq m} \sup_{t\in[0,1]^m} |1\{\Phi(Z_i) \leq t_1\} \cdots 1\{\Phi(Z_{i+m-1}) \leq t_m\}| \leq 1,
$$

for all $i = 1, \ldots, N_m$. Therefore, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}[\bar{M}^{2+\delta}] = 1 < \infty,$$

for some $\delta > 0$. The third condition is also satisfied because for all $t \in [0,1]^m$,

$$\begin{aligned} \sup_{1\le i\le N, N\ge 1} & \left\{ \mathbb{E}\left[\sup_{t_1\in[0,1]^m:|t_1-t|<\delta} |1\{Z_{i_m} \le t_1\} - 1\{Z_{i_m} \le t\}|^p \right] \right\}^{1/p} \\ \le & \int_{t_1-\delta^{1/m}}^{t_1+\delta^{1/m}} \cdots \int_{t_m-\delta^{1/m}}^{t_m+\delta^{1/m}} dU(t) \\ \le & C\delta \end{aligned}$$

for all $\delta > 0$ small and $U(t) = t_1 \cdots t_m$, and from lhopital's theorem. Therefore, we have

$$b_N(t) \Rightarrow B(t). \tag{A.6}$$

The case for $w_N(t)$ is exactly same as the case for $b_N(t)$ except that the empirical process $w_N(t)$ has domain of $[0,1]$. Since r_i is also m-dependent process, we can apply the central limit theorems to find the finite dimensional convergence. Showing stochastic equicontinuity in this case is similar. Note that for all $t \in [0,1]$, we have

$$\begin{aligned} \sup_{1\le i\le N, N\ge 1} \left\{ \mathbb{E}\left[\sup_{t_1\in[0,1]^m:|t_1-t|<\delta} |1\{r_i \le t_1\} - 1\{r_i \le t\}|^p \right] \right\}^{1/p} & \le \int_{t-\delta}^{t+\delta} du(t) \\ & \le C\delta \end{aligned}$$

for all $\delta > 0$ small and $u(t) = t$. Therefore, we have

$$w_N(t) \Rightarrow W(t),$$

where the covariance kernel of $W(t)$ is given by

$$k_W(s,t) = \lim_{N\to\infty} \mathbb{E}[w_N(s) w_N(t)]. \tag{A.7}$$

From Equation (A.6) and (A.7), the asymptotic distributions of D_m, W_m^2, and J_m are followed by the continuous mapping theorem.

The asymptotic distribution of $I_{m,P}$ largely relies on the property of Hermite polynomial. Especially, the Hermite polynomials are orthogonal with respect to the standard normal distribution

$$\int_{-\infty}^{\infty} H_i(x) H_j(x) \phi(x) dx = \begin{cases} 0 & \text{if } i \ne j \\ i! & \text{if } i = j \end{cases}$$

Each component $H_{m,p}$ of $I_{m,P}$ defined by Equation (3.9) is a quadratic form of the vector $h_{m,p}$ of a normalized sample analogue estimator for $\mathbb{E}[\prod_{j=1}^{m} H_{s_j}(z_{i+j-1})]$ with $\sum_{j=1}^{m} s_j = p$ while $s_1 \ne 0$. If we rearrange all the elements of the vector as an increasing order of s_1, and label them as h. Then, there will be $q(p) = \binom{m+p-1}{p} - \binom{m+p-2}{p}$ such elements and each elements are asymptotically independent because of the independence of (z_i) and the orthogonality of the Hermite polynomials $H_r(\cdot)$. Therefore, the quadratic form of $h_{m,p}$ will have the chi-square distribution with degrees of freedom $q(p)$. Moreover, it is easy to verify that all the elements in $h_{m,p}$ is also asymptotically independent with the elements in $h_{m,q}$ with $q \neq p$. Therefore, we have

$$I_{m,p} \to_d \chi\left(\sum_{p=1}^{P} q(p)\right). \tag{A.8}$$

Proof of Theorem 5.2 The stated result follows immediately from Lemma B.

VITA

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